The p-Airy distribution

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THE p-AIRY DISTRIBUTION

SERGIO CARACCIOLO, VITTORIO ERBA, AND ANDREA SPORTIELLO

Abstract. In this manuscript we consider the set of Dyck paths equipped with the uniform measure, and we study the statistical properties of a deformation of the observable "area below the Dyck path" as the size N of the path goes to infinity. The deformation under analysis is apparently new: while usually the area is constructed as the sum of the heights of the steps of the Dyck path, here we regard it as the sum of the lengths of the connected horizontal slices under the path, and we deform it by applying to the lengths of the slices a positive regular function $\omega(\ell)$ such that $\omega(\ell) \sim \ell^p$ for large argument. This shift of paradigm is motivated by applications to the Euclidean Random Assignment Problem in Random Combinatorial Optimization, and to Tree Hook Formulas in Algebraic Combinatorics.

For $p \in \mathbb{R}^+ \setminus \{\frac{1}{2}\}$, we characterize the statistical properties of the deformed area as a function of the deformation function $\omega(\ell)$ by computing its integer moments, finding a generalization of a well-known recursion for the moments of the area-Airy distribution, due to Takács. Most of the properties of the distribution of the deformed area are *universal*, meaning that they depend on the deformation parameter p, but not on the microscopic details of the function $\omega(\ell)$. We call *p-Airy distribution* this family of universal distributions.

Finally, we briefly study the limits $p \to \frac{1}{2}$, and $p \to 0$ (that is, with function $\omega(\ell) \sim \log(\ell)$), in which the recursion is singular in the sense of L'Hôpital, and the determination of the moments is more subtle. A more detailed derivation of the singular cases will be developed in a companion paper.

1. Introduction

The statistics of the area enclosed by a standard Brownian excursion[1] and the real axis is well studied, and is governed by the so-called *area-Airy distribution* $f_{\mathrm{Ai}}(x)$ (we follow the naming convention of [1]). The area-Airy distribution owes its name to the fact that its density and its Laplace transform admit a spectral representation related to the zeros of the Airy function $\mathrm{Ai}(x)$ [2, 3]. An interesting fact, dating back to the pionereeing work of Mark Kac [4], is that the area-statistics of a Brownian excursion, a classical problem in Probability Theory, is related to the study, in Quantum Mechanics, of a one-dimensional particle subject to a linear potential and to a hard wall in the origin, explaining the presence of the Airy function.

The area-Airy distribution has recently attracted attention in Statistical Physics, where it appears to model a large number of phenomena. A non-comprehensive list (more can be found in [1]) features the *maximum height of fluctuating interfaces* [5], the *size of avalanches in sandpile models* [6], the *size of ring polymers* [7] and the *anomalous diffusion in cold atoms* [8]. Very recently, the area-Airy distribution function was measured experimentally for the first time in a dilute colloidal system [9]. Generalizations of the area-Airy distribution to the statistics of the area of other Brownian processes, and to other properties of such Brownian processes, can be found in [10–17].

The present paper is aimed to the definition and study of a one-parameter family of distributions, that we call *p-Airy distributions*, which generalise the area-Airy case (corresponding to $p = 1$) to the range $p \in \mathbb{R}^+$. In the remaining of this introduction we give a list of probabilistic problems in which our generalisation arises naturally. As we will see, similarly to the original Airy distribution, our generalisation is "universal", that is, it does not depend on the microscopic details of the system that leads to its definition, and, for this reason, we expect that it can arise in a variety of applications in Statistical Mechanics and Probability, on a similar basis of the list of applications of the Airy distribution presented above. A *different* generalization of the Airy distribution, based on a phenomenon of coalescence of multiple saddle points, has been pursued in [18, 19]. Our definition and list of examples passes through a *détour* from stochastic processes to discrete combinatorics, along the line of Donsker's theorem (in reverse), and not dissimilar in spirit from what is done in [3] for the case $p = 1$.

Brownian excursions are the continuum limit of a class of discrete lattice paths called *Dyck paths*. A Dyck path of size N is a sequence of N up- and N down-steps, that is steps $(+1, +1)$ and $(+1, -1)$ on the two-dimensional integer lattice, starting at $(0, 0)$, ending at $(2N, 0)$ and never reaching negative heights. The area between a Dyck path w and the real axis can be interpreted in two natural ways, both as a sum of half-integers $\{h_w(i)\}_{1 \le i \le 2N}$ associated to the heights of the $2N$ vertical slices of the walk (analogous to a Riemann-like integral approximation), or as a sum of integers $\{\ell_w(e)\}_{1 \le e \le N}$ associated to the lengths of the N connected

[1] I.e. a Brownian motion that starts at $(0, 0)$, ends at $(1, 0)$ and never reaches negative heights.

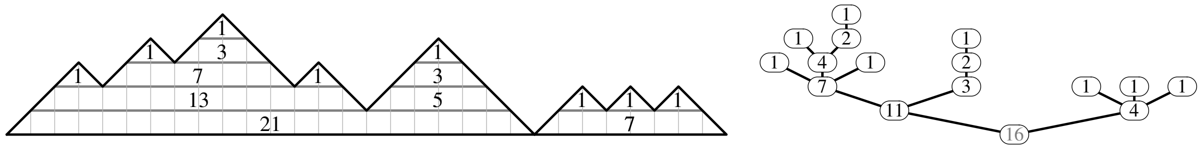


FIGURE 1. **Left:** a Dyck path of length $2N = 30$, subdivided into connected horizontal slices. In the middle of each strip e, it is indicated the value of its length $\ell_w(e)$. **Right:** the corresponding rooted planar tree $t(w)$. On each of the N non-root vertices v, it is indicated the value of $h_t(v)$, while the root vertex r has $h_t(r) = N+1$.

horizontal slices $\{e\} = E(w)$ of w (analogous to a Lebesgue-like integral approximation; see Figure 1). That is,

$$A(w) = \sum_{i=1}^{2N} h_w(i) = \sum_{e \in w} \ell_w(e)\,, \tag{1}$$

where with a slight abuse of notation we write $e \in w$ meaning $e \in E(w)$. The area-Airy distribution must be recovered by taking the continuum limit $N \to \infty$ of the distribution of $A(w)$ (induced by the uniform measure over Dyck paths), with a rescaling factor $N^{\frac{3}{2}}$. Indeed, also in [3] the characterization of the area-Airy distribution is based on the continuum limit of the distribution of the area of Dyck paths.

When considering a stochastic quantity which is the sum of many local contributions, it is natural to consider the distribution of the *moments* of the local terms. For example, the study of $A_p^{\mathrm{BM}}(w) := \sum_{i=1}^{2N} h_w(i)^p$, where the upper-script BM refers to Brownian Motion, has a long tradition [20]. However, the analogous generalisation $A_p = \sum_{e \in w} \ell_w(e)^p$ appears to be new, and is indeed the generalisation we are interested in.

One aspect of the aforementioned universality is that, in a sense that we make precise later on, we can replace $\ell_w(e)^p$ by any function $f(\ell)$ such that the large-ℓ behaviour is $f(\ell) \sim \ell^p$. Thus, we define the generalization of Equation (1) as

$$A_{(\omega_p)}(w) = \sum_{e \in w} \omega_p\left(\frac{\ell_w(e) - 1}{2}\right) \tag{2}$$

where $p \geq 0$, w is a Dyck path, $(\ell_w(e) - 1)/2$ is the semi-length of the horizontal slice e in w and ω_p is a positive regular function on the integers such that

$$\omega_p(k) \sim k^p \left(1 + \mathcal{O}\left(k^{-\eta}\right)\right) \tag{3}$$

for large k, and some $\eta > 0$. [2]

We want to compute the statistics of $A_{(\omega_p)}$ induced by the uniform measure on two classes of lattice paths, namely the Dyck paths that we have just described (or *excursions* in the following) and the Dyck bridges (which are Dyck paths without the non-negative height constraint).[3] As a result, the usual area of a Dyck path, or the unsigned area of a Dyck bridge, is recovered with the choice $\omega_p(k) = k + \frac{1}{2}$, up to the trivial overall factor of 2.

This generalization defines a deformation of the area-Airy distribution and of its analogue for Brownian bridges, and it is motivated by (at least) two rather different applications that we summarize below.

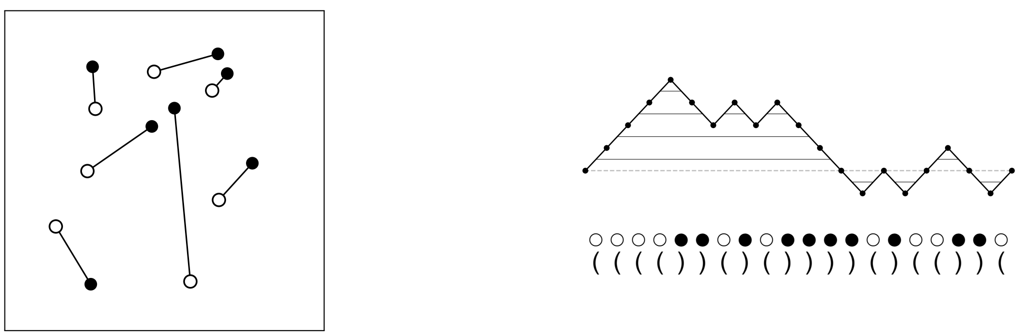


FIGURE 2. **Left:** example of optimum matching with $N = 7$ and $p = 2$ in $[0, 1]^2$. **Right:** an example of the Dyck bridge of length $N = 10$ associated to a sequence of white and black points. Horizontal solid lines represent the Dyck matching associated to this configuration of points.

[2] The subscript p in the notation $\omega_p(k)$ is to stress the asymptotic behaviour of this function, and to prepare the notation for the case of interest in which $\{\omega_p(k)\}_{p \in \mathbb{R}^+}$ is a smooth family of functions. For the moment though, p is considered fixed and $\omega_p(k)$ is just one function of a single argument, k.

[3] The names are slightly unusual in combinatorics, but are induced by the fact that the continuum limit of Dyck bridges and excursions, as we have called them, is the Brownian bridge and excursion, respectively.

1.1. **Euclidean Random Assignment Problem.** The Euclidean Assignment Problem (EAP) is a combinatorial optimization problem in which one has to pair N white points to N black points minimizing a certain cost function, depending on the Euclidean distance among the points. To be more precise, consider N white points with coordinates $\{w_i\}_{i=1}^N$ and N black points with coordinates $\{b_i\}_{i=1}^N$, and let $\tilde{c}(x)$ be a real function (in this work we will consider the case $\tilde{c}(x) = |x|^p$). We call $\tilde{c}(x)$ the *cost function* of the problem.

Solving one instance of the EAP corresponds to finding the perfect matching[4] between the white and black points which minimizes the sum of the cost function of the distances among the paired points, that is, a permutation $\pi \in \mathcal{S}_N$ such that the total cost

$$H[\pi] = \sum_{i=1}^{N} \tilde{c}(w_i - b_{\pi(i)}) \tag{4}$$

is minimal.

The random version of the EAP (that we shall denote by ERAP) is the probabilistic problem in which we study the measure over the instances obtained by extracting the positions of the points according to some probability law. The total cost of the minimum matching becomes a random variable and its statistical properties can be investigated also as a function of the exponent p. Plenty of results exist in one dimension for the cases $p \geq 1$ [21–23], also in a non compact domain [24] and some for $p < 0$ in the repulsive regime [25], due to the convexity of the cost function $\tilde{c}$. There have been remarkable extensions in higher dimensions [26, 27], particularly in the two-dimensional case when $p = 2$ [28–31]. Instead, for $0 < p < 1$, where the cost function is concave, the problem is not equally-well understood. It is known that the optimal matching must satisfy certain nesting properties [32, 33], but those are not restrictive enough to fully characterize it.

Recently, two independent upper bounds to the average optimal cost for $0 < p < 1$ were investigated by combinatorial [21, 34] and measure-theoretic [35] means. In particular, in [34] the authors studied the cost of the *Dyck matching* (in the following, the *Dyck cost*), and computed the asymptotic properties of its average value in the limit of $N \to \infty$. The Dyck matching associated to a certain configuration of points is determined by the N horizontal slices $\{e\} = E(w)$ of the Dyck bridge w generated by scanning over the points ordered by increasing coordinate, and performing an up (resp. down) step for each white (resp. black) point encountered (see Figure 2). This relation between matchings and lattice paths has already been exploited in the statistical mechanics literature to study, for example, the secondary structure of folded RNA in the models considered in [36, 37].

In the case in which the points are equispaced, the distance between two matched points in the Dyck matching and the length of the corresponding horizontal slice of the Dyck bridge are equivalent, so that the Dyck cost is exactly given by

$$H^{(p)}_{\mathrm{Dyck}}(w) = \sum_{e \in w} (\ell_w(e))^p = 2^p A_{(\omega_p)}(w) \quad \text{with} \quad \omega_p(k) = \left(k + \frac{1}{2}\right)^p , \tag{5}$$

where w is the Dyck bridge associated to the position of the points. Thus, studying the statistics of the Dyck cost in a model of equispaced points is a special case of our original problem of determining the distribution of $A_{(\omega_p)}(w)$ induced by the uniform measure over Dyck bridges of length N.

We conclude this section by pointing out that in [34] the authors highlighted a powerful universality for the statistics of the Dyck cost. If $p \geq \frac{1}{2}$, the average Dyck cost is vastly independent on the model of spacings between the points; in other words, it depends only on the long-distance behaviour of the cost function $\tilde{c}(k) \sim k^p$, and not on its microscopic details at short distance. We will find again this property, now at the level of the distribution of this stochastic quantity, not only of the average.

1.2. **Tree Hook Formula, and the statistics of star subtrees.** It is well known that Dyck paths of length $2N$ and rooted planar trees with $N+1$ vertices are in bijection, and this bijection relates the non-root vertices v of the tree and the horizontal slices e of the path (see e.g. [38, sec. I.5], and Figure 1).

