Sharp Estimates of Radial Dunkl and Heat Kernels in the Complex Case A n

Piotr Graczyk, Patrice Sawyer

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Abstract

In this article, we consider the radial Dunkl geometric case $k = 1$ corresponding to flat Riemannian symmetric spaces in the complex case and we prove exact estimates for the positive valued Dunkl kernel and for the radial heat kernel.

Dans cet article, nous considérons le cas géométrique radial de Dunkl $k = 1$ correspondant aux espaces symétriques riemanniens plats dans le cas complexe et nous prouvons des estimations exactes pour le noyau de Dunkl à valeur positive et pour le noyau de chaleur radial.

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1 Introduction and notations

Finding good estimates of Dunkl heat kernels is a challenging and important subject, developed recently in [1]. Establishing estimates of the heat kernels is equivalent to estimating the Dunkl kernel as demonstrated by equation (2.3) below.

In this paper we prove exact estimates in the W-radial Dunkl geometric case of multiplicity $k = 1$, corresponding to Cartan motion groups and flat Riemannian symmetric spaces with the ambient group complex G, the Weyl group W and the root system A_n.

We study for the first time the non-centered heat kernel, denoted $p_t^W(X, Y)$, on Riemannian symmetric spaces and we provide its sharp estimates. Exact estimates were obtained in [2] in the centered case $Y = 0$ for all Riemannian symmetric spaces.

We provide exact estimates for the spherical functions $\psi_\lambda(X)$ in the two variables X, λ when λ is real and, consequently, for the heat kernel $p_t^W(X, Y)$ in the three variables t, X, Y.

We recall here some basic terminology and facts about symmetric spaces associated to Cartan motion groups.

Let G be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of G. We recall the definition of the Cartan motion group and the flat symmetric space associated with the semisimple Lie group G with maximal compact subgroup K. The Cartan motion group is the semi-direct product $G_0 = K \rtimes \mathfrak{p}$ where the multiplication is defined by $(k_1, X_1) \cdot (k_2, X_2) =$

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$(k_1\, k_2, \mathrm{Ad}(k_1)(X_2)+X_1)$. The associated flat symmetric space is then $M=\mathfrak{p}\simeq G_0/K$ (the action of G_0 on $\mathfrak{p}$ is given by $(k,X)\cdot Y=\mathrm{Ad}(k)(Y)+X$).

The spherical functions for the symmetric space M are then given by

$$\psi_\lambda(X)=\int_K e^{\lambda(\mathrm{Ad}(k)(X))}\,dk$$

where λ is a complex linear functional on $\mathfrak{a}\subset\mathfrak{p}$, a Cartan subalgebra of the Lie algebra of G. To extend λ to $X\in\mathrm{Ad}(K)\mathfrak{a}=\mathfrak{p}$, one uses $\lambda(X)=\lambda(\pi_{\mathfrak{a}}(X))$ where $\pi_{\mathfrak{a}}$ is the orthogonal projection with respect to the Killing form (denoted throughout this paper by $\langle\cdot,\cdot\rangle$). Note that in [6, 7, 8], λ is replaced by $i\,\lambda$.

Throughout this paper, we usually assume that G is a semisimple complex Lie group. The complex root systems are respectively A_n for $n\geq 1$ (where $\mathfrak{p}$ consists of the $n\times n$ hermitian matrices with trace 0), B_n for $n\geq 2$ (where $\mathfrak{p}=i\,\mathfrak{so}(2\,n+1)$), C_n for $n\geq 3$ (where $\mathfrak{p}=i\,\mathfrak{sp}(n)$) and D_n for $n\geq 4$ (where $\mathfrak{p}=i\,\mathfrak{so}(2\,n)$) for the classical cases and the exceptional root systems E_6, E_7, E_8, F_4 and G_2.

The radial heat kernel is considered with respect to the invariant measure $\mu(dY)=\pi^2(Y)\,dY$ on M, where $\pi(Y)=\prod_{\alpha>0}\alpha(Y)$.

Note also that in the curved case $M_0=G/K$, the spherical functions for the symmetric space M_0 are then given by

$$\phi_\lambda(e^X)=\int_K e^{(\lambda-\rho)H(e^X\,k)}\,dk$$

where ρ is the half-sum of the roots counted with their multiplicities and $H(g)$ is the abelian component in the Iwasawa decomposition of g: $g=k\,e^{H(g)}\,n$.

2 Estimates of spherical functions and of the heat kernel

We will be developing a sharp estimate for the spherical function $\psi_\lambda(X)$. We introduce the following useful convention. We will write

$$f(t,X,\lambda)\asymp g(t,X,\lambda)$$

in a given domain of f and g if there exists constants $C_1>0$ and $C_2>0$ independent of t, X and λ such that $C_1\,f(t,X,\lambda)\leq g(t,X,\lambda)\leq C_2\,g(t,X,\lambda)$ in the domain of consideration.

We conjecture the following global estimate for the spherical function in the complex case.

Conjecture 2.1. *On flat Riemannian symmetric spaces with complex group G, we have*

$$\psi_\lambda(X)\asymp\frac{e^{\langle\lambda,X\rangle}}{\prod_{\alpha>0}\,(1+\alpha(\lambda)\alpha(X))},\qquad \lambda\in\overline{\mathfrak{a}}^+, X\in\overline{\mathfrak{a}}^+.$$

Remark 2.2. *Recall that, denoting $\delta(X)=\prod_{\alpha>0}\sinh^2\alpha(X)$, we have*

$$\phi_\lambda(e^X)=\frac{\pi(X)}{\delta^{1/2}(X)}\,\psi_\lambda(X).\tag{2.1}$$

Since $\delta^{1/2}(X) \asymp e^{\rho(X)}\,\pi(X)/\prod_{\alpha>0}(1+\alpha(X))$ *in the complex case, Conjecture 2.1 therefore becomes*

$$\phi_\lambda(e^X) \asymp e^{(\lambda-\rho)(X)} \prod_{\alpha>0} \frac{1+\alpha(X)}{1+\alpha(\lambda)\alpha(X)} \tag{2.2}$$

in the curved complex case.

