NEWTON NON-DEGENERATE μ-CONSTANT DEFORMATIONS ADMIT SIMULTANEOUS EMBEDDED RESOLUTIONS

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Abstract. Let $\mathbb{C}_o^{n+1}$ denote the germ of $\mathbb{C}^{n+1}$ at the origin. Let V be a hypersurface germ in $\mathbb{C}_o^{n+1}$ and W a deformation of V over $\mathbb{C}_o^m$. Under the hypothesis that W is a Newton non-degenerate deformation, in this article we will prove that W is a μ-constant deformation if and only if W admits a simultaneous embedded resolution. This result gives a lot of information about W, for example, the topological triviality of the family W and the fact that the natural morphism $(\mathrm{W}(\mathbb{C}_o)_m)_{red} \to \mathbb{C}_o$ is flat, where $\mathrm{W}(\mathbb{C}_o)_m$ is the relative space of m-jets. On the way to the proof of our main result, we give a complete answer to a question of Arnold on the monotonicity of Newton numbers in the case of convenient Newton polyhedra.

1. Introduction

Before stating and discussing the main problem of this article we will give some brief preliminaries and introduce the notation that will be used in the article.

1.0.1. *Preliminaries on μ-constant deformations.* Let

$$\mathcal{O}_{n+1}^x := \mathbb{C}\{x_1, ..., x_{n+1}\}, \quad n \geqslant 0,$$

be the $\mathbb{C}$-algebra of analytic function germs at the origin o of $\mathbb{C}^{n+1}$ and $\mathbb{C}_o^{n+1}$ the complex-analytic germ of $\mathbb{C}^{n+1}$. By abuse of notation we denote by o the origin of $\mathbb{C}_o^{n+1}$. Let V be a hypersurface of $\mathbb{C}_o^{n+1}$, $n \geqslant 1$, given by an equation $f(x) = 0$, where f is irreducible in $\mathcal{O}_{n+1}^x$. Assume that V has an isolated singularity at o. One of the important topological invariants of the singularity $o \in V$ is the Milnor number $\mu(f)$, defined by

$$\mu(f) := \dim_{\mathbb{C}} \mathcal{O}_{n+1}^x / J(f),$$

where $J(f) := (\partial_1 f, ..., \partial_{n+1} f) \subset \mathcal{O}_{n+1}^x$ is the Jacobian ideal of f. In this article we will consider deformations of f that preserve the Milnor number. Let F be a deformation of f:

$$F(x, s) := f(x) + \sum_{i=1}^{l} h_i(s) g_i(x)$$

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where $h_i \in \mathcal{O}_m^s := \mathbb{C}\{s_1, ..., s_m\}$, $m \geqslant 1$, and $g_i \in \mathcal{O}_{n+1}^x$ satisfy

$$h_i(o) = g_i(o) = 0.$$

Take a sufficiently small open set $\Omega \subset \mathbb{C}^m$ containing o, and representatives of the analytic function germs $h_1, ..., h_l$ in Ω. By a standard abuse of notation we will denote these representatives by the same letters $h_1, ..., h_l$. We use the notation $F_{s'}(x) := F(x, s')$ when $s' \in \Omega$ is fixed. We will say that the deformation F is μ-constant if the open set Ω can be chosen so that $\mu(F_{s'}) = \mu(f)$ for all $s' \in \Omega$.

Let $\mathcal{E} := \{e_1, e_2, ..., e_{n+1}\} \subset \mathbb{Z}_{\geqslant 0}^{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$. Let us write the convergent power series $g_1, \ldots, g_l \in \mathbb{C}\{x_1, ..., x_{n+1}\}$ as

$$g(x) = \sum_{\alpha \in Z} a_\alpha x^\alpha, \quad Z := \mathbb{Z}_{\geqslant 0}^{n+1} \backslash \{o\},$$

in the multi-index notation. The *Newton polyhedron* $\Gamma_+(g)$ is the convex hull of the set $\bigcup_{\alpha \in \mathrm{Supp}(g)} (\alpha + \mathbb{R}_{\geqslant 0}^n)$, where $\mathrm{Supp}(g)$ (short for "the support of g") is defined by $\mathrm{Supp}(g) := \{\alpha \mid a_\alpha \neq 0\}$. The *Newton boundary* of $\Gamma_+(g)$, denoted by $\Gamma(g)$, is the union of the compact faces of $\Gamma_+(g)$. We will say that $g(x) = \sum_{\alpha \in Z} a_\alpha x^\alpha$, $Z := \mathbb{Z}_{\geqslant 0}^{n+1} \backslash \{o\}$, is *non-degenerate with respect to its Newton boundary (or Newton non-degenerate)* if for every compact face γ of the Newton polyhedron $\Gamma_+(g)$ the polynomial $g_\gamma = \sum_{\alpha \in \gamma} a_\alpha x^\alpha$ does not have singularities in $(\mathbb{C}^\star)^{n+1}$.

We say that a deformation of F of f is *non-degenerate* if the neighborhood Ω of o in $\mathbb{C}^m$ can be chosen so that for all $s' \in \Omega$ the germ $F_{s'}$ is non-degenerate with respect to its Newton boundary $\Gamma(F_{s'})$.

1.0.2. *Preliminaries on Simultaneous Embedded Resolutions.* Let us keep the notation from the previous section. We denote $S := \mathbb{C}_o^m$, and W the deformation of V given by F. Then we have the following commutative diagram:

$$\begin{array}{ccccc} V & \hookrightarrow & W & \hookrightarrow & \mathbb{C}_o^{n+1} \times S \\ \downarrow & \square & \downarrow \varrho & \swarrow & \\ o & \hookrightarrow & S & & \end{array}$$

where the morphism ϱ is flat. We use the notation $W_{s'} := \varrho^{-1}(s')$, $s' \in S$.

In what follows we will define what we mean by *Simultaneous Embedded Resolution* of W.

We consider a proper bimeromorphic morphism $\varphi : \widetilde{\mathbb{C}_o^{n+1} \times S} \to \mathbb{C}_o^{n+1} \times S$ such that $\widetilde{\mathbb{C}_o^{n+1} \times S}$ is formally smooth over S, and we denote by $\widetilde{W}^s$ and $\widetilde{W}^t$ the strict and the total transform of W in $\widetilde{\mathbb{C}_o^{n+1} \times S}$, respectively.

Definition 1.1. *The morphism $\widetilde{W}^s \to W$ is a very weak simultaneous resolution if $\widetilde{W}^s_{s'} \to W_{s'}$ is a resolution of singularities for each $s' \in S$.*

Definition 1.2. *We say that $\widetilde{W}^t$ is a normal crossing divisor relative to S if the induced morphism $\widetilde{W}^t \to S$ is flat and for each $p \in \widetilde{W}^t$ there exists an open neighborhood $U \subset \widetilde{\mathbb{C}_0^{n+1} \times S}$ of p and a map ϕ,*

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \mathbb{C}_0^{n+1} \times S \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

biholomorphic onto its image, such that $\widetilde{W}^t \cap U$ is defined by the ideal $\phi^\star \mathcal{I}$, where $\mathcal{I} = \left(y_1^{a_1} \cdots y_{n+1}^{a_{n+1}}\right)$, $y_1, ...y_{n+1}$ is a coordinate system at o in $\mathbb{C}_0^{n+1}$ and the a_i are non-negative integers. If $p \in \widetilde{W}^s$, we require that $a_{n+1} = 1$ and that $\widetilde{W}^s \cap U$ be defined by the ideal $\phi^\star \mathcal{I}'$, where $\mathcal{I}' = (y_{n+1})$.

Definition 1.3. *We will say ϕ is a simultaneous embedded resolution if, in the above notation, the morphism $\widetilde{W}^s \to W$ is a very weak simultaneous resolution and $\widetilde{W}^t$ is a normal crossing divisor relative to S.*

Let us recall that W is defined by

$$F(x,s) := f(x) + \sum_{i=1}^{l} h_i(s) g_i(x)$$

where $h_i \in \mathcal{O}_m^s$, $m \geqslant 1$, and $g_i \in \mathcal{O}_{n+1}^x$ such that $h_i(o) = g_i(o) = 0$.

Let $\epsilon > 0$ (resp. $\epsilon' > 0$) be small enough so that $f, g_1, ..., g_l$ (resp. $h_1, ..., h_l$) are defined in the open ball $B_\epsilon(o) \subset \mathbb{C}^{n+1}$ (resp. $B_{\epsilon'}(o) \subset \mathbb{C}^m$), and the singular locus of W is $\{o\} \times B_{\epsilon'}(o)$. We will say that the deformation of W is *topologically trivial* if, in addition, there exists a homeomorphism ξ that commutes with the projection

$$pr_2 : B_\epsilon(o) \times B_{\epsilon'}(o) \to B_{\epsilon'}(o) :$$

$$\begin{array}{ccc} B_\epsilon(o) \times B_{\epsilon'}(o) & \xrightarrow{\xi} & B_\epsilon(o) \times B_{\epsilon'}(o) \\ & {\scriptstyle pr_2} \searrow \quad \swarrow {\scriptstyle pr_2} & \\ & B_{\epsilon'}(o) & \end{array}$$

such that $\xi(W) = V' \times B_{\epsilon'}(o)$, where $V' := \xi(V)$, that is to say, ξ trivializes W. The following Proposition relates Simultaneous Embedded Resolutions, topologically trivial deformations and μ-constant deformations.

Proposition 1.4. *Let V and W be as above. Assume that W admits a simultaneous embedded resolution. Then:*

1. *The deformation W is topologically trivial.*
2. *The deformation W is μ-constant.*

Proof. The Milnor number μ is a topological invariant, hence (1) implies (2). As W admits a simultaneous embedded resolution, there exists a proper bimeromorphic morphism $\varphi : \widetilde{\mathbb{C}_0^{n+1} \times S} \to \mathbb{C}_0^{n+1} \times S$ such that $\widetilde{\mathbb{C}_0^{n+1} \times S}$ is formally smooth over S and W^t is a normal crossing divisor relative to S. In the topological context this translates into the existence of a proper bimeromorphic morphism $\varphi : B_\epsilon(o) \widetilde{\times} B_{\epsilon'}(o) \to B_\epsilon(o) \times B_{\epsilon'}(o)$ such that for

all $p \in \varphi^{-1}(o)$ there exists $\epsilon'' > 0$, and a diffeomorphism

$$\phi_p : B_{\epsilon''}(p) \subset B_\epsilon(o) \widetilde{\times} B_{\epsilon'}(o) \to B_{\epsilon''}(o) \widetilde{\times} B_{\epsilon'}(o)$$

that trivializes $W^t \cap B_{\epsilon''}(p)$. Using partitions of unity and the projection ξ, we obtain the desired trivialization. □

1.0.3. *On the main result of the article.* Keep the notation of the previous sections. Recall that W is a deformation of V over $S := \mathbb{C}^m_o$ given by F. In the article [Oka89] the author proves that if W is a non-degenerate μ-constant deformation of V that induces a negligible truncation of the Newton boundary then W admits a very weak simultaneous resolution. However if the method of proof used is observed with detail, what is really proved is that W admits a simultaneous embedded resolution in the special case when $n = 2$, $l = m = 1$, $h_1(s) = s$ and $g_1(x)$ is a monomial in x. Intuitively one might think that the condition that W admit a simultaneous embedded resolution is more restrictive than the condition that W is a μ-constant deformation. However, this intuition is wrong in the case of Newton non-degenerate μ-constant deformations. More precisely, in this article we prove the following result:

Theorem. *Assume that W is a Newton non-degenerate deformation. Then the deformation W is μ-constant if and only if W admits a simultaneous embedded resolution.*

Observe that if W admits a simultaneous embedded resolution it follows directly from Proposition 1.4 that W is a μ-constant deformation. The converse of this is what needs to be proved.

From the above theorem and Proposition 1.4 we obtain the following corollary.

Corollary 1.5. *Let W be a Newton non-degenerate μ-constant deformation. Then W is topologically trivial.*

The result of the corollary is already known (see [Abd16]). In the general case, for $n \neq 2$ it is known that if W is a μ-constant deformation, then the deformation W is topologically trivial, (see [LDR76]). The case $n = 2$ is a conjecture (the Lê–Ramanujan conjecture).

The theorem has an interesting implication to spaces of m-jets. Let $\mathbb{K}$ be a field and Y a scheme over $\mathbb{K}$. We denote by $Y\!-\!\mathcal{S}ch$ (resp. $\mathcal{S}et$) the category of schemes over Y (resp. sets), and let X be a Y-scheme. It is known that the functor $Y\!-\!\mathcal{S}ch \to \mathcal{S}et :\ Z \mapsto \mathrm{Hom}_Y(Z \times_{\mathbb{K}} \mathrm{Spec}\,\mathbb{K}[t]/(t^{m+1}), X)$, $m \geqslant 1$, is representable. More precisely, there exists a Y-scheme, denoted by $\mathrm{X(Y)}_m$, such that $\mathrm{Hom}_Y(Z \times_{\mathbb{K}} \mathrm{Spec}\,\mathbb{K}[t]/(t^{m+1}), X) \cong \mathrm{Hom}_Y(Z, \mathrm{X(Y)}_m)$ for all Z in $Y\!-\!\mathcal{S}ch$. The scheme $\mathrm{X(Y)}_m$ is called the *space of m-jets of X relative to Y*. For more details see [Voj07] or [LA18]. Let us assume that Y is a reduced $\mathbb{K}$-scheme, and let Z be a Y-scheme. We denote by Z_{red} the reduced Y-scheme associated to Z.

