On primary pseudo-polynomials (Around Ruzsa's Conjecture)

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Abstract

Every polynomial $P(X) \in \mathbb{Z}[X]$ satisfies the congruences $P(n+m) \equiv P(n) \mod m$ for all integers $n, m \geq 0$. An integer valued sequence $(a_n)_{n\geq 0}$ is called a pseudo-polynomial when it satisfies these congruences. Hall characterized pseudo-polynomials and proved that they are not necessarily polynomials. A long standing conjecture of Ruzsa says that a pseudo-polynomial a_n is a polynomial as soon as $\limsup_n |a_n|^{1/n} < e$. Under this growth assumption, Perelli and Zannier proved that the generating series $\sum_{n=0}^{\infty} a_n z^n$ is a G-function. A primary pseudo-polynomial is an integer valued sequence $(a_n)_{n\geq 0}$ such that $a_{n+p} \equiv a_n \mod p$ for all integers $n \geq 0$ and all prime numbers p. The same conjecture has been formulated for them, which implies Ruzsa's, and this paper revolves around this conjecture. We obtain a Hall type characterization of primary pseudo-polynomials and draw various consequences from it. We prove that any primary pseudo-polynomial with an algebraic generating series is a polynomial. We make the Perelli-Zannier Theorem effective and show that its conclusion does not necessarily hold if $\limsup_n |a_n|^{1/n} \leq e$. We prove a Pólya type result: if there exists a function F analytic in a right-half plane with not too large exponential growth (in a precise sense) and such that for all large n the primary pseudo-polynomial $a_n = F(n)$, then a_n is a polynomial. Finally, we show how to construct a non-polynomial primary pseudo-polynomial starting from any primary pseudo-polynomial generated by a G-function different of $1/(1-x)$.

1 Introduction

Given a sequence $(a_n)_{n\geq 0} \in \mathbb{C}^{\mathbb{N}}$, we define its *binomial transform* $(b_n)_{n\geq 0} \in \mathbb{C}^{\mathbb{N}}$ as

$$\forall n \geq 0, \quad b_n := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k. \tag{1.1}$$

It is well-known that $(a_n)_{n\geq 0}$ can be recovered from $(b_n)_{n\geq 0}$ by

$$\forall n \geq 0, \quad a_n = \sum_{k=0}^{n} \binom{n}{k} b_k, \tag{1.2}$$

ie, the binomial transform is "almost" involutive. An important property of the binomial transform is that "there exists $P(X) \in \mathbb{C}[X]$ such that $a_n = P(n)$ for all n large enough" if and only if "$b_n = 0$ for all n large enough". In the sequel, we will say that a sequence a_n is "eventually in $\mathbb{A}[n]$" (where $\mathbb{A}$ is a given ring) when there exist a polynomial $P(X) \in \mathbb{A}[X]$ and an integer N such that $a_n = P(n)$ for all $n \geq N$.

A sequence $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is said to be a *pseudo-polynomial* when the following property holds: for any integers $n, k \geq 0$, $a_{n+k} \equiv a_n \mod k$. The terminology comes from the fact that for any $P(X) \in \mathbb{Z}[X]$, the sequence $(P(n))_{n\geq 0}$ is a pseudo-polynomial. Note that if $P(X) \in \mathbb{C}[X]$ is such that the sequence $(P(n))_{n\geq 0}$ is a pseudo-polynomial, then $P(X) \in \mathbb{Q}[X]$, but $P(X)$ does not necessarily belong to $\mathbb{Z}[X]$ as $P(X) = \frac{1}{2}X(X+1)$ shows. Pseudo-polynomials have long been studied for themselves, but they have also found recent applications in analytic number theory [8, 9].

We now set $d_n := \mathrm{lcm}\{1, 2, \ldots, n\}$ for $n \geq 1$ and $d_0 := 1$. We recall that $d_n = \prod_{p\leq n} p^{\lfloor \log_p(n)\rfloor} \leq 3^n$ for all n and that $d_n^{1/n} \to e$ as $n \to +\infty$ (by the Prime Number Theorem). Hall [5] proved the following fundamental property: a sequence $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is a pseudo-polynomial if and only if its binomial transform $(b_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ satisfies

$$\forall n \geq 0, \ d_n \,|\, b_n. \tag{1.3}$$

Using this characterization with $b_n := n!$ in (1.3), we see from (1.2) that the resulting pseudo-polynomial a_n is simply equal to $\lfloor n!e \rfloor$ for $n \geq 1$ (but not for $n = 0$), which is obviously not a polynomial. With $b_n := d_n$ in (1.3), we obtain from (1.2) another pseudo-polynomial of slower growth $a_n := \sum_{k=0}^{n} \binom{n}{k} d_n \leq 4^n$; since $a_n \geq 2^n$ for all $n \geq 0$, this is not a polynomial either.

The search of minimal growth conditions that can be attained by non-polynomial pseudo-polynomials has been the subject of many papers. Hall [5] and Ruzsa [16] independently proved that if

$$\limsup_{n\to+\infty} |a_n|^{1/n} < e - 1 \tag{1.4}$$

then a_n is eventually in $\mathbb{Q}[n]$. Ruzsa proposed the following

Conjecture 1 (Ruzsa)**.** *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a pseudo-polynomial such that*

$$\limsup_{n\to+\infty} |a_n|^{1/n} < e. \tag{1.5}$$

Then a_n is eventually in $\mathbb{Q}[n]$.

Using his characterization (1.3), Hall [5, p. 76] sketched an inductive construction of a non-polynomial pseudo-polynomial $(a_n)_{n\geq 0}$ such that $\limsup_{n\to+\infty} |a_n|^{1/n} \leq e$, showing that the upper bound $< e$ is best possible in Ruzsa's conjecture. Perelli and Zannier [13] then proved a highly non-trivial property: under the growth condition (1.5) in Ruzsa's Conjecture, the pseudo-polynomial sequence $(a_n)_{n\geq 0}$ satisfies a linear recurrence with polynomials coefficients; see (1.9) below. In other words, the generating function $f_a(x) := \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]]$ satisfies a linear differential equation with coefficients in $\mathbb{Z}[x]$.

Hence, f_a is a G-function ([1]). Perelli and Zannier also proved a form of Ruzsa's Conjecture 1 under a stronger assumption than (1.5), ie with e replaced by $e^{0.66}$.

In fact, many results towards Ruzsa's conjecture have been proven for sequences we shall call *primary pseudo-polynomial* (for lack of better terminology).

Definition 1. *A sequence $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is said to be a primary pseudo-polynomial when the following property holds: for any integer $n \geq 0$ and any prime number p, $a_{n+p} \equiv a_n \mod p$.*

The set of primary pseudo-polynomials is a ring for the term by term sum and product of sequences in $\mathbb{Z}^{\mathbb{N}}$, with the null and unit sequences defined with all terms equal to 0 and all terms equal to 1 respectively. A pseudo-polynomial is a primary pseudo-polynomial but the converse is false (see the comments following Theorem 1 below). Many authors delt with the following conjecture, the truth of which would imply that of Ruzsa.

Conjecture 2. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo-polynomial such that*

$$\limsup_{n\to+\infty} |a_n|^{1/n} < e. \tag{1.6}$$

Then a_n is eventually in $\mathbb{Q}[n]$.

As in the case of pseudo-polynomials, $< e$ cannot be replaced by $\leq e$, and we refer again to the comments following Theorem 1 for a proof of this. In fact, the above quoted results of Perelli and Zannier hold more generally for primary pseudo-polynomials, and Zannier [19] was even able to replace $e^{0.66}$ by $e^{0.75}$.

In this paper, we are interested in the properties of primary pseudo-polynomials and of their generating functions. We now present our four main results, make comments about their significance and mention further open problems.

• ***A Hall type characterization of primary pseudo-polynomials.*** We shall first prove an analogue (ie Eq. (1.7) below) of Hall's characterization for pseudo-polynomials and deduce some consequences of it. We set $P_n := \prod_{p\leq n} p$ for $n \geq 2$ and $P_0 = P_1 := 1$, where the product is over prime numbers. By the Prime Number Theorem, $P_n^{1/n} \to e$ as $n \to +\infty$.

Theorem 1. *(i) A sequence $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is primary pseudo-polynomial if and only if its binomial transform $(b_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is such that*

$$\forall n \geq 0,\ P_n \,|\, b_n. \tag{1.7}$$

[1] A power series $\sum_{n=0}^{\infty} a_n x^n \in \overline{\mathbb{Q}}[[x]]$ is said to be a G-function when it is solution of a non-zero linear differential equation over $\overline{\mathbb{Q}}(x)$ (D-finiteness), and the maximum of the modulus of all the Galoisian conjuguates of $a_0, \ldots, a_n$ as well as the positive denominator of $a_0, \ldots, a_n$ are both bounded for all $n \geq 0$ by C^{n+1}, for some $C \geq 1$. For instance, any D-finite series in $\mathbb{Z}[[x]]$ with positive radius of convergence is a G-function. A power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^x \in \overline{\mathbb{Q}}[[x]]$ is said to be an E-function when $\sum_{n=0}^{\infty} a_n x^n$ is a G-function. See [1, 17] for the properties satisfied by these functions.