Let us compare the estimate (2.2) *we conjecture for* ϕ_λ *with the one obtained in [9], cf. also [13]. The estimates in [9] apply in all the generality of hypergeometric functions of Heckman and Opdam. The authors show that there exists constants* $C_1(\lambda) > 0$, $C_2(\lambda) > 0$ *such that*

$$C_1(\lambda)\, e^{(\lambda-\rho)(X)} \prod_{\substack{\alpha>0,\\ \alpha(\lambda)=0}} (1+\alpha(X)) \le \phi_\lambda(e^X) \le C_2(\lambda)\, e^{(\lambda-\rho)(X)} \prod_{\substack{\alpha>0,\\ \alpha(\lambda)=0}} (1+\alpha(X)).$$

Given (2.1)*, corresponding estimates clearly also hold in the flat case for* $\psi_\lambda(X)$*. The interest of our result, in the case* A_n*, lies in the fact that our estimate is universal in both* λ *and* X*.*

The results of [9, 13] and our estimates in the A_n case strongly suggest that the Conjecture 2.1 is true for any complex root system.

Note that asymptotics of $\psi_\lambda(t\,X)$ when λ and X are singular and $t \to \infty$ were proven in [4] for all classical complex root systems and the systems F_4 and G_2.

Consider the relationship between the Dunkl kernel $E_k(X,Y)$ and the Dunkl heat kernel $p_t(X,Y)$, as given in [10, Lemma 4.5]

$$p_t(X,Y) = \frac{1}{2^{\gamma+d/2}c_k}\, t^{-\frac{d}{2}-\gamma}\, e^{\frac{-|X|^2-|Y|^2}{4t}}\, E_k\left(X, \frac{Y}{2t}\right), \tag{2.3}$$

where γ is the number of positive roots and the constant c_k is the Macdonald–Mehta–Selberg integral. The formula (2.3) remains true for the W-invariant kernels p_t^W and E^W. In the geometric cases $k = \frac{1}{2}, 1$ and 2, by [3], the W-invariant formula (2.3) translates in a similar relationship between the spherical function ψ_λ and the heat kernel $p_t^W(X,Y)$:

$$p_t^W(X,Y) = \frac{1}{2^{\gamma+d/2}c_k}\, t^{-\frac{d}{2}-\gamma}\, e^{\frac{-|X|^2-|Y|^2}{4t}}\, \psi_X\left(\frac{Y}{2t}\right). \tag{2.4}$$

A simple direct proof of (2.4) for $k=1$ is given in [4, Remark 2.9].

Equation (2.4) and Conjecture 2.1 bring us to an equivalent conjecture for the heat kernel $p_t^W(X,Y)$.

Conjecture 2.3. *We have*

$$p_t^W(X,Y) \asymp t^{-\frac{d}{2}} \frac{e^{\frac{-|X-Y|^2}{4t}}}{\prod_{\alpha>0}(t+\alpha(X)\,\alpha(Y))}.$$

Consider also the relationship between the heat kernel $p_t^W(X,Y)$ and the heat kernel $\tilde{p}_t^W(X,Y)$ in the curved case. We have

$$\tilde{p}_t^W(X,Y) = e^{-|\rho|^2 t}\, \frac{\pi(X)\,\pi(Y)}{\delta^{1/2}(X)\,\delta^{1/2}(Y)}\, p_t^W(X,Y). \tag{2.5}$$

This relation follows directly from the fact that the respective radial Laplacians and radial measures are $\pi^{-1}\, L_{\mathfrak{a}} \circ \pi$ and $\pi(X)\, dX$ in the flat case and $\delta^{-1/2}\, (L_{\mathfrak{a}} - |\rho|^2) \circ \delta^{1/2}$ and $\delta(X)\, dX$ in the curved case ($L_{\mathfrak{a}}$ stands for the Euclidean Laplacian on $\mathfrak{a}$).

In the curved complex case, Conjecture 2.3 becomes

$$\tilde{p}_t^W(X,Y) \asymp e^{-|\rho|^2 t} t^{-\frac{d}{2}}\, e^{-\rho(X+Y)} \prod_{\alpha>0} \frac{(1+\alpha(X))\,(1+\alpha(Y))}{(t+\alpha(X)\,\alpha(Y))}\, e^{\frac{-|X-Y|^2}{4t}}.$$

Remark 2.4. *In [5], sharp estimates of W-invariant Poisson and Newton kernels in the complex Dunkl case were obtained, by exploiting the method of construction of these W-invariant kernels by alternating sums. When a root system Σ acts in $\mathbf{R}^d$, the sharp estimates of [5] have the common form*

$$\mathcal{K}^W(X,Y) \asymp \frac{\mathcal{K}^{\mathbf{R}^d}(X,Y)}{\prod_{\alpha>0}\,(|X-Y|^2 + \alpha(X)\,\alpha(Y))}, \qquad X, Y \in \overline{\mathfrak{a}}^+, \tag{2.6}$$

where $\mathcal{K}^W(X,Y)$ is the W-invariant kernel in Dunkl setting and $\mathcal{K}^{\mathbf{R}^d}(X,Y)$ is the classical kernel on $\mathbf{R}^d$. Let us observe a common pattern in the appearance of the classical kernels $\mathcal{K}^{\mathbf{R}^d}$ and of products of roots $\alpha(X)\,\alpha(Y)$ in formulas (2.6) and of the Fourier kernel $e^{\langle\lambda,X\rangle}$ and the classical Gaussian heat kernel and of products $\alpha(\lambda)\alpha(X)$ in the estimates given in Conjecture 2.1 and Conjecture 2.3.

2.1 Proof of Conjecture 2.1 in some cases

We start with a practical result.