Let w be a path of length $2N$, and $t = t(w)$ the corresponding tree. The *hook* $h_t(v)$ at a vertex $v \in V(t)$ is defined as the number of vertices in the sub-tree of t rooted at v (including v). Under the bijection we just have that, if the (non-root) vertex v of the tree is in bijection with the edge e, then

$$\ell_w(e) = 2h_t(v) - 1 \,, \tag{6}$$

while obviously the root vertex r has $h_t(r) = N + 1$, for all trees of size $N + 1$. The product of the hooks enters the celebrated "tree hook formula" [39, sec. 5.1.4, ex. 20] (see e.g. [40] for an introduction to the tree-hook formula and its generalisations), which counts the fraction $L(t)$ of labellings of the vertices of a rooted tree

[4] A *perfect matching* is a bijection between the black and the white points, and a permutation π describes the matching such that w_i and b_j are matched if and only if $\pi(i) = j$. See Figure 2.

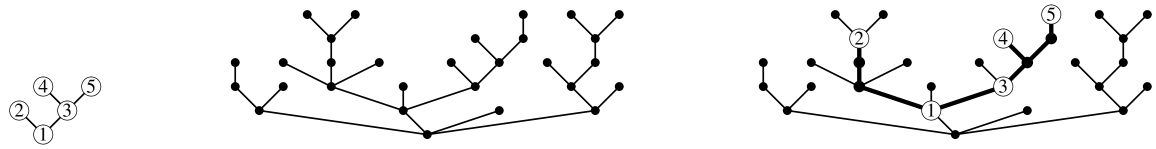


FIGURE 3. **Left:** a planar rooted labeled tree u. **Center:** a planar rooted tree t. **Right:** a proper embedding of u into t.

which are *increasing*:

$$L(t) = \prod_v h_t(v)^{-1} = \exp\left[-\sum_v \ln h_t(v) \right] = \lim_{p\to 0} \exp\left[-\frac{1}{p}\left(\sum_v h_t(v)^p - (N+1) \right)\right]. \tag{7}$$

So, the distribution of $L(t)$ induced by the uniform distribution on rooted planar trees with $N+1$ vertices is related to the statistics of the quantity

$$H^{(p)}_{\text{tree}}(t) = \sum_v (h_t(v))^p = A_{(\omega_p)}(w(t)) + (N+1)^p \qquad \text{with} \quad \omega_p(k) = (k+1)^p \tag{8}$$

in the limit $p \to 0$.

Moreover, the quantity $H^{(p)}_{\text{tree}}(t)$, for p integer, counts the $(p+1)$-tuples $(v_0, v_1, \dots, v_p)$ of vertices of t such that $v_0 \preceq v_i$ (i.e. v_0 is on the path connecting v_i to the root) for all $i = 1, \dots, p$, while the analogous quantity with $(h_t(v))^p$ replaced by $(h_t(v)-1)^p$ counts the $(p+1)$-tuples as above, with $\preceq$ replaced by $\prec$.

This is a special case of the statistics $H(t,u)$, which, for t and u two rooted trees, counts the number of *embeddings* of u in t (or *proper embeddings*, for the $\prec$ case), where a (proper) embedding of a rooted tree u in the rooted tree t is a map from the vertices of u into those of t that preserves the order relation $\preceq$ (or $\prec$), see an example in Figure 3. The statistics of the number of embeddings of a given rooted tree u, of size $\mathcal{O}(1)$, into random uniform trees t taken uniformly among all planar rooted trees of size N, in the limit of large N, is a problem of separate interest. In the case in which u is the 'star tree', composed of a root connected to p children, this is related to the statistics of $A_{(\omega_p)}(w)$. In this case, w should be uniformly distributed over Dyck paths of length N, and $\omega_p(k) = (k+1)^p$ or k^p in the case of embeddings or proper embeddings, respectively.

1.3. **A glance to our results.** In this manuscript, we fully characterize the distribution of the random variable $A^{(\text{E/B})}_{(\omega_p)}(w)$ induced by the uniform distribution over Dyck excursions or bridges for a generic cost function ω_p as above, in the limit of $N \to \infty$.

The statistical ensembles of Dyck excursions and Dyck bridges are treated with minor differences, and the results are analogous. So, for sake of compactness, we state here the main results for the case of excursions (and drop the 'E' superscript), and refer the reader to the following Sections for the details in the case of Dyck bridges. As anticipated by some facts already presented in [34], we identify three different behaviours as the parameter p varies.

For $p > \frac{1}{2}$, we find that

$$A_{(\omega_p)} \overset{d}{=} x_p N^{p+\frac{1}{2}} \left(1 + \mathcal{O}\left(N^{-\min\left(p-\frac{1}{2}, \eta, \frac{1}{2}\right)}\right)\right) \tag{9}$$

where '$\overset{d}{=}$' means 'distributed as', η controls the error term in the asymptotic behaviour of $\omega_p(k)$ (as in Equation (3)) and x_p is a random variable whose integer moments are given by

$$\langle x_p^s \rangle = \frac{4\sqrt{\pi} s!}{\Gamma\left(s\left(p+\frac{1}{2}\right) - \frac{1}{2}\right)} \mu_s(p) \tag{10}$$

where the coefficients $\mu_s(p)$ satisfy the quadratic recursion

$$\begin{aligned} \mu_0(p) &= -\frac{1}{2} \\ \mu_s(p) &= \mu_{s-1}(p) \frac{\Gamma\left(s\left(p+\frac{1}{2}\right)-1\right)}{2\Gamma\left(s\left(p+\frac{1}{2}\right)-1-p\right)} + \sum_{k=1}^{s-1} \mu_k(p)\mu_{s-k}(p) \qquad \text{for} \quad s \geq 1\,. \end{aligned} \tag{11}$$

This sequence of moments defines a unique distribution on $[0, +\infty)$, that we call $\rho^{(\text{E})}(x;p)$, (where '(E)' stands for 'excursions'), and for $p = 1$ reduces to a known recursion for the moments of the area-Airy distribution $\rho^{(\text{E})}(x;1)$, due to Takács. We recall some interesting facts on the area-Airy distribution in Appendix A. In particular, the Takács recursion for the moments is given in Equation (95). We stress again that the behaviour of $A_{(\omega_p)}$ in this regime is *universal*, in the sense that it depends only on the asymptotic behaviour of the cost function, but not on its details. For bridges, an analogous result holds, with a different moment recursion, see Proposition 4 later on. This leads to the definition of a different family of distributions $\rho^{(\text{B})}(x;p)$.

For $0 < p < \frac{1}{2}$, we find that the distribution of $A_{(\omega_p)}(w)$ peaks around its average value $\alpha(\omega_p)N$, where $\alpha(\omega_p)$ is a non-universal constant that depends on the details of ω_p that will be defined in detail in Equation (37), and which we anticipate here to be determined by

$$\sum_{k\geq 0} \frac{\Gamma(k+\frac{1}{2})}{2\sqrt{\pi}\,\Gamma(k+2)}\omega_p(k) z^k = \alpha(\omega_p) + \frac{\Gamma(p-\frac{1}{2})}{2\sqrt{\pi}}(1-z)^{\frac{1}{2}-p} + \cdots \tag{12}$$

where the dots stand for higher powers of $(1-z)$.

Nonetheless, the typical fluctuations around the mean are again of order $N^{p+\frac{1}{2}}$, and universal, and their distribution is given by the family of $\rho^{(\mathrm{E})}(x;p)$, which is determined by the same formulas (10) and (11), here for $0 < p < \frac{1}{2}$. An important difference is that the distributions $\rho^{(\mathrm{E})}(x;p)$ for $p < \frac{1}{2}$ have support on $\mathbb{R}$, contrarily to the case $p > \frac{1}{2}$, where the support is $\mathbb{R}^+$. For bridges we have the same picture, with average value $\alpha(\omega_p)N$ (the same as for the case of excursions), and with the distribution of fluctuations given by $\rho^{(\mathrm{B})}(x;p)$.

These results can be summarized by saying that, for $p \neq \frac{1}{2}$

$$A^{(\mathrm{E/B})}_{(\omega_p)} \overset{d}{=} \alpha(\omega_p)N + x_p^{(\mathrm{E/B})} N^{p+\frac{1}{2}}\left(1 + \mathcal{O}\big(N^{-\min(p,\eta,\frac{1}{2})}\big)\right), \tag{13}$$

with $x_p^{(\mathrm{E/B})}$ distributed with law $\rho^{(\mathrm{E})}(x;p)$ for excursions, and with law $\rho^{(\mathrm{B})}(x;p)$ for bridges. Note that, even just for $p > \frac{1}{2}$, this claim is slightly stronger than (9), as we have explicitated the possible correction term scaling as $N^{p-\frac{1}{2}}$. The proof of these claims is detailed in Section 2.1, for what concerns the explicit formulas, and in Section 2.2, for what concerns the uniqueness.

The integer moments of $\rho^{(\mathrm{E})}(x;p)$ have a nice combinatorial interpretation in terms of a sum over rooted planar trees, weighted in a peculiar way. We prove this connection in Section 3. For what we know, this fact, which remains non-trivial also for the $p = 1$ ordinary Airy distribution, and which, in this case, could have been evinced also from Takács recursion, was not previously observed.

If $\{\omega_p(k)\}_{p\in\mathbb{R}^+}$ is a family of functions that depend smoothly on p, we can perform the limit $p \to \frac{1}{2}$, but the involved procedure is delicate. In fact, all the moments of $\rho^{(\mathrm{E})}(x;p)$ diverge as p tends to $\frac{1}{2}$, and so does $\alpha(\omega_p)$. We anticipate the fact, to be proven in a forthcoming companion paper, that there exists a family of functions $t(p)$ such that all the moments of $\rho^{(\mathrm{E})}(x;p)$ are simultaneously regularized by shifting our random variable, $x \to x - t(p)$, and also such that $\alpha(\omega_p) + t(p)$ is a regular function of p. The family of $t(p)$'s is an affine space and has the following characterisation: t can be extended to a meromorphic function in $\mathbb{C}$, and it has a unique simple pole on the real-positive axis at $p = \frac{1}{2}$, with coefficient

$$t^* := \frac{1}{2\sqrt{\pi}} \tag{14}$$

(see Equation (70)). We discuss these claims in Section 4.

In light of this claim on the $p = \frac{1}{2}$ case, when $\{\omega_p(k)\}_{p\in\mathbb{R}^+}$ is a family of functions that depend smoothly on p, Equation (13) is better rewritten as

$$A^{(\mathrm{E/B})}_{(\omega_p)} \overset{d}{=} \left(\alpha(\omega_p) + t(p)N^{p-\frac{1}{2}}\right) N + \tilde{x}_p^{(\mathrm{E/B})} N^{p+\frac{1}{2}}\left(1 + \mathcal{O}\big(N^{-\min(p,\eta,\frac{1}{2})}\big)\right), \tag{15}$$

where $\tilde{x}_p^{(\mathrm{E/B})}$ is stochastic, distributed with the shifted law $\tilde{\rho}^{(\mathrm{E/B})}(x;p)$ (implicitly, depending on $t(p)$) such that

$$\tilde{\rho}^{(\mathrm{E/B})}(x;p) = \rho^{(\mathrm{E/B})}(x - t(p);p)\,, \tag{16}$$

and $\alpha(\omega_p)$ and $t(p)$ are deterministic constants (for a given p and ω_p). Analogously, we will call $\tilde{\mu}_s(p)$ the (purportedly finite) shifted moments.

The limit $p \to \frac{1}{2}$ of Equation (15) is slightly non-trivial, as it involves the use of L'Hôpital's rule, and gives

$$A_{(\omega_{\frac{1}{2}})} \overset{d}{=} t^*\,N\log N + \left(\alpha_0 + t_0 + \tilde{x}_{\frac{1}{2}}\right) N + o(N)\,, \tag{17}$$

where $t^* = 1/(2\sqrt{\pi})$ (as defined in (14)), and α_0, t_0 are the constant terms of the Laurent expansion at $p = \frac{1}{2}$ of the corresponding quantities, i.e.

$$t(p) = \frac{t^*}{p-\frac{1}{2}} + t_0 + \mathcal{O}(p-\tfrac{1}{2}) \qquad\qquad \alpha(\omega_p) = -\frac{t^*}{p-\frac{1}{2}} + \alpha_0 + o(1)\,. \tag{18}$$

Using this representation, it is easy to see that at $p = \frac{1}{2}$ the distribution $A_{(\omega_p)}$ concentrates around its average value $t^* N\log N$ with typical fluctuations of order N. The fluctuations are distributed with universal law $\tilde{\rho}^{(\mathrm{E/B})}\left(x;\frac{1}{2}\right)$, apart from a non-universal shift $\alpha_0 + t_0$ that depends on the details of the cost function ω_p and on the regularization. Expressions for the shifted moments $\tilde{\mu}_s(\frac{1}{2})$ are given, without proof, in Section 4.

For the sake of concreteness, it is tempting to choose once and for all a 'canonical' regularization $t(p)$, within the family of possible choices. A natural choice is to shift the distributions in order to have zero mean for each value of p, which corresponds to the choice

$$t(p) = \frac{\Gamma(p-\frac{1}{2})}{2\,\Gamma(p)} = \frac{1}{2\sqrt{\pi}\,(p-\frac{1}{2})} + \frac{\ln 2}{\sqrt{\pi}} + \frac{p-\frac{1}{2}}{\sqrt{\pi}}\left((\ln 2)^2 - \frac{\pi^2}{12}\right) + \mathcal{O}\big((p-\tfrac{1}{2})^2\big)\,. \tag{19}$$

In this paper we will call the distributions $\tilde{\rho}^{(\mathrm{E/B})}(x;p)$ resulting from the shift (19) 'the' *p-Airy distribution* (for excursions and bridges respectively), leaving aside the forementioned arbitrariness of the regularisation. We stress again that, under this choice, $\tilde{\rho}^{(\mathrm{E})}(x;1)$ is not the ordinary area-Airy distribution, but rather its centered version, that is, the distribution shifted by $-\sqrt{\pi}/2$.

