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 A new framework is proposed for the study of collisions between very heavy ions which lead to the synthesis of Super-Heavy Elements (SHE), to address the fusion hindrance phenomenon. The dynamics of the reaction is studied in terms of collective degrees of freedom undergoing relaxation processes with different time scales. The Nakajima-Zwanzig projection operator method is employed to eliminate fast variable and derive a dynamical equation for the reduced system with only slow variables. There, the time evolution operator is renormalised and an inhomogeneous term appears, which represents a propagation of the given initial distribution. The term results in a slip to the initial values of the slow variables. We expect that gives a dynamical origin of the so-called "injection point s" introduced by Swiatecki et al in order to reproduce absolute values of measured cross sections for SHE. A formula for the slip is given in terms of physical parameters of the system, which confirms the results recently obtained with a Langevin equation, and permits us to compare various incident channels.
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1. Introduction

Finding the limit of existence of nuclei is one of the challenging research programs in nuclear physics. vIt gives the limit of chemical elements, with which all the matter of the world is made. Within the Liquid Drop Model (LDM), the limit of the atomic charge Z is around 100, where the fission barrier becomes negligibly small. Beyond this limit, nuclei only owe their existence to the extra-stability provided by the shell correction energy due to quantum-mechanical structure effects in finite many-body Fermion systems. They are called superheavy elements (SHE). Their quests in nature turned out to be unsuccessful, and then the prediction of their existence is to be justified by synthesis with nuclear reactions, but the attempts face a challenge due to the extremely low cross sections, at the order of the picobarn or even smaller for the heaviest elements produced by fusion-evaporation reactions [1]. The reasons are twofold: First, fragility of the compound nuclei (CN) formed, that are supported by a very low fission barrier, and thus, undergo fission before cooling down through neutron evaporation. Second, the fusion process is hindered with respect to models developed for the

fusion of light nuclei. Fusion hindrance is observed in systems with $Z_1 \cdot Z_2$ being larger than $1\,600$ [2]. There is no commonly accepted explanation of the origin of the latter, while the former is quantitatively well described by the conventional statistical theory of decay [3], with some ambiguities in physical parameters such as nuclear masses [4].

An unveiling of the origin of the hindrance and its correct description is necessary to improve the predictive power of models describing the whole reaction leading to the synthesis of SHE. In all models, the fusion process consists in a sequence of two steps, i.e., overcoming of the Coulomb barrier and then formation of the CN, starting from the di-nucleus configuration of the hard contact of projectile and target nuclei. The necessity for the latter process is due to the fact that the contact configuration in heavy systems locates outside of the conditional saddle point, and thus, the system has to overcome the saddle point, after overcoming the Coulomb barrier [5–12]. In other words, the fusion probability is given by the product of the contact and the formation probabilities. The former factor is often called the capture probability or cross-section, while the latter is called formation factor or hindrance factor as in the Fusion-by-Diffusion model (FBD) [10, 11].

The dynamics of the latter process cannot be simply described by Newtonian mechanics, or its quantum version. For, at the hard contact of two nuclei, the di-nucleus system is internally excited. Most of the kinetic energy carried in by the incident channel is supposed to be already transformed into those of nucleonic motions, i.e., to be dissipated into heat. Therefore, the collective motions of the system are to be described by dissipation-fluctuation dynamics like in fission decay [13–18].

However, the first cross sections calculated with the Langevin model [19–21] as well as the Fusion-by-Diffusion model [10, 11], turned out to be still too large, by a few orders of magnitude, compared with the measured cross sections in so-called cold fusion path for SHE. In order to reproduce the absolute value of the measured cross sections, an arbitrary "injection point parameter s" was introduced in the latter model, shifting the contact configuration outward by about 2.0 fm. Such a shift makes the hindrance factor stronger and thus, the cross section even smaller by a few order of magnitude as desired [10, 11]. A systematics over several systems exhibited a regular behaviour [22, 23]. A similar analysis was conducted on the so-called hot fusion path with an injection point parameter found to be incident-channel dependent or incident-energy dependent for systematic reproduction of the experiments [24–26]. Those phenomenological results suggest that something is missing in theories of fusion mechanism hitherto developed.

First attempts to explain the shift of the injection parameter to dynamical process from the di-nucleus to a mono-nucleus shape were related to the fast disappearance of the neck degree of freedom that affects the injection point and gives rise to an additional hindrance [27–32]. Later, combining of the elimination of both neck degree of freedom and momentum ones in case of strong friction lead to the incident-energy dependence of the initial slip, thus explaining the behaviour of the injection point parameter, systematically both for the cold and the hot fusion paths [33]. This is encouraging for theoretical predictions on on-going and/or future experiments for synthesis of heavier elements with various incident channels.