(*ii*) *Given a primary pseudo-polynomial* $(a_n)_{n\geq 0}$, *if there is no* $Q(X) \in \mathbb{Q}[X]$ *such that* $a_n = Q(n)$ *for all* n *large enough, then*

$$\liminf_{n\to+\infty} |b_n|^{1/n} \geq e.$$

(*iii*) *If a primary pseudo-polynomial* $(a_n)_{n\geq 0}$ *is such that* $\limsup_{n\to+\infty} |a_n|^{1/n} < e-1$, *then* a_n *is eventually a polynomial.*

(*iv*) *Given any function* $\varphi : \mathbb{N} \to \mathbb{R}$ *with* $\varphi(0) = 1$, *there exists a non-polynomial primary pseudo-polynomial* $(A_n)_{n\geq 0}$ *such that* $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$ *for all* $n \in \mathbb{N}$.

We recall that $d_n = \prod_{p\leq n} p^{\lfloor \log_p(n) \rfloor}$. Hence, for all $n \geq 0$, P_n divides d_n, but obviously d_n divides P_n for no $n \geq 4$. Choosing $b_n := P_n$ in (1.7), the resulting sequence in (1.2) $a_n := \sum_{k=0}^{n} \binom{n}{k} P_k$ is a primary pseudo-polynomial, but not a pseudo-polynomial because it is does not satisfy Hall's criterion (1.3). Under the assumption in (*ii*), if we also assume that b_n is eventually of the same sign, then $\liminf_{n\to+\infty} |a_n|^{1/n} \geq e+1$ because $a_n = \sum_{k=0}^{n} \binom{n}{k} b_k$. Consequently, any putative counter-example $(a_n)_{n\geq 0}$ to Conjecture 2 must be such that its binomial transform $(b_n)_{n\geq 0}$ changes sign infinitely often. A similar remark applies to Ruzsa's Conjecture 1.

Assertion (*iii*) is the analogue of the Hall-Ruzsa result recalled at Eq. (1.4).

In (*iv*), given φ, the existence of sequence $(A_n)_{n\geq 0}$ is proved constructively by an inductive process. An important consequence of (*iv*) is the existence of a non-polynomial primary pseudo-polynomial of any growth $\varphi(n) > 0$ provided $\liminf_n \varphi(n)^{1/n} \geq e$. In particular, with $\varphi(n) = P_n$, we deduce that $< e$ cannot be replaced by $\leq e$ on the right-hand side of (1.6) in Conjecture 2.

• ***Primary pseudo-polynomials with an algebraic generating series.*** The generating functions f_a and f_b of the sequences $(a_n)_{n\geq 0}$ and its binomial transform $(b_n)_{n\geq 0}$ satisfy the relations

$$f_b(x) = \frac{1}{1+x} f_a\Big(\frac{x}{1+x}\Big), \quad f_a(x) = \frac{1}{1-x} f_b\Big(\frac{x}{1-x}\Big). \tag{1.8}$$

In particular $f_a(x)$ is algebraic over $\mathbb{Q}(x)$ if and only if $f_b(x)$ is algebraic over $\mathbb{Q}(x)$.

Theorem 2. *Let* $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ *be a primary pseudo-polynomial, and let* $(b_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ *be its binomial transform.*

(*i*) *If* $f_a(x)$ *is algebraic over* $\mathbb{Q}(x)$, *then* a_n *is eventually in* $\mathbb{Q}[n]$ *(and thus* $f_a(x) \in \mathbb{Q}(x)$*).*

(*ii*) *If* $f_b(x)$ *is algebraic over* $\mathbb{Q}(x)$, *then* $f_b(x)$ *is in* $\mathbb{Z}[x]$.

Theorem 2 implies in particular that the Taylor coefficients of an irrational algebraic function over $\mathbb{Q}(x)$ do not define a primary pseudo-polynomial, and *a fortiori* not a pseudo-polynomial either. It also implies that if there exists a counter example to Conjecture 2 or to Ruzsa's Conjecture 1, then its generating function $f_a(x)$ is transcendental over $\mathbb{C}(x)$. A slight generalization of Theorem 2(*i*) is presented in §3.

By definition, the diagonal of a multivariate power series $\sum_{n_1,\dots,n_k\geq 0} u_{n_1,\dots,n_k} z_1^{n_1}\cdots z_k^{n_k}$ is the univariate power series $\sum_{n=0}^{\infty} u_{n,n,\dots,n} z^n$. A classical result of Furstenberg [4] says that algebraic series coincide with diagonals of rational functions in two variables. It would be very interesting to know if the conclusions of Theorem 2 also hold when f_a and f_b are assumed to be diagonals of rational functions of an arbitrary (but fixed) number of variables (which they are, or not, simultaneously). Moreover, diagonal of rational functions are globally bounded G-functions in the sense of Christol, who conjectured that the converse holds (see [2]). In particular, G-functions with integer coefficients are globally bounded. Therefore given the Perelli-Zannier Theorem quoted just below, Christol's Conjecture and the above putative generalization of Theorem 2 would together imply Conjecture 2 and Ruzsa's Conjecture 1. See also related comments in [19, pp. 392–393].

• ***An effective version of a result of Perelli and Zannier.*** In [13], Perelli and Zannier sketched the proof of the following result.

Theorem (Perelli-Zannier)**.** *Let $(a_n)_{n\geq 0}$ be a primary pseudo-polynomial such that there exist $c>0$ and $1<\delta<e$ such that $|a_n|\leq c\delta^n$ for all $n\geq 0$. Then there exist an integer $S\geq 0$ and $S+1$ polynomials $p_0(X),\dots,p_S(X)\in\mathbb{Z}[X]$ not all zero such that*

$$\sum_{j=0}^{S} p_j(n)a_{n+j}=0. \tag{1.9}$$

In other words, $f_a(x)$ is D-finite, and even a G-function.

Perelli and Zannier mentioned that it would be possible to provide upper bounds for S and the degree/height of the $p_j(X)$ in terms of δ, but they did not write them down. We make more precise their theorem as follows, where given $Q(X)=\sum_j q_jX^j\in\mathbb{C}[X]$, we set $H(Q):=\max_j|q_j|$.

Theorem 3. *In the conditions and notations of the Perelli-Zannier Theorem, for any $\delta\in(1,e)$, there exists an effectively computable constant $H(\delta)\geq 1$ such that a non-trivial linear recurrence for $(a_n)_{n\geq 0}$ as in* (1.9) *holds with*

$$\begin{cases}\max_j\deg(p_j)\leq\max\left(0,\left\lceil\frac{5\log(\delta)-1}{1-\log(\delta)}\right\rceil\right),\\ \max_j H(p_j)\leq H(\delta),\\ S\leq\log\big(H(\delta)\big)/\log(\delta).\end{cases} \tag{1.10}$$

Moreover, the Perelli-Zannier Theorem is best possible in the sense that its conclusion does not necessarily hold if $\delta=e$.

To prove the final statement in Theorem 3, we take $\varphi(n):=P_n$ in Theorem 1(iv): we obtain a non-polynomial primary pseudo-polynomial A_n such that $|A_n|^{1/n}\to e$. Hence

$(A_n)_{n\geq 0}$ does not satisfy a non-zero linear recurrence with coefficients in $\mathbb{Q}[n]$. ([2])

Lower and upper bounds for the function $H(\delta)$ are given in (4.8) and (4.10) respectively in §4.3. Our bound for $\max\deg(p_j)$ in (1.10) is obviously not optimal; in fact, at the cost of more complicated computations, Perelli-Zannier [13] and then Zannier [19] obtained better bounds when $\delta \leq e^{0.66}$ and $\delta \leq e^{0.75}$.

The classification of primary pseudo-polynomials with a D-finite generating series is an open problem. As shown by the above example $(A_n)_{n\geq 0}$, Theorem 1 rules out the possibility that *every* primary pseudo-polynomial satisfies a linear recurrence with coefficients in $\mathbb{Q}[n]$. Another example is the primary pseudo-polynomial $D_n := \sum_{k=0}^{n}\binom{n}{k}P_k$: it cannot satisfy such a linear recurrence because otherwise $P_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}D_k$ would satisfy one as well, which is not possible because $P_n^{1/n} \to e$. Since $P_n \geq 0$ and $P_n = e^{n+o(n)}$, a simple analytic argument shows that $D_n = (e+1)^{n+o(n)}$. More specifically, it would be interesting to know if there is any non-polynomial primary pseudo-polynomial $(a_n)_{n\geq 0}$ such that $f_a(x)$ is a G-function. There exist primary pseudo-polynomials $(a_n)_{n\geq 0}$ such that f_a is D-finite but is not a G-function. For instance, $e_n := \lfloor (n+1)!e \rfloor = \sum_{k=0}^{n+1}\binom{n+1}{k}k!$ $(n \geq 0)$ is a primary pseudo-polynomial by Theorem 1, and for all $n \geq 0$, $e_{n+2} = (n+4)e_{n+1} - (n+2)e_n$ $(e_0 = 2, e_1 = 5)$, so that $\sum_{n=0}^{\infty} e_n x^n$ is D-finite. A method to obtain further examples is presented at the end of §6.

