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LACK OF NULL CONTROLLABILITY OF ONE DIMENSIONAL LINEAR COUPLED TRANSPORT-PARABOLIC SYSTEM WITH VARIABLE COEFFICIENTS

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ABSTRACT. In this article, we study the null controllability of linear coupled transport-parabolic systems with variable coefficients in one space dimension. We consider coupled systems with coupling of order zero, one and two. The systems are considered with homogeneous Dirichlet boundary conditions and with localized interior controls acting on both transport and parabolic equations. We show that coupled systems are not null controllable at small time. This time depends on the transport velocity and the support of the controls. When the transport velocity is identically zero, the systems are not null controllable at any time. To achieve these results, we construct highly localized solutions, known as *Gaussian beams*, corresponding to the adjoint systems, and using them, we show that the corresponding observability inqualities fail. However, these systems are null controllable at any time by controls acting everywhere in the parabolic equation, under suitable assumptions on the initial data and the coefficients.

1. INTRODUCTION AND MAIN RESULTS

The study of the controllability of coupled hyperbolic-parabolic systems has been an active area of research over the last few years. The coupled system arises to describe the physical phenomena in fluid dynamics, plasma physics, aeronautics, weather prediction and so on. Our motivation to study such coupled system comes from viscous compressible fluid models. The Navier-Stokes system of a viscous, compressible, isothermal barotropic fluid (density is function of pressure only), in a bounded domain $(0, L)$ is given by

$$\begin{cases} \partial_t \widehat{\rho} + \partial_x(\widehat{\rho}\widehat{u}) = 0 & \text{in } (0,T)\times(0,L), \\ \widehat{\rho}\,(\partial_t \widehat{u} + \widehat{u}\partial_x \widehat{u}) - \mu \partial_{xx}\widehat{u} + \partial_x \widehat{p} = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{1.1}$$

where $\widehat{\rho}(t,x)$ is the density of the fluid and $\widehat{u}(t,x)$ is its velocity, and the positive constant μ represents the fluid viscosity. The pressure $\widehat{p}$ satisfies the following constitutive law

$$\widehat{p}(\widehat{\rho}) = a\widehat{\rho}^{\gamma}, \qquad a > 0, \gamma \geqslant 1.$$

We assume that $(\rho_s(x), u_s(x))\,, x \in [0, L]$ is a stationary trajectory to the system (1.1). By setting

$$\widehat{\rho}(t,x) = \rho(t,x) + \rho_s(x), \qquad \widehat{u}(t,x) = u(t,x) + u_s(x), \quad x \in (0,L),\ t \in (0,T),$$

and collecting the linear terms in ρ and u, we obtain the following linear system :

$$\begin{cases} \partial_t \rho + \partial_x(\rho_s u) + \partial_x(u_s \rho) = 0 & \text{in } (0,T)\times(0,L), \\ \rho_s \partial_t u - \mu\partial_{xx} u + \rho_s \partial_x(u_s u) + (u_s\partial_x u_s + a\gamma\partial_x(\rho_s^{\gamma-1}))\rho + a\gamma\rho_s^{\gamma-1}\partial_x\rho = 0 & \text{in } (0,T)\times(0,L). \end{cases} \tag{1.2}$$

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Motivated by the above example, in this article, we first consider the following linear coupled transport-parabolic system with coupling of order zero and one, and with controls f_1 and f_2:

$$\begin{cases} \partial_t \rho + a_0 \partial_x \rho + a_1 \rho + c_1 \partial_x u + c_2 u = \mathbb{1}_{\mathcal{O}1} f_1 & \text{in } (0,T)\times(0,L), \\ \partial_t u - b_0 \partial_{xx} u + b_1 \partial_x u + b_2 u + d_1 \partial_x \rho + d_2 \rho = \mathbb{1}_{\mathcal{O}2} f_2 & \text{in } (0,T)\times(0,L), \end{cases} \tag{1.3}$$

where $\mathbb{1}_{\mathcal{O}_j}$ is the characteristic function of an open set $\mathcal{O}_j \subseteq (0,L)$, $j=1,2$. We complete the system (1.3) with the following initial condition

$$\rho(0,\cdot) = \rho^0, \qquad u(0,\cdot) = u^0 \text{ in } (0,L), \tag{1.4}$$

and the boundary conditions

$$\begin{cases} u(t,0) = u(t,L) = 0 & \text{in } (0,T), \\ \rho(t,0) = 0 & \text{in } (0,T), \text{ if } a_0(0) > 0, \\ \rho(t,L) = 0 & \text{in } (0,T), \text{ if } a_0(L) < 0. \end{cases} \tag{1.5}$$

Throughout this article we make the following assumptions on the coefficients :

$$\begin{aligned} a_i, b_j, c_i, d_i \in C^\infty([0,L]), \qquad &\text{for all } i=0,1, \qquad \text{for all } j = 0,1,2, \\ b_0(x) \geqslant \bar{b} > 0 \qquad &\text{for all } x \in [0,L]. \end{aligned} \tag{1.6}$$

In Section 2, we will show that the system (1.3)-(1.5) is well-posed in $\left(L^2(0,L)\right)^2$ (see Proposition 2.2). We are interested in the null controllability of the system (1.3)-(1.5).

Definition 1.1. *The system (1.3)-(1.5) is null controllable in $\left(L^2(0,L)\right)^2$ at time $T>0$ if for any $\left(\rho^0,u^0\right) \in \left(L^2(0,L)\right)^2$, there exist controls $f_i \in L^2(0,T;L^2(0,L))$, $i=1,2$, such that, (ρ,u), the solution to the system (1.3)-(1.5) satisfies*

$$(\rho,u)(T,x) = 0 \textit{ for all } x \in (0,L).$$

For later purpose, we introduce the spaces

$$L^2_{\mathrm{m}}(0,L) = \left\{ f \in L^2(0,L) \mid \int_0^L f \, \mathrm{d}x = 0 \right\}, \qquad H^s_{\mathrm{m}}(0,L) = H^s(0,L) \cap L^2_{\mathrm{m}}(0,L), \quad s>0.$$

Before stating our main results, let us mention some related works in this direction from the literature. As mentioned above, the compressible Navier-Stokes system linearized around a constant trajectory (ρ_s,u_s) for $\rho_s>0$, yields a coupled system with constant coefficients: in particular, a coupled ODE-parabolic system for $u_s=0$ and a coupled transport-parabolic system for $u_s \neq 0$ (see (1.2)). The controllability of such systems with constant coefficients in one dimension has been extensively studied. In [7], the linearized compressible Navier-Stokes system around $(\rho_s,0)$, i.e., the coupled ODE-parabolic system, is considered in $(0,L)$ with Dirichlet boundary conditions. In that paper, the authors proved that the system is not null controllable at any time $T>0$ by a localized interior control acting only in the parabolic equation. However, the system is null controllable in $H^1_{\mathrm{m}}(0,L)\times L^2(0,L)$ at any time $T>0$ using everywhere L^2-control in the parabolic equation. The case $u_s \neq 0$ was considered in [6, 5]. In both the articles, the system was considered in $(0,2\pi)$ with periodic boundary conditions, and with localized control acting only in the parabolic equation. In [5], using moment method, the authors proved the null controllability in $H^{s+1}_{\mathrm{m}}(0,2\pi)\times H^s(0,2\pi)$, $s>6.5$, at time $T>\frac{2\pi}{|u_s|}$. This result was improved in [6] by showing that the null controllability holds for any initial data in $H^1_{\mathrm{m}}(0,2\pi)\times L^2(0,2\pi)$. Moreover, it was also proved that, the system in consideration is not null controllable in $H^s_{\mathrm{m}}(0,2\pi)\times L^2(0,2\pi)$, $0\leqslant s<1$, at any time $T>0$ by L^2-control acting in the parabolic equation. Thus $H^1_{\mathrm{m}}(0,2\pi)\times L^2(0,2\pi)$ is the largest space in which the system is null controllable by a L^2-parabolic control. It is worth mentioning that, all the above works consider only the case where control is active on the parabolic equation only. Furthermore, the proofs are based on explicit computation of the eigenvalues and eigenfunctions of the linear operator, and thus restricted to certain boundary conditions.

Later in [20], the lack of null controllability issues associated to the linearized compressible Navier-Stokes systems have been studied in detail. In this article, the author studied compressible Navier-Stokes-Fourier systems for non-barotropic fluid linearized around constant steady states (ρ_s, u_s, θ_s) with $\rho_s > 0, \theta_s > 0$ in $(0, L)$. The systems are coupled between two parabolic equations and an ODE (if $u_s = 0$) or an transport equation (if $u_s \neq 0$). If $u_s = 0$, the system is not null controllable in $(L^2(0, L))^3$, by a localized control acting on the ODE/transport component and parabolic controls, at any time $T > 0$. If $u_s \neq 0$, the same result holds only for small time. Moreover, in the case $u_s = 0$, the system is null controllable in $H^1_{\mathrm{m}}(0, L) \times L^2(0, L) \times L^2(0, L)$ at any finite time $T > 0$ using controls in both parabolic equations acting everywhere in the domain. And, this result is optimal in the sense that, null controllability cannot be achieved by localized parabolic controls. Let us mention that the proofs in [20] do not require the knowledge of the spectrum. Thus the results can be extended to any suitable boundary conditions. Recently, in [1], the above results have been extended to more general constant coefficient coupled transport-parabolic systems. They considered coupling of several transport and parabolic equations in one dimensional torus. They proved null controllability in optimal time. Moreover, an algebraic necessary and sufficient condition, on the coupling term, was proved when controls act only on the parabolic or transport components.

In the context of compressible Navier-Stokes systems, the local exact controllability around constant states was studied in [10, 9, 24, 23, 25], and an analogous result around variable trajectories was obtained in [11].

We also note that such coupled system may arise to model parabolic equations with memory terms, damped wave equations, visco-elastic flows and so on, for instance see [22, 28, 3, 4, 18, 13, 2, 21]. Using a change of variables, for a suitable memory term, parabolic equation with memory can be written as a coupled ODE-parabolic system. An extensive study of controllability of evolution equations with memory term has been done in [4]. In particular, the lack of null controllability has been studied in [15, 16, 17] (heat equation with memory), [21, 28, 27] (viscous-elastic flows) and references therein.

In this present work, our first aim is to show the lack of null controllability of (1.3)-(1.5), when the coefficients are not necessarily constant. To state our results, we need to introduce some notations. We take an extension of a_0 on $\mathbb{R}$, still denoted by a_0. We introduce the characteristics X associated with a_0:

$$\begin{cases} \partial_t X(t,x) = -a_0\left(X(t,x)\right) \qquad (t \geqslant 0), \\ X(0,x) = x \qquad x \in \mathbb{R}. \end{cases} \tag{1.7}$$

Let $\mathcal{O} \subset (0, L)$ be such that $(0, L) \backslash \overline{\mathcal{O}}$ is a nonempty open subset of $(0, L)$. For each $x \in (0, L) \backslash \overline{\mathcal{O}}$, we set

$$T_{x,\mathcal{O}} := \sup\left\{\tau \mid X(t,x) \in (0,L) \setminus \overline{\mathcal{O}} \text{ for all } t \in [0,\tau)\right\}, \tag{1.8}$$

and

$$T_{\mathcal{O}} := \sup_{x \in (0,L) \backslash \overline{\mathcal{O}}} T_{x,\mathcal{O}}. \tag{1.9}$$

If $\mathcal{O}$ is the empty set of $(0, L)$, we set for each $x \in (0, L)$,

$$T_{x,\emptyset} = \sup\left\{\tau \mid X(t,x) \in (0,L) \text{ for all } t \in [0,\tau)\right\}, \quad \text{and} \quad T_{\emptyset} = \sup_{x \in (0,L)} T_{x,\emptyset}. \tag{1.10}$$

Our first main result indicates the lack of null controllability of (1.3)-(1.5) for initial data in $L^2(0, L) \times L^2(0, L)$:

Theorem 1.2. *Assume* (1.6)*, and*

$$\mathcal{O}_1 \subset (0, L), \qquad \mathcal{O}_2 \subseteq (0, L),$$

be such that $(0, L) \setminus \overline{\mathcal{O}}_1$ is a nonempty open subset of $(0, L)$. Then the system (1.3)-(1.5) *is not null controllable in $L^2(0, L) \times L^2(0, L)$, at any time $0 < T < T_{\mathcal{O}_1}$, by interior controls $f_1 \in L^2(0, T; L^2(0, L))$ supported in $\mathcal{O}_1$ and $f_2 \in L^2(0, T; L^2(0, L))$ with support in $\mathcal{O}_2$.*

Remark 1.3. *Let us make the following remarks:*

(1) *In the above theorem, for the controllability of* (1.3)-(1.5)*, the minimal time* $T_{\mathcal{O}_1}$ *can be either finite or infinite. Whenever* $T_{\mathcal{O}_1} = +\infty$*, we say that the system is not null controllable at any time. For instance, if* $a_0 = 0$*, then the system is coupled between an ODE and a parabolic equation. In this case,* $T_{\mathcal{O}_1} = +\infty$*, for* $\mathcal{O}_1$*, any proper open subset of* $(0, L)$*, and the system is not null controllable at any time* $T > 0$*. Furthermore, if* $a_0(x) = 0$ *for some* $x \in (0, L) \setminus \overline{\mathcal{O}_1}$*, then also* $T_{\mathcal{O}_1} = +\infty$*, and the corresponding system is not null controllable at any time* $T > 0$.

(2) *If* $a_0 = \bar{a}$*(constant) and* $\mathcal{O}_1 = (\ell_1, \ell_2) \subset (0, L)$ *then* $T_{\mathcal{O}_1} = \max\left\{\frac{\ell_1}{|\bar{a}|}, \frac{L-\ell_2}{|\bar{a}|}\right\}$*. Note that, this minimal time coincides with minimal time obtained in* [1, Eq (3)]*. Moreover, according to* [1, Theorem 2]*, the system* (1.3)-(1.4)*, with constant coefficients and periodic boundary conditions, is null controllable at time* $T > T_{\mathcal{O}_1}$*. This indicates that, perhaps in general, the minimal time obtained here is sharp. More precisely, we may expect null controllability of the system* (1.3)-(1.5) *if* $T > T_{\mathcal{O}_1}$*. However, to the best of our knowledge, this is still not known, even in the constant coefficient case with homogeneous Dirichlet boundary conditions.*

(3) *In Theorem 1.2, the time* $T_{\mathcal{O}_1}$ *depends only on* a_0 *and* $\mathcal{O}_1$*, and is independent of the choice of* $\mathcal{O}_2$*. In particular, in the theorem, the control* f_2 *may act everywhere in* $(0, L)$.

Let us now give some special attention to the case where control is not active in the transport equation $(1.3)_1$, i.e., $f_1 \equiv 0$. In this case, according to Theorem 1.2, the system (1.3)-(1.5) is not null controllable at any time $0 < T < T_{\emptyset}$ (see (1.10)). In particular, the time to obtain the controllability depends only on the transport velocity a_0. However, if there is no inflow, or if $a_0(x) = 0$ for some $x \in (0, L)$, then the system is not null controllable at any time $T > 0$. This is precisely stated in the next theorem.

Theorem 1.4. *Assume* (1.6)*,* $f_1 \equiv 0$ *and* $\mathcal{O}_2 \subseteq (0, L)$.

(i) *Then the system* (1.3)-(1.5) *is not null controllable in* $L^2(0, L) \times L^2(0, L)$*, at any time* $0 < T < T_{\emptyset}$*, by interior control* $f_2 \in L^2(0, T; L^2(0, L))$ *with support in* $\mathcal{O}_2$.

(ii) *Assume further that*

$$\text{either } a_0(x_*) = 0 \text{ for some } x_* \in (0, L) \text{ or } a_0(0) \leqslant 0,\ a_0(L) \geqslant 0. \tag{1.11}$$

Then the system (1.3)-(1.5) *is not null controllable in* $L^2(0, L) \times L^2(0, L)$*, at any time* $T > 0$*, by interior control* $f_2 \in L^2(0, T; L^2(0, L))$ *with support in* $\mathcal{O}_2$.