Proposition 2.5. *Let α_i be the simple roots and let A_{α_i} be such that $\langle X, A_{\alpha_i}\rangle = \alpha_i(X)$ for $X \in \mathfrak{a}$. Suppose $X \in \mathfrak{a}^+$ and $w \in W \setminus \{id\}$. Then we have*

$$Y - w\,Y = \sum_{i=1}^{r} 2\,\frac{a_i^w(Y)}{|\alpha_i|^2}\, A_{\alpha_i} \tag{2.7}$$

where a_i^w is a linear combination of positive simple roots with non-negative integer coefficients for each i.

Proof. Refer to [5]. □

Remark 2.6. *Note that $a_i^w(Y)/|\alpha_i|^2$ is bounded by $C\, \max_k\, |\alpha_k(Y)|$ where C is a constant depending only on $w \in W$ and, ultimately, on W.*

Corollary 2.7. *Let $Y \in \overline{\mathfrak{a}^+}$ and $w \in W$. Consider the decomposition (2.7) of $Y - wY$. If $a_k^w(Y) \neq 0$ then α_k appears in a_k^w, i.e. $a_k^w = \sum_{i=1}^r n_i\alpha_i$ with $n_k > 0$.*

Proof. Refer to [5]. □

Proposition 2.8. *Let $\delta > 0$. Suppose $\alpha_i(\lambda)\,\alpha_j(X) \leq \delta$ for all i, j. Then $\psi_\lambda(X) \asymp e^{\lambda(X)}$ (the constants involved only depend on δ).*

Proof. Let $K(X,Y)$ be the kernel of the Abel transform. Recall that $K(X,Y)\,dY$ is a probability measure supported on $C(X)$, the convex envelope of the orbit $W\cdot X$. Notice that

$$e^{w_{min}\,\lambda(X)} \le \psi_\lambda(X) = \int_{C(X)} e^{\lambda(Y)}\,K(X,Y)\,dY \le e^{\lambda(X)} \tag{2.8}$$

where w_{min} is the element of the Weyl group giving the minimum value of $w\,\lambda(X)$. Now, using Proposition 2.5 and Remark 2.6 with $Y=\lambda$, we see that for any $w\in W$

$$\begin{aligned} e^{\lambda(X)} \ge e^{w\,\lambda(X)} &= e^{\langle w\lambda-\lambda,X\rangle}\,e^{\langle\lambda,X\rangle} = \prod_{i=1}^{r} e^{-2\,\frac{a_i^w(\lambda)}{|\alpha_i|^2}\,\alpha_i(X)}\,e^{\langle\lambda,X\rangle} \\ &\ge \prod_{i=1}^{r} e^{-2\,C\,(\max_k\,\alpha_k(\lambda))\,\alpha_i(X)}\,e^{\langle\lambda,X\rangle} \ge \prod_{i=1}^{r} e^{-2\,C\,\delta}\,e^{\langle\lambda,X\rangle}. \end{aligned}$$

□

Remark 2.9. *This case and this method apply for any radial Dunkl case; it suffices to replace $K(X,Y)\,dY$ by the so-called Rösler measure $\mu_X(dY)$ in the integral in* (2.8)*, see [11].*

Proposition 2.10. *A spherical function $\psi_\lambda(X)$ on M is given by the formula*

$$\psi_\lambda(X) = \frac{\pi(\rho)}{2^\gamma\pi(\lambda)\,\pi(X)}\sum_{w\in W}\epsilon(w)e^{\langle w\lambda,X\rangle}, \tag{2.9}$$

where $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\alpha = \sum_{\alpha\in\Sigma^+}\alpha$ and $\gamma=|\Sigma^+|$ is the number of positive roots (refer to [8, Chap. IV, Proposition 4.8 and Chap. II, Theorem 5.35]).

Proposition 2.11. *Suppose $\alpha(\lambda)\,\alpha(X)\ge(\log|W|)/2$ for all $\alpha>0$. Then*

$$\psi_\lambda(X) \asymp \frac{e^{\lambda(X)}}{\pi(\lambda)\pi(X)}.$$

We are assuming here that $|\alpha_i|\ge 1$ for each i.

Proof. Suppose $w\in W$ is not the identity. In that case, $a_i^w(\lambda)$ is not equal to 0 for some i. By Proposition 2.5 with $y=\lambda$ and Corollary 2.7, $\lambda(X)-w\,\lambda(X)\ \ge\ 2\,a_i^w(\lambda)\,\alpha_i(X)/|\alpha_i|^2\ \ge\ 2\,\alpha_i(\lambda)\,\alpha_i(X)\ge\log|W|$. Each term $e^{\langle w\lambda,X\rangle}$ in the alternating sum (2.9) corresponding to $w\ne\mathrm{id}$ is bounded by $e^{-\log|W|}\,e^{\lambda(X)}=e^{\lambda(X)}/|W|$. Hence, since only half the terms in the sum are negative,

$$|W|\,e^{\lambda(X)} \ge \sum_{w\in W}\epsilon(w)e^{\langle w\lambda,X\rangle} \ge e^{\lambda(X)} - \frac{|W|}{2}\,e^{\lambda(X)}/|W| = \frac{1}{2}\,e^{\lambda(X)}.$$

□

3 The conjecture in the case of the root system A_n

We will prove the conjecture in the case of the root system of type A.