Corollary 1.6. *Let $S = \mathbb{C}_0$ and let W be a non-degenerate μ-constant deformation. The structure morphism $(\mathrm{W(S)}_m)_{red} \to S$ is flat for all $m \geqslant 1$.*

Proof. By the previous theorem W admits an embedded simultaneous resolution. Hence the corollary is an immediate consequence of Theorem 3.4 of [LA18]. □

The main result of this article initiates a new approach to the Lê-Ramanujam conjecture. To wit, in characteristic 0 every singularity can be embedded in a higher dimensional affine space in such a way that it is either Newton non-degenerate or Schön (this is due to Tevelev, answering a question of Teissier, see [Tei14], [Tev14] and [Mou17]). Note that Schön is the notion that generalizes Newton non-degenerate singularities to higher codimensions and guarantees the existence of embedded toric resolutions for singularities having this property. The idea is to prove a generalization of the main theorem of this article for an adapted embedding and then to apply the first part of Proposition 1.4.

Finally, we comment on the organization of the article. In section 2 we study geometric properties of pairs of Newton polyhedra that have the same Newton number. This will allow us to construct the desired simultaneous resolution. In this section we give an affirmative answer to the conjecture presented in article [BKW19]. This result together with Theorem 2.2 (see [Fur04]) is a complete solution to an Arnold problem (No. 1982-16 in his list of problems, see [Arn04]) in the case of convenient Newton polyhedra. In section 3 we prove the main result of the article. Finally, in section 4 we study properties of degenerate μ-constant deformations.

2. Preliminaries on Newton Polyhedra

In this section we study geometric properties of pairs of Newton Polyhedra having the same Newton number, one contained in the other.

Given an affine subspace H of $\mathbb{R}^n$, a convex polytope in H is a non-empty set P given by the intersection of H with a finite set of half spaces of $\mathbb{R}^n$. In particular, a compact convex polytope can be seen as the convex hull of a finite set of points in $\mathbb{R}^n$. The dimension of a convex polytope is the dimension of the smallest affine subspace of $\mathbb{R}^n$ that contains it. We will say that P is a polyhedron (resp. compact polyhedron) if P can be decomposed into a finite union of convex (resp. compact convex) polytopes. We will say that P is of pure dimension n if P is a finite union of n-dimensional convex polytopes. An n-dimensional simplex Δ is a compact convex polytope generated by $n+1$ points of $\mathbb{R}^n$ in general position. Given an n-dimensional compact polyhedron $P \subset \mathbb{R}^n_{\geqslant 0}$, the Newton number of P is defined by

$$\nu(P) := n!V_n(P) - (n-1)!V_{n-1}(P) \cdots (-1)^{n-1}V_1(P) + (-1)^n V_0(P),$$

where $V_n(P)$ is the volume of P, $V_k(P)$, $1 \leqslant k \leqslant n-1$, is the sum of the k-dimensional volumes of the intersection of P with the coordinate planes of dimension k, and $V_0(P) = 1$ (resp. $V_0(P) = 0$) if $o \in P$ (resp. $o \notin P$), where o is the origin of $\mathbb{R}^n$.

Let $I \subset \{1, 2, ..., n\}$. We define the following sets:

$$\mathbb{R}^I := \{(x_1, ..., x_n) \in \mathbb{R}^n : \ x_i = 0 \text{ if } i \notin I\}$$
$$\mathbb{R}_I = \{(x_1, ..., x_n) \in \mathbb{R}^n : \ x_i = 0 \text{ if } i \in I\}$$

Given a polyhedron P in $\mathbb{R}^n$, we write $P^I := P \cap \mathbb{R}^I$. Consider an n-dimensional simplex $\Delta \subset \mathbb{R}^n_{\geqslant 0}$. A *full supporting coordinate subspace of* Δ is a coordinate subspace $\mathbb{R}^I \subset \mathbb{R}^n$ such that $\dim \Delta^I = |I|$. In the article [Fur04] the author proves that there exists a unique full-supporting coordinate subspace of Δ of minimal dimension. We will call this subspace *the minimal full-supporting coordinate subspace of* Δ. We denote by $\mathrm{Ver}(P)$ the set of vertices of P.

The next result gives us a way of calculating the Newton number of certain polyhedra using projections.

Proposition 2.1. *(See [Fur04]) Let $o \notin P \subset \mathbb{R}^n_{\geqslant 0}$ be a compact polyhedron that is a finite union of n-simplices Δ_i, $1 \leqslant i \leqslant m$, that satisfy*

$$\mathrm{Ver}(\Delta_i) \subset \mathrm{Ver}(P).$$

Assume that there exists $I \subset \{1, 2, ..., n\}$ such that $\mathbb{R}^I$ is the minimal full-supporting coordinate subspace of Δ_i and $P^I = \Delta_i^I$ for all $1 \leqslant i \leqslant m$. Then $\nu(P) = |I|! V_{|I|}\left(P^I\right) \nu(\pi_I(P))$ where $\pi_I : \mathbb{R}^n \to \mathbb{R}_I$ is the projection map.

Let $\mathcal{E} := \{e_1, e_2, ..., e_n\} \subset \mathbb{Z}^n_{\geqslant 0}$ be the standard basis of $\mathbb{R}^n$. Let

$$P \subset \mathbb{R}^n_{\geqslant 0}$$

be a polyhedron of pure dimension n. Consider the following conditions:

(1) $o \in P$
(2) P^J is topologically equivalent to a $|J|$-dimensional closed disk for each $J \subset \{1, ..., n\}$.
(3) Let $I \subset \{1, ..., n\}$ be a non-empty subset. If $(\alpha_1, .., \alpha_n) \in \mathrm{Ver}(P)$ then for each $i \in I$ we must have either $\alpha_i \geqslant 1$ or $\alpha_i = 0$ (recall that the α_i are real numbers that need not be integers).

We will say that P is *pre-convenient* (resp. *I-convenient*) if it satisfies (1) and (2) (resp. (1), (2), and (3)). In the case when $I := \{1, ..., n\}$ we will simply say that P is *convenient* instead of I-convenient.

Given a discrete set $S \subset \mathbb{R}^n_{\geqslant 0} \backslash \{o\}$, denote by $\Gamma_+(S)$ the convex hull of the set $\bigcup_{\alpha \in S} (\alpha + \mathbb{R}^n_{\geqslant 0})$. The polyhedron $\Gamma_+(S)$ is called the *Newton polyhedron* associated to S. The *Newton boundary* of $\Gamma_+(S)$, denoted by $\Gamma(S)$, is the union of the compact faces of $\Gamma_+(S)$. Let $\mathrm{Ver}(S) := \mathrm{Ver}(\Gamma(S))$ denote the set of vertices of $\Gamma(S)$.

We say that the discrete set $S \subset \mathbb{R}^n_{\geqslant 0} \backslash \{o\}$ is *pre-convenient* (resp. *I-convenient*) if $\Gamma_-(S) := \overline{\mathbb{R}^n_{\geqslant 0} \backslash \Gamma_+(S)}$ is pre-convenient (resp. I-convenient). The Newton number of a pre-convenient discrete set $S \subset \mathbb{R}^n_{\geqslant 0} \backslash \{o\}$ is

$$\nu(S) := \nu(\Gamma_-(S)).$$

Note that this number can be negative.

In the case when P is the polyhedron $\Gamma_-(S)$ associated to a discrete set S, condition (1) holds automatically and condition (2) can be replaced by the following:

$(2')$ For each $e \in \mathcal{E}$ there exists $m > 0$ such that $me \in \mathrm{Ver}(S)$.

Consider a convergent power series $g \in \mathbb{C}\{x_1, ..., x_n\}$:

$$g(x) = \sum_{\alpha \in Z} a_\alpha x^\alpha, Z := \mathbb{Z}^n_{\geqslant 0} \backslash \{o\}.$$

We define $\Gamma_+(g) = \Gamma_+(\mathrm{Supp}(g))$ and $\Gamma(g) = \Gamma(\mathrm{Supp}(g))$. We say that g is a *convenient* power series if for all $e \in \mathcal{E}$ there exists $m > 0$ such that $me \in \mathrm{Supp}(g)$. Observe that the discrete set $\mathrm{Supp}(g)$ is convenient if and only if the power series g is convenient. We will use the following notation: $\mathrm{Ver}(g) := \mathrm{Ver}(\mathrm{Supp}(g))$, and $\nu(g) = \nu(\mathrm{Supp}(g))$.

Theorem 2.2. *([Fur04]) Let $P' \subset P$ be two convenient polyhedra. We have $\nu(P) - \nu(P') = \nu\left(\overline{P \backslash P'}\right) \geqslant 0$, and $\nu(P') \geqslant 0$.*

Corollary 2.3.

(1) Let S and S' be two convenient discrete subsets of $\mathbb{R}^n_{\geqslant 0}\backslash\{o\}$, and suppose that $\Gamma_+(S) \subsetneq \Gamma_+(S')$. We have

$$0 \leqslant \nu(S) - \nu(S') = \nu\left(\overline{\Gamma_-(S) \backslash \Gamma_-(S')}\right).$$

(2) Let S, S', and S'' be three convenient discrete subsets of $\mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ such that their Newton polyhedra satisfy

$$\Gamma_+(S) \subset \Gamma_+(S') \subset \Gamma_+(S'')$$

and $\nu(S) = \nu(S'')$. Then $\nu(S) = \nu(S') = \nu(S'')$.

For a set $I \subset \{1, ..., n\}$, we write $I^c := \{1, ..., n\} \backslash I$. The following result gives us a criterion for the positivity of the Newton number of certain polyhedra.

Proposition 2.4. *Let $o \notin P$ be a pure n-dimensional compact polyhedron such that there exists $I \subset \{1, ..., n\}$ such that $\dim\left(P^J\right) < |J|$ (resp. P^J is topologically equivalent to a $|J|$-dimensional closed disk) for all $I \not\subset J$ (resp. $I \subset J$). Assume that if*

$$(\beta_1, .., \beta_n) \in \mathrm{Ver}(P)$$

then for each $i \in I^c$ we have $\beta_i \geqslant 1$ or $\beta_i = 0$. Then there exists a sequence of sets $I \subset I_1, I_2, ..., I_m \subset \{1, ..., n\}$, and of polyhedra Z_i, $1 \leqslant i \leqslant m$, such that:

(1) $P = \bigcup_{i=1}^m Z_i$,

(2) $\nu(P) = \sum_{i=1}^m \nu(Z_i)$,

(3) $\nu(Z_i) = |I_i|! V_{|I_i|}\left(Z_i^{I_i}\right) \nu(\pi_{I_i}(Z_i)) \geqslant 0$.

In particular, $\nu(P) \geqslant 0$.

Remark 1. *Let S be a discrete subset of $\mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ and $\alpha \in \mathbb{R}^I_>$, $I \subset \{1, ..., n\}$.*

If $\alpha \notin \Gamma_+(S)$, then $P := \overline{\Gamma_+(S(\alpha)) \backslash \Gamma_+(S)}$ is topologically equivalent to an $|n|$-dimensional closed disk. What's more, by induction on n we obtain that

$\dim\left(P^J\right) = |J|$ *for all* $J \supset I$ *if and only if* P^J *is topologically equivalent to a* $|J|$*-dimensional closed disk. In addition, we observe that* $\dim\left(P^J\right) < |J|$ *for all* $J \not\ni I$.

Proof. The method of proof that we use is similar to the proof of Theorem 2.3 of [Fur04].

As P is a pure n-dimensional compact polyhedron, there exists a finite subdivision Σ of P such that:

(1) If $\Delta \in \Sigma$, then $\dim \Delta = n$.
(2) For all $\Delta \in \Sigma$, $\mathrm{Ver}(\Delta) \subset \mathrm{Ver}(P)$.
(3) Given $\Delta, \Delta' \in \Sigma$, we have $\dim(\Delta \cap \Delta') < n$ whenever $\Delta \neq \Delta'$.

Let S be the set formed by all the subsets $I' \subset \{1, ..., n\}$ such that there exists $\Delta \in \Sigma$ such that its minimal full-supporting coordinate subspace (m.f.-s.c.s.) is $\mathbb{R}^{I'}$.

As $\dim P^J < |J|$ for all $J \not\ni I$, we obtain that $I' \supset I$ for all $I' \in S$. We define:

$$\Sigma\left(I'\right) = \left\{\Delta \in \Sigma \ : \ \text{the m.f.-s.c.s. of } \Delta \text{ is } \mathbb{R}^{I'}\right\}.$$

Let us consider the set

$$\Sigma^{I'} := \left\{\Delta^{I'} : \ \Delta \in \Sigma\left(I'\right)\right\} = \left\{\sigma_1, ..., \sigma_{l(I')}\right\}.$$

Given $\sigma_i \in \Sigma^{I'}$, let $C_i := \left\{\Delta \in \Sigma(I') : \ \Delta^I = \sigma_i\right\}$. Consider the closed set

$$Z_{(i,I')} := \bigcup_{\Delta \in C_i} \Delta.$$

Observe that given $\alpha \in \sigma_i^\circ$ (where σ_i° is the relative interior of σ_i), there exists $\epsilon > 0$ such that for each $J \supset I'$, we have $B_\epsilon(\alpha) \cap Z^J_{(i,I')} = B_\epsilon(\alpha) \cap \mathbb{R}^J_{\geqslant 0}$. Indeed, as P^J is topologically equivalent to a $|J|$-dimensional closed disk for all $J \supset I'$, there exits $\epsilon > 0$ such that $B_\epsilon(\alpha) \cap \mathbb{R}^J_{\geqslant 0} \subset P^J$. Making ϵ smaller we may assume that $B_\epsilon(\alpha) \cap \mathbb{R}^J_{\geqslant 0} \subset Z^J_{(i,I')}$. This implies that $\pi_{I'}\left(Z_{(i,I')}\right)$ is a convenient polyhedron in $\mathbb{R}_{I'}$ (remember that if $(\beta_1, .., \beta_n) \in \mathrm{Ver}(P)$ then for each $i \in I^c$ we have $\beta_i \geqslant 1$ or $\beta_i = 0$), from which it follows that $\nu\left(\pi_{I'}\left(Z_{(i,I')}\right)\right) \geqslant 0$ (see Theorem 2.2). Now using Proposition 2.1 we obtain $\nu\left(Z_{(i,I')}\right) = |I|!V_{|I|}(\sigma_i)\nu\left(\pi_{I'}\left(Z_{(i,I')}\right)\right) \geqslant 0$.

