Inconsistency in pairwise comparisons as an Abelian Yang-Mills theory

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***Abstract*—We precise the correspondence between (classical) pairwise comparisons and some inconsistency indicators with the Abelian Yang-Mills theory in quantum physics. This work has already been performed in a more general framework by the author, who feels the need of a more explicit description of this correspondence in the classical framework used in decision theory. This communication is addressed to non-mathematician/physicists, and intends to provide a more comprehensive description of already published results.**

***Index Terms*—Pairwise comparisons, Abelian Yang-Mills theory, inconsitency.**

INTRODUCTION

We have recently linked explicitely inconsistency in pairwise comparisons (PC) with Yang-Mills theories, showing a possible (non-Abelian) generalization of the classical framework [6], [7]. After exchanges and comments, it appears that our works, are ill-received due to the very general framework considered, envolving mild mathematical notions which are not familiar to specialists of operations research. In this communication, we intend to propose a travel guide to understand in which way the (non generalized, classical) PCs fit with an Abelian Yang-Mills theory. We finish by an open problem, from the viewpoint of decision theory, which seems important to us. The same problem can be addressed in physics, namely in Electro-Magnetic fields encoded by a Yang-Mills action functional, but this version will be developped elsewhere in the future.

More precisely, we present a pedestrian approach of the announced correspondence the following way.

- We first recall basics on inconsistency in pairwise comaprisons. Our approach here is partial by choice. Indeed, there exists many non-equivalent ways to measure the inconsistency of a pairwise comparisons matrix. We concentrate our attention on one of them, Koczkodaj(s inconsistency indicator, which realizes the most obvious link with Yang-Mills theory.
- After that, we briefly discuss the consitencization of a pairwise comparisons matrix, mostly along the lines of a recent paper [4]. We address a brief critique of this method. Along our explanations, we need to describe the correspondence between multiplicative and additive pairwise comparisons matrices.
- Then, the link between multiplicative and additive pairwise comparisons matrices is analyzed from the viewpoint of Lie theory, which directly leads to the gauge theoric framework of Yang-Mills fields. The notions of holonomy and discretized connection are outlined. This section also intends to be an introduction to [7] for the reader who is not specialist of gauge theories.
- We conclude by focusing on the notion of inconsistency indicator, which apears to us as central (while most people working in decision theory seem to not care about the choice of the inconsistency indicator they use). Here, we precise that, to our opinion and considering the goal of a consistencization procedure, the prefered method should be a gradient method, which is not the most employed method in the classical litterature for decision making with pairwise comparisons. If so, the choice of the inconsistency indicator appears as the main feature in a modelization procedure. These non consensual considerations should be explored in future studies.

I. PAIRWISE COMPARISONS MATRICES WITH COEFFICIENTS IN $\mathbb{R}_+^*$

A paiswise comparisons matrix $(a_{i,j})$ is a $n \times n$ matrix with coefficients in $\mathbb{R}_+^* = \{x \in \mathbb{R} \,|\, x > 0\}$ such that

$$\forall i, j, \quad a_{j,i} = a_{i,j}^{-1}.$$

It is easy to explain the inconsistency in pairwise comparisons when we consider cycles of three comparisons, called triad and represented here as (x, y, z), which do not have the "morphism of groupoid" property such as

$$x * z \neq y,$$

which reads as

$$xz \neq y$$

in the multiplicative group $\mathbb{R}_+^*$. Evidently, the inconsistency in a triad $(x, y, z) \in \mathbb{R}^3$ is somehow (not linearly) proportional to $y - xz$. In the linear space, the inconsistency is measured by the "approximate flatness" of the triangle. The triad is consistent if the triangle is flat. For example, $(1,2,1)$ and $(10,101,10)$ have the difference $y - xz = 1$ but the inconsistency in the first triad is unacceptable. It is acceptable in the second triad. In order to measure inconsistency, one usually considers coefficients $a_{i,j}$ with values in an abelian group G, with at least 3 indexes i,j,k. The use of "inconsistency" has a meaning of a measure of inconsistency in this study; not the concept itself. The approach to inconsistency (originated in [2] and generalized in [**?**]) can be reduced to a simple observation:

- search all triads (which generate all 3 by 3 PC sub matrices) and locate the worst triad with a so-called inconsistency indicator (ii),
- ii of the worst triad becomes ii of the entire PC matrix.