The limit $p \to 0$ is trivial, especially under the choice $\omega_0(k) = 1$, that gives at sight in (2) $A_{(\omega_0)}(w) = N$ for all w. We just get $\alpha(\omega_0) = 1$ and $\rho^{(\mathrm{E})}(x;0) = \delta(x)$. Nonetheless, given a family of functions $\{\omega_p\}_{p\in\mathbb{R}^+}$, one can study a more subtle limit of the quantity $A_{(\omega_p)}(w)$, namely

$$\lim_{p\to 0^+} \frac{A_{(\omega_p)}(w) - N}{p} \tag{20}$$

which amounts to studying the observable $A_{(\omega_{0^+})}$, where ω_{0^+} denotes a function such that $\omega_{0^+}(k) \sim \log k$ for large argument k. We prove that, in this case, the distributions peak around their average value in the limit of $N \to \infty$, with Gaussian fluctuations:

$$A_{(\omega_{0^+})} \overset{d}{=} \alpha(\omega_{0^+})N + \sqrt{\gamma_E N \log N}\, z + \mathcal{O}\big(\sqrt{N}\big) \tag{21}$$

where γ_E is the Euler's gamma constant, and z is a standard Gaussian random variable. We present these results in Section 5.

2. The distribution of $A_{(\omega_p)}$ for $p \neq \frac{1}{2}$

Our strategy to prove Equation (13), namely

$$A_{(\omega_p)} \overset{d}{=} \alpha(\omega_p)N + x_p N^{p+\frac{1}{2}}\left(1 + \mathcal{O}\big(N^{-\min(p,\eta,\frac{1}{2})}\big)\right)\,, \tag{22}$$

goes through the derivation of the asymptotic behaviour of the integer moments of $A_{(\omega_p)}$. This will be achieved in Section 2.1 by studying their generating functions, and then applying singularity analysis. The identification of the distribution, by the classical Carleman's condition, will follow in Section 2.2.

2.1. The integer moments of $A_{(\omega_p,\epsilon)}$.

As a first step, it is useful to consider a slight generalization of $A_{(\omega_p)}(w)$, that is:

$$A_{(\omega_p,\epsilon)}(w) = \sum_{e\in w}\left[\omega_p\left(\frac{\ell_w(e)-1}{2}\right) - \epsilon\right] = A_{(\omega_p)} - \epsilon N\,. \tag{23}$$

The idea is that, in the following, ϵ can be tuned to cancel exactly the deterministic term $\alpha(\omega_p)N$ in Equation (22), that dominates when $0 < p < \frac{1}{2}$, thus allowing for a unified treatment of the two regimes $0 < p < \frac{1}{2}$ and $p > \frac{1}{2}$.

We shall consider the integer moments of $A_{(\omega_p,\epsilon)}(w)$ defined as

$$M_s^{(\mathrm{E})}(N;\omega_p,\epsilon) = C_N^{-1}\sum_{w\in\mathcal{C}_N}\left(A_{(\omega_p,\epsilon)}(w)\right)^s\,, \tag{24}$$

$$M_s^{(\mathrm{B})}(N;\omega_p,\epsilon) = B_N^{-1}\sum_{w\in\mathcal{B}_N}\left(A_{(\omega_p,\epsilon)}(w)\right)^s\,, \tag{25}$$

where $\mathcal{C}_N$ and $\mathcal{B}_N$ are the sets of Dyck paths and Dyck bridges, and $C_N = |\mathcal{C}_N| = \frac{1}{N+1}\binom{2N}{N}$ and $B_N = |\mathcal{B}_N| = \binom{2N}{N}$ denote the respective cardinalities.

To compute these moments, we will proceed as follows:

(1) we introduce two generating functions, $E_s(z)$ and $B_s(z)$, for the moments $M_s^{(\mathrm{E/B})}(N;\omega_p,\epsilon)$, see Equation (26) and Equation (27). The analysis of the leading singular behaviour of $E_s(z)$ and $B_s(z)$ around their dominant pole determines the asymptotic behaviour for large N of $M^{(\mathrm{E/B})}(N;\omega_p,\epsilon)$.
(2) we show in Proposition 1 how the generating functions $E_s(z)$ and $B_s(z)$ can be computed recursively, in terms of a suitable linear operator $\hat{L}$ acting on the space of formal power series, and determined by ω_p;
(3) in the perspective of the asymptotic analysis mentioned before, we provide some tools, namely Proposition 2 and Proposition 3, to study the recursion relations of Proposition 1 at their leading singular order.

Summing all these efforts up, we will be able to prove Proposition 4, where we determine the coefficient of the leading singular behaviour of $E_s(z)$ and $B_s(z)$, obtaining that the integer moments of the random variable x_p satisfy the Equations (10) and (11) already presented in the Introduction (and the respective versions for bridges).

So, we start by introducing the generating functions (the dependence on ω_p and ϵ is understood)

$$E_s(z) = \frac{1}{2} \sum_{N\geq 0} \frac{C_N}{s!} M_s^{(\mathrm{E})}(N;\omega_p,\epsilon) \left(\frac{z}{4}\right)^N = \frac{1}{2} \sum_{N\geq 0} \sum_{w\in\mathcal{C}_N} \frac{(A_{(\omega_p,\epsilon)}(w))^s}{s!} \left(\frac{z}{4}\right)^N , \tag{26}$$

$$B_s(z) = \sum_{N\geq 0} \frac{B_N}{s!} M_s^{(\mathrm{B})}(N;\omega_p,\epsilon) \left(\frac{z}{4}\right)^N = \sum_{N\geq 0} \sum_{w\in\mathcal{B}_N} \frac{(A_{(\omega_p,\epsilon)}(w)^s}{s!} \left(\frac{z}{4}\right)^N . \tag{27}$$

Let $\odot$ denote the Hadamard product between formal power series (see e.g. [38, V.3.2])

$$\left(\sum_{N\geq 0} g_N z^N\right) \odot \left(\sum_{N\geq 0} h_N z^N\right) = \sum_{N\geq 0} g_N h_N z^N \tag{28}$$

and let

$$L(z;\omega_p,\epsilon) = \sum_{k\geq 0} \left[\, \omega_p\,(k) - \epsilon\,\right] z^k . \tag{29}$$

We will need the following definition:

Definition 1. *Let $\hat{L}_{(\omega_p,\epsilon)}$ be the linear operator on formal power series defined by*

$$\hat{L}_{(\omega_p,\epsilon)}[f](z) = L(z;\omega_p,\epsilon) \odot f(z) = \sum_{N\geq 0} f_N \left[\, \omega_p\,(N) - \epsilon\,\right] z^N = \sum_{N\geq 0} f_N \omega_p\,(N)\, z^N - \epsilon f(z) . \tag{30}$$

Two small remarks are in order. First, we can define the integer powers $\hat{L}^k_{(\omega_p,\epsilon)}$ as

$$\hat{L}^k_{(\omega_p,\epsilon)}[f](z) = L(z;\omega_p,\epsilon)^{\odot k} \odot f(z) = \sum_{N\geq 0} f_N \left[\, \omega_p\,(N) - \epsilon\,\right]^k z^N . \tag{31}$$

Furthermore, calling $\hat{L}_\omega = \hat{L}_{(\omega,0)}$, we have $\hat{L}_{(\omega_p,\epsilon)} = \hat{L}_{\omega_p} - \epsilon\mathbb{I}$, where $\mathbb{I}$ is the identity operator on generating functions, that can be represented either by usual multiplication by 1, or by Hadamard multiplication by $(1-z)^{-1}$.

Proposition 1. *The generating functions $E_s(z)$ and $B_s(z)$ satisfy, respectively,*

$$\begin{aligned} E_0(z) &= \frac{1-\sqrt{1-z}}{z} , \\ E_s(z) &= \frac{z}{2\sqrt{1-z}} \sum_{\substack{s_1,s_2,s_3\geq 0 \\ s_1+s_2+s_3=s \\ s_1,s_2<s}} E_{s_1}(z) \frac{1}{s_3!} \hat{L}^{s_3}_{(\omega_p,\epsilon)}[E_{s_2}](z) \qquad \text{for} \quad s\geq 1 . \end{aligned} \tag{32}$$

and

$$\begin{aligned} B_0(z) &= \frac{1}{\sqrt{1-z}} , \\ B_s(z) &= \frac{z}{\sqrt{1-z}} \sum_{\substack{s_1,s_2,s_3\geq 0 \\ s_1+s_2+s_3=s \\ s_1<s}} B_{s_1}(z) \frac{1}{s_3!} \hat{L}^{s_3}_{(\omega_p,\epsilon)}[E_{s_2}](z) \qquad \text{for} \quad s\geq 1 . \end{aligned} \tag{33}$$

Notice that the range of the sums of Equation (32) and Equation (33) are slightly different: in the second case, $s_2 = s$ is allowed.

The proof can be found in Appendix D. For excursions, the idea is to decompose a Dyck path w as $w = (+, w_1, -, w_2)$, where, for some $0 \leq m \leq N-1$, $w_1 \in \mathcal{C}_m$, $w_2 \in \mathcal{C}_{N-m-1}$ and $+/-$ are up/down step. This allows to decompose $A_{(\omega_p,\epsilon)}(w)$ into two independent terms relative to w_1 and w_2 respectively. A similar decomposition holds for Dyck bridges, leading to the second set of Equations. Note that (32) is quadratic in the E_s's, while (33) is linear in the E_s's and B_s's, so that a way to proceed is first solve the non-linear recursive relation (32), and then deduce an inhomogeneous linear equation for the B_s's from (33). This is one of the reasons why we mostly concentrate on the case of excursions, and only sketch the minor modifications required for bridges, when there are no substantial differences.

The exact treatment of Equation (32) and Equation (33) and the expansion of $E_s(z)$ and $B_s(z)$ are not possible in general, mainly due to the fact that the operator $\hat{L}$ is complicated. Nonetheless, we are able to obtain exact informations for the asymptotics of their coefficients $[z^N]E_s(z)$ and $[z^N]B_s(z)$ at leading order for $N \to \infty$. In fact, the asymptotic behaviour of the coefficients of a generating function is strictly related to the behaviour of the generating function around its dominant singularity, i.e. its pole of smallest modulus. For a complete review on singularity analysis refer to [38], and for a summary of the basic results that we will need in the following see Appendix B.

First of all, we need to study how the operator $\hat{L}_{(\omega_p,\epsilon)}$ alters the singular behaviour of a generating function. Let $\tilde{\mathcal{O}}$ be the usual big-O notation, for expansions of functions of z near $z=1$, but with possible additional logarithmic factors $\log(1-z)$ that we are not interested in tracking (that is, $f(z)=g(z)+\tilde{\mathcal{O}}((1-z)^b)$ means that there exists a finite c such that $f(z)=g(z)+\mathcal{O}((1-z)^b(\ln(1-z))^c)$, which in particular implies that $f(z)=g(z)+\mathcal{O}((1-z)^{b+\delta})$ for all $\delta>0$)

Proposition 2. *Let $p>0$, $k\in\mathbb{Z}^+$ and let $f(z)=\sum_{N\geq 0} f_N z^N$ be a generating function with unit radius of convergence. Suppose that f admits the singular expansion*

$$f(z)=f^{\text{reg}}(z)+a\,(1-z)^{-\alpha}\left(1+\tilde{\mathcal{O}}\left((1-z)^{\xi}\right)\right)\qquad \text{for } z\to 1\,. \tag{34}$$

with $f^{\text{reg}}(z)$ analytic in a neighbourhood of $z=1$ and $\xi>0$ (in particular, $f(z)$ has a unique dominant singularity, at $z=1$).

Suppose further that

$$\omega_p(N)\sim N^p(1+\mathcal{O}(N^{-\eta}))\qquad \text{for } N\to\infty \tag{35}$$

for some $\eta>0$.

If α and $\alpha+kp\neq 0,-1,-2,\dots$ then

$$\hat{L}^k_{(\omega_p,\epsilon)}[f](z)=\tilde{f}^{\text{reg}}(z;\omega_p,\epsilon,f,k)+a\frac{\Gamma(\alpha+kp)}{\Gamma(\alpha)}(1-z)^{-(\alpha+kp)}\left(1+\tilde{\mathcal{O}}\left((1-z)^{\min(\eta,\xi)}\right)\right)\qquad \text{for } z\to 1 \tag{36}$$

with a new, a priori *unknown regular part $\tilde{f}^{\text{reg}}(z)$.*

Proof. As $\omega_p(N)\sim N^p$ for large N, the coefficients of $\hat{L}^k_{(\omega_p,\epsilon)}[\sum_{N\geq 0} f_N z^N]$ scale as $f_N N^{kp}$ for $N\to\infty$. The fact that f_N gets corrected sub-exponentially (in particular, algebraically) means that the radius of convergence is not changed by $\hat{L}_q$, so that also $\hat{L}_q[f]$ has singular expansion around $z=1$, and no other singularities of smaller radius. Now, Corollary 1 of Appendix B implies the result at leading order.

An analogous procedure allows to estimate the order of the remainder. As we are not requiring any condition related to ξ and η (namely, that suitable linear combinations are not in a certain range of integers), there may be additional logarithmic factors, that we treat consistently by means of the $\tilde{\mathcal{O}}$ notation. □

The regular part $\tilde{f}^{\text{reg}}(z;\omega_p,\epsilon,f,k)$ is not determined by the asymptotic behaviour of f and $L(z;\omega_p,\epsilon)$ alone, and its calculation requires the full functions ω_p and f, besides the parameter ϵ (see [38] for more information on the singularity analysis of Hadamard products). In the following, we will be interested in the determination of this elusive quantity only in the case $k=1$, $\epsilon=0$ and $f(z)=E_0(z)$, i.e. the regular part at $z=1$ of

$$\hat{L}_{(\omega_p,0)}\left[E_0\right](z)=\sum_{N\geq 0} c_N\omega_p(N)z^N=:\alpha(\omega_p)(1+\mathcal{O}(1-z))+\frac{\Gamma(p-\frac{1}{2})}{2\sqrt{\pi}}(1-z)^{\frac{1}{2}-p}(1+\mathcal{O}((1-z)^{\min\left(\eta,\frac{1}{2}\right)})) \tag{37}$$

where $c_k=2^{-2k-1}C_k=\frac{\Gamma(k+\frac{1}{2})}{2\sqrt{\pi}\,\Gamma(k+2)}$ are the normalized Catalan numbers (that is, $\sum_{k\geq 0} c_k=1$).