Theorem 3.1. *In the case of the root system of type A_n in the complex case, we have*

$$\psi_\lambda(e^X) \asymp \frac{e^{\langle\lambda,X\rangle}}{\prod_{i<j}(1+(\lambda_i-\lambda_j)(x_i-x_j))}, \qquad \lambda,\ X \in \overline{\mathfrak{a}}^+. \tag{3.1}$$

Corollary 3.2.

$$\phi_\lambda(e^X) \asymp e^{(\lambda-\rho)(X)} \prod_{i<j} \frac{1+x_i-x_j}{1+(x_i-x_j)(\lambda_i-\lambda_j)},$$

$$p_t^W(X,Y) \asymp t^{-\frac{d}{2}} \frac{e^{\frac{-|X-Y|^2}{4t}}}{\prod_{i<j}(t+(x_i-x_j)(y_i-y_j))},$$

$$\tilde{p}_t^W(X,Y) \asymp e^{-|\rho|^2 t}\, t^{-\frac{d}{2}}\, e^{-\rho(X+Y)} \prod_{i<j} \frac{(1+x_i-x_j)(1+y_i-y_j)}{(t+(x_i-x_j)(y_i-y_j))}\, e^{\frac{-|X-Y|^2}{4t}}.$$

We recall (refer to [12]) the following iterative formula for the spherical functions of type A in the complex case. Here we do not assume that the elements of the Lie algebra have trace 0. Here the Cartan subalgebra $\mathfrak{a}$ for the root system A_{n-1} is isomorphic to $\mathbf{R}^n$. For $\lambda, X \in \overline{\mathfrak{a}}^+ \subset \mathbf{R}^n$, we have

$$\begin{aligned} &\psi_\lambda(e^X) = e^{\lambda(X)} \text{ if } n=1 \text{ and} \\ &\psi_\lambda(e^X) = (n-1)!\, e^{\lambda_n \sum_{k=1}^n x_k} \big(\prod_{i<j}(x_i-x_j)\big)^{-1} \int_{x_n}^{x_{n-1}} \cdots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y) \\ &\qquad\qquad \prod_{i<j<n}(y_i-y_j)\, dy_1 \cdots dy_{n-1} \end{aligned} \tag{3.2}$$

where $\lambda_0(U) = \sum_{k=1}^{n-1}(\lambda_k-\lambda_n)\, u_k$.

Remark 3.3. *Formula* (3.2) *represents the action of the root system A_{n-1} on $\mathbf{R}^n$. If we assume $\sum_{k=1}^n x_k = 0 = \sum_{k=1}^n \lambda_k$, we have then the action of the root system A_{n-1} on $\mathbf{R}^{n-1}$. We can also consider the action of A_{n-1} on any $\mathbf{R}^m$ with $m \geq n-1$ by considering formula* (2.9) *and deciding on which entries x_k, the Weyl group $W = S_n$ acts. These considerations do not affect the conclusion of Theorem 3.1.*

3.1 Approximate factorization for A_n

Before proving the conjecture in the case A_n, we will prove an interesting "factorization".

Proposition 3.4. *For $n \geq 1$, consider the root system A_n on $\mathbf{R}^{n+1}$. Let $\lambda, X \in \overline{\mathfrak{a}}^+ \subset \mathbf{R}^{n+1}$ and $X' = [X_1, \ldots, X_n]$. Define*

$$\begin{aligned} I^{(n)} = I^{(n)}(\lambda; X) = &\int_{x_{n+1}}^{x_n} \int_{x_n}^{x_{n-1}} \cdots \int_{x_3}^{x_2} \int_{x_2}^{x_1} e^{-\lambda_0(X'-Y)} \\ &\prod_{i<j<n} \frac{(y_i-y_j)(\lambda_i-\lambda_j)}{1+(y_i-y_j)(\lambda_i-\lambda_j)}\, dy_1\, dy_2 \cdots dy_n. \end{aligned}$$

Then the following approximate factorization holds

$$I^{(n)} \asymp \prod_{k=1}^{n} I_k^{(n)} \tag{3.3}$$

where

$$I_1^{(n)} = \int_{x_2}^{x_1} e^{-(\lambda_1-\lambda_{n+1})\,(x_1-y_1)}\, dy_1 \text{ and}$$

$$I_k^{(n)} = \int_{x_{k+1}}^{x_k} e^{-(\lambda_k-\lambda_{n+1})\,(x_k-y_k)} \prod_{j=1}^{k-1} \frac{(x_j-y_k)\,(\lambda_j-\lambda_k)}{1+(x_j-y_k)\,(\lambda_j-\lambda_k)}\, dy_k \text{ for } 1<k\le n.$$

Proof. Since $u/(1+u)$ is an increasing function, we clearly have

$$I^{(n)} \le \int_{x_{n+1}}^{x_n}\int_{x_n}^{x_{n-1}}\cdots\int_{x_3}^{x_2}\int_{x_2}^{x_1} e^{-\lambda_0(X'-Y)} \prod_{i<j<n} \frac{(x_i-y_j)\,(\lambda_i-\lambda_j)}{1+(x_i-y_j)\,(\lambda_i-\lambda_j)}\, dy_1\, dy_2\,\cdots\, dy_n.$$

On the other hand,

$$\begin{aligned}
I^{(n)} \ge{}& \int_{(x_n+x_{n+1})/2}^{x_n}\int_{(x_{n-1}+x_n)/2}^{x_{n-1}}\cdots\int_{(x_2+x_3)/2}^{x_2}\int_{(x_1+x_2)/2}^{x_1} e^{-\lambda_0(X'-Y)}\\
&\prod_{i<j<n} \frac{(y_i-y_j)\,(\lambda_i-\lambda_j)}{1+(y_i-y_j)\,(\lambda_i-\lambda_j)}\, dy_1\, dy_2\,\cdots\, dy_n\\
\ge{}& \int_{(x_n+x_{n+1})/2}^{x_n}\int_{(x_{n-1}+x_n)/2}^{x_{n-1}}\cdots\int_{(x_2+x_3)/2}^{x_2}\int_{(x_1+x_2)/2}^{x_1} e^{-\lambda_0(X'-Y)}\\
&\prod_{i<j<n} \frac{((x_i+x_{i+1})/2-y_j)\,(\lambda_i-\lambda_j)}{1+((x_i+x_{i+1})/2-y_j)\,(\lambda_i-\lambda_j)}\, dy_1\, dy_2\,\cdots\, dy_n\\
\asymp{}& \int_{(x_n+x_{n+1})/2}^{x_n}\int_{(x_{n-1}+x_n)/2}^{x_{n-1}}\cdots\int_{(x_2+x_3)/2}^{x_2}\int_{(x_1+x_2)/2}^{x_1} e^{-\lambda_0(X'-Y)}\\
&\quad\prod_{i<j<n} \frac{(x_i-y_j)\,(\lambda_i-\lambda_j)}{1+(x_i-y_j)\,(\lambda_i-\lambda_j)}\, dy_1\, dy_2\,\cdots\, dy_n =\\
={}& \prod_{k=1}^{n}\int_{(x_k+x_{k+1})/2}^{x_k} e^{-(\lambda_k-\lambda_{n+1})\,(x_k-y_k)} \prod_{j=1}^{k-1} \frac{(x_j-y_k)\,(\lambda_j-\lambda_k)}{1+(x_j-y_k)\,(\lambda_j-\lambda_k)}\, dy_k = \prod_{k=1}^{n} A_k^{(n)}
\end{aligned}$$