More generally, the elimination of fast variables in a multidimensional system to reduce the model to few slow variables is addressed in several seminal articles [34, 35] and textbooks [36–38]. It always leads to a slip of the initial condition of the slow variables [39–44]. The purpose of the present article is to propose another framework for this problem.

In contrast with Ref. [33] which is based on the Langevin formalism [45], we shall rather use here the so-called Fokker-Planck [46, 47] formalism which is based on partial differential equations describing the time evolution of the probability density function, or the distribution function in the phase space. To look at the problem from two viewpoints often provides deeper physical understanding of the dynamics. We, thus, expect that the present approach sheds another light on the problem how the fast neck degree affects the motion of the slow variables and gives rise to a new additional hindrance.

There are various Fokker-Planck type equations to study the diffusion of collective degrees of freedom dragged by potential forces. In this study, we shall consider special forms describing the Brownian motion in an external field. When applied to position distributions, it is better known as Smoluchowski equation [48], and, when it also takes into account the conjugated momenta or velocity, it is called Klein-Kramers equation [13, 49]. The former is an approximation of the latter in the high viscosity limit.

Applied to collective degrees of freedom describing the shape of the colliding nuclei from contact to compound shape, dragged by a macroscopic potential map based on the LDM, we aim to give a global base-line theory of fusion of heavy systems. The fate of the system is either fusion (CN formation) or to re-separation (quasi-fission decay). Since the neck degree is distinctly (by factor several to one order of magnitude) faster than the other degrees of freedom [28–32], the neck rapidly reaches the equilibrium point, or the equilibrium distribution at the very beginning of the dynamical processes.

In order to rigorously eliminate the fast variable, we employ Nakajima-Zwanzig projection operator method [50, 51] and derive a dynamical equation only for the slow variables and see how the initial slip arises. In Section 2, the derivation of the reduced dynamical equation is given in the case of Smoluchowski equation with the fast (neck) and the slow (radial and mass-asymmetry) coordinates. For simplicity and facilitation of analytic calculations, we assume the potential (LDM energy surface) to be approximated by multi-dimensional parabolic saddle (Taylor expansion at the saddle point up to the second order) and the friction tensor for the collective degrees to be coordinate-independent, i.e., be constant. The coupling between the fast and the slow variables is supposed to be weak. Perturbative approximation with respect to the coupling enables us to obtain a reduced Fokker-Planck equation with simple analytic expressions for the evolution operator and the inhomogeneous term. The latter term is mostly neglected in the scientific literature with a few exceptions [39–44], but turns out to play a key role in the present subject, giving rise to the *slip* of the initial point and thus, an additional hindrance. In Section 3, we summarise the results, including those for the Klein-Kramers equation, which is equivalent to a full Langevin equation.

2. Elimination of fast variables by Nakajima-Zwanzig projection method and initial slip

As mentioned in the introduction, the elimination of the fast variables with the Nakajima-Zwanzig (N-Z) projection method [50, 51] is general and can be applied to either Klein-Kramers or Smoluchowski equations. Here, we shall only consider the Smoluchowski equation for general N-dimensional collective coordinates which describe shapes of the di-nucleus system. As an illustration, we eliminate the neck variable to derive a dynamical equation for the system with only slow variables and investigate possible effects of the eliminated variable to the motion of the slow ones. The N-Z method is amenable to analytic calculations

practically for cases for which a coupling is separable with respect to fast and slow variables, like in Caldeira-Leggett model [52]. In the present model, however, the coupling is in Smoluchowski type, as given below in section 2.1. Nevertheless, it turns out to be effectively bi-linear, helped by the quadratic potential and the constant coefficients. Thus, it is feasible to calculate the projection, i.e., the integration or trace over the fast variable analytically, which results in a simple dynamical equation for the reduced system, as shown in detail in section 2.2.

2.1. Projectors

The N-dimensional Smoluchowski equation for the distribution function $w(\vec{q}, t)$ is

$$\frac{\partial}{\partial t} w(\vec{q}, t) = \mathcal{L} \cdot w(\vec{q}, t), \tag{1}$$

with

$$\mathcal{L} = \sum_{ij} \frac{\partial}{\partial q_i} \mu_{ij} \cdot \left(\frac{\partial V}{\partial q_j} + T \frac{\partial}{\partial q_j} \right), \tag{2}$$

where indexes (i, j) denote all the fast and the slow variables, while indexes (α, β) used later only denote slow variables, i.e., the radial and mass-asymmetry coordinates. V denotes LDM potential, while μ is the inverse of the friction tensor γ. T is the temperature of the system.

