Differential transcendence of Bell numbers and relatives: a Galois theoretic approach

Alin Bostan, Lucia Di Vizio, Kilian Raschel

Differential transcendence of Bell numbers and relatives: a Galois theoretic approach

Alin Bostan

Inria, Université Paris-Saclay, 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France

Lucia Di Vizio

CNRS, Université Paris-Saclay, UVSQ, Laboratoire de mathématiques de Versailles, 78000, Versailles, France

Kilian Raschel

CNRS, Institut Denis Poisson, Université de Tours et Université d'Orléans, Parc de Grandmont, 37200 Tours, France

Abstract

We show that Klazar's results on the differential transcendence of the ordinary generating function of the Bell numbers over the field $\mathbb{C}(\{t\})$ of meromorphic functions at 0 is an instance of a general phenomenon that can be proven in a compact way using difference Galois theory. We present the main principles of this theory in order to prove a general result of differential transcendence over $\mathbb{C}(\{t\})$, that we apply to many other (infinite classes of) examples of generating functions, including as very special cases the ones considered by Klazar. Most of our examples belong to Sheffer's class, well studied notably in umbral calculus. They all bring concrete evidence in support to the Pak-Yeliussizov conjecture according to which a sequence whose both ordinary and exponential generating functions satisfy nonlinear differential equations with polynomial coefficients necessarily satisfies a *linear* recurrence with polynomial coefficients.

Keywords: Combinatorial power series, differential transcendence, Galois theory, Sheffer sequences, umbral calculus, Bell numbers, Bernoulli numbers, Euler numbers, Genocchi numbers.

2010 MSC: 11B68, 05A40, 12H05, 33B15, 33C45, 39A10, 30D30

1. Introduction

In this paper we deal with what we will call the *strong differential transcendence* of some power series in $\mathbb{C}[[t]]$, i.e., with their differential transcendence over the field $\mathbb{C}(\{t\})$ of germs of meromorphic functions at 0. More precisely, we prove that the solutions $f \in \mathbb{C}[[t]]$ of some first-order linear functional equations must be either rational, i.e., in $\mathbb{C}(t)$, or differentially transcendental over $\mathbb{C}(\{t\})$, i.e., for any non-negative integer n and any polynomial $P \in \mathbb{C}(\{t\})[X_0, X_1, \ldots, X_n]$ we must have $P(f, f', \ldots, f^{(n)}) \neq 0$, where we consider the derivation f' of f with respect to t. We conclude in this way that many well-known power series in $\mathbb{C}[[t]]$ with a combinatorial origin are strongly differentially transcendental.

State of the art

We have come to consider this problem, influenced by three previous works, by Klazar [36], by Pak [41] and by Adamczewski, Dreyfus and Hardouin [1].

Email addresses: alin.bostan@inria.fr (Alin Bostan), lucia.di-vizio@uvsq.fr (Lucia Di Vizio), raschel@math.cnrs.fr (Kilian Raschel)

URL: http://specfun.inria.fr/bostan (Alin Bostan), https://divizio.joomla.com (Lucia Di Vizio), http://www.lmpt.univ-tours.fr/~raschel/ (Kilian Raschel)

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under the Grant Agreement No 759702, and from the ANR project DeRerumNatura, ANR-19-CE40-0018.

First of all, in [36], Klazar considers the ordinary generating function (OGF) of the Bell numbers $B(t) := 1 + \sum_{n\geq 1} B_n t^n$, where B_n is the number of partitions of a set of cardinality $n \geq 1$, and proves that $B(t)$ is differentially transcendental over $\mathbb{C}(\{t\})$. To do so, he uses a functional equation satisfied by $B(t)$, namely:

$$B\left(\frac{t}{1+t}\right) = tB(t) + 1. \tag{1.1}$$

A classical and important property of the Bell numbers is that their *exponential generating function* (EGF)

$$\hat{B}(t) := 1 + \sum_{n\geq 1} \frac{B_n}{n!} t^n$$

satisfies

$$\hat{B}(t) = \exp(\exp t - 1).$$

As a consequence, $\hat{B}(t)$ is D-algebraic, meaning that it satisfies an algebraic differential equation over $\mathbb{Q}(t)$ (or equivalently over $\mathbb{C}(t)$). However, $\hat{B}(t)$ is not D-finite, that is, it does not satisfy any *linear* differential equation with coefficients in $\mathbb{Q}(t)$. This can be seen either analytically, using the asymptotics of B_n [39, Eq. (5.47)], or algebraically, using [47] and the fact that the power series $\exp(t)$ is not algebraic.

Secondly, starting from the example of the Bell numbers, Pak and Yeliussizov formulated the following ambitious conjecture as an "advanced generalization of Klazar's theorem":

Conjecture 1 ([41, Open Problem 2.4])**.** *If for a sequence of rational numbers $(a_n)_{n\geq 0}$ both ordinary and exponential generating functions $\sum_{n\geq 0} a_n t^n$ and $\sum_{n\geq 0} a_n \frac{t^n}{n!}$ are D-algebraic, then both are D-finite (equivalently, $(a_n)_{n\geq 0}$ satisfies a linear recurrence with polynomial coefficients in $\mathbb{Q}[n]$).*

Thirdly, a very recent work by Adamczewski, Dreyfus and Hardouin in difference Galois theory establishes the following general statement:

Theorem 2 ([1, Thm. 1.2])**.** *Let $f \in \mathbb{C}((t))$ be a Laurent power series satisfying a linear functional equation of the form*

$$\alpha_0 y + \alpha_1 \tau(y) + \cdots + \alpha_n \tau^n(y) = 0,$$

where $\alpha_i \in \mathbb{C}(t)$, not all zero, and τ is one of the following operators:

- $\tau(f(t)) = f\left(\frac{t}{t+1}\right)$*;*
- $\tau(f(t)) = f(qt)$ *for some $q \in \mathbb{C}^*$, not a root of unity;*
- $\tau(f(t)) = f(t^m)$ *for some positive integer m.*

Then either $f \in \mathbb{C}(t^{1/r})$ for some positive integer r, or f is D-transcendental over $\mathbb{C}(t)$. Moreover, in the case of the first operator, r is necessarily equal to 1.

The juxtaposition of the three works above rises immediately three remarks. The first one is that Klazar is concerned with the *strong* differential transcendence, i.e. over $\mathbb{C}(\{t\})$, while Pak and Yeliussizov's conjecture and the main theorem of [1] are concerned with the differential transcendence over the field of rational functions. The second one is that Theorem 2 would prove Conjecture 1 if we were able to rephrase the differential properties of the series $\sum_{n\geq 0} a_n \frac{t^n}{n!}$ in terms of a difference operator acting on $\sum_{n\geq 0} a_n t^n$. The third is that we could not find in the literature any another functional equation of the form (1.1), satisfied by other generating functions of combinatorial interest, with the exception of the generating function of the Bernoulli numbers, considered in [55]. In this paper, we address these three issues.

Combinatorial examples for the Pak-Yeliussizov conjecture

We consider families of polynomials $(P_n(x))_{n\geq 0}$ in $\mathbb{C}[x]$, with $\deg(P_n) = n$, and whose exponential generating functions (EGFs) are defined in terms of power series $u, v, f, g, h \in \mathbb{C}[[t]]$ by

$$\sum_{n\geq 0} P_n(x) \frac{t^n}{n!} = u(x)\, v(t)\, f\big(g(x)h(t)\big). \tag{1.2}$$

A surprisingly large amount of classical polynomials fits into the framework of Eq. (1.2). An important special case is provided by *Sheffer sequences*, very classical in umbral calculus [44, Ch. 2], whose EGFs have the form

$$\sum_{n\geq 0} P_n(x) \frac{t^n}{n!} = v(t)\, \mathrm{e}^{x\, h(t)}. \tag{1.3}$$

	polynomial $P_n(x)$	$v(t)$	$h(t)$
D-finite examples	Laguerre $L_n^{\alpha}(x)$	$(1-t)^{-1-\alpha}$	$-t(1-t)^{-1}$
	Hermite $H_n(x)$	$\exp(-t^2)$	$2t$
	Mott $M_n(x)$	1	$(\sqrt{1-t^2}-1)/t$
	Bessel $p_n(x)$	1	$1-\sqrt{1-2t}$
	Falling factorial $(x)_n$	1	$\log(1+t)$
exponential functions	Euler $E_n^{(\alpha)}(x)$	$2^{\alpha}(e^t+1)^{-\alpha}$	t
	Bernoulli $B_n^{(\alpha)}(x)$	$t^{\alpha}(e^t-1)^{-\alpha}$	t
	Bell-Touchard $\phi_n(x)$	1	$\exp(t)-1$
	Mahler $s_n(x)$	1	$1+t-\exp(t)$
	Actuarial $a_n^{(\beta)}(x)$	$\exp(\beta t)$	$1-\exp(t)$
logarithmic functions	Bernoulli, 2nd kind $b_n(x)$	$t/\log(1+t)$	$\log(1+t)$
	Poisson-Charlier $c_n(x;a)$	$\exp(-t)$	$\log(1+t/a)$
	Narumi $N_n^{(a)}(x)$	$t^a\log(1+t)^{-a}$	$\log(1+t)$
	Peters $P_n^{(\lambda,\mu)}(x)$	$(1+(1+t)^{\lambda})^{-\mu}$	$\log(1+t)$
	Meixner-Pollaczek $P_n^{(\lambda)}(x;\phi)$	$(1+t^2-2t\cos\phi)^{-\lambda}$	$i\log(\frac{1-te^{i\phi}}{1-te^{-i\phi}})$
	Meixner $m_n(x;\beta,c)$	$(1-t)^{-\beta}$	$\log(\frac{1-t/c}{1-t})$
	Krawtchouk $K_n(x;p,N)$	$(1+t)^N$	$\log(\frac{p-(1-p)t}{p(1+t)})$

Table 1: Examples of various families of polynomials $P_n(x)$ of the Sheffer type [44, Ch. 2], with the corresponding v, h as in (1.3). The blue entries correspond to D-finite examples. We focus on the red entries, also called *exponential functions* [22, §19.7]. The entries in black are sometimes called *logarithmic functions*, and are not covered by our methods.