Remark 1.5. *Let us point out that, when the coefficients in* (1.3) *are constant, similar results to Theorem 1.2 and Theorem 1.4, were already proved in* [7, 20, 1]*. More precisely, we refer the reader to* [20, Theorem 1.1, Theorem 1.5]*,* [7, Theorem 5.10] *and* [1, Theorem 2(i)] *for precise statements of the results when the coefficients are constant.*

In (1.3)-(1.5), we have dealt with the coupled transport-parabolic system with coupling of order one or zero. Next, we consider the following coupled transport-parabolic system with coupling of order two in the transport equation:

$$\begin{cases} \partial_t \rho + a_0 \partial_x \rho + a_1 \rho + c_0 \partial_{xx} u + c_1 \partial_x u + c_2 u = \mathbb{1}_{\mathcal{O}_1} f_1 & \text{in } (0, T) \times (0, L), \\ \partial_t u - b_0 \partial_{xx} u + b_1 \partial_x u + b_2 u + d_2 \rho = \mathbb{1}_{\mathcal{O}_2} f_2 & \text{in } (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = 0 & \text{in } (0, T), \\ \rho(t, 0) = 0 & \text{in } (0, T), \text{ if } a_0(0) > 0, \\ \rho(t, L) = 0 & \text{in } (0, T), \text{ if } a_0(L) < 0, \\ \rho(0, x) = \rho^0(x), \qquad u(0, x) = u^0(x) & \text{in } (0, L). \end{cases} \tag{1.12}$$

We make the following assumptions on the coefficients :

$$\begin{aligned} a_i, b_j, c_j, d_2 \in C^\infty([0, L]), \text{ for all } i = 0, 1, \quad \text{for all } j = 0, 1, 2, \\ b_0(x) \geqslant \bar{b} > 0, \quad c_0(x) \neq 0 \text{ for all } x \in [0, L]. \end{aligned} \tag{1.13}$$

The above type of systems arises in many physical phenomena, mostly to model fluid flows with visco-elastic effects; for example heat equation with memory terms, linearized Burgers equation with memory terms etc. The well-posedness of (1.12) is studied in Theorem 2.8. Regarding the lack of null controllability of (1.12), we obtain the following result:

Theorem 1.6. *Assume* (1.13)*, and*

$$\mathcal{O}_1 \subset (0,L), \qquad \mathcal{O}_2 \subset (0,L),$$

be such that $(0,L)\backslash\overline{\mathcal{O}_1 \cup \mathcal{O}_2}$ *is a nonempty open subset of* $(0,L)$*. Then the system* (1.12) *is not null controllable in* $L^2(0,L) \times L^2(0,L)$*, at any time* $0 < T < T_{\mathcal{O}_1 \cup \mathcal{O}_2}$*, by interior controls* $f_1 \in L^2(0,T;L^2(0,L))$ *supported in* $\mathcal{O}_1$ *and* $f_2 \in L^2(0,T;L^2(0,L))$ *with support in* $\mathcal{O}_2$*.*

In contrast to Theorem 1.2, the minimal time in the above theorem also depends on the support of the parabolic control. This is essentially due to different order of coupling (see Remark 3.4 for more details).

Remark 1.7. (1) *In this article, for simplicity, we have assumed all the coefficients are smooth. However, a careful reading of the proofs indicate that it is enough to assume that the coefficients belong to* C^4*.*

(2) *The results mentioned above can also be extended to the case where coefficients depend on both time and space, and the inflow boundary is independent of time. For the time independent case, in Section 2, we have used semigroup theory for well-posedness of the systems. Also, in this case the systems are autonomous, and we can directly use the duality between controllability and observability (see for instance*[8, Chapter 2.3]*) to derive required observability inequalities. However, in Section 3, the main tool to prove the main results i.e. Gaussian beam solutions are constructed for coefficients depending on both space and time variables. Thus all the results mentioned above also hold for coefficients depending on both time and space, provided one has well-posedness for all the systems considered above and duality between controllability and observability of such systems. In Section 6, we briefly indicate how to extend the above results to the non-autonomous case.*

When the coefficients are constant, the lack of null controllability of a system similar to (1.12) was studied in [20] (see Eq. (2.58) of [20]). Moreover, using this result, the lack of null controllability of (1.3)-(1.5) in $H^1(0,L) \times L^2(0,L)$ was proved in [20, Theorem 1.4], in the case where the coefficients are constant. Thus we may expect a similar behaviour for our case also. In fact, as a consequence of Theorem 1.6, we prove the lack of null controllability of (1.3)-(1.5) in small time in $H^1(0,L) \times L^2(0,L)$ under suitable assumptions on the coefficients. This is precisely stated in Theorem 4.3.

We note that the null controllability property for (1.12) is different from that of (1.3)-(1.5). In contrast to Theorem 1.2, the above theorem holds if $\mathcal{O}_2$, the support of the control for the parabolic component, is a proper subset of $(0,L)$. Therefore, one may expect null controllability of the system (1.12) if the control is active everywhere. In fact, under some assumptions on the coefficients we have obtained such a result in Section 5 (see for instance Theorem 5.4). Moreover, as a consequence of the results obtained in Section 5, we prove Theorem 5.5 which shows null controllability of (1.3)-(1.5) in $H^1_{\mathrm{m}}(0,L) \times L^2(0,L)$ by everywhere control in the parabolic equation.

The proof of these results is based on duality arguments. It is well known that the null controllability of a linear system is equivalent to a certain observability inequality satisfied by the solution of the corresponding adjoint problem (see [8, Chapter 2]). Thus to prove Theorem 1.2, Theorem 1.4 and Theorem 1.6, we construct special solutions to the corresponding adjoint problems violating the observability inequality. The idea is to construct highly localized solutions known as "Gaussian beam". These are high frequency solutions to PDEs which are concentrated on a single ray (characteristic) through space-time. This kind of construction has been used for hyperbolic systems in [26] to study the propagation of singularities of solution.

In order to observe these Gaussian beams, the observation set must intersect every ray. If not, one could construct a Gaussian beam along a ray that would not hit the observation set and clearly that Gaussian beam could not be observed, since the estimate of the observation would be negligible outside an arbitrarily small neighborhood of the ray. Exploiting this idea, in this article, we construct Gaussian beam solutions for coupled transport-parabolic systems with variable coefficients. It is clear that our systems are not strictly hyperbolic. Thus the Gaussian beam construction of [26] cannot be applied directly here. We also refer the reader to [19] for construction of such solutions and its application to prove the lack of observability for wave equation with variable coefficients (transmission problems, oscillating coefficients etc).

The novelty of our work is that we thoroughly study the null controllability of the transport-parabolic coupled systems with variable coefficients and with different orders of coupling, for example, zero, first and second order of coupling. The behavior of the null controllability of the coupled system may change according to the order of the coupling. Moreover, our results are a generalization of results available for the coupled systems with constant coefficients. It is clear that our systems are not strictly hyperbolic and they consist of the properties of both hyperbolic and parabolic equations. Further, if the initial conditions and controls are regular, then we may get the null controllability of the coupled system provided the coefficients satisfy some conditions (see Section 5). The results can be applicable to the coupled system with different boundary conditions or even to study the null controllability using boundary controls (see Section 7). This study leads to some interesting questions regarding the controllability of the coupled systems (see Section 7.3), which we plan to study in our future work.

The outline of the paper is as follows. In Section 2, we study the well-posedness of the coupled systems using semigroup theory, and we determine the adjoints of the linear operators. Section 3 is devoted towards the constructions of Gaussian beam solutions for the coupled systems. Our main results, Theorem 1.2, Theorem 1.4 and Theorem 1.6, are proved in Section 4. In Section 5, we prove the null controllability of the coupled system by parabolic control acting everywhere in the domain, under some extra assumptions on the coefficients. In Section 6, we extend the results to the case where coefficients depend both on space and time variables. Finally, in Section 7, we mention some further extensions of our main results to the system with periodic boundary conditions and to the system with boundary controls along with some interesting comments on some open problems.

2. Well-posedness of the linear systems

In this section, we will study well-posedness of the systems in consideration via semigroup theory. We will show that the associated unbounded linear operator generates a C^0-semigroup in a suitable Hilbert space. Moreover, we will determine the adjoint of the associated linear operators.

2.1. Well-posedness of the system (1.3) – (1.5).

Our aim is to prove the following result.

Theorem 2.1. *For any $(\rho^0, u^0) \in \Big(L^2(0,L)\Big)^2$ and $f_i \in L^2(0,T;L^2(0,L))$, $i=1,2$, the system (1.3)-(1.5) admits a unique solution $(\rho, u) \in C\Big([0,T];\Big(L^2(0,L)\Big)^2\Big)$.*

Let us set

$$\mathcal{Z} = L^2(0,L) \times L^2(0,L).$$

We define the unbounded operator $(\mathcal{A}, \mathcal{D}(\mathcal{A};\mathcal{Z}))$ in $\mathcal{Z}$ by

$$\begin{aligned}\mathcal{D}(\mathcal{A};\mathcal{Z}) = \Big\{&(\rho,u) \in L^2(0,L) \times H^1_0(0,L) \ \mid \ a_0\rho' \in L^2(0,L),\\ &\rho(0)=0 \text{ if } a_0(0)>0,\ \rho(L)=0 \text{ if } a_0(L)<0,\ b_0u' - d_1\rho \in H^1(0,L)\Big\},\end{aligned}$$

and

$$\mathcal{A} = \begin{bmatrix} -a_0\dfrac{d}{dx} - a_1 & -c_1\dfrac{d}{dx} - c_2 \\ -d_1\dfrac{d}{dx} - d_2 & b_0\dfrac{d^2}{dx^2} - b_1\dfrac{d}{dx} - b_2 \end{bmatrix}. \tag{2.1}$$

We introduce the input space $\mathcal{U} = \mathcal{Z}$ and the control operator $\mathcal{B} \in \mathcal{L}(\mathcal{U}; \mathcal{Z})$ defined by

$$\mathcal{B}f = (\mathbb{1}_{\mathcal{O}_1} f_1, \mathbb{1}_{\mathcal{O}_2} f_2), \qquad f = (f_1, f_2) \in \mathcal{U}. \tag{2.2}$$

With the above notations, the system (1.3) - (1.5) can be rewritten as

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}f(t), \quad t \in (0,T), \qquad z(0) = z^0,$$

where $z(t) = (\rho(t,\cdot), u(t,\cdot))$, $z^0 = (\rho^0, u^0)$ and $f(t) = (f_1(t,\cdot), f_2(t,\cdot))$.

The well-posedness of the system (1.3) - (1.5), in particular Theorem 2.1 follows as soon as we prove the following result.

Proposition 2.2. *The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}; \mathcal{Z}))$ is the infinitesimal generator of a strongly continuous semigroup $\mathbb{T}$ on $\mathcal{Z}$.*

Proof. The proof is similar to Proposition I.5 and Theorem III.2 in [14]. For the sake of completeness, we give the proof of the proposition.
Step 1. Quasi-dissipativity: Using a density argument (see for instance [14, Proposition I.1]), integration by parts and Young's inequality, one can easily verify that

$$\operatorname{Re}\left\langle \mathcal{A}\begin{pmatrix}\rho\\ u\end{pmatrix}, \begin{pmatrix}\rho\\ u\end{pmatrix}\right\rangle_{\mathcal{Z}} \leqslant \omega\, \|(\rho,u)\|_{\mathcal{Z}}^2, \qquad \text{for all } (\rho,u) \in \mathcal{D}(\mathcal{A};\mathcal{Z}),$$

for some $\omega > 0$ sufficiently large. Therefore, there exists $\omega > 0$ such that $\mathcal{A} - \omega I$ is dissipative on $\mathcal{Z}$.

Step 2. Maximality : We want to show that $\operatorname{Range}(\lambda I - \mathcal{A}) = \mathcal{Z}$, for λ large enough. We consider only the case, where

$$a_0(0) > 0, \quad a_0(L) > 0.$$

The other cases can be treated in a similar manner. We take $\lambda > \omega$. We need to show for any $(f,g) \in \mathcal{Z}$, there exists a unique $(\rho,u) \in \mathcal{D}(\mathcal{A};\mathcal{Z})$ such that

$$\begin{cases} \lambda\rho + a_0\rho' + a_1\rho + c_1u' + c_2u = f & \text{in } (0,L), \\ \lambda u - b_0u'' + b_1u' + b_2u + d_1\rho' + d_2\rho = g & \text{in } (0,L), \\ \rho(0) = 0, \quad u(0) = u(L) = 0. \end{cases} \tag{2.3}$$

The idea is to consider a regularised resolvent equation, where we add $-\varepsilon\rho''$ in $(2.3)_1$. More precisely, for $\varepsilon > 0$, we consider the following system

$$\begin{cases} \lambda\rho_\varepsilon - \varepsilon\rho_\varepsilon'' + a_0\rho_\varepsilon' + a_1\rho_\varepsilon + c_1u_\varepsilon' + c_2u_\varepsilon = f & \text{in } (0,L), \\ \lambda u_\varepsilon - b_0u_\varepsilon'' + b_1u_\varepsilon' + b_2u_\varepsilon + d_1\rho_\varepsilon' + d_2\rho_\varepsilon = g & \text{in } (0,L), \\ \rho_\varepsilon(0) = \rho_\varepsilon'(L) = 0, \quad u_\varepsilon(0) = u_\varepsilon(L) = 0, \end{cases} \tag{2.4}$$

where $(f,g) \in \mathcal{Z}$. The above system is a coupled parabolic system. Using Lax-Milgram theorem we show that, for every $\varepsilon > 0$, the above system admits a unique solution $\rho_\varepsilon \in H^1_{\{0\}}(0,L) \cap H^2(0,L)$ and $u_\varepsilon \in H^1_0(0,L) \cap H^2(0,L)$, satisfying some estimates uniform in ϵ, where $H^1_{\{0\}}(0,L) = \{f \in H^1(0,L) \mid f(0) = 0\}$.

To this aim, we set $\mathcal{V} = H^1_{\{0\}}(0,L) \times H^1_0(0,L)$ with the inner-product

$$\left\langle \begin{pmatrix}\rho\\ u\end{pmatrix}, \begin{pmatrix}\sigma\\ v\end{pmatrix}\right\rangle_{\mathcal{V}} := \int_0^L \rho'\bar{\sigma}'\,\mathrm{d}x + \int_0^L u'\bar{v}'\,\mathrm{d}x,$$

where $\bar\sigma$ and $\bar v$ are the complex conjugates of σ and v, respectively. We define the functional $B : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ by

$$B\left(\begin{pmatrix}\rho\\u\end{pmatrix},\begin{pmatrix}\sigma\\v\end{pmatrix}\right) = \lambda\int_0^L \rho\bar\sigma\,\mathrm{d}x + \varepsilon\int_0^L \rho'\bar\sigma'\,\mathrm{d}x + \int_0^L \left(a_0\rho' + a_1\rho + c_1u' + c_2u\right)\bar\sigma\,\mathrm{d}x$$
$$+\lambda\int_0^L u\bar v\,\mathrm{d}x + \int_0^L u'(b_0\bar v)'\,\mathrm{d}x + \int_0^L \left(b_1u' + b_2u + d_1\rho' + d_2\rho\right)\bar v\,\mathrm{d}x. \tag{2.5}$$

We can verify that, B is a continuous, sesquilinear form on $\mathcal{V} \times \mathcal{V}$ and coercive, i.e,

$$\begin{aligned}&\mathrm{Re}\left(B\left(\begin{pmatrix}\rho\\u\end{pmatrix},\begin{pmatrix}\rho\\u\end{pmatrix}\right)\right)\\ &\geqslant (\lambda-\omega)\Big(\|\rho\|^2_{L^2(0,L)} + \|u\|^2_{L^2(0,L)}\Big) + \varepsilon\|\rho'\|^2_{L^2(0,L)} + \frac{\bar b}{2}\|u'\|^2_{L^2(0,L)} + \frac{a_0(L)}{2}|\rho(L)|^2\\ &\geqslant C(\varepsilon)\|(\rho,u)\|^2_{\mathcal{V}}, \qquad \text{for all } (\rho,u)\in\mathcal{V},\end{aligned} \tag{2.6}$$

for some positive constant C depending on ε. Thus using Lax-Milgram theorem, for each $\varepsilon > 0$, for any $(f, g) \in \mathcal{Z}$, there exists a unique $(\rho_\varepsilon, u_\varepsilon) \in \mathcal{V}$ satisfying

$$B\left(\begin{pmatrix}\rho_\varepsilon\\u_\varepsilon\end{pmatrix},\begin{pmatrix}\sigma\\v\end{pmatrix}\right) = \int_0^L f\bar\sigma\,dx + \int_0^L g\bar v\,dx, \qquad \text{for all } (\sigma,v)\in\mathcal{V}. \tag{2.7}$$

Further, from (2.7), it can be derived that $\rho''_\varepsilon \in L^2(0, L)$ and $u''_\varepsilon \in L^2(0, L)$. Now multiplying $(2.4)_1$ by $\bar\sigma \in H^1_{\{0\}}(0, L)$ and using an integration by parts, we get

$$\lambda\int_0^L \rho_\varepsilon\bar\sigma\,dx + \varepsilon\int_0^L \rho'_\varepsilon\bar\sigma'\,dx - \varepsilon\rho'_\varepsilon(L)\bar\sigma(L) + \int_0^L \left(a_0\rho'_\varepsilon + a_1\rho_\varepsilon + c_1u'_\varepsilon + c_2u_\varepsilon\right)\bar\sigma\,dx = \int_0^L f\bar\sigma\,dx.$$

Then, using (2.7) for $(\sigma, 0) \in \mathcal{V}$, the above identity yields $\rho'_\epsilon(L) = 0$. Thus for all $\varepsilon > 0$, $(\rho_\varepsilon, u_\varepsilon)$ belongs to $\Big(H^1_{\{0\}}(0, L) \cap H^2(0, L)\Big) \times \Big(H^1_0(0, L) \cap H^2(0, L)\Big)$ and it satisfies (2.4).