By construction we obtain

$$P = \bigcup_{I' \in S} \bigcup_{i=1}^{l(I')} Z_{(i,I')}$$

and

$$\dim\left(Z^{J'}_{(i,I')} \cap Z^{J'}_{(i',I'')}\right) < |J'| \text{ for all } (i, I') \neq (i', I'').$$

This implies that

$$\nu(P) = \sum_{I' \in S} \sum_{i=1}^{l(I')} \nu\left(Z_{(i,I')}\right).$$

Rearranging the indices, we obtain the desired subdivision. □

Let S and S' be two discrete subsets of $\mathbb{R}^n_{\geqslant 0}\backslash\{0\}$ such that

$$\Gamma_+(S) \subset \Gamma_+(S').$$

We define $\mathrm{Ver}(S', S) := \mathrm{Ver}(S')\backslash \mathrm{Ver}(S)$. The following result tells us where the vertices $\mathrm{Ver}(S', S)$ are found.

Proposition 2.5. *Let S, S' be two convenient discrete subsets of $\mathbb{R}^n_{\geqslant 0}\backslash\{o\}$. Suppose that $\Gamma_+(S) \subsetneqq \Gamma_+(S')$ and $\nu(S) = \nu(S')$. Then*

$$\mathrm{Ver}(S', S) \subset \left(\mathbb{R}^n_{\geqslant 0}\backslash\mathbb{R}^n_{>0}\right).$$

Proof. Let us suppose that $\mathrm{Ver}(S', S) \nsubseteq (\mathbb{R}^n_{\geqslant 0}\backslash\mathbb{R}^n_{>0})$. Let

$$W = \mathrm{Ver}(S', S) \cap (\mathbb{R}^n_{\geqslant 0}\backslash\mathbb{R}^n_{>0})$$

and $\alpha \in \mathrm{Ver}(S', S)\backslash W$. Let us consider $S'' := S \cup \{\alpha\}$. As the discrete sets S, S', and S'' are convenient and

$$\Gamma_+(S) \subset \Gamma_+(S'') \subset \Gamma_+(S'),$$

we obtain $\nu(S'') = \nu(S) = \nu(S')$ (see Corollary 2.3). Let us prove that this is a contradiction. In effect, by definition of Newton number we have

$$\begin{aligned}\nu(S) &= n!V_n - (n-1)!V_{n-1} + \cdots (-1)^{n-1}V_1 + (-1)^n,\\ \nu(S'') &= n!V''_n - (n-1)!V''_{n-1} + \cdots (-1)^{n-1}V''_1 + (-1)^n,\end{aligned}$$

where $V_k := V_k(\Gamma_-(S))$ and $V''_k := V_k(\Gamma_-(S''))$ are the k-dimensional Newton volumes of $\Gamma_-(S)$ and $\Gamma_-(S'')$, respectively. By construction $V''_n < V_n$ and $V'_k = V_k$, $1 \leqslant k \leqslant n-1$, which implies that $\nu(S'') < \nu(S)$. □

If we suppose that $\nu(S') = \nu(S)$, it is not difficult to verify that this equality is not preserved by homothecies of $\mathbb{R}^n_{\geqslant 0}$. The following result describes certain partial homothecies of $\mathbb{R}^n_{\geqslant 0}$ which preserve the equality of the Newton numbers.

Let us consider $D(S, S') = \{I \subset \{1, 2, ..., n\} : \Gamma_-(S) \cap \mathbb{R}^I \neq \Gamma_-(S') \cap \mathbb{R}^I\}$ and $I(S, S') = \bigcap_{I \in D(S,S')} I$. It may happen that

$$I(S, S') \notin D(S, S')$$

or

$$I(S, S') = \varnothing.$$

Proposition 2.6. *Let $S, S' \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be two pre-convenient discrete sets such that $\Gamma_+(S) \subset \Gamma_+(S')$. Suppose that $\{1, 2, .., k\} \subset I(S, S')$, and consider the map*

$$\varphi_\lambda(x_1, .., x_n) = (\lambda x_1, .., \lambda x_k, x_{k+1}, ..., x_n),\ \lambda \in \mathbb{R}_{>0}.$$

Then $\nu(\varphi_\lambda(S')) - \nu(\varphi_\lambda(S)) = \lambda^k(\nu(S') - \nu(S))$.

Proof. We will use the notation $V_m(S) := V_m(\Gamma_-(S))$. Recall that

$$V_m(S) = \sum_{|I|=m} \mathrm{Vol}_m(\Gamma_-(S) \cap \mathbb{R}^I),$$

where $\mathrm{Vol}_m(\cdot)$ is the m-dimensional volume.

Let $J = \{1, 2, ..., k\}$. Observe that if $J \not\subset I$, then

$$\Gamma_-(S) \cap \mathbb{R}^I = \Gamma_-\left(S'\right) \cap \mathbb{R}^I,$$

which implies that $\mathrm{Vol}_{|I|}(\Gamma_-(\varphi_\lambda(S)) \cap \mathbb{R}^I) = \mathrm{Vol}_{|I|}(\Gamma_-(\varphi_\lambda(S')) \cap \mathbb{R}^I)$. In particular, if $m < k$ we have $V_m(\varphi_\lambda(S)) = V_m(\varphi_\lambda(S'))$. Let us suppose that $m \geqslant k$. Then:

$$V_m(\varphi_\lambda(S')) - V_m(\varphi_\lambda(S)) = \sum_{\substack{|I| = m \\ J \subset I}} (\mathrm{Vol}_m(\Gamma_-(\varphi_\lambda(S')) \cap \mathbb{R}^I) - \mathrm{Vol}_m(\Gamma_-(\varphi_\lambda(S)) \cap \mathbb{R}^I)).$$

From this we obtain that $V_m(\varphi_\lambda(S')) - V_m(\varphi_\lambda(S)) = \lambda^k(V_m(S') - V_m(S))$ and $\nu(\varphi_\lambda(S')) - \nu(\varphi_\lambda(S)) = \lambda^k(\nu(S') - \nu(S))$. □

The following Corollary is an analogue of Proposition 2.5 in the pre-convenient case.

Given $S \subset \mathbb{R}^n_{\geqslant}\backslash\{o\}$ and $R \subset \mathbb{R}^n_{\geqslant 0}$, we denote $S(R) := S \cup R$.

Corollary 2.7. *Let $S \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be a pre-convenient discrete set, and $\alpha \in \mathbb{R}^n_{>0}$, such that $\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha))$. Then $\nu(S(\alpha)) < \nu(S)$.*

Proof. Observe that there exists $\lambda > 0$ such that the discrete sets $\varphi_\lambda(S)$, $\varphi_\lambda(S(\alpha))$ are convenient where φ_λ is the homothety consisting of multiplication by λ. As $I(S, S(\alpha)) = \{1, ..., n\}$, we have

$$\nu(\varphi_\lambda(S)) - \nu(\varphi_\lambda(S(\alpha)) = \lambda^n(\nu(S) - \nu(S(\alpha))$$

(see Proposition 2.6). By Theorem 2.2, we have $\nu(\varphi_\lambda(S(\alpha))) \leqslant \nu(\varphi_\lambda(S))$, hence $\nu(S(\alpha)) \leqslant \nu(S)$. If

$$\nu(S(\alpha)) = \nu(S)$$

then $\nu(\varphi_\lambda(S)) = \nu(\varphi_\lambda(S(\alpha))$. This contradicts Proposition 2.5. □

Take a set $I \subset \{1, \ldots, n\}$.

Corollary 2.8. *Let S, S', and S'' be three I^c-convenient discrete sets such that $\Gamma_+(S) \subset \Gamma_+(S') \subset \Gamma_+(S'')$. Suppose that*

$$I \subset I(S, S') \cap I(S', S'')$$

Then $\nu(S) \geqslant \nu(S') \geqslant \nu(S'')$.

Proof. Without loss of generality, we may take $I = \{1, ..., k\}$. As S, S', and S'' are I^c-convenient, there exists $\lambda > 0$ such that after applying the map φ_λ given by $\varphi_\lambda(x_1, .., x_n) = (\lambda x_1, .., \lambda x_k, x_{k+1}, ..., x_n)$, the discrete sets $\varphi_\lambda(S)$, $\varphi_\lambda(S')$, and $\varphi_\lambda(S'')$ are convenient.

As $I \subset I(S, S') \cap I(S', S'')$, we have

$$\nu(\varphi_\lambda(S)) - \nu(\varphi_\lambda(S')) = \lambda^k(\nu(S) - \nu(S')),$$

and

$$\nu(\varphi_\lambda(S')) - \nu(\varphi_\lambda(S'')) = \lambda^k(\nu(S) - \nu(S'')).$$

By Theorem 2.2, we obtain $0 \leqslant \nu(S) - \nu(S')$ and $0 \leqslant \nu(S') - \nu(S'')$. □

Convention. From now till the end of the paper, whenever we talk about a vertex γ of a certain polyhedron and an edge of this polyhedron denoted by E_γ, it will be understood that γ is one of the endpoints of E_γ.

Given $I \subset \{1,2,...,n\}$, let $\mathbb{R}^I_{>0} := \{(x_1,x_2,...,x_n) \in \mathbb{R}^I : \ x_i > 0 \text{ if } i \in I\}$. Let $S \subset \mathbb{R}^n_{\geqslant}\backslash\{0\}$ be a discrete set, and let $\alpha \in \mathbb{R}^I_{>0}$ be such that

$$\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha)).$$

Let E_α be an edge of $\Gamma(S(\alpha))$ such that α is one of its endpoints. Given a set J with

$$I \subsetneqq J \subset \{1,2,...,n\},$$

we will say that E_α is (I,J)-*convenient* if for all

$$\beta := (\beta_1,...,\beta_n) \in (E_\alpha \cap \mathrm{Ver}(S))$$

we have $\beta_i \geqslant 1$ for $i \in J\backslash I$ and $\beta_i = 0$ for $i \in J^c$. We will say that E_α is *strictly* (I,J)-*convenient* if E_α is (I,J)-convenient and whenever

$$\beta \in (E_\alpha \cap \mathrm{Ver}(S)),$$

there exists $i \in J\backslash I$ such that $\beta_i > 1$.

The following Proposition will allow us to eliminate certain vertices.

Proposition 2.9. *Let $S \subset \mathbb{R}^n_{\geqslant 0}\backslash\{0\}$ be an I^c-convenient discrete set, J a set such that $I \subsetneqq J \subset \{1,...,n\}$, and $\alpha \in \mathbb{R}^I_{>0}$ such that $\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha))$ and $\nu(S(\alpha)) = \nu(S)$. Suppose that some of the following conditions are satisfied:*

(1) $\alpha' \in \overline{\Gamma_+(S(\alpha))\backslash\Gamma_+(S)} \cap \mathbb{R}^I$.

(2) $\alpha' \in \overline{\Gamma_+(S(\alpha))\backslash\Gamma_+(S)} \cap \mathbb{R}^J_{>0}$ *and there exists a strictly (I,J)-convenient edge E_α of $\Gamma(S(\alpha))$.*

Then $\nu(S(\alpha')) = \nu(S)$.

Proof. Let us suppose that $\alpha' \in \overline{\Gamma_+(S(\alpha))\backslash\Gamma_+(S)} \cap \mathbb{R}^I$. We may assume that $\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha')) \subsetneqq \Gamma_+(S(\alpha))$ (otherwise there is nothing to prove).

Observe that the discrete sets S, $S(\alpha')$, and $S(\alpha)$ are I^c-convenient and

$$I \subset I(S,S(\alpha')) \cap I(S(\alpha'),S(\alpha)).$$

Using Corollary 2.8, we obtain $\nu(S) = \nu(S(\alpha')) = \nu(S(\alpha))$. This completes the proof in Case (1).

Next, assume that (2) holds. Consider a strictly (I,J)-convenient edge E_α of $\Gamma(S(\alpha))$. Let $\beta := (\beta_1,.,,,\beta_n) \in E_\alpha \cap \mathrm{Ver}(S)$. Let $E' \subset E_\alpha$ be the line segment with endpoints α and β. Without loss of generality we may assume that $E' \cap \mathrm{Ver}(S) = \{\beta\}$. As E_α is strictly (I,J)-convenient, there exists $i \in J\backslash I$ such that $\beta_i > 1$. Let $\delta > 0$ be sufficiently small so that $\beta_i - \delta \geqslant 1$ and let $\beta' \in \mathbb{R}^I_{\geqslant 0}$ be such that $\gamma := \beta - \delta e_i + \beta' \in \Gamma(S(\alpha)) \cap \mathbb{R}^J_{>0}$. Then

$$\Gamma_+(S) \subsetneqq \Gamma_+(S(\gamma)) \subsetneqq \Gamma_+(S(\alpha)).$$

Observe that the discrete sets S, $S(\gamma)$, and $S(\alpha)$ are I^c-convenient and

$$I \subset I(S,S(\gamma)) \cap I(S(\gamma),S(\alpha)).$$

Then $\nu(S(\gamma)) = \nu(S(\alpha)) = \nu(S)$.