An important subclass is that of *Appell polynomials* [6], for which $h(t) = t$. Other interesting examples (see Table 1) include some families of classical orthogonal polynomials (Hermite, Laguerre, Bessel, ...), for which both ordinary and exponential generating functions are D-algebraic, since they are even D-finite (always over $\mathbb{C}(t)$ unless we clearly state otherwise). Even more interesting examples, for our purpose, are those for which D-finiteness does not hold: for instance the Bell-Touchard and the Bernoulli polynomials. This is a consequence of the fact that their EGF possess an infinite number of complex singularities, which is incompatible with D-finiteness. In these cases, a natural question is whether the corresponding OGF

$$F(x,t) = \sum_{n\geq 0} P_n(x)t^n$$

can still be D-algebraic, at least when evaluated at special values $x_0 \in \mathbb{C}$ of x.

As Klazar in his D-transcendence proof for the OGF of the Bell numbers [36] (which correspond to the evaluation at $x = 1$ of the Bell-Touchard polynomials $\phi_n(x)$ in Table 1), we focus on the case where $F(x,t)$ satisfies a functional equation of the form

$$F\left(x, \frac{t}{1+t}\right) = R(x,t)\cdot F(x,t) + S(x,t), \tag{1.4}$$

where R and S are non-zero rational functions in $\mathbb{C}(x,t)$. In §2.1, we explain a *recipe* to deduce a functional equation *à la Klazar* from the closed form of the EGFs in Table 2 (more precisely, from the differential equations with exponential coefficients that they satisfy). Therefore, all our examples bring further evidence and reinforce Conjecture 1, thanks to Theorem 2. This is the first contribution of our paper. To our knowledge, Klazar's examples of the Bell and of the (related) Uppuluri-Carpenter numbers were the only known combinatorial examples on which Conjecture 1 was proved prior to our work.

Main Galois theoretic result

Our aim in this article is to demonstrate, using difference Galois theory, that Klazar's result is a very particular instance of a general phenomenon. To do so, we equip $\mathbb{C}$ with the usual absolute value so that it makes sense to consider the field $\mathbb{C}(\{t\})$, which coincides with the field of fractions of the ring of convergent series at 0 with coefficients in $\mathbb{C}$. We can finally state the second contribution of this paper (see §4 below), which generalizes (from D-transcendence to strong D-transcendence) the first instance of Theorem 2, in the case of first-order inhomogeneous difference equations:

polynomial $P_n(x)$	EGF $\sum_{n\geq 0} P_n(x)\frac{t^n}{n!}$	R	S	ref.
Bernoulli $B_n(x)$	$\frac{t}{e^t-1}\cdot \exp(xt)$	$1+t$	$-\frac{(1+t)t}{(xt-t-1)^2}$	[5]
Glaisher $U_n(x)$	$\frac{t}{e^t+1}\cdot \exp(xt)$	$1+t$	$\frac{(1+t)t}{(xt-t-1)^2}$	[26, §229, §234]
Apostol-Bernoulli $A_n^{(\gamma)}(x)$	$\frac{t}{\gamma\, e^t-1}\cdot \exp(xt)$	$\gamma\,(1+t)$	$-\frac{(1+t)t}{(xt-t-1)^2}$	[4]
Imschenetsky $S_n(x)$	$\frac{t}{e^t-1}\cdot(\exp(xt)-1)$	$1+t$	$\frac{t^2x(xt-2\,t-2)}{(1+t)(xt-t-1)^2}$	[22, p. 254, (38)]
Euler $E_n(x)$	$\frac{2}{e^t+1}\cdot \exp(xt)$	$-(1+t)$	$\frac{2(1+t)}{1+t-xt}$	[16]
Genocchi $G_n(x)$	$\frac{2\,t}{e^t+1}\cdot \exp(xt)$	$-(1+t)$	$\frac{2\,(1+t)t}{(1+t-xt)^2}$	[31]
Carlitz $C_n^{(\gamma)}(x)$	$\frac{1-\gamma}{1-\gamma\, e^t}\cdot \exp(xt)$	$\gamma\,(1+t)$	$\frac{(1-\gamma)(1+t)}{1+t-xt}$	[17]
Fubini $F_n(x)$	$1/(1-x(e^t-1))$	$\frac{x}{x+1}\cdot(1+t)$	$\frac{1}{x+1}$	[49]
Bell-Touchard $\phi_n(x)$	$\exp(x(e^t-1))$	xt	1	[12, 51]
Mahler $s_n(x)$	$\exp(x(1+t-e^t))$	$\frac{x(1+t)t}{xt-t-1}$	$\frac{1+t}{1+t-xt}$	[38]
Toscano's actuarial $a_n^{(\gamma)}(x)$	$\exp(-xe^t+\gamma t+x)$	$\frac{x(1+t)t}{\gamma t-t-1}$	$\frac{1+t}{1+t-\gamma t}$	[50]

Table 2: Examples of power series $F(x,t)=\sum_{n\geq 0} P_n(x)t^n \in \mathbb{C}[x][[t]]$ with D-algebraic exponential generating functions (EGF) and satisfying first-order difference equations of the form $F\big(x,\frac{t}{1+t}\big) = R(x,t)\cdot F(x,t)+S(x,t)$, for some rational functions $R,S\in\mathbb{C}(x,t)$. In the cases of Apostol-Bernoulli, Carlitz and Toscano, γ is assumed to be a fixed parameter in $\mathbb{C}$.

Theorem 3. *Let $\alpha,\beta\in\mathbb{C}\setminus\{0\}$. Let $w\in\mathbb{C}((t))\setminus\mathbb{C}(t)$ be any solution of*

$$w\left(\frac{\alpha t}{1+\beta t}\right) = a(t)w(t)+f(t),$$

where $a,f\in\mathbb{C}(t)\setminus\{0\}$. Then w is differentially transcendental over $\mathbb{C}(\{t\})$.

This result will be proven in §4 in the particular case $\alpha=\beta=1$. Note that its general form is an easy consequence of this particular case, by a rational change of variable.

The theorem above applied to the OGFs of Table 2, including the infinite families obtained varying x and γ, allows to immediately obtain their strong differential transcendence:

Corollary 4. *Let $w(t)$ be any $F(x,t)$ in Table 2 evaluated at some $x,\gamma\in\mathbb{C}\setminus\{0\}$. Then $w(t)$ is differentially transcendental over $\mathbb{C}(\{t\})$.*

In particular, we deduce the strong D-transcendence of interesting combinatorial OGFs, among which the two main examples in Klazar's paper (see §2.2 for a few other combinatorial examples):

- the OGF of the *Bell numbers* [36, Prop. 3.3] (A000110)

$$\sum_{n\geq 0}\phi_n(1)t^n = 1+t+2\,t^2+5\,t^3+15\,t^4+52\,t^5+203\,t^6+\cdots$$

- the OGF of the *Uppuluri-Carpenter numbers* [36, Thm. 3.5] (A000587)

$$\sum_{n\geq 0}\phi_n(-1)t^n = 1-t+t^3+t^4-2\,t^5-9\,t^6-9\,t^7+50\,t^8+\cdots$$

- the OGF of the *bicolored partitions* [9, Tab. 2] (A001861)

$$\sum_{n\geq 0}\phi_n(2)t^n = 1+2\,t+6\,t^2+22\,t^3+94\,t^4+454\,t^5+\cdots$$

- the OGF of the number of *set partitions without singletons* (A000296)

$$\sum_{n\geq 0}s_n(-1)t^n = 1+t^2+t^3+4\,t^4+11\,t^5+41\,t^6+162\,t^7+715\,t^8+\cdots$$

- the OGF of the *Genocchi numbers* [20] (A001469)

$$\sum_{n\geq 0} G_n(1)t^n = t + t^2 - t^4 + 3\,t^6 - 17\,t^8 + 155\,t^{10} - \cdots$$

- the OGF of the *surjection numbers* (also, *preferential arrangements*) [23, p. 109] (A000670)

$$\sum_{n\geq 0} F_n(1)t^n = 1 + t + 3\,t^2 + 13\,t^3 + 75\,t^4 + 541\,t^5 + \cdots$$

While Theorem 3 and Corollary 4 demonstrate that Klazar's theorem is an instance of a general phenomenon, they raise more questions than they answer. First, it is striking that so many concrete combinatorial objects are enumerated by strongly differentially transcendental functions. Secondly, although there is a huge gap between the D-transcendental and the strongly differentially transcendental classes, Theorem 3 and Corollary 4 show that their intersections with solutions of difference equations of order 1 coincide. Thirdly, it is natural to inquire whether an extension of Conjecture 1 might hold with *differentially algebraic over* $\mathbb{Q}(t)$ replaced by *differentially algebraic over* $\mathbb{C}(\{t\})$. We feel that this paper should provide a motivation to look further into these questions.

We should also point out that the (strong) D-transcendence of very natural combinatorial examples such as the OGF of labeled rooted trees $\sum_{n\geq 1} n^{n-1}t^n$, or the OGF of the logarithmic functions in Table 1, does not fit into our framework, and actually escapes any other attempt of proof.

Finally, we have examples satisfying higher order linear τ-equations or linear q-difference equations, which we plan to consider in a subsequent publication, together with the differential transcendence with respect to the parameters x and γ appearing in Table 2. We expect that similar techniques will allow us to obtain more general results, such as the algebraic-differential independence of the families of power series in Table 2.

Content of the paper

In Section 2, we explain how to deduce a functional equation from a closed form of an exponential type as the EGFs in Table 2. As a corollary, we find functional equations satisfied by several combinatorial examples, on which Theorem 3 will be applied.

Section 3, and in particular §3.1, may be considered as a quick and gentle introduction to the parameterized Galois theory of difference equations, starting from the point of view of the usual Galois theory of difference equations. In §3.2 we have included some proofs, since we have stated some results in the exact form needed to prove Theorem 3. They are quite similar to some statements in [29], but under a weaker assumption on the field of constants, necessary in our applications. Besides, we have tried to provide a user-friendly exposition, avoiding as much as possible the use of the more sophisticated parameterized Galois theory, since we do not want to restrict the audience of the paper to specialists.

Theorem 3 is our main technical result. Its proof in given is Section 4, and it relies on results from Section 3.

2. From exponential generating functions to difference equations: setting and examples

2.1. From exponential generating functions to τ-equations

Let C be a field of characteristic zero. As before, we consider the substitution map $\tau : f(t) \mapsto f\left(\frac{t}{t+1}\right)$ and the derivation $\partial := \frac{d}{dt} : f(t) \mapsto f'(t)$. It is not difficult to see that τ defines an automorphism of $C((t))$, $C(\{t\})$ and $C(t)$. We will informally call *difference τ-equation*, or simply *τ-equation*, a linear functional equation with respect to τ.

