Invariant integrals and asymptotic fields near the front of a curved planar crack

Claude Stolz

"Int. Journal of fracture mechanics" manuscript No.
(will be inserted by the editor)

Invariant integrals and asymptotic fields near the front of a curved planar crack

C. Stolz

Received: date / Accepted: date

Abstract A plane crack is considered and the influence of local curvature of the crack front on the local mechanical fields is studied. The main goal is to determine the stress intensity factors along a curved planar crack in linear elasticity with accuracy. This is obtained by determination of new test fields and the use of bilinear forms, issued from invariant integrals, which separate the local modes of fracture.

Keywords Stress intensity factors, J-Integral, Asymptotic fields, Curved crack front

1 Introduction

The displacement asymptotic expansion is well known in the two dimensional case. The fundamental solution has been given by Williams (1957). Near the front of a crack it is well known that the most singular term of the expansion can be simply obtained by superposition of plane strain and the anti-plane strain solutions (Bui, 1975,1977).

However, it it important to consider additional higher order terms in order to study the possible path of propagation using developments of Leblond (1989) and also to ensure with more accuracy the condition of equilibrium in a finite domain near the crack tip.

The first step of the analysis is to characterize the local J-integral or the G-θ integral, by the use of the energy momentum tensor (Eshelby,1951). The invariant integral has the same property than the J-integral of Rice (1968) taking account of additional terms due to the curvature of the crack front by the way of a surface integral. This term is necessary to ensure the invariance with respect to the path of integration. This term vanishes for a rectilinear front, it is proportional to the curvature.

A similar idea is used to generalize the G-θ integral of Destuynder and Djaoua (1981).

The invariant integrals are quadratic forms with respect to the displacement gradient, and the associated bilinear form is introduced. In the case of linear elasticity, the obtained bilinear form is a generalisation of the Chen-Shield (1977) invariant integral M.

The invariant integrals satisfy the Irwin's value.

In a finite domain of integration of radius R, the fields obtained by Williams do not any more satisfy the equilibrium, and consequently the value of invariant integral taken with this field don't satisfy Irwin's formula.

The second step is to define a correction of the William's solution to take the curvature Γ into account. These corrected fields are compared with the stress development obtained in (Leblond & Torlai, 1982) and the fields proposed in (Yosibash et al., 2011)

The equations are established in a local moving frame associated to the crack front.

The additional terms are proposed for the three local mode of rupture up to order two, the approximated value of invariant integral is given to this order of approximation.

The use of bilinear forms with the new test fields on the analytical solutions due to Fabrikant (1988,1989) for mode I,II and III, gives an approximation of local stress intensity factors. The precision of the results depends on $\eta = R\Gamma$, this give an evaluation of the radius R of the domain of integration to guarantee a given precision with respect of the local curvature Γ of the crack front.

C. Stolz
IMSIA, UMR CNRS
E-mail: claude.stolz@ensta-paristech.fr

2 Some preliminaries

A point M_o of the crack front is defined by its arc-length s, the tangent vector to the crack front is then e_s

$$\mathrm{d}M_o = \mathrm{d}s\, e_s = \mathrm{d}s(\cos\phi e_x + \sin\phi e_y) = \frac{e_s}{\Gamma}\mathrm{d}\phi, \quad \Gamma = \frac{\mathrm{d}\phi}{\mathrm{d}s} \tag{1}$$

and Γ is the curvature of the crack front.

The normal to the plane of the crack is e_z and N is the normal vector to the crack front. In the frame of Fresnet, we have

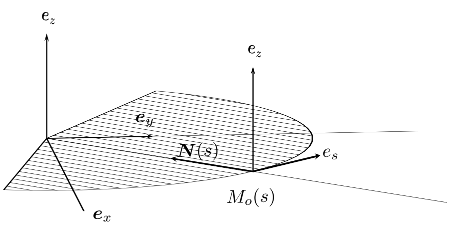

Fig. 1 The crack front

$$\frac{\mathrm{d}e_s}{\mathrm{d}s} = \Gamma N \tag{2}$$

A point of the plane (M_o, e_z, N) has cylindrical coordinates (r, θ), $M = M_o(s) + r\, e_r(s, \theta)$.

$$e_r = -\cos\theta\, N(s) + \sin\theta\, e_z, \quad e_\theta = \sin\theta\, N(s) + \cos\theta\, e_z \tag{3}$$

Then,

$$\mathrm{d}M = \mathrm{d}r\, e_r + r\mathrm{d}\theta\, e_\theta + \gamma\, e_s\, \mathrm{d}s, \quad \gamma = 1 + r\Gamma\cos\theta \tag{4}$$

The elementary volume is $\mathrm{d}\omega = \mathrm{d}r\; r\mathrm{d}\theta\; \gamma\, \mathrm{d}s$. Due to the moving frame with respect to s, the gradient of the displacement u depends on the curvature Γ, and accordingly to the local strain ε in the basis (e_r, e_θ, e_s) can be decomposed in three contributions, first a plane strain, an anti-plane strain, and additional terms proportional to Γ, see appendix B.

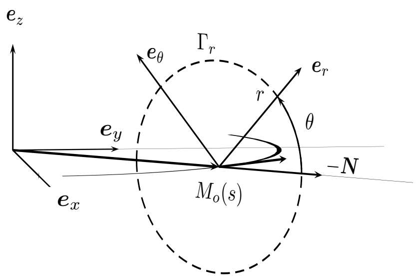

Fig. 2 The local frame

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \tag{5}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r} \tag{6}$$

$$2\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{7}$$

$$\varepsilon_{ss} = \frac{\Gamma}{\gamma}\frac{\partial w}{\partial\phi} + \frac{\Gamma}{\gamma}(u\cos\theta - v\sin\theta) \tag{8}$$

$$2\varepsilon_{rs} = \frac{\partial w}{\partial r} + \frac{\Gamma}{\gamma}\frac{\partial u}{\partial\phi} - w\cos\theta\frac{\Gamma}{\gamma} \tag{9}$$

$$2\varepsilon_{s\theta} = \frac{1}{r}\frac{\partial w}{\partial\theta} + \frac{\Gamma}{\gamma}\frac{\partial v}{\partial\phi} + w\sin\theta\frac{\Gamma}{\gamma} \tag{10}$$

The local stress σ in linear elasticity can be also decomposed in the same spirit and must satisfy

- the local equilibrium $\mathrm{div}\,\sigma = 0$, expression of divergence is given in appendix C,
- the boundary conditions on the lips $\Sigma^\pm$ of the cracks $\sigma^\pm.e_z = 0$.

