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HIGH ORDER HOMOGENIZED STOKES MODELS CAPTURE ALL THREE REGIMES*

FLORIAN FEPPON[†] AND WENJIA JING[‡]

Abstract. This article is a sequel to our previous work [13] concerned with the derivation of high-order homogenized models for the Stokes equation in a periodic porous medium. We provide an improved asymptotic analysis of the coefficients of the higher order models in the low-volume fraction regime whereby the periodic obstacles are rescaled by a factor η which converges to zero. By introducing a new family of order k corrector tensors with a controlled growth as $\eta \rightarrow 0$ uniform in $k \in \mathbb{N}$, we are able to show that both the infinite order and the finite order models converge in a coefficient-wise sense to the three classical asymptotic regimes. Namely, we retrieve the Darcy model, the Brinkman equation or the Stokes equation in the homogeneous cubic domain depending on whether η is respectively larger, proportional to, or smaller than the critical size $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$. For completeness, the paper first establishes the analogous results for the perforated Poisson equation, considered as a simplified scalar version of the Stokes system.

Key words. Homogenization, higher order models, perforated Poisson problem, Stokes system, low volume fraction asymptotics, strange term.

AMS subject classifications. 35B27, 76M50, 35330

1. Introduction. The homogenization of the Stokes system has attracted a lot of attention recently, regarding random or complex domains [17, 10], extensions to inhomogeneous viscosity or different kinds of boundary conditions [7, 16, 15], and new unified and quantitative homogenization approaches [21, 19] in the periodic setting.

The goal of this paper is to show that higher order effective models provide a unified understanding for the homogenization for the Stokes system in a periodic porous medium:

$$(1.1) \quad \begin{cases} -\Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f} & \text{in } D_\varepsilon \\ \operatorname{div}(\mathbf{u}_\varepsilon) = 0 & \text{in } D_\varepsilon \\ \mathbf{u}_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon \\ (\mathbf{u}_\varepsilon, p_\varepsilon) & \text{is } D\text{-periodic} \end{cases}$$

where $D_\varepsilon = D \setminus \overline{\omega_\varepsilon}$ is a d -dimensional cubic domain $D = (0, L)^d$ perforated with periodic obstacles $\omega_\varepsilon := \varepsilon(\mathbb{Z}^d + \eta T) \cap D$ (represented on Figure 1) and the right-hand side $\mathbf{f} \in C_{\text{per}}^\infty(D, \mathbb{R}^d)$ is a smooth D -periodic vector field. D_ε is the union of periodic cells of size $\varepsilon := L/N$ where $N \in \mathbb{N}$ is a large integer. Each cell contains an obstacle $\varepsilon\eta T$ where $\eta > 0$ is a rescaling of the obstacles. This parameter η allows to consider the so-called low volume fraction regime corresponding to the situation where the obstacles disappear at a rate $\eta \rightarrow 0$ which possibly depends on ε . We assume the total fluid domain D_ε to be connected, as well as the fluid component $Y = P \setminus (\eta T)$

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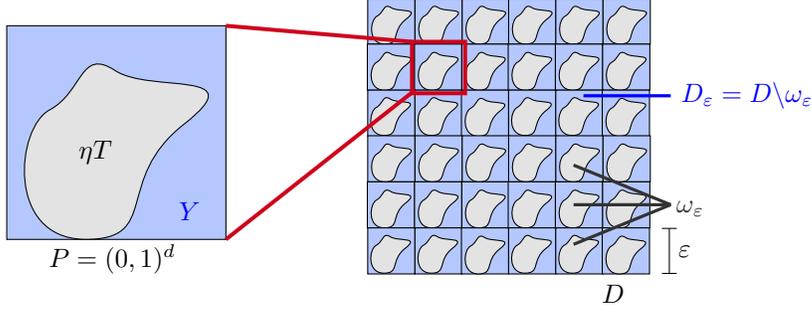


FIG. 1. The perforated domain $D_\varepsilon = D \setminus \overline{\omega_\varepsilon}$ and the unit cell $Y = P \setminus (\eta\overline{T})$.

34 of the rescaled unit cell $P := (-1/2, 1/2)^d$. The first assumption ensures that the
 35 pressure variable p_ε of (1.1) is uniquely determined up to a single additive constant
 36 while the second is used when considering cell problems in Y . For simplicity, the
 37 domain is assumed to be at least three-dimensional: $d \geq 3$.

38 In [13], we have derived a formal “infinite-order” homogenized system for (1.1)
 39 which reads in terms of averaged velocity and pressure $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ as

$$40 \quad (1.2) \quad \begin{cases} \sum_{k=0}^{+\infty} \varepsilon^{k-2} M^k \cdot \nabla^k \mathbf{u}_\varepsilon^* + \nabla p_\varepsilon^* = \mathbf{f} \text{ in } D \\ \operatorname{div}(\mathbf{u}_\varepsilon^*) = 0 \text{ in } D \\ (\mathbf{u}_\varepsilon^*, p_\varepsilon^*) \text{ is } D\text{-periodic.} \end{cases}$$

In (1.2), $(M^k)_{k \in \mathbb{N}}$ is a family of matrix valued tensors which can be explicitly constructed by a procedure involving cell problems that we review below, and k denotes the order of the tensor M^k . For a given $k \in \mathbb{N}$, $M^k \cdot \nabla^k$ is the differential operator defined for any $\mathbf{v} \in C^\infty(\mathbb{R}, \mathbb{R}^d)$ by

$$(M^k \cdot \nabla^k \mathbf{v})_l := M_{i_1 \dots i_k, lm}^k \partial_{i_1 \dots i_k}^k v_m$$

41 where we assume the Einstein summation convention over the repeated indices $1 \leq$
 42 $i_1 \dots i_k \leq d$ and $1 \leq l, m \leq d$.

43 In order to obtain effective models suitable for numerical computations, we have
 44 proposed a truncation procedure for (1.2) inspired from [27]. For any integer $K \in \mathbb{N}$,
 45 it yields a well-posed higher order homogenized model of *finite* order $2K + 2$, which
 46 reads

$$47 \quad (1.3) \quad \begin{cases} \sum_{k=0}^{2K+2} \varepsilon^{k-2} \mathbb{D}_K^k \cdot \nabla^k \mathbf{v}_{\varepsilon, K}^* + \nabla q_{\varepsilon, K}^* = \mathbf{f} \text{ in } D \\ \operatorname{div}(\mathbf{v}_{\varepsilon, K}^*) = 0 \text{ in } D \\ (\mathbf{v}_{\varepsilon, K}^*, q_{\varepsilon, K}^*) \text{ is } D\text{-periodic,} \end{cases}$$

48 where the coefficients $(\mathbb{D}_K^k)_{0 \leq k \leq 2K+2}$ is another family of matrix valued tensors. The
 49 system (1.3) is indeed a truncated version of (1.2) because the first half of the coefficients coincide, namely $\mathbb{D}_K^k = M^k$ for $0 \leq k \leq K$. The remaining higher order
 50 coefficients $(\mathbb{D}_K^k)_{K+1 \leq k \leq 2K+2}$ are in general different from $(M^k)_{K+1 \leq k \leq 2K+2}$; they
 51

52 ensure that (1.3) is well-posed. It is then possible to show that, for a fixed $\eta > 0$,
 53 $\mathbf{v}_{\varepsilon,K}^*$ and $q_{\varepsilon,K}^*$ yield approximations of \mathbf{u}_ε and p_ε at orders $O(\varepsilon^{K+3})$ and $O(\varepsilon^{K+1})$ in
 54 the $L^2(D_\varepsilon)$ norm respectively. Similar results hold for the Laplace problem with a
 55 smooth periodic right-hand side $f \in \mathcal{C}_{\text{per}}^\infty(D)$,

$$56 \quad (1.4) \quad \begin{cases} -\Delta u_\varepsilon = f & \text{in } D_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon \\ u_\varepsilon & \text{is } D\text{-periodic,} \end{cases}$$

57 which we considered in [12]. In fact, it turns out that in scalar context of (1.4), free of
 58 the divergence constraint, the approximation error on the solution u_ε committed by
 59 the homogenized model of order $2K + 2$ improves rather surprisingly up to the order
 60 $O(\varepsilon^{2K+4})$.

61 Still in [13], we have analyzed the asymptotic behaviors of the tensors M^k and
 62 \mathbb{D}_K^k in the low volume fraction regime $\eta \rightarrow 0$. Assuming $d \geq 3$ for simplicity, we have
 63 found (see Corollary 5.5 of this reference)

$$64 \quad (1.5) \quad M^0 \sim \eta^{d-2} F$$

$$65 \quad (1.6) \quad M^1 = o(\eta^{d-2})$$

$$66 \quad (1.7) \quad M^2 \rightarrow -I$$

$$67 \quad (1.8) \quad \forall k \geq 2, M^{2k} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right),$$

$$68 \quad (1.9) \quad \forall k \geq 1, M^{2k+1} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right),$$

69 as well as equivalent results for the tensors (\mathbb{D}_K^k) . The first result (1.5) has been known
 70 since the work of Allaire on the continuity of the Darcy equation [3], it involves a $d \times d$
 71 dimensional matrix $F \equiv (F_{ij})_{1 \leq i,j \leq d}$ which can be retrieved by solving an exterior
 72 problem in $\mathbb{R}^d \setminus T$ (the definition is recalled in (4.10) below). In the scalar case, the
 73 same results hold with F being replaced by the capacity $\text{Cap}(\partial T)$ of the obstacle.
 74

75 The motivation for seeking these asymptotics in [13] was to investigate whether
 76 the high order models (1.2) and (1.3) have the potential to unify the three classical ho-
 77 mogenized regimes acknowledged by the literature. Standard homogenization theory
 78 [26, 24, 9, 2, 4, 1, 5, 22, 23] states that $(\mathbf{u}_\varepsilon, p_\varepsilon)$ (or a suitable rescaling) converges in
 79 some sense to the solution (\mathbf{u}^*, p^*) to three possible limit equations as $\varepsilon \rightarrow 0$, depend-
 80 ing on how η compares with respect to the critical size $\eta_{\text{crit}} := \varepsilon^{2/(d-2)}$. The limiting
 81 equation is either the Darcy, the Brinkman or the Stokes model in the homogeneous
 82 domain D .

83 As far as we are concerned with the present periodic setting, we can read from
 84 (1.5)–(1.9), the following coefficient-wise convergences of (1.2) (or (1.3)) as $\eta \rightarrow 0$ and
 85 $\varepsilon \rightarrow 0$:

- 86 • if $1 \gg \eta \gg \varepsilon^{2/(d-2)}$, namely the holes are large, then the limiting equation
 87 for $(\eta^{d-2}\varepsilon^{-2}\mathbf{u}_\varepsilon, p_\varepsilon)$ is the Darcy problem

$$88 \quad (1.10) \quad \begin{cases} F\mathbf{u}^* + \nabla p^* = \mathbf{f} & \text{in } D \\ \text{div}(\mathbf{u}^*) = 0 & \text{in } D \\ \mathbf{u}^* & \text{is } D\text{-periodic;} \end{cases}$$

- 89 • if $\eta \sim c\varepsilon^{2/(d-2)}$, namely the holes are exactly proportional to the critical
 90 diameter $a_{\text{crit}} := \eta_{\text{crit}}\varepsilon = \varepsilon^{d/(d-2)}$, then the limiting equation for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ is
 91 the Brinkman problem

$$92 \quad (1.11) \quad \begin{cases} -\Delta \mathbf{u}^* + cF\mathbf{u}^* + \nabla p^* = \mathbf{f} & \text{in } D \\ \operatorname{div}(\mathbf{u}^*) = 0 & \text{in } D \\ (\mathbf{u}^*, p^*) \text{ is } D\text{-periodic,} \end{cases}$$

93 where in both (1.10) and (1.11), F is the matrix appearing in (1.5).
 94 The coefficient-wise convergence of (1.2) towards either (1.10) and (1.11) is consistent
 95 with the literature which asserts that the solutions (\mathbf{u}^*, p^*) to either (1.10) or (1.11)
 96 is the limit of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ in the corresponding regimes. This allowed us to conclude
 97 in [13] that the high order homogenization process captures both the Darcy and the
 98 Brinkman regimes (1.10) and (1.11).

99 Finally, the literature states that in the subcritical regime $\eta = o \ll \varepsilon^{2/(d-2)}$,
 100 $(\mathbf{u}_\varepsilon, p_\varepsilon)$ converges in some sense, as $\varepsilon \rightarrow 0$, to the solution (\mathbf{u}^*, p^*) of the Stokes
 101 equation in the homogeneous domain D (without holes):

$$102 \quad (1.12) \quad \begin{cases} -\Delta \mathbf{u}^* + \nabla p^* = \mathbf{f} & \text{in } D \\ \operatorname{div}(\mathbf{u}^*) = 0 & \text{in } D \\ (\mathbf{u}^*, p^*) \text{ is } D\text{-periodic.} \end{cases}$$

103 Intuitively, this means that when $\eta \ll \varepsilon^{2/(d-2)}$, the holes are too small to be actu-
 104 ally sensed by the effective model. However, the analysis that we performed in [11]
 105 is not sufficient to retrieve this result as a coefficient-wise convergence of the higher
 106 order models (1.2) or (1.3) to the homogeneous Stokes system (1.12). Indeed, al-
 107 though (1.5)–(1.7) allows to infer that the right convergence holds for the first three
 108 coefficients $M^0\varepsilon^{-2}$, $M^1\varepsilon^{-1}$ and M^2 , the asymptotic bounds (1.8) and (1.9) only en-
 109 able to obtain that the coefficient $\varepsilon^{2k-2}M^{2k}$ is bounded when $k \geq 2$ by the quantity
 110 $(\varepsilon^{2/(d-2)}/\eta)^{(k-1)(d-2)}$ which grows to infinity as $\eta \rightarrow 0$.