If $\alpha' \in \overline{\Gamma_+(S(\gamma))\backslash\Gamma_+(S)} \cap \mathbb{R}^J$, we have

$$\Gamma_+(S) \subset \Gamma_+(S(\alpha')) \subset \Gamma_+(S(\gamma)).$$

The discrete sets S, $S(\alpha')$, and $S(\gamma)$ are J^c-convenient and

$$J \subset I(S, S(\alpha')) \cap I(S(\alpha'), S(\gamma)).$$

Then $\nu(S(\alpha')) = \nu(S(\gamma)) = \nu(S)$.

We still need to study the case $\alpha' \in (\Gamma_+(S(\alpha))\backslash\Gamma_+(S(\gamma))) \cap \mathbb{R}^J_{>0}$.

Consider the compact set $C := \overline{(\Gamma_+(S(\alpha'))\backslash\Gamma_+(S))} \cap \mathbb{R}^J$, and the map

$$\nu_S : C \to \mathbb{R};\ \tau \mapsto \nu_S(\tau) := \nu(S(\tau)) = \sum_{m=0}^{n} (-1)^{n-m} m! V_m(S(\tau))$$

where $V_m(S(\tau)) := V_m(\Gamma_-(S(\tau)))$. The map ν_S is continuous in C. In effect, recall that $V_m(S(\tau)) = \sum_{|I'|=m} \mathrm{Vol}_m(\Gamma_-(S(\tau)) \cap \mathbb{R}^{I'})$. Hence

$$V_m(S(\tau)) = V(\tau) + V'(\tau),$$

where

$$V(\tau) := \sum_{\substack{|I'|=m \\ I' \supseteq J}} \mathrm{Vol}_m(\Gamma_-(S(\tau)) \cap \mathbb{R}^{I'}),$$

$$V'(\tau) := \sum_{\substack{|I'|=m \\ I' \not\supseteq J}} \mathrm{Vol}_m(\Gamma_-(S(\tau)) \cap \mathbb{R}^{I'}).$$

The function $V : \mathbb{R}^J \to \mathbb{R}; \tau \mapsto V(\tau)$ is continuous, since each summand is continuous in $\mathbb{R}^J$. The function $V' : C \to \mathbb{R}; \tau \mapsto V'(\tau)$ is constant, since $\Gamma_-(S(\alpha')) \cap (\mathbb{R}^J_{\geqslant 0}\backslash\mathbb{R}^J_{>0}) = \Gamma_-(S) \cap (\mathbb{R}^J_{\geqslant 0}\backslash\mathbb{R}^J_{>0})$. Then each $V_m(S(\tau))$ is continuous in $\tau \in C$, which implies that the function ν_S is continuous in C.

Let us assume that $\alpha' \in (\Gamma_+(S(\alpha))\backslash\Gamma_+(S(\gamma))) \cap \mathbb{R}^J_{>0}$ and $\alpha' \notin \Gamma(S(\alpha))$. Let us suppose that $\nu_S(\alpha') = \nu(S(\alpha')) \neq \nu(S)$. Let us consider the set $\mathcal{C} := \{\tau \in C :\ \nu_S(\tau)) = \nu_S(\alpha'))\}$. The continuity of ν_S implies that $\mathcal{C}$ is compact. We define the following partial order on $\mathcal{C}$. For $\tau, \tau' \in \mathcal{C}$ we will say that $\tau \leqslant \tau'$ if $\Gamma_+(S(\tau')) \subset \Gamma_+(S(\tau))$. Let us consider an ascending chain

$$\tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n \leqslant \cdots$$

We will prove that this chain is bounded above in $\mathcal{C}$. Let us consider the convex closed set

$$\Gamma = \bigcap_{i \geqslant 1} \Gamma_+(S(\tau_i)).$$

As $\mathcal{C}$ is compact, the sequence $\{\tau_1, \tau_2, ..., \tau_n, ...\}$ has a convergent subsequence $\{\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_n}, ...\}$. Observe that

$$\Gamma_+(S(\tau)) = \bigcap_{n \geqslant 1} \Gamma_+(S(\tau_{i_n})),$$

where $\tau := \lim_{n\to\infty} \tau_{i_n} \in \mathcal{C}$. By definition, $\Gamma \subset \Gamma_+(S(\tau))$, and by construction for each $i \geqslant 1$ there exists $n \geqslant 1$ such that $\Gamma_+(S(\tau_{i_n})) \subset \Gamma_+(S(\tau_i))$. Then

$\Gamma = \Gamma_+(S(\tau))$, which implies that $\tau_i \leqslant \tau$ for all $i \geqslant 1$. By Zorn's lemma $\mathcal{C}$ contains at least one maximal element. Let $\tau \in \mathcal{C}$ be a maximal element. Recall that we consider $\alpha' \notin \Gamma(S(\alpha))$, and we made the assumption that $\nu(S(\alpha')) \neq \nu(S)$. Hence $\tau \notin (\overline{\Gamma_+(S(\gamma))\backslash\Gamma_+(S)}) \cap \mathbb{R}^J$.

Observe that for all $\alpha'' \in \Gamma_+(S(\alpha))$ we have

$$\Gamma_+(S) \subset \Gamma_+(S(\alpha'')) \subset \Gamma_+(S(\gamma,\alpha'')) \subset \Gamma_+(S(\alpha)).$$

As the discrete sets S, $S(\alpha'')$, and $S(\gamma,\alpha'')$ are I^c-convenient and

$$I \subset I(S(\alpha''), S(\gamma,\alpha'')) \cap I(S(\gamma,\alpha''), S(\alpha)),$$

we obtain $\nu(S(\gamma,\alpha'')) = \nu(S(\alpha'')) = \nu(S)$.

As $\tau \notin \overline{\Gamma_+(S(\gamma))\backslash\Gamma_+(S)} \cap \mathbb{R}^J$ and $\gamma \in \Gamma(S(\alpha))$, there exists a relatively open subset Ω of the relative interior of $\overline{\Gamma_+(S(\alpha))\backslash\Gamma_+(S)} \cap \mathbb{R}^I$ such that τ belongs to the relative interior of

$$\left(\Gamma_+\left(S(\gamma,\alpha'')\right)\backslash\Gamma_+\left(S\left(\alpha''\right)\right)\right) \cap R^J$$

for all $\alpha'' \in \Omega$. We obtain

$$\Gamma_+\left(S\left(\alpha''\right)\right) \subsetneqq \Gamma_+\left(S\left(\tau,\alpha''\right)\right) \subsetneqq \Gamma_+\left(S\left(\gamma,\alpha''\right)\right).$$

The discrete sets $S(\alpha'')$, $S(\tau,\alpha'')$, and $S(\gamma,\alpha'')$ are J^c-convenient, and

$$J \subset I\left(S\left(\alpha''\right), S\left(\tau,\alpha''\right)\right) \cap I\left(S\left(\tau,\alpha''\right), S\left(\gamma,\alpha''\right)\right).$$

Hence $\nu(S(\tau,\alpha'')) = \nu(S(\alpha'')) = \nu(S)$.

Given an edge E_τ of $\Gamma(S(\tau))$ that connects τ with a vertex in $\mathrm{Ver}(\Gamma(S))$, let E'_τ be the subsegment of E_τ containing τ such that $|E'_\tau \cap \mathrm{Ver}(S)| = 1$. We choose $\alpha'' \in \Omega'$ such that for each edge E_τ of $\Gamma(S(\tau))$ connecting τ with an element of $\mathrm{Ver}(\Gamma(S))$ we have $\dim(E'_\tau \cap \Gamma(S(\alpha''))) = 0$. In other words, no subsegment of E'_τ is contained in the Newton boundary $\Gamma(S(\alpha''))$.

Let us consider the compact polyhedron $P := \overline{\Gamma_+(S(\tau,\alpha''))\backslash\Gamma_+(S(\alpha''))}$. Observe that $\nu(P) = 0$ (see Theorem 2.2).

Given the choice of α'', there exists $\tau' \in P$ such that

$$\Gamma_+\left(S\left(\tau'\right)\right) \subsetneqq \Gamma_+(S(\tau))$$

and $Q_0 := \overline{(\Gamma_+(S(\tau))\backslash\Gamma_+(S(\tau')))} \subset P$ (it is for achieving the last inclusion that the choice of α'' is really important).

Let $Q_1 := \overline{P\backslash Q_0}$. As $\dim\left(Q_0^{J'} \cap Q_1^{J'}\right) < |J'|$, for all $J' \subset \{1,...,n\}$, we obtain $\nu(P) = \nu(Q_0) + \nu(Q_1)$. The polyhedra Q_0 and Q_1 satisfy the hypotheses of Proposition 2.4. In effect:

(1) By construction Q_0 and Q_1 are pure n-dimensional compact polyhedra and $o \notin P = Q_0 \cup Q_1$.

(2) Recall that $\tau \in \mathbb{R}^J_{>0}$. The polyhedron P satisfies

$$\dim\left(P^{J'}\right) < |J'| \quad \text{for all } J' \supsetneq J,$$

which implies $\dim\left(Q_0^{J'}\right) < |J'|$ and $\dim\left(Q_1^{J'}\right) < |J'|$ for all $J' \supsetneq J$.

(3) Now we will verify that $Q_0^{J'}$ is topologically equivalent to a $|J'|$-dimesional closed disk for all $J' \supset J$. As S is I^c-convenient and $\tau \in \mathbb{R}^J_{>0}$, we have $\dim\left(Q_0^{J'}\right) = |J'|$ for each $J' \supset J$. By Remark 1 we obtain that $Q_0^{J'}$ is

topologically equivalent to a $|J'|$-dimensional closed disc. The proof for $Q_1^{J'}$ is analogous to the proof for $Q_0^{J'}$.

(4) As S is I^c-convenient (in particular J^c-convenient), we obtain that if $(\beta_1, .., \beta_n) \in \mathrm{Ver}(P)$ then for each $i \in J^c$ we have $\beta_i \geqslant 1$ or $\beta_i = 0$. This property is inherited by Q_0 and Q_1.

By Proposition 2.4 we have $\nu(Q_0) \geqslant 0$, $\nu(Q_1) \geqslant 0$. As $\nu(P) = 0$, we obtain $\nu(Q_0) = \nu(Q_1) = 0$. We have $\tau < \tau' \in \mathcal{C}$, which contradicts the maximality of τ in $\mathcal{C}$. As a consequence we obtain $\nu(S(\alpha')) = \nu(S)$.

Now let us suppose that $\alpha' \in \Gamma(S(\alpha)) \cap \mathbb{R}^J_{>0}$, and let $v \in \mathbb{R}^J_{>0}$. For $\epsilon > 0$ small enough $\alpha_\epsilon := \alpha' + \epsilon v$ belongs to the relative interior of $\Gamma_+(S(\alpha'))\backslash\Gamma_+(S)$. For the continuity of ν_S in $C := \overline{(\Gamma_+(S(\alpha'))\backslash\Gamma_+(S))} \cap \mathbb{R}^J$ we obtain

$$\lim_{\epsilon \to 0} \nu_S(\alpha_\epsilon) = \nu(S(\alpha')),$$

which implies that $\nu(S(\alpha')) = \nu(S)$. □

Corollary 2.10. *Let $I \subsetneqq J := \{1, ..., n\}$. Let $S, S' \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be two convenient discrete sets such that $\Gamma_+(S) \subsetneqq \Gamma_+(S')$ and $\nu(S) = \nu(S')$. Suppose that there exists $\alpha \in \mathrm{Ver}(S', S) \cap \mathbb{R}^I_{>0}$ and an edge E_α of $\Gamma(S)$ that is (I, J)-convenient. Then there exists*

$$(\beta_1, ..., \beta_n) \in \mathrm{Ver}(S) \cap E_\alpha$$

such that $\beta_i = 1$ for all $i \in I^c$.

Proof. Let $R := \mathrm{Ver}(S', S)\backslash\{\alpha\}$ and $S(R) = S \cup R$. The discrete sets S, $S(R)$ and S' are convenient, and $\Gamma_+(S) \subset \Gamma_+(S(R)) \subsetneqq \Gamma_+(S')$. Then

$$\nu(S(R)) = \nu(S').$$

We argue by contradiction. If there is no $(\beta_1, ..., \beta_n)$ as in the Corollary then the edge E_α is strictly (I, J)-convenient. By Proposition 2.9, for all $\alpha' \in \overline{\Gamma_+(S(\alpha)\backslash\Gamma_+(S)} \cap \mathbb{R}^n_{>0}$ we have

$$\nu(S(R)) = \nu(S(R \cup \{\alpha'\})),$$

which contradicts Proposition 2.5. □

The following Proposition allows us to fix a special coordinate hyperplane and gives information about the edges not contained in the hyperplane that contain a vertex of interest belonging to the hyperplane.