Consider the first term of Williams expansion, $u_o = \sqrt{r}\, u_w^o$

$$u_w^o = K_u(s)U_o(\theta)e_r + K_v(s)V_o(\theta)e_\theta + K_w(s)W_o(\theta)\; ez \tag{11}$$

the associated strain $\varepsilon(u_o)$ satisfies first the conditions of plane strain :

$$\varepsilon_{rr} = K_u\frac{1}{2\sqrt{r}}U_o = \varepsilon^o_{rr}, \tag{12}$$

$$\varepsilon_{\theta\theta} = K_v\frac{1}{\sqrt{r}}\frac{\partial V_o}{\partial\theta} + \frac{1}{\sqrt{r}}K_uU_o = \varepsilon^o_{\theta\theta}, \tag{13}$$

$$2\varepsilon_{r\theta} = \sqrt{r}K_u\frac{1}{r}\frac{\partial U_o}{\partial\theta} - K_v\frac{1}{2\sqrt{r}}V_o = 2\varepsilon^o_{r\theta} \tag{14}$$

anti-plane and normal components depend on the curvature

$$\begin{aligned} 2\varepsilon_{rs} &= K_w\frac{1}{2\sqrt{r}}W_o + \sqrt{r}\left(\frac{\mathrm{d}K_u}{\mathrm{d}\phi}U_o - K_wW_o\cos\theta\right)\frac{\Gamma}{\gamma} \\ &= 2\varepsilon^o_{rs} + 2\Gamma\varepsilon^1_{rs}, \\ 2\varepsilon_{s\theta} &= K_w\frac{1}{\sqrt{r}}\frac{\partial W_o}{\partial\theta} + \sqrt{r}\left(\frac{\mathrm{d}K_w}{\mathrm{d}\phi}V_o + K_wW_o\sin\theta\right)\frac{\Gamma}{\gamma} \\ &= 2\varepsilon^o_{s\theta} + 2\Gamma\varepsilon^1_{s\theta} \end{aligned}$$

and

$$\varepsilon_{ss} = \sqrt{r}\left(\frac{\mathrm{d}K_w}{\mathrm{d}\phi}W_o + K_uU_o\cos\theta - K_vV_o\sin\theta\right)\frac{\Gamma}{\gamma} = \Gamma\varepsilon^1_{ss} \tag{15}$$

We recover the singular part of the strain ε^o, and a additional term ε^1 which is proportional to $\sqrt{r}$.

The same decomposition exists for the stress,

$$\sigma = \sigma_o + \Gamma\sigma^1 \tag{16}$$

with order 0 in Γ singular as $1/\sqrt{r}$, taking these values in the equilibrium equation we recover that the dominant term when r vanishes corresponds exactly to the plane-strain and anti-plane strain solutions.

However, for mode I, the singular field σ_o is not in equilibrium, in a torus of finite radius ; the equilibrium equations depends on the curvature of the crack front:

$$\operatorname{div}\sigma_o = \begin{array}{l} \dfrac{K_{\mathrm{I}}}{4(2\nu-1)}\dfrac{\Gamma}{\sqrt{r}}\Big((8\nu-3)\cos\dfrac{3\theta}{2}-\cos\dfrac{\theta}{2}\Big)e_r \\ -\dfrac{K_{\mathrm{I}}}{4(2\nu-1)}\dfrac{\Gamma}{\sqrt{r}}\Big((8\nu-1)\sin\dfrac{3\theta}{2}-\sin\dfrac{\theta}{2}\Big)e_\theta \end{array} + \ldots$$

The same thing holds in mode II, and III.

3 Invariants integral J, G_θ

It is well known that the Eshelby's momentum tensor Ψ satisfies a conservation law in the case of homogeneous material

$$\operatorname{div}\Psi^T = 0, \quad \Psi = W(\varepsilon)\mathbb{I} - \sigma.\nabla u \tag{17}$$

where $\mathbb{I}$ is the identity, $W(\varepsilon)$ is the strain energy and $\sigma = \dfrac{\partial W}{\partial \varepsilon}$ is divergence free, then

$$\int_\Omega \operatorname{div}\Psi^T.\Theta \, \mathrm{d}\Omega = 0, \quad \forall\Theta \tag{18}$$

Consider Ω a section of the torus between $s, s+ds$, with external radius R_S and internal radius R_I, as presented in figure 3. Let A_i, (reps. A_s) be the disk with center M_o and radius R_I (resp. R_S).

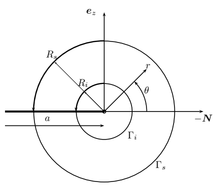

Fig. 3 A section of the torus

Chose a particular Θ : $\Theta = -\Theta(r,\theta)N$

$$\begin{aligned} 0 &= \int_\Omega \operatorname{div}\Psi^T.N\Theta \, \mathrm{d}\Omega \\ &= \int_s^{s+ds} \Big(\int_{A_s\setminus A_i} \operatorname{div}\Psi^T \, \Theta(r,\theta)\, \gamma \mathrm{d}A\Big).N(s)\, \mathrm{d}s \end{aligned}$$

therefore we obtain ($\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$)

$$0 = \int_{A_s\setminus A_i} \operatorname{div}\Psi^T.N(s)\, \Theta(r,\theta)\, \gamma \mathrm{d}A \tag{19}$$

By integration by part with respect to (r,θ) we have

$$\begin{aligned} 0 = &\int_{\Gamma_{R_S}} n.\Psi.N\, \Theta\, \gamma R_S \mathrm{d}\theta - \int_{\Gamma_{R_I}} n.\Psi.N\, \Theta\, \gamma R_I \mathrm{d}\theta \\ &- \int_{A_s\setminus A_i} N.\frac{\partial\Psi}{\partial\phi}.e_s\, \Theta\, \Gamma\gamma \mathrm{d}A + \int_{A_s\setminus A_i} \Psi : \nabla\Theta\gamma\, \mathrm{d}A \end{aligned} \tag{20}$$

Define $J_{\Gamma_r} = \int_{\Gamma_r} n.\Psi.N\, \Theta\, \gamma r\mathrm{d}\theta$, where Γ_r is the circle of radius r, the J-integral is exactly:

$$J = \lim_{r\to 0} J_{\Gamma_r} \tag{21}$$

we have whith the help of (20) the property

$$J = J_{\Gamma_{R_S}} + \lim_{R_I\to 0}\int_{A_s\setminus A_i} (\Psi : \nabla\Theta - N.\frac{\partial\Psi}{\partial\phi}.e_s\, \Gamma\, \Theta)\gamma \mathrm{d}A \tag{22}$$

then

$$J = J_{\Gamma_{R_S}} + \int_{A_s} (\Psi : \nabla\Theta - N.\frac{\partial\Psi}{\partial\phi}.e_s\, \Gamma\, \Theta)\gamma \mathrm{d}A \tag{23}$$

Integral J. For the particular choice $\Theta(r,\theta) = -N$ we obtain

$$J = -\int_{\Gamma_{R_S}} n.\Psi.N\, \gamma\, R_S \mathrm{d}\theta - \int_{A_s} (e_s.\Psi + N.\frac{\partial\Psi}{\partial\phi}).e_s\, \Gamma\, \gamma\, \mathrm{d}A \tag{24}$$

Integral G_Θ. For $\Theta = -\Theta(r,\theta)N$ with

$$\Theta(r,\theta) = \begin{cases} (R_S - r)/(R_S - R_I) & ,R_I \le r \le R_S \\ 1 & ,0 \le r \le R_I \end{cases} \tag{25}$$

$$G_\Theta = \int_{A_s} (\Psi : \nabla\Theta - \Theta N.\frac{\partial\Psi}{\partial\phi}.e_s\Gamma)\mathrm{d}A \tag{26}$$

This expression is used essentially in computational estimation on characteristic of fracture.