111 In this perspective, the purpose of this article is to propose a different asymptotic
 112 analysis of [13] which allows to substantially improve the asymptotic convergences of
 113 (1.5)–(1.9). Our main results are stated in Corollary 4.6 and Proposition 4.10 where
 114 we obtain that in fact, $M^k \rightarrow 0$ and $\mathbb{D}_K^k \rightarrow 0$ for any $k > 2$ with a convergence rate
 115 not bigger than $O(\eta^{d-2})$. This implies in particular the coefficient-wise convergence
 116 of the high-order models (1.2) and (1.3) towards the Stokes equation (1.12) not only
 117 in the subcritical regime $\eta = o(\varepsilon^{2/(d-2)})$ as $\varepsilon \rightarrow 0$, but also in the situation where the
 118 size of the periodic cell ε (and so their number) is *fixed* while the holes disappear as
 119 $\eta \rightarrow 0$.

120 All in all, this paper demonstrates that at least in the sense of coefficient-wise
 121 convergence, the effective models (1.2) and (1.3) have indeed the potential to yield high
 122 order homogenized approximations of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ that are valid in all possible regimes
 123 of size of holes. A more formal statement would require to improve the error bounds
 124 of [13] involving \mathbf{u}_ε and \mathbf{u}_ε^* , so as to obtain error results with bounding constants
 125 uniform with respect to η . We expect this could be done by using e.g. the unified
 126 approach proposed in [18] in the context of the homogenization of the Poisson system;
 127 a precise treatment is left for future works.

128 For completeness and in a pedagogical purpose, we prove the results first in the
 129 context of the Laplace problem (1.4), which can be considered as a simplified scalar

130 version of the full Stokes system (1.1). In a second part, we shall state how the results
 131 actually extend to (1.1) with an emphasis on the differences that occur due to the
 132 vectorial context and to the zero divergence constraint.

133 The paper outlines as follows. Notation conventions and the definitions of various
 134 families of tensors (including M^k and \mathbb{D}_K^k) related to the high order homogenization
 135 process are reviewed in section 2 for both the Poisson equation (1.4) and the Stokes
 136 system (1.1). Section 3 provides our new asymptotic analysis for the tensors M^k and
 137 \mathbb{D}_K^k in the context of the Poisson equation (1.4). Treating first the scalar case allows
 138 us to highlight the key arguments in a simplified setting, namely the introduction of
 139 a new family of cell tensors $(\mathcal{Y}^k(y))_{k \in \mathbb{N}}$ in $P \setminus (\eta T)$ whose averages $(\mathcal{Y}^{k*})_{k \in \mathbb{N}}$ remain of
 140 the same order $O(\eta^{2-d})$ uniformly in $k \in \mathbb{N}$ (Proposition 3.3). Finally, the Stokes case
 141 is treated in section 4. The main differences of the asymptotic analysis are related to
 142 the vectorial setting and the presence of the pressure, which require to consider vector
 143 and matrix valued tensors rather than scalar tensors. Furthermore, the asymptotic
 144 analysis of the coefficients \mathbb{D}_K^k requires an additional treatment due to the fact that,
 145 in contrast with the scalar case, half of the coefficients (for $K + 1 \leq k \leq 2K + 1$) do
 146 not coincide with the corresponding tensors M^k .

147 **2. Setting, notation and review of available results.** In this section, we
 148 review the notation conventions used for tensors and the definitions of the tensors
 149 $(M^k)_{k \in \mathbb{N}}$ and $(\mathbb{D}_K^k)_{0 \leq k \leq 2K+2}$ in both contexts of the Poisson equation (1.4) and the
 150 Stokes system (1.1). Both situations involve the solutions of partial differential equa-
 151 tions posed in the perforated unit cell $Y = P \setminus (\eta T)$ where $P = (-1/2, 1/2)^d$, T is an
 152 obstacle centered in the cell (i.e. $0 \in T$) and $\eta > 0$ is the rescaling. When consid-
 153 ering the low-volume fraction regime $\eta \rightarrow 0$, we also assume that the hole is strictly
 included in the cell for $0 < \eta \leq 1$: $T \subset P$. The setting is illustrated on Figure 2.

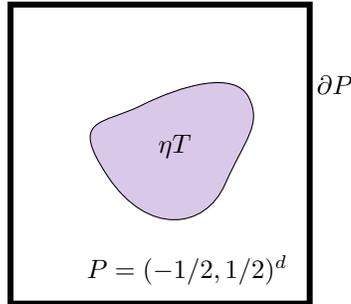


FIG. 2. Schematic of the cell P and the obstacle ηT .

154

155 **2.1. Notation conventions.** In the whole paper, we use the same notation
 156 conventions for tensor related operations as in our previous works [12, 13]. These are
 157 summarized in the nomenclature below. These notations allow us to systematically
 158 avoid writing indices for partial derivatives (e.g. $1 \leq i_1 \dots i_k \leq d$), and to distinguish
 159 them from spatial indices (e.g. $1 \leq l, m \leq d$) associated with vector or matrix
 160 components.

161 We recall that unless otherwise specified, the Einstein summation convention
 162 over repeated *subscript* indices is assumed (but never on *superscript* indices). Vectors
 163 $\mathbf{b} \in \mathbb{R}^d$ are written in bold face notation.

164 **Scalar, vector, and matrix valued tensors and their coordinates**

165	$\mathbf{b} \equiv (b_j)_{1 \leq j \leq d}$	Vector of \mathbb{R}^d
166	b^k	Scalar valued tensor of order k ($b_{i_1 \dots i_k}^k \in \mathbb{R}$ for $1 \leq i_1, \dots, i_k \leq d$)
167	\mathbf{b}^k	Vector valued tensor of order k ($\mathbf{b}_{i_1 \dots i_k}^k \in \mathbb{R}^d$ for $1 \leq i_1, \dots, i_k \leq d$)
168	B^k	Matrix valued tensor of order k ($B_{i_1 \dots i_k}^k \in \mathbb{R}^{d \times d}$ for $1 \leq i_1, \dots, i_k \leq d$)
169		
170	$(b_j^k)_{1 \leq j \leq d}$	Coordinates of the vector valued tensor \mathbf{b}^k (b_j^k is a <i>scalar</i> tensor of order k).
171		
172	$(B_{lm}^k)_{1 \leq l, m \leq d}$	Coefficients of the matrix valued tensor B^k (B_{lm}^k is a <i>scalar</i> tensors of order k).
173		
174	$b_{i_1 \dots i_k, j}^k$	Coefficient of the vector valued tensor \mathbf{b}^k ($1 \leq i_1, \dots, i_k, j \leq d$).
175	$B_{i_1 \dots i_k, lm}^k$	Coefficients of the matrix valued tensor B^k ($1 \leq i_1, \dots, i_k, l, m \leq d$).
176		

177 **Tensor products**

178	$b^p \otimes c^{k-p}$	Tensor product of scalar tensors b^p and c^{k-p} :
179	(2.1)	$(b^p \otimes c^{k-p})_{i_1 \dots i_k} := b_{i_1 \dots i_p}^p c_{i_{p+1} \dots i_k}^{k-p}$.
180	$B^p \otimes C^{k-p}$	Tensor product of matrix valued tensors B^p and C^{k-p} :
181	(2.2)	$(B^p \otimes C^{k-p})_{i_1 \dots i_k, lm} := B_{i_1 \dots i_p, lj}^p C_{i_{p+1} \dots i_k, jm}^{k-p}$.
182		Hence a matrix product is implicitly assumed in the notation $B^p \otimes C^{k-p}$.
183		
184	$\mathbf{b}^p \cdot \mathbf{c}^{k-p}$	Tensor product and inner product of vector valued tensors \mathbf{b}^p and \mathbf{c}^{k-p} :
185		

$$(2.3) \quad (\mathbf{b}^p \cdot \mathbf{c}^{k-p})_{i_1 \dots i_k} := b_{i_1 \dots i_p, m}^p c_{i_{p+1} \dots i_k, m}^{k-p}.$$

187 $B^p \cdot \mathbf{c}^{k-p}$ Tensor product of a matrix tensor B^p and a vector tensors \mathbf{c}^{k-p} :

$$(2.4) \quad (B^p \cdot \mathbf{c}^{k-p})_{i_1 \dots i_k, l} := B_{i_1 \dots i_p, lm}^p c_{i_{p+1} \dots i_k, m}^{k-p}.$$

189 Hence a matrix-vector product is implicitly assumed in $B^p \cdot \mathbf{c}^{k-p}$.

190 **Contraction with partial derivatives**

191 $b^k \cdot \nabla^k$ Differential operator of order k associated with a scalar tensor b^k :
 192 for any smooth scalar field $v \in C_{per}^\infty(D, \mathbb{R}^d)$,

$$(2.5) \quad b^k \cdot \nabla^k v := b_{i_1 \dots i_k}^k \partial_{i_1 \dots i_k}^k v.$$

194 $\mathbf{b}^k \cdot \nabla^k$ Differential operator of order k associated with a vector tensor \mathbf{b}^k :
 195 for any smooth vector field $\mathbf{v} \in C_{per}^\infty(D, \mathbb{R}^d)$,

$$(2.6) \quad \mathbf{b}^k \cdot \nabla^k \mathbf{v} = b_{i_1 \dots i_k, l}^k \partial_{i_1 \dots i_k}^k v_l.$$

197 $B^k \cdot \nabla^k$ Differential operator of order k associated with a matrix valued tensor B^k : for any smooth vector field $\mathbf{v} \in C_{per}^\infty(D, \mathbb{R}^d)$,

$$(2.7) \quad (B^k \cdot \nabla^k \mathbf{v})_l = B_{i_1 \dots i_k, lm}^k \partial_{i_1 \dots i_k}^k v_m.$$

200 In (2.5)–(2.7) above, the reader may equivalently think $\nabla^k v$ and
 201 $\nabla^k \mathbf{v}$ as scalar valued and vector valued tensors of order k and the
 202 dot \cdot notation as the contraction operator of two order k tensors.

203 **Special tensors**

- 204 $(e_j)_{1 \leq j \leq d}$ Vectors of the canonical basis of \mathbb{R}^d .
 205 e_j Scalar valued tensor of order 1 given by $e_{j,i_1} := \delta_{i_1 j}$ (with $1 \leq j \leq d$).
 206 δ_{ij} Kronecker symbol: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.
 207 I Scalar-valued identity tensor of order 2:

$$I_{i_1 i_2} = \delta_{i_1 i_2}.$$

207 The identity tensor is another notation for the Kronecker tensor and
 208 it holds $I = e_j \otimes e_j$ with summation on the index $1 \leq j \leq d$. With a
 209 small abuse of notation and when the context is clear, we also denote
 210 by I the *matrix-valued* second order tensor $I \equiv (I_{i_1 i_2, lm})_{1 \leq i_1, i_2, l, m \leq d}$
 211 defined by

212 (2.8)
$$I_{i_1 i_2, lm} := \delta_{i_1 i_2} \delta_{lm}.$$

213 This notation is used in (1.7), (4.18), and (4.32).

214 In the whole paper, we consider zeroth order tensors which are scalar, vector or
 215 matrices devoid of partial derivative indices; e.g. $b^0 \in \mathbb{R}$ if b^0 is scalar, $\mathbf{b}^0 \in \mathbb{R}^d$ if
 216 $\mathbf{b}^0(y)$ is a vector field, and so on. Then the various possible tensor products involving
 217 of a zero-th order tensor make sense and follow the same conventions as in eqn. (2.1)
 218 to (2.4).

Since a k -th order tensor b^k (scalar, vector or matrix valued) truly makes sense when contracted with k partial derivatives, as in (2.5)–(2.7), all the tensors considered throughout this work are identified to their symmetrization:

$$b_{i_1 \dots i_k}^k \equiv \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} b_{i_{\sigma(1)} \dots i_{\sigma(k)}}^k,$$

219 where \mathfrak{S}_k is the permutation group of order k . Consequently, the order in which the
 220 derivative indices i_1, \dots, i_k are written in $b_{i_1 \dots i_k}^k$ does not matter. This alleviates the
 221 need for specifying the order of the indices in tensor product notations such as in
 222 (2.15) below.

In the paper, the star-“*”- symbol is used to indicate that a quantity is “macroscopic” in the sense that it does not depend on the fast variable x/ε ; such as $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ or $(\mathbf{v}_{\varepsilon, K}^*, q_{\varepsilon, K}^*)$ in (1.2) and (1.3). In the particular case where a quantity $\mathcal{X}(y)$ is given as a P -periodic function of $Y = P \setminus (\eta T)$ extended by 0 on the obstacle $\partial(\eta T)$, then \mathcal{X}^* denotes the average of $y \mapsto \mathcal{X}(y)$ with respect to the y variable:

$$\mathcal{X}^*(y) := \int_P \mathcal{X}(y) dy = \int_{P \setminus (\eta T)} \mathcal{X}(y) dy.$$

At some places we find occasionally more convenient to write the cell average with the more usual angle bracket symbols:

$$\langle \mathcal{X} \rangle := \int_P \mathcal{X}(y) dy.$$

223 Finally, we write C or C_K to denote universal constants that do not depend on ε
 224 or η but whose values may be redefined from lines to lines.

225 *Remark 2.1.* In a limited number of places, the superscript or subscript indices
 226 $p, q \in \mathbb{N}$ are used. Naturally, these are not to be confused with the pressure variables
 227 p_ε or $q_{\varepsilon, K}^*$ introduced in (1.1) and (1.3).

228 *Remark 2.2.* In all what follows, the various tensors coming at play such as \mathcal{X}^k ,
 229 \mathcal{X}^{k*} , M^k , \mathbb{D}_K^k etc., depend on the scaling of the obstacle η , but this dependence is
 230 made implicit for notational simplicity.