since

$$\frac{((x_i+x_{i+1})/2-y_j)\,(\lambda_i-\lambda_j)}{1+((x_i+x_{i+1})/2-y_j)\,(\lambda_i-\lambda_j)} \le \frac{(x_i-y_j)\,(\lambda_i-\lambda_j)}{1+(x_i-y_j)\,(\lambda_i-\lambda_j)}$$

while

$$\frac{((x_i+x_{i+1})/2-y_j)\,(\lambda_i-\lambda_j)}{1+((x_i+x_{i+1})/2-y_j)\,(\lambda_i-\lambda_j)} \ge \frac{((x_i-y_j)/2\,(\lambda_i-\lambda_j)}{1+(x_i-y_j)/2\,(\lambda_i-\lambda_j)} \ge \frac{1}{2}\,\frac{(x_i-y_j)\,(\lambda_i-\lambda_j)}{1+(x_i-y_j)\,(\lambda_i-\lambda_j)}.$$

Now, let

$$B_k^{(n)} = \int_{x_{k+1}}^{(x_k+x_{k+1})/2} e^{-(\lambda_k-\lambda_{n+1})\,(x_k-y_k)} \prod_{j=1}^{k-1} \frac{(x_j-y_k)\,(\lambda_j-\lambda_k)}{1+(x_j-y_k)\,(\lambda_j-\lambda_k)}\,dy_k$$

and note that $I_k^{(n)} = A_k^{(n)} + B_k^{(n)}$.

Now, using the change of variable $2w = x_k - y_k$, we have

$$\begin{aligned} B_k^{(n)} &= 2\int_{(x_k-x_{k+1})/4}^{(x_k-x_{k+1})/2} e^{-2\,(\lambda_k-\lambda_{n+1})\,w} \prod_{j=1}^{k-1} \frac{(x_j-x_k+2\,w)\,(\lambda_j-\lambda_k)}{1+(x_j-x_k+2\,w)\,(\lambda_j-\lambda_k)}\,dw \\ &\le 4\int_{(x_k-x_{k+1})/4}^{(x_k-x_{k+1})/2} e^{-2\,(\lambda_k-\lambda_{n+1})\,w} \prod_{j=1}^{k-1} \frac{(x_j-x_k+w)\,(\lambda_j-\lambda_k)}{1+(x_j-x_k+w)\,(\lambda_j-\lambda_k)}\,dw \\ &\le 4\int_{0}^{(x_k-x_{k+1})/2} e^{-(\lambda_k-\lambda_{n+1})\,w} \prod_{j=1}^{k-1} \frac{(x_j-x_k+w)\,(\lambda_j-\lambda_k)}{1+(x_j-x_k+w)\,(\lambda_j-\lambda_k)}\,dw = 4\,A_k^{(n)}, \end{aligned}$$

where the last equality comes from the change of variable $w = x_k - y_k$ in the expression for $A_k^{(n)}$. Therefore $I_k^{(n)} = A_k^{(n)} + B_k^{(n)} \le 5\,A_k^{(n)}$. The result follows. □

The next proposition gives an inductive way of estimating $I^{(n+1)}$, knowing $I^{(n)}$ and $I^{(n-1)}$.

Proposition 3.5. *Consider the root system A_{n+1} on $\mathbf{R}^{n+2}$. Let $\lambda, X \in \overline{\mathfrak{a}}^+ \subset \mathbf{R}^{n+2}$. Assume $\alpha_1(X) \ge \alpha_{n+1}(X)$. Then*

$$I^{(n+1)}(\lambda;X) \asymp I^{(n)}(\lambda_1,\dots,\lambda_n,\lambda_{n+2};x_1,\dots,x_{n+1}) \frac{(x_1-x_{n+1})(\lambda_1-\lambda_{n+1})}{1+(x_1-x_{n+1})(\lambda_1-\lambda_{n+1})} \frac{I^{(n)}(\lambda_2,\dots,\lambda_{n+1},\lambda_{n+2};x_2,\dots,x_{n+2})}{I^{(n-1)}(\lambda_2,\dots,\lambda_n,\lambda_{n+2};x_2,\dots,x_{n+1})}.$$

Proof. We start with an outline of the proof.

(i) $I^{(n+1)}$ is estimated by a product of $n+1$ factors $I_k^{(n+1)}(\lambda;X)$.

(ii) The product of the first n factors $I_1^{(n+1)}(\lambda;X)$, …, $I_n^{(n+1)}(\lambda;X)$ give an estimate of the term $I^{(n)}(\lambda_1,\dots,\lambda_n,\lambda_{n+2};X')$ by Proposition 3.4.

(iii) In the last factor $I_{n+1}^{(n+1)}(\lambda;X)$, we "draw off" one term from under the integral, using the additional hypothesis $\alpha_1(X) \ge \alpha_{n+1}(X)$. The remaining integral corresponds to $I_n^{(n)}(\lambda_2,\dots,\lambda_{n+2};x_2,\dots,$

(iv) The last factor $I_n^{(n)}$ of $I^{(n)}$ is estimated by $I^{(n)}/I^{(n-1)}$, up to a change of variables (we re-use the idea of (ii)).