We want to tackle the relation between the closed forms of the EGFs of Table 2 and the existence of a difference τ-equation for the corresponding OGFs. First of all, we recall the definition of the Borel transform $\mathfrak{B} : C[[t]] \to C[[t]]$:

$$g := \sum_{n\geq 0} g_n t^n \mapsto \hat{g} := \mathfrak{B}(g) = \sum_{n\geq 0} g_n \frac{t^n}{n!}.$$

Moreover, we consider the map $\Phi_\tau : C[[t]] \to C[[t]]$ defined by:

$$f(t) \mapsto (\Phi_\tau f)(t) := \frac{1}{1-t} \cdot f\left(\frac{t}{1-t}\right).$$

The maps $\mathfrak{B}$ and Φ_τ intertwine in the following way, that reminds the formal Fourier transform:

Lemma 5. *For any $f, g \in C[[t]]$, we have:*

1. $\Phi_\tau(f) = g$ *if and only if* $\hat{g} = \hat{f} \cdot e^t$;
2. $\frac{d}{dt}(\mathfrak{B}(f)) = \mathfrak{B}\left(\frac{f(t)-f(0)}{t}\right)$.

Proof. The second statement is straightforward, therefore we will only prove the first one. We set $f := \sum_{n\geq 0} f_n t^n$ and $g := \sum_{h\geq 0} g_h t^h$. The identity $\hat{g} = \hat{f} \cdot e^t$ is clearly equivalent to the fact that $g_h = \sum_{n=0}^{h} \binom{h}{n} f_n$ for all $h \geq 0$, hence the following calculation

$$\Phi_\tau(f) = \sum_{n\geq 0} f_n \frac{t^n}{(1-t)^{n+1}} = \sum_{n\geq 0} f_n t^n \frac{1}{n!} \cdot \partial^n \left(\sum_{h\geq 0} t^h\right) = \sum_{h\geq 0} t^h \sum_{n=0}^{h} \binom{h}{n} f_n$$

ends the proof. □

Example 6. *The case of Bell numbers, considered by Klazar and mentioned in (1.1), coincides with the Bell-Touchard polynomials $\phi_n(x)$ in Table 2 evaluated at $x = 1$. The associated EGF is $\hat{B}(t) = \exp(\mathrm{e}^t - 1)$, which satisfies*

$$\partial(\hat{B}) - e^t \cdot \hat{B} = 0,$$

thus $\frac{B-1}{t} = \Phi_\tau(B)$, by Lemma 5. We find that B satisfies the first-order τ-equation $\tau(B) = tB + 1$, as expected. This provides an alternative proof to [36, Prop. 2.1].

In general, an iteration of the same argument allows to show:

Proposition 7. *Let $f \in C[[t]]$. If $\hat{f}$ satisfies a linear differential equation of order r, of the form:*

$$a_0(e^t)\hat{f} + a_1(e^t)\partial(\hat{f}) + \cdots + a_r(e^t)\partial^r(\hat{f}) = P(t), \tag{2.1}$$

with $a_0(t), \ldots, a_r(t), P(t) \in C[t]$, then f satisfies a linear inhomogeneous difference τ-equation of order at most $\max_i(\deg a_i)$, with coefficients in $C[t]$ of degree at most $\max_i(\deg a_i, \deg P)$.

Proof. An iterated application of the two properties of the previous lemma allows us to obtain a difference τ^{-1}-equation. We conclude by applying a convenient power of τ to the whole expression. □

The examples of Table 2 are proved to satisfy the τ-equations given in the table in a very similar fashion. Let us just prove in detail that one can obtain higher order difference equations in some other interesting cases:

Example 8. *We consider the EGF $\hat{f} = \exp((\mathrm{e}^t - 1)^2/2)$. The associated OGF is $f = 1 + t^2 + 3\,t^3 + 10\,t^4 + 45\,t^5 + \cdots$, the generating function of the numbers of simple labeled graphs on n nodes in which each component is a complete bipartite graph, see A060311. Then $\partial(\hat{f}) = \hat{f} \cdot \mathrm{e}^t(\mathrm{e}^t - 1)$, thus $\mathfrak{B}(\frac{f-1}{t}) = \mathfrak{B}(\Phi_\tau^2(f) - \Phi_\tau(f))$. Therefore, f satisfies the second-order τ-equation $\tau^2(f) + \frac{t}{t+1}\tau(f) - tf = 1$.*

Moreover, f does not satisfy any first-order τ-equation, since $\hat{f}$ does not satisfy any differential equation of the form (2.1) with all coefficients $a_j \in \mathbb{C}[t]$ of degree at most 1 in t.

2.2. Other combinatorial examples

As promised in the introduction, we finish this section by considering the series in Table 2 plus a few more examples of τ-equations arising from combinatorial generating functions.

As far as Table 2 is concerned, we will only detail the case of the family of Bernoulli polynomials $B_n(x) = B_n^{(0)}(x)$ (see [5]), defined by their exponential generating function as follows:

$$\frac{t}{\mathrm{e}^t - 1} \cdot \exp(xt) = \sum_{n\geq 0} B_n(x) \frac{t^n}{n!}.$$

We apply our recipe to find a linear τ-equation for $B(x,t)$.

Lemma 9. *The OGF of the Bernoulli polynomials $B(x,t) := \sum_{n\geq 0} B_n(x)t^n$ satisfies the functional equation*

$$\tau(B) = (1+t) \cdot B - \frac{t(1+t)}{(1+t-tx)^2}. \tag{2.2}$$

Proof. We have that $\hat{B} \cdot \exp(t) - \hat{B} = t \cdot \exp(xt)$, thus $\mathfrak{B}(\phi_\tau(B) - B - t/(xt-1)^2) = 0$, so that $B(t/(1-t)) = (1-t) \cdot B + t(1-t)/(xt-1)^2$, and the result follows after replacing t bt $t/(1+t)$. □

All the other functional equations for the OGFs in Table 2 are calculated in the same way. We complete our list with a few more examples from combinatorics.

2.2.1. Tangent numbers

The sequence of tangent numbers $(f_n)_{n\geq 0} = (1, 2, 16, 272, 7936, 353792, \ldots)$ (A000182) admits as EGF the D-algebraic tangent function:

$$\tan(t) = 1\,\frac{t}{1!} + 2\,\frac{t^3}{3!} + 16\,\frac{t^5}{5!} + 272\,\frac{t^7}{7!} + 7936\,\frac{t^9}{9!} + 353792\,\frac{t^{11}}{11!} + \cdots.$$

A slight variation of Lemma 5 based on the exponential form of the tangent function proves that the corresponding OGF, $F(t) = t + 2\,t^3 + 16\,t^5 + 272\,t^7 + 7936\,t^9 + \cdots$, satisfies the difference τ-equation

$$F\left(\frac{t}{1+2\,it}\right) + (1+2\,it)\,F(t) = 2\,t,$$

and Theorem 3 implies that $F(t)$ is strongly D-transcendental.

An alternative way to deduce this fact is to use the classical relation $f_n = 2^{2n+1}(2^{2n+2}-1)\frac{(-1)^n}{n+1}B_{2n+2}$ between the tangent and the Bernoulli numbers, together with Corollary 4.

2.2.2. Alternating permutations

Alternating permutations are counted by the sequence $(a_n)_{n\geq 0} = (1, 1, 1, 2, 5, 16, 61, 272, \ldots)$ (A000111). By a famous result due to André [3], its EGF is known to be $\tan(t) + \sec(t)$, which is clearly D-algebraic. A variation of Lemma 5 proves that the corresponding OGF, $A(t) := \sum_{n\geq 0} a_n t^n$, satisfies the τ-equation

$$A\left(\frac{t}{1+it}\right) = (t-i)\,A(t) + 1 + i + it,$$

and Theorem 3 implies that $A(t)$ is strongly D-transcendental.

2.2.3. Springer numbers

The sequence $(s_n)_{n\geq 0} = (1, 1, 3, 11, 57, 361, 2763, \ldots)$ (A001586) of the Springer numbers [48] bears several combinatorial interpretations; for instance, it counts the topological types of odd functions with $2n$ critical values [8, 7]. By a result due to Glaisher [26, 28], its EGF is $1/(\cos(t) - \sin(t))$, which is D-algebraic. A variation of Lemma 5 proves that the corresponding OGF, $S(t) = \sum_{n\geq 0} s_n t^n$, satisfies the τ-equation

$$S\left(\frac{t}{1+2it}\right) = (2\,t-i)\,S(t) + \frac{(1+i)\,(2\,t-i)}{t-i},$$

and Theorem 3 implies that $S(t)$ is strongly D-transcendental.

An alternative way to deduce this fact is to use the relations [26, §253]

$$s_{2n} = (-1)^n \frac{4^{2n+1}}{4n+2}\cdot U_{2n+1}(1/4), \quad s_{2n-1} = (-1)^n \frac{4^{2n}}{4n}\cdot U_{2n}(1/4)$$

between the Springer numbers and the values at $x = 1/4$ of the Glaisher polynomials $U_n(x)$ in Table 2.

2.2.4. Various other sequences

Many other examples of D-algebraic EGFs, whose OGFs can be shown to satisfy τ-equations, and are thus (strongly) D-transcendental, can be found in various references in [26, 27, 24, 22, 17, 13, 33, 49, 25, 46, 52, 37, 55, 15, 40, 2, 19, 32, 35, 45, 34, 11, 43], although these ones almost never mention explicitly the corresponding τ-equations.

2.3. An elementary treatment of one example: Bernoulli polynomials

We give here an elementary proof of the D-transcendence of $B(x,t)$, based on Hölder's theorem on the D-transcendence of the Euler gamma function [30, 10]. Notice that the conclusion (which can alternatively be deduced by combining Theorem 2 and Lemma 9), is weaker than the one in Theorem 3.