• ***A Pólya type result for primary pseudo-polynomials.*** Perelli and Zannier also proved in [12] that if a (primary) pseudo-polynomial $a_n = F(n)$ for some entire function F such that $\limsup_{R\to\infty} \frac{1}{R}\log\max_{|x|=R}|F(x)| < \log(e+1)$, then a_n is in $\mathbb{Q}[n]$. We prove here a result of a similar flavor with a different analyticity condition.

Theorem 4. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo-polynomial. Let us assume that there exists $F(x)$ analytic in a right-half plane $\Re(x) > u$ such that $a_n = F(n)$ for all $n > u$, and $c > 0, 0 < \rho < \log(2\sqrt{e})$ such that*

$$\big|F(x)\big| \leq c \cdot e^{\rho\Re(x)} \tag{1.11}$$

for $\Re(x) > u$. Then $F(x)$ is in $\mathbb{Q}[x]$ and a_n is eventually in $\mathbb{Q}[n]$.

We have $\log(2\sqrt{e}) \approx 1.193$ while $\log(e+1) \approx 1.313$. In the proof, we shall obtain that a_n is eventually a polynomial before proving that F is a polynomial. Because of the Perelli-Zannier Theorem recalled before Theorem 3, the assumptions of Theorem 4 are in fact natural in the context of Ruzsa's Conjecture 1. Indeed, for any given G-function $f(x) := \sum_{n=0}^{\infty} v_n x^n \in \overline{\mathbb{Q}}[[x]]$, there exists a function $\lambda(x) := \sum_{j=1}^{p} c_j(x)\cdot e^{\rho_j x}$ for some functions $c_j(x)$ analytic in $\Re(x) > u$ and of polynomial growth (at most), and such that $v_n = \lambda(n)$ for all $n > u$; the numbers $e^{-\rho_j}$ are the finite singularities of $f(x)$ (see [14, §7.1] for details). Notice that a bound involving $e^{\rho_j\Re(x)}$ is *a priori* different of a bound involving

[2] Indeed, if a solution $(a_n)_{n\geq 0}$ of a linear recurrence with coefficients in $\mathbb{Q}[n]$ is such that $|a_n|^{1/n} \to \alpha$ finite, then α is an algebraic number. Moreover, if a sequence of *rational numbers* satisfies a non-zero linear recurrence of minimal order with coefficients in $\mathbb{C}[n]$, then these coefficients are necessarily in $\mathbb{Q}[n]$, up to a common non-zero multiplicative constant. Hence $(A_n)_{n\geq 0}$, $(d_n)_{n\geq 0}$ and $(P_n)_{n\geq 0}$ do not satisfy any non-zero linear recurrence with coefficients in $\mathbb{C}[n]$.

$|e^{\rho_j x}| = e^{\Re(\rho_j x)}$, but they are the same when $\rho_j \in \mathbb{R}$. In particular, if all the singularities of $f(x)$ are positive real numbers, then a bound as in (1.11) holds for $\lambda(x)$ for some $\rho \in \mathbb{R}$.

In Theorem 1(iv), take $\varphi(n) = \delta^n$ with $e < \delta < 2\sqrt{e}$. This yields of non-polynomial primary pseudo-polynomial $(a_n)_{n\geq 0}$ such that $a_n = \delta^{n+o(n)}$ as $n \to +\infty$. Theorem 4 thus implies that there is no function $F(x)$ analytic in a right-half plane on which (1.11) holds, and such that $a_n = F(n)$ for all large n.

Perelli-Zannier's result and Theorem 4 are similar to Pólya's celebrated theorem: if an entire function $F(x)$ is such that $\limsup_{R\to+\infty} \frac{1}{R} \max_{|x|=R} |F(x)| < \log(2)$ and $F(\mathbb{N}) \subset \mathbb{Z}$, then $F(x)$ is a polynomial. See [18] for a recent survey on Pólya type results, where a connection with Rusza's Conjecture 1 is also mentioned.

Theorems 1, 2, 3 and 4 are proved in §2, §3, §4 and §5 respectively. In §6, we present a method to construct a non-polynomial primary pseudo-polynomial starting from every primary pseudo-polynomial generated by a G-function (Theorem 6).

2 Proof of Theorem 1

(i) Let $(a_n)_{n\geq 0}$ be a sequence of integers and consider its binomial transform $(b_n)_{n\geq 0}$.

Assume that for every non-negative integer n, P_n divides b_n. Let p be a fixed prime number. Hence, for every integer $n \geq p$, p divides b_n. For every non-negative integer n, it yields

$$\begin{aligned} a_{n+p} &= \sum_{k=0}^{n+p} \binom{n+p}{k} b_k \\ &\equiv \sum_{k=0}^{p-1} \binom{n+p}{k} b_k \mod p \\ &\equiv \sum_{k=0}^{p-1} \binom{n}{k} b_k \mod p \\ &\equiv a_n \mod p, \end{aligned}$$

where we used Lucas' congruence for binomial coefficients: for every u, v in $\{0, \dots, p-1\}$ and every non-negative integers m and ℓ, we have

$$\binom{u+mp}{v+\ell p} \equiv \binom{u}{v}\binom{m}{\ell} \mod p.$$

It follows that $(a_n)_{n\geq 0}$ is a primary pseudo-polynomial.

Conversely, assume that $(a_n)_{n\geq 0}$ is a primary pseudo-polynomial. Let p be a prime number and $n \geq p$ be an integer. It suffices to show that p divides b_n.

Write $n = v + mp$ with v in $\{0, \ldots, p-1\}$ and $m \geq 1$. We obtain that

$$\begin{aligned}
&b_{v+mp} \\
&= \sum_{k=0}^{v+mp} (-1)^{v+mp-k} \binom{v+mp}{k} a_k \\
&= \sum_{u=0}^{p-1} \sum_{\ell=0}^{m-1} (-1)^{v-u}(-1)^{(m-\ell)p} \binom{v+mp}{u+\ell p} a_{u+\ell p} + \sum_{u=0}^{v} (-1)^{v-u} \binom{v+mp}{u+mp} a_{u+mp} \\
&\equiv \sum_{u=0}^{p-1} \sum_{\ell=0}^{m-1} (-1)^{v-u}(-1)^{(m-\ell)p} \binom{v}{u} \binom{m}{\ell} a_u + \sum_{u=0}^{v} (-1)^{v-u} \binom{v}{u} a_u \mod p \\
&\equiv \sum_{u=0}^{v} (-1)^{v-u} \binom{v}{u} a_u \sum_{\ell=0}^{m} (-1)^{(m-\ell)p} \binom{m}{\ell} \mod p \quad \left(\binom{v}{u} = 0 \text{ for } u = v+1, \ldots, p-1\right) \\
&\equiv \sum_{u=0}^{v} (-1)^{v-u} \binom{v}{u} a_u (1 + (-1)^p)^m \mod p \\
&\equiv 0 \mod p \qquad (m \geq 1).
\end{aligned}$$

Hence p divides b_n as expected. It follows that, for every non-negative integer n, P_n divides b_n. The first equivalence in Theorem 1 is proved.

(ii) The binomial transform $(b_n)_{n \geq 0}$ is eventually 0 if, and only if there exists a polynomial $Q(X)$ in $\mathbb{Q}[X]$ such that $a_n = Q(n)$ for every large enough non-negative integer n. Hence, if the latter is false, then $(b_n)_{n\geq 0}$ is not eventually 0 and, since P_n divides b_n, it follows that

$$\liminf_{n \to +\infty} |b_n|^{1/n} \geq e$$

because $P_n^{1/n} \to e$ as $n \to +\infty$.

(iii) If the primary pseudo-polynomial a_n is not eventually a polynomial, then by (ii) above, $\liminf_{n\to+\infty} |b_n|^{1/n} \geq e$. But since $b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k$, the assumption $\limsup_{n\to+\infty} |a_n|^{1/n} < e - 1$ implies that

$$\limsup_{n\to+\infty} |b_n|^{1/n} \leq \limsup_{n\to+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} |a_k| \right)^{1/n} < e.$$

This proves that a_n is eventually a polynomial.

(iv) The following argument generalizes Hall's (sketchy) construction of a non-polynomial pseudo-polynomial with growth $\leq e^{n+o(n)}$ in [5, p. 76]. By (i), we know that any sequence of integers $(B_n)_{n\geq 0}$ such that $P_n | B_n$ defines a primary pseudo-polynomial $A_n := \sum_{k=0}^{n} \binom{n}{k} B_k$, which is not eventually a polynomial if (and only if) $B_n \neq 0$ for infinitely many n. Since A_{n-1} depends only on $B_0, B_1, \ldots, B_{n-1}$, we will recursively construct $B_n \neq 0$ and thus A_n. We have $A_0 = B_0$: choosing $B_0 = 1$, we have $\varphi(0) = A_0 = \varphi(0) + 2P_0$. Let $n \geq 0$ and let us assume that we have constructed $B_0, B_1, \ldots, B_{n-1}$ all non-zero and such that $P_k | B_k$ and $\varphi(k) \leq A_k \leq \varphi(k) + 2P_k$ for all $k \in \{0, \ldots, n-1\}$.