231 **2.2. High order effective models for the perforated Poisson equation.**
 232 For the Poisson equation, the homogenized equations of respectively “infinite” order
 233 and of order $2K + 2$ read respectively

$$234 \quad (2.9) \quad \begin{cases} \sum_{k=0}^{+\infty} \varepsilon^{2k-2} M^{2k} \cdot \nabla^{2k} u_\varepsilon^* = f \text{ in } D \\ u_\varepsilon^* \text{ is } D\text{-periodic,} \end{cases}$$

235

$$236 \quad (2.10) \quad \begin{cases} \sum_{k=0}^{K+1} \varepsilon^{2k-2} \mathbb{D}_K^{2k} \cdot \nabla^{2k} v_{\varepsilon, K}^* = f \text{ in } D \\ v_{\varepsilon, K}^* \text{ is } D\text{-periodic.} \end{cases}$$

237 Note that in this scalar context, (2.9) and (2.10) feature no *odd* order differential
 238 operators, i.e. $M^{2k+1} = 0$ and $\mathbb{D}_K^{2k+1} = 0$. The coefficients $(M^k)_{k \in \mathbb{N}}$ and $(\mathbb{D}_K^k)_{k \in \mathbb{N}}$
 239 are defined by a procedure involving cell tensors $(\mathcal{X}^k(y))_{k \in \mathbb{N}}$ and $(N^k(y))_{k \in \mathbb{N}}$ with
 240 $y \in Y$.

241 **DEFINITION 2.3.** *The cell tensors $(\mathcal{X}^k(y))_{k \in \mathbb{N}}$ are defined recursively as the solu-*
 242 *tions to the following cascade of equations:*

$$243 \quad (2.11) \quad \begin{cases} -\Delta \mathcal{X}^0 = 1 \text{ in } P \setminus (\eta T) \\ -\Delta \mathcal{X}^1 = 2\partial_j \mathcal{X}^0 \otimes e_j \text{ in } P \setminus (\eta T) \\ -\Delta \mathcal{X}^{k+2} = 2\partial_j \mathcal{X}^{k+1} \otimes e_j + \mathcal{X}^k \otimes I \text{ in } P \setminus (\eta T), \quad k \geq 0 \\ \mathcal{X}^k = 0 \text{ on } \partial(\eta T), \quad k \geq 0 \\ \mathcal{X}^k \text{ is } P\text{-periodic.} \end{cases}$$

244 We then denote by \mathcal{X}^{k*} the average of the tensor field \mathcal{X}^k :

$$245 \quad (2.12) \quad \mathcal{X}^{k*} := \int_{P \setminus (\eta T)} \mathcal{X}^k(y) dy.$$

Remark 2.4. Owing to our notation convention of subsection 2.1, the third equa-
 tion of (2.11) can be equivalently written

$$\begin{aligned} -\Delta \mathcal{X}_{i_1 \dots i_{k+2}}^{k+2} &= 2\partial_j \mathcal{X}_{i_1 \dots i_{k+1}}^{k+1} \delta_{j i_{k+2}} + \mathcal{X}_{i_1 \dots i_k}^k \delta_{i_{k+1} i_{k+2}} \\ &= 2\partial_{i_{k+2}} \mathcal{X}_{i_1 \dots i_{k+1}}^{k+1} + \mathcal{X}_{i_1 \dots i_k}^k \delta_{i_{k+1} i_{k+2}}. \end{aligned}$$

246 In particular, the repeated index k in the equation is not summed over, but the
 247 repeated index j is.

248 DEFINITION 2.5. *The family of constant scalar tensors $(M^k)_{k \in \mathbb{N}}$ is defined by the*
 249 *following recursive formula*

$$250 \quad (2.13) \quad M^k := \begin{cases} (\mathcal{X}^{0*})^{-1} & \text{if } k = 0 \\ -(\mathcal{X}^{0*})^{-1} \sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^p & \text{if } k \geq 1. \end{cases}$$

251 The definition (2.13) is valid because the tensor $\mathcal{X}^{0*} = \int_Y |\nabla \mathcal{X}^0|^2 dy > 0$ is a positive
 252 number; it rewrites equivalently as

$$253 \quad (2.14) \quad \sum_{p=0}^k \mathcal{X}^{p*} \otimes M^{k-p} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

254 For $k \geq 1$, M^k can be computed by the following explicit formula, see [12] (Proposition
 255 6):

$$256 \quad (2.15) \quad M^k = \sum_{p=1}^k (-1)^p \sum_{\substack{i_1 + \dots + i_p = k \\ 1 \leq i_1 \dots i_p \leq k}} (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1}.$$

257 In [12] (Proposition 3), we have found that the definitions (2.11)–(2.13) imply that
 258 the odd order coefficient tensors vanish, namely $\mathcal{X}^{2k+1*} = 0$ and $M^{2k+1} = 0$ for all
 259 $k \geq 0$.

260 The Cauchy product of the tensors $(M^k)_{k \in \mathbb{N}}$ and $(\mathcal{X}^k(y))_{k \in \mathbb{N}}$ then yields an ad-
 261 ditional and important family of cell tensors $(N^k(y))_{k \in \mathbb{N}}$.

262 DEFINITION 2.6. *For any $k \in \mathbb{N}$, we define the k -th order cell tensor N^k by*

$$263 \quad (2.16) \quad N^k(y) := \sum_{p=0}^k M^p \otimes \mathcal{X}^{k-p}(y), \quad y \in Y.$$

264 Remark 2.7. Equation (2.14) states that the averages of the tensors $(N^k)_{k \in \mathbb{N}}$ are
 265 given respectively by

$$266 \quad (2.17) \quad \int_Y N^k(y) dy = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

267 The tensors $(N^k(y))_{k \in \mathbb{N}}$ allow to reconstruct the oscillating solution u_ε of (1.4) from
 268 its high order homogenized approximations u_ε^* or $v_{\varepsilon,K}^*$ given by (2.9) and (2.10).
 269 Indeed, the following identity holds at least in a formal sense,

$$270 \quad (2.18) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k u_\varepsilon^*(x), \quad x \in D_\varepsilon$$

and likewise, we proved in [12] (Corollary 5) that the reconstructed function

$$W_{\varepsilon,2K+1}(v_{\varepsilon,K}^*) := \sum_{k=0}^{2K+1} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k v_{\varepsilon,K}^*(x), \quad x \in D_\varepsilon$$

271 approximates u_ε up to a remainder of order $O(\varepsilon^{2K+4})$ in the $L^2(D_\varepsilon)$ norm. The
 272 identity (2.18) relating u_ε to u_ε^* is somewhat remarkable. We have called it a “criminal
 273 ansatz” based on similar observations which hold in the context of the conductivity
 274 or wave equation [8, 6].

275 Finally, the tensors $(N^k(y))_{k \in \mathbb{N}}$ determine the coefficients $(\mathbb{D}_K^k)_{0 \leq k \leq 2K+2}$ of (2.10)
 276 (see [12], Proposition 13).

277 **DEFINITION 2.8.** *For any $K \geq 0$ and $0 \leq k \leq 2K + 2$, the coefficient \mathbb{D}_K^k is*
 278 *defined by:*

$$279 \quad (2.19) \quad \mathbb{D}_K^k = M^k \text{ for any } 0 \leq k \leq 2K + 1$$

$$280 \quad (2.20) \quad \mathbb{D}_K^{2K+2} = (-1)^{K+1} \int_Y N^K(y) \otimes N^K(y) \otimes Idy$$

282 where $N^K(y)$ is the cell tensor given by (2.16).

283 **2.3. High order effective models for the Stokes system in a porous**
 284 **medium.** The construction of the tensors $(M^k)_{k \in \mathbb{N}}$ and $(\mathbb{D}_K^k)_{0 \leq k \leq 2K+2}$ for the effec-
 285 tive Stokes systems (1.2) and (1.3) follow the same construction as in the scalar case,
 286 up to the following differences:

- 287 1. due to the vectorial nature of \mathbf{u}_ε , the tensors M^k , \mathbb{D}_K^k , $\mathcal{X}^k(y)$, \mathcal{X}^{k*} and
 288 $N^k(y)$ become *matrix valued*. They include therefore k partial derivatives
 289 indices $i_1 \dots i_k$, and two spatial indices $1 \leq l, m \leq d$ which follow the notation
 290 conventions of subsection 2.1;
- 291 2. the presence of the pressure p_ε and of the divergence constraint $\operatorname{div}(\mathbf{u}_\varepsilon) = 0$
 292 in (1.1) reflects in the introduction of *vector valued* tensorial pressure fields
 293 $\boldsymbol{\alpha}^k(y)$, $\boldsymbol{\beta}^k(y)$ coming along $\mathcal{X}^k(y)$ and $N^k(y)$. The vector valued tensors
 294 $\boldsymbol{\alpha}^k(y)$ and $\boldsymbol{\beta}^k(y)$ are therefore characterized by k partial derivative indices
 295 $1 \leq i_1 \dots i_k \leq d$ and one spatial index $1 \leq l \leq d$.

296 The starting point is the definition of the solution tensors $(\mathcal{X}^k(y), \boldsymbol{\alpha}^k(y))$ to a hierarchy
 297 of Stokes systems analogous to (2.11):

298 **DEFINITION 2.9.** *For any $k \geq 0$, we define respectively the vector valued tensors*
 299 *$(\mathcal{X}_j^k(y))_{1 \leq j \leq d}$ and the scalar valued tensors $(\alpha_j^k(y))_{1 \leq j \leq d}$ to be the unique solutions*
 300 *in $H_{\text{per}}^1(Y, \mathbb{R}^d) \times L^2(Y)$ to the following cell problems:*

$$301 \quad (2.21) \quad \begin{cases} -\Delta_{yy} \mathcal{X}_j^0 + \nabla_y \alpha_j^0 = \mathbf{e}_j \text{ in } Y, \\ \operatorname{div}_y(\mathcal{X}_j^0) = 0 \text{ in } Y \end{cases}$$

$$302 \quad (2.22) \quad \begin{cases} -\Delta_{yy} \mathcal{X}_j^1 + \nabla_y \alpha_j^1 = (2\partial_l \mathcal{X}_j^0 - \alpha_j^0 \mathbf{e}_l) \otimes \mathbf{e}_l \text{ in } Y \\ \operatorname{div}_y(\mathcal{X}_j^1) = -(\mathcal{X}_j^0 - \langle \mathcal{X}_j^0 \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \text{ in } Y, \end{cases}$$

$$303 \quad (2.23) \quad \begin{cases} -\Delta_{yy} \mathcal{X}_j^{k+2} + \nabla_y \alpha_j^{k+2} = (2\partial_l \mathcal{X}_j^{k+1} - \alpha_j^{k+1} \mathbf{e}_l) \otimes \mathbf{e}_l + \mathcal{X}_j^k \otimes I \text{ in } Y \\ \operatorname{div}_y(\mathcal{X}_j^{k+2}) = -(\mathcal{X}_j^{k+1} - \langle \mathcal{X}_j^{k+1} \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \text{ in } Y \end{cases} \quad \forall k \geq 0,$$

304
 305 *supplemented with the following boundary conditions:*

$$306 \quad (2.24) \quad \begin{cases} \int_Y \alpha_j^k dy = 0 \\ \mathcal{X}_j^k = 0 \text{ on } \partial(\eta T) \\ (\mathcal{X}_j^k, \alpha_j^k) \text{ is } P\text{-periodic} \end{cases} \quad \forall k \geq 0.$$

The k -th order matrix valued tensor field $\mathcal{X}^k(y)$ is then assembled by gathering the d vector valued tensors $(\mathcal{X}_j^k(y))_{1 \leq j \leq d}$ into columns:

$$\mathcal{X}^k(y) := \begin{bmatrix} \mathcal{X}_1^k(y) & \dots & \mathcal{X}_d^k(y) \end{bmatrix}, \quad \forall y \in Y, \quad \forall k \geq 0,$$

or in other words $\mathcal{X}_{ij}^k(y) = \mathcal{X}_j^k(y) \cdot e_i$. Similarly, we define $\alpha^k(y)$ to be the k -th order vector valued tensor whose coordinates are the scalar tensors $\alpha_j^k(y)$:

$$\alpha^k(y) := (\alpha_j^k(y))_{1 \leq j \leq d}, \quad \forall y \in Y, \quad \forall k \geq 0.$$

307 Following (2.12), the matrix valued tensor \mathcal{X}^{k*} is then defined as the average of $\mathcal{X}^k(y)$
 308 over the perforated cell:

$$309 \quad (2.25) \quad \mathcal{X}^{k*} := \int_Y \mathcal{X}^k(y) dy, \quad \forall k \geq 0.$$

310 Note that by the definition (2.24), $\alpha^k(y)$ is of zero average for any $k \geq 0$. Similarly,
 311 the porosity matrix \mathcal{X}^{0*} is known to be symmetric definite positive [26]. Therefore,
 312 the following definition makes sense:

313 DEFINITION 2.10. *The family of matrix valued tensors $(M^k)_{k \in \mathbb{N}}$ is defined by the*
 314 *following recursive formula:*

$$315 \quad (2.26) \quad \begin{cases} M^0 = (\mathcal{X}^{0*})^{-1} \\ M^k = -(\mathcal{X}^{0*})^{-1} \sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^p, \quad \forall k \geq 1. \end{cases}$$

316 Note that in contrast with the scalar case, matrix products take place between the
 317 tensors \mathcal{X}^{k-p*} and M^p in (2.26). The explicit formula (2.15) still holds under the
 318 same convention.

319 Contrarily to the scalar case, odd order tensors \mathcal{X}^{2k+1*} and M^{2k+1} do not vanish
 320 in general (they do in case the obstacle ηT is symmetric with respect to the cell axes).
 321 Instead, we find the following symmetry properties (Proposition 3.5 of [13]):

322 PROPOSITION 2.11. *The k -th order tensors \mathcal{X}^{k*} and M^k are symmetric and an-*
 323 *tisymmetric matrix valued for respectively even and odd values of k ; i.e.*

$$324 \quad \begin{aligned} \mathcal{X}_{i_1 \dots i_k, lm}^{2k*} &= \mathcal{X}_{i_1 \dots i_k, ml}^{2k*}, & M_{i_1 \dots i_k, lm}^{2k} &= M_{i_1 \dots i_k, ml}^{2k}, \\ \mathcal{X}_{i_1 \dots i_k, lm}^{2k+1*} &= -\mathcal{X}_{i_1 \dots i_k, ml}^{2k+1*}, & M_{i_1 \dots i_k, lm}^{2k+1} &= -M_{i_1 \dots i_k, ml}^{2k+1}, \end{aligned}$$

327 where $1 \leq i_1, \dots, i_k \leq d$ and $1 \leq l, m \leq d$ denote respectively the partial derivative
 328 and the spatial indices.