Expressing it a bit more formally in terms of triads (the upper triangle of a PC sub matrix 3×3), we have:

$$Kii(x,y,z) = 1 - \min\left\{\frac{y}{xz}, \frac{xz}{y}\right\}. \tag{1}$$

According to [5], it is equivalent to:

$$ii(x,y,z) = 1 - e^{-\left|\ln\left(\frac{y}{xz}\right)\right|}$$

The expression $|\ln(\frac{y}{xz})|$ is the distance of the triad T from 0. When this distance increases, the $ii(x,y,z)$ also increases. It is important to notice here that this definition allows us to localize the inconsistency in the matrix PC and it is of a considerable importance for most applications.

Another possible definition of the inconsistency indicator can also be defined as:

$$Kii_n(A) = \max_{1 \le i < j \le n}\left(1 - e^{-\left|\ln\left(\frac{a_{ij}}{a_{i,i+1}a_{i+1,i+2}\cdots a_{j-1,j}}\right)\right|}\right) \tag{2}$$

The first Koczkodaj's indicator Kii_3 allows us not only to find the localization of the worst inconsistency but to reduce the inconsistency by a step-by-step process which is crucial for practical applications. The second Koczkodaj's indicator Kii_n is useful when the global inconsistency indicator is needed for acceptance or rejection of the PC matrix. Following [7], these $[0;1]-$valued inconsistency indicators (notice that the terminilogy fixed in [3] imposes to such maps to be normalized, i.e. bounded by 0 and 1) can be gathered in an "inconsistency map" with values in formal series, by assuming that each Kii_n furnishes the coefficient of the $n-$th monomial.

II. From multiplication to addition: the basic viewpoint for making consistent a PC matrix

This is well-known that the map $\ln$ is a **morphism of groups** from the multiplicative group $\mathbb{R}_+^*$ to the additive group $\mathbb{R}$, that is,

$$\forall (x,y) \in (\mathbb{R}_+^*)^2, \ln(xy) = \ln(x) + \ln(y).$$

Its inverse, the exponential map, satisfies also a similar property:

$$\forall (x,y) \in \mathbb{R}^2, \exp(x+y) = \exp(x)\exp(y).$$

Hence, PC matrices for multiplication can be transformed bi-univoqely, to PC matrices for addition. This correspondence is the key motivation for [4] where an apparently natural way to link any PC matrix with a "prefered" consistent one is described. This impression of simplicity and clarity is obtained by the use of basic geometrical mathods, mostly the use of orthogonal vector spaces. However, we have to mention that this method my appear as very arbitrary in a multiplicative language. Indeed, one should prefer to make the inconsistency indicator, for example Kii_3, as small as possible and with as few iterations as possible in the method.

III. Multiplication versus addition: the viewpoint of Lie theory and quantum gravity

In a Lie theoric viewpoint [1], $\mathbb{R}_+^*$ is called an Abelian Lie group, due to the smoothness of the multiplication operation. Its tangent space at 1 is called its Lie algebra, and in this case the Lie algebra can be identified with $\mathbb{R}$. The exponential map $\exp : \mathbb{R} \to \mathbb{R}_+^*$ is here a (smooth) diffeomorphism. Now, following [6], [7], let n be the dimension of the PC matrices under consideration, and let us consider the $n-$simplex

$$\Delta_n = \Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \left(\sum_{i=0}^{n} x_i = 1\right) \wedge (\forall i \in \{0, \ldots .n\}, x_i \ge 0)\Big\}$$