Equation (37), which corresponds to (12) with a more precise description of the error terms, implicitly defines the constant $\alpha(\omega_p)$ as the limit for $z\to 1$ of the regular part of $\hat{L}_{(\omega_p,0)}\left[E_0\right](z)$. For some special choices of ω_p it is possible to compute explicitly this regular part, and we provide some examples in Appendix C. In any case, we can formally get rid of the unknown term $\alpha(\omega_p)$ by reabsorbing it into the trivial constant ϵ.

Indeed, for $k=1$, we recall that $\hat{L}_{(\omega_p,\epsilon)}[f](z)=\hat{L}_{(\omega_p,0)}[f](z)-\epsilon f(z)$. Thus, we have that

$$\tilde{f}^{\text{reg}}(z;\omega_p,\epsilon,f,1)=\tilde{f}^{\text{reg}}(z;\omega_p,0,f,1)-\epsilon f^{\text{reg}}(z)\,. \tag{38}$$

As a result we have:

Proposition 3. *By an appropriate choice of ϵ, namely*

$$\epsilon(\omega_p,f)=\frac{\tilde{f}^{\text{reg}}(1;\omega_p,0,f,1)}{f^{\text{reg}}(1)}\,, \tag{39}$$

we can guarantee that $\tilde{f}^{\text{reg}}(z;\omega_p,\epsilon(\omega_p,f),f,1)=\mathcal{O}(1-z)$ for $z\to 1$. In the case of $f=E_0$, this reduces to

$$\epsilon=\alpha(\omega_p)\,. \tag{40}$$

We are now ready to study the asymptotic behaviour of the moments in our models. By induction, it is easy to prove that the location of the dominant singularity of $E_s(z)$ and of $B_s(z)$ must be at $z=1$. Our strategy is to expand Equations (32) and (33) to the leading singular order in $1-z$, to obtain the leading singular order of $E_s(z)$ and $B_s(z)$.

Proposition 4. *Let* $\epsilon = \epsilon(\omega_p, E_0) = \alpha(\omega_p)$ *and* $p > 0$, $p \neq \frac{1}{2}$. *Suppose that*

$$\omega_p(N) \sim N^p(1 + \mathcal{O}(N^{-\eta})) \quad \textit{for } N \to \infty \tag{41}$$

for some $\eta > 0$, *and let*

$$\eta_s = \begin{cases} 1 & s = 0 \\ \min\left(\eta, \frac{1}{2}\right) & s = 1 \\ \min\left(\eta, p, \frac{1}{2}\right) & s \geq 2 \end{cases}. \tag{42}$$

Then, for $s \geq 1$

$$E_s(z) = 2\mu_s^{(\mathrm{E})}(p)(1-z)^{-\left(p+\frac{1}{2}\right)s+\frac{1}{2}}(1 + \tilde{\mathcal{O}}((1-z)^{\eta_s})) \quad \textit{for } z \to 1\,, \tag{43}$$

and

$$B_s(z) = \ \mu_s^{(\mathrm{B})}(p)(1-z)^{-\left(p+\frac{1}{2}\right)s-\frac{1}{2}}(1 + \tilde{\mathcal{O}}((1-z)^{\eta_s})) \quad \textit{for } z \to 1\,. \tag{44}$$

The coefficients $\mu_s^{(\mathrm{E})}(p)$ *satisfy*

$$\begin{aligned} \mu_0^{(\mathrm{E})}(p) &= -\frac{1}{2}\,, \\ \mu_1^{(\mathrm{E})}(p) &= \frac{1}{8\sqrt{\pi}}\Gamma\left(p - \tfrac{1}{2}\right)\,, \\ \mu_s^{(\mathrm{E})}(p) &= \mu_{s-1}^{(\mathrm{E})}(p)\frac{\Gamma\left(s\left(p+\frac{1}{2}\right)-1\right)}{2\Gamma\left(s\left(p+\frac{1}{2}\right)-p-1\right)} + \sum_{k=1}^{s-1}\mu_k^{(\mathrm{E})}(p)\mu_{s-k}^{(\mathrm{E})}(p) \qquad \textit{for} \quad s \geq 2\,. \end{aligned} \tag{45}$$

The coefficients $\mu_s^{(\mathrm{B})}(p)$ *satisfy*

$$\begin{aligned} \mu_0^{(\mathrm{B})}(p) &= 1 \\ \mu_s^{(\mathrm{B})}(p) &= 2\sum_{k=0}^{s-1}\mu_k^{(\mathrm{B})}(p)\mu_{s-k}^{(\mathrm{E})}(p) \qquad \textit{for} \quad s \geq 1\,. \end{aligned} \tag{46}$$

Note in particular that, if $\omega_p(k) = k^p + \mathcal{O}(1, k^{p-\frac{1}{2}})$, we can bound all error-term exponents by $\eta_s = \min(p, \frac{1}{2})$.

A detailed proof can be found in Appendix E. The idea is the following. For $s \in \{0, 1\}$, Equation (43) and (44), and the starting conditions for the recursions in Equation (45) and (46), can be verified explicitly. The particular choice of $\epsilon = \alpha(\omega_p)$ is crucial for the ansatz to be correct at $s = 1$, as otherwise the non-null regular part of $\hat{L}_{(\omega_p,0)}[E_0](z)$ would dominate when p is in the range $0 < p < \frac{1}{2}$. Then, one proceeds by induction, using Proposition 2 and the expansions of Equation (32) and Equation (33) to the leading singular order in $(1-z)$.

Finally, using again Corollary 1 we get that

$$M_s^{(\mathrm{E})}(N; \omega_p, \alpha(\omega_p)) \sim \frac{4\sqrt{\pi}\, s!}{\Gamma\left(\left(p+\frac{1}{2}\right)s - \frac{1}{2}\right)}\mu_s^{(\mathrm{E})}(p)N^{s\left(p+\frac{1}{2}\right)}(1 + \tilde{\mathcal{O}}(N^{-\eta_s})) \tag{47}$$

and

$$M_s^{(\mathrm{B})}(N; \omega_p, \alpha(\omega_p)) \sim \frac{\sqrt{\pi}\, s!}{\Gamma\left(\left(p+\frac{1}{2}\right)s + \frac{1}{2}\right)}\mu_s^{(\mathrm{B})}(p)N^{s\left(p+\frac{1}{2}\right)}(1 + \tilde{\mathcal{O}}(N^{-\eta_s}))\,. \tag{48}$$

Equations (47) and (48) show that, for both ranges $p \in (0, \frac{1}{2})$ and $p \in (\frac{1}{2}, +\infty)$, the stochastic leading term in our quantity of interest scales as $N^{p+\frac{1}{2}}$. Let us define the asymptotic rescaled moments

$$\overline{M}_s := \lim_{N\to\infty} M_s(N; \omega_p, \alpha(\omega_p))\, N^{-s\left(p+\frac{1}{2}\right)}\,, \tag{49}$$

that is

$$\overline{M}_s^{(\mathrm{E})}(\omega_p, \alpha(\omega_p)) = \frac{4\sqrt{\pi}\, s!}{\Gamma\left(\left(p+\frac{1}{2}\right)s - \frac{1}{2}\right)}\mu_s^{(\mathrm{E})}(p) \tag{50}$$

$$\overline{M}_s^{(\mathrm{B})}(\omega_p, \alpha(\omega_p)) = \frac{\sqrt{\pi}\, s!}{\Gamma\left(\left(p+\frac{1}{2}\right)s + \frac{1}{2}\right)}\mu_s^{(\mathrm{B})}(p)\,, \tag{51}$$

These are the moments of the candidate distribution our random variable x_p introduced in Equation (9). However, we need to prove that these moments, determined by the recursions in Equation (45) and Equation (46), define *uniquely* two families of distributions, that we shall call $\rho^{(\mathrm{E/B})}(x; p)$. The uniqueness of these distributions is discussed in the following section.

As a final remark, let us rephrase the role of the constant $\epsilon = \alpha(\omega_p)$. Recall that

$$A_{(\omega_p)}(w) = \epsilon N + A_{(\omega_p,\epsilon)}(w)\,. \tag{52}$$

We proved that the integer moments of $A_{(\omega_p,\epsilon)}(w)$ scale as $N^{s(p+\frac{1}{2})}$, so that, at leading order in N and for $0<p<\frac{1}{2}$, the moments of $A_{(\omega_p)}(w)$ are given by

$$\left\langle \left[A_{(\omega_p)}(w)\right]^s \right\rangle \sim [\alpha(\omega_p)]^s\, N^s\,. \tag{53}$$

Thus, for $0<p<\frac{1}{2}$, the distribution of the rescaled variable $N^{-1}A_{(\omega_p)}$ converges, as N grows, to a Dirac's delta distribution, with average value $\alpha(\omega_p)$. In this regime, the fluctuations around the mean of the rescaled variable are of order $N^{\frac{1}{2}-p}$, and their distribution, after an appropriate rescaling, is described by the non-trivial moments $\overline{M}(\omega_p, \alpha(\omega_p))$.

Notice also that, for $p>\frac{1}{2}$, the candidate distributions are supported on $[0,+\infty)$ as they describe a positive random costs, while for $0<p<\frac{1}{2}$ the distributions are supported on $(-\infty,+\infty)$, which is compatible with the presence of the shift at leading order.

2.2. Uniqueness of the distributions $\rho^{(\mathrm{E/B})}(x;p)$.

The problem of determining whether a moment sequence defines uniquely a distribution goes under the name of *moment problem* [41]. In particular, if the distribution is supported on $[0,+\infty)$, the problem is called *Stieltjes moment problem*, while for distributions supported on $(-\infty,+\infty)$ it is called *Hamburger moment problem*. In both cases, a sufficient condition for the uniqueness of the distribution is given by Carleman's condition. In the case of the Hamburger moment problem, the distribution is uniquely determined if

$$\sum_{n\geq 1} m_{2n}^{-\frac{1}{2n}} = +\infty\,, \tag{54}$$

where $\{m_n\}_{n=1}^{+\infty}$ is the moment sequence. In the case of the Stieltjes moment problem, the distribution is uniquely determined if

$$\sum_{n\geq 1} m_{n}^{-\frac{1}{2n}} = +\infty\,. \tag{55}$$

In both cases, we see that the uniqueness of the distribution $\rho^{(\mathrm{E/B})}(x;p)$ can be determined by an asymptotic analysis of $\mu_s^{(\mathrm{E/B})}(p)$ for large order s. If the moment sequences do not grow too fast, then Carleman's condition will grant the uniqueness of the distribution.

Proposition 5. *If $p\in\mathbb{R}^+\setminus\{\frac{1}{2}\}$, there exist A_p and $R_p\in\mathbb{R}^+$ such that*

$$\begin{aligned} |\mu_s^{(\mathrm{E})}(p)| &\leq R_p\, A_p^s\, \Gamma\left(ps+1\right) C_{s-1}\,, \quad \forall s\geq 1\\ |\mu_s^{(\mathrm{B})}(p)| &\leq A_p^s\, \Gamma\left(ps+1\right) C_{s}\,, \quad \forall s\geq 0\,. \end{aligned} \tag{56}$$

A proof can be found in Appendix F, along with explicit expressions for A_p and R_p.

By combining Proposition 5 with the normalizations of Equation (47) and Equation (48), we find that the moments $M^{(\mathrm{E/B})}$ grow slower than $\exp\left(\frac{s}{2}\log s\right)$ for large s. Thus, Carleman's condition immediately implies that both distributions $\rho^{(\mathrm{E/B})}(x;p)$ are uniquely determined by their moment sequences for all $p\in\mathbb{R}^+\setminus\{\frac{1}{2}\}$.

3. A combinatorial interpretation of the coefficients $\mu_s^{(\mathrm{E})}(p)$

In this section we show that Equation (45) is solved by a diagrammatic expansion involving rooted planar trees, with a peculiar form of the weight.

Let us call $\mathcal{T}(s)$ the set of rooted planar trees with s non-root vertices, and, for a tree $T\in\mathcal{T}(s)$ and a vertex $v\in T$, call ℓ_v its out-going degree (i.e., the number of 'children' in the tree), and k_v the number of 'descendents' (that is, w.r.t. the notion of hook of a rooted tree introduced in Section 1.2, it is the hook at v, minus one).

Proposition 6.

$$\mu_s^{(\mathrm{E})}(p) = \frac{1}{b(s)} \sum_{T\in\mathcal{T}(s)} \prod_{v\in T} a(\ell_v) b(k_v) \tag{57}$$

where the range of the product $v\in T$ stands for the $s+1$ vertices of T,

$$a(\ell) = \frac{4^{\ell-1}\,\Gamma\left(\ell-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\ell+1\right)} \tag{58}$$

and

$$b(k) = \frac{\Gamma\left((k+1)(p-\frac{1}{2})+k\right)}{2\,\Gamma\left(k(p-\frac{1}{2})+k-\frac{1}{2}\right)}\,. \tag{59}$$

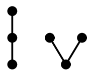
s	diagrams	$X(z)$	$Y(z)$
0		b_0a_0z	0
1		$b_0a_0\,b_1a_1\,z^2$	$b_0a_0\,a_1\,z$
2		$\big(b_0a_0\,b_1a_1\,b_2a_1 + (b_0a_0)^2b_2a_2\big)z^3$	$\big(b_0a_0\,b_1a_1^2 + (b_0a_0)^2a_2\big)z^2$

TABLE 1. The first terms of the series $X(z)$ and $Y(z)$ in (62) and (63). Note that Equation (67) is satisfied up to the order z^2, provided that $a_1 = a_2 = 1$, which is in agreement with the prescription (66).