Since $x_{n+2} \le y_{n+1} \le x_{n+1}$ and $x_{n+1} - x_{n+2} \le x_1 - x_2$, we get $x_1 - x_{n+1} \le x_1 - y_{n+1} \le x_1 - x_{n+2} \le 2(x_1 - x_{n+1})$ and we have

$$\begin{aligned}
I_{n+1}^{(n+1)} &\asymp \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1}-\lambda_{n+2})\,(x_{n+1}-y_{n+1})} \frac{(x_1 - y_{n+1})\,(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - y_{n+1})\,(\lambda_1 - \lambda_{n+1})} \\
&\prod_{j=2}^{n} \frac{(x_j - y_{n+1})\,(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1}}) \,(\lambda_j - \lambda_{n+1})\, dy_{n+1} \\
&\asymp \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1}-\lambda_{n+2})\,(x_{n+1}-y_{n+1})} \\
&\prod_{j=2}^{n} \frac{(x_j - y_{n+1})\,(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})\,(\lambda_j - \lambda_{n+1})}\, dy_{n+1}.
\end{aligned}$$

Hence, noting that $I_1^{(n+1)}(\lambda; X) \cdots I_n^{(n+1)}(\lambda; X) \asymp I^{(n)}(\lambda_1, \ldots, \lambda_n, \lambda_{n+2}; X')$, we have

$$\begin{aligned}
I^{(n+1)}(\lambda; X) &\asymp I^{(n)}(\lambda_1, \ldots, \lambda_n, \lambda_{n+2}; X') \frac{(x_1 - x_{n+1})\,(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})\,(\lambda_1 - \lambda_{n+1})} \\
&\qquad \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1}-\lambda_{n+2})\,(x_{n+1}-y_{n+1})} \prod_{j=2}^{n} \frac{(x_j - y_{n+1})\,(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})\,(\lambda_j - \lambda_{n+1})}\, dy_{n+1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1}-\lambda_{n+2})\,(x_{n+1}-y_{n+1})} \prod_{j=2}^{n} \frac{(x_j - y_{n+1})\,(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})\,(\lambda_j - \lambda_{n+1})}\, dy_{n+1} \\
&= \frac{\prod_{k=1}^{n} \int_{x_{k+2}}^{x_{k+1}} e^{-(\lambda_{k+1}-\lambda_{n+2})\,(x_{k+1}-y_{k+1})} \prod_{j=1}^{k-1} \frac{(x_{j+1}-y_{k+1})\,(\lambda_{j+1}-\lambda_{k+1})}{1+(x_{j+1}-y_{k+1})\,(\lambda_{j+1}-\lambda_{k+1})}\, dy_{k+1}}{\prod_{k=1}^{n-1} \int_{x_{k+2}}^{x_{k+1}} e^{-(\lambda_{k+1}-\lambda_{n+2})\,(x_{k+1}-y_{k+1})} \prod_{j=1}^{k-1} \frac{(x_{j+1}-y_{k+1})\,(\lambda_{j+1}-\lambda_{k+1})}{1+(x_{j+1}-y_{k+1})\,(\lambda_{j+1}-\lambda_{k+1})}\, dy_{k+1}} \\
&= \frac{I^{(n)}(\lambda_2, \ldots, \lambda_{n+1}, \lambda_{n+2}; x_2, \ldots, x_{n+2})}{I^{(n-1)}(\lambda_2, \ldots, \lambda_n, \lambda_{n+2}; x_2, \ldots, x_{n+1})}.
\end{aligned}$$

□

Remark 3.6. *When $n = 1$, the result of Proposition 3.5 remains valid if we set $I^{(0)} = 1$.*

We now prove our main result.

Proof of Theorem 3.1. We use induction on the rank. In the case of A_1, we have

$$\begin{aligned}
\psi_\lambda(e^X) &= e^{\lambda_2\,(x_1+x_2)}\,(x_1 - x_2)^{-1} \int_{x_2}^{x_1} e^{(\lambda_1-\lambda_2)\,y}\, dy \\
&= e^{\lambda_2\,(x_1+x_2)}\,(x_1 - x_2)^{-1} \frac{e^{(\lambda_1-\lambda_2)\,x_1} - e^{(\lambda_1-\lambda_2)\,x_2}}{\lambda_1 - \lambda_2} \\
&= e^{\lambda_1\,x_1+\lambda_2\,x_2} \frac{1 - e^{-(\lambda_1-\lambda_2)\,(x_1-x_2)}}{(\lambda_1 - \lambda_2)\,(x_1 - x_2)} \asymp e^{\lambda_1\,x_1+\lambda_2\,x_2} \frac{1}{1 + (\lambda_1 - \lambda_2)\,(x_1 - x_2)}
\end{aligned}$$

since $1 - e^{-u} \asymp u/(1+u)$ for $u \geq 0$.

Assume that the result is true for A_r, $1 \leq r \leq n$, $n \geq 1$. Using (3.2) and the induction hypothesis, we have for $r = 1, \ldots, n+1$ and λ, X in positive Weyl chamber in $\mathbf{R}^{r+1}$

$$\begin{aligned}
&\pi(X)\,\pi(\lambda')\,e^{-\lambda(X)}\,\psi_\lambda([x_1,\ldots,x_r,x_{r+1}])\\
&= r!\,\pi(\lambda')\,e^{-\lambda(X)}\,e^{\lambda_{r+1}\sum_{k=1}^{r+1} x_k}\int_{x_{r+1}}^{x_r}\cdots\int_{x_2}^{x_1}\psi_{\lambda_0}(e^Y)\prod_{i<j<r+1}(y_i-y_j)\,dy_1\cdots dy_r\\
&\asymp \int_{x_{r+1}}^{x_r}\int_{x_r}^{x_{r-1}}\cdots\int_{x_3}^{x_2}\int_{x_2}^{x_1} e^{-\lambda_0(X'-Y)}\prod_{i<j<r+1}\frac{(y_i-y_j)\,(\lambda_i-\lambda_j)}{1+(y_i-y_j)\,(\lambda_i-\lambda_j)}\,dy_1\,dy_2\,\cdots\,dy_r
\end{aligned}$$

where $X' = \operatorname{diag}[x_1,\ldots,x_r]$ and $\lambda' = [\lambda_1,\ldots,\lambda_r]$. Using the notation introduced in Proposition 3.4, we have

$$\pi(X)\,\pi(\lambda')\,e^{-\lambda(X)}\,\psi_{[\lambda_1,\ldots,\lambda_{r+1}]}([x_1,\ldots,x_{r+1}]) = r!\,I^{(r)}(\lambda_1,\ldots,\lambda_{r+1},x_1,\ldots,x_{r+1}).$$