Proposition 10. *The ordinary generating function $B(x,t)$ is D-transcendental over $\mathbb{C}(t)$ for any $x \in \mathbb{C}$.*

Proof. We first prove the result for $x = 0$, that is for the OGF of the sequence of *Bernoulli numbers* $(B_n)_{n\geq 0}$, classical in number theory,

$$B_0(t) = B(0,t) = 1 - \frac{1}{2}t + \frac{1}{6}t^2 - \frac{1}{30}t^4 + \frac{1}{42}t^6 - \frac{1}{30}t^8 + \cdots.$$

The idea is to use the "logarithmic Stirling formula": as $t \to \infty$,

$$\log \Gamma(t) \sim \left(t - \frac{1}{2}\right) \log t - t + \frac{\log(2\pi)}{2} + \sum_{n \geq 1} \frac{B_{2n}}{2n(2n-1)} t^{2n-1},$$

more precisely its consequence on the asymptotic expansion as $t \to \infty$ of the derivative of the digamma function $\Psi(t) := \Gamma'(t)/\Gamma(t)$, see [21, Sec. 1.18]:

$$\Psi'(t) \sim_{t \to \infty} \frac{1}{t} + \frac{1}{2t^2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{t^{2n+1}}, \tag{2.3}$$

and the fact that $\Gamma(t)$ is D-transcendental. More formally, let us introduce the power series in $\mathbb{C}[[t]]$ defined by

$$S(t) := t + \frac{t^2}{2} + \sum_{n=1}^{\infty} B_{2n} t^{2n+1}.$$

As $\Gamma(t)$ is D-transcendental, it follows that $\Psi(t)$ and $\Psi'(t)$ are also D-transcendental, hence $S(t)$ is also D-transcendental. Since $S(t) = tB_0(t) + t^2$, it also follows that $B_0(t)$ is D-transcendental.

Now, let us treat the case of a general $x \in \mathbb{C}$. The key is the following equality, which holds in $\mathbb{C}[x][[t]]$:

$$B(x,t) = \frac{1}{t} \cdot S\left(\frac{t}{1+t-tx}\right),$$

and which readily implies that $B(x,t)$ is D-transcendental. To prove this last equality, it is sufficient to check that both sides are power series in $\mathbb{C}[x][[t]]$ that satisfy the same τ-equation from Lemma 9. This is an easy consequence of the fact that B_0 satisfies the equation $\tau(B_0) = (t+1) \cdot B_0(t) - \frac{t}{1+t}$. □

2.4. Why the situation is not so easy in general

The proof of Proposition 10 might look a bit magical, and pulled out of a hat, however it is the result of an educated guess. We know from [42, Thm. 1] that any linear functional equation of the form

$$a_0(t)y + a_1(t)\tau(y) + \cdots + a_n(t)\tau^n(y) = 0,$$

with $a_0(t), \ldots, a_n(t) \in \mathbb{C}(t)$, has a basis of meromorphic solutions at ∞, i.e., in $\mathbb{C}\left(\left\{\frac{1}{t}\right\}\right)$. This means that the functional equation (2.2)

$$\tau(B) = (1+t) \cdot B - \frac{t(1+t)}{(1+t-tx)^2}$$

satisfied by the OGF of the Bernoulli polynomials, has a meromorphic solution at ∞, that we can find explicitly with some lucky and elementary manipulations. These manipulations are described in the next lemma. Once the solution is found, checking its correctness simply amounts to using the classical identity $\psi'(t) - \psi'(1+t) = \frac{1}{t^2}$.

Lemma 11. *The meromorphic function $F(x,t) = \frac{1}{t}\Psi'\left(\frac{1+t-tx}{t}\right)$ at ∞ is a solution of (2.2), for any $x \in \mathbb{C}$.*

Proof. We first remark that $z(t) = 1/t$ is solution to the homogeneous equation $\tau(z) = (1+t) \cdot z(t)$. Setting $G(x,t) := F(x,t)/z(t) = tF(x,t)$ yields a simpler functional equation, amenable to telescopic summation:

$$\tau(G) = G - \left(\frac{t}{1+t-tx}\right)^2. \tag{2.4}$$

Iterating identity (2.4), we obtain that G satisfies the following equation for any $n \geq 1$

$$G - \tau^n(G) = G(x,t) - G\left(x, \frac{t}{1+nt}\right) = \sum_{k=0}^{n-1} \tau^k \left(\frac{t}{1+t-tx}\right)^2 = \sum_{k=1}^{n} \left(\frac{t}{1+kt-tx}\right)^2.$$

We remind that $\psi'(t) = \sum_{k \geq 0} \frac{1}{(t+k)^2}$. Now, letting $n \to \infty$, we conclude that $G(x,t) = G(x,0) + \Psi'\left(\frac{1+t-tx}{t}\right)$. Since $\lim_{t \to 0^+} \psi'(1/t) = 0$ we can choose $G(x,0) = 0$, hence we have found a solution of the functional equation in Lemma 9 satisfied by $B(x,t)$, namely

$$F(x,t) = \frac{1}{t}\Psi'\left(\frac{1+t-tx}{t}\right).$$

This ends the proof. □

The solution $F(x,t)$ is a "nice" solution, due to its link with the gamma function. Hölder's theorem implies immediately that it is D-transcendental over $\mathbb{C}(t)$, for any $x \in \mathbb{C}$. One could try to use (2.3) to deduce that $B(x,t)$ is the expansion of $F(x,t)$, but this may be a delicate procedure. Galois theory comes into the picture to solve this problem. One can even say that the solution of such a problem is the core of Galois theory, namely recognizing the properties of solutions that depend on the equation, and therefore being able to transfer a property from a solution to another, in spite of the fact that they live in very different algebras of functions. For Eq. (2.4), the question is treated in Proposition 26, where the differential transcendence of $G(x,t)$, or of $t\,B(x,t)$, is proven to be equivalent to the differential properties of the rational function $\left(\frac{t}{1+t-tx}\right)^2$.

3. Differential transcendence via Galois theory

3.1. A survey of difference Galois theory

We quickly review the basic concepts of Galois theory for difference equations. We follow [53], in order to state the two main theorems of the theory: the Galois correspondence (see Theorem 20) and the theorem on the dimension of the Galois group (see Theorem 17). All the criteria for the applications we have in mind are their consequences; they form the object of the subsequent sections.

Let us consider a field $\mathbb{K}$ of characteristic zero equipped with an automorphism $\tau : \mathbb{K} \to \mathbb{K}$. We call C the subfield of $\mathbb{K}$ of τ-invariant elements of $\mathbb{K}$, i.e., $C := \mathbb{K}^\tau := \{f \in \mathbb{K} : \tau(f) = f\}$. The elements of C are the "constants of the theory", therefore they are also called τ-constants or simply constants, when the meaning is clear from the context.

Example 12. *We can take for instance $\mathbb{K} = \mathbb{C}((t))$ and set $\tau(f(t)) := f\left(\frac{t}{t+1}\right)$, for any $f \in \mathbb{C}((t))$. We claim that the field of constants $\mathbb{C}((t))^\tau$ of $\mathbb{C}((t))$ coincides with $\mathbb{C}$. Let us assume that there exists $\sum_{n\geq -N} a_n t^n \in \mathbb{C}((t))$, for some positive integer N, with $a_{-N} \neq 0$, which is invariant by τ. We have*

$$\sum_{n=1}^{N} a_{-n}\left(1+\frac{1}{t}\right)^n + \sum_{n\geq 0} a_n \frac{t^n}{(t+1)^n} = \sum_{n=1}^{N} \frac{a_{-n}}{t^n} + \sum_{n\geq 0} a_n t^n.$$

By identifying the coefficients of t^{1-N}, one sees that $Na_{-N} + a_{-N+1} = a_{-N+1}$, hence $a_{-N} = 0$. This is in contradiction with our assumptions and therefore we conclude that we must have $N \leq 0$. A similar argument allows to exclude the case of a formal power series with positive valuation, and to conclude that the only τ-constants are the actual constants. Notice that τ induces an automorphism of $\mathbb{C}(t)$ and of $\mathbb{C}(\{t\})$ as well. Therefore we also have $\mathbb{C}(t)^\tau = \mathbb{C}(\{t\})^\tau = \mathbb{C}$.

We consider a linear functional system $\tau(\vec{y}) = A\vec{y}$, where A is an invertible square matrix of order ν with coefficients in $\mathbb{K}$, $\vec{y}$ is a vector of unknowns and τ acts on vectors (and later also on matrices) componentwise.

Picard-Vessiot rings. The Galois theory of difference equations follows the general structure of classical Galois theory. Picard-Vessiot rings play the role of the splitting fields, where we can find abstract solutions that can be manipulated in the proofs.

Definition 13 ([53, Def. 1.5])**.** *A Picard-Vessiot ring for $\tau(\vec{y}) = A\vec{y}$ over $\mathbb{K}$ is a $\mathbb{K}$-algebra R equipped with an automorphism extending the action of τ, that we still call τ, and such that:*

1. *R does not have any non-trivial proper ideal invariant by τ, i.e., R is a simple τ-ring;*
2. *there exists $Y \in \mathrm{GL}_\nu(R)$ such that $\tau(Y) = AY$ and R is generated by the entries of Y and the inverse of $\det Y$, that is $R = \mathbb{K}\left[Y, \det Y^{-1}\right]$.*

Proposition 14 ([53, §1.1])**.** *A Picard-Vessiot ring always exists. If C is algebraically closed, then $R^\tau = C$ and R is unique up to an isomorphism of rings commuting to τ.*

Remark 15. *We will not need the explicit construction of R, but it is quite simple and it may be helpful to have it in mind: one considers the ring of polynomials in the ν^2 variables $X = (x_{i,j})$ with coefficients in $\mathbb{K}$. Inverting $\det X$ and setting $\tau(X) = AX$, we obtain a ring $\mathbb{K}[X, \det X^{-1}]$ with an automorphism τ. Any of its quotients by a maximal τ-invariant ideal is a Picard-Vessiot ring of $\tau(\vec{y}) = A\vec{y}$ over $\mathbb{K}$.*

It is important to notice that the ring R does not need to be a domain. It can be written as a direct sum $R_1 \oplus \cdots \oplus R_r$, such that: $R_i = e_i R$, for some $e_i \in R$ with $e_i^2 = e_i$; R_i is a domain; there exists a permutation σ of $\{1, \ldots, r\}$ such that $\tau(R_i) \subset R_{\sigma(i)}$. See [53, Cor. 1.16].