Smoluchowskian is rewritten as follows, with the premise of perturbative approximation with respect to the coupling between the fast and the slow coordinates,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \quad \text{with} \quad \mathcal{L}_0 = \mathcal{L}_f + \mathcal{L}_s \quad \text{and} \quad \mathcal{L}_1 = \mathcal{L}_{fs} + \mathcal{L}_{sf}, \tag{3}$$

where

$$\mathcal{L}_f = \frac{\partial}{\partial q_f} \mu_f \left[\frac{\partial V}{\partial q_f} + T \frac{\partial}{\partial q_f} \right], \tag{4}$$

$$\mathcal{L}_s = \sum_{\alpha\beta} \frac{\partial}{\partial q_\alpha} \mu_{\alpha\beta} \left[\frac{\partial V}{\partial q_\beta} + T \frac{\partial}{\partial q_\beta} \right], \tag{5}$$

$$\mathcal{L}_{fs} = \sum_{\alpha} \frac{\partial}{\partial q_f} \mu_{f\alpha} \left[\frac{\partial V}{\partial q_\alpha} + T \frac{\partial}{\partial q_\alpha} \right], \tag{6}$$

$$\mathcal{L}_{sf} = \sum_{\alpha} \frac{\partial}{\partial q_\alpha} \mu_{\alpha f} \left[\frac{\partial V}{\partial q_f} + T \frac{\partial}{\partial q_f} \right], \tag{7}$$

where μ_f represents the μ_{ff}. For the sake of simplicity, we have only considered a single fast variable here. This could be easily generalised to any number of fast variables.

Assuming that the fast variable quickly converges towards an equilibrium distribution $\phi_0(q_f)$, we introduce the projection operators [50, 51],

$$P = \phi_0(q_f) \cdot \int dq_f, \quad \text{and} \quad Q = 1 - P. \tag{8}$$

Since q_f becomes rapidly in equilibrium, non-equilibrium component disappears rapidly, and thus, life time of Q space is short, compared with time scale of the slow variables.

It is easy to show that projected distributions Pw and Qw satisfy

$$\frac{\partial}{\partial t}Pw = P\mathcal{L}Pw + P\mathcal{L}Qw, \tag{9}$$

$$\frac{\partial}{\partial t}Qw = Q\mathcal{L}Pw + Q\mathcal{L}Qw, \tag{10}$$

which is rewritten into a closed equation for Pw,

$$\frac{\partial}{\partial t}Pw = P\mathcal{L}Pw + P\mathcal{L}\cdot e^{Q\mathcal{L}\cdot t}\int_0^t dt'\, e^{-Q\mathcal{L}\cdot t'}Q\mathcal{L}Pw(t') + P\mathcal{L}\cdot e^{Q\mathcal{L}\cdot t}\cdot Qw(\vec{q},0). \tag{11}$$

So far, no approximation is made and therefore this equation is equivalent to the original one, but is amenable to perturbative approximation.

The last term of Eq. (11) carries the information of the Q space component of the initial distribution function. Of course, it is natural to disregard it, because its effect quickly disappears as time goes. However, in systems with bifurcation or in open systems, it can dramatically change the fate of the system. Actually, in our case, the elimination results effectively in a *slip* of the initial points of the slow variables, thus affecting the formation probability by a few order of magnitude, as is shown below.

The second term of Eq. (11) exhibits a memory effect, which systematically appears when some variables are eliminated [34–37]. The method has been generally discussed in the field of statistical mechanics to mostly determine to what dynamics the reduced system obeys. We, however, are also interested in the inhomogeneous term in Eq. (11), which carries the Q space component in the initial distribution.

2.2. Renormalised Smoluchowski operator and initial slip

To facilitate the calculations, we first diagonalise the coefficient matrix of the potential, although it is quite close to be diagonal in case of Two-Center-Parameterization (TCP) of di-nucleus system [53, 54]. Thus, the potential is as follows,

$$V = \frac{1}{2}\sum_{ij}c_{ij}q_iq_j \Rightarrow \frac{1}{2}c_fq_f^2 + \frac{1}{2}\sum_{\alpha}c_{\alpha}q_{\alpha}^2, \tag{12}$$

and then,

$$\frac{\partial V}{\partial q_i} = c_iq_i. \tag{13}$$

Note that $\mathcal{L}_f$ and $\mathcal{L}_s$ are operators solely in the fast and the slow variables, respectively, while $\mathcal{L}_1$ is the coupling between them.