We want to construct an integer $B_n \neq 0$ such that $P_n | B_n$ and $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$. To do this, we first set $C_n := \sum_{k=0}^{n-1} \binom{n}{k} B_k$, so that we will have $A_n = B_n + C_n$. We now perform the euclidean division of C_n by P_n: we have $C_n = u_n P_n + v_n$ with $u_n \in \mathbb{Z}, v_n \in \mathbb{N}$ and $0 \leq v_n < P_n$. We set $B_n := w_n P_n \neq 0$ where the non-zero integer w_n is defined as follows: if $\lceil \frac{\varphi(n) - v_n}{P_n} \rceil \neq u_n$, we take

$$w_n = \left\lceil \frac{\varphi(n) - v_n}{P_n} \right\rceil - u_n,$$

while if $\lceil \frac{\varphi(n) - v_n}{P_n} \rceil = u_n$, we take $w_n = 1$. Since $A_n = (u_n + w_n) P_n + v_n$, we see that $\varphi(n) \leq A_n \leq \varphi(n) + P_n$ in the former case, while $\varphi(n) + P_n \leq A_n \leq \varphi(n) + 2P_n$ in the latter case. This finishes the recursive construction of a non-polynomial primary pseudo-polynomial $(A_n)_{n\geq 0}$ such that $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$ for all integer $n \geq 0$.

3 Proof of Theorem 2

From the proof of Theorem 1(i), we see that for any given prime number p, the assertions "for all $n \geq 0$, $a_{n+p} \equiv a_n \mod p$" and "for all $n \geq p$, $p \,|\, b_n$" are equivalent. It follows that the assertions "for all $p \in \mathcal{P}$ and all $n \geq 0$, $a_{n+p} \equiv a_n \mod p$" and "for all $p \in \mathcal{P}$ and all $n \geq p$, p divides b_n" are equivalent, where $\mathcal{P}$ is a same set of prime numbers, and this generalizes Theorem 1(i). We shall in fact prove Theorem 2 under the weaker assumption that there exists an infinite set $\mathcal{P}$ of prime numbers such that for all $p \in \mathcal{P}$ and all $n \geq 0$, $a_{n+p} \equiv a_n \mod p$.

Given $u \in \mathbb{Z}$ and a prime number p, we set $u_{|p} := u \mod p$. Given a power series $F(x) := \sum_{n=0}^{\infty} u_n x^n \in \mathbb{Z}[[x]]$, we set $F_{|p}(x) := \sum_{n=0}^{\infty} u_{n|p} x^n \in \mathbb{F}_p[[x]]$.

We shall first prove (ii) for the series $f_b(x) := \sum_{n=0}^{\infty} b_n x^n$. Let $\mathcal{P}$ denote an infinite set of prime numbers such that for all $n \geq 0$ and all $p \in \mathcal{P}$, we have $a_{n+p} \equiv a_n \mod p$. As already said, this is equivalent to the fact that for all $p \in \mathcal{P}$ and all $n \geq p$, p divides b_n. It follows in particular that for any $p \in \mathcal{P}$, $f_{b|p}(x)$ is a polynomial in $\mathbb{F}_p[x]$ of degree at most $p-1$. For simplicity, we denote by $Q_p(x)$ this polynomial, and by q_p its degree.

Let us now assume that $f_b(x)$ is algebraic over $\mathbb{Q}(x)$. If $f_b(x)$ is a constant, there is nothing else to prove. We now assume that $f_b(x)$ is not a constant so that it has degree $d \geq 1$. There exist an integer $\delta \in \{1, \ldots, d-1\}$, some integers $0 \leq j_1 < j_2 < \ldots < j_\delta \leq d-1$, and some polynomials $A_d(x), A_{j_1}(x), \ldots, A_{j_\delta}(x) \in \mathbb{Z}[x]$ all not identically zero such that

$$A_d f_b^d = \sum_{\ell=1}^{\delta} A_{j_\ell} f_b^{j_\ell} \tag{3.1}$$

in $\mathbb{Z}[[x]]$.

We fix $p \in \mathcal{P}$ such that $p > H$ where H is the maximum of the modulus of the coefficients of $A_d(x)$, $A_{j_1}(x), \ldots, A_{j_\delta}(x)$. It follows that

$$\deg(A_{d|p}) = \deg(A_d),\ \deg(A_{j_1|p}) = \deg(A_{j_1}),\ \ldots,\ \deg(A_{j_\delta|p}) = \deg(A_{j_\delta}). \tag{3.2}$$

We deduce from the reduction of (3.1) mod p that

$$A_{d|p}Q_p^d = \sum_{\ell=1}^{\delta} A_{j_\ell|p}Q_p^{j_\ell} \tag{3.3}$$

in $\mathbb{F}_p[[x]]$, and in fact in $\mathbb{F}_p[x]$ because $Q_p(x) \in \mathbb{F}_p[x]$.

Case 1). If Q_p is identically zero, this means that p divides the coefficients b_n for all $n \geq 0$.

Case 2). If Q_p is not identically zero, we deduce from (3.2) and (3.3) that

$$\begin{aligned}\deg(A_d) + dq_p &\leq \max\big(\deg(A_{j_1}) + j_1q_p, \deg(A_{j_2}) + j_2q_p, \ldots, \deg(A_{j_\delta}) + j_\delta q_p\big)\\ &\leq \max\big(\deg(A_{j_1}), \deg(A_{j_2}), \ldots, \deg(A_{j_\delta})\big) + (d-1)q_p.\end{aligned}$$

Hence

$$q_p \leq \max\big(\deg(A_{j_1}), \deg(A_{j_2}), \ldots, \deg(A_{j_\delta})\big) - \deg(A_d) =: N.$$

It follows that for any $n > N$, p divides b_n, where N is *independent* of p.

Since $p \in \mathcal{P}$ was simply assumed larger than a quantity H depending only on f_b, the conclusion of Case 1 and Case 2 is that for any $p \in \mathcal{P}$ such that $p > H$ and any $n > N$, p divides b_n. Since $\mathcal{P}$ is infinite, b_n is divisible by infinitely many primes when $n > N$. Hence $b_n = 0$ for all $n > N$ and $f_b(x) = \sum_{n=0}^{N} b_n x^n \in \mathbb{Z}[x]$, as expected.

Let us now prove (i). If $f_a(x)$ is algebraic over $\mathbb{Q}(x)$, then $f_b(x)$ as well by (1.8). Hence $f_b(x) \in \mathbb{Z}[x]$ by (ii) just proven, *i.e.* there exists an integer M such that $b_n = 0$ if $n > M$. Since, for all $n \geq 0$,

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k = \sum_{k=0}^{\min(n,M)} \binom{n}{k} b_k,$$

it follows that for all $n \geq M$, we have $a_n = Q(n)$ with $Q(X) = \sum_{k=0}^{M} \binom{X}{k} b_k \in \mathbb{Q}[X]$.

The first part of Theorem 2 can be generalized has follows.

Theorem 5. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo-polynomial. Assume there exists $m \geq 0$ such that $f_a^{(m)}(x)$ is algebraic over $\mathbb{Q}(x)$. Then a_n is eventually in $\mathbb{Q}[n]$, and thus $f_a(x) \in \mathbb{Q}(x)$.*

The proof uses the following simple lemma.

Lemma 1. *Let $R(X) \in \mathbb{Q}(X)$ be such that $R(n) \in \mathbb{Z}$ for infinitely many integers. Then, $R(X) \in \mathbb{Q}[X]$.*

Proof. We write $R = \frac{A}{B}$ with $A, B \in \mathbb{Q}[X]$. We assume that $\deg(B) \geq 1$ otherwise there is nothing to prove. There exist $U, V \in \mathbb{Q}[X]$ such that $A = UB + V$ and $\deg(V) < \deg(B)$. Let $w \in \mathbb{Z} \setminus \{0\}$ be such that $wU, wV \in \mathbb{Z}[X]$. Let $\mathcal{N}$ be the infinite set of integers n such that $R(n) \in \mathbb{Z}$; without loss of generality, we can assume that $\mathcal{N}$ contains infinitely many positive integers. For every $n \in \mathcal{N}$, we have $\frac{wV(n)}{B(n)} = wR(n) - wU(n) \in \mathbb{Z}$. But $\lim_{x\to+\infty} \frac{wV(x)}{B(x)} = 0$. Hence there exists M such that $n \in \mathcal{N}$ and $n \geq M$ imply that $\frac{wV(n)}{B(n)} = 0$. Therefore, wV has infinitely many roots: it must be the null polynomial, so that $R = U \in \mathbb{Q}[X]$. □

Proof of Theorem 5. We have

$$f_a^{(m)}(x) = \sum_{n=0}^{\infty} (n+m)(n+m-1)\cdots(n+1)a_{n+m}x^n.$$

Since $(a_n)_{n\geq 0}$ is a primary pseudo-polynomial, this is also the case of $((n+m)(n+m-1)\cdots(n+1)a_{n+m})_{n\geq 0}$ because it is a product of two primary pseudo-polynomials. Since $f_a^{(m)}(x)$ is algebraic over $\mathbb{Q}(x)$, $(n+m)(n+m-1)\cdots(n+1)a_{n+m}$ is eventually in $\mathbb{Q}[n]$ by Theorem 2. Hence a_{n+m} is eventually in $\mathbb{Q}(n)$, so that by Lemma 1, a_{n+m} is eventually in $\mathbb{Q}[n]$. This completes the proof. □

4 Proof of Theorem 3

The proof of the Perelli-Zannier Theorem is based on the following lemma proved in [13]. We shall also use it.