Proof. Given two sequences $\{a(\ell)\}_{\ell\geq 0}$ and $\{b(k)\}_{k\geq 0}$, we define the quantities

$$\nu(s) = \sum_{T\in\mathcal{T}(s)}\prod_{v\in T} b(k_v)a(\ell_v)\,. \tag{60}$$

Let us introduce the generating functions:

$$A(z) = \sum_{\ell} a(\ell)z^{\ell}\,, \tag{61}$$

$$X(z) = \sum_{s\geq 1}\nu(s-1)z^s \tag{62}$$

$$Y(z) = \sum_{s\geq 1}\frac{\nu(s)}{b(s)}z^s\,. \tag{63}$$

Then, the recursive combinatorial definition of rooted planar trees gives $\nu(0) = b(0)a(0)$ and, for $s\geq 1$,

$$\nu(s) = b(s)\sum_{\ell\geq 1}a(\ell)\sum_{\substack{s_1\ldots s_\ell\geq 1\\ \sum_i s_i = s}}\prod_{i=1}^{\ell}\nu(s_i-1)\,, \tag{64}$$

that is, multiplying both sides by $z^s/b(s)$, and summing over $s\geq 1$, we get

$$Y(z) = A(X(z)) - A(0)\,. \tag{65}$$

In the specific case in which $a(\ell)$ are the (shifted) Catalan numbers,

$$a(\ell) = \frac{4^{\ell-1}\Gamma\left(\ell-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\ell+1\right)} \tag{66}$$

(i.e. $a(0) = -\frac{1}{2}$ and $a(\ell) = C_{\ell-1}$ for $\ell\geq 1$), we get $A(z) = -\frac{1}{2}\sqrt{1-4z}$, and Equation (65) can be rewritten as

$$Y(z) = X(z) + Y(z)^2\,. \tag{67}$$

(See Table 1 for the first few terms of these series.)

By substituting the definitions (62) and (63), we recognise Equation (45), under the identifications

$$b(s) = \frac{\Gamma\left((s+1)(p-\frac{1}{2})+s\right)}{2\Gamma\left(s(p-\frac{1}{2})+s-\frac{1}{2}\right)}\,, \tag{68}$$

and $\nu(s) = b(s)\,\mu_s^{(\mathrm{E})}(p)$. □

4. The case of $p = \frac{1}{2}$

As already mentioned in the introduction, the moments of $\rho^{(\mathrm{E})}(x;p)$, as well as the constant $\alpha(\omega_p)$, diverge in the limit $p\to\frac{1}{2}$. In particular, it is easy to prove by induction that

$$\overline{M}_s^{(\mathrm{E})}(\omega_p,\alpha(\omega_p)) = \left(\frac{t^*}{p-\frac{1}{2}}\right)^s\left(1+\mathcal{O}\left(p-\tfrac{1}{2}\right)\right) \tag{69}$$

where

$$t^* = 4\lim_{p\to\frac{1}{2}}\left[\mu_1^{(\mathrm{E})}(p)\left(p-\tfrac{1}{2}\right)\right] = \frac{1}{2\sqrt{\pi}} \tag{70}$$

is the residue of $\overline{M}_1^{(\mathrm{E})}(\omega_p,\alpha(\omega_p))$ at its simple pole $p = \frac{1}{2}$ (the factor of 4 comes from the prefactor of the moments).

s	diagrams	contribution to $\widetilde{M}_s$
1		$\frac{\mathrm{d}}{\mathrm{d}y_1}\frac{e^{\xi y_1}}{\Gamma(1-y_1)}a(1)b(1-y_1)\Big\vert_{y_1=0} = 0$
2		$-2\frac{\mathrm{d}}{\mathrm{d}y_1}\frac{e^{\xi y_1}}{\Gamma(2-y_1)}a(1)^2b(1-y_1)b(2-y_1)\Big\vert_{y_1=0} = \frac{4(\ln 2-1)}{\pi}$
		$\frac{1}{4\sqrt{\pi}}\frac{\mathrm{d}^2}{\mathrm{d}y_1\mathrm{d}y_2}\frac{e^{\xi(y_1+y_2)}}{\Gamma(2-y_1-y_2)}a(2)b(2-y_1-y_2)\Big\vert_{y_1=y_2=0} = \frac{4}{\pi}-\frac{\pi}{4}$

TABLE 2. The first diagrams involved in the evaluation of the moments in Equation (76). Here the functions a and b are as in (58) and (59), evalued at $p=\frac{1}{2}$, and $\xi = 2\ln 2+\gamma_E$.

Moreover, the explicit examples for the computation of $\alpha(\omega_p)$ presented in Appendix C, plus the easy universality argument at the end of the same appendix, imply that, independently from the microscopic details of the function ω_p,

$$\alpha(\omega_p) = -\frac{t^*}{p-\frac{1}{2}}\left(1+\mathcal{O}\left(p-\tfrac{1}{2}\right)\right) . \tag{71}$$

Thus, at least at the level of the first moment, the divergences cancel out, and the average of $A_{(\omega_p)}$ has a finite limit at $p=\frac{1}{2}$.

The apparent mechanism is that the two divergences are a spurious effect of the way we decided to separate the random variable $A_{(\omega_p)}$ into a deterministic part $\alpha(\omega_p)N$ and a probabilistic part $x_pN^{p+\frac{1}{2}}$ (see Equation (13)). In this case, both divergences could be regularized by simply shifting the random variable $x_pN^{p+\frac{1}{2}} \to (x_p - t(p))\,N^{p+\frac{1}{2}}$, and reabsorbing the shift into the deterministic part $\alpha(\omega_p)N \to \alpha(\omega_p)N + t(p)N^{p+\frac{1}{2}}$. The shift function $t(p)$ must satisfy

$$t(p) = \frac{t^*}{p-\frac{1}{2}}\left(1+\mathcal{O}\left(p-\tfrac{1}{2}\right)\right) . \tag{72}$$

in order to regularize at sight both the deterministic component of $A_{(\omega_p)}$ and the average value of its probabilistic part.

If this intuition is right, it remains to be proven that this shift regularizes all the higher-order moments as well. However, in this paper we have chosen to use only methods from singularity analysis, which are badly adapted to non-linear shifts (in N), so that all the claims in the remaining part of this section shall be proven in a second companion paper in which we perform an asymptotic probabilistic analysis at fixed N that produces a tree-diagram perturbative expansion in the spirit of Section 3 (but with "coloured" trees, in order to keep into account the shift terms).

An excerpt of the resulting theory is the following fact:

Definition 2. *Define the operator $\widehat{H}^\circ_U$, acting on functions $f(\{y_i\}_{i\in U})$, as a multi-dimensional finite-difference operator:*

$$\widehat{H}^\circ_U[f(y)] := \sum_{\{y_j\}\in\{0,1\}^U} (-1)^{\sum_j y_j} f(y_1,\dots,y_{|U|}) , \tag{73}$$

and let $\widehat{H}_U[f(y)] = \lim_{\delta\to 0}(-\delta)^{-|U|}\widehat{H}^\circ_U[f(\delta y)]$ be the corresponding multi-dimensional L'Hôpital evaluation:

$$\widehat{H}_U[f] := \frac{\mathrm{d}^{|U|}}{\prod_{i\in U}\mathrm{d}y_i} f(y_1,\dots,y_{|U|})\Big\vert_{y_1=\dots=y_{|U|}=0} . \tag{74}$$

Proposition 7. *For T a rooted tree, calling r the root vertex, and $U=U(T)$ the set of leaf vertices. For a vertex $v\in V(T)$ call U_v the set of leaves which are strictly below v.[5] Introduce one variable y_i per leaf vertex i, and call $y_v = \sum_{i\in U_v} y_i$ (in particular, $y_\mathrm{r} = \sum_{i\in U(T)} y_i$).*

Under our "canonical" choice (19) of shift function $t(p)$, the shifted moments $\widetilde{M}_s$ are given by

$$\widetilde{M}^{(\mathrm{E})}_s(\tfrac{1}{2}+\delta) = 8\sqrt{\pi}\,s!\sum_{T\in\mathcal{T}(s)}\widehat{H}^\circ_{U(T)}\left[\frac{\left(\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}+\delta)\right)^{y_\mathrm{r}}}{\Gamma\big(s+\delta(s+1-y_\mathrm{r})\big)}\prod_{v\in V(T)} a(\ell_v)b\big(k_v - \tfrac{\delta}{1+\delta}y_v\big)\right] \tag{75}$$

[5] In tree order, u is strictly below v if $u\neq v$ and the path from r to u goes through v.

$s=2$	$2^3\ell - 3\zeta_2$	$0.610375\dots$
$s=3$	$2^4\ell(\ell-1) - 8\zeta_2 + 14\zeta_3$	$0.266217\dots$
$s=4$	$\frac{2^6}{15}\ell(8\ell^2+39\ell+18) - \frac{2^4}{5}\zeta_2(61\ell-8) + \frac{2^8}{5}\zeta_3 - 9\zeta_2^2$	$1.28827\dots$
$s=5$	$\frac{2^8}{315}\ell(90\ell^3+1663\ell^2-1461\ell-657) - \frac{2^5}{105}\zeta_2(2295\ell^2+157\ell+544)$ $+\frac{2^5}{105}\zeta_3(5115\ell-728) + \frac{4944}{35}\zeta_2^2 - 420\zeta_2\zeta_3 + 744\zeta_5$	$1.73555\dots$

TABLE 3. Regularized (and rescaled) moments $(2\sqrt{\pi})^s\langle(x_p - t(p))^s\rangle = (2\sqrt{\pi})^s\widetilde{M}_s = 2(2\sqrt{\pi})^{s+1}\frac{\Gamma(s+1)}{\Gamma(s-\frac{1}{2})}\tilde{\mu}_s$ for $t(p)$ as in (19), at $p=\frac{1}{2}$. Here ℓ is a shortcut for $\ln 2$, and ζ_n denotes the Riemann's zeta function $\zeta(n)$. Note that, under the formal replacement $\zeta_n \to \ell^n$, the expressions above are rational polynomials in ℓ, of degree s.

where $a(\ell)$ and $b(k)$ are as in (58) and (59), evalued at $p=\frac{1}{2}+\delta$. In particular, for $\delta=0$,

$$\begin{aligned}\widetilde{M}_s^{(\mathrm{E})}(\tfrac{1}{2}) &= 8\sqrt{\pi}\,s!\sum_{T\in\mathcal{T}(s)}(-1)^{|U(T)|}\widehat{H}_{U(T)}\left[\frac{e^{(2\ln 2+\gamma_E)y_{\mathrm{r}}}}{\Gamma(s-y_{\mathrm{r}})}\prod_{v\in V(T)}a(\ell_v)b^{(\mathrm{reg})}(k_v-y_v)\right]\\ &= 8\sqrt{\pi}\,s!\sum_{T\in\mathcal{T}(s)}(-C)^{|U(T)|}\widehat{H}_{U(T)}\left[\frac{e^{(2\ln 2+\gamma_E)y_{\mathrm{r}}}}{\Gamma(s-y_{\mathrm{r}})}\prod_{v\in V(T)\smallsetminus U(T)}a(\ell_v)b(k_v-y_v)\right]\end{aligned} \tag{76}$$

where $b^{(\mathrm{reg})}=b$ for a non-leaf node, and $C=a(0)b^{(\mathrm{reg})}(0)=\frac{1}{8\sqrt{\pi}}=\lim_{\delta\to0}\delta\,a(0)b(0)$ for a leaf node.

Note that, indeed, $\frac{\mathrm{d}}{\mathrm{d}p}\ln(\Gamma(\frac{1}{2})/\Gamma(p))\Big|_{p=\frac{1}{2}} = 2\ln 2+\gamma_E$, and that $b(0)$ is the only weight in the expression (75) that is singular for $\delta\to0$ (while all $a(\ell)$ and derivatives $\partial^h b^{(\mathrm{reg})}(k)$ for $h\le k$ are regular in the limit).

See Table 2 for the first few terms of these series. As a check of the proposition above, we computed exactly the moments $\langle(x_p-t(p))^s\rangle$ using Equation (11) and the software *Mathematica 12*, with the choice (19) of $t(p)$, up to order $s=5$, finding that they are all regular at $p=\frac{1}{2}$, and consistent with Equation (76) (see Table 3).

5. The limit of $p\to0$, and logarithmic cost functions

It is interesting to study the limiting behaviour of $A_{(\omega_p)}$ when p goes to zero along a family of functions $\{\omega_p\}_{p\in\mathbb{R}^+}$. As discussed in the Introduction, as the limit itself is trivially $A_{(\omega_p)}(w)=N$, we shall instead focus on the behaviour of the rescaled shifted quantity

$$B_{(\omega_p)}(w) = \frac{A_{(\omega_p)}(w)-N}{p} \tag{77}$$

as $p\to0^+$. Notice that this is equivalent to the study of the original observable $A_{(\omega_{0^+})}$, where by ω_{0^+} we denote a positive regular function such that $\omega_{0^+}(k)\sim\log(k)$ for large k. The strategy of Section 2 remains valid, but, as our treatment was assuming that the function ω has a power-law asymptotics, in this section we have to repeat our analysis. For simplicity of notation, in the remainder of this section we drop the $+$ superscript of ω_{0^+}.

In the case of Dyck excursions, we will prove that, in the limit of large N,

$$A_{(\omega_0)} \stackrel{d}{=} \alpha(\omega_0)N + \sqrt{\gamma_E N\log N}\,z + \mathcal{O}(\sqrt{N}) \tag{78}$$

where z is a standard Gaussian random variable, γ_E is the Euler's gamma constant and $\alpha(\omega_0)$ is a non-universal constant that depends on the details of $\omega_0(k)$ away from the large k regime (see Equation (37)). To this end, we will compute the integer moments of the observable $A_{(\omega_0,\epsilon)}$, where ϵ will be tuned to directly access the fluctuations around the mean, and we will find that they are equal to the moments of a Gaussian random variable. The case of Dyck bridges can be treated analogously.

We follow very closely the line of thought of Section 2, taking for given the definitions and the results presented therein. In particular, Proposition 1 holds in the logarithmic case as well. Thus, we will start our treatment of the problem by generalizing Proposition 2 to the case of logarithmic functions ω_0. In the following, we define

$$L(z) = \log\left(\frac{1}{1-z}\right). \tag{79}$$

Proposition 8. *Let $k\in\mathbb{Z}^+$ and $f(z)=\sum_{N\ge0}f_N z^N$ be a generating function with unit radius of convergence. Suppose that f admits the singular expansion*

$$f(z) = f^{\mathrm{reg}}(z) + a\,(1-z)^{-\alpha}\left[\log\left(\frac{1}{1-z}\right)\right]^{\beta}\left(1+\mathcal{O}\left([L(z)]^{-1}\right)\right) \qquad \textit{for } z\to1\,. \tag{80}$$

with $f^{\text{reg}}(z)$ analytic in a neighbourhood of $z = 1$ and $\beta > 0$ (in particular, $f(z)$ has a unique dominant singularity, at $z = 1$).