In the following calculations, we use the basic properties of $\mathcal{L}_f$:

$$\int dq_f\, \mathcal{L}_f\cdot X = 0, \quad \forall X. \tag{14}$$

Naturally, distribution functions are supposed to be vanishing at the boundaries of q_f variable. For the equilibrium distribution $\phi_0(q_f)$, we have that

$$\mathcal{L}_f\phi_0(x) = 0, \quad \text{with} \quad \phi_0(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, \quad \text{and} \quad x = \sqrt{\frac{c_f}{T}}q_f. \tag{15}$$

The eigenvalues and eigenfunctions are

$$\mathcal{L}_f\phi_n(x) = -\lambda_n\phi_n(x), \quad \text{with} \quad \lambda_n = n\mu_f c_f \quad \text{and} \quad \phi_n(x) = H_n(x)\cdot\phi_0(x). \tag{16}$$

Here, $H_n(x)$ are the Hermite polynomials.

Hereafter, we calculate the terms in r.h.s. of Eq. (11) at the lowest order with respect to the coupling $\mathcal{L}_1$ between the fast and the slow variables. For the first term, we simply have that

$$PLPw = \phi_0 \mathcal{L}_s \cdot w(\vec{q}_\alpha, t), \quad \text{with} \quad w(\vec{q}_\alpha, t) = \int dq_f w(\vec{q}_i, t). \tag{17}$$

This is just the Smoluchowski operator for the $(N-1)$ slow variables only.

The second term is already at the second order with respect to $\mathcal{L}_1$, due to the pre- and post-factors of the propagator. Therefore, we approximate the full Smoluchowski operator $\mathcal{L}$ in the propagator by $\mathcal{L}_0$, so that, $e^{Q\mathcal{L}\cdot t} \simeq e^{Q\mathcal{L}_0 \cdot t}$. Noticing that $Q\mathcal{L}_s$ and $Q\mathcal{L}_f$ commute, $e^{Q(\mathcal{L}_f + \mathcal{L}_s)\cdot(t-t')} = e^{Q\mathcal{L}_f\cdot(t-t')} \cdot e^{Q\mathcal{L}_s\cdot(t-t')}$. Thus, with the properties of $\mathcal{L}_f$ given above, the memory kernel is governed by the factor $e^{-\mu_f \cdot c_f \cdot (t-t')}$. Thus, the duration of the memory effect is determined by the relaxation time of the fast neck degree. In the t' integration, we use Laplace approximation, taking into account that the integrand is slowly varying function,

$$e^{-\mu_f \cdot c_f \cdot (t-t')} \simeq \delta(t'-t)/(\mu_f \cdot c_f). \tag{18}$$

Then, the second term is rewritten as follows,

$$-\phi_0 \frac{\partial}{\partial q_\alpha} \mu_{\alpha f} \frac{1}{\mu_f} \mu_{f\beta} \cdot \left[c_\beta q_\beta + T \frac{\partial}{\partial q_\beta} \right] w(\vec{q}, t). \tag{19}$$

The first two terms of Eq. (11) are summed up into

$$\phi_0 \frac{\partial}{\partial q_\alpha} \mu_{\alpha\beta}^{\text{eff}} \left[c_\beta q_\beta + T \frac{\partial}{\partial q_\beta} \right] w(\vec{q}, t), \quad \text{with} \quad \mu_{\alpha\beta}^{\text{eff}} = \mu_{\alpha\beta} - \mu_{\alpha f} \frac{1}{\mu_f} \mu_{f\beta}. \tag{20}$$

This means that the elimination of the fast variable results in the renormalisation of Smoluchowski operator for the slow variables with $\mu_{\alpha\beta}^{\text{eff}}$:

$$\mathcal{L}^{\text{eff}} = \frac{\partial}{\partial q_\alpha} \mu_{\alpha\beta}^{\text{eff}} \left[c_\beta q_\beta + T \frac{\partial}{\partial q_\beta} \right]. \tag{21}$$

Now, we analyse the last term in Eq. (11) which is the purpose of the present study. Using a perturbative expansion of the propagator $e^{Q\mathcal{L}\cdot t} = e^{Q(\mathcal{L}_0 + \mathcal{L}_1)\cdot t}$ with respect to $\mathcal{L}_1$, the third term is given as follows,

$$PL \cdot e^{Q\mathcal{L}\cdot t} Qw(0) = P(\mathcal{L}_0 + \mathcal{L}_1) \cdot e^{Q\mathcal{L}_0 \cdot t} \cdot \left[1 + \int_0^t dt' \, e^{-Q\mathcal{L}_0 \cdot t'} Q\mathcal{L}_1 e^{Q\mathcal{L}_0 t'} + \cdots \right] \cdot Qw(0). \tag{22}$$

The zeroth order term is simply zero, and the first order term which does not vanish is,

$$I_1 = PL_1 \cdot e^{Q\mathcal{L}_0 \cdot t} Qw(0) = P\mathcal{L}_{sf} \cdot e^{Q\mathcal{L}_0 \cdot t} Qw(0) \tag{23}$$

$$= \frac{\partial}{\partial q_\alpha} \mu_{\alpha f} \cdot c_f \phi_0 \int dq_f \, q_f e^{Q\mathcal{L}_s \cdot t'} \cdot e^{Q\mathcal{L}_f \cdot t} \cdot Qw(0). \tag{24}$$