Example 16. *Let a, f be non-zero elements of $\mathbb{K}$ and let us consider the functional equation $\tau(y) = ay + f$. It can be rewritten as $\tau(\vec{y}) = \begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix} \vec{y}$. This system has two linearly independent solution vectors, so that an invertible matrix of solutions has the form $Y = \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix}$, where w, z are elements of a Picard-Vessiot ring R, with $\tau(z) = az$ and $\tau(w) = aw + f$. Then $R = \mathbb{K}[z, z^{-1}, w]$.*

The Galois group. We suppose that the field of constants C of $\mathbb{K}$ is algebraically closed. The Galois group G of $\tau(\vec{y}) = A\vec{y}$ over $\mathbb{K}$ is defined to be the group $\mathrm{Aut}^\tau(R/\mathbb{K})$ of automorphisms of rings $\varphi : R \to R$ that commute to τ and such that $\varphi_{|\mathbb{K}}$ is the identity.

Since $\varphi \in G$ leaves $\mathbb{K}$ invariant, the matrix $\varphi(Y)$ is another invertible matrix of solutions of $\tau(\vec{y}) = A\vec{y}$. It follows that $\tau(Y^{-1}\varphi(Y)) = Y^{-1}\varphi(Y)$ and hence that $Y^{-1}\varphi(Y) \in \mathrm{GL}_\nu(C)$. In other words, we have a natural group morphism $G \to \mathrm{GL}_\nu(C)$. It depends on the choice of Y and a different choice gives a conjugated map. The theorem below contains two crucial pieces of information: First of all, G is a geometric object, more precisely it can be identified with the C-points of a linear algebraic group. Roughly, this is another way of saying that G can be identified with a subgroup of matrices of $\mathrm{GL}_\nu(C)$, whose entries and their determinant are exactly the points in an algebraic variety of the affine space $\mathbb{A}_C^{\nu^2+1}$. Secondly, an algebraic relation among the entries of Y exists if and only if the dimension of G is "smaller than expected". These ideas can be formalized as follows:

Theorem 17 ([53, Thm. 1.13 and Cor. 1.18])**.** *The morphism $G \to \mathrm{GL}_\nu(C)$, $\varphi \mapsto Y^{-1}\varphi(Y)$, represents G as the group of the C-points of a linear algebraic subgroup of $\mathrm{GL}_\nu(C)$. Moreover, the dimension of G over C as an algebraic variety is equal to the transcendence degree of R over $\mathbb{K}$, i.e., $\mathrm{tr.deg}_{\mathbb{K}} R = \dim_C G$.*

Example 18. *Let us go back to Example 16. We consider the associated Galois group G over C. Any $\varphi \in G$ must map a matrix of solutions of the functional equation into another matrix of solutions, therefore there exist $c_\varphi, d_\varphi \in C$ such that $\varphi(z) = c_\varphi z$ and $\varphi(w) = w + d_\varphi z$. In other words, we must have: $\varphi \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\varphi & d_\varphi \\ 0 & 1 \end{pmatrix}$. Therefore, G is a subgroup of $\widetilde{G} := \left\{ \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} : c, d \in C, c \neq 0 \right\} \subset \mathrm{GL}_2(C)$. According to whether z and w are algebraically dependent or not, either G will be a proper linear algebraic subgroup of $\widetilde{G}$, or $G = \widetilde{G}$.*

The total Picard-Vessiot ring. We have seen in Remark 15 that R is not a domain, therefore we cannot consider its field of fractions. However, it is a direct sum of domains, so that we can consider its ring $\mathbb{L}$ of total fractions, which is the direct sum of the fields of fractions $\mathbb{L}_i$ of the R_i's. This means that $\mathbb{L} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_r$, where, for any $i = 1, \ldots, r$, $\mathbb{L}_i$ is a field and τ^r induces an automorphism of $\mathbb{L}_i$. Moreover, there exists a permutation σ of $\{1, \ldots, r\}$ such that $\tau(\mathbb{L}_i) = \mathbb{L}_{\sigma(i)}$. We will call $\mathbb{L}$ the total Picard-Vessiot ring of $\tau(\vec{y}) = A\vec{y}$.

The action of the Galois group $\mathrm{Aut}^\tau(R/\mathbb{K})$ naturally extends from R to $\mathbb{L}$. See [53, §1.3] for details. For further reference, we recall the following characterization of total Picard-Vessiot rings.

Proposition 19 ([53, Prop. 1.23])**.** *In the notation above, the total Picard-Vessiot ring $\mathbb{L}$ of $\tau(\vec{y}) = A\vec{y}$ is uniquely determined, up to a morphism of ring commuting to τ, by the following properties:*

1. *$\mathbb{L}$ has no nilpotent element and any non-zero divisor of $\mathbb{L}$ is invertible.*
2. *$\mathbb{L}^\tau = C$.*
3. *The system $\tau(\vec{y}) = A\vec{y}$ has a matrix solution in $\mathrm{GL}_\nu(\mathbb{L})$.*
4. *$\mathbb{L}$ is minimal (for the inclusion) with respect to the three previous properties.*

In particular, any τ-ring satisfying the first three properties above contains a copy of the Picard-Vessiot ring R of $\tau(\vec{y}) = A\vec{y}$ over $\mathbb{K}$.

The Galois correspondence. We consider the set $\mathcal{F}$ of all τ-stable rings $F \subset \mathbb{L}$, such that $\mathbb{K} \subset F$ and that any element of F is either a zero divisor or a unit, and the set $\mathcal{G}$ of all linear algebraic subgroups of G. For any $H \in \mathcal{G}$, we set $\mathbb{L}^H = \{f \in \mathbb{L} : \varphi(f) = f \text{ for all } \varphi \in H\}$ and, for any $F \in \mathcal{F}$, we set $H_F := \{\varphi \in \mathbb{G} : \varphi(f) = f \text{ for all } f \in F\}$.

Theorem 20 (Galois correspondence [53, Thm. 1.29 and Cor. 1.30])**.** *In the notation above, the following two maps are one the inverse of the other:*

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{F} \\ H & \mapsto & \mathbb{L}^H \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F} & \to & \mathcal{G} \\ F & \mapsto & H_F \end{array}.$$

In particular, $\mathbb{L}^H = \mathbb{K}$ if and only if $H = G$. Moreover, the group H is a normal subgroup of G if and only if $\mathbb{L}^H$ is the total Picard-Vessiot ring of a linear system of τ-equations whose Galois group coincides with G/H.

We start mentioning the following immediate consequence of the theorem above, which illustrates how the transcendental nature of a solution to the homogeneous equation transfers to solutions of the inhomogeneous system. This statement is well known to specialists.

Corollary 21. *In the notation of Example 18, if z is transcendental over $\mathbb{K}$ and $w \notin \mathbb{K}$, then also w is transcendental over $\mathbb{K}$.*

Proof. We suppose that w is algebraic over $\mathbb{K}$. Since $w \notin \mathbb{K}$, there exists $\varphi \in G$, such that $\varphi(w) = d_\varphi z + w \neq w$, i.e., $d_\varphi \neq 0$. The element $\varphi(w)$ is necessarily algebraic over $\mathbb{K}$, and therefore $\varphi(w) - w = d_\varphi z$ is also algebraic over $\mathbb{K}$. □

The purpose of the following subsections is to give a proof of an analogous statement for differential transcendence.

3.2. Application to differential transcendence

The ∂-Picard-Vessiot ring. Now, we suppose that there exists a derivation ∂ on $\mathbb{K}$ commuting to τ.

Example 22. *In the situation of Example 12, for $\tau(f(t)) = f\left(\frac{t}{1+t}\right)$, we can take $\partial := t^2 \frac{d}{dt}$.*

Proposition 23 ([54] and [18, Prop. 1.16, Rem. 1.18, Cor. 1.19])**.** *For any linear system of the form $\tau(\vec{y}) = A\vec{y}$, with $A \in \mathrm{GL}_\nu(\mathbb{K})$, there exists a $\mathbb{K}$-algebra $\mathcal{R}$, equipped with an extension of τ and of ∂, preserving the commutation, such that:*

1. *there exists $Z \in \mathrm{GL}_\nu(\mathcal{R})$ such that $\tau(Z) = AZ$;*
2. *$\mathcal{R}$ is generated over $\mathbb{K}$ by the entries of Z, $\frac{1}{\det(Z)}$ and all their derivatives;*
3. *$\mathcal{R}$ is τ-simple.*

Moreover, the total ring of fractions of $\mathcal{R}$ satisfies the first three properties of Proposition 19.

We will call the ring $\mathcal{R}$ the ∂-Picard-Vessiot ring of $\tau(\vec{y}) = A\vec{y}$ over $\mathbb{K}$, without giving a formal definition. Applying ∂^n to the system $\tau(\vec{y}) = A\vec{y}$ for any positive integer n, we can consider the difference system:

$$\tau(\vec{y}) = \begin{pmatrix} A & \partial(A) & \cdots & \partial^n(A) \\ 0 & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \partial(A) \\ 0 & \cdots & 0 & A \end{pmatrix} \vec{y}, \text{ with solution } \begin{pmatrix} \partial^n(Z) & \partial^{n-1}(Z) & \cdots & Z \\ 0 & \partial^n(Z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \partial^{n-1}(Z) \\ 0 & \cdots & 0 & \partial^n(Z) \end{pmatrix}. \tag{3.1}$$

The following fact is explicitly mentioned and proved in the proof of [18, Prop. 1.16]. Anyhow, since $\mathcal{R}$ is generated by Z and its derivatives, it is quite natural:

Corollary 24. *For any integer $n \geq 0$, the ring $\mathcal{R}$ contains a copy of the Picard-Vessiot ring of (3.1).*

Before dealing with the general case of linear τ-difference equations $\tau(y) = ay + f$ of order 1, we study the particular cases $f = 0$ and $a = 1$.

Notation 25. *From now on, F will be a $\mathbb{K}$-algebra with no nilpotent elements, and such that any element is either a zero divisor, or invertible. We suppose that F is equipped with an extension of τ and of ∂, preserving the commutation, and that $F^\tau = C$. Notice that, if there exists $Z \in \mathrm{GL}_\nu(F)$ such that $\tau(Z) = AZ$, then F contains a copy of the (total) Picard-Vessiot ring of (3.1) for any $n \geq 0$, and hence a copy of $\mathcal{R}$. Of course, it does not need to be the same algebra in all the statements.*

We say that an element of F is differentially algebraic over $\mathbb{K}$ if it satisfies an algebraic differential equation with coefficients in $\mathbb{K}$, and that it is differentially transcendental otherwise.