Suppose further that

$$\omega_0(N) \sim (\log N)(1 + \mathcal{O}(N^{-\eta})) \quad \textit{for } N \to \infty \tag{81}$$

for some $\eta > 0$.

If $\alpha \neq 0, -1, -2, \dots$ then

$$\hat{L}^k_{(\omega_0,\epsilon)}[f](z) = \tilde{f}^{\text{reg}}(z; \omega_0, \epsilon, f, k) + a(1-z)^{-\alpha} [L(z)]^{\beta+k} \left(1 + \mathcal{O}\left([L(z)]^{-1}\right)\right) \quad \textit{for } z \to 1 \tag{82}$$

with a new, a priori *unknown regular part $\tilde{f}^{\text{reg}}(z)$. If, instead, $\alpha = 0$, then*

$$\hat{L}^k_{(\omega_0,\epsilon)}[f](z) = \tilde{f}^{\text{reg}}(z; \omega_0, \epsilon, f, k) + a\,[L(z)]^{\beta+k+1} \left(1 + \mathcal{O}\left([L(z)]^{-1}\right)\right) \quad \textit{for } z \to 1\,. \tag{83}$$

Proof. See the proof of Proposition 2. Here one must pay attention to logarithmic factors; see [38, Sec. VI.2]. Notice that the final error term here is independent on the error terms on $f(z)$ and on $\omega_0(k)$ as long as they are algebraic. □

Now that we have specified the leading behaviour of $\hat{L}_{(\omega_0,\epsilon)}$, we can again study the singular behaviour of $E_s(z)$ inductively.

Proposition 9. *Let $\epsilon = \epsilon(\omega_0, E_0) = \alpha(\omega_0)$. Suppose that*

$$\omega_0(N) \sim (\log N)(1 + \mathcal{O}(N^{-\eta})) \quad \textit{for } N \to \infty \tag{84}$$

for some $\eta > 0$. Then,

$$E_s(z) = \begin{cases} -\frac{1}{2} L(z) \left(1 + \mathcal{O}\left([L(z)]^{-1}\right)\right) & s = 1 \\ \tau_s (1-z)^{\frac{1-s}{2}} [L(z)]^{\frac{s}{2}} + \mathcal{O}\left((1-z)^{\frac{1-s}{2}} [L(z)]^{\frac{s-2}{2}}\right) & s \geq 2 \end{cases} \tag{85}$$

for $z \to 1$. The coefficients τ_s satisfy $\tau_2 = \frac{\gamma_E}{4}$, and

$$\tau_s = \begin{cases} C_{\ell-1} 2^{1-\ell} \tau_2^\ell & s = 2\ell, \quad \ell \geq 2 \\ 0 & s = 2\ell+1, \quad \ell \geq 1 \end{cases}. \tag{86}$$

Implicitly, $\tau_1 = 0$, because $E_1(z) \sim L(z)$ instead of $L(z)^{\frac{1}{2}}$.

Proof. We prove the result by induction. The expression for $E_1(z)$ can be easily obtained by using Equation (32) and Proposition 8. Here it is crucial to set $\epsilon = \alpha(\omega_0)$ to discard a contribution to $E_1(z)$ of order $(1-z)^{\frac{1}{2}}$ that would otherwise dominate the entire induction process.

For $s = 2$, to obtain the leading singular order of $E_2(z)$, one must expand the term $\hat{L}_{(\omega_0,\epsilon)}[E_1](z)$ to its next-to-leading order. This verifies that the scaling given in Equation (85) and the value of τ_2 are correct.

Finally, for $s \geq 3$, it is easy to verify, using Equation (32), that the leading singular order of $E_s(z)$ has the correct scaling, and that

$$\tau_s = \frac{1}{2} \sum_{k=2}^{s-2} \tau_k \tau_{s-k}\,. \tag{87}$$

This recursion is consistent with the ansatz that $\tau_k = 0$ for all odd k, which is indeed implied by the fact that $\tau_1 = 0$. Moreover, by introducing the generating function

$$T(z) = \sum_{s=1}^{\infty} \tau_{2s}\, z^s \tag{88}$$

it is easy to recognise in (87) the classical quadratic relation for Catalan numbers, and in turns see that

$$T(z) = 1 - \sqrt{1 - 2\tau_2 z}\,, \tag{89}$$

whose expansion gives the explicit expression for τ_{2s} given in Equation (86). □

Thus, by using Corollary 1, we find that the even integer moments $M^{(\mathrm{E})}_{2\ell}(N; \omega_0, \alpha(\omega_0))$ satisfy

$$M^{(\mathrm{E})}_{2\ell}(N; \omega_0, \alpha(\omega_0)) \sim (2\ell-1)!!(\gamma_E N \log N)^\ell \tag{90}$$

where $s!!$ denotes the double factorial, and that the odd integer moments are null at order $(N \log N)^{\frac{s}{2}}$. Thus, the rescaled moments $\overline{M}^{(\mathrm{E})}_s$ satisfy

$$\overline{M}^{(\mathrm{E})}_{2\ell}(\omega_0, \alpha(\omega_0)) = \lim_{N\to\infty} \frac{M^{(\mathrm{E})}_{2\ell}(N; \omega_0, \alpha(\omega_0))}{(N \log N)^\ell} = \gamma_E^\ell (2\ell-1)!! \tag{91}$$

which are precisely the integer moments of a Gaussian random variable with variance γ_E.

We finally prove Equation (77) by observing that:

- the $\epsilon = \alpha(\omega_0)$ shift accounts for the deterministic term of Equation (77);
- Proposition 9, along with Carleman's condition, accounts for the stochastic term of Equation (77);
- the dominant error term is not induced by the error on the higher moments, but by the leading behaviour of the average value, that scales as $\sqrt{N}$.

The computations of this section can be easily generalized to Dyck bridges, and to cost functions with asymptotic behaviour $(\log k)^p$ with $p > 0$, but we do not detail this here.

Appendix A. Some facts about the area-Airy distribution

We recall the main known facts about the area-Airy distribution, following the review [5]. The area-Airy distribution $f(x) = f_{\mathrm{Ai}}(x)$ has support on $\mathbb{R}^+$, and Laplace transform [2, 42]

$$\tilde{f}_{\mathrm{Ai}}(\lambda) = \int_0^\infty f_{\mathrm{Ai}}(x) e^{-\lambda x} dx = \lambda\sqrt{2\pi} \sum_{k=1}^{\infty} e^{-a_k \lambda^{2/3} 2^{-1/3}} \tag{92}$$

where a_k is the value of the k-th zero of the standard Airy function $\mathrm{Ai}(x)$. A closed expression for the density was found in [3]

$$f_{\mathrm{Ai}}(x) = \frac{2\sqrt{6}}{x^{10/3}} \sum_{k=1}^{\infty} e^{-b_k/x^2} b_k^{2/3} U\left(-\frac{5}{6}, \frac{4}{3}, \frac{b_k}{x^2}\right) \tag{93}$$

where $b_k = 2a_k^3/27$ and $U(a, b, z)$ is the confluent hypergeometric function. Moreover, a recursion for the moments of $f_{\mathrm{Ai}}(x)$ is known [3]: define K_s as

$$M_s^{\mathrm{Ai}} = \int_0^\infty f_{\mathrm{Ai}}(x) x^s dx =: \sqrt{\pi}\, 2^{(4-s)/2} \frac{\Gamma(s+1)}{\Gamma\left(\frac{3s-1}{2}\right)} K_s\,. \tag{94}$$

Then

$$\begin{aligned} K_s &= \frac{3s-4}{4} K_{s-1} + \sum_{j=1}^{s-1} K_j K_{s-j} \quad \forall s \geq 1 \\ K_0 &= -\frac{1}{2}\,. \end{aligned} \tag{95}$$

Finally, $f_{\mathrm{Ai}}(x)$ has asymptotic behaviours [3, 12]

$$\begin{aligned} f_{\mathrm{Ai}}(x) &\sim x^{-5} e^{-2a_1^3/27x^2} && \text{as } x \to 0 \\ f_{\mathrm{Ai}}(x) &\sim e^{-6x^2} && \text{as } x \to \infty\,. \end{aligned} \tag{96}$$

Appendix B. Some facts about singularity analysis

The main result that we will need is the following theorem (here stated informally, see [38] for a precise statement):

Theorem 1. *Let $f(z) = \sum_{N\geq 0} f_N z^N$ and $g(z) = \sum_{N\geq 0} g_N z^N$ be two generating functions with radius of convergence r. Then*

$$f(z) \sim g(z) \quad \text{for } z \to r \quad \Longleftrightarrow \quad f_N \sim g_N \quad \text{for } N \to \infty\,. \tag{97}$$

In particular, we will need two special cases:

Corollary 1. *Let $f(z) = \sum_{N\geq 0} f_N z^N$ be a generating function with unit radius of convergence. If f admits the singular expansion*

$$f(z) = f^{\mathrm{reg}}(z) + a\,(1-z)^{-\alpha} \left(\log\frac{1}{1-z}\right)^{\beta} (1 + \mathcal{O}(1-z)) \quad \text{for } z \to 1 \tag{98}$$

with $\alpha \neq 0, -1, -2, \ldots$ and $f^{\mathrm{reg}}(z)$ analytic in a neighbourhood of $z = 1$, then

$$f_N = \frac{a}{\Gamma(\alpha)} N^{\alpha-1} (\log N)^{\beta} (1 + \mathcal{O}(N^{-1})) \sim \frac{a}{\Gamma(\alpha)} N^{\alpha-1} (\log N)^{\beta} \quad \text{for } N \to \infty\,, \tag{99}$$

and the viceversa is also true. If $\alpha = 0$, $\beta = 1$ the same statement holds with

$$f_N \sim \frac{1}{N}\,. \tag{100}$$

Proof. Follows from the expansions given in [38] for products of polynomial and logarithmic singularities. □

Similar statements hold if the error term has a different form. Namely, if in (98) we replace $\mathcal{O}(1-z)$ by $\mathcal{O}\left((1-z)^{-a}\left(\log\frac{1}{1-z}\right)^b\right)$, in (99) we get an error term of the form $1+\mathcal{O}(N^{-a}(\ln N)^b)$ if $\alpha+a\neq 0,-1,-2,\dots$ Cases in in which $\alpha+a=0,-1,-2,\dots$ must be treated separately; see [38].

Appendix C. Examples of cost functions

In this Appendix, we provide some examples of families of cost functions ω_p for which we can compute analytically the quantity $\alpha(\omega_p)$ defined in (37), that is the regular part at $z=1$ of

$$\hat{L}_{(\omega_p,0)}\left[E_0\right](z)=\sum_{N\geq 0} c_N\omega_p(N)z^N\,. \tag{101}$$

As a result, in these cases it is possible to compute exactly the shift $\epsilon(\omega_p,E_0)$, that just coincides with $\alpha(\omega_p)$, and have a complete control over the asymptotic behaviour of the random variable $A_{(\omega_p)}$ given in (13). Recall that $c_N=2^{-2N-1}C_N$ are the normalized Catalan numbers, and that the regular part at $z=1$ of $E_0(z)$ equals 1.

As a first example, we set $\omega_p^{(\frac12)}(k)=\frac{\Gamma\left(k+p+\frac12\right)}{\Gamma\left(k+\frac12\right)}$. In this case

$$\hat{L}_{(\omega_p^{(\frac12)},0)}\left[E_0\right](z)=\frac{\Gamma\left(p-\frac12\right)}{2\sqrt{\pi}\,z}\left((1-z)^{\frac12-p}-1\right) \tag{102}$$

whose regular part at $z=1$ equals $-\frac{\Gamma\left(p-\frac12\right)}{2\sqrt{\pi}\,z}$, giving

$$\alpha(\omega_p^{(\frac12)})=\epsilon(\omega_p^{(\frac12)},E_0)=-\frac{\Gamma\left(p-\frac12\right)}{2\sqrt{\pi}}=\frac{\Gamma\left(p-\frac12\right)}{\Gamma\left(-\frac12\right)}\,. \tag{103}$$

As a second example, we set $\omega_p^{(1)}(k)=\frac{\Gamma(k+p+1)}{\Gamma(k+1)}$. In this case

$$\hat{L}_{(\omega_p^{(1)},0)}\left[E_0\right](z)=\frac{\Gamma\left(p+1\right)}{2}\,{}_2F_1\left(\left.\begin{matrix}\frac12,1+p\\2\end{matrix}\right|z\right) \tag{104}$$

where ${}_2F_1$ is the hypergeometric function, whose regular part at $z=1$ can be extracted by using the inversion formula [43, Equation 15.3.6, pg. 559], giving

$$\alpha(\omega_p^{(1)})=-\frac{\Gamma\left(p\right)\Gamma\left(\frac12-p\right)}{\Gamma\left(\frac12\right)\Gamma\left(-p\right)}\,. \tag{105}$$

More generally, for $\omega_p^{(a)}(k)=\frac{\Gamma(k+p+a)}{\Gamma(k+a)}$, defined for $a+p$ not a negative integer, and positive for all integer k when $a>0$, we get

$$\hat{L}_{(\omega_p^{(a)},0)}\left[E_0\right](z)=\frac{\Gamma\left(a+p-1\right)}{z\Gamma\left(a-1\right)}\left(1-{}_2F_1\left(\left.\begin{matrix}-\frac12,a+p-1\\a-1\end{matrix}\right|z\right)\right)\,. \tag{106}$$

Again, the regular part at $z=1$ of the hypergeometric function can be extracted by using the inversion formula, giving

$$\alpha(\omega_p^{(a)})=\Gamma\left(a+p-1\right)\left(\frac{1}{\Gamma\left(a-1\right)}-\frac{\Gamma\left(\frac12-p\right)}{\Gamma\left(a-\frac12\right)\Gamma\left(-p\right)}\right) \tag{107}$$

Note that, within this family, only the case $a=\frac12$ gives an expression for $\alpha(\omega_p^{(a)})$ which is finite for all $p\in\mathbb{R}^+\setminus\{\frac12\}$.