329 From the Cauchy product of M^k and $\mathcal{X}^k(y)$, we define matrix and vector valued
 330 cell tensors $N^k(y)$ and $\beta^k(y)$ (Proposition 3.9 in [13]).

DEFINITION 2.12. *For any $k \in \mathbb{N}$, let N^k and β^k be respectively the k -th order*
matrix valued and vector valued tensors defined by

$$N^k(y) := \sum_{p=0}^k \mathcal{X}^{k-p}(y) \otimes M^p, \quad \beta^k(y) := \sum_{p=0}^k (-1)^p M^p \cdot \alpha^{k-p}(y), \quad \forall y \in Y.$$

331 Remark that a matrix product and a matrix-vector product take place in the respec-
 332 tive definitions of $N^k(y)$ and $\beta^k(y)$. We have the following property analogous to
 333 (2.17) in this vectorial context:

$$334 \quad (2.27) \quad \int_Y N^k(y) dy = \begin{cases} I & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

335 It is useful to consider $(\mathbf{N}_j^k)_{1 \leq j \leq d}$ and $(\beta_j^k)_{1 \leq j \leq d}$ which are respectively the column
 336 vectors and the coefficients of $N^k(y)$ and $\beta^k(y)$:

$$337 \quad (2.28) \quad \forall 1 \leq i, j \leq d, \mathbf{N}_j^k(y) := N^k(y) \mathbf{e}_j \quad \text{and} \quad \beta_j^k(y) := \beta^k(y) \cdot \mathbf{e}_j, \quad y \in Y.$$

338 Similar to the scalar case, these new tensors allow to reconstruct the oscillating
 339 velocity and pressure $(\mathbf{u}_\varepsilon, p_\varepsilon)$ solutions to (1.1) from their homogenized approxima-
 340 tions $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ or $(\mathbf{v}_{\varepsilon, K}^*, q_{\varepsilon, K}^*)$ given by (1.2) and (1.3). We have indeed the following
 341 formal identities

$$342 \quad (2.29) \quad \begin{cases} \mathbf{u}_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^i N^i(x/\varepsilon) \cdot \nabla^i \mathbf{u}_\varepsilon^*(x) \\ p_\varepsilon(x) = p_\varepsilon^*(x) + \sum_{i=0}^{+\infty} \varepsilon^{i-1} \beta^i(x/\varepsilon) \cdot \nabla^i \mathbf{u}_\varepsilon^*(x), \end{cases} \quad \forall x \in D_\varepsilon.$$

343 Likewise, we proved in [13] that the reconstructed velocity and pressures

$$344 \quad (2.30) \quad \mathbf{W}_{\varepsilon, K}(\mathbf{v}_{\varepsilon, K}^*)(x) := \sum_{k=0}^K \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k \mathbf{v}_{\varepsilon, K}^*(x), \quad x \in D_\varepsilon$$

(2.31)

$$345 \quad Q_{\varepsilon, K-1}(\mathbf{v}_{\varepsilon, K}^*, q_{\varepsilon, K}^*)(x/\varepsilon) := q_{\varepsilon, K}^*(x) + \sum_{k=0}^{K-1} \varepsilon^{k-1} \beta^k(x/\varepsilon) \cdot \nabla^k \mathbf{v}_{\varepsilon, K}^*(x), \quad x \in D_\varepsilon$$

347 yield approximations of \mathbf{u}_ε and p_ε of respective order $O(\varepsilon^{K+3})$ and $O(\varepsilon^{K+1})$ in the
 348 $L^2(D_\varepsilon)$ norm. Unfortunately and in contrast with the scalar case, we do not obtain
 349 an error estimate of order $O(\varepsilon^{2K+4})$ for the velocity as it could have been expected,
 350 because only half of the coefficients \mathbb{D}_K^k obtained from the well-posed truncation
 351 process of [13] turn to be equal to the M^k .

352 The latter coefficients $(\mathbb{D}_K^k)_{0 \leq k \leq 2K+2}$ are indeed given by the following formulas
 353 (Proposition 4.10 of [13]):

354 DEFINITION 2.13. For any $K \geq 0$ and $0 \leq k \leq 2K+2$, the coefficient \mathbb{D}_K^k is
 355 defined by

$$356 \quad (2.32) \quad \mathbb{D}_{K, ij}^k = \begin{cases} M^k & \text{if } 0 \leq k \leq K \\ M^k + \mathbb{A}_K^k & \text{if } K+1 \leq k \leq 2K+1 \\ (-1)^{K+1} \int_Y \mathbf{N}_i^K \cdot \mathbf{N}_j^K \otimes Idy & \text{if } k = 2K+2. \end{cases}$$

357 where the matrix valued tensor \mathbb{A}_K^k is given for any $K+1 \leq k \leq 2K+1$ by

$$358 \quad (2.33) \quad \mathbb{A}_{K, ij}^k := (-1)^{K+1} \int_Y (\nabla \beta_j^{k-K-1} \cdot \mathbf{N}_i^{K+1} + (-1)^k \nabla \beta_i^{k-K-1} \cdot \mathbf{N}_j^{K+1}) dy,$$

359 remembering the definition (2.28) of the vector valued and scalar valued tensors $\mathbf{N}^k(y)$
 360 and $\beta_i^k(y)$.

361 **3. Low volume fraction asymptotic of the high order homogenized**
 362 **Laplace model.** In this section, we are concerned with the scalar context of the
 363 perforated Laplace problem (1.4); the setting is therefore the one considered in sub-
 364 section 2.2. We aim at establishing the coefficient-wise convergence of both higher
 365 order models (2.9) and (2.10) of respectively infinite and finite orders, in the low-
 366 volume fraction regime $\eta \rightarrow 0$, or in other words, the convergence of the tensors M^k
 367 and \mathbb{D}_K^k .

368 The main results of this section are Corollary 3.8 and Proposition 3.12 where we
 369 effectively obtain the asymptotics of these coefficient tensors.

370 **3.1. Cell tensors $\mathcal{Y}^k(y)$ of controlled growth.** The key ingredient which was
 371 missing in our previous analysis [12, 13] is the introduction of a new family of cell
 372 tensors $(\mathcal{Y}^k(y))_{k \in \mathbb{N}}$ of controlled growth with respect to k .

373 DEFINITION 3.1. We define the family of cell tensors $(\mathcal{Y}^k(y))_{k \in \mathbb{N}}$ by induction as
 374 the solutions to the following cascade of equations:

$$375 \quad (3.1) \quad \begin{cases} -\Delta \mathcal{Y}^0 = 1 \text{ in } P \setminus (\eta T) \\ -\Delta \mathcal{Y}^1 = 2\partial_j \mathcal{Y}^0 \otimes e_j \\ -\Delta \mathcal{Y}^{k+2} = 2\partial_j \mathcal{Y}^{k+1} \otimes e_j + (\mathcal{Y}^k - \mathcal{Y}^{k*}) \otimes I \text{ in } P \setminus (\eta T) \\ \mathcal{Y}^k = 0 \text{ on } \partial(\eta T) \\ \mathcal{Y}^k \text{ is } P\text{-periodic.} \end{cases}$$

376 where for any $k \in \mathbb{N}$, we denote by \mathcal{Y}^{k*} the average of these tensors in the unit cell:

$$377 \quad (3.2) \quad \mathcal{Y}^{k*} := \int_{P \setminus (\eta T)} \mathcal{Y}^k(y) dy.$$

378 The benefit of introducing $\mathcal{Y}^k(y)$ lies in the fact that the mean \mathcal{Y}^{k*} remains not bigger
 379 than $O(\eta^{2-d})$ as $\eta \rightarrow 0$ uniformly in $k \in \mathbb{N}$. The proof relies on the following classical
 380 Poincaré estimates in the perforated cell [3, 18] which is recalled in the next lemma.

381 LEMMA 3.2. For any $v \in H^1(P \setminus (\eta T))$ which is P -periodic and vanishes on the
 382 hole $\partial(\eta T)$, the following Poincaré inequality holds:

$$383 \quad (3.3) \quad \|v\|_{L^2(P \setminus (\eta T))} \leq C\eta^{1-d/2} \|\nabla v\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}$$

384 for a constant $C > 0$ independent of η and v . Furthermore, for any $v \in H^1(P \setminus (\eta T))$,
 385 the following Poincaré-Wirtinger inequality holds:

$$386 \quad (3.4) \quad \|v - \langle v \rangle\|_{L^2(P \setminus (\eta T))} \leq C \|\nabla v\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}$$

387 These inequalities entail the following result for the tensors $(\mathcal{Y}^k(y))_{k \in \mathbb{N}}$:

388 PROPOSITION 3.3. For any integer $k \geq 0$, there exists a constant $C_k > 0$ inde-
 389 pendent of η such that

$$390 \quad (3.5) \quad \|\nabla \mathcal{Y}^k\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)} \leq C_k \eta^{1-d/2}$$

$$391 \quad (3.6) \quad \|\mathcal{Y}^k - \mathcal{Y}^{k*}\|_{L^2(P \setminus (\eta T))} \leq C_k \eta^{1-d/2}$$

where, with a little abuse of notation, it is understood that every component $\mathcal{Y}_{i_1 \dots i_k}^k$ with $1 \leq i_1 \dots i_k \leq d$ satisfies (3.5) and (3.6). In addition, there exists a constant $\alpha > 0$ independent of k and η such that

$$0 < C_k < \alpha(1 + \sqrt{2})^k C^k$$

where C is the Poincaré constant of (3.3) and (3.4).

Proof. We proceed by induction.

Case $k = 0$: multiply the first equation of (3.1) by \mathcal{Y}^0 , then integrate by parts to obtain

$$\begin{aligned} \|\nabla \mathcal{Y}^0\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}^2 &= \int_{P \setminus (\eta T)} \mathcal{Y}^0 dy \leq \|\mathcal{Y}^0\|_{L^2(P \setminus (\eta T))} \\ &\leq C\eta^{1-d/2} \|\nabla \mathcal{Y}^0\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}. \end{aligned}$$

Case $k = 1$: multiply the second equation of (3.1) by \mathcal{Y}^1 , then integrate by parts to obtain

$$\begin{aligned} \|\nabla \mathcal{Y}^1\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}^2 &\leq 2\|\nabla \mathcal{Y}^0\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)} \|\mathcal{Y}^1 - \mathcal{Y}^{1*}\|_{L^2(P \setminus (\eta T))} \\ &\leq 2C^2\eta^{1-d/2} \|\nabla \mathcal{Y}^1\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}. \end{aligned}$$

Case $k + 2$ with $k \geq 0$: assuming the result is true till rank $k + 1$, multiply the third equation of (3.1) by \mathcal{Y}^{k+2} , then integrate by parts to obtain

$$\begin{aligned} \|\nabla \mathcal{Y}^{k+2}\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}^2 &\leq (2\|\nabla \mathcal{Y}^{k+1}\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)} + \|\mathcal{Y}^k - \mathcal{Y}^{k*}\|_{L^2(P \setminus (\eta T))}) \|\mathcal{Y}^{k+2} - \mathcal{Y}^{k+2*}\|_{L^2(P \setminus (\eta T))} \\ &\leq (2C_{k+1} + C_k)C\eta^{1-d/2} \|\nabla \mathcal{Y}^{k+2}\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}. \end{aligned}$$

This implies (3.5). Then (3.6) follows from (3.3) and (3.4). \square

Using the Cauchy-Schwarz inequality and (3.5), we can infer from the above result that $|\mathcal{Y}^{k*}| \leq C_k\eta^{2-d}$. The next proposition provides more precise asymptotics for the mean \mathcal{Y}^{k*} (eqn. (3.2)). In particular, we find that in fact, $\mathcal{Y}^{k*} = o(\eta^{2-d})$ for $k \geq 1$.

PROPOSITION 3.4. *The following asymptotic convergences hold for the mean tensors $(\mathcal{Y}^{k*})_{k \in \mathbb{N}}$ as $\eta \rightarrow 0$:*

$$(3.7) \quad \mathcal{Y}^{0*} \sim \frac{\eta^{2-d}}{\text{Cap}(\partial T)} \text{ and } \mathcal{Y}^{k*} = o(\eta^{2-d}) \text{ for } k \geq 1.$$

Proof. Since $\mathcal{Y}^{0*} = \mathcal{X}^{0*}$, the result for $k = 0$ is standard and can be found in [3, 18]. The case $k = 1$ (and in fact for any odd value of k) is trivial since $\mathcal{Y}^{1*} = \mathcal{X}^{1*} = 0$. In order to prove that $\mathcal{Y}^{k+2*} = o(\eta^{2-d})$ for any $k \geq 0$, we follow the lines of the proof of [12], Proposition 14.