The following proposition is the analogue in our setting of [29, Prop. 3.1]. Here we assume that the field of constants is algebraically closed, while in [29] the authors assume that the field of constants is differentially closed, which is much stronger[1] than our assumption. The main difference is in the proof: We have given a proof based on the usual Galois theory of difference equations rather than the parameterized one, which has avoided the use of some more sophisticated technology.

Proposition 26. *Let f be a non-zero element of $\mathbb{K}$, and let $w \in F$ be such that $\tau(w) = w + f$. Then the following assertions are equivalent:*

1. *w is differentially algebraic over $\mathbb{K}$.*
2. *There exist a non-negative integer n, $\alpha_0, \ldots, \alpha_n \in C$ (not all zero) and $g \in \mathbb{K}$ such that $\alpha_0 f + \alpha_1 \partial(f) + \cdots + \alpha_n \partial^n(f) = \tau(g) - g$.*
3. *There exist a non-negative integer n and $\alpha_0, \ldots, \alpha_n \in C$ (not all zero) such that $g := \sum_{i=0}^n \alpha_i \partial^i(w) \in \mathbb{K}$.*

Proof. It follows from Corollary 24 that F contains a copy of the Picard-Vessiot ring $R_{f,n}$ of the system $\{\tau(y_i) = y_i + \partial^i(f), i = 0, \ldots, n\}$. Notice that the latter can be written in the form of a linear τ-difference system as follows (where diagonal(A, B, C) indicates a block diagonal matrix, having the blocks A, B and C on the diagonal):

$$\tau(\vec{y}) = \text{diagonal}\left(\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \partial(f) \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & \partial^n(f) \\ 0 & 1 \end{pmatrix}\right)\vec{y}. \tag{3.2}$$

For any $i \geq 0$, $\partial^i(w)$ is a solution of $\tau(y_i) = y_i + \partial^i(f)$, so that $R_{f,n} = \mathbb{K}[w, \partial(w), \ldots, \partial^n(w)]$, as in Example 16 and Example 18.

The element $w \in \mathcal{R}$ is differentially algebraic over $\mathbb{K}$ if and only if there exists $n \geq 0$ such that $w, \partial(w), \ldots, \partial^n(w)$ are algebraically dependent over $\mathbb{K}$, therefore if and only if there exists $n \geq 0$ such that $\operatorname{tr.deg}_{\mathbb{K}} R_{f,n} \leq n$. Moreover, for any $\varphi \in G_n := \operatorname{Aut}^\tau(R_{f,n}/\mathbb{K})$, we must have $\varphi(\partial^i(w)) = \partial^i(w) + d_{\varphi,i}$, for some $d_{\varphi,i} \in C$. The composition of two automorphisms φ and ψ in the Galois group is represented by the sum $d_{\varphi,i} + d_{\psi,i}$. This means that we can identify the Galois group to a subgroup of the vector space $(C, +)^{n+1}$. Theorem 17 implies that $\operatorname{tr.deg}_{\mathbb{K}} R_{f,n} \leq n$ if and only if $\dim_C G_n \leq n$, hence if and only if G_n is a proper linear subgroup of $(C, +)^{n+1}$. Thus, the property $\operatorname{tr.deg}_{\mathbb{K}} R_{f,n} \leq n$ is equivalent to the existence of $\alpha_0, \ldots, \alpha_n \in C$ (not all zero) such that for any $\varphi \in G$ we have $\sum_{i=0}^n \alpha_i d_{\varphi,i} = 0$. The last linear relation is equivalent to the fact that $g := \sum_{i=0}^n \alpha_i \partial^i(w)$ is G-invariant, that is $\varphi(\sum_{i=0}^n \alpha_i \partial^i(w)) = \sum_{i=0}^n \alpha_i \partial^i(w)$ for any $\varphi \in G$. Theorem 20 implies that the latter condition is equivalent to the fact that g belongs to $\mathbb{K}$ and we obtain:

$$\tau(g) - g = \tau\left(\sum_{i=0}^n \alpha_i \partial^i(w)\right) - \sum_{i=0}^n \alpha_i \partial^i(w) = \sum_{i=0}^n \alpha_i \partial^i(f).$$

The proof is completed. □

Example 27. *We consider the OGF of the family of Bernoulli polynomials, which satisfies the functional equation*

$$\tau(B) = (1 + t) \cdot B - \frac{t(1 + t)}{(1 + t - tx)^2},$$

as proved in Lemma 9. Setting $G(x, t) := B(x, t)/z(t) = tB(x, t)$, one obtains the functional equation:

$$\tau(G) = G - \left(\frac{t}{1 + t - tx}\right)^2.$$

We need to prove that G is differentially transcendental over $\mathbb{C}(t)$, for any fixed value of $x \in \mathbb{C}$, to conclude the differential transcendence of B. To do so, one can use Proposition 26 and show that for any non-negative integer n, there do not exist any constants $a_0, \ldots, a_n$ (not all zero) and any function $g \in \mathbb{C}(t)$ such that

$$\alpha_0 f + \alpha_1 \partial(f) + \cdots + \alpha_n \partial^n(f) = \tau(g) - g, \tag{3.3}$$

[1] Indeed, the commutativity of τ and ∂ implies that C is stable by ∂. Saying that C is differentially closed means that any algebraic differential equation with respect to ∂ and with coefficients in C admits a solution in C as soon as it has a solution in *an* extension of C equipped with an extension of ∂. Of course, this means in particular that C is huge, and thus discards fields like $C = \mathbb{Q}$, $C = \overline{\mathbb{Q}}$ or $C = \mathbb{C}$.

with $f = \left(\frac{t}{1+t-tx}\right)^2$. *We easily compute, for all* $k \geq 0$*,*

$$\partial^k(f) = (k+1)! \left(\frac{t}{1+t-tx}\right)^{k+2}. \tag{3.4}$$

Let us first deal with the case $x \neq 1$*. By* (3.4)*, the left-hand side of* (3.3) *has a unique pole, namely, at* t *such that* $1 + t - tx = 0$ *(i.e.,* $t = \frac{1}{x-1}$*). This proves that the right-hand side of* (3.3) *should have at least this pole. But having a pole, the right-hand side of* (3.3) *should actually admit at least two poles, which is not compatible with* (3.3)*.*

In the case $x = 1$*, the left-hand side of* (3.3) *becomes a polynomial with no constant term. One first proves that* $g \in \mathbb{C}(t)$ *must also be a polynomial and finally gets a contradiction, showing that it can only be a constant, in contradiction with the fact that the left-hand side is non-zero.*

Let a be a non-zero element of $\mathbb{K}$. We consider the equation $\tau(y) = ay$ and the Picard-Vessiot ring $R_{a,n}$ over $\mathbb{K}$ of the linear difference system obtained as in (3.1) for $A = (a)$. The following proposition is proved using the same trick as in [29, Cor. 3.3] of taking the logarithmic derivatives.

Proposition 28. *Let* $a \in \mathbb{K}$ *be non-zero,* F *be a* $\mathbb{K}$*-algebra as in Notation* 25 *and let* $z \in F$ *be a non-zero solution of* $\tau(y) = ay$*. With the notation introduced above, the following assertions are equivalent:*

1. z *is differentially algebraic over* $\mathbb{K}$*.*
2. *there exists a non-negative integer* n *such that* $\operatorname{tr.deg}_{\mathbb{K}} R_{a,n} \leq n$*;*
3. *there exists a non-negative integer* n *such that there exist* $\alpha_0, \ldots, \alpha_n \in C$ *(not all zero) and* $g \in \mathbb{K}$ *satisfying*

$$\alpha_0 \frac{\partial(a)}{a} + \alpha_1 \partial\left(\frac{\partial(a)}{a}\right) + \cdots + \alpha_n \partial^n \left(\frac{\partial(a)}{a}\right) = \tau(g) - g. \tag{3.5}$$

Proof. We have $R_{a,n} = \mathbb{K}[z, \partial(z), \ldots, \partial^n(z), z^{-1}] \subset F$, up to an automorphism commuting to τ. The first equivalence follows immediately from the definition of differential algebraicity and Theorem 17. We notice that z is differentially algebraic over $\mathbb{K}$ if and only if $w := \frac{\partial(z)}{z} \in F$ is differentially algebraic over $\mathbb{K}$. Since we have $\tau(w) = w + \frac{\partial(a)}{a}$, we conclude by applying Proposition 26. □

Example 29. *Let us consider the setting of Examples* 12 *and* 22*, with* $\tau(f(t)) = f\left(\frac{t}{1+t}\right)$ *and* $\partial = t^2\frac{d}{dt}$*. The function* $z(t) := \Gamma\left(\frac{1}{t}\right)^{-1}$*, which lives in the algebra of analytic functions over* $\mathbb{C}^* \cup \{\infty\}$*, is a solution of the equation* $\tau(y) = ty$*, that is the homogenous equation associated to Klazar's example* (1.1)*. Since* $\frac{\partial(t)}{t} = t$ *and* $\partial^n(t) = n!t^{n+1}$*, one deduces that for any* $\alpha_0, \ldots, \alpha_n \in \mathbb{C}$ *(not all zero) the linear combination*

$$\alpha_0 t + \alpha_1 \partial(t) + \cdots + \alpha_n \partial^n(t) = \alpha_0 t + \alpha_1 t^2 + \cdots + \alpha_n\, n! t^{n+1}$$

cannot be written as $g\left(\frac{t}{1+t}\right) - g(t)$*, for any* $g \in \mathbb{C}(t)$*. This is easily proved by reasoning on the poles of the potential rational functions* g*. We deduce from Proposition* 28 *that* $z(t) := \Gamma\left(\frac{1}{t}\right)^{-1}$ *is D-transcendental over* $\mathbb{K} := \mathbb{C}(t)$*, which reproves Hölder's theorem [30, 10] on the D-transcendence of the gamma function. We also deduce from Proposition* 28 *that any non-zero solution of* $\tau(y) = ty$ *in any algebra* F *as above is differentially transcendental over* $\mathbb{C}(t)$*.*

Differential transcendence: the equation $\tau(y) = ay + f$*.* We want to prove a generalization of Corollary 21 to differential transcendence (see Theorem 30 below). Let f and a be non-zero elements of $\mathbb{K}$, and let us consider the difference equation $\tau(y) = ay + f$. It can be transformed in a linear system as in the examples above: $\tau(\vec{y}) = A\vec{y}$, with $A := \begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix}$.