Since $e^{Q\mathcal{L}_f \cdot t} = e^{\mathcal{L}_f \cdot t}$, and $e^{Q\mathcal{L}_s \cdot t} \cdot e^{Q\mathcal{L}_f \cdot t} \cdot Qw(0) = e^{\mathcal{L}_s \cdot t} \cdot e^{\mathcal{L}_f \cdot t} \cdot Qw(0)$,

$$I_1 = \frac{\partial}{\partial q_\alpha} \mu_{\alpha f} \cdot c_f e^{\mathcal{L}_s \cdot t} \phi_0 \int dq_f \, q_f \left[e^{\mathcal{L}_f \cdot t} \phi(0) - \phi_0 \right] w(\vec{q}, 0) \tag{25}$$

$$= \frac{\partial}{\partial q_\alpha} \mu_{\alpha f} c_f e^{\mathcal{L}_s \cdot t} \phi_0 \int dq_f \, q_f \cdot \phi(q_f, t) w(\vec{q}_a, 0), \tag{26}$$

where $w(q_i, 0) = w(q_f, 0)w(q_\alpha, 0)$ is assumed. And

$$\phi(q_f, t) = e^{\mathcal{L}_f \cdot t} \cdot \phi(q_f, t = 0) \tag{27}$$

denotes the solution of 1-D Smoluchowski equation, which is well known [34–37]:

$$\phi(q_f, t) = \frac{1}{\sqrt{2\pi\sigma(t)}} \exp\left[-\frac{(q_f - \langle q_f(t)\rangle)^2}{2\sigma(t)}\right], \tag{28}$$

with

$$\langle q_f(t)\rangle = q_{f0}e^{-\mu_f c_f \cdot t} \quad \text{and} \quad \sigma(t) = \sigma(0)e^{-2\mu_f c_f \cdot t} + \frac{T}{c_f}\left[1 - e^{-2\mu_f c_f \cdot t}\right], \tag{29}$$

where q_{f0} is the initial value of the fast variable. Finally, integrated over q_f, the term I_1 becomes

$$\vec{B}(t)^T \cdot \frac{\partial}{\partial \vec{q}} e^{\mathcal{L}_s \cdot t} \cdot w(\vec{q}_\alpha, 0) \quad \text{where} \quad B_\alpha(t)^T = \mu_{\alpha f} c_f \cdot q_{f0} \cdot e^{-\mu_f c_f \cdot t}, \tag{30}$$

which denotes α-th component of the vector $\vec{B}(t)^T$. It clearly shows that this term rapidly diminishes with the time scale of the fast variable, as expected.

Eventually, Eq. (11) turns out to be

$$\frac{\partial}{\partial t} w(\vec{q}_\alpha, t) = \mathcal{L}^{\text{eff}} \cdot w(\vec{q}_\alpha, t) + \vec{B}(t)^T \cdot \frac{\partial}{\partial \vec{q}_\alpha} e^{\mathcal{L}_s \cdot t} w(\vec{q}_\alpha, 0). \tag{31}$$

The question is what the last term in r.h.s. of Eq. (31), i.e., the inhomogeneous term, gives rise to. As usual, a formal solution is written down,

$$w(\vec{q}, t) = e^{\mathcal{L}^{\text{eff}} \cdot t} \cdot w^{\text{eff}}(\vec{q}, t), \tag{32}$$

with

$$w^{\text{eff}}(q, t) = \left[1 + \int_0^t dt'\, e^{-\mathcal{L}_s \cdot t'} \cdot \vec{B}(t')^T \cdot \frac{\partial}{\partial \vec{q}}\right] w(\vec{q}, t = 0). \tag{33}$$

This means that the reduced system for only slow variables is described by the renormalised Smoluchowski operator $\mathcal{L}^{\text{eff}}$ with the t-dependent "initial distribution". The function $w^{\text{eff}}(\vec{q}, t)$ is equal to $w(\vec{q}, 0)$ at $t = 0$, while at $t \gg 1/(\mu_f \cdot c_f)$, i.e., after the equilibration of the fast variable,

$$w^{\text{eff}}(\vec{q}, t) = \left[1 + \bar{B} \cdot \frac{\partial}{\partial \vec{q}}\right] w(\vec{q}, 0) \cong \exp\left[\bar{B} \cdot \frac{\partial}{\partial \vec{q}}\right] \cdot w(\vec{q}, 0), \tag{34}$$

where we use again Laplace approximation in the time integration to calculate the 2nd term in Eq. (33). $\bar{B}$ denotes a constant shift vector whose component is given explicitly,

$$\bar{B}_\alpha = \frac{\mu_{\alpha f}}{\mu_f} q_{f0}, \quad \text{and the initial value is effectively shifted as} \quad q_{\alpha 0}^{\text{eff}} = q_{\alpha 0} - \bar{B}_\alpha. \tag{35}$$

In other words, after the beginning short time, the system evolves as if it has started with the *slipped initial point*. Now, it should be noted that the slip which gives the additional hindrance is given in terms of physical parameters of the system.