as a graph, made of $\frac{n(n-1)}{2}$ edges linking n vertices, equipped with a pre-fixed numerotation. Then there is a one-to-one and onto correspondence between edges and the positions of the coefficients in the PC matrix. Hence, each PC matrix $(a_{i,j})$ assigns the coefficient $a_{i,j}$ to the (oriented) edge from the $i-$th vertex to the $j-$th vertex. These numbers are called **holonomies** in mathematics and physics, and the holonomy of a path is the product of the holonomies of its edges. Then one can read the formula (2) as the maximum the holonomy of a loop with

n edges. Minimizing holonomies is required in a model in physics called Yang-Mills theory [8], [9], [10]. There exists various approaches, but the one which is more of interest for us is the quantum gravity approach [8], where the seek of minimization of the distance between the holonomies and 1 is outlined. Let us only mention one apparent difference: the classical approaches of Abelian Yang-Mills theory very often consider the circle

$$U(1) = \{z \in \mathbb{C} \,|\, |z| = 1\}$$

instead of $\mathbb{R}_+^*$ as the suitable group for the coefficients (called structure group in the litterature). We have to precise that the presence of $\mathbb{R}_+^*$ simplifies technical issues.

In particular, it is possible to consider the (additive) PC matrix $(\ln(a_{ij}))$ without any restrictive additional assumption. This PC matrix, used in the "naive" approach [4] represents the so called **discretized connection** along the lines of [11], which encodes (first order) infinitesimal holonomies. In this picture, a distance between the family $\sum_{i,j} \ln(a_{ij})$ and 0; where the double indexes i, j run through the coefficients along a loop, needs to be minimized.

IV. Open problem: "which distance" means "how to minimize inconsistency"

One of the most efficient ways to minimize a functionnal is the well-knwon **gradient method**. This method consists in determining in which direction (the gradient) the decay to the map is faster. The gradient is directly derived from the first derivative of the map with values in $\mathbb{R}$. Other methods may work too, but if one requires the most efficient changes in the evaluations of inconsistency, with the less changes in the coefficients of the initial PC matrix, the best method uses the gradient.

Let us now turn back to Koczkodaj's inconsistency indicators, slighly reformulated:

$$Kii_n(A) =$$
$$1 - exp\left(- \max_{1 \le i < j \le n} \left| \ln\left(\frac{a_{ij}}{a_{i,i+1}a_{i+1,i+2}\dots a_{j-1,j}}\right)\right|\right).$$

The presence of the $\max$ operator refers to the intrinsic use of the norm $||.||_\infty$ defined by

$$\forall x = (x_1, ..., x_m) \in \mathbb{R}^m, ||x||_\infty = \max\{|x_1|, ..., |x_m|\}.$$

This norm does not admit a derivative at each point of its domain. By the way, the "smooth" approach of [4] proposes a way to minimize inconsistency, but this cannot the one with the less changes in the coefficients.

In order to recover now the quemtum gravity analog, one replaces

$$\max_{1 \le i < j \le n}(...)$$

by a quadratic mean, one recovers the so-called Yang-Mills action functional, which is smooth and on which the gradient method applies.

In such a context, our question is methodological and not mathematical. Minimization of the inconsistencies can be performed by many ways, and with one which has to be preferred, namely the gradient method. Even if for Koczkodaj's inconsistency indicator (for example) the non-differentiability of the formulas requires more technical attention, one may circumvent these difficulties via refined applications of the gradient method, which is largely generalized to many classes of not-so-regular functions. But before that, we have to open the question of what is meant by "minimizing inconsistency". This meaning is encoded in the inconsistency indicator under consideration (one may say that the inconsistency indicator is a modelization of the preferences of the users) and there exists many of them, very differents. This question is strengthened by the link with Yang-Mills in quantum gravity, where theoretical physicists prefer a mean of the holonomies to the maximal one.

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