Evaluation of J-integrals on a torus. Consider the theoretical value J_{th} given by the Irwin's formula :

$$J_{th} = \frac{1-\nu^2}{E}(\mathrm{K}_{\mathrm{I}}^2 + \mathrm{K}_{\mathrm{II}}^2) + \frac{1}{2\mu}\mathrm{K}_{\mathrm{III}}^2 \tag{27}$$

this value is compared with the approximation of J or G_Θ when the displacement is the first singular terms of Williams expansion

$$u_o = \sqrt{r}\Big(\frac{\mathrm{K}_{\mathrm{I}}}{2\mu}u_o^{\mathrm{I}}(\theta) + \frac{\mathrm{K}_{\mathrm{II}}}{2\mu}u_o^{\mathrm{II}}(\theta) + \frac{\mathrm{K}_{\mathrm{III}}}{2\mu}u_o^{\mathrm{III}}(\theta)\Big) \tag{28}$$

where

$$\begin{aligned} u_o^{\mathrm{I}}(\theta) = &\frac{1}{2}(-\cos\frac{3\theta}{2} + (5-8\nu)\cos\frac{\theta}{2})e_r \\ &+ \frac{1}{2}(\sin\frac{3\theta}{2} + (8\nu-7)\sin\frac{\theta}{2})e_\theta \end{aligned} \tag{29}$$

$$u_o^{\mathrm{II}}(\theta) = \frac{1}{2}(3\sin\frac{3\theta}{2} + (8\nu - 5)\sin\frac{\theta}{2})e_r + \frac{1}{2}(3\cos\frac{3\theta}{2} + (8\nu - 7)\cos\frac{\theta}{2})e_\theta \tag{30}$$

$$u_o^{\mathrm{III}}(\theta) = \sin\frac{\theta}{2}e_z \tag{31}$$

The J-integral is evaluated for each mode of fracture, on a torus of external radius R_S

3.1 Mode I

The obtained value of J differs from the theoretical value as

$$J_o^{\mathrm{I}} = J_{th}(1 + J_{o1}^{\mathrm{I}}\eta + J_{o2}^{\mathrm{I}}\eta^2 + J_{o3}^{\mathrm{I}}\eta^3 + ...) \tag{32}$$

$$G_\Theta = J_{th}(1 + G_1^0\eta_i + G_2^0\eta_i^2 + G_3^0\eta_i^3) \tag{33}$$

with $k = \dfrac{R_S}{R_I}, \eta_i = \Gamma R_I, \eta = \Gamma R_S$ and

$$J_{o1}^{\mathrm{I}} = \frac{32\nu^2 - 32\nu + 5}{8(2\nu - 1)},\ G_1^0 = \frac{k+1}{2}J_{o1}^{\mathrm{I}} \tag{34}$$

$$J_{o2}^{\mathrm{I}} = -\frac{64\nu^2 - 64\nu + 15}{32(2\nu - 1)},\ G_2^0 = \frac{k^3 - 1}{3(k-1)}J_{o2}^{\mathrm{I}} \tag{35}$$

$$J_{o3}^{\mathrm{I}} = \frac{64\nu^2 - 56\nu + 11}{96(2\nu - 1)},\ G_3^0 = \frac{k^4 - 1}{4(k-1)}J_{o3}^{\mathrm{I}} \tag{36}$$

3.2 Mode II

In the same spirit for mode II,

$$\frac{J_o^{\mathrm{II}}}{J_{th}} = 1 + \eta\frac{32\nu^2 - 32\nu + 9}{8(2\nu - 1)} + \eta^2\frac{64\nu^2 - 128\nu + 63}{32(2\nu - 1)}... \tag{37}$$

3.3 Mode III

The evaluation of integral J for $w_o(r,\theta)$ is then

$$J_o^{\mathrm{III}} = J_{th}(1 - \eta - \eta^2 - \frac{\eta^3}{3} + ...) \tag{38}$$

3.4 Comments on the results of order 0

The difference with the theoretical value is essentially due to the fact that the plane and anti-plane fields are not statically admissible in the torus of radius R_S. When η or η_i tends to zero, the theoretical value is recovered, that is conformed to the fact that the singular part of the displacement is the plane-strain or anti-plane strain solution.

The fact that each terms of the expansion of G_Θ depends only of the corresponding term of J multiply by a function of $k = R_S/R_I$ is due to the choice of Θ as a function of r, then the integration on the domain can be decomposed into separate integration with respect to r and to θ.

To have a best approximation of the J integral, we propose to define displacement fields which satisfy the equilibrium equation as the best possible. The correction is issued from the plane-strain and anti-plane-strain solutions and is obtained as an asymptotic expansion with respect to the local curvature Γ of the crack front.

4 Construction of more consistent fields

To converge to the exact solution, asymptotic field is now build with respect to the equilibrium equations and boundary conditions for higher order in Γ. Consider the expansion of the displacement, where the generalized stress intensity factors K_α are considered as uniform :

$$u = \sqrt{r}\sum_\alpha \mathrm{K}_\alpha \sum_j r^j\Gamma^j u_j^\alpha(\theta) \tag{39}$$

For mode α and terms of order j in Γ, the equilibrium and the boundary conditions up to order j must be satisfied that is

$$\operatorname{div}\sigma = o(\Gamma^j), \quad \sigma.n = o(\Gamma^j) \tag{40}$$

We study successively each mode of rupture.

4.1 Mode I

Order one. As we have seen previously (2), the equilibrium at order 0 is not satisfied. In order to cancel the linear term in Γ in equilibrium equations, we consider a correction of order one :

$$u_1 = \sqrt{r}\frac{\mathrm{K_I}}{2\mu}(u_o^{\mathrm{I}} + r\Gamma u_1^{\mathrm{I}}) \tag{41}$$

where

$$u_1^{\mathrm{I}} = U_1^{\mathrm{I}}(\theta)e_r + V_1^{\mathrm{I}}(\theta)e_\theta \tag{42}$$

depends only on θ

$$U_1^{\mathrm{I}} = U_{ct}^{\mathrm{I}}\cos\frac{3\theta}{2} + U_{cu}^{\mathrm{I}}\cos\frac{\theta}{2} \tag{43}$$

$$V_1^{\mathrm{I}} = V_{st}^{\mathrm{I}}\sin\frac{3\theta}{2} + V_{su}^{\mathrm{I}}\sin\frac{\theta}{2} \tag{44}$$

With respect to equilibrium equations and boundary conditions on the lips, the constants are determined as

$$U_{ct}^{\mathrm{I}} = \frac{8\nu - 3}{8}, \quad U_{cu}^{\mathrm{I}} = \frac{128\nu^2 - 96\nu + 13}{24} \tag{45}$$

$$V_{st}^{\mathrm{I}} = -\frac{8\nu - 5}{8}, \quad V_{su}^{\mathrm{I}} = \frac{128\nu^2 - 192\nu + 55}{24} \tag{46}$$

This solution is not unique, a additional term issued form the Williams solution (39) can be considered.

The local stresses $\sigma = \sigma_0 + \Gamma \sigma_1$ for order one is

$$
\begin{aligned}
\sigma_1^{rr} &= (\nu - \frac{9}{16}) \cos\frac{3\theta}{2} - (\frac{13}{16} - \nu) \cos\frac{\theta}{2} \\
\sigma_1^{r\theta} &= \frac{7-16\nu}{16} (\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}) \\
\sigma_1^{\theta\theta} &= \frac{9-16\nu}{16} (\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}) \\
\sigma_1^{ss} &= -\frac{1}{2}\Big((1+\nu)\cos\frac{3\theta}{2} + (16\nu^2 - 2\nu - 5)\cos\frac{\theta}{2}\Big)
\end{aligned}
$$

The development of Leblond & Torlai (1982) is recovered.