Lemma 2. *For $\underline{k} = (k_j)_{j\geq 0} \in (\mathbb{R}^+)^{\mathbb{N}}$, an integer $N \geq 1$, set*

$$A(N, \underline{k}) := \{(x_1, \ldots, x_N) \in \mathbb{Z}^N : |x_j| \leq k_j \quad \text{and} \quad \forall p, \forall n \leq N-p,\ x_{n+p} \equiv x_n \mod p\}.$$

Then

$$\#A(N, \underline{k}) \leq \prod_{j=1}^{N} \left(1 + \frac{2k_j}{P_{j-1}}\right).$$

Let $R \geq 1, H \geq 0, D \geq 1$ be integers. Let $q_0, \ldots, q_R \in \mathbb{Z}[X]$ with $\max_j H(q_j) \leq H$ and $\max_j \deg(q_j) \leq D-1$. Considering the coefficients of the q_j's as indeterminates, there are $(2H+1)^{RD}$ functions F of the form

$$F(n) := \sum_{j=0}^{R-1} q_j(n)a_{n+j}. \tag{4.1}$$

Such a function satisfies $|F(n)| \leq cRDH(n+1)^D\delta^{n+R}$ for all $n \geq 0$ and $F(n+p) \equiv F(n) \mod p$ for all prime number p and all $n \geq 0$. Hence for all $N \geq 1$, $(F(0), \ldots, F(N-1)) \in$

$A(N, \underline{K})$ where $K_j := cRDHj^D\delta^{j+R-1}$. Therefore, given N, if

$$\prod_{j=1}^{N}\left(1+\frac{2K_j}{P_{j-1}}\right) < (2H+1)^{RD} \tag{4.2}$$

there exists *two* different functions F_1 and F_2 of the form (4.1) such that $F_1(n) = F_2(n)$ for all $n \in \{0, 1, \dots, N-1\}$. Hence the function $G_N := F_1 - F_2$ is of the form (4.1) $G_N(n) = \sum_{j=0}^{R-1} q_j(n)a_{n+j}$ with $q_0(X), \dots, q_{R-1}(X) \in \mathbb{Z}[X]$ not all identically zero, with $|G_N(n)| \leq 2cRDH(n+1)^D\delta^{n+R}$ for all $n \geq 0$ and $G_N(n) = 0$ for every integer n in $\{0, 1, \dots, N-1\}$. Note that G_N depends on N which is fixed but can be as large as desired in this construction.

Eq. (4.2) holds if we assume the stronger condition

$$\prod_{j=1}^{\infty}\left(1+\frac{2K_j}{P_{j-1}}\right) \leq H^{RD}, \tag{4.3}$$

because the assumption $\delta < e$ implies the convergence of the product

$$\Phi(D, x) := \prod_{j=1}^{\infty}\left(1+x\frac{j^D\delta^j}{P_{j-1}}\right)$$

for all $x \geq 0$ and all $D \geq 0$, and obviously $1 + x\frac{j^D\delta^j}{P_{j-1}} \geq 1$. We shall provide an upper bound for $\Phi(D, x)$ in §4.2, from which we shall deduce values of H, R, D such that (4.3) holds. It is important to observe here that (4.3) does not depend on N.

4.1 Proof that $G_N(n) = 0$ for all $n \geq 0$

Following the Perelli-Zannier method, we now want to prove that, provided N is large enough, $G_N(n)$ vanishes for all $n \geq 0$. Assume this is not the case. Then for every N, let $M_N \geq N$ denote the largest integer such that $G_N(0) = G_N(1) = \ldots = G_N(M_N) = 0$ but $G_N(M_N + 1) \neq 0$. We fix $\alpha \in \left(0, \frac{2}{\log(\delta)} - 2\right)$.

We shall first prove that $G_N(m) = 0$ for m in $I := [2M_N, (2+\alpha)M_N]$. Let $m \in I$. First assume that p is a prime and $p < M_N$: since $G_N(0) = \cdots = G_N(p) = 0$ and $G_N(n+p) \equiv G_N(n) \mod p$, p divides $G_N(m)$. Assume now that $M_N \leq p \leq m/2$, so that

$$0 \leq m - 2p \leq m - 2M_N \leq \alpha M_N \leq M_N,$$

hence $G_N(m) \equiv G_N(m-2p) = 0 \mod p$. Assume to finish that $m - M_N < p \leq m$ (such primes have not yet been considered) so that $0 \leq m-p < M_N$ and $G_N(m) \equiv G_N(m-p) \equiv 0 \mod p$. It follows that $G_N(m)$ si divisible by $P_{m/2}P_m/P_{m-M_N}$.

Therefore, if $G_N(m) \neq 0$ for some $m \in I$, then

$$|G_N(m)| \geq e^{m/2+M_N+o(M_N)} \geq e^{2M_N+o(M_N)},$$

where $o(M_N)$ denotes a term such that $o(M_N)/M_N$ becomes arbitrarily small when N is taken arbitrarily large. But on the other hand, we know that, for any $m \in I$,

$$|G_N(m)| \leq 2cRDH(m+1)^D \delta^{m+R} \leq 2cRDH(2M_N + \alpha M_N + 1)^D \delta^R \delta^{(2+\alpha)M_N}.$$

We recall that we assume that H, R, D are such that (4.3) holds, which is independent of N. Hence we can let $N \to +\infty$, hence *a fortiori* $M_N \to +\infty$ so that the above lower and upper bounds for $G_N(m) \neq 0$ imply that $e^2 \leq \delta^{2+\alpha}$, *i.e.* that $\alpha \geq \frac{2}{\log(\delta)} - 2$, which is contrary to the assumption on α. Hence, provided N is large enough, we have $G_N(m) = 0$ for all $m \in I$.

We thus have $G_N(m) = 0$ for all integers m in $[0, \dots, M_N]$ or $[2M_N, (2+\alpha)M_N]$. It follows that for any $p \leq (1+\alpha)M_N - 1$, p divides $G_N(M_N+1)$. Indeed, if $p \leq M_N$, we write $M_N + 1 = n + p$ for some $n \leq M_N - 2$ so that $G_N(M_N+1) \equiv G(n) = 0 \mod p$, while if $M_N < p \leq (1+\alpha)M_N - 1$, we have $0 = G_N(M_N + 1 + p) \equiv G(M_N+1) \mod p$ because $M_N + 1 + p \in [2M_N, (2+\alpha)M_N]$. Hence, because $G_N(M_N+1) \neq 0$, we have

$$|G_N(M_N+1)| \geq P_{(1+\alpha)M_N - 1} \geq e^{(1+\alpha)M_N + o(M_N)}.$$

On the other hand,

$$|G_N(M_N+1)| \leq c2RDH(M_N+2)^D \delta^{M_N+1+R}.$$

As above, we take N large enough so that these two bounds imply that $\alpha \leq \log(\delta) - 1$, which is impossible because $\log(\delta) - 1 < 0$ while α was chosen positive.

Therefore, there is no such M_N such that $G(M_N+1) \neq 0$, so that $G(n) = 0$ for all integer $n \geq 0$.