The following statement generalizes [29, Prop. 3.8, 1.] to the case of an algebraically closed field of constants. We remind that in [29] the authors assume that the field of constants is differentially closed. The advantage of the statement below, compared to [29], is that one can look for the solutions of the inhomogeneous equation and the associated homogeneous equation in two different algebras. As we have already observed in §2.4, this is particularly useful in our setting. We will illustrate further the situation in Remark 32 below.

Theorem 30. *Let us consider an equation of the form $\tau(y) = ay + f$, with $a, f \in \mathbb{K}$, such that $a \neq 0, 1$ and $f \neq 0$. Let $F/\mathbb{K}$ be a field extension such that there exists $w \in F \setminus \mathbb{K}$ satisfying the equation $\tau(w) = aw + f$. Moreover, let F_a be a $\mathbb{K}$-algebra as in Notation 25, such that there exists $z \in F_a$ satisfying the equation $\tau(z) = az$.*

If z is differentially transcendental over $\mathbb{K}$, then w is differentially transcendental over $\mathbb{K}$.

Proposition 28 and Theorem 30 imply directly the following corollary:

Corollary 31. *In the notation of the theorem above, if for any $n \geq 0$, any $\alpha_0, \ldots, \alpha_n \in C$ (not all zero) and any $g \in \mathbb{K}$, we have*

$$\alpha_0 \frac{\partial(a)}{a} + \alpha_1 \partial \left(\frac{\partial(a)}{a} \right) + \cdots + \alpha_n \partial^n \left(\frac{\partial(a)}{a} \right) \neq \tau(g) - g, \tag{3.6}$$

then w is differentially transcendental over $\mathbb{K}$.

Remark 32. *The corollary above allows us to conclude immediately that the generating function of the Bell numbers is differentially transcendental over $\mathbb{C}(t)$, as a consequence of Hölder's theorem. See Example 29. If we want to apply Theorem 30, it is enough to take $F = \mathbb{C}((t))$ and F_a the field of meromorphic functions at ∞. Notice that there is no common natural extension of the fields F and F_a.*

We will rather prove the following statement, which is obviously equivalent to the Theorem 30:

Proposition 33. *In the notation of Theorem 30, if $w \in F \setminus \mathbb{K}$ is differentially algebraic over $\mathbb{K}$, then $z \in F_a$ is differentially algebraic over $\mathbb{K}$.*

Proof. Let R_n be the Picard-Vessiot ring over $\mathbb{K}$ of the system obtained from $\tau(\vec{y}) = \begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix} \vec{y}$ by derivation as in (3.1) and $\mathcal{R}$ be its ∂-Picard-Vessiot ring over F. It follows from Proposition 23 and Proposition 19 that we can suppose without loss of generality that $R_n \subset \mathcal{R}$, because $\mathbb{K} \subset F$, by assumption. Moreover, $R_n = \mathbb{K}[z, \partial(z), \ldots, \partial^n(z), w, \partial(w), \ldots, \partial^n(w), z^{-1}]$.

Let us assume that w is differentially algebraic over $\mathbb{K}$. This means that w satisfies an algebraic differential equation of order κ with coefficients in $\mathbb{K}$. Let $\varphi \in \mathrm{Aut}^{\tau}(R_n/\mathbb{K})$, for some $n > 2(\kappa + 1)$. Then $\varphi(w)$ is also solution of an algebraic differential equation or order κ, with coefficients in $\mathbb{K}$. We deduce that $w - \varphi(w)$ satisfies an algebraic differential equation of order at most $2(\kappa + 1)$, see [14, Thm. 2.2]. Since $\widetilde{z} := w - \varphi(w) \in R_n$ is a solution of $\tau(y) = ay$, we deduce by Proposition 28, applied to $\widetilde{z} \in R_n$, that there exist $\alpha_0, \ldots, \alpha_n \in C$ (not all zero) and $g \in \mathbb{K}$ such that $\sum_{i=0}^{n} \alpha_i \partial^i \left(\frac{\partial(a)}{a} \right) = \tau(g) - g$. We conclude, by applying again Proposition 28 to $z \in F_a$, that $z \in F_a$ is itself differentially algebraic over $\mathbb{K}$. □

4. Main result: Strong differential transcendence for first-order difference equations

We consider a field $\mathbb{K}_0 := C(t)$, where C is an algebraically closed field of characteristic zero, equipped with a norm $|\cdot|$. Typically, we will choose C to be the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ or $\mathbb{C}$, with the usual norm.

We will look for solutions in $\mathbb{F} := C((t))$ and we will establish their differential transcendence over the field $\mathbb{K} := C(\{t\})$, where the convergence is considered with respect to $|\cdot|$. We remind that the derivation $\partial := t^2 \frac{d}{dt}$ commutes to $\tau : f(t) \mapsto f(t/(t+1))$ and that establishing the differential transcendence with respect to ∂ is equivalent to establishing it with respect to $\frac{d}{dt}$.

The next theorem generalizes [1, Thm. 1.2] for order-1 difference equations, in the case called S_∞ in *loc.cit.* It should be the first step of a generalization of the results in [1] to higher-order difference equations (in case S_∞).

Theorem 34. *Let $a, f \in \mathbb{K}_0$, with $a, f \neq 0$, and let $w \in \mathbb{F} \setminus \mathbb{K}_0$ verify the difference equation $\tau(w) = aw + f$. Then w is differentially transcendental over $\mathbb{K}$.*

A simple rational change of variable shows the following corollary:

Corollary 35. *The theorem above holds if we replace τ with the endomorphism associated to any homography with only one fixed point t_0, $\mathbb{K}$ with the field of germs of meromorphic functions at t_0 and F with the field of formal Laurent power series based at t_0.*

The proof of the theorem relies on the following lemmata. The first lemma shows that the theorem above is true for $a = 1$.

Lemma 36. *Let $f \in \mathbb{K}_0$, $f \neq 0$, and let $w \in \mathbb{F} \setminus \mathbb{K}_0$ satisfy $\tau(w) = w + f$. Then w is differentially transcendental over $\mathbb{K}$.*

Proof. Notice that we can replace w by $w + r$ for any $r \in \mathbb{K}_0$, and change the equation accordingly, since this will change neither the hypotheses nor the conclusion of the lemma. Indeed, the element $w + r$ of $\mathbb{F}$ cannot be an element of C, since $w \notin \mathbb{K}_0$, therefore f cannot be 0. With this in mind, one sees that for any $\alpha \in C$, $\alpha \neq 0, 1$, and any $m \in \mathbb{Z}$ we have:

$$\tau\left(\frac{1}{(t-\alpha)^m}\right) - \frac{1}{(t-\alpha)^m} = \frac{(1+t)^m}{((1-\alpha)t-\alpha)^m} - \frac{1}{(t-\alpha)^m}.$$

Notice that $\tau\left(\frac{\alpha}{1-\alpha}\right) = \alpha$, hence the poles of the right-hand side of the expression above are on the same τ-orbit. Therefore, replacing w by $w + r$, where r is a rational function, we can "shift the poles $\neq 0, 1$ of f along their τ-orbit". The poles of the form $\frac{1}{n}$, for a positive integer n, are all moved in the pole 1, which is in their orbit. We conclude that, adding a convenient rational function to w, we can suppose that f has only one pole in each τ-orbit, that does not contain 0.

We now assume by contradiction that w is differentially algebraic over $\mathbb{K}$. It follows from Proposition 26 that there exist a non-negative integer n, $\alpha_0, \dots, \alpha_n \in C$ (not all zero) and $g \in \mathbb{K}$ such that $\alpha_0 f + \alpha_1 \partial(f) + \dots + \alpha_n \partial^n(f) = \tau(g) - g$. By assumption, g is analytic in a punctured disk around zero. Let us suppose by absurdum that there exists $t_0 \in C^*$ on the border of the domain of analyticity of g, i.e. that g is not analytic on C^*. We notice that $\tau^{-m}(t_0) = \frac{t_0}{1-mt_0}$ for any $m \in \mathbb{Z}$, therefore the orbit of t_0 has an accumulation point at 0 as $m \to \infty$. Since g is analytic in the punctured disk at 0, we conclude that there must exist t_1 in the τ-orbit of t_0 such that t_1 is a singularity of g and that, for any positive integer m, no $\tau^{-m}(t_1)$ is a singularity of g. Then $\tau^{-1}(t_1)$ is a singularity of $\tau(g)$ and $\tau(g) - g$ is forced to have at least two singularities in the τ-orbit of t_1, against the assumptions on f. We conclude that the domain of analyticity of g is the whole C^* and that f cannot have any pole other than 0 and ∞. In other words, f is a Laurent polynomial and g is analytic over C^*, with a pole at 0.

We now notice that for any integer $m \geq 1$, we have

$$\tau\left(\frac{1}{t^m}\right) - \frac{1}{t^m} = \sum_{j=0}^{m-1} \binom{m}{m-j} \frac{1}{t^j}.$$

This means that replacing w by $w + r$ for a convenient choice of $r \in \mathbb{K}_0$, we can suppose that f is a non-zero polynomial in t without constant coefficient.