Proposition 2.11. *Let $S, S' \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be two convenient discrete sets such that $\Gamma_+(S) \subsetneqq \Gamma_+(S')$ and $\nu(S) = \nu(S')$. Let us suppose that*

$$\alpha \in \mathrm{Ver}(S', S) \cap \mathbb{R}^I_{>0}, \qquad I \subsetneqq \{1, .., n\}.$$

Then there exists $i \in I^c$ such that for all the edges E_α of $\Gamma(S')$ not contained in $\mathbb{R}_{\{i\}}$ there exists $(\beta_1, ..., \beta_n) \in \mathrm{Ver}(S) \cap E_\alpha$ such that $\beta_i = 1$.

Proof. First we will prove the following Lemma.

Lemma 2.12. *Let $S \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be an I^c-convenient discrete set and*

$$\alpha \in \mathbb{R}^I_{>0}, \qquad I \subset \{1, .., n\},$$

such that $\nu(S) = \nu(S(\alpha))$. Then there exists $i \in I^c$ such that for each edge E_α of $\Gamma(S(\alpha))$ not contained in $\mathbb{R}_{\{i\}}$ there exists $(\beta_1, ..., \beta_n) \in \mathrm{Ver}(S) \cap E_\alpha$ such that $\beta_i = 1$.

Proof of the Lemma. By Corollary 2.7, we have $|I| < n$. Let k be the greatest element of $\{1, \dots, n-1\}$ such that the Lemma is false for some I with $|I| = k$. In other words, for all $i \in I^c$ there exists an edge E_α, not contained in $\mathbb{R}_{\{i\}}$, such that for all $(\beta_1, ..., \beta_n) \in \mathrm{Ver}(S) \cap E_\alpha$ we have $\beta_i > 1$. Let $J \subset \{1, ..., n\}$ be a set of the smallest cardinality such that $E_\alpha \subset \mathbb{R}^J$. Then E_α is a strictly (I, J)-convenient edge. Using Proposition 2.9 we obtain that for all $\alpha' \in \overline{\Gamma_+(S(\alpha) \backslash \Gamma_+(S)} \cap \mathbb{R}^J_{>0}$ we have $\nu(S(\alpha')) = \nu(S)$. Now let us choose α' sufficiently close to α so that for each edge $E_{\alpha'}$ of $\Gamma(S(\alpha'))$, and $\beta \in E_{\alpha'} \cap \mathrm{Ver}(S)$ adjacent to α' in $E_{\alpha'}$, there exists an edge E_α of $\Gamma(S(\alpha))$ such that $\beta \in E_\alpha$. Then the discrete sets S, $S(\alpha')$ are J^c-convenient and do not satisfy the conclusion of the Lemma, which is a contradiction, since $|J| > k$. □

The proof of the Proposition is by induction on the cardinality of $\mathrm{Ver}(S', S)$. Lemma 2.12 says that the Proposition is true whenever $|\mathrm{Ver}(S', S)| = 1$. Let us assume that the Proposition is true for all S, S' such that

$$|\mathrm{Ver}(S', S)| \leqslant m - 1.$$

Let S, S' with $|\mathrm{Ver}(S', S)| = m \geqslant 2$ be such that the Proposition is false. Then there exists $\alpha \in \mathrm{Ver}(S', S)$ such that for each $i \in I^c$ there exists an edge E_α of $\Gamma_+(S')$, not contained in $\mathbb{R}_{\{i\}}$, that satisfies the following condition:

(*) for all $\beta = (\beta_1, ..., \beta_n) \in \mathrm{Ver}(S) \cap E_\alpha$ we have $\beta_i > 1$;

note that condition (*) is vacuously true if

$$\mathrm{Ver}(S) \cap E_\alpha = \varnothing. \tag{1}$$

Observe that for each $\alpha' \in \mathrm{Ver}(S', S) \backslash \{\alpha\}$, we have $|\mathrm{Ver}(S', S(\alpha'))| = m - 1$ and, by Corollary 2.8, $\nu(S(\alpha')) = \nu(S')$.

First, let us suppose that there exists $i \in I^c$ such that (1) does not hold for the corresponding edge E_α. Let us fix $\alpha' \in \mathrm{Ver}(S', S) \backslash \{\alpha\}$. Then E_α connects α with a vertex β of S, hence $\alpha' \notin E'$, where $E' \subset E_\alpha$ is the line segment with endpoints α and β. We obtain that the polyhedra $\Gamma_+(S(\alpha')) \subsetneqq \Gamma_+(S')$ do not satisfy the conclusion of the Proposition, which contradicts the induction hypothesis.

Next, let us suppose that there exists $i \in I^c$ such that (1) is satisfied for the corresponding edge E_α. Then $|E_\alpha \cap \mathrm{Ver}(S', S)| = 2$. Now, take

$$\alpha'' = (\alpha'_1, ..., \alpha'_n) \in E_\alpha \cap \mathrm{Ver}(S', S)$$

such that $\alpha' \neq \alpha$. If $\alpha'_i > 1$, then the Newton polyhedra

$$\Gamma_+(S(\alpha')) \subsetneqq \Gamma_+(S')$$

do not satisfy the Proposition and (1) does not hold, which is a contradiction. Hence $\alpha'_i = 1$. Let $\epsilon > 0$ be such that

$$\alpha'_\epsilon := \alpha' + \epsilon e_i \in \left(\Gamma_+\left(S'\right) \backslash \Gamma_+(S)\right).$$

Put

$$R := (\mathrm{Ver}(S', S) \backslash \{\alpha'\}) \cup \{\alpha'_\epsilon\}.$$

Then $\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha'_\epsilon))) \subsetneqq \Gamma_+(S(R))) \subsetneqq \Gamma_+(S')$.

The discrete sets S, $S(\alpha'_\epsilon)$, $S(R)$, and S' are convenient. We have

$$\nu(S(\alpha'_\epsilon)) = \nu(S(R)) = \nu(S).$$

Let us assume that ϵ is small enough so that there exists an edge $E'_\alpha \ni \alpha$ of $\Gamma(S(R))$ such that $\alpha'_\epsilon \in E'_\alpha$. Then the Newton polyhedra $\Gamma_+(S) \subsetneqq \Gamma_+(S(R))$ satisfy the preceding case (namely, $\alpha'_i > 1$). This completes the proof of the Proposition. □

Corollary 2.13. *Assume given two convenient discrete sets $S, S' \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ such that $\Gamma_+(S) \subsetneqq \Gamma_+(S')$ and $\nu(S) = \nu(S')$. Assume that*

$$\alpha \in \mathrm{Ver}(S', S) \cap \mathbb{R}^I_{>0}.$$

The for the $i \in I^c$ of Proposition 2.11 there exists an edge E_α of $\Gamma(S')$, and $(\beta_1, ..., \beta_n) \in E_\alpha \cap \mathrm{Ver}(S)$, such that $\beta_j = \delta_{ij}$, $j \in I^c$, where δ_{ij} is the Kronecker delta.

Proof of the Corollary. By Proposition 2.11 there exists $i \in I^c$ such that for all the edges E_α of $\Gamma(S')$, not contained in $\mathbb{R}_{\{i\}}$, there exists

$$(\beta_1, ..., \beta_n) \in \mathrm{Ver}(S) \cap E_\alpha$$

such that $\beta_i = 1$. Since the set S is convenient, there exists $m > 1$ such that $me_i \in \mathrm{Ver}(S)$. Let $J = I \cup \{i\}$. Since $\alpha, me_i \in \mathbb{R}^J$, there exists a chain of edges of $\Gamma(S')$ connecting α with me_i, contained in $\mathbb{R}^J$. The edge E_α belonging to this chain and containing α satisfies the conclusion of the Corollary. □

Remark 2. *Using the same idea as in Corollary 2.13, but using Lemma 2.12 instead of Proposition 2.11, we can prove the following fact. Let $I \subsetneqq \{1, ..., n\}$ and let $S \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be an I-convenient discrete set. Let $\alpha \in \mathbb{R}^I_{>0}$ be such that $\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha))$, and $\nu(S) = \nu(S')$. Then for the $i \in I^c$ of Lemma 2.12 there exists an edge E_α of $\Gamma(S(\alpha))$, and $(\beta_1, ..., \beta_n) \in E_\alpha \cap \mathrm{Ver}(S)$, such that $\beta_j = \delta_{ij}$, $j \in I^c$, where δ_{ij} is the Kronecker delta.*

The following Theorem generalizes to all dimensions the main theorem of [BKW19]. In [BKW19] this result is conjectured.

Definition 2.14. *Let $S, S' \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be two discrete sets such that*

$$\Gamma_+(S) \subsetneqq \Gamma_+(S'),$$

$I \subset \{1, ..., n\}$ and $\alpha \in \mathrm{Ver}(S', S) \cap \mathbb{R}^I_{>0}$. We will say that α has an apex if:

1. *$I \subsetneqq \{1, ..., n\}$*
2. *There exists $i \in I^c$ and a unique edge of E_α of $\Gamma(S')$ that contains α and is not contained in $\mathbb{R}_{\{i\}}$.*

In this case the point $\beta \in \mathrm{Ver}(S) \cap E_\alpha$ adjacent to α in E_α is called the apex of α. We will say that an apex, $\beta := (\beta_1, ..., \beta_n)$, is good if $\beta_j = \delta_{ij}$, $j \in I^c$.

Remark 3. *Let $S \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be a convenient discrete set,$I \subsetneqq \{1, ..., n\}$ and $\alpha \in \mathbb{R}^I_{>0}$ such that $\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha))$. The condition that α has a good apex $\beta \in \mathbb{R}^{I\cup\{i\}}$, $i \in I^c$, is equivalent to $P := \overline{\Gamma_+(S(\alpha))\backslash\Gamma_+(S)}$ being a pyramid with apex β and base $P \cap \mathbb{R}_{\{i\}}$.*

Theorem 2.15. *Let $S, S' \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be two convenient discrete sets such that $\Gamma_+(S) \subsetneqq \Gamma_+(S')$. Then $\nu(S) = \nu(S')$ if and only if each $\alpha \in \mathrm{Ver}(S', S)$ has a good apex.*

Proof. First we will prove the following Lemma.

Lemma 2.16. *Let $S \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be a discrete set and $\alpha \in \mathbb{R}^I_{>0}$, $I \subsetneqq \{1,...,n\}$, such that $\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha))$, and α has a good apex. Then*

$$\nu(S(\alpha)) = \nu(S).$$

Proof of the lemma. Let β be a good apex of α. Let $i \in I^c$ be such that $\beta \in \mathbb{R}^{I\cup\{i\}}_{>0}$.

Given an element $m \in \{1,\dots,n\}$ and $J \subset \{1,...,n\}$ such that $|J| = m$, we will use the notation

$$V_m(\alpha, J) = \mathrm{Vol}_m(\Gamma_-(S(\alpha)) \cap \mathbb{R}^J) - \mathrm{Vol}_m(\Gamma_-(S) \cap \mathbb{R}^J)$$

As $\alpha \in \mathbb{R}^I_{>0}$, we have

$$\begin{aligned}\nu(S(\alpha)) - \nu(S) = & \sum_{m=|I|}^{n} (-1)^m \sum_{\substack{|J|=m \\ I\subset J}} |J|!V_m(\alpha, J) \\ = & \sum_{m=|I|}^{n-1} (-1)^m \sum_{\substack{|J|=m \\ i\notin J,\, I\subset J}} (|J|!V_m(\alpha, J) - (|J|+1)!V_{m+1}(\alpha, J\cup\{i\}))\end{aligned}$$

As the apex of α is good, we obtain

$$|J|!V_m(\alpha, J) = (|J|+1)!V_{m+1}(\alpha, J\cup\{i\})$$

which implies that $\nu(S) = \nu(S(\alpha))$. □

Now we will prove that if each $\alpha \in \mathrm{Ver}(S', S)$ has a good apex, then

$$\nu(S) = \nu(S').$$

The proof is by induction on the cardinality of $\mathrm{Ver}(S', S)$. Let us assume that the implication is true for all S and S' such that $|\mathrm{Ver}(S', S)| < m$. To verify the implication for $|\mathrm{Ver}(S', S)| = m$, let $\alpha \in \mathrm{Ver}(S', S)$ and $R = \mathrm{Ver}(S', S)\backslash\{\alpha\}$. By the induction hypothesis $\nu(S(R)) = \nu(S)$ and by Lemma 2.16 we have $\nu(S') = \nu(S(R))$. This proves that $\nu(S') = \nu(S)$.

To finish the proof of the Theorem we need the following Lemma.

Lemma 2.17. *Let $S \subset \mathbb{R}^n_{\geqslant 0}\backslash\{o\}$ be a discrete set and let $\alpha \in \mathbb{R}^I_{>0}$, $I \subsetneqq \{1,...,n\}$, be such that $\Gamma_+(S) \subsetneqq \Gamma_+(S(\alpha))$. Let us suppose that $S(\alpha)$ is I^c-convenient and that $\nu(S(\alpha)) = \nu(S)$. Then α has a good apex.*

Proof of the Lemma. Let $i \in I^c$ be as in Remark 2. Then there exists E_α of $\Gamma(S(\alpha))$, and $\beta := (\beta_1, ..., \beta_n) \in E_\alpha \cap \mathrm{Ver}(S)$, such that $\beta_j = \delta_{ij}$, $j \in I^c$. We want to prove that β is a (necessarily good) apex of α. Let us assume that β is not an apex of α, aiming for contradiction. Then there exits another edge $\alpha \in E'_\alpha$ de $\Gamma(S(\alpha))$, and $\beta' := (\beta'_1, ..., \beta'_n) \in E'_\alpha \cap \mathrm{Ver}(S)$ adjacent to α in E'_α such that $\beta'_i = 1$.