Still using the induction hypothesis, we have

$$\begin{aligned}
\pi(X)\,\pi(\lambda')\,e^{-\lambda(X)}\,\psi_{[\lambda_1,\ldots,\lambda_{r+1}]}([x_1,\ldots,x_{r+1}] &= r!\,I^{(r)}(\lambda_1,\ldots,\lambda_{r+1};x_1,\ldots,x_{r+1})\\
&\asymp \frac{\pi(X)\,\pi(\lambda')}{\prod_{i<j\leq r+1}(1+(\lambda_i-\lambda_j)\,(x_i-x_j))}
\end{aligned} \tag{3.4}$$

for $r = 1, \ldots, n$.

It remains to show that (3.4) holds for $r = n+1$, *i.e.* that

$$I^{(n+1)}(\lambda_1,\ldots,\lambda_{n+2};x_1,\ldots,x_{n+2}) \asymp \frac{\pi(X)\,\pi(\lambda')}{\prod_{i<j\leq n+2}(1+(\lambda_i-\lambda_j)\,(x_i-x_j))}.$$

It is sufficient to prove the last formula under the hypothesis that $\alpha_1(X) \geq \alpha_{n+1}(X)$ since the case $\alpha_1(X) \leq \alpha_{n+1}(X)$ is symmetric. Now, according to Proposition 3.5 and (3.4),

$$\begin{aligned}
I^{(n+1)}(\lambda;X) &\asymp \frac{(x_1-x_{n+1})(\lambda_1-\lambda_{n+1})}{1+(x_1-x_{n+1})(\lambda_1-\lambda_{n+1})}\,I^{(n)}(\lambda_1,\ldots,\lambda_n,\lambda_{n+2};x_1,\ldots,x_{n+1})\\
&\qquad I^{(n)}(\lambda_2,\ldots,\lambda_{n+1},\lambda_{n+2};x_2,\ldots,x_{n+2})\,\big(I^{(n-1)}(\lambda_2,\ldots,\lambda_n,\lambda_{n+2};x_2,\ldots,x_{n+1})\big)^{-1}\\
&\asymp \frac{(x_1-x_{n+1})(\lambda_1-\lambda_{n+1})}{1+(x_1-x_{n+1})(\lambda_1-\lambda_{n+1})}\\
&\qquad \frac{\prod_{i<j\leq n+1}(x_i-x_j)\,\prod_{i<j<n+1}(\lambda_i-\lambda_j)}{\prod_{i<j\leq n}(1+(x_i-x_j)\,(\lambda_i-\lambda_j))\,\prod_{i=1}^{n}(1+(x_i-x_{n+1})\,(\lambda_i-\lambda_{n+2}))}\\
&\qquad \frac{\prod_{1<i<j\leq n+2}(x_i-x_j)\,\prod_{1<i<j\leq n+1}(\lambda_i-\lambda_j)}{\prod_{1<i<j\leq n+2}(1+(x_i-x_j)\,(\lambda_i-\lambda_j))}\\
&\qquad \frac{\prod_{1<i<j\leq n}(1+(x_i-x_j)\,(\lambda_i-\lambda_j))\,\prod_{i=2}^{n}(1+(x_i-x_{n+1})\,(\lambda_i-\lambda_{n+2}))}{\prod_{1<i<j\leq n+1}(x_i-x_j)\,\prod_{1<i<j<n+1}(\lambda_i-\lambda_j)}\\
&= \frac{x_1-x_{n+1}}{x_1-x_{n+2}}\,\frac{1+(x_1-x_{n+1})\,(\lambda_1-\lambda_2)}{1+(x_1-x_{n+1})\,(\lambda_1-\lambda_2)}\,\frac{\prod_{i<j\leq n+2}(x_i-x_j)\,\prod_{i<j\leq n+1}(\lambda_i-\lambda_j)}{\prod_{i<j\leq n+2}(1+(\lambda_i-\lambda_j)\,(x_i-x_j))}.
\end{aligned}$$

The result follows since $x_1 - x_{n+1} \asymp x_1 - x_{n+2}$ given that $x_1 - x_2 \geq x_{n+1} - x_{n+2}$. $\square$

4 Comparison with the estimates of Anker et al. in [1]. Conjecture for Dunkl setting

In [1, Theorems 4.1 p. 2372 and 4.4, p. 2377] the following estimates were proven for the heat kernel $p_t(X,Y)$ in the Dunkl setting on $\mathbf{R}^n$. There exists positive constants c_1, c_2, C_1 and C_2 such that for all $X,Y\in\overline{\mathfrak{a}^+}$

$$\frac{C_1 e^{-c_1|X-Y|^2/t}}{\min\{w(B(X,\sqrt{t})),w(B(Y,\sqrt{t}))\}}\le p_t(X,Y)\le \frac{C_2 e^{-c_2|X-Y|^2/t}}{\max\{w(B(X,\sqrt{t})),w(B(Y,\sqrt{t}))\}} \tag{4.1}$$

where w is the W-invariant reference measure (in our paper $w=\pi(X)^2dX$) and the w-volume of a ball satisfies the estimate ([1, p. 2365])

$$w(B(X,r))\asymp r^n\prod_{\alpha>0}(r+\alpha(X))^{2k(\alpha)}.$$

The same estimates follow for $p_t^W(X,Y)$. Our sharp estimates in Corollary 3.2 for $k(\alpha)=1$ in the W-radial case A_n suggest that $c_1=c_2=1/4$ in (4.1) and that products of terms $(t+\alpha(X)\alpha(Y))^{k(\alpha)}$ are natural in place of separate terms $w(B(X,\sqrt{t}))$ and $w(B(Y,\sqrt{t}))$. On the other hand, estimates (4.1) and in Corollary 3.2 suggest that the following conjecture is true in the Dunkl setting.