Applied to heavy-ion fusion and the slow relative distance between the ions at contact, the initial slip $-\bar{B}_R$ can be estimated for the system ^{209}Bi $+^{70}$Zn, for example. Using the TCP of LDM energy [53, 54] and the one-body dissipation [55], a preliminary value obtained is about 2.3 fm. Considering the approximations and the different parameterization of nuclear shapes, the result is similar to the phenomenological "injection point parameter s" which is about 2 fm in the FBD model [10, 11]. Systematic analyses will be given elsewhere.

2.3. Formation Probability with the initial shift: A New Dynamical Hindrance

Now we obtain explicit expressions of Formation Probability (Hindrance Factor), taking into the initial slip. The solution of Smoluchowski equation with parabolic potential and given initial conditions is known as,

$$w(\vec{q}, t) = \frac{1}{(\sqrt{2\pi})^N} \frac{1}{\sqrt{\det(\Sigma(t))}} \cdot \exp\left[-\frac{1}{2}(\vec{q} - e^{-\mu \cdot c \cdot t}\vec{q}_0)^T \cdot \Sigma(t)^{-1} \cdot (\vec{q} - e^{-\mu \cdot c \cdot t}\vec{q}_0)\right], \quad (36)$$

with

$$\Sigma(t) = \frac{T}{c}\left(1 - e^{-2c \cdot \mu \cdot t}\right). \quad (37)$$

Here note that the exponent is $2 \cdot c \cdot \mu$, not $\mu \cdot c$, and that the matrix Σ is symmetric, so Σ^{-1} as well. The initial vector $\vec{q}_0$ should be the slipped one due to the effective initial value given in Eq. (35) and the matrix μ should be the effective one given in Eq. (20), though it is supposed to be close to the original one.

We now consider the system of the two slow variables (R, α) after the elimination of the fast neck variable ε. The relative distance R faces a parabolic barrier and the asymmetry α, a parabolic well. The system is considered as fused when the relative distance has crossed the barrier. If not, the system reseparates. With the Smoluchowski approach, the formation of CN is solely due to the thermal diffusion over the barrier [10, 11]. Note that an initial slip affects both variables, R and α. For the latter, this has no effect on the final equilibration because the potential well confines it. However, for the former, it increases the height of the potential barrier faced by the system and reduces the formation probability.

The formation probability is obtained by integrating over both variables. However, for the radial coordinate R, to only count the tail of the distribution function that crosses the barrier whose top is set at $R = 0$, the integration is restricted to the interval that goes from minus infinity to zero. Finally, we obtain the formation probability as a function of time t,

$$P_f(t) = \int_{-\infty}^{0} dR \, w(R, t) = \frac{1}{2}\text{erfc}\left[r(t)\right], \quad (38)$$

with

$$r(t) = \frac{1}{\sqrt{2\Sigma(t)_{RR}}} \exp\left(-\mu \cdot c \cdot t\right) \cdot \begin{pmatrix} R_0 \\ \alpha_0 \end{pmatrix}_1, \quad (39)$$

where the subscript 1 denotes the first component of the vector which is the product of the exponential matrix and the vector in two dimensions. The complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt. \quad (40)$$

The formation probability is explicitly obtained by taking the time limit $t \to \infty$ which gives a simple approximate formula [56, 57],

$$r(t) \underset{t\to\infty}{\simeq} \sqrt{\frac{V_R}{T}}, \quad \text{with} \quad V_R = \frac{1}{2}|c_{11}| \cdot \bar{R}_D^2 \quad \text{and} \quad \bar{R}_D = R_0 - \frac{\mu_{11}c_{12} + \mu_{12}c_{22}}{\mu_{11}|c_{11}| + \mu_{22}c_{22}}\alpha_0. \quad (41)$$

Then,

$$P_{\text{form}} \simeq \frac{1}{2\sqrt{\pi}}\sqrt{\frac{T}{V_R}} e^{-V_R/T}, \quad (42)$$

where the asymptotic expansion of the error function is used for large argument. If the second term in r.h.s of Eq. (41) for the expression of $\bar{R}_D$ which depends on α_0, stemming

from the coupling to the asymmetry α is negligibly small, the above formula is the same as a simple 1-D problem. However, it is worth noticing here that R_0, the initial point above, is the slipped one given in Eq. (35), leading to a larger barrier height V_R, like in FBD model.