Taking $(\sigma, v) = (\rho_\varepsilon, u_\varepsilon)$ in (2.7), using a similar estimate as in (2.6) along with the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\frac{\lambda-\omega}{2}\Big(\|\rho_\varepsilon\|^2_{L^2(0,L)} + \|u_\varepsilon\|^2_{L^2(0,L)}\Big) + \varepsilon\|\rho'_\varepsilon\|^2_{L^2(0,L)} + \frac{\bar b}{2}\|u'_\varepsilon\|^2_{L^2(0,L)} + \frac{a_0(L)}{2}|\rho_\varepsilon(L)|^2\\ &\qquad\qquad\leqslant \frac{1}{2(\lambda-\omega)}\Big(\|f\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)}\Big).\end{aligned}$$

From the above estimate, it follows that, there exist $\rho \in L^2(0, L)$, $u \in H^1_0(0, L)$ and $\ell \in \mathbb{C}$ such that, upto a subsequence, as $\varepsilon \to 0$,

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \text{ in } H^1_0(0,L), &\qquad \rho_\varepsilon &\rightharpoonup \rho \text{ in } L^2(0,L),\\ \varepsilon(\rho_\varepsilon)' &\to 0 \text{ in } L^2(0,L), &\qquad a_0(L)\rho_\varepsilon(L) &\to \ell \text{ in } \mathbb{C}.\end{aligned} \tag{2.8}$$

Note that from (2.7), for $\sigma \in C^\infty_c(0, L)$ and $v = 0$, we obtain

$$\lambda\int_0^L \rho_\varepsilon\bar\sigma\,\mathrm{d}x + \varepsilon\int_0^L \rho'_\varepsilon\bar\sigma'\,\mathrm{d}x - \int_0^L \rho_\varepsilon(a_0\bar\sigma)'\,\mathrm{d}x + \int_0^L \left(a_1\rho_\varepsilon + c_1u'_\varepsilon + c_2u_\varepsilon\right)\bar\sigma\,\mathrm{d}x = \int_0^L f\bar\sigma\,dx. \tag{2.9}$$

Taking $\varepsilon \to 0$ in the above identity, we get that (ρ, u) satisfies $(2.3)_1$ in the distribution sense and hence from $(2.3)_1$, it follows that $a_0\rho' \in L^2(0, L)$. Similarly, from (2.7), for $\sigma = 0$ and $v \in C^\infty_c(0, L)$, we obtain

$$\lambda\int_0^L u_\varepsilon\bar v\,\mathrm{d}x + \int_0^L u'_\varepsilon(b_0\bar v)'\,\mathrm{d}x - \int_0^L \rho_\varepsilon(d_1\bar v)'\,\mathrm{d}x + \int_0^L \left(b_1u'_\varepsilon + b_2u_\varepsilon + d_2\rho_\varepsilon\right)\bar v\,\mathrm{d}x = \int_0^L g\bar v\,dx. \tag{2.10}$$

Taking $\varepsilon \to 0$ in the above identity, we get that (ρ, u) satisfies $(2.3)_2$ in the distribution sense and hence from $(2.3)_2$, it follows that $(b_0 u' - d_1\rho) \in H^1(0,L)$. Next, multiplying $(2.4)_1$ by $\bar\sigma \in C_c^\infty(0,L)$ and using an integration by parts along with (2.8) and (2.9), we obtain

$$\int_0^L a_0\rho_\varepsilon'\bar\sigma\,dx \to \int_0^L a_0\rho'\bar\sigma\,dx, \qquad \text{for all } \bar\sigma \in C_c^\infty(0,L),$$

and using a density argument, we derive that

$$a_0\rho_\varepsilon' \rightharpoonup a_0\rho', \quad \text{in} \quad L^2(0,L), \quad \text{as} \quad \varepsilon \to 0. \tag{2.11}$$

Similarly, multiplying $(2.3)_2$ by $\bar v \in C_c^\infty(0,L)$ and using an integration by parts along with (2.8) and (2.10), it can be derived that

$$(b_0u_\varepsilon' - d_1\rho_\varepsilon) \rightharpoonup (b_0u' - d_1\rho), \quad \text{in} \quad H^1(0,L), \quad \text{as} \quad \varepsilon \to 0, \tag{2.12}$$

by noticing that as $\varepsilon \to 0$, $(b_0u_\varepsilon' - d_1\rho_\varepsilon) \rightharpoonup (b_0u' - d_1\rho)$ in $L^2(0,L)$ and $(b_0u_\varepsilon' - d_1\rho_\varepsilon)' \rightharpoonup (b_0u' - d_1\rho)'$ in $L^2(0,L)$.

It only remains to show $(\rho,u) \in \mathcal{D}(\mathcal{A};\mathcal{Z})$. To this aim, we set

$$\mathcal{W} = \Big\{(\rho,u) \in L^2(0,L)\times H_0^1(0,L) \ \mid \ a_0\rho' \in L^2(0,L), \quad (b_0u' - d_1\rho) \in H^1(0,L)\Big\}.$$

Note that, $\mathcal{W}$ is a Hilbert space with the graph norm, and $\mathcal{D}(\mathcal{A};\mathcal{Z})$ is a closed subspace of $\mathcal{W}$. Thus, the weak closure and the strong closure of $\mathcal{D}(\mathcal{A};\mathcal{Z})$ in $\mathcal{W}$ are same. In the calculation above we have actually shown that (ρ,u) lies in the weak closure of $\mathcal{D}(\mathcal{A};\mathcal{Z})$. Therefore $(\rho,u) \in \mathcal{D}(\mathcal{A};\mathcal{Z})$, and in particular, $\rho(0) = 0$.Thus, for any given $(f,g) \in \mathcal{Z}$, there exists a $(\rho,u) \in \mathcal{D}(\mathcal{A};\mathcal{Z})$ satisfying (2.3).

Finally, we show uniqueness of solution to (2.3). For given $(f,g) \in \mathcal{Z}$, let $(\rho_1,u_1), (\rho_2,u_2) \in \mathcal{D}(\mathcal{A};\mathcal{Z})$ be two solutions of (2.3). Setting $\rho = \rho_1 - \rho_2$, $u = u_1 - u_2$, it is enough to show that $(\rho,u) = (0,0)$. To do it, we note that $(\rho,u) \in \mathcal{D}(\mathcal{A};\mathcal{Z})$ satisfying $(\lambda I - \mathcal{A})\begin{bmatrix}\rho\\u\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. Now taking the inner product with (ρ,u) and using quasi-dissipativity property from Step 1, we get

$$\lambda\,\|(\rho,u)\|_{\mathcal{Z}}^2 = \ \mathrm{Re}\ \left\langle \mathcal{A}\begin{pmatrix}\rho\\u\end{pmatrix}, \begin{pmatrix}\rho\\u\end{pmatrix}\right\rangle_{\mathcal{Z}} \leqslant \omega\,\|(\rho,u)\|_{\mathcal{Z}}^2\,.$$

Since $\lambda > \omega$ in (2.3), it yields $(\rho,u) = (0,0)$. This completes the proof of the proposition. ☐

Remark 2.3. *In the proof above, we can actually show that $a_0(L)\rho(L) = \ell$. In fact, multiplying $(2.3)_1$ by $\bar\sigma \in H^1_{\{0\}}(0,L)$, we obtain*

$$\lambda\int_0^L \rho\bar\sigma\,\mathrm{d}x - \int_0^L \rho(a_0\bar\sigma)'\,\mathrm{d}x + a_0(L)\rho(L)\bar\sigma(L) + \int_0^L \left(a_1\rho + c_1u' + c_2u\right)\bar\sigma\,\mathrm{d}x = \int_0^L f\bar\sigma\,dx. \tag{2.13}$$

From (2.7), for $\sigma \in H^1_{\{0\}}(0,L)$ and $v = 0$, it follows that

$$\lambda\int_0^L \rho_\varepsilon\bar\sigma\,\mathrm{d}x + \varepsilon\int_0^L \rho_\varepsilon'\bar\sigma'\,\mathrm{d}x - \int_0^L \rho_\varepsilon(a_0\bar\sigma)'\,\mathrm{d}x$$
$$+ a_0(L)\rho_\epsilon(L)\bar\sigma(L) + \int_0^L \left(a_1\rho_\varepsilon + c_1u_\varepsilon' + c_2u_\varepsilon\right)\bar\sigma\,\mathrm{d}x = \int_0^L f\bar\sigma\,dx.$$

Taking $\varepsilon \to 0$ in the above identity, along with (2.13) we get $a_0(L)\rho(L) = \ell$.

It is well known that the null controllability of the pair $(\mathcal{A},\mathcal{B})$ is equivalent to the final-state observability of the pair $(\mathcal{A}^*,\mathcal{B}^*)$, where $\mathcal{A}^*$ and $\mathcal{B}^*$ are the adjoint operators of $\mathcal{A}$ and $\mathcal{B}$ respectively (see for instance [8, Chapter 2.3],[29, Section 11.2]). Thus it is important to determine the adjoint of the operator $\mathcal{A}$:

Proposition 2.4. *The adjoint of* $(\mathcal{A}, \mathcal{D}(\mathcal{A}; \mathcal{Z}))$ *in* $\mathcal{Z}$ *is defined by*

$$\mathcal{D}(\mathcal{A}^*; \mathcal{Z}) = \Big\{(\sigma, v) \in L^2(0,L) \times H^1_0(0,L) \ \mid \ a_0\sigma' \in L^2(0,L),$$
$$\sigma(L) = 0 \text{ if } a_0(0) > 0,\ \sigma(0) = 0 \text{ if } a_0(L) < 0,\ b_0 v' + c_1\sigma \in H^1(0,L)\Big\}, \quad (2.14)$$

and

$$\mathcal{A}^* = \begin{bmatrix} a_0\dfrac{d}{dx} - (a_1 - a_0') & d_1\dfrac{d}{dx} + (d_1' - d_2) \\ c_1\dfrac{d}{dx} - (c_2 - c_1') & b_0\dfrac{d^2}{dx^2} + (2b_0' + b_1)\dfrac{d}{dx} + (b_0'' + b_1' - b_2) \end{bmatrix}. \quad (2.15)$$

For future purpose, we also need to study the well-posedness of the adjoint system with non-homogenous source terms and boundary data. More precisely, we consider the following non-homogeneous system

$$\begin{cases} \partial_t\sigma - a_0\partial_x\sigma + (a_1 - a_0')\sigma - d_1\partial_x v - (d_1' - d_2)v = \zeta_1 & \text{in } (0,T)\times(0,L), \\ \partial_t v - b_0\partial_{xx}v - (2b_0' + b_1)\partial_x v + (b_2 - b_1' - b_0'')v & \\ \qquad\qquad\qquad - c_1\partial_x\sigma + (c_2 - c_1')\sigma = \zeta_2 & \text{in } (0,T)\times(0,L), \\ v(t,0) = h_0(t), \quad v(t,L) = h_L(t) & \text{in } (0,T), \\ \sigma(t,0) = g_0(t) \text{ if } a_0(L) < 0, \qquad \sigma(t,L) = g_L(t) \text{ if } a_0(0) > 0 & \text{in } (0,T), \\ \sigma(0,x) = \sigma^0(x), \qquad v(0,x) = v^0(x), & \text{in } (0,L). \end{cases} \quad (2.16)$$

From the well-posedness of the adjoint operator $\mathcal{A}^*$, the result below can be obtained easily.

Proposition 2.5. *Let* $T > 0$. *Then for any* $(\sigma^0, v^0) \in \mathcal{Z}$, $(\zeta_1, \zeta_2) \in \Big(L^2(0,T; L^2(0,L))\Big)^2$ *and* $(h_0, h_L, g_0, g_L) \in \Big(H^1(0,T)\Big)^4$, *the system* (2.16) *admits a unique solution* $(\sigma, v) \in C([0,T]; \mathcal{Z})$, *satisfying*

$$\|(\sigma,v)\|_{C([0,T];\mathcal{Z})} \leqslant C\Big(\left\|(\sigma^0, v^0)\right\|_{\mathcal{Z}} + \|(\zeta_1,\zeta_2)\|_{(L^2(0,T;L^2(0,L)))^2} + \|(h_0,h_L,g_0,g_L)\|_{(H^1(0,T))^4} \Big),$$

where the positive constant C *depends only on* T, L *and the coefficients of the system.*

We also need to show the operator $\mathcal{A}$ generates a C^0-semigroup on $\mathcal{H} := H^1(0,L) \times L^2(0,L)$, under some suitable assumptions on the coefficients. We assume (1.6) and

$$a_0(0) \leqslant 0, \quad a_0(L) \geqslant 0. \quad (2.17)$$

Note that, under the above assumption on a_0, we do not need to provide any boundary condition for ρ in (1.5). We consider the unbounded operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}; \mathcal{H}))$ in $\mathcal{H}$ with

$$\mathcal{D}(\mathcal{A}; \mathcal{H}) = \Big\{(\rho, u) \in H^1(0,L) \times H^2(0,L) \cap H^1_0(0,L) \ \mid \ a_0\rho' \in H^1(0,L)\Big\}.$$

We prove the following result:

Theorem 2.6. *Assume* (1.6) *and* (2.17). *The operator* $(\mathcal{A}, \mathcal{D}(\mathcal{A}; \mathcal{H}))$ *is the infinitesimal generator of a strongly continuous semigroup* $\mathbb{T}$ *on* $\mathcal{H}$.