At order 1, we obtain an approximation of J-integral at order two:

$$\frac{J_1^{\mathrm{I}}}{J_{th}} = 1 - J_{12}^{\mathrm{I}}\eta^2 - J_{13}^{\mathrm{I}}\eta^3 + ...) \tag{47}$$

$$J_{12}^{\mathrm{I}} = \frac{192\nu^2 - 72\nu + 3}{64(1-2\nu)} \tag{48}$$

$$J_{13}^{\mathrm{I}} = \frac{4096\nu^4 - 4608\nu^3 + 1072\nu^2 + 168\nu - 41}{192(2\nu - 1)} \tag{49}$$

and for G_θ

$$\frac{G_\theta}{J_{th}} = 1 + \frac{k^3-1}{3(k-1)} J_2^1 \eta_i^2 + \frac{k^4-1}{4(k-1)} J_3^1 \eta^3 \tag{50}$$

At order two. A correction of order 2 is obtained by the same way, balancing the Γ^2 terms of the equilibrium equations with a displacement $u_2^{\mathrm{I}} = U_2^{\mathrm{I}}(\theta)e_r + V_2^{\mathrm{I}}(\theta)e_\theta$ where

$$
\begin{aligned}
U_2^{\mathrm{I}} &= U_{cq}^{\mathrm{I}} \cos\frac{5\theta}{2} + U_{ct}^{\mathrm{I}} \cos\frac{3\theta}{2} + U_{cu}^{\mathrm{I}} \cos\frac{\theta}{2}, \\
V_2^{\mathrm{I}} &= V_{sq}^{\mathrm{I}} \sin\frac{5\theta}{2} + V_{st}^{\mathrm{I}} \sin\frac{3\theta}{2} + U_{su}^{\mathrm{I}} \sin\frac{\theta}{2}.
\end{aligned}
$$

The constants are given by the static conditions

$$
\begin{aligned}
U_{cq}^{\mathrm{I}} &= \tfrac{3-24\nu}{64}, & V_{sq}^{\mathrm{I}} &= \tfrac{24\nu-9}{64} \\
U_{ct}^{\mathrm{I}} &= \tfrac{-512\nu^3+64\nu^2+80\nu+53}{360}, & V_{st}^{\mathrm{I}} &= \tfrac{-512\nu^3+704\nu^2+80\nu-137}{360} \\
U_{cu}^{\mathrm{I}} &= \tfrac{67-256\nu^2}{192}, & V_{su}^{\mathrm{I}} &= \tfrac{256\nu^2-192\nu-37}{192}
\end{aligned}
$$

For this new field,

$$u_2 = \mathrm{K_I}\sqrt{r}(u_o^{\mathrm{I}} + r\Gamma u_1^{\mathrm{I}} + r^2\Gamma^2 u_2^{\mathrm{I}}) \tag{51}$$

an evaluation of J-integral up to order 3 is obtained.

$$J_2^{\mathrm{I}} = J_{th}\ (1 + \eta^3 J_{23}^{\mathrm{I}} + ...) \tag{52}$$

with

$$J_{23}^{\mathrm{I}} = \frac{6144\nu^3 - 256\nu^2 - 4224\nu + 871}{1536(2\nu - 1)} \tag{53}$$

and in a similar way

$$G = J_{th}\ (1 + \eta_i^3 J_{23}^{\mathrm{I}} \frac{k^4-1}{4(k-1)} + ...) \tag{54}$$

For a practical view point, taking account of order n correction for the displacement make a correction of order $n+1$ for J-integral.

4.2 Mode II

The displacement u_o for mode II corresponds to the plane strain singular field

$$u_o^{\mathrm{II}} = \sqrt{r}\mathrm{K_{II}}\Big(u_o^{\mathrm{II}}(\theta)e_r + v_o^{\mathrm{II}}(\theta)e_\theta\Big) \tag{55}$$

with

$$
\begin{aligned}
u_o^{\mathrm{II}} &= 3\sin\frac{3\theta}{2} + (8\nu - 5)\sin\frac{\theta}{2}, \\
v_o^{\mathrm{II}} &= 3\cos\frac{3\theta}{2} + (8\nu - 7)\sin\frac{\theta}{2}
\end{aligned}
$$

Then the value of J becomes

$$J_o^{\mathrm{II}} = J_{th}(1 + \eta J_{o1}^{\mathrm{II}} + \eta^2 J_{o2}^{\mathrm{II}} + ...) \tag{56}$$

with

$$J_{o1}^{\mathrm{II}} = \frac{32\nu^2 - 32\nu + 9}{8(2\nu - 1)}, \quad J_{o2}^{\mathrm{II}} = \frac{64\nu^2 - 128\nu + 63}{32(2\nu - 1)}$$

Ordre one. The correction of equilibrium at the order 1 gives

$$u_1^{\mathrm{II}} = U_{st}^{\mathrm{II}} \sin\frac{3\theta}{2} + U_{su}^{\mathrm{II}} \sin\frac{\theta}{2}, \quad v_1^{\mathrm{II}} = V_{ct}^{\mathrm{II}} \cos\frac{3\theta}{2} + V_{su}^{\mathrm{II}} \sin\frac{\theta}{2}$$

with

$$
\begin{aligned}
U_{st}^{\mathrm{II}} &= (3 - 8\nu)/4, \quad U_{su}^{\mathrm{II}} = (128\nu^2 - 96\nu - 107)/60, \\
V_{ct}^{\mathrm{II}} &= (5 - 8\nu)/4, \quad V_{cu}^{\mathrm{II}} = -(128\nu^2 - 192\nu + 79)/60
\end{aligned}
$$

The local stresses $\sigma = \sigma_0 + \Gamma \sigma_1$ for order one is

$$
\begin{aligned}
\sigma_1^{rr} &= (\nu - \frac{9}{16}) \sin\frac{3\theta}{2} - (\frac{107}{80} - \frac{7}{5}\nu) \sin\frac{\theta}{2} \\
\sigma_1^{r\theta} &= -\frac{1}{80}\Big((80\nu - 35)\cos\frac{3\theta}{2} + (31 - 16\nu)\cos\frac{\theta}{2}\Big) \\
\sigma_1^{\theta\theta} &= -\frac{9-16\nu}{80} (\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}) \\
\sigma_1^{ss} &= -\frac{1}{10}\Big(5(1+\nu)\sin\frac{3\theta}{2} + (16\nu^2 + 14\nu - 45)\sin\frac{\theta}{2}\Big)
\end{aligned}
$$

The development of Leblond & Torlai (1982) is recovered for mode II at order 1.