Let us consider $\beta'_\epsilon := \beta' + \epsilon e_i$, and the discrete set $S^\epsilon = (S\backslash\{\beta'\}) \cup \{\beta'_\epsilon\}$, $\epsilon > 0$. Let us assume that ϵ is small enough so that:

(1) $\mathrm{Ver}(S^\epsilon) = (\mathrm{Ver}(S)\backslash\{\beta'\}) \cup \{\beta'_\epsilon\}$
(2) There exists an edge E^ϵ_α of $\Gamma(S^\epsilon(\alpha))$ such that $\beta_\epsilon \in E^\epsilon_\alpha \cap \mathrm{Ver}(S^\epsilon)$ is adjacent to α in E^ϵ_α.

Let $P^\epsilon = \overline{(\Gamma_+(S^\epsilon(\alpha)\backslash\Gamma_+(S^\epsilon))}$. Let Q_0 be the convex hull of the set

$$\{\beta\} \cup \left(P^\epsilon \cap \mathbb{R}_{\{i\}}\right)$$

(observe that Q_0 does not depend on ϵ) and $Q_1^\epsilon := \overline{P^\epsilon\backslash Q_0}$. Recall that β satisfies $\beta_j = \delta_{ij}$, $j \in I^c$. Then, using the same idea as in the proof of Lemma 2.16 we obtain $\nu(Q_0) = 0$. As $\dim\left(Q_0^J \cap (Q_1^\epsilon)^J\right) < |J|$ for all $J \subset \{1, ..., n\}$, we have

$$\nu(P^\epsilon) = \nu(Q_0) + \nu(Q_1^\epsilon). \text{ Then } \nu(P^\epsilon) = \nu(Q_1^\epsilon).$$

As $S^\epsilon(\alpha)$ is I^c-convenient, Q_1^ϵ satisfies the hypotheses of Proposition 2.4 (to prove this statement use the same idea as in the proof of Proposition 2.9). Let us consider the sequence

$$I \cup \{i\} \subset I_1, I_2, ..., I_m \subset \{1, ..., n\},$$

and the polyhedra Z_j^ϵ, $1 \leqslant j \leqslant m$, such that

(1) $Q_1^\epsilon = \bigcup_{j=1}^m Z_j^\epsilon$

(2) $\nu(Q_1^\epsilon) = \sum_{j=1}^m \nu\left(Z_j^\epsilon\right) = 0$

(3) $\nu\left(Z_j^\epsilon\right) = |I_j|!V_{|I_j|}\left(\left(Z_j^\epsilon\right)^{I_j}\right)\nu\left(\pi_{I_j}\left(Z_j^\epsilon\right)\right) \geqslant 0$

(the existence of these objects is given by Proposition 2.4). For each j, $1 \leqslant j \leqslant m$, we may choose the family Z_j^ϵ of polyhedra to vary continuously with ϵ. More precisely, we can choose the Z_j^ϵ to satisfy the following additional condition: for each j, $1 \leqslant j \leqslant m$, either $Z_j^\epsilon = Z_j^0$ for all small ϵ or $\text{Ver}(Z_j^\epsilon)$ differs from $\text{Ver}(Z_j^0)$ in exactly one element, $\beta'_\epsilon \neq \beta'$, for all small $\epsilon > 0$. Since $i \in I_j$, we have $\pi_{I_j}(\beta'_\epsilon) = \pi_{I_j}(\beta')$. This implies that $\nu\left(\pi_{I_j}\left(Z_j^\epsilon\right)\right)$ is independent of ϵ for all $1 \leqslant j \leqslant m$. For $\epsilon = 0$, we have

$$\nu\left(\pi_{I_j}\left(Z_j^0\right)\right) = 0.$$

Hence $\nu(P^\epsilon) = \nu(Q_1^\epsilon) = 0$ for ϵ small enough. Then there exists a0 set J, $\{i\} \cup I \subset J \subset \{1, 2, ..., n\}$, such that the edge E_α^ϵ is strictly (I, J)-convenient. By Proposition 2.9, given $\alpha' \in \overline{\Gamma_+(S^\epsilon(\alpha))\backslash\Gamma_+(S^\epsilon)} \cap \mathbb{R}_{>0}^J$ we have

$$\nu(S^\epsilon(\alpha')) = \nu(S^\epsilon).$$

This proves that $|I| < n - 1$: indeed, if $|I| = n - 1$, then $\alpha' \in \mathbb{R}_{>0}^n$, which contradicts Proposition 2.7.

Let r be the largest element of $\{1, \ldots, n-1\}$ such that the Lemma is true for all I such that $|I| > r$. Now let us assume that $|I| = r$. Let us choose α' sufficiently close to α so that for each edge $E_{\alpha'}$ of $\Gamma(S^\epsilon(\alpha'))$ and

$$\beta \in E_{\alpha'} \cap \text{Ver}(S^\epsilon)$$

adjacent to α' in $E_{\alpha'}$, there exists an edge E_α of $\Gamma(S^\epsilon(\alpha))$ such that $\beta \in E_\alpha$. This implies that α' does not have a good apex, which contradicts the choice of r, since $|J| > r$. This completes the proof of the Lemma. □

Now we can finish the proof of the Theorem. We will prove that if

$$\nu(S) = \nu(S')$$

then each $\alpha \in \mathrm{Ver}(S', S)$ has a good apex. The proof is by induction on the cardinality of $\mathrm{Ver}(S', S)$. Lemma 2.17 shows that the implication is true for $|\mathrm{Ver}(S', S)| = 1$. Let us assume that this is true for every pair (S, S') of convenient discrete sets such that $|\mathrm{Ver}(S', S)| < m$. Let us prove the result for $|\mathrm{Ver}(S', S)| = m$. Let $\alpha \in \mathrm{Ver}(S', S)$, $R = \mathrm{Ver}(S', S)\backslash\{\alpha\}$ and $\alpha_\epsilon = (1+\epsilon)\alpha$, where $\epsilon > 0$. Then

$$\Gamma_+(S) \subseteq \Gamma_+(S(\alpha_\epsilon)) \subsetneqq \Gamma_+(S(\alpha_\epsilon)(R)) \subseteq \Gamma_+(S').$$

By Corollary 2.3 we have $\nu(S(\alpha_\epsilon)(R)) = \nu(S(\alpha_\epsilon))$. Observe that

$$|\mathrm{Ver}(S(\alpha_\epsilon)(R), S(\alpha_\epsilon))| \leqslant m - 1.$$

By the induction hypothesis, each $\alpha' \in R$ has a good apex $\beta \in \mathrm{Ver}(S(\alpha_\epsilon))$ for the inclusion $\Gamma_+(S(\alpha_\epsilon)) \subsetneqq \Gamma_+(S(\alpha_\epsilon)(R))$ of Newton polyhedra. Since all the non-zero coordinates of α_ϵ are strictly greater than 1, we have $\beta \neq \alpha_\epsilon$, so that $\beta \in \mathrm{Ver}(S)$. We take ϵ small enough so that for every $\alpha' \in R$ every edge $E_{\alpha'}$ of $\Gamma(S(\alpha_\epsilon)(R))$ that connects α' with a vertex in $\mathrm{Ver}(S)$ is an edge of $\Gamma(S')$. Thus every $\alpha' \in R$ has a good apex for the inclusion

$$\Gamma_+(S) \subset \Gamma_+(S')$$

of Newton polyhedra.

Now it suffices to verify that α has a good apex for the inclusion

$$\Gamma_+(S) \subset \Gamma_+(S') \tag{2}$$

of Newton polyhedra. Let $\epsilon > 0$ and put $R_\epsilon := \{(1+\epsilon)\alpha' : \alpha' \in R\}$. Then

$$\Gamma_+(S) \subset \Gamma_+(S(R_\epsilon)) \subsetneqq \Gamma_+(S(R_\epsilon)(\alpha)) \subset \Gamma_+(S').$$

By Corollary 2.3 we have $\nu(S(R_\epsilon)) = \nu(S(R_\epsilon)(\alpha)) = \nu(S)$. Observe that $\mathrm{Ver}(S(R_\epsilon)(\alpha), S(R_\epsilon)) = \{\alpha\}$. By Lemma 2.17, α has a good apex

$$\beta \in \mathrm{Ver}(S(R^\epsilon)).$$

Since every non-zero coordinate of every element of R_ϵ is strictly greater than 1, we have $\beta \notin R_\epsilon$, so that $\beta \in \mathrm{Ver}(S)$. Take ϵ small enough so that every edge E_α of $\Gamma(S(R_\epsilon)(\alpha))$ that connects α with a vertex in $\mathrm{Ver}(S)$ is an edge of $\Gamma(S')$. Then β is a good apex of α for the inclusion (2), as desired. This completes the proof of the Theorem. □

We end this section by recalling a result that relates the Milnor number to the Newton number.

If the formal power series g is not convenient, we can define the Newton number $\nu(g)$ of g ($\nu(g)$ could be ∞) in the following way. Let $\mathcal{E}' \subset \mathcal{E}$ such that there does not exist $m \in \mathbb{Z}_{>0}$, such that $me \in \mathrm{Ver}(g)$. We define the Newton number of g as

$$\nu(g) := \sup_{m \in \mathbb{Z}_{>0}} \nu(\mathrm{Supp}(g) \cup {\mathcal{E}'}_m),$$

where ${\mathcal{E}'}_m := \{me : e \in \mathcal{E}'\}$.

Theorem 2.18 (See [Kou76]). *Let $h \in \mathcal{O}^x_{n+1}$. Then $\mu(h) \geqslant \nu(h)$. Moreover, $\mu(h) = \nu(h)$ if h is non-degenerate.*

Remark 4. *Let $h \in \mathcal{O}^x_{n+1}$ be non-degenerate and convenient. Then $\mu(h) < \infty$, which implies that h has, at most, an isolated singularity in the origin o.*

Example 1. *Consider the following families of non-degenerate deformations:*

$$F^\lambda(x,y,z,s) := x^{5\lambda} + y^{7\lambda}z + z^{15} + y^{8\lambda} + sx^\lambda y^{6\lambda},\ \lambda \geqslant 1.$$

Observe that F^1 is a μ-constant deformation of Briançon-Speder (convenient version), see [BS75]. By virtue of Theorem 2.18 and Proposition 2.6, for each $\lambda \geqslant 1$ the deformation F^λ is non-degenerate and μ-constant.

3. Characterization of Newton non-degenerate μ-constant deformations

First let us recall some information regarding the Newton fan and toric varieties. Given $S \subset \mathbb{Z}^{n+1}_{\geqslant 0}\backslash\{o\}$, consider the support function

$$\mathrm{h}_{\Gamma_+(S)} : \Delta \to \mathbb{R};\ \alpha \mapsto \mathrm{h}_{\Gamma_+(S)}(\alpha) := \inf\{\langle \alpha, p\rangle \mid p \in \Gamma_+(S)\},$$

where $\Delta := \mathbb{R}^{n+1}_{\geqslant 0}$ is the standard cone, and $\langle\cdot,\cdot\rangle$ is the standard scalar product. Let $1 \leqslant i \leqslant n$, and let F be an i-dimensional face of the Newton polyhedron $\Gamma_+(S)$. The set $\sigma_F := \{\alpha \in \Delta :\ \langle \alpha, p\rangle = \mathrm{h}_{\Gamma_+(S)}(\alpha),\ \forall p \in F\}$ is a cone, and $\Gamma^\star(S) := \{\sigma_F :\ F \text{ is a face of } \Gamma_+(S)\}$ is a subdivision of the fan Δ (by abuse of notation we will denote for Δ the fan induced by the standard cone Δ). The fan $\Gamma^\star(S)$ is called the Newton fan of S. Let $\Delta' \lneq \Delta$ be a strict face of the standard cone Δ , and $(\Delta')^\circ$ is the interior relative to Δ'. Observe that if there exists $\alpha \in (\Delta')^\circ$ such that $\mathrm{h}_{\Gamma_+(\alpha)} = 0$, then Δ' is a cone of the fan $\Gamma^\star(S)$. We will say that Σ is an *admissible subdivision* of $\Gamma^\star(S)$ if Σ is a subdivision that preserves the previous property, which is to say that if there exists $\alpha \in (\Delta')^\circ$ such that $\mathrm{h}_{\Gamma_+(S)}(\alpha) = 0$, then $\Delta' \in \Sigma$. In the case that the discrete set S is convenient, an admissible subdivision of $\Gamma^\star(S)$ is a fan where there are not subdivisions of the strict faces of Δ.

Given a fan Σ, we denote X_Σ the toric variety associated to the fan Σ. Given $\sigma \in \Sigma$, we denote X_σ as the open affine of X_Σ associated to the cone σ. Let Σ' be a subdivision of Σ, it is known that there exists a proper, birational and equivariant morphism,$\pi : X_{\Sigma'} \to X_\Sigma$, induced by the subdivision. Given $\sigma' \in \Sigma'$, we denote $\pi_{\sigma'} := \pi|_{X_{\sigma'}}$.

Now we will use the notations from Section 1.0.1. Let V be a hypersurface of $\mathbb{C}^{n+1}_o$, provided by a unique isolated singularity at the point o. Let us assume that V is given by the equation $f(x) = 0$, where $f \in \mathcal{O}^x_{n+1}$ is irreducible, and let $\varrho : W \to \mathbb{C}^m_0$ be a deformation of V given by $F(x,s) \in \mathbb{C}\{x_1,...,x_{n+1},s_1,...,s_m\}$.