Proof. We rewrite $\mathcal{A} := \mathcal{A}_1 + \mathcal{A}_2$, with

$$\mathcal{A}_1 = \begin{bmatrix} -a_0\dfrac{d}{dx} & -c_1\dfrac{d}{dx} - c_2 \\ 0 & b_0\dfrac{d^2}{dx^2} - b_1\dfrac{d}{dx} - b_2 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} -a_1 & 0 \\ -d_1\dfrac{d}{dx} - d_2 & 0 \end{bmatrix}.$$

Note that, there exists a positive constant C such that

$$\left\| \mathcal{A}_2 \begin{bmatrix} \rho \\ u \end{bmatrix} \right\|_{\mathcal{H}} \leqslant C \left\| \begin{bmatrix} \rho \\ u \end{bmatrix} \right\|_{\mathcal{H}} \quad \text{for all } \begin{bmatrix} \rho \\ u \end{bmatrix} \in \mathcal{H}.$$

Thus $\mathcal{A}_2$ is a bounded perturbation of the operator $\mathcal{A}_1$ on $\mathcal{H}$ (for details, see [29, Theorem 2.11.2]). Therefore, it is sufficient to show that $\mathcal{A}_1$ generates a C^0-semigroup on $\mathcal{H}$. Considering $a_1 = 0, d_1 = 0 = d_2$ in Proposition 2.2, we get that $\mathcal{A}_1$, with $\mathcal{D}(\mathcal{A}_1;\mathcal{Z})$ defined by

$$\mathcal{D}(\mathcal{A}_1;\mathcal{Z}) = \Big\{(\rho,u) \in L^2(0,L) \times H^2(0,L) \cap H_0^1(0,L) \mid a_0\rho' \in L^2(0,L)\Big\},$$

generates a C^0-semigroup $\widetilde{\mathbb{T}}$ on $\mathcal{Z}$. For any $(\rho^0,u^0) \in \mathcal{Z}$ and $T > 0$, we set

$$\begin{bmatrix}\rho(t,\cdot)\\ u(t,\cdot)\end{bmatrix} = \widetilde{\mathbb{T}}_t \begin{bmatrix}\rho^0\\ u^0\end{bmatrix} \quad (t \in [0,T]).$$

Then $\rho \in C([0,T];L^2(\Omega))$ and $u \in C([0,T];L^2(\Omega))$, and (ρ,u) satisfies the following system

$$\begin{cases}\partial_t\rho + a_0\partial_x\rho + c_1\partial_x u + c_2 u = 0 & \text{in } (0,T)\times(0,L),\\ \partial_t u - b_0\partial_{xx}u + b_1\partial_x u + b_2 u = 0 & \text{in } (0,T)\times(0,L),\\ u(t,0) = u(t,L) = 0 & \text{on } (0,T),\\ \rho(0,x) = \rho^0(x), \qquad u(0,x) = u^0(x) & \text{in } (0,L).\end{cases} \tag{2.18}$$

Moreover, using standard results for parabolic equations (see for instance [12, Chapter 7.1]), we obtain $u \in L^2(0,T;H_0^1(0,L)) \cap C([0,T];L^2(0,L))$.

In view of [29, Proposition 2.4.4], we need to show $\mathcal{H}$ is invariant under $\widetilde{\mathbb{T}}$ and the restriction of $\widetilde{\mathbb{T}}$ to $\mathcal{H}$ is a strongly continuous on $\mathcal{H}$. To this aim, we first show that for any $(\rho^0,u^0) \in \mathcal{H}$, (ρ,u), the solution to (2.18) belongs to $C([0,T];\mathcal{H})$. We define

$$\eta = \frac{c_1}{b_0}u + \partial_x\rho.$$

Then η solves

$$\begin{cases}\partial_t\eta + a_0\partial_x\eta + a_0'\eta = g & \text{in } (0,T)\times(0,L),\\ \eta(0) = \dfrac{c_1}{b_0}u^0 + (\rho^0)' & \text{in } (0,L),\end{cases} \tag{2.19}$$

where $\eta(0) \in L^2(0,L)$, and

$$g = \left(\left(\frac{a_0c_1}{b_0}\right)' - \frac{c_1b_2}{b_0} - c_2'\right)u + \left(\frac{c_1a_0}{b_0} - c_1' - c_2 - \frac{c_1b_1}{b_0}\right)\partial_x u \in L^2(0,T;L^2(0,L)).$$

Thus, for $(\rho^0,u^0) \in \mathcal{H}$, it yields $\eta \in C([0,T];L^2(0,L))$ and, hence $\rho \in C([0,T];H^1(0,L))$. This gives that $\mathcal{H}$ is invariant under $\widetilde{\mathbb{T}}$, and the restriction of the semigroup $\widetilde{\mathbb{T}}$ is a strongly continuous semigroup in $\mathcal{H}$. It is easy to verify that its domain is $\mathcal{D}(\mathcal{A}_1;\mathcal{Z}) \cap \mathcal{H} = \mathcal{D}(\mathcal{A};H)$. ☐

Remark 2.7. *Let us remark that, under the condition* (2.17)*, we do not need to provide any boundary conditions for η in* (2.19)*. This is the reason why we have assumed the condition* (2.17)*. Otherwise, we need to recover the boundary condition for η from* $(2.18)_1$*, which requires $\partial_x u(t,0)$ to be well-defined. This seems possible if we have more regular initial data for the parabolic component. Alternatively, if we consider* (2.18) *with periodic boundary conditions and with all the coefficients and quantities being periodic with respect to x, we obtain η is also periodic with respect to x. Thus in this case, any extra assumption on a_0 is not required to obtain our results.*

2.2. **Well-posedness of the system (1.12).** In this subsection we prove the following result

Theorem 2.8. *Let us assume* (1.13)*. For any $(\rho^0,u^0) \in \Big(L^2(0,L)\Big)^2$ and $f_i \in L^2(0,T;L^2(0,L))$, $i = 1,2$, the system* (1.12) *admits a unique solution $(\rho,u) \in C\Big([0,T];\Big(L^2(0,L)\Big)^2\Big)$.*

We define the unbounded operator $(\widehat{\mathcal{A}}, \mathcal{D}(\widehat{\mathcal{A}}; \mathcal{Z}))$ in $\mathcal{Z}$ by

$$\mathcal{D}(\widehat{\mathcal{A}}; \mathcal{Z}) = \Big\{(\rho, u) \in L^2(0,L) \times H^2(0,L) \cap H^1_0(0,L) \ \mid \ a_0\rho' \in L^2(0,L) \\ \rho(0) = 0 \text{ if } a_0(0) > 0, \ \rho(L) = 0 \text{ if } a_0(L) < 0\Big\},$$

and

$$\widehat{\mathcal{A}} = \begin{bmatrix} -a_0\dfrac{d}{dx} - a_1 & -c_0\dfrac{d^2}{dx^2} - c_1\dfrac{d}{dx} - c_2 \\ -d_2 & b_0\dfrac{d^2}{dx^2} - b_1\dfrac{d}{dx} - b_2 \end{bmatrix}. \tag{2.20}$$

We rewrite $\widehat{\mathcal{A}} := \widehat{\mathcal{A}}_1 + \widehat{\mathcal{A}}_2$, with

$$\widehat{\mathcal{A}}_1 = \begin{bmatrix} -a_0\dfrac{d}{dx} - a_1 & -c_0\dfrac{d^2}{dx^2} - c_1\dfrac{d}{dx} - c_2 \\ 0 & b_0\dfrac{d^2}{dx^2} - \widehat{b}_1\dfrac{d}{dx} - \widehat{b}_2 \end{bmatrix}, \quad \widehat{\mathcal{A}}_2 = \begin{bmatrix} 0 & 0 \\ -d_2 & (\widehat{b}_1 - b_1)\dfrac{d}{dx} + (\widehat{b}_2 - b_2), \end{bmatrix}$$

where

$$\widehat{b}_1 = \frac{a_0c_0 - b_0c_1}{c_0}, \quad \widehat{b}_2 = \frac{b_0a_1 - a_0b_0'}{b_0} + \frac{a_0c_0' - b_0c_2}{c_0}.$$

With the above notations, system (1.12) can be rewritten as

$$\dot{z}(t) = \widehat{\mathcal{A}}z(t) + \mathcal{B}f(t), \quad t \in (0,T), \qquad z(0) = z^0,$$

where $z(t) = (\rho(t,\cdot), u(t,\cdot))$, $z^0 = (\rho^0, u^0)$, $f(t) = (f_1(t,\cdot), f_2(t,\cdot))$, and $\mathcal{B}$ is defined in (2.2). We want to show that, the operator $\widehat{\mathcal{A}}$ generates a C^0-semigroup on $\mathcal{Z}$. To this aim, let us first consider the system

$$\frac{d}{dt}\begin{bmatrix}\rho \\ u\end{bmatrix} = \widehat{\mathcal{A}}_1\begin{bmatrix}\rho \\ u\end{bmatrix}, \qquad \begin{bmatrix}\rho \\ u\end{bmatrix}(0) = \begin{bmatrix}\rho^0 \\ u^0\end{bmatrix} \in \mathcal{Z}. \tag{2.21}$$

By setting $\eta = b_0\rho + c_0u$, we obtain the following system satisfied by (η, u) :

$$\begin{cases} \partial_t\eta + a_0\partial_x\eta + \widehat{a}_1\eta = 0 & \text{in } (0,T) \times (0,L), \\ \partial_t u - b_0\partial_{xx}u + \widehat{b}_1\partial_x u + \widehat{b}_2 u = 0 & \text{in } (0,T) \times (0,L), \\ u(t,0) = u(t,L) = 0 & \text{in } (0,T), \\ \eta(t,0) = 0 & \text{in } (0,T), \ \text{if } a_0(0) > 0, \\ \eta(t,L) = 0 & \text{in } (0,T), \ \text{if } a_0(L) < 0, \\ \eta(0) = b_0\rho^0 + c_0u^0, \qquad u(0) = u^0 & \text{in } (0,L), \end{cases}$$

where $\widehat{a}_1 = \dfrac{a_1b_0 - a_0b_0'}{b_0}$. We now define

$$\mathcal{D}(\mathcal{A}^{\text{tr}}) = \Big\{\eta \in L^2(0,L) \mid a_0\eta' \in L^2(0,L), \quad \eta(0) = 0 \text{ if } a_0(0) > 0, \quad \eta(L) = 0 \text{ if } a_0(L) < 0\Big\},$$
$$\mathcal{A}^{\text{tr}} = -a_0\eta' - \widehat{a}_1\eta,$$

and

$$\mathcal{D}(\mathcal{A}^{\text{p}}) = H^2(0,L) \cap H^1_0(0,L), \qquad \mathcal{A}^{\text{p}}u = b_0u'' - \widehat{b}_1u' - \widehat{b}_2u.$$

Note that, $(\mathcal{A}^{\text{tr}}, \mathcal{D}(\mathcal{A}^{\text{tr}}))$ generetes a C^0 semigroup $\mathbb{T}^{\text{tr}}$ on $L^2(0,L)$. Also, $(\mathcal{A}^{\text{p}}, \mathcal{D}(\mathcal{A}^{\text{p}}))$ generetes a C^0 semigroup $\mathbb{T}^{\text{p}}$ on $L^2(0,L)$. Then the solution of (2.21), can be written as

$$\rho(t) = \frac{1}{b_0}\Big(\mathbb{T}^{\text{tr}}_t(b_0\rho^0 + c_0u^0) - c_0\mathbb{T}^{\text{p}}_t u^0\Big), \quad u(t) = \mathbb{T}^{\text{p}}_t u^0.$$

This motivates us to define a semigroup $\widehat{\mathbb{T}}^1$ on $\mathcal{Z}$ as follows:

$$\widehat{\mathbb{T}}^1_t\begin{bmatrix}\rho \\ u\end{bmatrix} = \begin{bmatrix}\dfrac{1}{b_0}\Big(\mathbb{T}^{\text{tr}}_t(b_0\rho + c_0u) - c_0\mathbb{T}^{\text{p}}_t u\Big) \\ \mathbb{T}^{\text{p}}_t u\end{bmatrix}. \tag{2.22}$$

The following result can be obtained easily. Thus the proof is omitted here.

Lemma 2.9. *The family of operators $\widehat{\mathbb{T}}^1$ defined in (2.22) is a C^0-semigroup on $\mathcal{Z}$. Moreover, its generator is $(\widehat{\mathcal{A}}_1, \mathcal{D}(\widehat{\mathcal{A}};\mathcal{Z}))$.*

We are now in a position to prove Theorem 2.8.

Proof of Theorem 2.8. We want to show that the operator $(\widehat{\mathcal{A}}, \mathcal{D}(\widehat{\mathcal{A}};\mathcal{Z}))$ generates a C^0-semigroup on $\mathcal{Z}$. In view of [29, Theorem 5.4.2], it is enough to show that $\widehat{\mathcal{A}}_2 : \mathcal{D}(\widehat{\mathcal{A}};\mathcal{Z}) \to \mathcal{Z}$ is an admissible observation operator for $\widehat{\mathbb{T}}^1$. Using the standard regularity results for parabolic equation, it is easy to see that, there exists a positive constant C, such that

$$\int_0^T \left\|\widehat{\mathcal{A}}_2 \widehat{\mathbb{T}}^1_t(\rho^0, u^0)\right\|^2_{\mathcal{Z}} \, dt \leqslant C \left\|(\rho^0, u^0)\right\|^2_{\mathcal{Z}}, \quad \forall\, (\rho^0, u^0) \in \mathcal{D}(\widehat{\mathcal{A}};\mathcal{Z}).$$

Therefore, $\widehat{\mathcal{A}}_2$ is an admissible observation operator for $\widehat{\mathbb{T}}^1$ (see for instance [29, Definition 4.3.1]). This completes the proof. ☐

We now determine the adjoint of the operator $\widehat{\mathcal{A}}$.

Proposition 2.10. *The adjoint of $(\widehat{\mathcal{A}}, \mathcal{D}(\widehat{\mathcal{A}};\mathcal{Z}))$ in $\mathcal{Z}$ is defined by*

$$\begin{aligned}\mathcal{D}(\widehat{\mathcal{A}}^*;\mathcal{Z}) = \Big\{ &(\sigma, v) \in L^2(0,L) \times L^2(0,L) \;\mid\; a_0\sigma' \in L^2(0,L), \\ &\sigma(L) = 0 \text{ if } a_0(0) > 0,\ \sigma(0) = 0 \text{ if } a_0(L) < 0,\ b_0 v - c_0\sigma \in H^1_0(0,L), \\ &\partial_{xx}(b_0 v - c_0\sigma) + \partial_x(b_1 v + c_1\sigma) \in L^2(0,L) \Big\},\end{aligned} \tag{2.23}$$

and

$$\widehat{\mathcal{A}}^* \begin{bmatrix} \sigma \\ v \end{bmatrix} = \begin{bmatrix} \partial_x(a_0\sigma) - a_1\sigma - d_2 v \\ \partial_{xx}(b_0 v - c_0\sigma) + \partial_x(b_1 v + c_1\sigma) - (b_2 v + c_2\sigma) \end{bmatrix}. \tag{2.24}$$

3. Gaussian beam construction

In this section, we construct Gaussian beam solutions for coupled ODE-parabolic and coupled transport-parabolic systems with variable coefficients in one space dimension. These solutions are highly localized near certain curves in space-time. In the case, where the coefficients are constant, such solutions were constructed in [20] using Fourier transform. However, for the system with variable coefficients, we cannot use the Fourier transform to construct such solutions. Our approach is inspired by the ideas in [26], where such solutions were constructed for strictly hyperbolic equations. We adapt them to the case for our coupled systems.

3.1. **Coupled ODE-parabolic systems : coupling of order zero or one.** We consider the following coupled ODE-parabolic system, that will be used to prove our main results. More precisely, we will construct Gaussian beam for the following operator:

$$\mathcal{L}_1 \begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{pmatrix} \partial_t\sigma + \alpha_1\sigma + \gamma_1\partial_x v + \gamma_2 v \\ \partial_t v - \beta_0\partial_{xx} v + \beta_1\partial_x v + \beta_2 v + \delta_1\partial_x\sigma + \delta_2\sigma \end{pmatrix}, \quad \text{in } [0,T] \times \mathbb{R}. \tag{3.1}$$

We assume that the coefficients satisfy the following conditions

$$\inf_{[0,T]\times\mathbb{R}} \beta_0 > 0, \qquad \alpha_1, \beta_i, \gamma_j, \delta_j \in C^\infty_b([0,T] \times \mathbb{R}), \quad \text{for all } i = 0,1,2, \quad \text{for all } j = 1,2. \tag{3.2}$$

We prove the following result.

Theorem 3.1. *Assume (3.2) is satisfied by the coefficients in $\mathcal{L}_1$, introduced in (3.1). Let $T > 0$, $x_0 \in \mathbb{R}$ and $k \in \mathbb{N}$. Then, there exist a positive constant C, which may depend on T but independent of k, and a sequence of functions $(\sigma_k, v_k)_{k\in\mathbb{N}}$ satisfying*

$$\sigma_k \in C^1([0,T]; C^1_b(\mathbb{R})), \quad v_k \in C^1([0,T]; C^2_b(\mathbb{R})), \tag{3.3}$$

such that the following holds:

$$\sup_{t\in[0,T]}\left\|\mathcal{L}_1\begin{pmatrix}\sigma_k\\ v_k\end{pmatrix}(t,\cdot)\right\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})} \leqslant Ck^{-1}, \tag{3.4}$$

$$\lim_{k\to\infty}\int_{\mathbb{R}}|\sigma_k(t,x)|^2\,dx = A(t) > 0 \qquad (t\in[0,T]), \tag{3.5}$$

$$\sup_{t\in[0,T]}\int_{|x-x_0|>k^{-1/4}}|\sigma_k(t,x)|^2\,\mathrm{d}x \leqslant Ce^{-\sqrt{k}/2}, \tag{3.6}$$

$$\sup_{t\in[0,T]}\int_{\mathbb{R}}|v_k(t,x)|^2\,\mathrm{d}x \leqslant Ck^{-2}, \tag{3.7}$$

where $A(t)$ is positive for all $t\in[0,T]$ and is independent of k.