The value of J-integral is now

$$J_1^{\mathrm{II}} = J_{th}\ (1 + J_{12}^{\mathrm{II}}\eta^2 + ...) \tag{57}$$

with

$$J_{12}^{\mathrm{II}} = -\frac{448\nu^2 - 5576\nu + 943}{320(1 - 2\nu)} \tag{58}$$

Order two. The correction of equilibrium at the order 2 gives

$$u_2^{\mathrm{II}} = U_2^{\mathrm{II}} e_r + V_2^{\mathrm{II}} e_\theta \tag{59}$$

$$U_2^{\mathrm{II}}(\theta) = U_{su}^{\mathrm{II}} \sin\frac{\theta}{2} + U_{sq}^{\mathrm{II}} \sin\frac{5\theta}{2} + U_{st}^{\mathrm{II}} \sin\frac{3\theta}{2}$$

$$V_2^{\mathrm{II}}(\theta) = V_{cu}^{\mathrm{II}} \cos\frac{\theta}{2} + V_{cq}^{\mathrm{II}} \cos\frac{5\theta}{2} + V_{ct}^{\mathrm{II}} \cos\frac{3\theta}{2}$$

with

$$V_{cu}^{\mathrm{II}} = (256\nu^2 + 320\nu - 581)/960$$
$$U_{su}^{\mathrm{II}} = (256\nu^2 + 512\nu - 803)/960$$
$$V_{cq}^{\mathrm{II}} = (24\nu - 9)/64$$
$$U_{sq}^{\mathrm{II}} = (24\nu - 3)/64$$
$$V_{ct}^{\mathrm{II}} = (512\nu^3 + 320\nu^2 - 3056\nu + 2749)/4200$$
$$U_{st}^{\mathrm{II}} = -(512\nu^3 + 960\nu^2 + 464\nu - 3161)/4200$$

Then the value of J-integral is

$$J_2^{\mathrm{II}} = J_{th}(1 + \eta^3 \frac{43008\nu^3 + 166144\nu^2 - 31550\nu + 139513}{53560(2\nu - 1)})$$

4.3 Mode III

The field $u_o = \mathrm{K}_{\mathrm{III}}\sqrt{r}\, W_o(\theta) e_z$ gives the singular anti-plane shear strain.

$$W_o = \sin\frac{\theta}{2}$$
$$J_o^{\mathrm{III}} = \frac{1}{2\mu} K_{III}^2\, (1 + J_{o1}^{\mathrm{III}}\eta + J_{o2}^{\mathrm{III}}\eta^2 + ..)$$
$$J_{o1}^{\mathrm{III}} = -1, \quad J_{o2}^{\mathrm{III}} = -1, \ldots$$

Ordre one. At the order one $u_1 = \mathrm{K}_{\mathrm{III}}\sqrt{r}(W_o + r\Gamma W_1) e_z$:

$$W_1 = \frac{1}{4}\sin\frac{\theta}{2}$$
$$J_1^{\mathrm{III}} = J_{th}(1 + J_{12}^{\mathrm{III}}\eta^2 + \ldots$$
$$J_{12}^{\mathrm{III}} = -\frac{3}{2}$$

The local stress σ_1 is

$$\sigma_1^{rs} = \frac{7}{8}\sin\frac{\theta}{2} \tag{60}$$

$$\sigma_1^{s\theta} = \frac{5}{8}\cos\frac{\theta}{2} \tag{61}$$

Order two. The displacement is now

$$u_2 = \mathrm{K}_{\mathrm{III}}\sqrt{r}(W_o + r\Gamma W_1 + r^2\Gamma^2 W_2) \tag{62}$$

with:

$$W_2 = \frac{1}{6}\sin(\theta/2) - \frac{3}{32}\sin(3\theta/2) \tag{63}$$

and

$$J_2^{\mathrm{III}} = J_{th}(1 + J_{23}^{\mathrm{III}}\eta^3 + \ldots), \quad J_{23}^{\mathrm{III}} = -\frac{5}{4} \tag{64}$$

4.4 General remark

We have build corrected fields, they are not unique. As the asymptotic expansion is defined as proposed in (39), we know that for each term in $\sqrt{r}\, r^j$ the corresponding terms u_j^w of the Williams expansion satisfies the equilibrium, and a correction of this field with respect to Γ is needed at order $\sqrt{r}\, r^{j+1}$.

5 Separation of the modes of rupture

To separate the three modes of rupture, the bilinear form associated to J or G_θ in terms of displacement is now introduced. Consider two displacements U, V which are admissible for the local problem of elasticity inside the torus almost locally, then

$$J(U+V) = J(U) + J(V) + 2\mathbb{J}(U,V) \tag{65}$$

where $\mathbb{J}$ is the bilinear form associated to the quadratic form J. It is easy to show now that

$$\mathbb{J}(U,V) = \frac{1}{4}\Big(J(U+V) - J(U-V)\Big) \tag{66}$$

this provides an extension of the biliear form proposed by Chen & Shield (1977).

To extract, the local K_α it is possible to use a test field

$$V = \sum_\alpha \mathrm{K}_\alpha^* \sqrt{r} \sum_{j=0}^{j=n} r^j \Gamma^j u_j^\alpha \tag{67}$$

where u_j^α are given as in preceding sections.

It can be noticed that for $n = 1$ for the bilinear form takes the value

$$\begin{aligned} 2\mu\mathbb{J}(u,V) = {} & \mathrm{K}_{\mathrm{I}}\mathrm{K}_{\mathrm{I}}^*(1-\nu)(1+\eta^2 J_{12}^{\mathrm{I}}) \\ & + \mathrm{K}_{\mathrm{II}}\mathrm{K}_{\mathrm{II}}^*(1-\nu)(1+\eta^2 J_{12}^{\mathrm{II}}) \\ & + \mathrm{K}_{\mathrm{III}}\mathrm{K}_{\mathrm{III}}^*(1+\eta^2 J_{12}^{\mathrm{III}}) + \ldots \end{aligned} \tag{68}$$

and it is used to extract the K_α along the crack front. Such an extraction is studied analytically in the next section using the particular solutions of Fabrikant (1988,1989).

6 Comparison with analytical solutions

Fabrikant gives the displacement solution of a circular crack under uniform pressure and shear.

6.1 For uniform pressure

For a circular crack under uniform pressure (Appendix.E.1), the displacement is given by

$$u_F = \sum_j r^j f(\theta) + \sqrt{r}\Big(\mathrm{K}_F u_o(\theta) + \sum_{\alpha\geq 1} r^\alpha u^F_\alpha(\theta)\Big) \tag{69}$$

where

$$u^F_1 = \mathrm{K}_F(u_1 + u_w(1-8\nu)/12) \tag{70}$$

and the value of $\mathbb{J}$

$$\begin{aligned} 2\mu\mathbb{J}(u_F, \mathrm{K}^*_\mathrm{I} u_1) &= \mathrm{K}^*_\mathrm{I}\mathrm{K}_\mathrm{I}(1-\nu)(1+J^\mathrm{I}_{12}\eta^2+...) \\ &= \mathrm{K}^*_\mathrm{I}\mathrm{K}_F(1-\nu) \end{aligned} \tag{71}$$

gives the value $\mathrm{K}_\mathrm{I} = \mathrm{K}_F$ up to order two in η.