4.2 Upper bound for $\Phi(D, x)$

In this section, $x > 0$ is a fixed real parameter and $D \geq 1$ is a fixed integer. We fix $\varepsilon > 0$ such that $\delta < e - \varepsilon$, and we let $\omega = \delta/(e-\varepsilon) < 1$. By the Prime Number Theorem, $P_{j-1} \geq (e-\varepsilon)^j$ for all $j > J = J(\varepsilon)$ so that

$$\Phi(D, x) := \prod_{j=1}^{\infty} \left(1 + xj^D \frac{\delta^j}{P_{j-1}}\right) \leq \prod_{j=1}^{J} \left(1 + xj^D \frac{\delta^j}{P_{j-1}}\right) \prod_{j=J+1}^{\infty} \left(1 + xj^D \omega^j\right).$$

We have

$$\prod_{j=1}^{J} \left(1 + xj^D \frac{\delta^j}{P_{j-1}}\right) \leq \prod_{j=1}^{J} \left(1 + xj^D \delta^j\right) \leq \prod_{j=1}^{J} \left(1 + (1+x)j^D \delta^j\right) \leq \prod_{j=1}^{J} \left(2(1+x)j^D \delta^j\right),$$

since $(1+x)j^D\delta^j \geq 1$ for every j in $\{1, \dots, J\}$. Hence,

$$\prod_{j=1}^{J} \left(2(1+x)j^D \delta^j\right) = 2^J (1+x)^J J!^D \delta^{J(J+1)/2} \leq 2^J (1+x)^J J!^D \delta^{J^2}.$$

In addition, we have

$$\prod_{j=J+1}^{\infty}\left(1+xj^D\omega^j\right)\leq\prod_{j=1}^{\infty}\left(1+xj^D\omega^j\right),$$

which yields

$$\Phi(D,x)\leq 2^J(1+x)^J J!^D\delta^{J^2}\prod_{j=1}^{\infty}\left(1+xj^D\omega^j\right).$$

We now bound the infinite product $\Psi(D,x):=\prod_{j=1}^{\infty}\left(1+xj^D\omega^j\right)$. The maximum of the function $t\mapsto t^D\omega^{t/2}$ is $m(D):=(2D/(e\log(1/\omega)))^D$, attained at $j_0:=2D/\log(1/\omega)$. Hence, for all $j\geq j_0$, we have $j^D\leq m(D)\omega^{-j/2}$. Moreover, $t\mapsto t^D\omega^{t/2}$ is increasing on $[0,j_0]$ and $m(D)\leq j_0^D$. Hence,

$$\begin{aligned}\Psi(D,x)&\leq\prod_{1\leq j<j_0}\left(1+xj^D\omega^j\right)\prod_{j\geq j_0}^{\infty}\left(1+xm(D)\omega^{j/2}\right)\\&\leq\left(1+(1+x)j_0^D\right)^{\lfloor j_0\rfloor}\prod_{j=1}^{\infty}\left(1+xm(D)\omega^{j/2}\right)\\&\leq\left(2(1+x)j_0^D\right)^{\lfloor j_0\rfloor}\prod_{j=1}^{\infty}\left(1+xj_0^D\omega^{j/2}\right).\end{aligned}$$

We now bound $\prod_{j=1}^{\infty}\left(1+xj_0^D\omega^{j/2}\right)$. We set $y:=xj_0^D$. Since $t\mapsto\omega^t$ is decreasing on $[0,\infty)$, we have

$$\begin{aligned}\log\left(\prod_{j=1}^{\infty}\left(1+y\omega^{j/2}\right)\right)&\leq\int_0^{+\infty}\log(1+y\omega^{t/2})dt\\&\leq\frac{2}{\log(1/\omega)}\int_0^y\frac{\log(1+u)}{u}du\qquad(u:=y\omega^{t/2})\\&\leq-\frac{2\mathrm{Li}_2(-y)}{\log(1/\omega)}.\end{aligned}$$

Here, we use the dilogarithm $\mathrm{Li}_2(z):=-\int_0^z\log(1-x)/xdx$ defined for $z\in\mathbb{C}\setminus(1,+\infty)$ using the principal branch of log in the integral; see [10, p. 1, (1.4)]. Here, we want to use it for large negative values $-y$. For this, we use the identity (see [10, p. 4, (1.7)])

$$\mathrm{Li}_2(-y)=-\frac{1}{2}\log(y)^2-\mathrm{Li}_2(-1/y)-\zeta(2),\quad y>0$$

which yields

$$-2\mathrm{Li}_2(-y)\leq\log(y)^2+4\zeta(2),\quad y\geq 1,$$

because $\mathrm{Li}_2(-1/y)+\zeta(2)\leq 2\zeta(2)$ when $y\geq 1$. We obtain that for $y\geq 1$,

$$\prod_{j=1}^{\infty}\left(1+y\omega^{j/2}\right)\leq c_0e^{\log(y)^2/\log(1/\omega)},$$

with $c_0 := \exp(4\zeta(2)/\log(1/\omega)) \geq 1$.

Putting all the pieces together, we finally obtain the bound

$$1 \leq \Phi(D,x) \leq 2^J(1+x)^J J!^D \delta^{J^2}(2(1+x)j_0^D)^{\lfloor j_0 \rfloor} c_0 e^{\log(xj_0^D)^2/\log(1/\omega)}, \tag{4.4}$$

where we recall that $\omega = \delta/(e-\varepsilon)$ where $\varepsilon > 0$ is such that $\delta < e - \varepsilon$.

4.3 Conclusion of the proof

For ease of reading, we set d, r, h for D, R, H. We want to find conditions on d, r and h such that $\Phi(d,x) \leq h^{rd}$ when $x = 2crdh\delta^{r-1}$ (which corresponds to (4.3)). It will be enough to find conditions on d, r, h and ε such that the right-hand side of (4.4) is $\leq h^{rd}$.

From now on, we set $\ell := \log(\delta) < 1$. We assume that d and ρ depend on ℓ but are independent of h, and we let $r := \lfloor \rho \log(h) \rfloor + 1$. Since J and j_0 are also fixed, when $h \to +\infty$, we have

$$\log\left(c_0 2^J (1+x)^J J!^d \delta^{J^2}(2(1+x)j_0^d)^{\lfloor j_0 \rfloor} e^{\log(xj_0^d)^2/\log(1/\omega)}\right) \sim \frac{(1+\rho\ell)^2}{\log(1/\omega)} \log(h)^2,$$

while

$$\log(h^{rd}) \sim d\rho \log(h)^2.$$

Hence for our goal, it suffices to choose d, ρ and ε such that

$$\frac{(1+\rho\ell)^2}{\log(1/\omega)} < d\rho. \tag{4.5}$$

Recall that $\omega = \delta/(e-\varepsilon)$ so that $\log(1/\omega) \to 1 - \ell$ as $\varepsilon \to 0$. So, by choosing d and ρ such that

$$\frac{(1+\rho\ell)^2}{1-\ell} \leq d\rho, \tag{4.6}$$

we can choose $\varepsilon > 0$ such that Eq. (4.5) holds true. Eq. (4.6) is equivalent to

$$\ell^2\rho^2 + (2\ell - d(1-\ell))\rho + 1 \leq 0,$$

which defines a polynomial in ρ whose discriminant is $\Delta := d(1-\ell)(d(1-\ell) - 4\ell)$. Taking $d := \max(1, \lceil \frac{4\ell}{1-\ell} \rceil)$ ensures that (4.6) holds true for any choice of $\rho > 0$ in

$$\left[\frac{d(1-\ell) - 2\ell - \sqrt{\Delta}}{2\ell^2}, \frac{d(1-\ell) - 2\ell + \sqrt{\Delta}}{2\ell^2}\right].$$

For simplicity, we also restrict ρ to be $\leq 1/\ell$ which is possible because the product of those roots is $1/\ell^2$ and, since $d(1-\ell) \geq 4\ell$, we have

$$\frac{d(1-\ell) - 2\ell + \sqrt{\Delta}}{2\ell^2} \geq \frac{1}{\ell} \quad \text{and} \quad \frac{d(1-\ell) - 2\ell - \sqrt{\Delta}}{2\ell^2} \leq \frac{1}{\ell}.$$

With such choices of ε, d and ρ, we now define $H(\delta)$ as the smallest integer $h \geq 1$ such that

$$c_0 2^J (1+x)^J J!^d \delta^{J^2} (2(1+x)j_0^d)^{\lfloor j_0 \rfloor} e^{\log(xj_0^d)^2/\log(1/\omega)} \leq h^{rd}, \tag{4.7}$$

where $x = 2crdh\delta^{r-1}$. We then obtain (1.10) with $\max\deg(p_j) \leq d-1 = \max(0, \lceil \frac{5\log(\delta)-1}{1-\log(\delta)} \rceil)$ and $S \leq r-1 \leq \lfloor \rho \log(h) \rfloor \leq \log(H(\delta))/\ell$. Notice that $H(\delta)$ also depends on the choice of ρ and we now explain how to bound it.

The left-hand side of (4.7) is an increasing function of $h \geq 1$, which appears in the expressions of $r := \lfloor \rho \log(h) \rfloor + 1$ and $x := 2crdh\delta^{r-1}$. Hence, $H(\delta)^{d(1+\lfloor \rho \log(H(\delta)) \rfloor)}$ is larger than the value A (which is ≥ 1) of the left-hand side of (4.7) at $h = 1$, in which case $r = 1$ and $x = 2cd$. It follows that $\log(A) \leq d\log(H(\delta)) + d\rho \log(H(\delta))^2$, so that

$$H(\delta) \geq \exp\Big(\frac{\sqrt{d^2 + 4d\rho\log(A)} - d}{d\rho}\Big). \tag{4.8}$$

Since $A \to +\infty$ when $\delta \to e$ and $\varepsilon \to 0$ (because of the term $1/\log(\omega)$), $H(\delta)$ can be very large.