Let us denote $\tilde{\mathcal{Y}}^k := \eta^{d-2}\mathcal{Y}^k$ for any $k \geq 0$. Then $\tilde{\mathcal{Y}}^k$ is the solution to

$$(3.8) \quad \begin{cases} -\Delta \tilde{\mathcal{Y}}^{k+2} = 2\eta \partial_j \tilde{\mathcal{Y}}^{k+1} \otimes e_j + \eta^2 (\tilde{\mathcal{Y}}^k - \langle \tilde{\mathcal{Y}}^k \rangle) \otimes I \text{ in } \eta^{-1}P \setminus T \\ \tilde{\mathcal{Y}}^{k+2} = 0 \text{ on } \partial T \\ \tilde{\mathcal{Y}}^{k+2} \text{ is } \eta^{-1}P\text{-periodic,} \end{cases}$$

with $\langle \tilde{\mathcal{Y}}^k \rangle := \eta^d \int_{\eta^{-1}P \setminus T} \tilde{\mathcal{Y}}^k dx$. From the previous proposition, there exists a constant $C > 0$ independent of η such that

$$\|\nabla \tilde{\mathcal{Y}}^{k+2}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)} \leq C \text{ and } |\langle \tilde{\mathcal{Y}}^{k+2} \rangle| \leq C.$$

Hence, up to extracting a subsequence, we may assume the existence of order $k+2$ field and scalar valued tensors $\Psi^{k+2}(x) \in H_{loc}^1(\eta^{-1}P \setminus T)$ and $\gamma^{k+2} \in \mathbb{R}$ such that

$$\tilde{\mathcal{Y}}^{k+2} \rightharpoonup \Psi^{k+2} \text{ weakly in } H_{loc}^1(\mathbb{R}^d \setminus T) \text{ and } \langle \tilde{\mathcal{Y}}^{k+2} \rangle \rightarrow \gamma^{k+2} \text{ as } \eta \rightarrow 0.$$

420 Furthermore, the lower-semi continuity of the $\mathcal{D}^{1,2}(\mathbb{R}^d \setminus T)$ norm (see the proof of
 421 Theorem 3.1 in [3] for a detailed justification) implies that $\Psi^{k+2} - \gamma^{k+2}$ belongs to
 422 $\mathcal{D}^{1,2}(\mathbb{R}^d \setminus T)$. Multiplying (3.8) by a compactly supported test function ϕ , integrating
 423 by part and passing to the limit implies that Ψ^{k+2} is the solution to the exterior
 424 problem

$$425 \quad (3.9) \quad \begin{cases} -\Delta \Psi^{k+2} = 0 & \text{in } \mathbb{R}^d \setminus T \\ \Psi^{k+2} = 0 & \text{on } \partial T \\ \Psi^{k+2} \rightarrow \gamma^{k+2} & \text{at infinity.} \end{cases}$$

426 Therefore $\Psi^{k+2} = \gamma^{k+2} \phi^*$ where ϕ^* is the solution to

$$427 \quad (3.10) \quad \begin{cases} -\Delta \phi^* = 0 & \text{in } \mathbb{R}^d \setminus T \\ \phi^* = 0 & \text{on } \partial T \\ \phi^* \rightarrow 1 & \text{at infinity.} \end{cases}$$

To identify γ^{k+2} , we multiply (3.8) by the constant function 1 and we integrate by part to obtain that

$$0 = - \int_{\partial T} \frac{\partial \tilde{\mathcal{Y}}^{k+2}}{\partial \mathbf{n}} dy$$

because the right-hand side of (3.8) is of average zero. Using now the continuity of the normal flux with respect to the $H_{loc}^1(\mathbb{R}^d \setminus T)$ weak convergence, we obtain by passing to the limit as $\eta \rightarrow 0$:

$$0 = - \lim_{\eta \rightarrow 0} \int_{\partial T} \gamma^{k+2} \frac{\partial \phi^*}{\partial \mathbf{n}} dy = \text{Cap}(\partial T) \gamma^{k+2},$$

428 whence $\gamma^{k+2} = 0$. This implies that the whole sequence $(\langle \tilde{\mathcal{Y}}^{k+2} \rangle)_{\eta > 0}$ converges to
 429 zero, and then (3.7) by rescaling. \square

430 We now find that the tensors $(\mathcal{X}^k(y))_{k \in \mathbb{N}}$ to $(\mathcal{Y}^k(y))_{k \in \mathbb{N}}$ are related by a Cauchy
 431 product identity.

432 **PROPOSITION 3.5.** *The tensors $\mathcal{Y}^k(y)$ can be rewritten in terms of the tensors*
 433 *$\mathcal{X}^k(y)$ and \mathcal{X}^{k*} according to the following recursive formula:*

$$434 \quad (3.11) \quad \begin{cases} \mathcal{Y}^0(y) = \mathcal{X}^0(y) \\ \mathcal{Y}^1(y) = \mathcal{X}^1(y) \\ \mathcal{Y}^k(y) = \mathcal{X}^k(y) - \sum_{l=0}^{k-2} \mathcal{Y}^l(y) \otimes \mathcal{X}^{k-l-2*} \otimes I \text{ for } k \geq 2, \end{cases} \quad y \in P \setminus (\eta T).$$

Proof. Let us denote by $\mathcal{Y}^k(y)$ the tensors defined according to (3.11). We prove that the tensors \mathcal{Y}^k referring to this definition solve the cascade of partial differential equations (3.1), which implies the result by uniqueness. Obviously (3.1) is true for \mathcal{Y}^k with $k = 0$ or $k = 1$. Assuming the third equation is true till $k - 1$ with $k \geq 0$ (with the convention $\mathcal{Y}^{-1} = 0$), we then prove that it still holds at rank k . We compute

$$\begin{aligned}
-\Delta \mathcal{Y}^{k+2} &= -\Delta \mathcal{X}^{k+2} - \sum_{l=0}^k (-\Delta \mathcal{Y}^l) \otimes \mathcal{X}^{k-l*} \otimes I \\
&= 2\partial_j \mathcal{X}^{k+1} \otimes e_j + \mathcal{X}^k \otimes I - \mathcal{X}^{k*} \otimes I - 2\partial_j \mathcal{Y}^0 \otimes e_j \otimes \mathcal{X}^{k-1*} \otimes I \\
&\quad - \sum_{l=2}^k (2\partial_j \mathcal{Y}^{l-1} \otimes e_j + (\mathcal{Y}^{l-2} - \mathcal{Y}^{l-2*}) \otimes I) \otimes \mathcal{X}^{k-l*} \otimes I \\
&= 2\partial_j \left(\mathcal{X}^{k+1} - \sum_{l=0}^{k-1} \mathcal{Y}^l \otimes \mathcal{X}_\eta^{k-l-1*} \otimes I \right) \otimes e_j \\
&\quad + \left(\mathcal{X}^k \otimes I - \mathcal{X}^{k*} \otimes I - \sum_{l=0}^{k-2} (\mathcal{Y}^l - \mathcal{Y}^{l*}) \otimes \mathcal{X}^{k-l-2*} \otimes I \otimes I \right) \\
&= 2\partial_j \mathcal{Y}^{k+1} \otimes e_j + (\mathcal{Y}^k - \mathcal{Y}^{k*}) \otimes I,
\end{aligned}$$

435 which implies the result. \square

436 *Remark 3.6.* Let us comment the consequences of Propositions 3.3 and 3.5. From
437 (3.11), we have obtained, for any $p \in \mathbb{N}$,

$$438 \quad (3.12) \quad \mathcal{X}^{p*} = \mathcal{Y}^{p*} + \sum_{l=0}^{p-2} \mathcal{Y}^{p-2-l*} \otimes \mathcal{X}^{l*} \otimes I.$$

439 Since $\mathcal{Y}^{p-2-l*} = O(\eta^{1-d/2})$, a simple recursive argument, (3.12) yields the following
440 asymptotic for the tensors \mathcal{X}^{2k*} for $k \geq 1$:

$$441 \quad (3.13) \quad \mathcal{X}^{2k*} = \frac{\eta^{(2-d)(k+1)}}{\text{Cap}(\partial T)^{k+1}} \underbrace{I \otimes I \otimes \cdots \otimes I}_{k \text{ times}} + o(\eta^{(2-d)k}), \quad k \geq 1,$$

442 which is a slight quantitative improvement of the result of [12], Proposition 14. In
443 our previous works [12, 13], our asymptotics (1.5) to (1.9) were obtained by inserting
444 the estimate (3.13) into the explicit formula (2.15) for the tensor M^k . However this
445 leads to suboptimal bounds due to the fact that the mean of \mathcal{X}^{2k} is growing with k
446 like $\mathcal{X}^{2k*} = O(\eta^{-(d-2)(k+1)})$.

447 Since from (3.7), \mathcal{Y}^{k*} has a controlled growth with respect to η (namely $\mathcal{Y}^{k*} =$
448 $O(\eta^{2-d})$ independently of k), we obtain in the next section improved asymptotic
449 estimates for the coefficient tensors M^k by relying on the *exact* identity (3.12). Note
450 that (3.12) can be interpreted as an asymptotic expansion for \mathcal{X}^{p*} , because the terms
451 $\mathcal{Y}^{p-2-2l*} \otimes \mathcal{X}^{2l*}$ of the expansion have an increasing magnitude $O(\eta^{-(d-2)(l+2)})$.

452 3.2. Low-volume fraction asymptotics of the infinite order homoge- 453 nized equation.

454 PROPOSITION 3.7. *The following identity holds for any $k \geq 1$:*

$$455 \quad (3.14) \quad \sum_{p=0}^k \mathcal{Y}^{p*} \otimes M^{k-p} = -\mathcal{Y}^{k-2*} \otimes I$$

456 with the convention $\mathcal{Y}^{-1*} = 0$.

457 *Proof.* Let us multiply (3.12) by M^{k-p} and compute the summation for $0 \leq p \leq k$:

$$\begin{aligned}
 (3.15) \quad \sum_{p=0}^k \mathcal{X}^{p*} \otimes M^{k-p} &= \sum_{p=0}^k \mathcal{Y}^{p*} \otimes M^{k-p} + \sum_{p=0}^k \sum_{l=0}^{p-2} \mathcal{Y}^{l*} \otimes \mathcal{X}^{p-2-l*} \otimes M^{k-p} \otimes I \\
 458 \quad &= \sum_{p=0}^k \mathcal{Y}^{p*} \otimes M^{k-p} + \sum_{l=0}^{k-2} \sum_{p=l+2}^k \mathcal{Y}^{l*} \otimes \mathcal{X}^{p-2-l*} \otimes M^{k-p} \otimes I \quad . \\
 &= \sum_{p=0}^k \mathcal{Y}^{p*} \otimes M^{k-p} + \sum_{l=0}^{k-2} \mathcal{Y}^{l*} \otimes \left(\sum_{p=0}^{k-l-2} \mathcal{X}^{p*} \otimes M^{k-l-2-p} \right) \otimes I
 \end{aligned}$$

Using now (2.14), the second terms of the above equation vanishes except for $k-l-2 = 0$ where it is equal to one. Since the above quantity is also zero for $k \geq 1$, we obtain therefore, for $k \geq 2$:

$$0 = \sum_{p=0}^k \mathcal{X}^{p*} \otimes M^{k-p} = \sum_{p=0}^k \mathcal{Y}^{p*} \otimes M^{k-p} + \mathcal{Y}^{k-2*} \otimes I$$

459 which is the result (3.14). \square

460 Identity (3.14) is a recursive formula for the tensors $(M^k)_{k \in \mathbb{N}}$. This allows to obtain
461 the following asymptotic estimates.

462 **COROLLARY 3.8.** *The tensors M^k satisfy the following asymptotics as $\eta \rightarrow 0$:*

$$463 \quad (3.16) \quad M^0 \sim \text{Cap}(\partial T) \eta^{d-2}$$

$$464 \quad (3.17) \quad M^2 = -I + o(\eta^{d-2})$$

$$465 \quad (3.18) \quad M^{2k} = o(\eta^{d-2}) \text{ for any } k \geq 2.$$

Proof. The first asymptotic is already known. For $k = 1$, (3.14) reads

$$M^2 = (\mathcal{Y}^{0*})^{-1} (-\mathcal{Y}^{0*} \otimes I - \mathcal{Y}^{2*} \otimes M^0) = -I + (M^0)^2 \mathcal{Y}^{2*}.$$

467 Since $M^0 = O(\eta^{d-2})$ and $\mathcal{Y}^{2*} = o(\eta^{2-d})$, we obtain (3.17).

Then for $k \geq 2$, we rewrite (3.14) as

$$\begin{aligned}
 M^{2k} &= -(\mathcal{Y}^{0*})^{-1} \left(\mathcal{Y}^{2k-2*} \otimes I + \sum_{p=1}^k \mathcal{Y}^{2p*} \otimes M^{2(k-p)} \right) \\
 &= -M^0 (\mathcal{Y}^{2k*} \otimes M^0 + \mathcal{Y}^{2k-2*} \otimes (M^2 + I) + \mathcal{Y}^{2k-4*} \otimes M^4 + \dots + \mathcal{Y}^{2*} \otimes M^{2k-2}).
 \end{aligned}$$

468 Assuming the results holds till the rank $k - 1$, we see that all the terms in the
469 parenthesis are of order $o(1)$. Therefore, (3.18) follows by induction, since $M^0 =$
470 $O(\eta^{d-2})$. \square

471 *Remark 3.9.* We now have the full picture of how (2.9) behaves in the low volume
472 fraction limit. Indeed, we have obtained, as $\eta \rightarrow 0$

$$473 \quad (3.19) \quad \varepsilon^{-2} M^0 \sim \eta^{d-2} \varepsilon^{-2} \text{Cap}(\partial T)$$

$$474 \quad (3.20) \quad \varepsilon^0 M^2 \rightarrow -I$$

$$(3.21) \quad \varepsilon^{2k-2} M^{2k} = o(\varepsilon^{2k-2} \eta^{d-2}) \text{ for } k \geq 2.$$

Therefore we obtain the coefficient-wise convergence of the infinite order homogenized equation (2.9) to the three classical limiting equations depending on how η compares with the critical scaling $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$:

- if $\eta \gg \varepsilon^{2/(d-2)}$, then the zero-th order term remains dominant and the limiting equation for $\varepsilon^{-2} \eta^{d-2} u_\varepsilon$ is the zero-th order model

$$(3.22) \quad \begin{cases} \text{Cap}(\partial T) u^* = f \text{ in } D \\ u^* \text{ is } D\text{-periodic,} \end{cases}$$

which is the scalar analogue of the Darcy equation (1.10);

- if $\eta = c\varepsilon^{2/(d-2)}$ for some constant $c > 0$, then $\varepsilon^{-2} M^0$ converges to $c\text{Cap}(\partial T)$ and (2.9) converges coefficient-wisely to the Poisson equation with “strange term”

$$(3.23) \quad \begin{cases} -\Delta u^* + c\text{Cap}(\partial T) u^* = f \text{ in } D \\ u^* \text{ is } D\text{-periodic.} \end{cases}$$

This is the analogue of the Brinkman regime (1.11).