6.2 Solution de Fabrikant Mode II-III

For a circular crack under uniform shear (Appendix.E.2):

$$\begin{aligned} u_F &= \sum_j r^j f_j(\theta) + \sqrt{r}\mathrm{K}_\mathrm{II}(\phi)\Big(u_o(\theta) + \sum_\alpha r^\alpha u^F_\alpha(\theta)) \\ &+ \sqrt{r}\mathrm{K}_\mathrm{III}(\phi) T\Big(w_o(\theta) + \sum_\alpha r^\alpha w^F_\alpha(\theta)\Big) \end{aligned} \tag{72}$$

with $\mathrm{K}_\mathrm{II}(\phi) = K_o\cos\phi/(1-\nu), \mathrm{K}_\mathrm{III}(\phi) = K_o\sin\phi$, this is conformed to the relations given in (Bui, 1975,1977). Then we can show that

$$\begin{aligned} u^F_1 &= \mathrm{K}_\mathrm{II}(\phi)\ (u_1 + u^F_w + U_1(w_o,\theta)) \\ w^F_1 &= \mathrm{K}_\mathrm{III}(\phi)\ (w_1 + w^F_w + W_1(u_o,\theta)) \end{aligned} \tag{73}$$

The William's components for $(r\sqrt{r}\Gamma)$ are

$$\begin{aligned} u_w &= 5\sin\frac{5\theta}{2} + (8\nu-3)\sin\frac{\theta}{2},\ u^F_w = (8\nu-13)u_w/120 \\ v_w &= 5\cos\frac{5\theta}{2} + (8\nu-9)\cos\frac{\theta}{2},\ v^F_w = (8\nu-13)v_w/120 \\ w_w &= \sin\frac{3\theta}{2},\ w^F_w = (2\nu-3)\ \frac{\cos\phi}{1-\nu} w_w \end{aligned}$$

The interaction terms between plane strain and anti-plane strain is evaluated as

$$\begin{aligned} U_1 &= -\mathrm{K}_\mathrm{II}(\phi)\ \sin\frac{\theta}{2}\ \frac{64\nu^2-48\nu-16}{15} \\ V_1 &= \mathrm{K}_\mathrm{II}(\phi)\ \cos\frac{\theta}{2}\ \frac{64\nu^2-48\nu-16}{15} \\ W_1 &= \frac{\mathrm{K}_\mathrm{III}(\phi)}{1-\nu}\ \sin\frac{\theta}{2} \end{aligned}$$

Separation of the modes II and III. To separate the stress intensity factor, we use the $\mathbb{J}$ bilinear forms and the proposed new test fields, then

$$\begin{aligned} (u^*, v^*) &= \sqrt{r}\mathrm{K}^*_\mathrm{II}(u^\mathrm{II}_o + r\Gamma u^\mathrm{II}_1, v^\mathrm{II}_o + r\Gamma v^\mathrm{II}_1), \\ w^* &= \mathrm{K}^*_\mathrm{III}\sqrt{r}(w_o + r\Gamma w_1) \end{aligned}$$

and

$$\begin{aligned} 2\mu\mathbb{J}(u_F, u^*) &= \mathrm{K}^*_\mathrm{II}\mathrm{K}_\mathrm{II}(1-\nu) + \mathrm{K}^*_\mathrm{III}\mathrm{K}_\mathrm{III} \\ &= \cos\phi\ K_o\ \mathrm{K}^*_\mathrm{II}(1+\eta^2 J^2_{II}) \\ &+ \sin\phi\ K_o\ \mathrm{K}^*_\mathrm{III}(1+\eta^2 J^2_{III}) \end{aligned} \tag{74}$$

The values of $\mathrm{K}_\mathrm{II}(\phi) = \dfrac{K_o\ \cos\phi}{1-\nu}$ and $\mathrm{K}_\mathrm{III}(\phi) = K_o\ \sin\phi$ are recovered (at order two) !

7 Conclusion

After the definition of generalized invariant integral J for a general curved crack, the influence of the curvature of the crack front on its value has been studied. Introduction of an asymptotic expansion of the displacement in terms of the curvature ensures that equilibrium state is described more precisely in the vicinity of the crack front. This development permits a best evaluation of the J-integral inside a finite domain, which is useful for numerical computation. The development is given at order two, which ensures that the value of the integral is correct up to order 3.

The correction of order one in curvature for the displacement gives the same correction in stresses given by Leblond & Torlai (1982).

To show the ability of the new fields to give a more precise evaluation of the stress intensity factors, analytical results are given using the local solutions of a circular crack under pressure or shear.

These results emphasize the possibility to characterize the local stress intensity factors along a planar crack front. For a computational point of view the correction depends on the parameter $\eta = R_S\Gamma$,which determines the radius R_S of domain of integration to be used with respect to the local curvature of the front of the crack in order to guarantee a given precision.

Appendises

A Derivation in the local frame

The local basis is given by e_r, e_θ, e_s for this moving frame we have the relations

$$\frac{\partial e_r}{\partial \theta} = e_\theta; \quad \frac{\partial e_r}{\partial s} = \Gamma \cos\theta e_s$$
$$\frac{\partial e_\theta}{\partial \theta} = -e_r; \quad \frac{\partial e_\theta}{\partial s} = -\Gamma \sin\theta e_s$$
$$\frac{\partial e_s}{\partial s} = \Gamma N = -\Gamma(\cos\theta e_r - \sin\theta e_\theta)$$

We can replace the derivative with respect to ds by dϕ taking account of $\frac{\mathrm{d}\phi}{\mathrm{d}s} = \Gamma$.

B Gradient of a vector

For a vector $u = ue_r + ve_\theta + we_s$ the gradient is

$$\begin{aligned}\nabla u =& \frac{\partial u}{\partial r} e_r \otimes e_r + (\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}) e_r \otimes e_\theta + (\frac{1}{\gamma}\frac{\partial u}{\partial s} - w\frac{\Gamma}{\gamma}\cos\theta) e_r \otimes e_s \\ &+ \frac{\partial v}{\partial r} e_\theta \otimes e_r + (\frac{1}{\gamma}\frac{\partial v}{\partial s} + w\frac{\Gamma}{\gamma}\sin\theta) e_\theta \otimes e_s + (\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}) e_\theta \otimes e_\theta \\ &+ \frac{\partial w}{\partial r} e_s \otimes e_r + \frac{1}{r}\frac{\partial w}{\partial \theta} e_s \otimes e_\theta \\ &+ (\frac{1}{\gamma}\frac{\partial w}{\partial s} + u\frac{\Gamma}{\gamma}\cos\theta - v\frac{\Gamma}{\gamma}\sin\theta) e_s \otimes e_s\end{aligned}$$

C Divergence of a second order tensor

$$\begin{aligned}P =& P^{rr} e_r \otimes e_r + P^{r\theta} e_r \otimes e_\theta + P^{rs} e_r \otimes e_s \\ &+ P^{\theta r} e_\theta \otimes e_r + P^{\theta\theta} e_\theta \otimes e_\theta + P^{\theta s} e_\theta \otimes e_s \\ &+ P^{sr} e_s \otimes e_r + P^{\theta s} e_s \otimes e_\theta + P^{ss} e_s \otimes e_s\end{aligned}$$

$$\begin{aligned}\mathrm{div}(P) =& (\frac{\partial P^{rr}}{\partial r} + \frac{1}{r}\frac{\partial P^{r\theta}}{\partial \theta} + \frac{P^{rr} - P^{\theta\theta}}{r}) e_r \\ &+ (\frac{1}{\gamma}\frac{\partial P^{rs}}{\partial s} + (P^{rr} - P^{ss})\cos\theta\frac{\Gamma}{\gamma} - P^{r\theta}\sin\theta\frac{\Gamma}{\gamma}) e_r \\ &+ (\frac{\partial P^{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial P^{\theta\theta}}{\partial \theta} + (P^{r\theta} + P^{\theta r})\frac{1}{r}) e_\theta \\ &+ (\frac{1}{\gamma}\frac{\partial P^{\theta s}}{\partial s} + P^{\theta r}\cos\theta\frac{\Gamma}{\gamma} + (P^{ss} - P^{\theta\theta})\sin\theta\frac{\Gamma}{\gamma}) e_\theta \\ &+ (\frac{\partial P^{sr}}{\partial r} + \frac{1}{r}\frac{\partial P^{\theta s}}{\partial \theta} + P^{sr}\frac{1}{r}) + \frac{1}{\gamma}\frac{\partial P^{ss}}{\partial s}) e_s \\ &+ ((P^{rs} + P^{sr})\cos\theta - (P^{\theta s} + P^{\theta s})\sin\theta)\frac{\Gamma}{\gamma} e_s\end{aligned}$$

D William's asymptotic expansion for the displacement in plane strain

The displacement takes the formal form

$$u = \sqrt{r}\sum_{p\alpha} k_{p\alpha} r^p (u_p^\alpha(\theta) e_r + v_p^\alpha(\theta) e_\theta + w_p^\alpha(\theta) e_s)$$

where the components $(u_p^\alpha, v_p^\alpha, w_p^\alpha)$ depend on the mode of fracture α.