Proof. In what follows, the positive constant C, which may change from line to line, is independent of $k\in\mathbb{N}$. The proof of Theorem 3.1 is divided into several steps.

Step 1: Construction of (σ_k, v_k) :

Let us set

$$\varphi(x) = \frac{i}{2}(x-x_0)^2 + (x-x_0) \quad (x\in\mathbb{R}). \tag{3.8}$$

Then for each $k\in\mathbb{N}$, we look for (σ_k, v_k) in the form

$$\sigma_k(t,x) = k^{1/4}e^{ik\varphi(x)}\eta(t,x) \qquad (t\in[0,T],\ x\in\mathbb{R}), \tag{3.9}$$

$$v_k(t,x) = k^{-3/4}e^{ik\varphi(x)}\Big[w_0(t,x) + \frac{w_1(t,x)}{k}\Big] \qquad (t\in[0,T],\ x\in\mathbb{R}). \tag{3.10}$$

Our aim is to choose η, w_0 and w_1 suitably so that (3.4) - (3.7) holds. Plugging the above expressions of σ_k and v_k in (3.1) and after some standard computations, we obtain

$$\mathcal{L}_1\begin{pmatrix}\sigma_k\\ v_k\end{pmatrix} = k^{-3/4}e^{ik\varphi(x)}\begin{pmatrix}kg_1 + g_0 + k^{-1}g_{-1}\\ k^2h_2 + kh_1 + h_0 + k^{-1}h_{-1}\end{pmatrix}, \tag{3.11}$$

where

$$g_1 = \partial_t\eta + \alpha_1\eta + i\gamma_1\varphi' w_0, \quad g_0 = \gamma_1\Big(i\varphi' w_1 + \partial_x w_0\Big) + \gamma_2 w_0, \quad g_{-1} = \gamma_1\partial_x w_1 + \gamma_2 w_1,$$

$$h_2 = \beta_0(\varphi')^2 w_0 + i\delta_1\varphi'\eta,$$

$$h_1 = \beta_0(\varphi')^2 w_1 - i\beta_0\varphi'' w_0 - 2i\beta_0\varphi'\partial_x w_0 + i\beta_1\varphi' w_0 + \delta_2\eta + \delta_1\partial_x\eta,$$

$$h_0 = \partial_t w_0 - i\beta_0\varphi'' w_1 - 2i\beta_0\varphi'\partial_x w_1 - \beta_0\partial_{xx}w_0 + i\beta_1\varphi' w_1 + \beta_1\partial_x w_0 + \beta_2 w_0,$$

$$h_{-1} = \partial_t w_1 - \beta_0\partial_{xx}w_1 + \beta_1\partial_x w_1 + \beta_2 w_1.$$

Since we want (σ_k, v_k) such that (3.4) holds, we choose η, w_0 and w_1 such that

$$g_1(t,x) = h_2(t,x) = h_1(t,x) = 0 \text{ for all } t\in[0,T], x\in\mathbb{R}. \tag{3.12}$$

The condition $h_2 = 0$ implies that $w_0 = -\dfrac{i\delta_1\eta}{\beta_0\varphi'}$. Using this in the expression of g_1 above, we obtain the following ODE for η :

$$\partial_t\eta(t,x) + \Big(\alpha_1(t,x) + \gamma_1(t,x)\frac{\delta_1(t,x)}{\beta_0(t,x)}\Big)\eta(t,x) = 0 \qquad (t\in[0,T],\ x\in\mathbb{R})\,. \tag{3.13}$$

Let $\zeta\in C_c^\infty(\mathbb{R})$ with $\zeta(x_0)\neq 0$. We choose

$$\eta(t,x) = \exp\left(-\int_0^t\left(\alpha_1(s,x) + \gamma_1(s,x)\frac{\delta_1(s,x)}{\beta_0(s,x)}\right)\mathrm{d}s\right)\zeta(x) \quad (t\in[0,T], x\in\mathbb{R})\,. \tag{3.14}$$

With the above choice of η, we take

$$w_0(t,x) = \frac{-i\delta_1(t,x)\eta(t,x)}{\beta_0(t,x)\varphi'(x)} \quad (t \in [0,T], x \in \mathbb{R}), \tag{3.15}$$

and

$$w_1(t,x) = \frac{1}{\beta_0(t,x)\left(\varphi'(x)\right)^2}\Big[i\beta_0(t,x)\varphi''(x)w_0(t,x) + 2i\beta_0(t,x)\varphi'(x)\partial_x w_0(t,x) \\ - i\beta_1(t,x)\varphi'(x)w_0(t,x) - \delta_2(t,x)\eta(t,x) - \delta_1(t,x)\partial_x\eta(t,x)\Big] \quad (t \in [0,T], x \in \mathbb{R}). \tag{3.16}$$

Using (3.2) it is easy to verify that

$$\eta \in C^1([0,T]; C_b^1(\mathbb{R})), \quad w_0, w_1 \in C^1([0,T]; C_b^2(\mathbb{R})),$$

so that (σ_k, v_k) satisfies (3.3). Moreover,

$$g_0, g_{-1}, h_0, h_{-1} \in C([0,T]; C_b(\mathbb{R})).$$

We are now in a position to show that (σ_k, v_k) defined in (3.9) - (3.10), with η, w_0, w_1 given by (3.14) - (3.16), satisfy the estimates (3.4) – (3.7).

Step 2: Proof of (3.4) *:* In view of (3.11) and (3.12), to prove (3.4), it is enough to show the following estimates

$$\int_{\mathbb{R}} \left|k^{-3/4} e^{ik\varphi(x)} g_0(t,x)\right|^2 \,\mathrm{d}x \leqslant k^{-2}\sqrt{\pi}\,\|g_0\|^2_{L^\infty((0,T)\times\mathbb{R})}, \tag{3.17}$$

$$\int_{\mathbb{R}} \left|k^{-7/4} e^{ik\varphi(x)} g_{-1}(t,x)\right|^2 \,\mathrm{d}x \leqslant k^{-4}\sqrt{\pi}\,\|g_{-1}\|^2_{L^\infty((0,T)\times\mathbb{R})}, \tag{3.18}$$

$$\int_{\mathbb{R}} \left|k^{-3/4} e^{ik\varphi(x)} h_0(t,x)\right|^2 \mathrm{d}x \leqslant k^{-2}\sqrt{\pi}\,\|h_0\|^2_{L^\infty((0,T)\times\mathbb{R})}, \tag{3.19}$$

$$\int_{\mathbb{R}} \left|k^{-7/4} e^{ik\varphi(x)} h_{-1}(t,x)\right|^2 \mathrm{d}x \leqslant k^{-4}\sqrt{\pi}\,\|h_{-1}\|^2_{L^\infty((0,T)\times\mathbb{R})}, \tag{3.20}$$

hold for any $t \in [0,T]$. We provide a proof of (3.17) only. The other estimates will follow in a similar fashion. Using (3.8), we deduce

$$\int_{\mathbb{R}} \left|k^{-3/4} e^{ik\varphi(x)} g_0(t,x)\right|^2 \,\mathrm{d}x \leqslant k^{-3/2}\,\|g_0\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} e^{-k(x-x_0)^2}\,\mathrm{d}x \\ = k^{-2}\,\|g_0\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} e^{-z^2}\mathrm{d}z = k^{-2}\sqrt{\pi}\,\|g_0\|^2_{L^\infty((0,T)\times\mathbb{R})}.$$

Step 3: Proof of the estimates (3.5)-(3.7) *:* The estimate (3.7) can be obtained similarly as Step 2 above. Using (3.8), (3.9) and (3.14), we have

$$\int_{|x-x_0|>k^{-1/4}} |\sigma_k(t,x)|^2\,\mathrm{d}x \leqslant k^{1/2}\,\|\eta\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{|x-x_0|>k^{-1/4}} e^{-k(x-x_0)^2}\mathrm{d}x \\ = \sqrt{2}\,\|\eta\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{|z|>\frac{k^{\frac14}}{\sqrt{2}}} e^{-2z^2}\mathrm{d}z \leqslant Ce^{-\sqrt{k}/2}\int_{\mathbb{R}} e^{-z^2}\mathrm{d}z = C\sqrt{\pi}e^{-\sqrt{k}/2}.$$

This proves (3.6). To prove (3.5), noting that $k^{\frac12}\int_{\mathbb{R}} e^{-k(x-x_0)^2}\mathrm{d}x = \sqrt{\pi}$, we first obtain

$$\int_{\mathbb{R}} |\sigma_k(t,x)|^2\,\mathrm{d}x = k^{\frac12}\int_{\mathbb{R}} e^{-k(x-x_0)^2}|\eta(t,x)|^2\mathrm{d}x \\ = \eta^2(t,x_0)k^{\frac12}\int_{\mathbb{R}} e^{-k(x-x_0)^2}\mathrm{d}x + k^{\frac12}\int_{\mathbb{R}} e^{-k(x-x_0)^2}\Big(\eta^2(t,x) - \eta^2(t,x_0)\Big)\mathrm{d}x \\ = \sqrt{\pi}\eta^2(t,x_0) + R_k(t), \tag{3.21}$$

where

$$R_k(t) = k^{\frac{1}{2}} \int\limits_{\mathbb{R}} e^{-k(x-x_0)^2} \Big(\eta^2(t,x) - \eta^2(t,x_0)\Big) \mathrm{d}x.$$

By proceeding similarly as the proof of (3.6) above, we obtain

$$\begin{aligned}|R_k(t)| &\leqslant Ck^{1/2} \int_{|x-x_0|>k^{-1/4}} e^{-k(x-x_0)^2} \mathrm{d}x + k^{\frac{1}{2}} \int_{|x-x_0|\leqslant k^{-1/4}} e^{-k(x-x_0)^2} \Big|\eta^2(t,x) - \eta^2(t,x_0)\Big| \mathrm{d}x \\ &\leqslant Ce^{-\sqrt{k}/2} + k^{\frac{1}{2}} \int_{|x-x_0|\leqslant k^{-1/4}} e^{-k(x-x_0)^2} \Big|\eta^2(t,x) - \eta^2(t,x_0)\Big| \mathrm{d}x.\end{aligned}$$

Since $\eta \in C^1([0,T]; C_b^1(\mathbb{R}))$, we have

$$\begin{aligned}\Big|\eta^2(t,x) - \eta^2(t,x_0)\Big| &= \Big|\eta(t,x) + \eta(t,x_0)\Big|\Big|\eta(t,x) - \eta(t,x_0)\Big| \\ &\leqslant C \left\|\frac{\partial \eta}{\partial x}\right\|_{L^\infty([0,T]\times\mathbb{R})} |x - x_0| \leqslant C|x - x_0|.\end{aligned}$$

Therefore, using the above estimate we get

$$|R_k(t)| \leqslant Ce^{-\sqrt{k}/2} + Ck^{-\frac{1}{4}} \int_{\mathbb{R}} k^{\frac{1}{2}} e^{-k(x-x_0)^2} \, \mathrm{d}x \leqslant C(e^{-\sqrt{k}/2} + \sqrt{\pi} k^{-\frac{1}{4}}).$$

Therefore, from (3.21) we obtain (3.5) with $A(t) = \sqrt{\pi}|\eta(t,x_0)|^2 \neq 0$ for all $t \in [0,T]$. This completes the proof of Theorem 3.1. $\square$

As a consequence of Theorem 3.1, we also have the following result.

Lemma 3.2. *Let $T > 0$, $x_0 \in (0,L)$, and let $(\sigma_k, v_k)_{k\in\mathbb{N}}$ be constructed as in Theorem 3.1.*

(i) *Let $\ell_1 : [0,T] \to \mathbb{R}$ be a smooth function such that*

$$|\ell_1(t) - x_0| > k_0^{-1/4} \text{ for all } t \in [0,T], \text{ for some } k_0 \in \mathbb{N}.$$

Then, for all $k \geqslant k_0$,

$$\|\sigma_k(\cdot, \ell_1(\cdot))\|_{H^1(0,T)} \leqslant Ce^{-\frac{\sqrt{k}}{4}},$$

where C is a positive constant, which may depend on k_0 and T, but independent of k.

(ii) *Let $\ell_2 : [0,T] \to \mathbb{R}$ be a smooth function. Then, for all $k \geqslant k_0$,*

$$\|v_k(\cdot, \ell_2(\cdot))\|_{H^1(0,T)} \leqslant Ck^{-3/4},$$

where C is a positive constant, which may depend on k_0 and T, but independent of k.

Proof. From (3.9), using $\eta \in C^1([0,T]; C_b^1(\mathbb{R}))$ along with the estimate $|\ell_1(t) - x_0| > k_0^{-1/4}$, for all $t \in [0,T]$, we have, for all $k \geqslant k_0$,

$$|\sigma_k(t, \ell_1(t))| \leqslant Ck^{1/4} e^{-\frac{\sqrt{k}}{2}}, \quad |\partial_t \sigma_k(t, \ell_1(t))| \leqslant Ck^{3/4} e^{-\frac{\sqrt{k}}{2}}, \qquad \text{for all } t \in [0,T],$$

for some constant C depending on k_0, T but independent of k. Therefore, using the fact that $k^{1/4} e^{-\frac{\sqrt{k}}{4}} \leqslant 1$ and $k^{3/4} e^{-\frac{\sqrt{k}}{4}} \leqslant 4$ for all $k \in \mathbb{N}$, we get

$$\|\sigma_k(\cdot, \ell_1(\cdot))\|_{H^1(0,T)} \leqslant Ce^{-\frac{\sqrt{k}}{4}}, \quad \forall\, k \geqslant k_0.$$

Similarly, the other estimates mentioned in the lemma can be obtained. $\square$

3.2. **Coupled ODE-parabolic systems : coupling of order two.** In this subsection, we construct Gaussian beam solutions for the following coupled ode-parabolic system with coupling of order two. More precisely, we consider

$$\mathcal{L}_2 \begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{pmatrix} \partial_t \sigma + \alpha_1 \sigma + \gamma_2 v \\ \partial_t v - \beta_0 \partial_{xx} v + \beta_1 \partial_x v + \beta_2 v + \delta_0 \partial_{xx} \sigma + \delta_1 \partial_x \sigma + \delta_2 \sigma \end{pmatrix}, \quad \text{in } [0,T] \times \mathbb{R}. \tag{3.22}$$

Let us assume that

$$\inf_{[0,T]\times\mathbb{R}} \beta_0 > 0, \qquad \delta_0 \neq 0, \qquad \alpha_1, \beta_i, \gamma_2, \delta_i \in C_b^\infty([0,T] \times \mathbb{R}) \text{ for all } i = 0,1,2. \tag{3.23}$$

We prove the following result.

Theorem 3.3. *Assume (3.23), $T > 0$, $x_0 \in \mathbb{R}$ and $k \in \mathbb{N}$. Then, there exist a positive constant C, which may depend on T but independent of k, and a sequence of functions $(\sigma_k, v_k)_{k\in\mathbb{N}}$ satisfying*

$$\sigma_k \in C^1([0,T]; C_b^2(\mathbb{R})), \quad v_k \in C^1([0,T]; C_b^2(\mathbb{R})),$$

such that the following holds:

$$\sup_{t\in[0,T]} \left\| \mathcal{L}_2 \begin{pmatrix} \sigma_k \\ v_k \end{pmatrix} (t,\cdot) \right\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})} \leqslant C k^{-1}, \tag{3.24}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}} |\sigma_k(t,x)|^2 \, dx = A(t) > 0 \qquad (t \in [0,T]), \tag{3.25}$$

$$\sup_{t\in[0,T]} \int_{|x-x_0|>k^{-1/4}} |\sigma_k(t,x)|^2 \, \mathrm{d}x \leqslant C e^{-\sqrt{k}/2}, \tag{3.26}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}} |v_k(t,x)|^2 \, dx = B(t) > 0 \qquad (t \in [0,T]), \tag{3.27}$$

$$\sup_{t\in[0,T]} \int_{|x-x_0|>k^{-1/4}} |v_k(t,x)|^2 \, \mathrm{d}x \leqslant C e^{-\sqrt{k}/2}, \tag{3.28}$$

where $A(t)$ and $B(t)$ are positive for all $t \in [0,T]$ and do not depend on k.