- Finally, if $\eta = o(\varepsilon^{2/(d-2)})$, then $\varepsilon^{-2} M^0 \rightarrow 0$, $\varepsilon^0 M^2 \rightarrow -I$ and $\varepsilon^{2k-2} M^{2k} \rightarrow 0$ for $k \geq 2$. We obtain therefore the Poisson equation in the homogeneous domain D as the limit model:

$$(3.24) \quad \begin{cases} -\Delta u^* = f \text{ in } D, \\ u^* \text{ is } D\text{-periodic,} \end{cases}$$

which is the analogue of the unperturbed Stokes regime (1.12).

3.3. Low volume fraction asymptotics of the truncated higher order homogenized equation. We finally terminate this section by showing that the homogenized model (2.10) of finite order $2K + 2$ has the same asymptotic behavior as (2.9) in the low-volume fraction regime $\eta \rightarrow 0$.

According to Definition 2.8, it is sufficient to examine the asymptotic of the coefficient \mathbb{D}_K^{2K+2} only, since $\mathbb{D}_K^k = M^k$ for $0 \leq k \leq 2K + 1$. From (2.20), this requires to estimate the tensor $N^K(y)$ defined in (2.16). This can be achieved by conveniently rewriting $N^K(y)$ in terms of the tensors $(\mathcal{Y}^k(y))_{k \in \mathbb{N}}$.

PROPOSITION 3.10. *For any $k \geq 0$, the tensor $N^k(y)$ reads in terms of $\mathcal{Y}^k(y)$ as follows:*

$$(3.25) \quad \begin{aligned} N^k(y) &= \sum_{p=0}^k \mathcal{Y}^p(y) \otimes M^{k-p} + \mathcal{Y}^{k-2}(y) \otimes I \\ &= \mathcal{Y}^k \otimes M^0 + \mathcal{Y}^{k-1} \otimes M^1 + \mathcal{Y}^{k-2} \otimes (M^2 + I) + \mathcal{Y}^{k-3} \otimes M^3 + \dots + \mathcal{Y}^0 \otimes M^k \end{aligned}$$

where $\mathcal{Y}^{k-2} := 0$ for $0 \leq k \leq 1$ by convention.

Proof. The proof is identical to that of Proposition 3.7: it suffices to replace $\mathcal{X}^{k-p}(y)$ with the formula given by (3.11) and to simplify the Cauchy product by using (2.14). \square

Remark 3.11. It is visible that the identity (3.14) can also be obtained by computing the average of (3.25) and by using (2.17).

511 The estimates of [Corollary 3.8](#) finally allow to prove that the truncated homogenized
 512 equation [\(2.10\)](#) of order $2K + 2$ has the same limiting behavior as the infinite order
 513 homogenized equation [\(2.9\)](#) as $\eta \rightarrow 0$.

514 **PROPOSITION 3.12.** *We have the following asymptotics for the tensor \mathbb{D}_K^{2K+2} as*
 515 *$\eta \rightarrow 0$:*

$$516 \quad (3.26) \quad \mathbb{D}_0^2 = -I + O(\eta^{d-2})$$

$$517 \quad (3.27) \quad \mathbb{D}_K^{2K+2} = O(\eta^{d-2}) \text{ for } K \geq 1.$$

519 *In particular, for any $K \in \mathbb{N}$, the coefficients $(\mathbb{D}_K^k)_{0 \leq k \leq 2K+2}$ of the higher homog-*
 520 *enized equation [\(2.10\)](#) of order $2K + 2$ satisfy the same asymptotics as the tensors*
 521 *(M^k) as $\eta \rightarrow 0$:*

$$522 \quad \mathbb{D}_K^0 \sim \text{Cap}(\partial T) \eta^{d-2}$$

$$523 \quad \mathbb{D}_K^2 = -I + O(\eta^{d-2})$$

$$524 \quad \mathbb{D}_K^{2k} = O(\eta^{d-2}) \text{ for any } k \geq 2.$$

Proof. Case $K = 0$: we have

$$\begin{aligned} \mathbb{D}_0^2 &= \left(-|M^0|^2 \int_Y |\mathcal{Y}^0|^2 dy \right) I \\ &= \left(- \left(|\mathcal{Y}^{0*}|^2 (1 - \eta^d |T|) + \|\mathcal{Y}^0 - \mathcal{Y}^{0*}\|_{L^2(P \setminus (\eta T))}^2 \right) |M^0|^2 \right) I \\ &= (-1 + O(\eta^{d-2})) I \end{aligned}$$

526 where the last estimate is a consequence of [\(3.6\)](#).
 Case $K \geq 1$: we have

$$|\mathbb{D}^{2K+2}| = \left| \int_Y N^K \otimes N^K \otimes I dy \right| \leq C_K \|N^K\|_{L^2(P \setminus (\eta T))}^2$$

for a constant $C_K > 0$ which depends only on K . Since N^K is of average zero for $K \geq 1$, we can rewrite [\(3.25\)](#) as

$$\begin{aligned} N^K &= \sum_{p=0}^K (\mathcal{Y}^p - \mathcal{Y}^{p*}) \otimes M^{K-p} + (\mathcal{Y}^{K-2} - \mathcal{Y}^{K-2*}) \otimes I \\ &= (\mathcal{Y}^K - \mathcal{Y}^{K*}) \otimes M^0 + (\mathcal{Y}^{K-1} - \mathcal{Y}^{K-1*}) \otimes M^1 + (\mathcal{Y}^{K-2} - \mathcal{Y}^{K-2*}) \otimes (M^2 + I) \\ &\quad + (\mathcal{Y}^{K-3} - \mathcal{Y}^{K-3*}) \otimes M^3 + \dots + (\mathcal{Y}^0 - \mathcal{Y}^{0*}) \otimes M^K. \end{aligned}$$

Therefore by using again [\(3.6\)](#) and [Corollary 3.8](#), we arrive at

$$\|N^K\|_{L^2(P \setminus (\eta T))}^2 = O(\eta^{d-2})$$

527 which yields the result by using [\(2.20\)](#). □

528 **Remark 3.13.** We lost a bit in terms of speed of convergence: the high order coef-
 529 ficients $(\mathbb{D}_K^k)_{3 \leq k \leq 2K+2}$ are only $O(\eta^{d-2})$ while $(M^k)_{k \geq 3}$ is of order $o(\eta^{d-2})$. However,
 530 since both quantities converge to zero due to our assumption $d \geq 3$, the conclusions of
 531 [Remark 3.9](#) remain valid. Therefore the truncated model [\(2.10\)](#) converge as well to ei-
 532 ther of the three regimes [\(3.22\)](#)–[\(3.24\)](#) depending on whether η is greater, proportional
 533 to or lower than the critical size $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$.

534 **4. The Stokes case.** In this final section, we extend the asymptotic analysis
 535 of the previous [section 3](#) to the Stokes system [\(1.1\)](#). We recall the homogenization
 536 setting reviewed in [subsection 2.3](#), and our goal is to prove the coefficient-wise conver-
 537 gence of both the infinite order and the finite order effective models [\(1.2\)](#) and [\(1.3\)](#).
 538 We recall the [Definitions 2.10](#) and [2.13](#) of their respective coefficients $(M^k)_{k \in \mathbb{N}}$ and
 539 $(\mathbb{D}^k)_{0 \leq k \leq 2K+2}$.

540 The asymptotics of these coefficient tensors are obtained in [Corollary 4.6](#) and
 541 [Proposition 4.10](#). The proof follow the lines of [section 3](#); the key ingredient is the in-
 542 troduction of matrix and vector valued cell tensors $(\mathcal{Y}^k(y), \omega^k(y))_{k \in \mathbb{N}}$ with controlled
 543 growth, which generalize the family of scalar valued tensors $(\mathcal{Y}^k(y))_{k \in \mathbb{N}}$ introduced in
 544 [subsection 3.1](#).

545 **4.1. Cell tensors $(\mathcal{Y}^k(y), \omega^k(y))_{k \in \mathbb{N}}$ of controlled growth.** Recall the hier-
 546 archy of corrector systems [\(2.21\)–\(2.23\)](#) defining the cell tensors $(\mathcal{X}_j^k(y), \alpha_j^k(y))_{k \in \mathbb{N}}$.
 547 We define the cell tensors $(\mathcal{Y}_j^k(y), \omega_j^k(y))_{k \in \mathbb{N}}$ by an analogous recurrence.

548 **DEFINITION 4.1.** *For any $1 \leq j \leq d$, we define a family of vector valued tensors*
 549 *$(\mathcal{Y}_j^k(y))$ and scalar valued tensors $(\omega_j^k(y))_{k \in \mathbb{N}}$ as the unique solutions in $H_{\text{per}}^1(Y, \mathbb{R}^d) \times$*
 550 *$L^2(Y)$ to the following recursive systems:*

$$551 \quad (4.1) \quad \begin{cases} -\Delta \mathcal{Y}_j^0 + \nabla \omega_j^0 = \mathbf{e}_j \text{ in } Y, \\ \operatorname{div}(\mathcal{Y}_j^0) = 0 \text{ in } Y, \end{cases}$$

$$552 \quad (4.2) \quad \begin{cases} -\Delta \mathcal{Y}_j^1 + \nabla \omega_j^1 = (2\partial_l \mathcal{Y}_j^0 - \omega_j^0 \mathbf{e}_l) \otimes \mathbf{e}_l \text{ in } Y, \\ \operatorname{div}(\mathcal{Y}_j^1) = -(\mathcal{Y}_j^0 - \langle \mathcal{Y}_j^0 \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \text{ in } Y, \end{cases}$$

$$553 \quad (4.3) \quad \begin{cases} -\Delta \mathcal{Y}_j^{k+2} + \nabla \omega_j^{k+2} = (2\partial_l \mathcal{Y}_j^{k+1} - \omega_j^{k+1} \mathbf{e}_l) \otimes \mathbf{e}_l + (\mathcal{Y}_j^k - \langle \mathcal{Y}_j^k \rangle) \otimes I, \text{ in } Y \\ \operatorname{div}(\mathcal{Y}_j^{k+2}) = -(\mathcal{Y}_j^{k+1} - \langle \mathcal{Y}_j^{k+1} \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \text{ in } Y, \end{cases}$$

555 *supplemented with the following boundary conditions:*

$$556 \quad (4.4) \quad \begin{cases} \int_Y \omega_j^k dy = 0 \\ \mathcal{Y}_j^k = 0 \text{ on } \partial(\eta T) \\ (\mathcal{Y}_j^k, \omega_j^k) \text{ is } P\text{-periodic} \end{cases} \quad \forall k \geq 0.$$

557 It is immediate to see that $(\mathcal{Y}_j^k(y), \omega_j^k(y))$ and $(\mathcal{X}_j^k(y), \alpha_j^k(y))$ coincide for $k = 0, 1$.
 558 In what follows, we also set $(\mathcal{Y}_j^{-1}(y), \omega_j^{-1}(y)) = (\mathcal{X}_j^{-1}(y), \alpha_j^{-1}(y)) = 0$ by convention,
 559 so that [\(4.3\)](#) becomes valid for $k = -1$.

560 Our goal next is to obtain controlled estimates for $(\mathcal{Y}_j^k(y), \omega_j^k(y))$ that are similar
 561 to those obtained in [Proposition 3.3](#) in the Laplace case. We rely on the following
 562 result which allows to estimate the pressure term.

563 **LEMMA 4.2.** *Consider $\mathbf{h} \in L^2(P \setminus (\eta T), \mathbb{R}^d)$ and $g \in L^2(P \setminus (\eta T))$ a function sat-*
 564 *isfying $\int_{P \setminus (\eta T)} g dx = 0$. Let $(\mathbf{v}, \phi) \in H^1(P \setminus (\eta T), \mathbb{R}^d) \times L^2(P \setminus (\eta T))$ be the unique*

565 *solution to the following Stokes system:*

$$566 \quad (4.5) \quad \begin{cases} -\Delta \mathbf{v} + \nabla \phi = \mathbf{h} & \text{in } P \setminus (\eta T) \\ \operatorname{div}(\mathbf{v}) = g & \text{in } P \setminus (\eta T) \\ \int_{P \setminus (\eta T)} \phi dx = 0 \\ \mathbf{v} = 0 & \text{on } \partial(\eta T) \\ \mathbf{v} \text{ is } P\text{-periodic.} \end{cases}$$

567 *There exists a constant $C > 0$ independent of (\mathbf{v}, ϕ) , η , \mathbf{h} and g such that*

$$568 \quad (4.6) \quad \|\nabla \mathbf{v}\|_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} + \|\phi\|_{L^2(P \setminus (\eta T))} \\ 569 \leq C(\|\mathbf{h} - \langle \mathbf{h} \rangle\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)} + \eta^{1-d/2} |\langle \mathbf{h} \rangle| + \|g\|_{L^2(P \setminus (\eta T))}). \\ 570$$

571 *Proof.* (4.6) is obtained by rescaling the estimates of Lemma 5.3 of [13] from the
572 growing domain $\eta^{-1}P \setminus T$ to the perforated cell $P \setminus (\eta T)$. \square

573 Using this lemma yields the fact that $(\mathbf{y}_j^k(y), \omega^k(y))$ has indeed a magnitude
574 controlled with respect to k .

575 PROPOSITION 4.3. *For any $k \geq 0$ and $1 \leq j \leq d$, there exists a constant $C_k > 0$
576 independent of η such that*

$$577 \quad (4.7) \quad \|\nabla \mathbf{y}_j^k\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)} + \|\omega_j^k\|_{L^2(P \setminus (\eta T))} \leq C_k \eta^{1-d/2},$$

$$578 \quad (4.8) \quad \|\mathbf{y}_j^k - \langle \mathbf{y}_j^k \rangle\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)} \leq C_k \eta^{1-d/2}. \\ 579$$

580 *Proof.* Again, we proceed by induction. Note that it is enough to prove (4.7)
581 since (4.8) follows from the Poincaré-Wirtinger inequality (3.4).

Case $k = 0$: applying Lemma 4.2 to (4.1) yields

$$\|\nabla \mathbf{y}_j^0\|_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} + \|\omega_j^0\|_{L^2(P \setminus (\eta T))} \leq C \eta^{1-d/2}$$

582 since $\mathbf{e}_j = \langle \mathbf{e}_j \rangle = 1 - \eta^d |T|$.