3. Concluding Remarks

We consider the dynamical evolution of di-nucleus systems of very heavy ions as relaxation processes of collective degrees of freedom with different time scales, starting with the initial conditions given by the incident channel. We have analysed effects of the elimination of the fast neck degree on the dynamics of the remaining slow variables (mass-asymmetry and radial), starting with N-dimensional Smoluchowski equation with the use of N-Z method and of perturbation approximation. We have found that Smoluchowski operator is renormalised and that the inhomogeneous term appears to result in a *slip* in the initial values of the slow variables, which provides a dynamical origin of the injection point phenomenologically introduced in FBD. Since the slip is given in terms of the physical parameters of the incident di-nuclear system, we can compare the hindrance effects among various incident channels, not only the cold and the hot fusion paths, but also other possible incident channels on-going now and in future.

It is noteworthy here that a constant neck value often assumed in numerical Langevin calculations is only valid after the relaxation time of the neck. Neglect of the dynamics in the beginning short time of the neck relaxation should be paid by the slip of the initial values of slow variables.

Instead of N-dimensional Smoluchowski equation, we can also start with a Klein-Kramers equation in $2N$-dimensional (q,p) phase space which is equivalent to the full Langevin equation used in Ref. [33]. For intuition, we take two steps, depending on time scales: Firstly, we eliminate the whole momentum space, to obtain N-dimensional Smoluchowski equation with memory kernel and an inhomogeneous term. Next, we follow the neck elimination given in the present article. We end up with three inhomogeneous terms, which, in the lowest order, gives rise to a sum of two shifts from the momentum and the neck eliminations, respectively. The results coincide with those obtained in [33]. This will be published elsewhere, together with systematic analysis of incident-channel dependence. The present dynamical approach shall be widely developed not only to fusion, but also to quasi-fission. It will give a novel vista to various dynamical aspects in heavy-ion collisions as well as to quantitative predictions of synthesis cross sections of SHE.

References

[1] Sigurd Hofmann, *Superheavy Elements*, Lect. Notes Phys. 764, 203-252 (2009)

[2] K.-H. Schmidt and W. Morawek, Reports on Progress in Physics **54**, 949 (1991).

[3] Hongliang Lü, Anthony Marchix, Yasuhisa Abe, and David Boilley, Comp. Phys. Comm. **200**, 381-399 (2016)

[4] Hongliang Lü, David Boilley, Yasuhisa Abe, and Caiwan Shen, Phys. Rev. C, **94**, 034616 (2016).

[5] G. Royer and B. Remaud, Nucl. Phys. A, **444**, 477 (1985).

[6] Caiwan Shen, Grigori Kosenko, and Yasuhisa Abe, Phys. Rev. C, **66**, 061602(R) (2002).

[7] Yasuhisa Abe, David Boilley, Grigori Kosenko, Jingdong Bao, Bertrand Giraud, and Takahiro Wada, Prog. Theor. Phys. Suppl., **146**, 104 (2002).

[8] Yasuhisa Abe, Eur. Phys. J. **A13**, 143 (2002)

[9] Yasuhisa Abe, David Boilley, Grigori Kosenko, and Caiwan Shen, Acta Phys. Pol. **B34**, 2091 (2003)

[10] W. J. Świątecki, K. Siwek-Wilczyńska, and J. Wilczyński, Acta Phys. Pol. **B34**, 2049 (2003)

[11] W. J. Świątecki, K. Siwek-Wilczyńska, and J. Wilczyński, Phys. Rev. C, **71**, 014602 (2005).

[12] Valery Zagrebaev and Walter Greiner, J. Phys. G, **31**, 825 (2005).

[13] H. A. Kramers, Physica (Amsterdam) **7**, 284 (1940).

[14] P. Grangé, Li Jun-Qing, and H. A. Weidenmüller, Phys. Rev. C, **27**, 2063 (1983).

[15] Yasuhisa Abe, Christian Grégoire, and Hugue Delagrange, J. de Physique **C4**, 329 (1986)

[16] David Boilley, Eric Suraud, Yasuhisa Abe, and Sakir Ayik, Nucl. Phys. A, **556**, 67 (1993).

[17] Y. Abe, S. Ayik, P.-G. Reinhard, and E. Suraud, Phys. Reports **275**, 49 (1996)

[18] P. Fröbrich, I.I. Gontchar, Phys. Rep. **292**, 131 (1998)

[19] Y. Aritomo, T. Wada, M. Ohta, and Y. Abe, Phys. Rev. C, **59**, 796 (1999).