We now explain how to bound $Y := H(\delta)$ from above. We assume that $H(\delta) \geq 2$ otherwise there is nothing else to do. The left-hand side of (4.7) is ≥ 1, so that we can take the logarithms of both sides. After some transformations, we obtain a function $S(h) \geq 0$ for all $h \geq 1$ (which could be explicited) such that Y is the smallest integer $h \geq 1$ such that

$$S(h) \leq \Big(\rho d - \frac{(1+\rho\ell)^2}{\log(1/\omega)}\Big) \log(h)^2. \tag{4.9}$$

Recall that $\gamma := \rho d - \frac{(1+\rho\ell)^2}{\log(1/\omega)} > 0$. Moreover, there exist α, β that depend on $\rho, d, \delta, \varepsilon$ (and could be explicited as well) such that $S(h) \leq \alpha \log(h) + \beta$ for all $h \geq 1$. Since $Y \geq 2$ is the smallest integer such that (4.9) holds, we have

$$\gamma \log(Y-1)^2 < s(Y-1) \leq \alpha \log(Y-1) + \beta.$$

Hence, $\log(Y-1)$ is smaller than the largest solution of the quadratic equation $\gamma X^2 - \alpha X - \beta = 0$, so that finally

$$H(\delta) \leq 1 + \exp\Big(\frac{\alpha + \sqrt{\alpha^2 + 4\beta\gamma}}{2\gamma}\Big). \tag{4.10}$$

5 Proof of Theorem 4

Let $F(z)$ be as in the theorem such that $F(n) = a_n$ for all $n > u$. Notice that $\widetilde{a}_n := a_{n+\lceil u \rceil + 2}$, $n \geq 0$, is a primary peudo-polynomial; it is eventually in $\mathbb{Q}[n]$ if and only if a_n is eventually in $\mathbb{Q}[n]$. The function $\widetilde{F}(z) := F(z + \lceil u \rceil + 2)$ is analytic in $\Re(z) > -2$, satisfies $|\widetilde{F}(z)| \leq \tilde{c} \cdot \exp(\rho \Re(z))$ in $\Re(z) > -2$ for some constant $\tilde{c} > 0$, and $\widetilde{a}_n = \widetilde{F}(n)$ for all $n \geq 0$. Moreover, $F(z)$ is in $\mathbb{Q}[z]$ if and only if $\widetilde{F}(z)$ is in $\mathbb{Q}[z]$. Therefore, without loss of generality, we can and will assume that $F(z)$ is analytic in $\Re(z) > -2$ and that $F(n) = a_n$ for all integers $n \geq 0$.

Let $\mathcal{C}_n$ denote the circle of center n and radius n oriented in the direct sense. The function $F(z-1)$ being analytic in $\Re(z) > -1$, the residue theorem yields

$$\frac{(n-1)!}{2i\pi}\int_{\mathcal{C}_n} \frac{F(z-1)}{(z-1)\cdots(z-n)}dz = \sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n-1}{k}a_k := b_{n-1} \in \mathbb{Z}.$$

We parametrize the circle $\mathcal{C}_n$ as $n + ne^{2ix} = 2n\cos(x)e^{ix}$ for $x \in [-\pi/2, \pi/2]$. Nörlund [11, p. 387] proved that

$$\left|\frac{(n-1)!}{(z-1)\cdots(z-n)}\right| \leq c_1(n)e^{-2n\cos(x)\psi(x)}, \quad z = 2n\cos(x)e^{ix} \in \mathcal{C}_n,$$

where $c_1(n) > 0$ is bounded above by some polynomial in n and

$$\psi(x) := \cos(x)\log(2\cos(x)) + x\sin(x).$$

(See also the proof given in [15].) The minimum on $[-\pi/2, \pi/2]$ of $\psi(x)$ is $\log(2)$ at $x = 0$.

We have

$$|F(z-1)| \leq c \cdot e^{\rho\Re(z-1)} = ce^{-\rho} \cdot e^{2n\rho\cos(x)^2}, \quad z = 2n\cos(x)e^{ix} \in \mathcal{C}_n.$$

Hence

$$|b_{n-1}| \leq c_2(n) \max_{x\in[-\pi/2,\pi/2]} e^{2n\cos(x)(\rho\cos(x)-\psi(x))}$$

where $c_2(n) > 0$ is bounded above by some polynomial in n. Notice that $0 \leq \cos(x) \leq 1$ on $[-\pi/2, \pi/2]$. Recall also that $\rho > 0$. Hence if x is such that $\rho\cos(x) - \psi(x) \geq 0$, then $\cos(x)(\rho\cos(x)-\psi(x)) \leq \rho-\psi(x) \leq \rho-\log(2)$, whereas if x is such that $\rho\cos(x)-\psi(x) \leq 0$, then $\cos(x)(\rho\cos(x) - \psi(x)) \leq 0$. Therefore, for all $x \in [-\pi/2, \pi/2]$,

$$e^{2\cos(x)(\rho\cos(x)-\psi(x))} \leq \max(1, e^{2(\rho-\log(2))})$$

and $|b_{n-1}| \leq c_2(n)\max(1, e^{2(\rho-\log(2))})^n$. Since $2(\rho - \log(2)) < 1$, it follows that

$$\limsup_{n\to+\infty} |b_n|^{1/n} < e.$$

But because $(a_n)_{n\geq 0}$ is a primary pseudo-polynomial, we know by Theorem 1(ii) that if b_n is not eventually equal to 0 then

$$\liminf_{n\to+\infty} |b_n|^{1/n} \geq e.$$

This implies that b_n is indeed eventually equal to 0, thus that there exist $P(X) \in \mathbb{Q}[X]$ and an integer $N \geq 0$ such that $a_n = P(n)$ for all $n \geq N$.

Consider now the function $g(z) := F(z+N)-P(z+N)$ which is analytic in $\Re(z) > -2$ (at least), and such that $g(n) = 0$ for every integer $n \geq 1$. Moreover, since $\rho > 0$, there exists a constant $d > 0$ such that $|g(z)| \leq d \cdot \exp(\rho|z|)$ for any z such that $\Re(z) > 0$. Since $\rho < \frac{1}{2} + \log(2) < \pi$, we can then apply a classical result of Carlson (see Hardy [6, p. 328]) and deduce that $g(z) = 0$ identically. Hence $F(z)$ reduces to a polynomial function in $\mathbb{Q}[z]$. This completes the proof of Theorem 4.

6 Construction of non-polynomial primary pseudo-polynomials

We conclude this paper by presenting a method to construct a non-polynomial primary pseudo-polynomial starting from a given primary pseudo-polynomial $(a_n)_{n\geq 0}$ such that $a_0 = 1$. The justification of the method uses a non-trivial property satisfied by E-functions.

Let as usual $(b_n)_{n\geq 0}$ be the binomial transform (1.1) of $(a_n)_{n\geq 0}$. Let $F_b(x) := \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$. We assume that $a_0 = 1$, so that $b_0 = 1$ as well. We define the sequence $(c_n)_{n\geq 0}$ formally by

$$F_c(x) := \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \frac{1}{F_b(x)}.$$

Let us now define $(u_n)_{n\geq 0}$ as the inverse binomial transform (1.2) of $(c_n)_{n\geq 0}$, *i.e.*

$$u_n := \sum_{k=0}^{n} \binom{n}{k} c_k.$$

Then we have the following.

Theorem 6. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo polynomial such that $a_0 = 1$. Then, $(u_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is a primary pseudo-polynomial. Moreover, assuming also that $f_a(x)$ is a G-function, if $\limsup_n |u_n|^{1/n} < e$, then $a_n = u_n = 1$ for all $n \geq 0$.*

Consequently, if $f_a(x)$ is a G-function not equal to $\frac{1}{1-x}$, then $(u_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is a primary pseudo-polynomial such that $\limsup_n |u_n|^{1/n} \geq e$ (hence not a polynomial). We explain after the proof of the theorem why u_n grows like $n!$ in this case.

So far, the assumption that $f_a(x)$ is a G-function is known to be satisfied only when $\limsup_n |a_n|^{1/n} < e$ (when the Perelli-Zannier Theorem can be applied), which in turn implies that a_n should eventually be in $\mathbb{Q}[n]$ by Conjecture 2. Hence, in practice the second assertion of Theorem 6 is useful only when a_n is already known to be eventually in $\mathbb{Q}[n]$ in which case $F_b(x) \in \mathbb{Q}[x]$: For instance, if $a_n = n + 1$, then

$$\sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \frac{1}{1+x} \quad \text{and} \quad u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k!. \tag{6.1}$$

Proof of Theorem 6. We first prove that for all $n \geq 0$, $c_n \in \mathbb{Z}$ and for every prime $p \leq n$, p divides c_n. By definition of the c_n's, we have

$$1 = \left(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n\right) \left(\sum_{n=0}^{\infty} \frac{c_n}{n!} x^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} b_k c_{n-k}\right) x^n,$$

so that $c_0 = 1$ and, for every integer $n \geq 1$, we have

$$\sum_{k=0}^{n} \binom{n}{k} b_k c_{n-k} = 0.$$

This yields the recursive relation (because $b_0 = 1$)

$$c_n = -\sum_{k=1}^{n} \binom{n}{k} b_k c_{n-k}, \quad (n \geq 1).$$

It follows that, for all $n \geq 0$, c_n is an integer.