Proof. The proof is similar to that of Theorem 3.1. We just provide the expressions of σ_k and v_k. For each $k \in \mathbb{N}$, we look for $(\sigma_k, v_k)_{k\in\mathbb{N}}$ in the form

$$\sigma_k(t,x) = k^{1/4} e^{ik\varphi(x)} \eta(t,x), \qquad (t \in [0,T],\ x \in \mathbb{R}),$$
$$v_k(t,x) = k^{1/4} e^{ik\varphi(x)} \Big[w_0(t,x) + \frac{w_1(t,x)}{k} + \frac{w_2(t,x)}{k^2} \Big], \qquad (t \in [0,T],\ x \in \mathbb{R}),$$

where φ is defined in (3.8) and η, w_0, w_1 and w_2 are given by

$$\eta(t,x) = \exp\left(- \int_0^t \left(\alpha_1(s,x) + \gamma_2(s,x) \frac{\delta_0(s,x)}{\beta_0(s,x)} \right) ds \right) \zeta(x) \quad (t \in [0,T], x \in \mathbb{R}),$$

for some $\zeta \in C_c^\infty(\mathbb{R})$ with $\zeta(x_0) \neq 0$ and

$$w_0(t,x) = \frac{\delta_0(t,x)}{\beta_0(t,x)} \eta(t,x) \quad (t \in [0,T], x \in \mathbb{R}),$$

$$\begin{aligned} w_1(t,x) = & \frac{1}{\beta_0(t,x) \left(\varphi'(x)\right)^2} \Big[2i\beta_0(t,x)\varphi'(x)\partial_x w_0(t,x) + i\beta_0(t,x)\varphi''(x) w_0(t,x) \\ & - i\beta_1(t,x)\varphi'(x) w_0(t,x) - 2i\delta_0(t,x)\varphi'(x)\partial_x\eta(t,x) - i\delta_0(t,x)\varphi''(x)\eta(t,x) \\ & - i\delta_1(t,x)\varphi'(x)\eta(t,x) \Big] \qquad (t \in [0,T], x \in \mathbb{R}), \end{aligned}$$

$$
\begin{aligned}
w_2(t,x) =& \frac{1}{\beta_0(t,x)\,(\varphi'(x))^2}\Big[-\partial_t w_0(t,x) + i\beta_0(t,x)\varphi''(x)w_1(t,x) + 2i\beta_0(t,x)\varphi'(x)\partial_x w_1(t,x)\\
&+ \beta_0(t,x)\partial_{xx}w_0(t,x) - i\beta_1(t,x)\varphi'(x)w_1(t,x) - \beta_1(t,x)\partial_x w_0(t,x) - \beta_2(t,x)w_0(t,x)\\
&- \delta_0(t,x)\partial_{xx}\eta(t,x) - \delta_1(t,x)\partial_x\eta(t,x) - \delta_2(t,x)\eta(t,x)\Big] \qquad (t\in[0,T], x\in\mathbb{R}).
\end{aligned}
$$

The rest of the proof is similar to that of Theorem 3.1. Moreover,

$$A(t) = \sqrt{\pi}|\eta(t,x_0)|^2 \neq 0 \text{ and } B(t) = \sqrt{\pi}|w_0(t,x_0)|^2 \neq 0 \text{ for all } t\in[0,T].$$

□

Remark 3.4. *Due to the different order of coupling in the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, the Gaussian beam solutions in Theorem 3.3 corresponding to the operator $\mathcal{L}_2$ are different from the solutions obtained in Theorem 3.1 for the operator $\mathcal{L}_1$. In contrast to (3.7), in Theorem 3.3, for all $t\in[0,T]$, $\|v_k(t,\cdot)\|^2_{L^2(\mathbb{R})}$ converges to a nonzero constant $B(t)$ as $k\to\infty$. It leads to a different behaviour of the null controllability of (1.12) than that of (1.3)-(1.5). In particular, the lack of null controllability of (1.3)-(1.5) is obtained in Theorem 1.2 and Theorem 1.4 using even everywhere control in the parabolic equation, whereas the lack of null controllability of (1.12) is obtained in Theorem 1.6, using localized control in the parabolic equation. In fact, using everywhere control in the parabolic equation, (1.12) with some assumptions on the coefficients is null controllable (see Theorem 5.4).*

3.3. **Coupled transport-parabolic systems : coupling of order one or zero.** In this subsection, we prove analogous result of Theorem 3.1, for the coupled transport-parabolic operator. We consider the following operator:

$$\mathcal{L}_3\begin{pmatrix}\sigma\\ v\end{pmatrix} = \begin{pmatrix}\partial_t\sigma + \alpha_0\partial_x\sigma + \alpha_1\sigma + \gamma_1\partial_x v + \gamma_2 v\\ \partial_t v - \beta_0\partial_{xx}v + \beta_1\partial_x v + \beta_2 v + \delta_1\partial_x\sigma + \delta_2\sigma\end{pmatrix}, \quad \text{in } [0,T]\times\mathbb{R}, \tag{3.29}$$

where the coefficients satisfy (3.2) along with

$$\alpha_0 \in C_b^\infty([0,T]\times\mathbb{R}). \tag{3.30}$$

We introduce the characteristics $\overline{X}$ associated with α_0:

$$\begin{cases}\partial_t\overline{X}(t,x) = \alpha_0\left(t,\overline{X}(t,x)\right), & (t>0),\\ \overline{X}(0,x) = x, \qquad x\in\mathbb{R}.\end{cases} \tag{3.31}$$

Note that for each $t\geq 0$, the mapping $x\mapsto\overline{X}(t,x)$ is a C^1 diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$ and the smoothness of $\overline{X}$ follows from that of α_0.

We set

$$\widetilde{\sigma}(t,x) = \sigma(t,\overline{X}(t,x)), \qquad \widetilde{v}(t,x) = v(t,\overline{X}(t,x)), \qquad (t\in[0,T], x\in\mathbb{R})\,. \tag{3.32}$$

Then the operator $\mathcal{L}_3$ transforms into

$$\mathcal{L}_3\begin{pmatrix}\sigma\\ v\end{pmatrix} = \begin{pmatrix}\partial_t\widetilde{\sigma} + \widetilde{\alpha}_1\widetilde{\sigma} + \widetilde{\gamma}_1\partial_x\widetilde{v} + \widetilde{\gamma}_2\widetilde{v}\\ \partial_t\widetilde{v} - \widetilde{\beta}_0\partial_{xx}\widetilde{v} + \widetilde{\beta}_1\partial_x\widetilde{v} + \widetilde{\beta}_2\widetilde{v} + \widetilde{\delta}_1\partial_x\widetilde{\sigma} + \widetilde{\delta}_2\widetilde{\sigma}\end{pmatrix}, \tag{3.33}$$

where

$$\widetilde{\alpha}_1(t,x) = \alpha_1(t,\overline{X}(t,x)), \quad \widetilde{\gamma}_1(t,x) = \gamma_1(t,\overline{X}(t,x))\left(\frac{\partial \overline{X}(t,x)}{\partial x}\right)^{-1}, \quad \widetilde{\gamma}_2(t,x) = \gamma_2(t,\overline{X}(t,x)),$$

$$\widetilde{\beta}_0(t,x) = \beta_0(t,\overline{X}(t,x))\left(\frac{\partial \overline{X}(t,x)}{\partial x}\right)^{-2}, \quad \widetilde{\beta}_2(t,x) = \beta_2(t,\overline{X}(t,x)), \quad \widetilde{\delta}_2(t,x) = \delta_2(t,\overline{X}(t,x)),$$

$$\begin{aligned}\widetilde{\beta}_1(t,x) = \beta_0(t,\overline{X}(t,x))\left(\frac{\partial \overline{X}(t,x)}{\partial x}\right)^{-3}\frac{\partial^2 \overline{X}(t,x)}{\partial x^2} \\ + (\beta_1(t,\overline{X}(t,x)) - \alpha_0(t,\overline{X}(t,x)))\left(\frac{\partial \overline{X}(t,x)}{\partial x}\right)^{-1},\end{aligned}$$

$$\widetilde{\delta}_1(t,x) = \delta_1(t,\overline{X}(t,x))\left(\frac{\partial \overline{X}(t,x)}{\partial x}\right)^{-1}.$$

According to Theorem 3.1, we can construct sequence of functions $(\widetilde{\sigma}_k, \widetilde{v}_k)_{k\in\mathbb{N}}$ satisfying (3.3)-(3.7). Gathering the above properties, we have obtained the following result:

Theorem 3.5. *Assume (3.2), (3.30), $T > 0$, $x_0 \in \mathbb{R}$ and $k \in \mathbb{N}$. Then, there exist a positive constant C, which may depend on T but independent of k, and a sequence of functions $(\sigma_k, v_k)_{k\in\mathbb{N}}$ satisfying*

$$\sigma_k \in C^1([0,T]; C_b^1(\mathbb{R})), \quad v_k \in C^1([0,T]; C_b^2(\mathbb{R})), \tag{3.34}$$

such that the following holds:

$$\sup_{t\in[0,T]} \left\| \mathcal{L}_3 \begin{pmatrix} \sigma_k \\ v_k \end{pmatrix}(t,\cdot) \right\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})} \leqslant Ck^{-1}, \tag{3.35}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}} |\sigma_k(t,x)|^2 \,\mathrm{d}x \geqslant \overline{A}(t) > 0 \qquad (t \in [0,T]), \tag{3.36}$$

$$\sup_{t\in[0,T]} \int_{|x-\overline{X}(t,x_0)|>k^{-1/4}} |\sigma_k(t,x)|^2 \,\mathrm{d}x \leqslant Ce^{-\sqrt{k}/2}, \tag{3.37}$$

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} |v_k(t,x)|^2 \,\mathrm{d}x \leqslant Ck^{-2}, \tag{3.38}$$

where $\overline{A}(t)$ is positive for all $t \in [0,T]$, and is independent of k.

In view of Lemma 3.2 and Theorem 3.5, we have the following result.

Lemma 3.6. *Let $x_0 \in (0,L)$, and let $(\sigma_k, v_k)_{k\in\mathbb{N}}$ be constructed as in Theorem 3.5. Let us define*

$$\begin{aligned} T_{x_0,0} &:= \sup\left\{\tau \mid \overline{X}(t,x_0) > 0 \text{ for all } t \in [0,\tau)\right\}, \\ T_{x_0,L} &:= \sup\left\{\tau \mid \overline{X}(t,x_0) < L \text{ for all } t \in [0,\tau)\right\}. \end{aligned} \tag{3.39}$$

Then there exists $k_0 \in \mathbb{N}$, such that for any $k \geqslant k_0$ the following estimates hold

$$\|v_k(\cdot,0)\|_{H^1(0,T)} \leqslant Ck^{-3/4}, \qquad \|v_k(\cdot,L)\|_{H^1(0,T)} \leqslant Ck^{-3/4} \quad \textit{for any } 0 < T < \infty,$$
$$\|\sigma_k(\cdot,0)\|_{H^1(0,T)} \leqslant Ce^{-\frac{\sqrt{k}}{4}} \quad \textit{for } T \in (0, T_{x_0,0}),$$
$$\|\sigma_k(\cdot,L)\|_{H^1(0,T)} \leqslant Ce^{-\frac{\sqrt{k}}{4}} \quad \textit{for } T \in (0, T_{x_0,L}),$$

where C is a positive constant, which may depend on k_0 and T, but independent of k.

Proof. Let $x_0 \in (0,L)$, and $0 < T < T_{x_0,0}$. Then there exists $k_0 \in \mathbb{N}$ such that

$$|\overline{X}(t,x_0)| > k_0^{-1/4} \text{ for all } t \in [0,T].$$

For each $t \geqslant 0$, we denote by $\overline{X}(t,\cdot)^{-1}$, the inverse of $\overline{X}(t,\cdot)$. We set $\ell_1(t) = \overline{X}(t,0)^{-1}$. Then, after suitably redefining k_0, we have

$$|\ell_1(t) - x_0| > k_0^{-1/4} \text{ for all } t \in [0,T].$$

Thus by Lemma 3.2(i) we get, for $k \geqslant k_0$

$$\|\widetilde{\sigma}_k(\cdot, \ell_1(\cdot))\|_{H^1(0,T)} \leqslant Ce^{-\frac{\sqrt{k}}{4}},$$

where the positive constant C may depend on k_0 and T but is independent of k. From the above estimate together with (3.32) and bounds of $\overline{X}(\cdot,\cdot)$, we infer that

$$\|\sigma_k(\cdot,0)\|_{H^1(0,T)} \leqslant Ce^{-\frac{\sqrt{k}}{4}} \quad \text{for } T \in (0, T_{x_0,0}),$$

for some positive constant C, independent of k. The other estimate of σ_k can be proved similarly. To prove the estimates of v_k, we simply take $\ell_2(t) = \overline{X}(t,0)^{-1}$ or $\ell_2(t) = \overline{X}(t,L)^{-1}$, and apply Lemma 3.2(ii). □

Lemma 3.7. *Let $x_0 \in (0,L)$ and $0 < T < \min\{T_{x_0,0}, T_{x_0,L}\}$, where $T_{x_0,0}$, $T_{x_0,L}$ are as defined in (3.39). Let $(\sigma_k, v_k)_{k\in\mathbb{N}}$ be constructed as in Theorem 3.5. It can be shown that*

$$\lim_{k\to\infty} \int_0^L |\sigma_k(T,x)|^2 \,\mathrm{d}x \geqslant \frac{\overline{A}(T)}{2},$$

for $\overline{A}(T) > 0$, same as in (3.36), independent of k.

Proof. Let $x_0 \in (0,L)$ and $0 < T < \min\{T_{x_0,0}, T_{x_0,L}\}$. By the definitions of $T_{x_0,0}$ and $T_{x_0,L}$, it follows that for any $0 < T < \min\{T_{x_0,0}, T_{x_0,L}\}$, $\overline{X}(T,x_0) \in (0,L)$. Thus, there exists a large $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$S = \{x \in \mathbb{R} \mid |x - \overline{X}(T,x_0)| < k^{-1/4}\} \subset (0,L),$$

and thus we have

$$\int_S |\sigma_k(T,x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} |\sigma_k(T,x)|^2 \,\mathrm{d}x - \int_{|x-\overline{X}(T,x_0)|\geq k^{-1/4}} |\sigma_k(T,x)|^2 \,\mathrm{d}x.$$

Now from (3.36), (3.37) and the above inequality, it follows that

$$\lim_{k\to\infty} \int_0^L |\sigma_k(T,x)|^2 \,\mathrm{d}x \geqslant \lim_{k\to\infty} \int_S |\sigma_k(T,x)|^2 \,\mathrm{d}x \geqslant \frac{\overline{A}(T)}{2},$$

for the positive constant $\overline{A}(T)$, same as in (3.36), independent of k. □

3.4. **Coupled transport-parabolic systems : coupling of order two.** We consider the following operator:

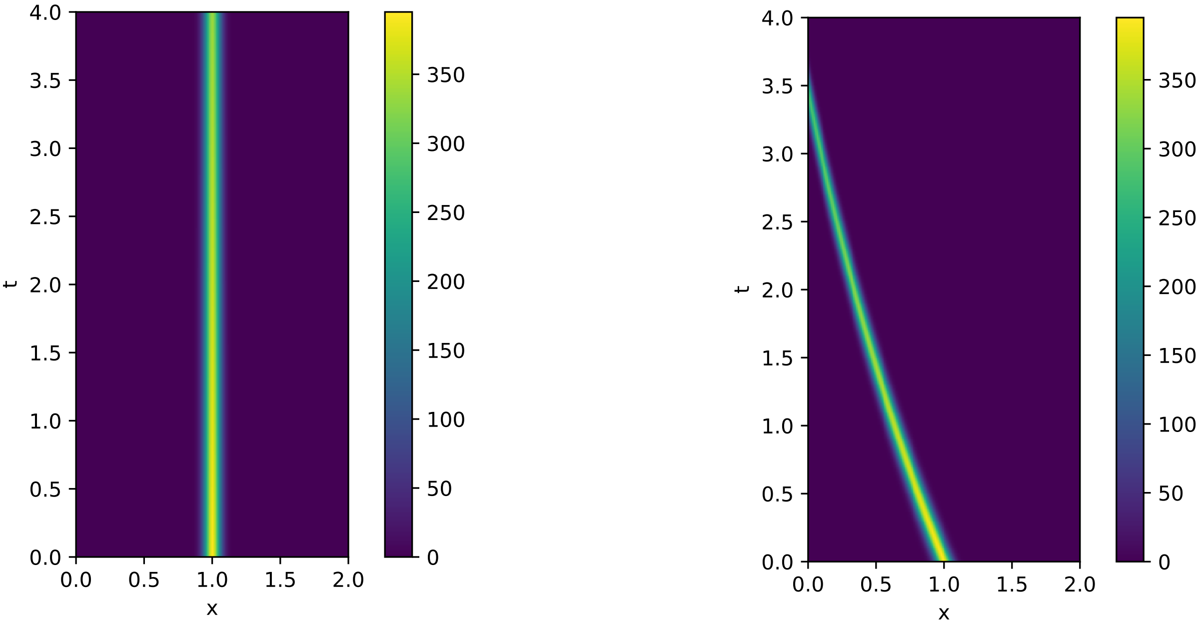
$$\mathcal{L}_4 \begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{pmatrix} \partial_t\sigma + \alpha_0\sigma + \alpha_1\sigma + \gamma_2 v \\ \partial_t v - \beta_0\partial_{xx}v + \beta_1\partial_x v + \beta_2 v + \delta_0\partial_{xx}\sigma + \delta_1\partial_x\sigma + \delta_2\sigma \end{pmatrix}, \quad \text{in } [0,T]\times\mathbb{R}, \tag{3.40}$$

where the coefficients satisfy (3.23) and (3.30).