Another simple family is $\omega_p^{(a,-\frac32)}(k)=\frac{\Gamma\left(k+p+a-\frac32\right)\Gamma(k+2)}{\Gamma(k+a)\Gamma\left(k+\frac12\right)}$, defined for $a+p-\frac32$ not a negative integer, and generalising $\omega^{(\frac12)}$ (as $\omega_p^{(2,-\frac32)}(k)=\omega_p^{(\frac12)}(k)$), that gives

$$\hat{L}_{(\omega_p^{(a,-\frac32)},0)}\left[E_0\right](z)=\frac{\Gamma\left(a+p-\frac32\right)}{2\sqrt{\pi}\,\Gamma\left(a\right)}\,{}_2F_1\left(\left.\begin{matrix}1,a+p-\frac32\\a\end{matrix}\right|z\right)\,, \tag{108}$$

and in turns

$$\alpha(\omega_p^{(a,-\frac32)})=\frac{\Gamma\left(a+p-\frac32\right)}{\Gamma\left(\frac12\right)\Gamma\left(a-1\right)(1-2p)}\,. \tag{109}$$

This quantity is finite for all $p\in\mathbb{R}^+\setminus\{\frac12\}$ whenever $a>\frac32$.

Finally, we observe that, in all the cases analysed here in detail, we have that, in the limit $p\to\frac12$,

$$\alpha(\omega_p)=\frac{1}{p-\frac12}\left(-\frac{1}{2\sqrt{\pi}}+o(1)\right)\,. \tag{110}$$

This is not a coincidence. Indeed, the operator $\hat{L}_{(\omega,0)}$ is linear: $\hat{L}_{(\omega'+\omega'',0)}f(z) = \hat{L}_{(\omega',0)}f(z) + \hat{L}_{(\omega'',0)}f(z)$, so that, if ω' is any of the families above, and ω'' is a family of functions of the form $\omega''_p(k) = k^p(1 + b(k,p)k^{-\eta})$, with $b(k,p)$ uniformly bounded and $\eta > 0$, then $(\omega'_p - \omega''_p)(k) = k^{p-\eta}b'(k,p)$, with $b'(k,p)$ uniformly bounded, and $\lim_{z\to 1} |\hat{L}_{(\omega'_{\frac{1}{2}}-\omega''_{\frac{1}{2}},0)}E(z)| < +\infty$. As the singular parts of $\hat{L}_{(\omega'_{\frac{1}{2}},0)}E(z)$ and of $\hat{L}_{(\omega''_{\frac{1}{2}},0)}E(z)$ are the same (because they are determined only by p), and are diverging, it must be the case that also the regular (in z) parts diverge (in p) with the same coefficient.

Appendix D. Proof of Proposition 1

By using the fact that a Dyck excursion w can always be decomposed as $w = (+, w_2, -, w_1)$, with w_1 and w_2 being Dyck excursions of length m and $N - m - 1$ (for some $0 \le m \le N - 1$) and $+/-$ an up/down step, one finds that

$$
\begin{aligned}
&\frac{1}{s!}\sum_{w\in\mathcal{C}_N}\left[A_{(\omega_p,\epsilon)}(w)\right]^s \\
&= \frac{1}{s!}\sum_{m=0}^{N-1}\sum_{w_2\in\mathcal{C}_m}\sum_{w_1\in\mathcal{C}_{N-m-1}}\left[A_{(\omega_p,\epsilon)}(w_1) + A_{(\omega_p,\epsilon)}(w_2) + (\omega_p(m)-\epsilon)\right]^s \\
&= \frac{1}{s!}\sum_{m=0}^{N-1}\sum_{w_2\in\mathcal{C}_m}\sum_{w_1\in\mathcal{C}_{N-m-1}}\sum_{\substack{s_1,s_2,s_3\ge 0\\ s_1+s_2+s_3=s}}\left[A_{(\omega_p,\epsilon)}(w_1)\right]^{s_1}\left[A_{(\omega_p,\epsilon)}(w_2)\right]^{s_2}\left[\omega_p(m)-\epsilon\right]^{s_3}\frac{s!}{s_1!s_2!s_3!} \\
&= \sum_{m=0}^{N-1}\sum_{\substack{s_1,s_2,s_3\ge 0\\ s_1+s_2+s_3=s}}\left[\frac{1}{s_1!}\sum_{w\in\mathcal{C}_{N-m-1}}\left[A_{(\omega_p,\epsilon)}(w)\right]^{s_1}\right]\frac{\left[\omega_p(m)-\epsilon\right]^{s_3}}{s_3!}\left[\frac{1}{s_2!}\sum_{w\in\mathcal{C}_m}\left[A_{(\omega_p,\epsilon)}(w)\right]^{s_2}\right]
\end{aligned}
\tag{111}
$$

implying that

$$
2E_s(z) = \delta_{s,0} + z\sum_{\substack{s_1,s_2,s_3\ge 0\\ s_1+s_2+s_3=s}} E_{s_1}(z)\frac{1}{s_3!}\hat{L}^{s_3}_{(\omega_p,\epsilon)}[E_{s_2}](z)\,, \qquad s\ge 0\,. \tag{112}
$$

The case $s = 0$ gives rise to an equation involving only $E_0(z)$, and no summations, which is nothing but the generating function of normalized Catalan numbers $c_k := 2^{-2k-1}C_k$. So we easily get

$$
E_0(z) = \frac{1-\sqrt{1-z}}{z}\,, \tag{113}
$$

For the general case $s \ge 1$, notice that the term $E_s(z)$ appears in both sides of the s-th Equation (112), and only linearly. If we isolate $E_s(z)$, we obtain after some simplifications

$$
E_s(z) = \frac{z}{2\sqrt{1-z}}\sum_{\substack{s_1,s_2,s_3\ge 0\\ s_1+s_2+s_3=s\\ s_1,s_2<s}} E_{s_1}(z)\frac{1}{s_3!}\hat{L}^{s_3}_{(\omega_p,\epsilon)}[E_{s_2}](z)\,. \tag{114}
$$

The proof for Dyck bridges is in the same spirit of the one for Dyck paths. Now, a Dyck bridge decomposes uniquely as $\pm w = (+, w_2, -, w_1)$, where w_2 is a Dyck excursion and w_1 is a Dyck bridge, hence the recursion involving E_{s_2}. Note that, as the first step of a Dyck bridge can be either a $+$ or a $-$, a factor of two must be taken into account when dealing with this decomposition.

Appendix E. Proof of Proposition 4

We give the proof for $E_s(z)$, the one for $B_s(z)$ being completely analogous, and we proceed by induction.

The case $s = 0$ is trivially verified from the explicit form of $E_0(z)$.

The case $s = 1$ can be computed explicitly by using Equation (32):

$$
E_1(z) = \frac{z}{2\sqrt{1-z}}E_0\hat{L}_{(\omega_p,\epsilon)}[E_0](z) = \frac{1-\sqrt{1-z}}{2\sqrt{1-z}}\hat{L}_{(\omega_p,\epsilon)}[E_0](z)\,. \tag{115}
$$

Using Proposition 2 with $k = 1$, $f(z) = E_0(z)$ and thus $f^{\text{reg}}(z) = \frac{1}{z}$, $\alpha = -\frac{1}{2}$ and $\xi = 1$ we obtain

$$
\begin{aligned}
\hat{L}_{(\omega_p,\epsilon)}[E_0](z) &= \tilde{E}_0^{\text{reg}}(z;\omega_p,\epsilon,E_0,1) - \frac{\Gamma\left(p-\frac{1}{2}\right)}{-2\sqrt{\pi}}(1-z)^{\frac{1}{2}-p}\left(1+\tilde{\mathcal{O}}((1-z)^{\min(\eta,1)})\right) \\
&= \frac{\Gamma\left(p-\frac{1}{2}\right)}{2\sqrt{\pi}}(1-z)^{\frac{1}{2}-p}\left(1+\tilde{\mathcal{O}}\left((1-z)^{\min(\eta,1,\frac{1}{2}+p)}\right)\right)
\end{aligned}
\tag{116}
$$

where the second passage is due to the choice $\epsilon = \epsilon(\omega_p, E_0)$ as in Equation (39) to eliminate the regular part at $z = 1$. Thus,

$$
\begin{aligned}
E_1(z) &= \frac{1-\sqrt{1-z}}{2\sqrt{1-z}} \frac{\Gamma\left(p-\frac{1}{2}\right)}{2\sqrt{\pi}} (1-z)^{\frac{1}{2}-p} \left(1+\tilde{\mathcal{O}}\left((1-z)^{\min(\eta,1,\frac{1}{2}+p)}\right)\right) \\
&= \frac{\Gamma\left(p-\frac{1}{2}\right)}{4\sqrt{\pi}} (1-z)^{-p} \left(1+\tilde{\mathcal{O}}\left((1-z)^{\min(\eta,\frac{1}{2})}\right)\right)
\end{aligned}
\tag{117}
$$

(where the exponents 1 and $\frac{1}{2}+p$ in the error term are always subleading w.r.t. the exponent $\frac{1}{2}$ coming from the algebraic prefactor).

Notice that if we had left ϵ free, the leading signularity of $E_1(z)$ would have had exponent $-\frac{1}{2}$ for $p < \frac{1}{2}$. Thus, the tuning of ϵ is crucial to allow for a unified treatment for all $p \neq \frac{1}{2}$.

For $s \geq 2$, we suppose that Equation (43) is correct for $E_m(z)$, $0 \leq m \leq s-1$, with an error term of the form $1 + \tilde{\mathcal{O}}((1-z)^{\eta_m})$, and we compute the singular expansion around $z = 1$ of Equation (32). First of all, Proposition 2 tells us that, for $s_3 \geq 1$,

$$
\hat{L}^{s_3}_{(\omega_p,\epsilon)}\left[E_{s_2}(z)\right] = \tilde{E}^{\text{reg}}_{s_2}(z) + \frac{2\mu^{(\text{E})}_{s_2}(p)\Gamma\left(\left(p+\frac{1}{2}\right)s_2 - \frac{1}{2} + ps_3\right)}{\Gamma\left(\left(p+\frac{1}{2}\right)s_2-\frac{1}{2}\right)} (1-z)^{-(p+\frac{1}{2})s_2+\frac{1}{2}-ps_3}(1+\tilde{\mathcal{O}}((1-z)^{\min(\eta,\eta_{s_2})}))
\tag{118}
$$

while for $s_3 = 0$ the specialisation of the RHS to this value holds, with the simpler error term $1 + \mathcal{O}((1-z)^{\eta_{s_2}})$.

Notice that all the non-integrity conditions for the singular exponents of Proposition 2 are satisfied under our ansatz because $s_2 < s$ and $s_3 \geq 0$, and that $\tilde{E}^{\text{reg}}_{s_2}(z)$ has various dependences, that we drop for simplicity. Then, Equation (32) reduces to

$$
\begin{aligned}
&2\mu^{(\text{E})}_s(p)(1-z)^{1-(p+\frac{1}{2})s}\left(1+\tilde{\mathcal{O}}\left((1-z)^{\eta_s}\right)\right) = \\
&= \sum_{\substack{s_1,s_2,s_3\geq 0\\ s_1+s_2+s_3=s\\ s_1,s_2<s}} \frac{1}{s_3!}\Big[E^{\text{reg}}_{s_1}(z)\tilde{E}^{\text{reg}}_{s_2}(z) + \\
&+ \tilde{E}^{\text{reg}}_{s_2}(z)\mu^{(\text{E})}_{s_1}(p)(1-z)^{\frac{1}{2}-(p+\frac{1}{2})s_1}\left(1+\mathcal{O}\left((1-z)^{\eta_{s_1}}\right)\right) + \\
&+ E^{\text{reg}}_{s_1}(z)\frac{\mu^{(\text{E})}_{s_2}(p)\Gamma\left(\left(p+\frac{1}{2}\right)s_2-\frac{1}{2}+ps_3\right)}{\Gamma\left(\left(p+\frac{1}{2}\right)s_2-\frac{1}{2}\right)}(1-z)^{\frac{1}{2}-(p+\frac{1}{2})s_2-ps_3}\left(1+\tilde{\mathcal{O}}\left((1-z)^{\min(\eta_{s_2},\eta)}\right)\right) + \\
&+2\frac{\mu^{(\text{E})}_{s_1}(p)\mu^{(\text{E})}_{s_2}(p)\Gamma\left(\left(p+\frac{1}{2}\right)s_2-\frac{1}{2}+ps_3\right)}{\Gamma\left(\left(p+\frac{1}{2}\right)s_2-\frac{1}{2}\right)}(1-z)^{1-(p+\frac{1}{2})(s_1+s_2)-ps_3}\left(1+\tilde{\mathcal{O}}\left((1-z)^{\min(\eta_{s_1},\eta_{s_2},\eta)}\right)\right)\Big]
\end{aligned}
\tag{119}
$$

where $E^{\text{reg}}_{s_1}(z)$ is the regular part of $E_{s_1}(z)$, and the equality is at leading order in powers of $(1-z)$.

In the RHS of Equation (119), only the third and fourth term contribute to the leading order, and the former only for $(s_1, s_2, s_3) = (0, s-1, 1)$, while the latter only for $(s_1, s_2, s_3) = (k, s-k, 0)$, with $1 \leq k \leq s-1$; this immediately gives the recursion in Equation (45) for the $\mu^{(\text{E})}_s(p)$ coefficients.

It is also easy to verify that our claim on the form of the error-term exponents η_s holds inductively, by explicitly analysing the subleading contributions of the various terms in Equation (119).