Mode I.

$$u_p(\theta) = -\frac{1}{2}[(1-2p)\cos\left((2p+3)\frac{\theta}{2}\right) + (8\nu - 5 + 2p)\cos\left((2p-1)\frac{\theta}{2}\right)]$$
$$v_p(\theta) = -\frac{1}{2}[(2p-1)\sin\left((2p+3)\frac{\theta}{2}\right) + (8\nu - 7 - 2p)\sin\left((2p-1)\frac{\theta}{2}\right)]$$

Mode II.

$$u_p(\theta) = -\frac{1}{2}[(3+2p)\sin\left((2p+3)\frac{\theta}{2}\right) + (5 - 8\nu - 2p)\sin\left((2p-1)\frac{\theta}{2}\right)]$$
$$v_p(\theta) = -\frac{1}{2}[(2p+3)\cos\left((2p+3)\frac{\theta}{2}\right) + (8\nu - 7 - 2p)\cos\left((2p-1)\frac{\theta}{2}\right)]$$

Mode III.

$$w_p = \sin(2p+1)\frac{\theta}{2}$$

E Fabrikant solution

E.1 Circular crack under uniform pressure

In the frame (e_x, e_z) the displacement is given as (Fabrikant, 1988)

$$u_x = \frac{p\rho}{2\pi\mu}\left\{(1-2\nu)\left[\frac{a\sqrt{l_2^2 - a^2}}{l_2^2} - \arcsin\frac{a}{l_2}\right] + \frac{2a^2|z|\sqrt{a^2 - l_1^2}}{l_2^2(l_2^2 - l_1^2)}\right\}$$
$$u_z = \frac{2p}{\pi\mu}\left\{2(1-\nu)\left[\frac{z}{|z|}\sqrt{a^2 - l_1^2} - z\arcsin\frac{a}{l_2}\right] + z\left[\arcsin\frac{a}{l_2} - \frac{a\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2}\right]\right\}$$

with

$$\rho = 1 + \eta\cos\theta, \quad z = \eta\sin\theta, \quad \eta = r\Gamma$$
$$l_1 = \sqrt{1 + \eta\cos\theta + \frac{\eta^2}{4}} - \frac{\eta}{2}$$
$$l_2 = \sqrt{1 + \eta\cos\theta + \frac{\eta^2}{4}} + \frac{\eta}{2}$$

The solution is developed up to order 2 in Γ

$$\begin{aligned}1 - l_1^2 &= \eta(2-\eta)\sin^2\frac{\theta}{2} \\ l_2^2 - 1 &= \eta(2+\eta)\cos^2\frac{\theta}{2} \\ A &= \sqrt{l_2^2 - 1 l_2^2} = \sqrt{\eta}\sqrt{1+\cos\theta}\,(1 - \frac{\eta}{4}(3 + 4\cos\theta)) \\ \arcsin\frac{1}{l_2} &= \frac{\pi}{2} + \sqrt{\eta}\sqrt{\cos\theta + 1}(-1 + \frac{\eta}{12}(1 + 4\cos\theta)) \\ \sqrt{A} - \arcsin\frac{1}{l_2} &= -\frac{\pi}{2} + \sqrt{\eta}\sqrt{\cos\theta + 1}\,(2 - \frac{\eta}{6}(5 + 8\cos\theta)) \\ \frac{2z}{l_2^2} &= \eta\sin\theta(2 - 2\eta(1 + \cos\theta) + \ldots)\end{aligned}$$

with these quantities we obtain

$$U_x = \frac{\pi}{2}(1 + \eta\cos\theta)(2\nu - 1) + \sqrt{\eta}U_x^0 + \eta\sqrt{\eta}U_x^1 + \ldots$$
$$U_z = \pi\eta\sin\theta(2\nu - 1) + \sqrt{\eta}U_z^0 + \eta\sqrt{\eta}U_z^1 + ..$$

the components of the expansion are obtained successively,

- at order 0 the plane strain solution

$$U_x^0 = \frac{1}{2}((5 - 8\nu)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2})$$
$$U_z^0 = \frac{1}{2}((8\nu - 7)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2})$$

- at the order 1

$$U_x^1 = \frac{1}{24}\Big(3\cos\frac{5\theta}{2} + (20 - 16\nu)\cos\frac{3\theta}{2} + 3(8\nu - 9)\cos\frac{\theta}{2}\Big)$$
$$U_z^1 = \frac{1}{24}\Big(-3(8\nu + 1)\sin\frac{\theta}{2} + 24(1 - 2\nu)\sin\frac{3\theta}{2} + 3\sin\frac{5\theta}{2}\Big)$$

These components are decomposed into two contributions, the correction for the order 0

$$u_x^1 = \frac{1}{24}\Big(3\cos\frac{5\theta}{2} + (128\nu^2 - 144\nu + 34)\cos\frac{3\theta}{2} + (72\nu - 33)\cos\frac{\theta}{2}\Big)$$
$$u_z^1 = \frac{1}{24}\Big(3\sin\frac{5\theta}{2} + (128\nu^2 - 144\nu + 34)\sin\frac{3\theta}{2} + (24\nu - 9)\sin\frac{\theta}{2}\Big)$$

and a plane strain term as Williams had proposed for $r\sqrt{r}$:

$$W_x^1 = (7 - 8\nu)\cos\frac{3\theta}{2} - 3\cos\frac{\theta}{2}$$
$$W_y^1 = (5 - 8\nu)\sin\frac{3\theta}{2} - 3\sin\frac{\theta}{2}$$

Then the order one displacement is

$$U^1 = u^1 + \frac{1 - 8\nu}{12} W^1 \quad (75)$$

E.2 Circular crack under uniform shear

For uniform shear, the decomposition is similar . On the crack an uniform shear is imposed

$$u_x = 2\frac{1-\nu}{2-\nu}\frac{\pi}{\mu}\tau_x\sqrt{a^2 - \rho^2}, u_y = 2\frac{1-\nu}{2-\nu}\frac{\pi}{\mu}\tau_y\sqrt{a^2 - \rho^2}, \quad (76)$$

ρ i the distance to the axis Oz. Then we introduce the stress intensity factors proposed by Williams ($\rho = a - r$)

$$K_{II} = 2\frac{\sqrt{2a}}{\pi}\frac{\tau_x}{2-\nu} \quad (77)$$
$$K_{III} = 2\frac{\sqrt{2a}}{\pi}\frac{1-\nu}{2-\nu}\tau_y \quad (78)$$