Combining Theorem 3.3 and the change of coordinates introduced in (3.31), we deduce the following result.

Theorem 3.8. *Assume (3.23), (3.30), $T > 0$, $x_0 \in \mathbb{R}$ and $k \in \mathbb{N}$. Then, there exist a positive constant C, which may depend on T but independent of k, and a sequence of functions $(\sigma_k, v_k)_{k\in\mathbb{N}}$ satisfying*

$$\sigma_k \in C^1([0,T]; C_b^2(\mathbb{R})), \quad v_k \in C^1([0,T]; C_b^2(\mathbb{R})),$$



(A) σ_k constructed in Theorem 3.1.

(B) σ_k constructed in Theorem 3.5.

FIGURE 1. Localization of σ_k for the operators $\mathcal{L}_1$ and $\mathcal{L}_3$. In the horizontal axis we represent the space variable, and the vertical one represents time variable. *Left:* Plot of $|\sigma_k(t,x)|^2$ constructed in Theorem 3.1 with suitable choice of coefficients and $x_0 = 1$. According to (3.5) and (3.6), σ_k is localized around the curve (t, x_0). *Right:* Plot of $|\sigma_k(t,x)|^2$ constructed in Theorem 3.5 with $\alpha_0(x) = -0.2(1+x)$ and $x_0 = 1$. In this case, $\overline{X}(t,x_0) = (1+x_0)e^{-.2t} - 1$. According to (3.36) and (3.37), σ_k is localized around the curve $(t, \overline{X}(t,x_0))$.

such that the following holds:

$$\sup_{t\in[0,T]} \left\| \mathcal{L}_4 \begin{pmatrix} \sigma_k \\ v_k \end{pmatrix} (t,\cdot) \right\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})} \leqslant Ck^{-1}, \tag{3.41}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}} |\sigma_k(t,x)|^2 \,\mathrm{d}x \geqslant \overline{A}(t) \neq 0 \qquad (t\in[0,T]), \tag{3.42}$$

$$\sup_{t\in[0,T]} \int_{|x-\overline{X}(t,x_0)|>k^{-1/4}} |\sigma_k(t,x)|^2 \,\mathrm{d}x \leqslant Ce^{-\sqrt{k}/2}, \tag{3.43}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}} |v_k(t,x)|^2 \,dx \geqslant \overline{B}(t) \neq 0 \qquad (t\in[0,T]), \tag{3.44}$$

$$\sup_{t\in[0,T]} \int_{|x-\overline{X}(t,x_0)|>k^{-1/4}} |v_k(t,x)|^2 \,\mathrm{d}x \leqslant Ce^{-\sqrt{k}/2}, \tag{3.45}$$

where $\overline{A}(t)$ and $\overline{B}(t)$ are positive for all $t\in[0,T]$ and do not depend on k.

4. Proof of the main results

In this section we prove Theorem 1.2 and Theorem 1.6. As mentioned earlier, null controllability of a pair $(\mathcal{A},\mathcal{B})$ is equivalent to the final-state observability of the pair $(\mathcal{A}^*,\mathcal{B}^*)$. We recall the final state observability of $(\mathcal{A}^*,\mathcal{B}^*)$:

Definition 4.1. *The pair $(\mathcal{A}^*, \mathcal{B}^*)$ is final-state observable at time T if there exists a positive constant $C_T > 0$ such that*

$$\int_0^T \|\mathcal{B}^* \mathbb{T}_t^* z\|_{\mathcal{U}}^2 \, \mathrm{d}t \geqslant C_T \, \|\mathbb{T}_T^* z\|_{\mathcal{Z}}^2 , \quad \forall \, z \in \mathcal{D}(\mathcal{A}^*),$$

where $\mathbb{T}^*$ is the C^0-semigroup generated by $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$ in the Hilbert space $\mathcal{Z}$.

For $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*; \mathcal{Z}))$ defined in (2.14)-(2.15) and $(\sigma^0, v^0) \in \mathcal{Z}$, we set

$$(\sigma(t), v(t)) = \mathbb{T}_t^*(\sigma^0, v^0) \qquad (t \geqslant 0),$$

where $\mathbb{T}^*$ is the C^0-semigroup generated by $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*; \mathcal{Z}))$ on $\mathcal{Z}$. In view of Proposition 2.4, (σ, v) belongs to $C([0,T]; \mathcal{Z})$ and satisfies:

$$\begin{cases} \partial_t \sigma - a_0 \partial_x \sigma + (a_1 - a_0')\sigma - d_1 \partial_x v - (d_1' - d_2) v = 0 & \text{in } (0,T) \times (0,L), \\ \partial_t v - b_0 \partial_{xx} v - (2b_0' + b_1)\partial_x v - (b_0'' + b_1' - b_2) v & \\ \qquad\qquad\qquad\qquad - c_1 \partial_x \sigma + (c_2 - c_1')\sigma = 0 & \text{in } (0,T) \times (0,L), \\ v(t,0) = v(t,L) = 0 & \text{in } (0,T), \\ \sigma(t,L) = 0 \text{ if } a_0(0) > 0, \qquad \sigma(t,0) = 0 \text{ if } a_0(L) < 0 & \text{in } (0,T), \\ \sigma(0,x) = \sigma^0(x), \quad v(0,x) = v^0(x) & \text{in } (0,L). \end{cases} \tag{4.1}$$

In view of [29, Theorem 11.2.1], null controllability of the system (1.3)-(1.5) is equivalent to the following observability inequality:

Proposition 4.2. *The system (1.3)-(1.5) is null controllable in $\mathcal{Z}$ at time $T > 0$ using two controls f_1 and f_2 in $L^2(0,T; L^2(0,L))$ with supports in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, if and only if, for $T > 0$, there exists a positive constant $C_T > 0$ such that for any $(\sigma^0, v^0) \in \mathcal{Z}$, (σ, v), the solution of (4.1), satisfies the following observability inequality:*

$$\int_0^L |\sigma(T,x)|^2 \, \mathrm{d}x + \int_0^L |v(T,x)|^2 \, \mathrm{d}x \leqslant C_T \Big(\int_0^T \int_{\mathcal{O}_1} |\sigma(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} |v(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \Big). \tag{4.2}$$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Recall the definition of $T_{\mathcal{O}_1}$ from (1.9) and fix $0 < T < T_{\mathcal{O}_1}$. In view of Proposition 4.2, it is enough to show that, there exists a sequence of initial conditions $\left(\sigma_k^0, v_k^0\right)_{k \in \mathbb{N}}$ in $\mathcal{Z}$, such that, the corresponding solution (σ_k, v_k) to the system (4.1) satisfy the following estimates

$$\lim_{k \to \infty} \left(\int_0^T \int_{\mathcal{O}_1} |\sigma_k(t,x)|^2 \, \mathrm{d}x \mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} |v_k(t,x)|^2 \, \mathrm{d}x \mathrm{d}t \right) = 0,$$

$$\lim_{k \to \infty} \left(\int_0^L |\sigma_k(T,x)|^2 \, \mathrm{d}x \; + \; \int_0^L |v_k(T,x)|^2 \, \mathrm{d}x \right) \geqslant \overline{A},$$

for some $\overline{A} > 0$, independent of k.

We take extensions of the functions a_i, b_i, c_i, d_i on $\mathbb{R}$, still denoted by the same notation. Further, for a_0 we take the same extension that we have used to define X in (1.7). From the definition of $T_{\mathcal{O}_1}$, we deduce that, there exist $x_0 \in (0,L) \setminus \overline{\mathcal{O}_1}$ and $k_0 \in \mathbb{N}$ such that

$$\left\{ x \mid |x - X(t,x_0)| < k_0^{-1/4} \right\} \subset (0,L) \backslash \overline{\mathcal{O}_1} \qquad \text{for all } t \in [0,T],$$

where X is defined in (1.7). Let us fix such x_0. Let $\left(\sigma_k^\sharp, v_k^\sharp\right)_{k \in \mathbb{N}}$ be sequence of functions constructed in Theorem 3.5, with

$$\begin{aligned} &\alpha_0 = -a_0, \quad \alpha_1 = a_1 - a_0', \quad \gamma_1 = -d_1, \quad \gamma_2 = -(d_1' - d_2), \\ &\beta_0 = b_0, \quad \beta_1 = -(2b_0' + b_1), \quad \beta_2 = -(b_0'' + b_1' - b_2), \quad \delta_1 = -c_1, \quad \delta_2 = c_2 - c_1'. \end{aligned} \tag{4.3}$$

Note that, with the above choice of α_0, $\overline{X}$ coincides with X. For $k > k_0$, let us define

$$
\begin{aligned}
g_{0,k}(t) &:= \sigma_k^\sharp(t,0), \quad g_{L,k}(t) := \sigma_k^\sharp(t,L) \qquad (t \in [0,T]),\\
h_{0,k}(t) &:= v_k^\sharp(t,0), \quad h_{L,k}(t) := v_k^\sharp(t,L) \qquad (t \in [0,T]),
\end{aligned}
$$

and

$$
\begin{pmatrix} \zeta_{1,k} \\ \zeta_{2,k} \end{pmatrix} (t,x) := \mathbb{1}_{(0,L)} \mathcal{L}_3 \begin{pmatrix} \sigma_k^\sharp \\ v_k^\sharp \end{pmatrix} (t,x) \quad (t \in [0,T],\ x \in [0,L]) \,,
$$

where $\mathcal{L}_3$ is the operator defind in (3.29) with coefficients as in (4.3). Next, for $k > k_0$, we consider the following system.

$$
\begin{cases}
\partial_t \sigma_k^\dagger - a_0 \partial_x \sigma_k^\dagger + (a_1 - a_0')\sigma_k^\dagger - d_1 \partial_x v_k^\dagger - (d_1' - d_2) v_k^\dagger = \zeta_{1,k} & \text{in } (0,T) \times (0,L),\\
\partial_t v_k^\dagger - b_0 \partial_{xx} v_k^\dagger - (2b_0' + b_1)\partial_x v_k^\dagger - (b_0'' + b_1' - b_2) v_k^\dagger & \\
\qquad\qquad\qquad\qquad - c_1 \partial_x \sigma_k^\dagger + (c_2 - c_1')\sigma_k^\dagger = \zeta_{2,k} & \text{in } (0,T) \times (0,L),\\
v_k^\dagger(t,0) = h_{0,k}(t), \quad v_k^\dagger(t,L) = h_{L,k}(t) & \text{in } (0,T),\\
\sigma_k^\dagger(t,L) = g_{L,k}(t) \text{ if } a_0(0) > 0, \qquad \sigma_k^\dagger(t,0) = g_{0,k}(t) \text{ if } a_0(L) < 0 & \text{in } (0,T),\\
\sigma_k^\dagger(0,x) = 0, \quad v_k^\dagger(0,x) = 0 & \text{in } (0,L).
\end{cases}
\tag{4.4}
$$

In view of Proposition 2.5, (3.35) and Lemma 3.6, for $k > k_0$, the system (4.4) admits a unique solution $(\sigma_k^\dagger, v_k^\dagger) \in C([0,T]; \mathcal{Z})$ together with the estimate

$$
\left\| (\sigma_k^\dagger, v_k^\dagger) \right\|_{C([0,T];\mathcal{Z})} \leqslant C k^{-3/4} \qquad (k \geqslant k_0) \,, \tag{4.5}
$$

where the positive constant C is independent of k. Finally, we set

$$
\sigma_k = \sigma_k^\sharp - \sigma_k^\dagger, \qquad v_k = v_k^\sharp - v_k^\dagger.
$$

Then (σ_k, v_k) satisfies the system (4.1) with the initial data $(\sigma_k^0, v_k^0) = \left(\sigma_k^\sharp(0), v_k^\sharp(0)\right)$. Note that for any given $0 < T < T_{\mathcal{O}_1}$, from the choice of x_0 and the definition of $T_{\mathcal{O}_1}$ in (1.9), we have that $X(t,x_0) \in (0,L)\backslash\overline{\mathcal{O}_1}$ for all $t \in [0,T]$. From the definitions of $T_{x_0,0}$ and $T_{x_0,L}$ in (3.39), it follows that $0 < T < \min\{T_{x_0,0},\, T_{x_0,L}\}$. Using Lemma 3.7 and (4.5), we deduce that

$$
\lim_{k\to\infty} \left(\int_0^L |\sigma_k(T,x)|^2 \,\mathrm{d}x \; + \; \int_0^L |v_k(T,x)|^2 \,\mathrm{d}x \right) \geqslant \lim_{k\to\infty} \int_0^L |\sigma_k(T,x)|^2 \,\mathrm{d}x \geqslant \frac{\overline{A}(T)}{2},
$$

for some positive constant $\overline{A}(T)$. Similarly, from (3.37), (3.38) and (4.5), it follows that

$$
\lim_{k\to\infty} \Big(\int_0^T \int_{\mathcal{O}_1} |\sigma_k(t,x)|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} |v_k(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \Big) = 0.
$$

This completes the proof of the theorem. □

Next, we prove Theorem 1.4.

Proof of Theorem 1.4. Let us take $\mathcal{O}_2 = (0,L)$. Then the system (1.3)-(1.5) is null controllable in $\mathcal{Z}$ at time $T > 0$ using a control f_2 in $L^2(0,T;L^2(0,L))$, if and only if, for $T > 0$, there exists a positive constant $C_T > 0$ such that for any $(\sigma^0, v^0) \in \mathcal{Z}$, (σ, v), the solution of (4.1), satisfies the following observability inequality:

$$
\int_0^L |\sigma(T,x)|^2 \,\mathrm{d}x \; + \; \int_0^L |v(T,x)|^2 \,\mathrm{d}x \leqslant C_T \int_0^T \int_0^L |v(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t.
$$

Noting (1.10), the proof of Theorem 1.4(i) follows similarly as that of Theorem 1.2. To prove Theorem 1.4(ii), we show that under the assumptions (1.11), $T_\emptyset = +\infty$. This is divided in several cases.