We are finally reduced to an $f \in tC[t]$, that satisfies an expression of the form $\alpha_0 f + \alpha_1 \partial(f) + \dots + \alpha_n \partial^n(f) = \tau(g) - g$, for some g having a pole at 0 and analytic over the whole C^*. Since $\partial^i(f) \in t^{i+1}C[t]$, reasoning recursively on the coefficients of g, we see first that $g \in tC\{t\}$ and secondly that no such non-zero g can exist. So $g = f = 0$, against the assumptions. We have found a contradiction, therefore we can conclude that, if $w \notin \mathbb{K}_0$, then w is differentially transcendental over $\mathbb{K}$. □

Lemma 37. *Let $a \in \mathbb{K}_0$, $a \neq 0$, and let F be a $\mathbb{K}$-algebra as in Notation 25 containing a solution z of $\tau(y) = ay$. Then either $w \in \mathbb{K}_0 \subset F$ or the solution z (and hence any solution) of $\tau(y) = ay$ is differentially transcendental over $\mathbb{K}$.*

Example 38. *We set $C = \mathbb{C}$ and we go back to Example 29. Since we know that the gamma function is not rational, we immediately obtain that $\Gamma\left(\frac{1}{t}\right)^{-1}$ is strongly differentially transcendental, being a solution of $\tau(y) = ty$. Notice that the statement makes sense since $\Gamma\left(\frac{1}{t}\right)^{-1}$ is analytic in any punctured disk around 0, hence we can consider the $\mathbb{K}$-field of functions $\mathbb{K}\left(\Gamma\left(\frac{1}{t}\right)^{-1}\right)$.*

Remark 39. *Of course, in the previous statement we can take $F = \mathbb{F}$, but such a result would not be enough to prove Theorem 34.*

Proof of Lemma 37. If z is differentially algebraic over $\mathbb{K}$, so is $\partial(z)/z$, which is solution of $\tau(y) = y + \partial(a)/a$. It follows from the previous lemma that $\partial(z)/z$ is differentially algebraic over $\mathbb{K}$ if and only if $b := \partial(z)/z \in \mathbb{K}_0$. We have $\partial(a)/a = \tau(b) - b$. Multiplying z by a factor of the form $(t-\alpha)^m$ boils down to replacing b by $b - \frac{m}{t-\alpha}$. By reasoning as in the previous lemma, we conclude that if z is differentially algebraic over $\mathbb{K}$, there exists $r \in \mathbb{K}_0$ such that, replacing z with rz, we can suppose that $b = 0$. Hence $rz \in C = F^\tau$, which is equivalent to $z \in \mathbb{K}_0$. □

Proof of Theorem 34. Let $\mathcal{R}$ be the ∂-Picard-Vessiot ring of $\tau(y) = ay$, constructed in Proposition 23 and let $z \in \mathcal{R}$ be a solution of $\tau(y) = ay$. It follows from Lemma 37 (for $F = \mathcal{R}$) that either $z \in \mathbb{K}_0$, and hence $\mathcal{R} = \mathbb{K}$, or z is differentially transcendental over $\mathbb{K}$.

If $z \in \mathbb{K}_0$ is a solution of $\tau(y) = ay$, then Lemma 36 implies that either $w/z \in \mathbb{K}_0$, and hence $w \in \mathbb{K}_0$, or w/z is differentially transcendental over $\mathbb{K}$, and so does w. If z is differentially transcendental over $\mathbb{K}$, then w is differentially transcendental over $\mathbb{K}$ because of Theorem 30. □

References

[1] B. Adamczewski, T. Dreyfus, and C. Hardouin. Hypertranscendence and linear difference equations. *Preprint arXiv:1910.01874*, 2019.

[2] C. Ahlbach, J. Usatine, and N. Pippenger. Barred preferential arrangements. *Electron. J. Combin.*, 20(2):Paper 55, 18, 2013.

[3] D. André. Sur les permutations alternées. *Journal de Mathématiques Pures et Appliquées*, 7:167–184, 1881.

[4] T. M. Apostol. On the Lerch zeta function. *Pacific J. Math.*, 1:161–167, 1951.

[5] T. M. Apostol. A primer on Bernoulli numbers and polynomials. *Math. Mag.*, 81(3):178–190, 2008.

[6] P. Appell. Sur une classe de polynômes. *Ann. Sci. École Norm. Sup. (2)*, 9:119–144, 1880.

[7] V. I. Arnol′d. Snake calculus and the combinatorics of the Bernoulli, Euler and Springer numbers of Coxeter groups. *Uspekhi Mat. Nauk*, 47(1(283)):3–45, 240, 1992.

[8] V. I. Arnol′d. Springer numbers and Morsification spaces. *J. Algebraic Geom.*, 1(2):197–214, 1992.

[9] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246(1-3):29–55, 2002. Formal power series and algebraic combinatorics (Barcelona, 1999).

[10] S. B. Bank and R. P. Kaufman. A note on Hölder's theorem concerning the gamma function. *Math. Ann.*, 232(2):115–120, 1978.

[11] H. Belbachir and N. Souddi. Some explicit formulas for the generalized Frobenius-Euler polynomials of higher order. *Filomat*, 33(1):211–220, 2019.

[12] E. T. Bell. Exponential polynomials. *Ann. of Math. (2)*, 35(2):258–277, 1934.

[13] R. P. Boas, Jr. and R. C. Buck. *Polynomial expansions of analytic functions.* Second printing, corrected. Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., Bd. 19. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin, 1964.

[14] M. Boshernitzan and L. A. Rubel. Coherent families of polynomials. *Analysis*, 6(4):339–389, 1986.

[15] K. N. Boyadzhiev. A series transformation formula and related polynomials. *Int. J. Math. Math. Sci.*, 2005(23):3849–3866, 2005.

[16] L. Carlitz. Eulerian numbers and polynomials. *Math. Mag.*, 32:247–260, 1958/59.

[17] L. Carlitz. Some generalized multiplication formulas for the Bernoulli polynomials and related functions. *Monatsh. Math.*, 66:1–8, 1962.

[18] L. Di Vizio and C. Hardouin. Descent for differential Galois theory of difference equations: confluence and q-dependence. *Pacific J. Math.*, 256(1):79–104, 2012.

[19] A. Dixit, V. H. Moll, and C. Vignat. The Zagier modification of Bernoulli numbers and a polynomial extension. Part I. *Ramanujan J.*, 33(3):379–422, 2014.

[20] D. Dumont. Interprétations combinatoires des nombres de Genocchi. *Duke Math. J.*, 41:305–318, 1974.

[21] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher transcendental functions. Vols. I, II.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. Based, in part, on notes left by Harry Bateman.

[22] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher transcendental functions. Vol. III.* Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1955 original.

[23] P. Flajolet and R. Sedgewick. *Analytic combinatorics.* Cambridge University Press, Cambridge, 2009.

[24] G. Frobenius. Über die Bernoullischen Zahlen und die Eulerschen Polynome. *Preussische Akademie der Wissenschaften Berlin: Sitzungsberichte der Preußischen Akademie der Wissenschaften zu Berlin*, pages 809–847, 1910.

[25] I. Gessel and R. P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.

[26] J. W. L. Glaisher. On the Bernoullian function. *Quart. J. Pure and Applied Math.*, 29:1–168, 1898.

[27] J. W. L. Glaisher. On a Set of Coefficients analogous to the Eulerian Numbers. *Proc. Lond. Math. Soc.*, 31:216–235, 1899.

[28] J. W. L. Glaisher. On the coefficients in the expansions of $\cos x/\cos 2x$ and $\sin x/\cos 2x$. *Quart. J. Pure and Applied Math.*, 45:187–222, 1914.

[29] C. Hardouin and M. F. Singer. Differential Galois theory of linear difference equations. *Mathematische Annalen*, 342(2):333–377, 2008.

[30] O. Hölder. Ueber die Eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen. *Math. Ann.*, 28:1–13, 1887.

[31] A. F. Horadam. Genocchi polynomials. In *Applications of Fibonacci numbers, Vol. 4 (Winston-Salem, NC, 1990)*, pages 145–166. Kluwer Acad. Publ., Dordrecht, 1991.

[32] L. Jiu, V. H. Moll, and C. Vignat. A symbolic approach to some identities for Bernoulli-Barnes polynomials. *Int. J. Number Theory*, 12(3):649–662, 2016.

[33] B. K. Karande and N. K. Thakare. On the unification of Bernoulli and Euler polynomials. *Indian J. Pure Appl. Math.*, 6(1):98–107, 1975.

[34] L. Kargın. Exponential polynomials and its applications to the related polynomials and numbers. *Preprint arXiv:1503.05444*, 2015.

[35] D. S. Kim, G.-W. Jang, H.-I. Kwon, and T. Kim. Two variable higher-order degenerate Fubini polynomials. *Proc. Jangjeon Math. Soc.*, 21(1):5–22, 2018.

[36] M. Klazar. Bell numbers, their relatives, and algebraic differential equations. *J. Combin. Theory Ser. A*, 102(1):63–87, 2003.

[37] K. Kozuka. On a p-adic interpolating power series of the generalized Euler numbers. *J. Math. Soc. Japan*, 42(1):113–125, 1990.

[38] K. Mahler. Zur Approximation der Exponentialfunktion und des Logarithmus. Teil I. *J. Reine Angew. Math.*, 166:118–136, 1932.

[39] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1063–1229. Elsevier Sci. B. V., Amsterdam, 1995.

[40] H. Ozden, Y. Simsek, and H. M. Srivastava. A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials. *Comput. Math. Appl.*, 60(10):2779–2787, 2010.

[41] I. Pak. Complexity problems in enumerative combinatorics. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures*, pages 3153–3180. World Sci. Publ., Hackensack, NJ, 2018.

[42] C. Praagman. Fundamental solutions for meromorphic linear difference equations in the complex plane, and related problems. *J. Reine Angew. Math.*, 369:101–109, 1986.

[43] F. Qi. Determinantal expressions and recurrence relations for Fubini and Eulerian polynomials. *Journal of Interdisciplinary Mathematics*, 22(3):317–335, 2019.

[44] S. Roman. *The umbral calculus*, volume 111 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.

[45] C.-S. Ryoo. Differential equations associated with twisted (h, q)-tangent polynomials. *J. Appl. Math. Inform.*, 36(3-4):205–212, 2018.

[46] K. Shiratani and S. Yamamoto. On a p-adic interpolation function for the Euler numbers and its derivatives. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 39(1):113–125, 1985.

[47] M. F. Singer. Algebraic relations among solutions of linear differential equations. *Trans. Amer. Math. Soc.*, 295(2):753–763, 1986.

[48] T. A. Springer. Remarks on a combinatorial problem. *Nieuw Arch. Wisk. (3)*, 19:30–36, 1971.

[49] S. M. Tanny. On some numbers related to the Bell numbers. *Canad. Math. Bull.*, 17(5):733–738, 1974/75.

[50] L. Toscano. Una classe di polinomi delia matematica attuariale. *Riv. Mat. Univ. Parma*, 1:459–470, 1950.

[51] J. Touchard. Nombres exponentiels et nombres de Bernoulli. *Canadian J. Math.*, 8:305–320, 1956.

[52] H. Tsumura. On a p-adic interpolation of the generalized Euler numbers and its applications. *Tokyo J. Math.*, 10(2):281–293, 1987.

[53] M. van der Put and M. F. Singer. *Galois theory of difference equations*. Springer-Verlag, Berlin, 1997.

[54] M. Wibmer. Existence of ∂-parameterized Picard-Vessiot extensions over fields with algebraically closed constants. *J. Algebra*, 361:163–171, 2012.

[55] D. Zagier. A modified Bernoulli number. *Nieuw Arch. Wisk. (4)*, 16(1-2):63–72, 1998.