Conjecture 4.1. *The Weyl-invariant heat kernel for a root system Σ in $\mathbf{R}^d$ satisfies the following estimates*

$$p_t^W(X,Y)\asymp t^{-\frac{d}{2}}\frac{e^{\frac{-|X-Y|^2}{4t}}}{\prod_{\alpha>0}(t+\alpha(X)\alpha(Y))^{k(\alpha)}}. \tag{4.2}$$

Formula (2.3) *then implies that the W-invariant Dunkl kernel satisfies the estimate*

$$E_k^W(X,Y)\asymp\frac{e^{\lambda(X)}}{\prod_{\alpha>0}(1+\alpha(X)\,\alpha(\lambda))^{k(\alpha)}}.$$

5 Additional formulas for $p_t^W(X,Y)$

Let us finish by giving formulas relating the heat kernel $p_t^W(X,Y)$ with the spherical functions $\psi_{i\lambda}$ and $\phi_{i\lambda}$. These formulas can be useful in further study of the kernel $p_t^W(X,Y)$.

Proposition 5.1. *(a) In the flat Riemannian symmetric case, the following formula holds:*

$$p_t^W(X,Y)=C\int_{\mathfrak{a}}e^{-|\lambda|^2t}\,\psi_{i\lambda}(X)\,\psi_{-i\lambda}(Y)\,\pi(\lambda)^2\,d\lambda,\qquad C>0. \tag{5.1}$$

(b) In the curved non-compact Riemannian symmetric case the following formula holds

$$p_t^W(X,Y)=C\int_{\mathfrak{a}^*}e^{-(|\lambda|^2+|\rho|^2)\,t}\,\phi_{i\lambda}(X)\,\phi_{-i\lambda}(Y)\,\frac{d\lambda}{|c(\lambda)|^2} \tag{5.2}$$

where $c(\lambda)$ is the Harish-Chandra c-function (refer to [8] for details). The constant C can be given explicitly.

Proof. We will prove (b). We show that the right hand side of equation (5.1) satisfies the definition of the heat kernel. For a test function f, consider

$$u(X,t) = C \int_{\mathfrak{a}} \int_{\mathfrak{a}} e^{-(|\lambda|^2+|\rho|^2)\,t}\, \phi_{i\,\lambda}(X)\, \phi_{-i\,\lambda}(Y)\, K\, |c(\lambda)|^{-2}\, d\lambda\, f(Y)\, |c(\lambda)|^{-2}\, dY$$

where $K\,|c(\lambda)|^{-2}\,d\lambda$ is Plancherel measure.

The fact that $\Delta\, u(X,t) = \frac{\partial}{\partial t}\, u(X,t)$ where Δ is the radial Laplacian follows easily from the fact that $\Delta\, \phi_{i\,\lambda}(X) = -(|\lambda|^2 + |\rho|^2)\, \phi_{i\,\lambda}(X)$ and $\frac{\partial}{\partial t}\, e^{-(|\lambda|^2+|\rho|^2)]\,t} = -(|\lambda|^2 + |\rho|^2)\, e^{-|\lambda|^2 t}$. Now, using Fubini's theorem,

$$\begin{aligned} u(X,t) &= C\, K \int_{\mathfrak{a}} e^{-(|\lambda|^2+|\rho|^2)\,t} \left[\int_{\mathfrak{a}} \phi_{-i\,\lambda}(Y)\, f(Y)\, |c(\lambda)|^{-2}\, dY\right] \phi_{i\,\lambda}(X)\, \pi(\lambda)^2\, d\lambda \\ &= C\, K \int_{\mathfrak{a}} e^{-(|\lambda|^2+|\rho|^2)\,t}\, \tilde{f}(\lambda)\, \phi_{i\,\lambda}(X)\, \pi(\lambda)^2\, d\lambda \end{aligned}$$

which tends to $f(X)$ as $t \to 0$ by the dominated convergence theorem. □

Remark 5.2. *The heat kernel estimates of $h_t^W(X) = p_t^W(X,0)$ on symmetric spaces ([2] and references therein) are based on the inverse spherical Fourier transform formula which is a special case of* (5.2) *when $Y = 0$. Thus one may hope that estimates of $p_t^W(X,Y)$ can be deduced from* (5.2)*.*

Remark 5.3. *The passage from $h_t^W(X)$ to $p_t^W(X,Y)$ is well understood at the group level:*

$$p_t^W(g,h) = h_t^W(h^{-1}g),$$

which is equivalent to

$$p_t^W(X,Y) = \int_K h_t^W(e^{-Y}\, k^{-1}\, e^X)\, dk$$

and to

$$p_t^W(X,Y) = \int_{\mathfrak{a}} h_t^W(H)\, k(H,-Y,X)\, \pi(H)\, dH, \tag{5.3}$$

where the last formula contains the product formula kernel k which is defined by

$$\int_{\mathfrak{a}} \psi_\lambda(e^H)\, k(H,X,Y)\, \pi(H)\, dH = \psi_\lambda(e^X)\, \psi_\lambda(e^Y) = \int_K \psi_\lambda(e^X\, k\, e^Y)\, dk.$$

Similarly,

$$\tilde{p}_t^W(X,Y) = \int_{\mathfrak{a}} \tilde{h}_t^W(H)\, \tilde{k}(H,-Y,X)\, \delta(H)\, dH, \tag{5.4}$$

where the last formula contains the product formula kernel $\tilde{k}$ which is defined by

$$\int_{\mathfrak{a}} \phi_\lambda(e^H)\, \tilde{k}(H,X,Y)\, \delta(H)\, dH = \phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_K \phi_\lambda(e^X\, k\, e^Y)\, dk.$$

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