Appendix F. Proof of Proposition 5

First of all, we study the case of Dyck excursions. We drop the superscript (E) and the dependence on p for simplicity. Equation (45) implies that

$$
|\mu_s| \leq \frac{\Gamma\left(s\left(p+\frac{1}{2}\right)-1\right)}{2\Gamma\left(s\left(p+\frac{1}{2}\right)-p-1\right)}|\mu_{s-1}| + \sum_{k=1}^{s-1}|\mu_k||\mu_{s-k}|\,.
\tag{120}
$$

We want to prove by induction that

$$
|\mu_s| \leq R\, A^s\, \Gamma\left(ps+1\right) C_{s-1} \qquad \forall s \geq 1
\tag{121}
$$

for some values of R and A (possibly depending on p). For $s = 1$, we obtain that R and A must satisfy the condition

$$
RA \geq f(p)
\tag{122}
$$

where

$$
f(p) = \frac{\left|\Gamma\left(p-\frac{1}{2}\right)\right|}{8\sqrt{\pi}\,\Gamma\left(p+1\right)}\,.
\tag{123}
$$

For the inductive step, we assume that the ansatz in Equation (121) is satisfied for all μ_k with $1 \leq k \leq s-1$ for some fixed values of R and A. This implies, using Equation (45), that

$$|\mu_s| \leq R\, A^s\, \Gamma(ps+1)\, C_{s-1} \left[\frac{\Gamma\left(s\left(p+\frac{1}{2}\right)-1\right)}{2\Gamma\left(s\left(p+\frac{1}{2}\right)-p-1\right)} \frac{1}{A} \frac{\Gamma(p(s-1)+1)}{\Gamma(ps+1)} \frac{C_{s-2}}{C_{s-1}} + \right. \\ \left. +R \sum_{k=1}^{s-1} \frac{C_{k-1}C_{s-k-1}}{C_{s-1}} \frac{\Gamma(pk+1)\,\Gamma(p(s-k)+1)}{\Gamma(ps+1)} \right] . \tag{124}$$

The first term in the parenthesis of Equation (124) must be treated separately for different values of s:

- for $s = 2$, it equals exactly $\frac{1}{4A}$;
- for s even and $s \geq 4$, it equals

$$\frac{1}{2A}\frac{s}{4s-6} \prod_{i=1}^{\lfloor s/2 \rfloor -2} \frac{sp + \lfloor s/2 \rfloor - i - 1}{(s-1)p + \lfloor s/2 \rfloor - i - 1} \leq \frac{1}{2A}\frac{s}{4s-6}\left(\frac{sp+1}{(s-1)p+1}\right)^{\frac{s-4}{2}} \tag{125}$$

  where the last term is bounded from above by its limit for $s, p \to \infty$,[6] that is $\frac{\sqrt{e}}{8A}$;
- for $s = 3$, it is monotone decreasing in p, and can be similarly bounded by its limit for $p \to 0$, i.e. $\frac{1}{4A}$;
- for s odd and $s \geq 5$, it can be bounded by

$$\frac{1}{2A}\frac{s}{4s-6}\frac{\Gamma\left(s\left(p+\frac{1}{2}\right)-\frac{1}{2}\right)}{\Gamma\left(s\left(p+\frac{1}{2}\right)-p-\frac{1}{2}\right)}\frac{\Gamma(p(s-1)+1)}{\Gamma(ps+1)} = \frac{1}{2A}\frac{s}{4s-6}\prod_{i=1}^{\lfloor s/2 \rfloor -2} \frac{sp + \lfloor s/2 \rfloor - i - 1}{(s-1)p + \lfloor s/2 \rfloor - i - 1} \tag{127}$$

  that, in turn, can be bounded by the same procedure used in the even case (which, incidentally, produces the same value for the bound).

The resulting bound is the maximum over the bounds of the various terms. As $\sqrt{e} < 2$, we have that the first term of Equation (124) can be bounded from above by $\frac{1}{4A}$, for all $s \geq 2$ and all $p > 0$.

The second term in the parentheses of Equation (124) can be simplified by using the fact that the Gamma function is logarithmically convex, giving in particular that

$$\log \Gamma(tx+1) \leq t \log \Gamma(x+1)\,, \quad \forall t \in [0,1]\,. \tag{128}$$

Thus,

$$\frac{\Gamma(pk+1)\,\Gamma(p(s-k)+1)}{\Gamma(ps+1)} \leq \frac{\Gamma(ps+1)^{\frac{k}{s}}\,\Gamma(ps+1)^{\frac{s-k}{s}}}{\Gamma(ps+1)} = 1\,. \tag{129}$$

At this point, the sum over k gives 1, thanks to the well-known recursion for Catalan numbers, i.e.

$$\sum_{i=0}^{n-1} C_i C_{n-i-1} = C_n\,. \tag{130}$$

As a result, we have obtained two condition that must be satisfied by R and A to confirm that the ansatz in Equation (121) is indeed true:

$$RA \geq f(p) \tag{131}$$

and

$$\frac{1}{4A} + R \leq 1\,. \tag{132}$$

For the case of Dyck bridges, Equation (46) implies

$$|\mu_s^{(\mathrm{B})}| \leq 2 \sum_{k=0}^{s-1} |\mu_k^{(\mathrm{B})}||\mu_{s-k}^{(\mathrm{E})}|\,. \tag{133}$$

[6] The supremum over p of the function at the right-hand side of Equation (125) is realised for $p \to \infty$, and equals

$$\frac{s}{4s-6}\left(\frac{s}{s-1}\right)^{\frac{s-4}{2}}\,. \tag{126}$$

This function changes monotonicity at the zeroes with odd multiplicity of the function

$$-3 + (2s-3)(s-1)\left(s \log\left(\frac{s}{s-1}\right) - 1\right)\,.$$

From the fact that $\log\left(\frac{s}{s-1}\right) \leq \frac{1}{s} + \frac{1}{2s^2}$ for $s \geq 2$, we get that there are no zeroes on the right of the right-most zero of the equation $6s = (2s-3)(s-1)$, which is slightly smaller than 6. Thus, the only candidate maxima are $s = 4$ and the limit $s \to +\infty$, and the latter is larger than the former.

We now have the ansatz

$$|\mu_s^{(\mathrm{B})}| \leq A^s \Gamma\left(sp+1\right) C_s\,, \quad \forall s \geq 0 \tag{134}$$

where A is the same as for Dyck excursions. Substituting the equation above, and (121), into (133), easily shows that the ansatz is verified if R satisfies

$$R \geq \frac{1}{2}\,. \tag{135}$$

One choice of functions R_p and A_p that satisfies all three conditions, Equations (131), (132) and (135) is given by

$$(A_p, R_p) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right) & \text{if} \quad 0 < f(p) \leq \frac{1}{4} \\ \left(\frac{1+4f(p)}{4}, \frac{4f(p)}{1+4f(p)}\right) & \text{if} \quad f(p) > \frac{1}{4} \end{cases}\,, \tag{136}$$

Notice that Equation (131) cannot be satisfied for $p = \frac{1}{2}$, as $f(p)$ (and thus also A_p) diverge in this limit.

References

[1] Cyril Banderier and Guy Louchard. Philippe Flajolet and the Airy Function. 2012.

[2] Donald A. Darling. On the supremum of a certain gaussian process. *The Annals of Probability*, 11(3):803–806, 1983.

[3] Lajos Takács. A Bernoulli excursion and its various applications. *Advances in Applied Probability*, 23(3):557–585, 1991.

[4] Mark Kac. On some connections between probability theory and differential and integral equations. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 189–215, Berkeley, Calif., 1951. University of California Press.

[5] Satya N. Majumdar and Alain Comtet. Airy Distribution Function: From the Area Under a Brownian Excursion to the Maximal Height of Fluctuating Interfaces. *Journal of Statistical Physics*, 119(3):777–826, 2005.

[6] Mollie A. Stapleton and Kim Christensen. One-Dimensional Directed Sandpile Models and the Area under a Brownian Curve. *Journal of Physics A: Mathematical and General*, 39(29):9107–9126, 2006.

[7] Shlomi Medalion, Erez Aghion, Hagai Meirovitch, Eli Barkai, and David A. Kessler. Size distribution of ring polymers. *Scientific Reports*, 6(1):1–8, 2016.

[8] Eli Barkai, Erez Aghion, and David A. Kessler. From the Area under the Bessel Excursion to Anomalous Diffusion of Cold Atoms. *Physical Review X*, 4(2):021036 (34pp), 2014.

[9] Tal Agranov, Pini Zilber, Naftali R. Smith, Tamir Admon, Yael Roichman, and Baruch Meerson. The Airy Distribution: Experiment, Large Deviations and Additional Statistics. *arXiv:1908.08354 [cond-mat]*, 2019.

[10] Lajos Takács. Random Walk Processes and their Applications in Order Statistics. *The Annals of Applied Probability*, 2(2):435–459, 1992.

[11] Lajos Takács. On the Distribution of the Integral of the Absolute Value of the Brownian Motion. *The Annals of Applied Probability*, 3(1):186–197, 1993.

[12] Lajos Takács. Limit distributions for the Bernoulli meander. *Journal of Applied Probability*, 32(2):375–395, 1995.

[13] Leonid Tolmatz. Asymptotics of the distribution of the integral of the absolute value of the Brownian bridge for large arguments. *The Annals of Probability*, 28(1):132–139, 2000.

[14] Leonid Tolmatz. On the Distribution of the Square Integral of the Brownian Bridge. *The Annals of Probability*, 30(1):253–269, 2002.

[15] Leonid Tolmatz. The saddle point method for the integral of the absolute value of the brownian motion. In Cyril Banderier and Christian Krattenthaler, editors, *Discrete Random Walks, DRW'03*, volume AC of *DMTCS Proceedings*, pages 309–324. Discrete Mathematics and Theoretical Computer Science, 2003.

[16] Michel Nguyen The. Area and Inertial Moment of Dyck Paths. *Combinatorics, Probability and Computing*, 13(4-5):697–716, 2004.

[17] Leonid Tolmatz. Asymptotics of the Distribution of the Integral of the Positive Part of the Brownian Bridge for Large Arguments. *Journal of Mathematical Analysis and Applications*, 304(2):668–682, 2005.

[18] Nils Haug, Adri Olde Daalhuis, and Thomas Prellberg. Higher-Order Airy Scaling in Deformed Dyck Paths. *Journal of Statistical Physics*, 166(5):1193–1208, March 2017.

[19] Nina Haug and Thomas Prellberg. Multicritical scaling in a lattice model of vesicles. *Journal of Physics A: Mathematical and Theoretical*, 2018.

[20] Roger Mansuy and Marc Yor. *Aspects of Brownian motion*. Springer Science & Business Media, 2008.

[21] Elena Boniolo, Sergio Caracciolo, and Andrea Sportiello. Correlation function for the grid-poisson euclidean matching on a line and on a circle. *Journal of Statistical Mechanics: Theory and Experiment*, 2014:P11023 (27pp), 2014.

[22] Sergio Caracciolo and Gabriele Sicuro. On the one dimensional euclidean matching problem: exact solutions, correlation functions and universality. *Physical Review E*, 90:042112 (8pp), 2014.

[23] Sergio Caracciolo, Andrea Di Gioacchino, Enrico M. Malatesta, and Luca G. Molinari. Selberg integrals in 1d random euclidean optimization problems. *Journal of Statistical Mechanics: Theory and Experiment*, 2019:063401 (10pp), 2019.

[24] Sergio Caracciolo, Matteo P. D'Achille, and Gabriele Sicuro. Anomalous scaling of the optimal cost in the one-dimensional random assignment problem. *Journal of Statistical Physics*, 174(4):846–864, 2019.

[25] Sergio Caracciolo, Matteo P. D'Achille, and Gabriele Sicuro. Random Euclidean matching problems in one dimension. *Physical Review E*, 96:042102 (20pp), 2017.

[26] Sergio Caracciolo, Carlo Lucibello, Giorgio Parisi, and Gabriele Sicuro. Scaling hypothesis for the euclidean bipartite matching problem. *Physical Review E*, 90:012118 (9pp), 2014.

[27] Sergio Caracciolo and Gabriele Sicuro. Scaling hypothesis for the euclidean bipartite matching problem. ii. correlation functions. *Physical Review E*, 91:062125 (8pp), 2015.

[28] Sergio Caracciolo and Gabriele Sicuro. Quadratic stochastic euclidean bipartite matching problem. *Physical Review Letters*, 115(23):230601, 2015.

[29] Luigi Ambrosio, Federico Stra, and Dario Trevisan. A PDE approach to a 2-dimensional matching problem. *Probab. Theory Relat. Fields*, 173:433–477, 2019.
[30] Luigi Ambrosio and Federico Glaudo. Finer estimates on the 2-dimensional matching problem. *J. Éc. Polytech. Math.*, 6:737–765, 2019.
[31] Luigi Ambrosio, Federico Glaudo, and Dario Trevisan. On the optimal map in the 2-dimensional random matching problem. *Discrete & Continuous Dynamical Systems - A*, 39:1078–0947, 2019.
[32] Robert J. McCann. Exact solutions to the transportation problem on the line. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 455(1984):1341–1380, 1999.
[33] Julie Delon, Julien Salomon, and Andrei N. Sobolevski. Local matching indicators for transport problems with concave costs. *SIAM J. Discret. Math.*, 26(2):801–827, 2012.
[34] Sergio Caracciolo, Matteo P. D'Achille, Vittorio Erba, and Andrea Sportiello. The Dyck bound in the concave 1-dimensional random assignment model. *Journal of Physics A: Mathematical and General*, 53(6):064001 (25pp), 2020.
[35] Sergey G. Bobkov and Michel Ledoux. Transport inequalities on euclidean spaces for non-euclidean metrics. *2019*.
[36] Mikhail V. Tamm and Serguei K. Nechaev. Necklace-cloverleaf transition in associating RNA-like diblock copolymers. *Phys. Rev. E*, 75:031904, 2007.
[37] Ivo L. Hofacker, Peter Schuster, and Peter F. Stadler. Combinatorics of RNA secondary structures. *Discrete Applied Mathematics*, 88(1-3):207–237, 1998.
[38] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge; New York, 2009.
[39] Donald E. Knuth. *The Art of Computer Programming*, volume 3: (2nd Ed.) Sorting and Searching. Addison Wesley Longman Publishing Co., Inc., USA, 1998.
[40] Valentin Féray and Ian P. Goulden. A multivariate hook formula for labelled trees. *Journal of Combinatorial Theory, Series A*, 120(4):944 – 959, 2013.
[41] Gwo Dong Lin. Recent developments on the moment problem. *Journal of Statistical Distributions and Applications*, 4(1):1–17, 2017.
[42] Guy Louchard. Kac's formula, Levy's local time and Brownian Excursion. *Journal of Applied Probability*, 21(3):479–499, 1984.
[43] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*. Applied mathematics series. Dover Publications, 1972.

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