These quantities varies along the front of the crack, a pure mode II along axis e_x becomes a pure anti-plane shear along e_y. This is conformed to the relations given in (Bui, 1975,1977). The Fabrikant solution (Fabrikant, 1989) is written as $(A = 1/(\pi\mu(2-\nu))$

$$F(r,\theta) = (-5+4\nu)z\arcsin\frac{a}{l_2} + 4(1-\nu)\sqrt{a^2 - l_1^2}$$
$$G(r,\theta) = \frac{z\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2}$$
$$\frac{u_x}{A} = F(r,\theta)\tau_x + aG(r,\theta)\big(\tau_x + \frac{l_1^2}{l_2^2}(\tau_x\cos 2\phi + \tau_y\sin 2\phi)\big)$$
$$\frac{u_y}{A} = F(r,\theta)\tau_y + aG(r,\theta)\big(\tau_y + +\frac{l_1^2}{l_2^2}(\tau_x\sin 2\phi - \tau_y\cos 2\phi)\big)$$
$$\frac{u_z}{A} = \rho(\tau_x\cos\phi + \tau_y\sin\phi)\Big(\frac{1-2\nu}{2}(\arcsin\frac{a}{l_2} - a\frac{\sqrt{l_2^2 - a^2}}{l_2^2}) + \frac{a^2}{l_2^2}G(r,\theta)\Big)$$

Now, we consider the normal plane to the crack front, with direction $T(\phi)$. In the frame $(-N, e_z)$, in this plane we have $(U,V,W) = u\frac{\sin\phi}{1-\nu}, v\frac{\sin\phi}{1-\nu}, w\cos\phi))$

$$u = -\frac{\Gamma}{12}r\sqrt{r}\Big((16\nu - 26)\sin\frac{5\theta}{2} + (24\nu - 9)\sin\frac{3\theta}{2} + (32\nu - 7)\sin\frac{\theta}{2}\Big)$$
$$+ \sqrt{r}\Big(3\sin\frac{3\theta}{2} + (8\nu - 5)\sin\frac{\theta}{2}\Big)$$
$$- \frac{1}{2}\pi r(2-\nu)\sqrt{2\Gamma}\sin 2\theta + (1-2\nu)\frac{\pi}{\sqrt{2\Gamma}}\sin\theta$$
$$v = -\frac{\Gamma}{12}r\sqrt{r}\Big(2(8\nu - 13)\cos\frac{5\theta}{2} + 3(8\nu - 5)\cos\frac{3\theta}{2} - (32\nu - 37)\cos\frac{\theta}{2}\Big)$$
$$- \sqrt{r}\Big(3\cos\frac{3\theta}{2} + (8\nu - 7)\cos\frac{\theta}{2}\Big)$$
$$+ \frac{1}{2}\pi r\sqrt{2\Gamma}((\nu - 2)\cos 2\theta - 3(1-\nu)\cos\theta) + (1-2\nu)\frac{\pi}{\sqrt{2\Gamma}}\cos\theta$$
$$w = 8\sqrt{r}\sin\frac{\theta}{2} + r\sqrt{r}\frac{2\Gamma}{1-\nu}((3-2\nu)\sin\frac{3\theta}{2} + (2-\nu)\sin\frac{\theta}{2})$$
$$- \frac{\pi(5-4\nu)}{\sqrt{2}(1-\nu)}r\sqrt{\Gamma}\sin\theta$$

Consider the first singular terms ($\sqrt{r}$), we recognize the mode II for (u,v) and mode III for w. The pure mode III valid for $\phi = 0$ is changed in pure mode II at $\phi = \pi/2$.

The displacement is decomposed into three contributions at order 1 in Γ :

- a plane strain correction
- an anti-plane strain correction,
- a correction with coupling into plane and anti-plane solutions,
- and additional Williams contribution proportional to $r\sqrt{r}$

The plane and anti-plane corrections have the form given in section 4.

For the third contribution, the anti-plane shear gives a contribution in plane strain, this is due to the fact that the stress intensity factors are non uniform and depend on ϕ. For given w_o, the equilibrium equation in direction e_r, e_θ must be satisfied, with the boundary conditions on the lips. Solving this system we obtain with ($K_2 = \sin\phi/(1-\nu), K_3 = \cos\phi$)

$$u_1 = -K_2\, r\sqrt{r}\,\Gamma\frac{64\nu^2 - 48\nu - 16}{15}\sin\frac{\theta}{2}$$
$$v_1 = K_2\, r\sqrt{r}\,\Gamma\frac{64\nu^2 - 96\nu + 32}{15}\cos\frac{\theta}{2}$$

In a similar way, the plane strain gives a contribution normal to the plane (equilibrium equation on direction (e_z))

$$w_1 = \frac{K_3}{1-\nu}\, r\sqrt{r}\,\Gamma\sin\frac{\theta}{2}$$

The terms of Williams are given for $r\sqrt{r}$ with $n = 3, p = 1$:

- plane strain mode II
$$u_w^{II} = r\sqrt{r}\,(5\sin\frac{5\theta}{2} + (8\nu - 3)\sin\frac{\theta}{2})$$
$$v_w^{II} = r\sqrt{r}\,(5\cos\frac{5\theta}{2} + (8\nu - 9)\cos\frac{\theta}{2})$$
- anti-plane mode III
$$w_w = r\sqrt{r}\sin\frac{3\theta}{2}$$

$$u_1^w = (8\nu - 13)u_w^{II}/120 \quad (79)$$
$$v_1^w = (8\nu - 13)v_w^{II}/120 \quad (80)$$
$$w_1^w = \frac{(2\nu - 3)}{1-\nu}w_w \quad (81)$$

It can be noticed that this type of decomposition is conformed to the asymptotic expansion proposed in (Leblond & Torlai, 1982).

References

1. Williams ML (1957) On the stress distribution at the base of a stationary crack. ASME J. Appl. Mech 24: 109-114
2. Bui HD (1975) Application des potentiels élastiques à l'étude des fissures de forme arbitraire en milieu tridimensionel. C.R. Acad. Sci. Paris, 280:1157:1161
3. Bui HD (1977) An integral equations method for solving the problem of a plane crack of arbitrary shape. J. Mech. Phys. Solids, 23:29-39
4. Leblond JB (1989) Crack path in plane situation 1: General form of the expansion of the stress intensity factor. Int. J. Solids Structures 25:1131-1325
5. Eshelby JD (1951) The force on an elastic singularity. Phil. Trans. Roy. Soc. London, A 244:376-396

6. Rice JR (1968) A path independent integral and the approximate analysis of strain concentration by notches and cracks. J. Appl. Mech. 35
7. Destuynder P, Djaoua M (1981) Sur une interprétation mathématique de l'intégrale de Rice en théorie de la rupture fragile. Math. Meth. in Appl. Sci, 3:70-87
8. Chen FHK, Shield RR (1977) Conservation laws in elasticity of the J-integral type. Journal of Applied Mathematics and Physics 28:122
9. Leblond JB, Torlai O (1982) The stress field near the front of an arbitrarily shaped crack in a three-dimensional elastic body. J. of Elasticity 29:97-131
10. Yosibash Z, Shannon S, Dauge M, Costabel M. (2011) Circular edge singularities for the Laplace equation and the elasticity system in 3-D domains. Int. J. Fract, 168:31-52.
11. Fabrikant VI (1988) Green's functions for a penny-shaped crack under normal loading. Engineering Fracture Mechanics 30(1):87-104
12. Fabrikant VI (1989) Flat crack under shear loading. Acta Mechanica 78:1-31