Without loss of generality we can assume that the germ of analytic function f is convenient. In effect the Milnor number, $\mu(f) := \dim_{\mathbb{C}} \mathcal{O}^x_{n+1}/J(f)$, is finite, then for each $e \in \mathcal{E}$ there exists $m >> 0$ such that x^{me} belongs to the ideal $J(f)$, which implies that the singularity of f and of $f + x^{me}$ have the same analytic type.

Let Σ be an admissible subdivision of $\Gamma^\star(F_s)$ (not necessarily regular), and we denote $\pi : X_\Sigma \to \mathbb{C}^{n+1}$ morphism given by the subdivision of Δ. Using the morphism $\mathbb{C}^{n+1}_o \to \mathbb{C}^{n+1}$ we can consider the base change of π and

X_Σ to the base $\mathbb{C}_o^{n+1}$. By abuse of notation we will note for $\pi : X_\Sigma \to \mathbb{C}_0^{n+1}$ the base change morphism.

Let us recall the following known fact. Let V' be a hypersurface of $\mathbb{C}_o^{n+1}$, $n \geqslant 1$, provided by a unique isolated singularity at the point o. Let us suppose that V' is given by the equation $g(x) = 0$, where $g \in \mathcal{O}_{n+1}^x$. Let us suppose that Σ is a regular admissible subdivision of a Newton fan $\Gamma^\star(g)$. If g is non-generate with respect to the Newton boundary, then the morphism between toric varieties $\pi : X_\Sigma \to \mathbb{C}_o^{n+1}$ defines an embedded resolution of V' in a neighborhood of $\pi^{-1}(o)$ (see [Var76], [Oka87] or [Ish07]). This shows that if $\Gamma_+(F_s) = \Gamma_+(f)$, where s is the generic point of $\mathbb{C}_o^m$, and F is a non degenerate deformation of f (μ-constant deformations of f for the Theorem 2.18 in particular), a regular admissible resolution of the Newton fan defines a simultaneous embedded resolution of W. Whereby for the rest of this section we will assume:

(1) $F(x, s) \in \mathbb{C}\{x_1, ..., x_{n+1}, s_1, ..., s_m\}$ is a non-degenerate μ-constant deformation of f.
(2) $\Gamma_+(F_s) \neq \Gamma_+(f)$. In particular $\mathrm{Ver}(F_s, f) := \mathrm{Ver}(F_s) \backslash \mathrm{Ver}(f) \neq \varnothing$.

Let morphism $\varphi : X_\Sigma \times \mathbb{C}_o^m \to \mathbb{C}_o^{n+1} \times \mathbb{C}_o^m$ be induced by π. Let s be the generic point of $\mathbb{C}_o^m$. Given $\alpha \in \mathrm{Ver}(F_s)$ we denote for σ_α the $(n+1)$-dimensional cone of $\Gamma^\star(F_s)$ generated by all the non-negative normal vectors to faces of $\Gamma_+(F_s)$ which contain to α. Denote for $\widetilde{W}^t$ the total transform of W by φ.

Proposition 3.1. *Let s be the generic point of $\mathbb{C}_o^m$, and let us assume that $\nu(F_s) = \nu(f)$. Then there exists an admissible subdivision, Σ, of $\Gamma^\star(F_s)$, such that for each $\alpha \in \mathrm{Ver}(F_s, f)$, the fan Σ defines a subdivision, $\{\sigma_\alpha^1, ..., \sigma_\alpha^r\}$, regular to σ_α, such that $\widetilde{W}^t \cap (X_{\sigma_\alpha^j} \times \mathbb{C}_o^m)$ is a normal crossings divisor relative to $\mathbb{C}_o^m$ for $j \in \{1, ..., r\}$.*

Proof. Let us recall that $\mathcal{E} := \{e_1, e_2, ..., e_{n+1}\} \subset \mathbb{Z}_{\geqslant 0}^{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. First we will construct a simplicial subdivision of $\Gamma^\star(F_s)$. Let $\Gamma^\star(F_s)(j)$ be the set of all the j-dimensional cones of $\Gamma^\star(F_s)$. Let us consider a compatible simplicial subdivision, ΣS, of $\bigcup_{j=1}^{n} \Gamma^\star(F_s)(j)$, such that if σ' is a simplicial j-dimensional cone of $\Gamma^\star(F_s)(j)$, $1 \leqslant j \leqslant n$, then $\sigma' \in \Sigma S$ and $\Sigma S(1) = \Gamma^\star(F_s)(1)$, where $\Sigma S(1)$ is the set of all the 1-dimensional cones of ΣS.

Let us consider the case

$$\alpha \in \mathrm{Ver}(F_s, f).$$

By Theorem 2.15, α has a good apex. Then there exists $I \subsetneqq \{1, ..., n+1\}$ such that $\alpha \in \mathbb{R}_{>0}^I$ and $i \in I^c$ such that there exists a single edge $E_\alpha \ni \alpha$, of $\Gamma(F_s)$ not contained in $\mathbb{R}_{\{i\}}$. Let $\beta = (\beta_1, ..., \beta_{n+1}) \in \mathrm{Ver}(F_s) \cap E_\alpha$ be the good apex, which is to say $\beta_i = \delta_{ij}$, $j \in I^c$.

Observe that $e_i \in \mathcal{E}$, is an extremal vector of σ_α. Let us consider the following simplicial subdivision of σ_α:

$$\Sigma^s(\sigma_\alpha) := \{\mathrm{Cone}(e_i, \tau) : \tau \in \Sigma S \text{ and } \tau \subset \sigma_\alpha\} \cup \{\tau \in \Sigma S : \tau \subset \sigma_\alpha\},$$

where cone $\mathrm{Cone}(\{\cdot\})$ is the cone generated by $\{\cdot\}$. Now let us consider the case

$$\alpha \in \mathrm{Ver}(F_s) \backslash \mathrm{Ver}(F_s, f).$$

let $\Sigma^s(\sigma_\alpha)$ be an arbitrary simplicial subdivision of σ_α that is compatible with ΣS. Then

$$\Sigma^s := \bigcup_{\alpha \in \mathrm{Ver}(F_s)} \Sigma^s(\sigma_\alpha)$$

is a simplicial subdivision of $\Gamma^\star(F_s)$. As F_s is convenient, the faces of σ_α, $\alpha \in \mathrm{Ver}(F_s)$, contained in a coordinate plane are simplicial cones, then Σ^s is an admissible subdivision.

Now we will define a subdivision of Σ^s to obtain the sought after fan.

Let $\alpha \in \mathrm{Ver}(F_s, f)$. By abuse of notation we will denote for σ_α a cone in $\Sigma^s(\sigma_\alpha)(n+1)$. Without loss of generality we can suppose $i = n+1$, in this manner we have that $\sigma_\alpha = \mathrm{Cone}(e_{n+1}, \tau)$ with $\tau \in \Sigma S$. We denote $H_0 = \mathbb{R}_{\{n+1\}} \cap \Gamma_+(F_s)$ and H_1,....,H_n the n-dimensional faces of $\Gamma_+(F_s)$ that define σ_α, then $\bigcap_{j=0}^{n} H_j = \{\alpha\}$. Then $E_\alpha := \bigcap_{j=1}^{n} H_j$

Let $p_1, ..., p_n$ be non-negative normal vectors to the faces $H_1, ..., H_n$. Then $\sigma_\sigma := \mathrm{Cone}(p_1, ..., p_n, e_{n+1})$. Now we will construct a regular subdivision of σ_α. Let us consider the cone $\tau := \mathrm{Cone}(p_1, ..., p_n) \subset \sigma_\alpha$, and a regular subdivision $\mathrm{RS}(\tau)$ of τ that does not subdivide regular faces of τ. Then $\mathrm{RS}(\tau)$ does not subdivide faces $\Delta' \precneqq \Delta$. Let $\tau' \in \mathrm{RS}(\tau)$, then there exists $q_1, ..., q_n \in \mathrm{Cone}(p_1, ..., p_n)$ such that $\tau' := \mathrm{Cone}(q_1, ..., q_n)$. Observe that the cones

$$(\star) \qquad \sigma'_\alpha := \mathrm{Cone}(q_1, ..., q_n, \mathrm{e}_{n+1})$$

define a subdivision of the cone σ_α that can be extended to a subdivision Σ of Σ^s that does not subdivide faces $\Delta' \precneqq \Delta$, which implies that Σ is admissible.

Now we will prove that $\sigma'_\alpha := \mathrm{Cone}(q_1, ..., q_n, e_{n+1})$ is regular. Looking at q_j as column vectors, and consider the matrix of the size $(n+1) \times n$:

$$A := \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix} = \begin{pmatrix} q_{1\,1} & \cdots & q_{n\,1} \\ \vdots & \vdots & \\ q_{1\,n+1} & \cdots & q_{n\,n+1} \end{pmatrix}$$

For each $j \in \{1, ..., n+1\}$ let A_j be the matrix of the size $n \times n$ obtained by deleting the row j of the matrix A. As $\tau' := \mathrm{Cone}(q_1, ..., q_n)$ is regular, we have that the greatest common divisor, $\gcd(d_1, ..., d_{n+1})$, where

$$d_j = |\det(A_j)|,$$

is equal to 1. Let us suppose that the cone

$$\sigma'_\alpha := \mathrm{Cone}(q_1, ..., q_n, e_{n+1})$$

is not regular, then $|\det(q_1, ..., q_n, e_{n+1})| = d_{n+1} \geqslant 2$. For each H_j, $1 \leqslant j \leqslant n$ we have that $\alpha, \beta \in H_j$, then $\langle \alpha, p_j \rangle = \langle \beta, p_j \rangle$ for all $1 \leqslant j \leqslant n$, which implies that $\langle \alpha, q_j \rangle = \langle \beta, q_j \rangle$ for all $1 \leqslant j \leqslant n$. With which we obtain that

$q_{j\,n+1} = \sum_{k=1}^{n}(\alpha_k - \beta_k)q_{jk}$ for all $1 \leqslant i \leqslant n$ (remember that β is the good apex of α). Then d_{n+1} divides to d_j for all $1 \leqslant j \leqslant n$, which contradicts the fact that $\gcd(d_1, ..., d_{n+1}) = 1$. Which implies that σ'_α is regular.

Observe that there exist coordinates $y_1, ..., y_{n+1}$ of $X_{\sigma'_\alpha} \cong \mathbb{C}^{n+1}$ (before the base change) such that the morphism

$$\pi_{\sigma'_\alpha}(y) := \pi_{\sigma'_\alpha}(y_1, ..., y_{n+1}) = (x_1, ..., x_{n+1})$$

is defined by:

$$x_{n+1} := y_1^{q_{1\,n+1}} \cdots y_n^{q_{n\,n+1}} y_{n+1} \text{ and } x_i := y_1^{q_{1i}} \cdots y_n^{q_{ni}},\ 1 \leqslant i \leqslant n.$$

From this we obtain

$$F(\pi_{\sigma'_\alpha}(y), s) = y_1^{m_1} \cdots y_n^{m_n} \overline{F}(y, s),\ \ m_i = \langle q_i, \alpha \rangle,\ 1 \leqslant i \leqslant n.$$

Let us suppose that $r = (r_1, ..., r_{n+1})$ is a singular point of $\overline{F}(y, o)$, then there exists $1 \leqslant j \leqslant n$ such that $r_j = 0$. Without loss of generality we can suppose that $r_n = 0$. We know that for each $\beta' \in E_\alpha \cap \mathrm{Ver}(f)$ we have that $\langle \alpha, q_i \rangle = \langle \beta', q_i \rangle$, for all $1 \leqslant i \leqslant n$, and as α has a good apex, we obtain that

$$\overline{F}(y, s) = c_0(s) + \overline{H}(\overline{y}, s) + \overline{K}(y_{n+1}, s) + y_n \overline{G}(y, s),$$

where $\overline{y} = (y_1, ..., y_{n-1})$, $c_0(o) = 0$, and

$$\overline{K}(y_{n+1}, s) = c_1(s) y_{n+1} + \cdots + c_l(s) y_{n+1}^l,\ c_1(o) \neq 0.$$

If $E_\alpha \cap \mathrm{Ver}(f) = \{\beta\}$, then $\overline{K}(y_{n+1}, s) = c_1(s) y_{n+1}$. Which shows that r cannot be a singular point of $\overline{F}$. If $|E_\alpha \cap \mathrm{Ver}(f)| > 1$, then the singular point $r = (r_1, ..., r_{n+1})$ satisfies that

$$\frac{d\overline{K}(r_{n+1}, 0)}{dy_{n+1}} = 0.$$

Which implies that $r_{n+1} \neq 0$. We will prove that this is contradiction.

Let $W = \mathrm{Ver}(F_s, f) \cap \mathbb{R}_{n+1}$ and we define

$$F'(x, s) = f(x) + \textstyle\sum_{\gamma \in W} d_\gamma(s) x^\gamma,\ d_\gamma(o) = 0 \text{ for all } \gamma \in W.$$

We can assume that F' is a non degenerate deformation of f. As $\Gamma_+(f) \subset \Gamma_+(F'_s) \subset \Gamma_+(F_s)$, we have that $\nu(F'_s) = \nu(f)$ (see corollary 2.3). By definition of F', the point α belongs to $\mathrm{Ver}(F'_s, f) = W$. We note σ_α the cone of $\Gamma^\star(F'_s)$ associated to α. By construction the cone σ_α of $\Gamma^\star(F'_s)$ is the cone σ_α of $\Gamma^\star(F_s)$ previously defined. Using the same regular subdivision of σ_α we can define a regular admissible subdivision Σ' of the fan $\Gamma^\star(F'_s)$.