[20] Bertrand Bouriquet, Grigori Kosenko and Yasuhisa Abe, Eur. Phys. J. **A22**, 9 (2004)

[21] Caiwan Shen, Yasuhisa Abe, David Boilley, Grigori Kosenko, and E. Zhao, Intern. J. Mod. Phys. E17, suppl, 66 (2008).

[22] T. Cap, K. Siwek-Wilczyńska, and J. Wilczyński, Phys. Rev. C, **83**, 054602 (2011).

[23] K. Siwek-Wilczyńska, T. Cap, M. Kowal, and J. Wilczyński, Phys. Script. T **154**, 014005 (2013)

[24] T. Cap, K. Siwek-Wilczyńska, I. Skwira-Chalot, and J. Wilczyński, Acta Phys. Pol. **B43**, 297 (2012)

[25] K. Siwek-Wilczyńska, T. Cap, M. Kowal, A. Sobiczewski, and J. Wilczyński, Phys. Rev. C, **86**, 04611 (2012).

[26] Kouichi Hagino, Phys. Rev. C, **98**, 014607 (2018).

[27] Yasuhisa Abe, Caiwan Shen, Grigori Kosenko, David. Boilley, and Bertrand G. Giraud, Int. J. Mod. Phys. E **17**, 2214(2008)

[28] Y. Abe, C. Shen, D. Boilley, and B. G. Giraud, Int. J. Mod. Phys. E **18**, 2169 (2009)

[29] D. Boilley, C. Shen, Y. Abe, and B. G. Giraud, EPJ Web of Conf. **17**, 10001 (2011)

[30] David Boilley, Hongliang Lü, Caiwan Shen, Yasuhisa Abe, and Bertrand G. Giraud, Phys. Rev. C, **84**, 054608 (2011).

[31] Yu-Jie Liang, Min Zhu, Zu-Hua Liu, and Wen-Zhong Wang, Eur. Phys. J. A **48**, 133 (2012)

[32] Min Zhu, Jun-Li Fu, Zhen Qu, Zu-Hua Liu, and Wen-Zhong Wang, Chin. Phys. Lett. **30**, 082401 (2013)

[33] David Boilley, Yasuhisa Abe, Bartholomé Cauchois, and Caiwan Shen, J. Phys. G, **46**, 115102 (2019). and J. Phys. G, **46**, 129601 (2019).

[34] S. Chandrasekhar, Rev. Mod. Phys., **15**, 1 (1943).

[35] N. G. Van Kampen, Phys. Reports, **124**(2), 69 (1985)

[36] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry*, North Holland (1981)

[37] H. Risken, *The Fokker-Planck equation*, Springer-Verlag (1984)

[38] C. W. Gardiner, *Handbook of Stochastic Methods*, Springer-Verlag (1990)

[39] Harold Grad, The Physics of Fluids, **6**, 147 (1963)

[40] James A. McLennan, Phys. Rev. A, **10**, 1272 (1974).

[41] B. C. Sanctuary and R. F. Snider, J. Chem. Phys. **67**, 5517 (1977)

[42] U.M. Titulaer, of a Brownian particle in the high friction case. Physica A **91**, 321 (1978)

[43] Fritz Haake and Maciej Lewenstein, Phys. Rev. A, **28**, 3606 (1983).

[44] Stephen M. Cox and A.J. Roberts Physica D **85**, 126 (1995)

[45] P. Langevin, C. R. Acad. Sci. (Paris) **146**, 530 (1908)

[46] A. D. Fokker, Ann d. Physik, **43**, 810 (1914)

[47] M. Planck, Sitzung der physikalisch-mathematischen Klasse vom 10. Mai 1917

[48] M. von Smoluchowski, Ann d. Physik, **21**, 757 (1906)

[49] O. Klein, Ark. Math. Astr. Fys. **16**, 1 (1921)

[50] Sadao Nakjima, Prog. Theor. Phys., **20**, 948 (1958).

[51] Robert Zwanzig, J. Chem. Phys. **33**, 13388 (1960)

[52] A.O. Caldeira and A.J. Leggett, Annals of Phys. 149, 374 (1983)

[53] J. Maruhn and W. Greiner, Z. Physik **251**, 431 (1972)

[54] K. Sato, S. Yamaji, K. Harada and S. Yoshida, Z. Physik **A290**, 149 (1979)

[55] J. Blocki, Y. Boneh, J. R. Nix, J. Randrup, M. Robel, A. Sierk, and W. J. Swiatecki, Ann. Phys. **113**, 330 (1978)

[56] Y. Abe, D. Boilley, B.G. Giraud, and T. Wada, Phys. Rev. E**61**, 1125 (2000)

[57] D. Boilley, Y. Abe, and J.-D. Bao, Eur. Phys. J. A **18**, 627 (2003)