Let n be a positive integer such that, for every positive integer $m < n$ and every prime $p \leq m$, p divides c_m. Consider a prime number $p \leq n$ and an integer k in $\{1, \ldots, n\}$. If $p \leq k$ then p divides b_k. If $p \leq n-k$, then p divides c_{n-k}. If $p > \max(k, n-k)$, then p divides $\binom{n}{k}$ because p divides $n!$ but neither $k!$ nor $(n-k)!$. In all cases, p divides $\binom{n}{k} b_k c_{n-k}$, so that p divides c_n. By strong induction on n, it follows that, for every integer $n \geq 0$ and every prime $p \leq n$, p divides c_n (this property holds trivially if $n = 0$ or 1). By Theorem 1(i), the sequence $(u_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is a primary pseudo-polynomial. Notice that so far we had no need to assume that f_a is a G-function.

We now complete the proof of Theorem 6 when f_a is also a G-function. We first observe that $F_b(z)$ is an E-function: indeed, $f_b(x)$ is a G-function because

$$f_b(x) := \sum_{n=0}^{\infty} b_n x^n = \frac{1}{1+x} f_a\left(\frac{x}{1+x}\right)$$

and $f_a(x)$ is a G-function. ([3]) Let us assume that $\limsup_n |u_n|^{1/n} < e$. By the Perelli-Zannier Theorem quoted in the Introduction, $f_u(x) := \sum_{n=0}^{\infty} u_n x^n$ is a G-function. From the equation $c_n := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} u_k$ $(\forall n \geq 0)$, we deduce that

$$f_c(x) := \sum_{n=0}^{\infty} c_n x^n = \frac{1}{1+x} f_u\left(\frac{x}{1+x}\right)$$

is also a G-function. Hence $F_c(x) := 1/F_b(z)$ is also an E-function. Therefore, $F_b(x)$ is a unit of the ring of E-functions, *i.e.* it is of the form $\beta e^{\alpha z}$ with $\alpha \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}^*$ by [1, p. 717] (see also Footnote 4). Therefore, $b_n = \beta\alpha^n$ for all $n \geq 0$, with the usual convention that $\alpha^0 = 1$ if $\alpha = 0$. Since all primes $p \leq n$ divide b_n, we deduce that $\alpha = 0$ is the only possibility. Hence $1 = b_0 = \beta$ and $b_n = 0$ for all $n \geq 0$, so that $a_n = 1$ for all $n \geq 0$. It now remains to observe the following facts: "$a_n = 1$ for all $n \geq 0$" is equivalent to "$b_0 = 1$ and $b_n = 0$ for all $n \geq 1$" which is equivalent to "$c_0 = 1$ and $c_n = 0$ for all $n \geq 1$", which in turn is equivalent to "$u_n = 1$ for all $n \geq 0$". □

To conclude, let us explain how to obtain the asymptotic behavior of u_n as $n \to +\infty$. Since $F_u(x) := \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n = e^x F_c(x) = e^x / F_b(x)$, the asymptotic behavior of $u_n/n!$ is determined by the zeroes of smallest modulus of $F_b(x)$, when it has at least one. Notice that an E-function $F(x)$ with no zero in $\mathbb{C}$ must be of the form $\beta e^{\alpha x}$ with $\alpha \in \overline{\mathbb{Q}}, \beta \in \overline{\mathbb{Q}}^*$. Indeed, an E-function is an entire function satisfying $|F(x)| \ll e^{\rho|x|}$ for some $\rho > 0$, so

[3]Given a G-function $f(x)$ and $\alpha(x)$ an algebraic function over $\overline{\mathbb{Q}}(x)$ regular at $x = 0$, $f(x\alpha(x))$ is a G-function.

that Hadamard's factorization theorem yields $F(x) = \beta x^m e^{\alpha x} \prod_{j\geq 1} \big((1-\frac{x}{x_j})e^{x/x_j}\big)$ where the x_j's are the zeroes of $F(x)$, $m \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{C}$. ([4]) Therefore, a "no zero" assumption implies that $F(x) = \beta e^{\alpha x}$ with $\beta \neq 0$, and since $F(x) \in \overline{\mathbb{Q}}[[x]]$, α, β must be in $\overline{\mathbb{Q}}$. Coming back to $F_b(x)$, we have seen during the proof of Theorem 6 that this case implies that $\alpha = 0, \beta = 1$, hence that $f_a(x) = \frac{1}{1-x}$.

Therefore, assuming that $f_a(x)$ is a G-function different of $\frac{1}{1-x}$, the E-function $F_b(x)$ has at least one zero. Let $x_1, \ldots, x_m$ denote the zeroes of $F_b(x)$ of the same modulus which is the smallest amongst all modulus of the zeroes. Classical transfer theorems in [3, Chapter VI] enable to deduce the asymptotic behavior of u_n. For instance, if the x_j's are simple zeroes of $F_b(x)$, then

$$u_n = n! \sum_{j=1}^{m} \frac{e^{x_j}}{F_b'(x_j)} \frac{1+o(1)}{x_j^n}.$$

This is coherent with (6.1) above (where $F_b(x) = 1 + x$) because we can rewrite it as

$$u_n = (-1)^n n! \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \sim \frac{(-1)^n}{e} n!, \quad n \to +\infty.$$

Because of the different arguments of the x_j's, oscillations can occur. In presence of zeroes of $F_b(x)$ of higher multiplicities, similar but more complicated expressions can be given. Finally, even though $F_b(x)/e^x$ is an E-function, we don't expect $e^x/F_b(x)$ to be D-finite in general, ([5]) but this is obviously the case if $F_b(x)$ is a polynomial.

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References

[1] Y. André, *Séries Gevrey de type arithmétique I. Théorèmes de pureté et de dualité*, Annals of Math. **151** (2000), 705–740.

[2] G. Christol, *Fonctions hypergéométriques bornées*, Groupe de travail d'analyse ultramétrique, Tome **14** (1986-1987) p. 1–16.

[3] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[4]This argument also explains the characterization of the units of the ring of E-functions: we simply have to observe that if an E-function $F(x)$ is a unit, *i.e.* that $1/F(x)$ is an E-function, then it does not vanish anywhere on $\mathbb{C}$ because an E-function is an entire function.

[5]A classical result of Harris-Sibuya [7] ensures that if y and $1/y$ are both holonomic, then y'/y is an algebraic function.

[4] H. Furstenberg, *Algebraic functions over finite fields*, J. Algebra **7** (1967), 271–277.

[5] R. R. Hall, *On pseudo-polynomials*, Mathematika **18** (1971), 71–77.

[6] G. H. Hardy, *On two theorems of F. Carlson and S. Wigert*, Acta Math. **42** (1920), 327–339.

[7] W. A. Harris, Y. Sibuya, *The reciprocals of solutions of linear ordinary differential equations*, Adv. Math. **58** (1985), 119–132.

[8] E. Kowalski and K. Soundararajan, *Equidistribution from the Chinese Remainder Theorem*, `https://arxiv.org/abs/2003.12965`

[9] V. Kuperberg, *A note on pseudo-polynomials divisible only by a sparse set of primes*, `https://arxiv.org/abs/2006.02527`

[10] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, xiii+359 pages, 1981.

[11] N. E. Nörlund, *Sur les formules d'interpolation de Stirling et de Newton*, Ann. sci. ENS **39** (1922), 343–403.

[12] A. Perelli, U. Zannier, *Su un teorema di Pólya*, Boll. UMI **5**, 18/A (1981).

[13] A. Perelli, U. Zannier, *On recurrent mod p sequences*, Crelle's Journal **348** (1984), 135–146.

[14] S. Fischler, T. Rivoal, *Linear independence of values of G-functions*, Journal of the EMS **22**.5 (2020), 1531–1576.

[15] T. Rivoal, M. Welter, *Sur les fonctions arithmétiques non entières*, Israel J. Math. **169**.1 (2009), 155–179.

[16] I. Z. Ruzsa, *On congruence preserving functions*, Mat. Lapok **22** (1971), 125–134.

[17] A. B. Shidlovskii, *Transcendental Numbers*, W. de Gruyter Studies in Mathematics **12**, 1989.

[18] M. Waldschmidt, *Integer-valued functions, Hurwitz functions and related topics: a survey*, preprint 2020, 27 pages, submitted to the proceedings of the 31th Journées Arithmétiques (Istanbul, June 2019).

[19] U. Zannier, *On periodic mod p sequences and G-functions (On a conjecture of Ruzsa)*, manuscripta math. **90** (1996), 391–402.

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