Let σ'_α be one of the two regular cones of the subdivision of σ_α (see $(\star)$). As we previously obtained

$$F'(\pi_{\sigma'_\alpha}(y), s) = y_1^{m_1} \cdots y_n^{m_n} \overline{F'}(y, s),\ \ m_i = \langle q_i, \alpha \rangle,\ 1 \leqslant i \leqslant n.$$

Then r is a singular point of $\overline{F}(y, o)$ if and only if r is a singular point of $\overline{F'}(y, o)$ (in fact $\overline{F}(y, o) = \overline{F'}(y, o)$). We recall that $|E_\alpha \cap \mathrm{Ver}(f)| > 1$, and that E_α is the only edge of $\Gamma_+(F_s)$ not contained in $\mathbb{R}_i$, which contains α and its good apex. Observe that E_α also is the unique edge $\Gamma_+(F'_s)$ which satisfies the previous properties. Let $\beta' \neq \alpha$ an end point of E_α, and $\sigma_{\beta'} \in \Gamma^\star(F_s)$ the cone associated to β'. As $|E_\alpha \cap \mathrm{Ver}(f)| > 1$, and $\mathrm{Ver}(F'_s, f) \subset \mathbb{R}_{n+1}$, we have that the cone $\sigma_{\beta'}$ belongs to $\Gamma^\star(f)$. Then the regular subdivision $\sigma^1_{\beta'}$, ...,$\sigma^t_{\beta'}$

of $\sigma_{\beta'}$ defined by the regular admissible subdivision Σ' can be extended to regular admissible subdivision Σ'' of $\Gamma_+(f)$. By construction there exists $1 \leqslant j \leqslant t$ such that $r \in X_{\sigma^j_{\beta'}} \cong \mathbb{C}^{n+1}$. But f is non degenerate, which implies that $\widetilde{V}^s \cap X_{\sigma^i_{\beta'}}$ is smooth, from where we obtain the sought after contradiction. This implies that $F(\pi_{\sigma'}(y), s)$, which is a normal crossings divisor relative to $\mathbb{C}^m_o$ around $\pi^{-1}_{\sigma'_\alpha}(o) \times \mathbb{C}^m_o$. □

The following theorem is the main result of this article. Let s be the generic point of $\mathbb{C}^m_o$. We will construct a regular admissible subdivision, Σ, of $\Gamma^\star(F_s)$ in the manner that $\rho : X_\Sigma \times (\mathbb{C}^m, o) \to \mathbb{C}^{n+1}_o \times \mathbb{C}^m_o$ is the sought after simultaneous embedded resolution. Observe that for the result commented upon previously, $\pi : X_\Sigma \to \mathbb{C}^{n+1}_o$ defines an embedded resolution of W_s.

Theorem 3.2. *Assume that W is a Newton non-degenerate deformation. The deformation W is μ-constant if and only if W admits a simultaneous embedded resolution.*

Proof. The "if" part is given by Proposition 1.4. We will prove "only if".

By Proposition 3.1 there exists an admissible subdivision, Σ, of $\Gamma^\star(F_s)$ such that for each $\alpha \in \mathrm{Ver}(F_s, f)$, the fan Σ defines a subdivision $\sigma^1_\alpha, ..., \sigma^r_\alpha$, regular of σ_α, such that $\widetilde{W}^t \cap X_{\sigma^i_\alpha} \times \mathbb{C}^m_o$ is a normal crossings divisor relative to $\mathbb{C}^m_o$ for $i \in \{1, ..., r\}$. Consider the set, $\Sigma(j)$, of all the cones of dimension j of Σ. Observe that given a regular admissible subdivision of $\Sigma(j)$, there exists a regular admissible subdivision of $\Sigma(j+1)$ compatible with the given subdivision. Using recurrence we have that there exists a regular admissible subdivision of Σ that does not subdivide its regular cones. By abuse of notation we will denote for Σ the regular admissible subdivision. To finish the proof we still need to consider $\alpha \in \mathrm{Ver}(F_s) \backslash \mathrm{Ver}(F_s, f)$. Let us consider the cone $\sigma \subset \mathbb{R}^{n+1}_{\geqslant 0}$ generated by all the non-negative normal vectors to faces of $\Gamma_+(F_s)$ which contain a α, and let $\sigma^1, ..., \sigma^r$ be the regular subdivision defined by Σ. Let us suppose that $p^i_1, ..., p^i_{n+1}$ are the extremal vectors of σ^i. As σ^i is regular, we have that $X_{\sigma^i} \cong \mathbb{C}^{n+1}$ (before the base change). Then we can associate the coordinates $y_1, ..., y_{n+1}$ to X_{σ^i} such that $\pi_{\sigma^i} := \pi|_{X_{\sigma^i}}$ is defined by

$$\pi_{\sigma^i}(y) := \pi_{\sigma^i}(y_1, ..., y_{n+1}) = x := (x_1, ..., x_{n+1}),$$

where $x_j := y_1^{p^i_{1j}} \cdots y_{n+1}^{p^i_{n+1\, j}}$, $p^i_j := (p^i_{j1}, ..., p^i_{j\, n+1})$, $1 \leqslant j \leqslant n+1$. Then

$$F(\pi_{\sigma^i}(y), s) = y_1^{m_1} \cdots y_{n+1}^{m_{n+1}} \overline{F}(y, s), \;\; m_j = \langle p^i_j, \alpha \rangle, \; 1 \leqslant j \leqslant n+1.$$

Let $c(s)$ be the coefficient of degree zero of $\overline{F}(y, s)$. As $\alpha \notin \mathrm{Ver}(F_s, f)$, then $c(s) \neq 0$ for all $s \in \mathbb{C}^m_o$. Then the property of non degeneracy of F_s implies that $F(\pi_{\sigma^i}(y), s)$ is a normal crossings divisor relative to $\mathbb{C}^m_o$ around $\pi^{-1}_{\sigma^i}(o) \times \mathbb{C}^m_o$. □

4. The Degenerate Case

Let us recall that F is a deformation of f:

$$F(x,s) := f(x) + \sum_{i=1}^{l} h_i(s)g_i(x),$$

where $h_i \in \mathcal{O}_m^s := \mathbb{C}\{s_1,...,s_m\}$, $m \geqslant 1$, and $g_i \in \mathcal{O}_{n+1}^x$ such that $h_i(o) = g_i(o) = 0$. Consider the relative Jacobian ideal

$$J_x(F) := \left(\partial_{x_1}F,...,\partial_{x_{n+1}}F\right) \subset \mathbb{C}\{s_1,...,s_m,x_1,...,x_{n+1}\}.$$

The following theorem gives a valuative criterion for the μ-constancy of a deformation.

Theorem 4.1 (See [Gre86], [LDuS73] and [Tei73])**.** *The following are equivalent:*

1. *F is a μ-constant deformation of f.*
2. *For all $i \in 1,....,m$ we have that $\partial_{s_i}F \in \overline{J_x(F)}$, where $\overline{J_x(F)}$ denotes the integral closure of the ideal $J_x(F)$.*
3. *For all analytic curve $\gamma : (\mathbb{C},o) \to (\mathbb{C}^{n+1}\times\mathbb{C}^m, o)$, $\gamma(o) = o$, and for all $i \in \{1,...,m\}$ we have that:*
$$\mathrm{Ord}_t(\partial_{s_i}F\circ\gamma(t)) > \min\{\mathrm{Ord}_t(\partial_{x_j}F\circ\gamma(t)) \mid 1 \leqslant j \leqslant n+1\}.$$

In the general case the following proposition is analogous to Corollary 2.13. In the rest of the section let us assume that F is a μ-constant deformation of f.

Proposition 4.2. *Let us conserve the hypothesis on f and F, and let us suppose that $\mathrm{Supp}(F_s,f) \cap \mathbb{R}^I_{>0} \neq \varnothing$, $I \subsetneq \{1,...,n+1\}$. Then given $I \subsetneq J \subset \{1,..,n+1\}$, F satisfies at least one of the following conditions:*

1. *$F|_{\mathbb{R}^J}$ is a μ-constant deformation of $f|_{\mathbb{R}^J}$ provided by a unique isolated singularity at the point o.*
2. *There exists $i \in J^c$ and $\beta := (\beta_1,...,\beta_{n+1}) \in \mathrm{Supp}(F_s)$ such that $\beta_i = \delta_{ij}$, for $j \in J^c$.*

A difference between the degenerate case and the non-degenerate cases is that we do not have, in general, that the point $\beta \in \mathrm{Supp}(F_s)$ of the previous proposition belongs to the set $\mathrm{Supp}(f)$.

Example 2. *Consider the following deformation*

$$F(x_1,x_2,x_3,s) := x_1^5 + x_2^6 + x_3^5 + x_2^3x_3^2 + 2sx_1^2x_2^2x_3 + s^2x_1^4x_2.$$

In the article [Alt87] it is shown that F is a μ-constant degenerate deformation of the non-degenerate polynomial $f(x_1,x_2,x_3) := x_1^5+x_2^6+x_3^5+x_2^3x_3^2$. In this example we have that $\mathrm{Ver}(F_s,f) := \{(4,1,0)\} \subset \mathbb{R}^{\{1,2\}}_{>0}$ and $\beta := (2,2,1)$. Observe that $\beta \notin \mathrm{Supp}(f)$.

Proof of the Proposition 4.2. There is not loss of generality in supposing that $J = \{1,...,k\}$, $k \leqslant n$. We can always write F in the following manner:

$$F(x_1,...,x_{n+1},s) = G(x_1,..,x_k,s) + \sum_{k<i} x_iG_i(x_1,...,x_k,s) + \sum_{k<i\leqslant j} x_ix_jG_{ij}(x_1,...,x_{n+1},s),$$

where $s = (s_1,...,s_m)$. Observe that $F|_{\mathbb{R}^J} = G$, and let $g := f|_{\mathbb{R}^J} = G|_{s=0}$. Let us suppose that (2) is not satisfied, then $G_i(x_1,...,x_k,s) \equiv 0$ for all $k < i \leqslant n+1$, then

$$F(x_1,...,x_{n+1},s) = G(x_1,..,x_k,s) + \sum_{k<i\leqslant j} x_i x_j G_{ij}(x_1,...,x_{n+1},s).$$

So we obtain that:

(1) $\partial_l F = \partial_l G + \sum_{k<i\leqslant j} x_i x_j \partial_l G_{ij}$, for $1 \leqslant l \leqslant k$,
(2) $\partial_l F = \sum_{k<i\leqslant l} x_i G_{il} + \sum_{l\leqslant j} x_j G_{lj} + \sum_{k<i\leqslant j} x_i x_j \partial_l G_{ij}$, for $k < l$,
(3) $\partial_{s_{j'}} F = \partial_{s_{j'}} G + \sum_{k<i\leqslant j} x_i x_j \partial_{s_{j'}} G_{ij}$, for $1 \leqslant j' \leqslant m$.

Let us suppose that the singularity of $g(x) = G(x_1,..,x_k,0)$ is not isolated in the origin o. Then for each open set $o \in \Omega \subset \mathbb{C}^{k'}$ there exists $(p_1,...,p_k) \in \Omega$ such that:

(i) $g(p_1,...,p_k) = 0$,
(ii) $\partial_l g(p_1,...,p_k) = 0$, for $1 \leqslant l \leqslant k$.

Then $(p_1,...,p_k,0,...,0) \in \mathbb{C}^{n+1}$ is a singularity of f, which is a contradiction.

Let us suppose that $G(x_1,...,x_k,s)$ is not a μ-constant deformation of g. Then by virtue of theorem 4.1 there exists $1 \leqslant j \leqslant m$, and an analytic curve

$$\gamma(t) := (t^{r_1}a_1(t),...,t^{r_k}a_k(t),t^{q_1}b_1(t),...,t^{q_m}b_m(t)),\ r_i,q_i \in \mathbb{Z}_{>0},$$

such that:

$$\mathrm{Ord}_t\, \partial_{s_j} G \circ \gamma(t) \leqslant \min_{1\leqslant i\leqslant k}\{\mathrm{Ord}_t\, \partial_i G \circ \gamma(t)\}.$$

Let us consider the following analytic curve:

$$\beta(t) := (t^{r_1}a_1(t),...,t^{r_{n+1}}a_{n+1}(t),t^{q_1}b_1(t),...,t^{q_m}b_m(t)).$$

Using the equations (1), (2) and (3), we observe that we can choose the large enough $r_{k+1},...r_{n+1}$, and the $a_{k+1}(t),...a_{n+1}(t)$, which are general enough in the manner that:

(1) $\mathrm{Ord}_t\, \partial_{s_j} F \circ \beta(t) = \mathrm{Ord}_t\, \partial_{s_j} G \circ \gamma(t)$ for $1 \leqslant j \leqslant m$,
(2) $\mathrm{Ord}_t\, \partial_i F \circ \beta(t) = \mathrm{Ord}_t\, \partial_i G \circ \gamma(t)$ for $1 \leqslant i \leqslant k$,
(3) $\mathrm{Ord}_t\, \partial_l F \circ \beta(t) \geqslant \max_{1\leqslant i\leqslant k}\{\mathrm{Ord}_t\, \partial_i F \circ \beta(t)\}$ for $k < l$.

This implies that

$$\mathrm{Ord}_t\, \partial_{s_j} F \circ \beta(t) \leqslant \min_{1\leqslant i\leqslant n+1}\{\mathrm{Ord}_t\, \partial_i F \circ \beta(t)\}.$$

This contradicts Theorem 4.1 since F defines a μ-constant deformation. Then $G(x_1,...,x_k,s)$ is a μ-constant deformation of g or there exists at least one non-zero G_i. □

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