Case $k = 1$: since the right-hand side of (4.2) is of zero average, applying Lemma 4.2
yields

$$\begin{aligned} & \|\nabla \mathbf{y}_j^1\|_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} + \|\omega_j^1\|_{L^2(P \setminus (\eta T))} \\ & \leq C(2\|\nabla \mathbf{y}_j^0\|_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} + \|\omega_j^0\|_{L^2(P \setminus (\eta T))} + \|\mathbf{y}_j^0 - \langle \mathbf{y}_j^0 \rangle\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}). \\ & \leq C_1 \eta^{1-d/2}. \end{aligned}$$

Case $k + 2$ with $k \geq 0$: similarly, the right-hand side of (4.3) is of average zero. There-
fore, assuming (4.7) and (4.8) holds till rank $k + 1$ with $k \geq 0$, applying Lemma 4.2
yields

$$\begin{aligned} & \|\nabla \mathbf{y}_j^{k+2}\|_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} + \|\omega_j^{k+2}\|_{L^2(P \setminus (\eta T))} \\ & \leq C'(2\|\nabla \mathbf{y}_j^{k+1}\|_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} + \|\omega_j^{k+1}\|_{L^2(P \setminus (\eta T))} + \|\mathbf{y}_j^k - \langle \mathbf{y}_j^k \rangle\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)} \\ & \quad + \|\mathbf{y}_j^{k+1} - \langle \mathbf{y}_j^{k+1} \rangle\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)}) \\ & \leq C_{k+2} \eta^{1-d/2}. \end{aligned}$$

In the sequel, we consider the matrix-valued tensors \mathcal{Y}^k and the vector-valued tensors ω^k obtained by gathering the vector valued tensors $(\mathcal{Y}_j^k(y))_{1 \leq j \leq d}$ as columns and the scalar valued components $(\omega_j^k(y))_{1 \leq j \leq d}$ as coordinates:

$$(\mathcal{Y}_{ij}^k(y))_{1 \leq i, j \leq d} := \left[\mathcal{Y}_1^k(y) \quad \dots \quad \mathcal{Y}_d^k(y) \right]_{ij}, \quad \forall y \in Y, \quad \forall k \geq 0.$$

$$\omega^k(y) := (\omega_j^k(y))_{1 \leq j \leq d}, \quad \forall y \in Y, \quad \forall k \geq 0.$$

As before, we introduce the mean matrix tensor \mathcal{Y}^{k*} defined by

$$\mathcal{Y}^{k*} := \int_{P \setminus (\eta T)} \mathcal{Y}^k(y) dy.$$

584

585 By using arguments similar to those of the proof of [Proposition 3.4](#), we can
586 precise the convergence of the mean \mathcal{Y}^{k*} . For any $1 \leq j \leq d$, let us consider the
587 unique solution (Ψ_j, σ_j) to the exterior Stokes problem

$$588 \quad (4.9) \quad \begin{cases} -\Delta \Psi_j + \nabla \sigma_j = 0 & \text{in } \mathbb{R}^d \setminus T \\ \operatorname{div}(\Psi_j) = 0 & \text{in } \mathbb{R}^d \setminus T \\ \Psi_j = 0 & \text{on } \partial T \\ \Psi_j \rightarrow e_j & \text{at } \infty \\ \sigma_j \in L^2(\mathbb{R}^d \setminus T). \end{cases}$$

589 The existence and uniqueness of a solution to [\(4.9\)](#) is standard by using layer potential
590 theory [\[20, 19\]](#) or variational arguments in homogeneous Sobolev spaces [\[14, 25\]](#) (also
591 called Deny-Lions or Beppo-Levi spaces). We denote by $F := (F_{ij})_{1 \leq i, j \leq d}$ the matrix
592 collecting the drag force components:

$$593 \quad (4.10) \quad F_{ij} := \int_{\mathbb{R}^d \setminus T} \nabla \Psi_i : \nabla \Psi_j dx = - \int_{\partial T} e_j \cdot (\nabla \Psi_i - \sigma_i I) \cdot \mathbf{n} ds,$$

594 where the normal \mathbf{n} is pointing *inward* T . The matrix F is the analogue of the
595 capacity $\operatorname{Cap}(\partial T)$ in the context of the Stokes equation. The following result holds.

596 **PROPOSITION 4.4.** *The mean matrix valued tensor \mathcal{Y}^{k*} satisfy the following as-*
597 *ymptotic convergences as $\eta \rightarrow 0$:*

$$598 \quad (4.11) \quad \mathcal{Y}^{0*} \sim \eta^{2-d} F^{-1} \quad \text{and} \quad \mathcal{Y}^{k*} = o(\eta^{2-d}) \quad \text{for } k \geq 1.$$

599 *Proof.* The convergence for \mathcal{Y}^{0*} is a classical result and a proof can be found in
600 [\[3\]](#). The second estimate result from the fact that the right-hand sides of [\(4.2\)](#) and
601 [\(4.3\)](#) are of zero average. The proof is obtained by repeating arguments similar to
602 those of [Proposition 3.4](#), see also the proof of Proposition 5.4 in [\[13\]](#). \square

603 The pairs $(\mathcal{Y}^k(y), \omega^k(y))$ and $(\mathcal{X}^k(y), \alpha^k(y))$ are related by Cauchy-product iden-
604 tities analogous to [\(3.11\)](#).

605 PROPOSITION 4.5. *The matrix valued tensors $(\mathcal{Y}^k(y))_{k \in \mathbb{N}}$ and $(\mathcal{X}^k(y))_{k \in \mathbb{N}}$ are re-*
 606 *lated through the following identities:*

$$607 \quad (4.12) \quad \begin{cases} \mathcal{Y}^0(y) = \mathcal{X}^0(y), & \omega^0(y) = \alpha^0(y), \\ \mathcal{Y}^1(y) = \mathcal{X}^1(y), & \omega^1(y) = \alpha^1(y), \\ \mathcal{Y}^k(y) = \mathcal{X}^k(y) - \sum_{l=0}^{k-2} \mathcal{Y}^l(y) \otimes \mathcal{X}^{k-l-2*} \otimes I, & y \in P \setminus (\eta T), \quad k \geq 2. \\ \omega^k(y) = \alpha^k(y) - \sum_{l=0}^{k-2} (-1)^{k-l} \mathcal{X}^{k-l-2*} \cdot \omega^l(y) \otimes I, \end{cases}$$

608 *In particular, we have the formula*

$$609 \quad (4.13) \quad \mathcal{X}^{k*}(y) = \mathcal{Y}^{k*} + \sum_{l=0}^{k-2} \mathcal{Y}^{l*} \otimes \mathcal{X}^{k-l-2*} \otimes I$$

610 *remembering Proposition 2.11 whereby \mathcal{X}^{k*} is symmetric when k is even and is anti-*
 611 *symmetric when k is odd.*

612 *Proof.* The identities for $(\mathcal{Y}^k(y), \omega^k(y)) = (\mathcal{X}^k(y), \alpha^k(y))$, with $k = 1, 2$, are
 613 obvious from the definitions (4.1) and (4.2). By induction, (4.12) is obtained as soon
 614 as we prove

$$615 \quad (4.14) \quad \begin{aligned} \mathcal{Y}_j^{k+2} &= \mathcal{X}_j^{k+2} - \sum_{l=0}^k \mathcal{Y}^l(y) \cdot \langle \mathcal{X}_j^{k-l} \rangle \otimes I = \mathcal{X}_j^{k+2} - \sum_{l=0}^k (\mathcal{X}_{ij}^{k-l*} \otimes I) \mathcal{Y}_i^l(y), \\ \omega_j^{k+2} &= \alpha_j^{k+2} - \sum_{l=0}^k \omega^l(y) \cdot \langle \mathcal{X}_j^{k-l} \rangle \otimes I = \alpha_j^{k+2} - \sum_{l=0}^k (\mathcal{X}_{ij}^{k-l*} \otimes I) \omega_i^l(y), \end{aligned}$$

for $k \geq 0$, assuming these identities hold for lower values of k (remind the symmetry and antisymmetry properties of Proposition 2.11). Note that we use the implicit summation convention over the repeated index $1 \leq i \leq d$. Let (\mathbf{v}, ϕ) be the right-hand sides of the above equations. We compute

$$\begin{aligned} -\Delta \mathbf{v} + \nabla \phi &= (-\Delta \mathcal{X}_j^{k+2} + \nabla \alpha_j^{k+2}) - \sum_{l=0}^k (\mathcal{X}_{ij}^{k-l*} \otimes I) (-\Delta \mathcal{Y}_i^l + \nabla \omega_i^l) \\ &= (2\partial_l \mathcal{X}_j^{k+1} - \alpha_j^{k+1} \mathbf{e}_l) \otimes \mathbf{e}_l + \mathcal{X}_j^k \otimes I - (\mathcal{X}_{ij}^{k*} \otimes I) \mathbf{e}_i \\ &\quad - (\mathcal{X}_{ij}^{k-1*} \otimes I) (2\partial_m \mathcal{X}_i^0 - \alpha_i^0 \mathbf{e}_m) \otimes \mathbf{e}_m \\ &\quad - \sum_{l=2}^k (\mathcal{X}_{ij}^{k-l*} \otimes I) [(2\partial_m \mathcal{Y}_i^{l-1} - \omega_i^{l-1}) \otimes \mathbf{e}_m + (\mathcal{Y}_i^{l-2} - \langle \mathcal{Y}_i^{l-2} \rangle) \otimes I] \\ &= (2\partial_m \mathcal{Y}_j^{k+1} - \omega_j^{k+1} \mathbf{e}_m) \otimes \mathbf{e}_m + (\mathcal{Y}_j^k - \langle \mathcal{Y}_j^k \rangle). \end{aligned}$$

616 In the last equality, we used the assumption that (4.14) holds when k is replaced by
 617 $k-1$ or $k-2$. By uniqueness of the defining problem for $(\mathcal{Y}_j^{k+1}(y), \omega_j^{k+1}(y))$, we
 618 obtain that (4.14) holds. \square

619 **4.2. Low-volume fraction asymptotic of the infinite order homogenized**
 620 **Stokes system.** We now obtain the asymptotic of the coefficients M^k by relating

621 them to the mean tensors \mathcal{Y}^{k*} . Recall that the recursive definition (2.26) of the tensors
 622 M^k states that

$$623 \quad \sum_{p=0}^k \mathcal{X}^{k-p*} \otimes M^p = \begin{cases} I, & \text{if } k = 0, \\ 0, & \text{if } k \geq 1. \end{cases}$$

624 Using this result and repeating the proof of Proposition 3.7, we obtain that the identity
 625 (3.14) remains valid in the present vectorial context:

$$626 \quad (4.15) \quad \sum_{p=0}^k \mathcal{Y}^{p*} \otimes M^{k-p} = -\mathcal{Y}^{k-2*} \otimes I \text{ for any } k \geq 2.$$

627 This identity implies the following results.

628 **COROLLARY 4.6.** *Let M^k be the tensors defined by (2.26) and $F \equiv (F_{ij})_{1 \leq i, j \leq d}$ the*
 629 *drag force matrix defined by (4.10). Then as $\eta \rightarrow 0$,*

$$630 \quad (4.16) \quad M^0 \sim \eta^{d-2} F$$

$$631 \quad (4.17) \quad M^1 = o(\eta^{d-2})$$

$$632 \quad (4.18) \quad M^2 = -I + o(\eta^{d-2})$$

$$633 \quad (4.19) \quad M^k = o(\eta^{d-2}) \text{ for any } k > 2.$$

635 *Proof.* The proof is identical to that of Corollary 3.8, except that some extra
 636 care must be taken because of non-commuting matrix products and non-zero odd
 637 order tensors. The result for $M^0 = (\mathcal{X}^{0*})^{-1}$ is a restatement of the first asymptotic
 638 convergence of (4.11). For $k = 1$, we have by definition

$$639 \quad M^1 = -(\mathcal{Y}^{0*})^{-1} \otimes \mathcal{Y}^{1*} \otimes M^0 = -M^0 \otimes \mathcal{Y}^{1*} \otimes M^0.$$

640 Since $\mathcal{Y}^{1*} = o(\eta^{2-d})$ and $M^0 = O(\eta^{d-2})$, we obtain $M^1 = o(\eta^{d-2})$. For $k = 2$, the
 641 identity (4.15) yields

$$642 \quad M^2 + I = -M^0 \otimes [\mathcal{Y}^{1*} \otimes M^1 + \mathcal{Y}^{2*} \otimes M^0]$$

643 which is also of order $o(\eta^{d-2})$. Finally, for $k > 2$, we rewrite (4.15) as

$$644 \quad M^k = -M^0 (\mathcal{Y}^{k*} \otimes M^0 + \mathcal{Y}^{k-1*} \otimes M^1 + \mathcal{Y}^{k-2*} \otimes (M^2 + I) + \dots + \mathcal{Y}^{1*} \otimes M^{k-1}).$$

645 By induction, we deduce from the above relation that $M^k = o(\eta^{d-2})$ for all $k \geq 2$,
 646 which completes the proof. \square

647 *Remark 4.7.* We recall that there is a slight abuse of notation in the notation I
 648 featured in (4.18) because I is here the second-order matrix-valued defined by (2.8)
 649 and not the scalar valued tensor I of the other equations.

650 *Remark 4.8.* We have therefore obtained the first main result of the paper, i.e.
 651 the coefficient-wise convergence of the infinite order homogenized Stokes system (1.2)
 652 towards either the Darcy, Brinkman or Stokes regimes (1.10)–(1.12) for the various
 653 scalings of η when compared to the critical size $\varepsilon^{2/(d-2)}$. Indeed, the coefficients of
 654 (1.2) satisfy as $\eta \rightarrow 0$:

$$655 \quad (4.20) \quad \varepsilon^{-2} M^0 \sim \eta^{d-2} \varepsilon^{-2} F$$

$$656 \quad (4.21) \quad \varepsilon^{-1} M^1 = o(\varepsilon^{-1} \eta^{d-2})$$

$$657 \quad (4.22) \quad \varepsilon^0 M^2 \rightarrow -I$$

$$658 \quad (4.23) \quad \varepsilon^k M^k = o(\varepsilon^k \eta^{d-2}) \text{ for } k > 2.$$

660 Reasoning as in [Remark 3.9](#) we obtain the coefficient-wise convergence of [\(1.2\)](#) to-
 661 wards the three regimes as $\varepsilon \rightarrow 0$ and $\eta \rightarrow 0$ for the three possible scalings of η . Note
 662 that we also obtain the coefficient-wise convergence of the infinite order model [\(1.2\)](#)
 663 towards the homogeneous Stokes system [\(1.12\)](#) if ε is fixed while $\eta \rightarrow 0$.

664 **4.3. Low-volume fraction asymptotic of the truncated homogenized**
 665 **Stokes system of order $2K + 2$.** We now come to the final result concerned with
 666 the coefficient-wise limit of the truncated homogenized model [\(1.3\)](#), or in other words
 667 with the limit of the tensors \mathbb{D}_K^k as $\eta \rightarrow 0$. Similarly as in [subsection 3.3](#) and by
 668 reading the definition [\(2.32\)](#), we need to find the asymptotic limits of the tensors
 669 $N^k(y)$ and $\beta^k(y)$ of [Definition 2.12](#). Using [\(4.13\)](#), we can represent them using the
 670 controlled tensors \mathcal{Y}^k and ω^k , as shown in the next result.

671 **PROPOSITION 4.9.** *For $k \geq 1$ and with the convention $\mathcal{Y}^{-2*} = \mathcal{Y}^{-1*} = 0$ and*
 672 *$\omega^{-2} = \omega^{-1} = 0$, the following identities hold:*

$$673 \quad (4.24) \quad N^k(y) = \sum_{p=0}^k \mathcal{Y}^{k-p}(y) \otimes M^p + \mathcal{Y}^{k-2}(y) \otimes I, \quad y \in Y,$$

$$674 \quad (4.25) \quad \beta^k(y) = \sum_{p=0}^k (-1)^p M^p \cdot \omega^{k-p}(y) + \omega^{k-2}(y) \otimes I, \quad y \in Y.$$

676 *Proof.* Both identities are proved following the arguments and computations of
 677 [Proposition 3.7](#). We only provide the proof for the second identity. We first multiply
 678 [\(4.12\)](#) by $(-1)^p M^p$ and sum over $0 \leq p \leq k$:

$$\begin{aligned} & \sum_{p=0}^k (-1)^p M^p \cdot \alpha^{k-p} \\ 679 \quad &= \sum_{p=0}^k (-1)^p M^p \cdot \omega^{k-p} + \sum_{p=0}^k \sum_{l=0}^{k-p-2} (-1)^{k-l-2} M^p \otimes \mathcal{X}^{k-p-l-2} \cdot \omega^l \otimes I \\ &= \sum_{p=0}^k (-1)^p M^p \cdot \omega^{k-p} + \sum_{l=0}^{k-2} (-1)^{k-l-2} \left[\sum_{p=0}^{k-l-2} M^p \otimes \mathcal{X}^{k-p-l-2} \right] \cdot \omega^l \otimes I, \end{aligned}$$

680 In view of [\(4.13\)](#), the summation in the brackets vanishes unless $l = k - 2$ when it
 681 sums to I . This leads to

$$682 \quad (4.26) \quad \beta^k = \sum_{p=0}^k (-1)^p M^p \cdot \alpha^{k-p} = \sum_{p=0}^k (-1)^p M^p \cdot \omega^{k-p} + \omega^{k-2} \otimes I$$

683 which is the desired result. \square

684 With those formulas, we finally obtain the low volume fraction asymptotics of the
 685 tensors $(\mathbb{D}_K^k)_{0 \leq k \leq 2K+2}$ of the high order truncated homogenized Stokes system [\(1.3\)](#).
 686 The analysis requires slightly more work than in the scalar case due to the presence
 687 of the tensors $(\mathbb{A}_K^k)_{K+1 \leq k \leq 2K+1}$ induced by the divergence constraint.

688 PROPOSITION 4.10. *We have the following asymptotics for the tensors \mathbb{D}_K^{2K+2} and*
 689 *$(\mathbb{A}_K^k)_{K+1 \leq k \leq 2K+1}$ defined by (2.32) as $\eta \rightarrow 0$:*

$$690 \quad (4.27) \quad \mathbb{D}_0^2 = -I + O(\eta^{d-2})$$

$$691 \quad (4.28) \quad \mathbb{D}_K^{2K+2} = O(\eta^{d-2}) \text{ for } K \geq 1$$

$$692 \quad (4.29) \quad \mathbb{A}_K^k = O(\eta^{d-2}) \text{ for } K \geq 0 \text{ and } K+1 \leq k \leq 2K+1.$$

694 *Therefore, for any $K \in \mathbb{N}$, the matrix-valued coefficient tensors $(\mathbb{D}_K^k)_{0 \leq k \leq 2K+2}$ of*
 695 *the truncated homogenized Stokes system (1.3) satisfy the following convergences as*
 696 *$\eta \rightarrow 0$:*

$$697 \quad (4.30) \quad \mathbb{D}_K^0 \sim \eta^{d-2} F,$$

$$698 \quad (4.31) \quad \mathbb{D}_K^1 = O(\eta^{d-2}),$$

$$699 \quad (4.32) \quad \mathbb{D}_K^2 = -I + O(\eta^{d-2}),$$

$$700 \quad (4.33) \quad \mathbb{D}_K^k = O(\eta^{d-2}) \text{ for any } k > 2.$$

Proof. 1. Asymptotic (4.27). By the definition (2.32) and by using (4.24), we have

$$\begin{aligned} \mathbb{D}_{ij}^2 &= - \int_Y \mathbf{N}_i^0 \cdot \mathbf{N}_j^0 \otimes \text{Id} y = -M_{mi}^0 M_{lj}^0 \int_Y \mathbf{y}_m^0 \cdot \mathbf{y}_l^0 \otimes \text{Id} y \\ &= -M_{mi}^0 M_{lj}^0 \left(\langle \mathbf{y}_m^0 \rangle \cdot \langle \mathbf{y}_l^0 \rangle (1 - \eta^d |T|) + \int_Y (\mathbf{y}_m^0 - \langle \mathbf{y}_m^0 \rangle) \cdot (\mathbf{y}_l^0 - \langle \mathbf{y}_l^0 \rangle) dy \right) \otimes I, \end{aligned}$$

with implicit summation over the repeated indices $1 \leq l, m \leq d$. Then, we observe that $M_{mi}^0 \langle \mathbf{y}_m^0 \rangle = \mathcal{X}^{0*} M e_i = e_i$, and similarly $M_{lj}^0 \langle \mathbf{y}_l^0 \rangle = e_j$; this implies

$$-M_{mi}^0 M_{lj}^0 (\langle \mathbf{y}_m^0 \rangle \cdot \langle \mathbf{y}_l^0 \rangle (1 - \eta^d |T|)) = -\delta_{ij} I + O(\eta^d).$$

702 Finally, using (4.8), (4.16) and the Cauchy-Schwarz inequality allows to obtain

703

$$\begin{aligned} 704 \quad & -M_{mi}^0 M_{lj}^0 \left(\int_Y (\mathbf{y}_m^0 - \langle \mathbf{y}_m^0 \rangle) \cdot (\mathbf{y}_l^0 - \langle \mathbf{y}_l^0 \rangle) dy \right) \otimes I \\ 705 \quad & = O(\eta^{d-2}) O(\eta^{d-2}) O(\eta^{2-d}) = O(\eta^{d-2}) \end{aligned}$$

707 which implies (4.27).

2. Asymptotic (4.28). We use (4.24) to rewrite, for any $k \geq 1$, \mathbf{N}_i^k as

$$\begin{aligned} \mathbf{N}_i^k &= \sum_{p=0}^k \mathbf{y}_m^{k-p} \otimes M_{mi}^p + \mathbf{y}_i^{k-2} \otimes I \\ &= \sum_{p=0}^k (\mathbf{y}_m^{k-p} - \langle \mathbf{y}_m^{k-p} \rangle) \otimes M_{mi}^p + (\mathbf{y}_i^{k-2} - \langle \mathbf{y}_i^{k-2} \rangle) \otimes I \\ &= (\mathbf{y}_m^k - \langle \mathbf{y}_m^{k-1} \rangle) \otimes M_{mi}^0 + (\mathbf{y}_m^{k-1} - \langle \mathbf{y}_m^{k-1} \rangle) \otimes M_{mi}^1 \\ &\quad + (\mathbf{y}_m^{k-2} - \langle \mathbf{y}_m^{k-2} \rangle) \otimes (M_{mi}^2 + \delta_{mi} I) \\ &\quad + (\mathbf{y}_m^{k-3} - \langle \mathbf{y}_m^{k-3} \rangle) \otimes M_{mi}^3 + \cdots + (\mathbf{y}_m^0 - \langle \mathbf{y}_m^0 \rangle) \otimes M_{mi}^k, \end{aligned}$$

where we used that $\langle \mathbf{N}_i^k \rangle = 0$ at the second equality. Therefore the result of [Corollary 4.6](#) and the bound of [\(4.8\)](#) controlling $\|\mathfrak{Y}_m^k - \langle \mathfrak{Y}_m^k \rangle\|_{L^2(P \setminus (\eta T))}$ imply that

$$\|\mathbf{N}_i^k\|_{L^2(P \setminus (\eta T))} = O(\eta^{d/2-1}) \text{ for } k \geq 1.$$

708 Then [\(4.28\)](#) follows from the definition [\(2.32\)](#) and the Cauchy-Schwarz inequality.

709 3. Asymptotics [\(4.29\)](#). By integration by parts, the formula [\(2.33\)](#) for $\mathbb{A}_{K,ij}^k$ with
710 $K+1 \leq k \leq 2K+1$ can be rewritten as

$$711 \quad \mathbb{A}_{K,ij}^k = (-1)^K \int_Y (\beta_j^{k-K-1} \otimes \operatorname{div}(\mathbf{N}_i^{K+1}) + (-1)^k \beta_i^{k-K-1} \otimes \operatorname{div}(\mathbf{N}_j^{K+1})) \, dy.$$

712 Therefore we need to control the L^2 norm of $\beta_j^k(y)$ for $0 \leq k \leq K$ and of $\operatorname{div} \mathbf{N}_i^k$ for
713 any $k \geq 1$ and $1 \leq i, j \leq d$. Using [\(4.24\)](#) to compute the divergence, we obtain for
714 any $k \geq 1$

$$\begin{aligned} \operatorname{div} \mathbf{N}_i^k &= \sum_{p=0}^k \operatorname{div} \mathfrak{Y}_m^{k-p} \otimes M_{mi}^p + \operatorname{div} \mathfrak{Y}_i^{k-2} \otimes I \\ &= - \sum_{p=0}^k (\mathfrak{Y}_m^{k-p-1} - \langle \mathfrak{Y}_m^{k-p-1} \rangle) \cdot \mathbf{e}_l \otimes e_l \otimes M_{mi}^p - (\mathfrak{Y}_i^{k-3} - \mathfrak{Y}_i^{k-3*}) \cdot \mathbf{e}_l \otimes e_l \otimes I \\ &= [(\mathfrak{Y}_m^{k-1} - \langle \mathfrak{Y}_m^{k-1} \rangle) \otimes M_{mi}^0 + (\mathfrak{Y}_m^{k-2} - \langle \mathfrak{Y}_m^{k-2} \rangle) \otimes M_{mi}^1 \\ &\quad + (\mathfrak{Y}_m^{k-3} - \langle \mathfrak{Y}_m^{k-3} \rangle) \otimes (M_{mi}^2 + \delta_{mi} I) \\ &\quad + (\mathfrak{Y}_m^{k-4} - \langle \mathfrak{Y}_m^{k-4} \rangle) \otimes M_{mi}^3 + \cdots + (\mathfrak{Y}_m^0 - \langle \mathfrak{Y}_m^0 \rangle) \otimes M_{mi}^{k-1}] \cdot \mathbf{e}_l \otimes e_l, \end{aligned}$$

still assuming the summation convention over the repeated index $1 \leq m \leq d$. By using the result of [Corollary 4.6](#) and the bound [\(4.8\)](#), we obtain therefore that

$$\|\operatorname{div} \mathbf{N}_i^K\|_{L^2(P \setminus (\eta T))} = O(\eta^{d/2-1}) \text{ for any } K \in \mathbb{N}.$$

Similarly, [\(4.24\)](#) allows to rewrite β_j^k as

$$\begin{aligned} \beta_j^k &= \sum_{p=0}^k \omega_m^{k-p} \otimes M_{mj}^p + \omega_j^{k-2} \otimes I \\ &= \omega_m^k \otimes M_{mj}^0 + \omega_m^{k-1} \otimes M_{mj}^1 + \omega_m^{k-2} \otimes (M_{mj}^2 + \delta_{mj} I) \\ &\quad + \omega_m^{k-3} \otimes M_{mj}^3 + \cdots + \omega_m^0 \otimes M_{mj}^k. \end{aligned}$$

716 Therefore, the bound [\(4.7\)](#) controlling $\|\omega_j^k\|_{L^2(P \setminus (\eta T))}$ and [Corollary 4.6](#) yield

$$\|\beta_j^k\|_{L^2(P \setminus (\eta T))} = O(\eta^{d/2-1}) \text{ for any } k \in \mathbb{N}.$$

717 Hence [\(4.29\)](#) follows by using the Cauchy-Schwarz inequality. \square

718 The result of [Proposition 4.10](#) implies that the conclusions of [Remark 4.8](#) still hold
719 for the truncated model [\(1.3\)](#), which converges therefore in the coefficient-wise sense
720 towards either of the three models [\(1.10\)](#)–[\(1.12\)](#) depending on how the scaling η
721 compares to the critical value $\eta_{\text{crit}} \sim \eta^{2/(d-2)}$ as claimed.

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