Case 1. $a_0(x_*) = 0$ *for some* $x_* \in (0, L)$: In this case, $X(t, x_*) = x_*$ for all $t \geqslant 0$, and hence $T_{x_*,\emptyset} = +\infty$.
Case 2. $a_0(0) < 0$ *and* $a_0(L) > 0$*:* In such case, there exist $x_* \in (0, L)$ such that $a_0(x_*) = 0$ and we are back to Case 1.
Case 3. $a_0(0) = 0 = a_0(L)$: In this case, $X(t, 0) = 0$ and $X(t, L) = L$ for all $t \geqslant 0$. Thus, for any $x \in (0, L)$, $X(t, x) \in (0, L)$ for $t \geqslant 0$.
Case 4. $a_0(0) < 0$, $a_0(L) = 0$ *and* $a_0 < 0$ *in* $(0, L)$*:* In this case, $X(t, L) = L$ for all $t \geqslant 0$, and $X(t, 0) \geqslant 0$ for all $t \geqslant 0$. Thus, there exists $x_* \in (0, L)$ such that $X(t, x_*) \in (0, L)$, for all $t \geqslant 0$. Therefore, $T_\emptyset = \infty$.

The other cases can be treated in a similar manner and thus the theorem is proved. □

Finally, we give a proof of Theorem 1.6.

Proof of Theorem 1.6. The proof of Theorem 1.6 is similar to that of Theorem 1.2. First of all, the null controllability of (1.12) is equivalent to the final-state observability of the pair $(\widehat{\mathcal{A}}^*, \mathcal{B}^*)$, where $\widehat{\mathcal{A}}^*$ is the adjoint of $\widehat{\mathcal{A}}$ defined in Proposition 2.10. Note that, the operator $\mathcal{L}_4$ defined in (3.40), corresponds to the adjoint operator $\widehat{\mathcal{A}}^*$. To prove Theorem 1.6, we can proceed in a similar manner as in the proof of Theorem 1.2 using Gaussian beams constructed in Theorem 3.8 instead of Theorem 3.5. □

As indicated in the introduction, the system (1.12) is related to the system (1.3)-(1.5), if initial data lies in $H^1(0, L) \times L^2(0, L)$. Therefore, as a consequence of Theorem 1.6, we obtain following result for the system (1.3)-(1.5).

Theorem 4.3. *Assume* (1.6)*, and*

$$a_0(0) \leqslant 0, \quad a_0(L) \geqslant 0, \tag{4.6}$$

$$d_2(a_0' d_1 - a_0 d_1') = d_1(d_1 a_1' - a_0 d_2'), \tag{4.7}$$

$$c_1 \neq 0, \quad d_1 \neq 0. \tag{4.8}$$

Further, let $f_1 \equiv 0$ *in* (1.3) *and* $\mathcal{O}_2 \subset (0, L)$ *be such that* $(0, L) \setminus \overline{\mathcal{O}}_2$ *is a nonempty open subset of* $(0, L)$*. Then the system* (1.3)*-*(1.5) *is not null controllable in* $H^1(0, L) \times L^2(0, L)$*, at any time* $0 < T < T_{\mathcal{O}_2}$*, by an interior control* $f_2 \in L^2(0, T; L^2(0, L))$ *supported in* $\mathcal{O}_2$*.*

Proof. The proof relies on a contradiction argument. Let us assume that, under the hypothesis of Theorem 4.3, the system (1.3)-(1.5) is null controllable in $H^1(0, L) \times L^2(0, L)$, at any time $0 < T < T_{\mathcal{O}_2}$, by an interior control $f_2 \in L^2(0, T; L^2(0, L))$ supported in $\mathcal{O}_2$, where $(0, L) \setminus \overline{\mathcal{O}}_2$ is a nonempty open subset of $(0, L)$. Thus $(\rho, u) \in C([0, T]; H^1(0, L) \times L^2(0, L))$ (see Theorem 2.6), and

$$\rho(T, x) = u(T, x) = 0, \qquad \text{for all } x \in (0, L). \tag{4.9}$$

Setting, $\eta = d_1 \partial_x \rho + d_2 \rho$, we observe

$$\eta(T, x) = u(T, x) = 0, \qquad \text{for all } x \in (0, L). \tag{4.10}$$

Using (4.6) - (4.8), it is easy to verify that, (η, u) solves the following system

$$\begin{cases} \partial_t \eta + a_0 \partial_x \eta + \bar{a}_1 \eta + \bar{c}_0 \partial_{xx} u + \bar{c}_1 \partial_x u + \bar{c}_2 u = 0 & \text{in } (0, T) \times (0, L), \\ \partial_t u - b_0 \partial_{xx} u + b_1 \partial_x u + b_2 u + \eta = \mathbb{1}_{\mathcal{O}2} f_2 & \text{in } (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = 0 & \text{in } (0, T), \\ \eta(0) = d_1 (\rho^0)' + d_2 \rho^0, \qquad u(0) = u^0 & \text{in } (0, L), \end{cases} \tag{4.11}$$

where

$$\bar{a}_1 = a_1 + \frac{a_0' d_1 - a_0 d_1'}{d_1}, \quad \bar{c}_0 = c_1 d_1, \quad \bar{c}_1 = (d_1 c_1' + d_1 c_2 + d_2 c_1), \quad \bar{c}_2 = d_1 c_2' + d_2 c_2.$$

Note that (4.11) is similar to (1.12) with the coefficients satisfying the assumptions in Theorem 1.6. Hence, using Theorem 1.6, we obtain that (4.11) is not null controllable in $L^2(0, L) \times L^2(0, L)$

at any time $0 < T < T_{\mathcal{O}_2}$ using a localized control in the parabolic equation, which contradicts (4.10). Hence the proof is complete. $\square$

Remark 4.4. *Let us compare the results of Theorem 1.4(ii) and Theorem 4.3, under the assumption* (4.6)*, as both correspond to the same system* (1.3)-(1.5) *with parabolic control only, i.e.,* $\mathcal{O}_1 = \emptyset$*. In Theorem 1.4(ii), the system is not null controllable in* $L^2(0,L) \times L^2(0,L)$ *at any time* $T > 0$ *using any control in the parabolic equation even with support everywhere in* $(0,L)$*. Thus, it is reasonable to ask whether the controllability property of the system improves for the regular initial data. In Theorem 4.3, under the additional assumptions* (4.7) *and* (4.8)*, for initial data in* $H^1(0,L) \times L^2(0,L)$*, the lack of null controllability of the system using any control in the parabolic equation with support in* $\mathcal{O}_2$ *is obtained at time* T*, where* $0 < T < T_{\mathcal{O}_2}$ *and* $\mathcal{O}_2$ *is a proper subset of* $(0,L)$*. Thus the results of Theorem 4.3, indicates the possibility to obtain the null controllability in* $H^1(0,L) \times L^2(0,L)$ *using only one control in the parabolic equation acting everywhere in* $(0,L)$*, whereas Theorem 1.4(ii) shows that no control acting only in the parabolic equation even with support everywhere in* $(0,L)$ *can give the null controllability of the system in* $L^2(0,L) \times L^2(0,L)$ *at time* $T > 0$*.*

5. Null controllability by parabolic control

In Theorem 1.6 and Theorem 4.3, we have seen that the corresponding systems are not controllable for small time with localized interior controls. Furthermore, if $a_0 \equiv 0$, as mentioned in Remark 1.3, the corresponding systems are not controllable at any finite time by localized interior controls. Nevertheless, if $a_0 \equiv 0$, the system can be null controllable by using even a control acting only in the parabolic equation but with support everywhere in the domain and the goal of this section is to prove the result. To this aim, we first study the null controllability of the following auxiliary system:

$$\begin{cases} \partial_t \rho + \alpha_0 \partial_{xx}(\gamma_0 u) = 0 & \text{in } (0,T) \times (0,L), \\ \partial_t u - \dfrac{b_0}{\gamma_0} \partial_{xx}\Big(\gamma_0 u\Big) + \rho = g & \text{in } (0,T) \times (0,L), \\ u(t,0) = u(t,L) = 0 & \text{in } (0,T), \\ \rho(0) = \rho^0, \qquad u(0) = u^0 & \text{in } (0,L), \end{cases} \tag{5.1}$$

where g is the control.

We assume

$$\alpha_0, \gamma_0, b_0 \in C^\infty([0,L]), \quad \min_{[0,L]} b_0 > 0, \quad \alpha_0 \neq 0, \quad \min_{[0,L]} \gamma_0 > 0. \tag{5.2}$$

The system (5.1) is well posed in $L^2(0,L) \times L^2(0,L)$ due to Theorem 2.8. We prove the following controllability result for the system (5.1).

Theorem 5.1. *Let us assume* (5.2) *and* $T > 0$*. For any* $(\rho^0, u^0) \in L^2(0,L) \times L^2(0,L)$*, there exists a control* $g \in L^2(0,T;L^2(0,L))$ *such that* (ρ, u)*, the corresponding solution to* (5.1)*, belongs to* $C([0,T];L^2(0,L)) \times \Big(L^2(0,T;H_0^1(0,L)) \cap C([0,T];L^2(0,L))\Big)$ *and satisfies*

$$\rho(T,x) = u(T,x) = 0, \quad \textit{for all} \quad x \in (0,L).$$

The null controllability of the system (5.1) is equivalent to the final-state observability of the adjoint system. According to Proposition 2.10, the corresponding adjoint system reads as:

$$\begin{cases} \partial_t \sigma + v = 0 & \text{in } (0,T) \times (0,L), \\ \partial_t v - \gamma_0 \partial_{xx}\Big(\dfrac{b_0}{\gamma_0} v - \alpha_0 \sigma\Big) = 0 & \text{in } (0,T) \times (0,L), \\ \Big(\dfrac{b_0}{\gamma_0} v - \alpha_0 \sigma\Big)(t,0) = \Big(\dfrac{b_0}{\gamma_0} v - \alpha_0 \sigma\Big)(t,L) = 0 & \text{in } \in (0,T), \\ \sigma(0) = \sigma^0, \qquad v(0) = v^0 & \text{in } (0,L). \end{cases} \tag{5.3}$$

In view of [29, Theorem 11.2.1], the statement of Theorem 5.1 is equivalent to the following theorem

Theorem 5.2. *Under the assumptions of Theorem 5.1, for any $T > 0$, there exists a positive constant C_T such that (σ, v), the solution to (5.3) with initial condition (σ^0, v^0), satisfies*

$$\int_0^L |\sigma(T,x)|^2 \,\mathrm{d}x \;+\; \int_0^L |v(T,x)|^2 \,\mathrm{d}x \leqslant C_T \int_0^T \int_0^L |v(x,t)|^2 \,\mathrm{d}x \,\mathrm{d}t \quad \text{for all } (\sigma^0, v^0) \in \mathcal{D}(\widehat{\mathcal{A}}^*), \tag{5.4}$$

where $\mathcal{D}(\widehat{\mathcal{A}}^*)$ is the domain of the linear operator corresponding to (5.3) as defined in (2.23).

Proof. Let $\widetilde{\mathcal{A}}$ denotes the linear operator associated with the system (5.3). In view of Proposition 2.10, this operator $\widetilde{\mathcal{A}}$ is the adjoint of the operator $\widehat{\mathcal{A}}$ defined in (2.20) with suitable choice of coefficients. In particular, we have $\widetilde{\mathcal{A}}$ generates a C^0-semigeoup $\widetilde{\mathbb{T}}$ in $\mathcal{Z}$. Moreover, there exist $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\left\|\widetilde{\mathbb{T}}_t\right\| \leqslant M e^{\omega t}, \quad t \geq 0.$$

Using the above property, it is easy to see that, for any $(\sigma^0, v^0) \in \mathcal{Z}$, the solution (σ, v) to the system (5.3) satisfies

$$\|(\sigma(T_2,\cdot), v(T_2,\cdot)\|_{\mathcal{Z}} \leqslant M e^{\omega(T_2 - T_1)} \|(\sigma(T_1,\cdot), v(T_1,\cdot)\|_{\mathcal{Z}},$$

for any $0 \leqslant T_1 < T_2 \leqslant T$. For any $t \in [0,T)$, choosing $T_2 = T$ and $T_1 = t$, and noting $(T-t) \leqslant T$, from the above estimate it follows that

$$\int_0^L |\sigma(T,x)|^2 \,\mathrm{d}x + \int_0^L |v(T,x)|^2 \mathrm{d}x \leqslant M^2 e^{2\omega T}\Big(\int_0^L |\sigma(t,x)|^2 \,\mathrm{d}x + \int_0^L |v(t,x)|^2 \,\mathrm{d}x\Big) \quad (0 \leqslant t < T).$$

Now integrating both side of the above inequality over $[0,T]$ with respect to t, we have the existence of a positive constant C such that

$$\int_0^L |\sigma(T,x)|^2 \,\mathrm{d}x + \int_0^L |v(T,x)|^2 \mathrm{d}x \leqslant C\Big(\int_0^T \int_0^L |\sigma(t,x)|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_0^L |v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t\Big). \tag{5.5}$$

The desired conclusion of this theorem holds provided that there exits a constant $C_T > 0$ such that

$$\int_0^T \int_0^L |\sigma(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \leqslant C_T \int_0^T \int_0^L |v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t. \tag{5.6}$$

Multiplying $(5.3)_1$ by σ we first obtain

$$\int_0^T \int_0^L |\sigma(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \leqslant C\Big(\int_0^L |\sigma^0(x)|^2 \,\mathrm{d}x + \int_0^T \int_0^L |v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t\Big), \tag{5.7}$$

for some $C > 0$ constant.

Let us consider $\zeta \in C_c^\infty(0,T)$ with $\zeta \geqslant 0$ on $(0,T)$. Setting $\Psi(t) = \displaystyle\int_t^T \zeta(s)\,\mathrm{d}s$, we note that $\Psi'(t) = -\zeta(t)$ for all $t \in (0,T)$ and $\Psi(T) = 0$, $\Psi(0) > 0$.

Multiplying $(5.3)_1$ by $\Psi(t)$ and using an integration by parts, we obtain

$$\sigma^0(x)\Psi(0) + \int_0^T \sigma(t,x)\Psi'(t)\,dt = \int_0^T v(t,x)\Psi(t)\,dt, \quad \forall x \in (0,L),$$

and then using the definition of Ψ and integrating over $[0,L]$ with respect to x variable along with the Cauchy-Schwarz inequality, we derive

$$\int_0^L |\sigma^0(x)|^2 \,\mathrm{d}x \leqslant C\Big(\int_0^T \int_0^L |v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^L \Big|\int_0^T \sigma(t,x)\zeta(t)\,\mathrm{d}t\Big|^2 \mathrm{d}x\Big), \tag{5.8}$$

for some positive constant C. Note that, $(5.3)_2$ can be written as

$$\frac{b_0}{\gamma_0} v - \alpha_0 \sigma = (-\Delta_D)^{-1}\left(-\frac{1}{\gamma_0}\partial_t v\right),$$

where $-\Delta_D : H_0^1(0,L) \to H^{-1}(0,L)$ is an isomorphism. Multiplying the above identity by $\zeta(t)$, after integration by parts, we have

$$\int_0^T \alpha_0(x)\sigma(t,x)\zeta(t)\,\mathrm{d}t = \int_0^T (-\Delta_D)^{-1}\left(-\frac{1}{\gamma_0}v\right)\zeta'(t)\,\mathrm{d}t + \int_0^T \frac{b_0(x)}{\gamma_0(x)}v(t,x)\zeta(t)\,\mathrm{d}t, \tag{5.9}$$

and it yields

$$\int_0^L \Big|\int_0^T \sigma(t,x)\zeta(t)\,\mathrm{d}t\Big|^2 \mathrm{d}x \leqslant C\int_0^T\int_0^L |v(t,x)|^2\,\mathrm{d}x\mathrm{d}t, \tag{5.10}$$

for some positive constant C. Combining the above estimate together with (5.8) and (5.7) we get (5.6). This completes the proof of the theorem. $\square$

Remark 5.3. *Let us point out that, by duality, the estimate (5.5) is equivalent to the fact that the system (5.1) is null controllable by controls acting everywhere in the both components. In fact, this can be proved directly by multiplying the free system, i.e. the system without any control, by a smooth function of time, which is 1 at $t=0$ and 0 at $t=T$. And, the rest of the proof after the estimate (5.5) is actually one way of removing control from the first component.*

As a consequence of Theorem 5.1, we obtain the following null controllability result for the system (1.12).

Theorem 5.4. *Assume (1.13), $a_0 = a_1 = 0$ and*

$$c_0 = \alpha_0\gamma_0, \qquad c_1 = 2\alpha_0\gamma_0', \qquad c_2 = \alpha_0\gamma_0'',$$

for any $\alpha_0, \gamma_0 \in C^\infty([0,L])$ with $\alpha_0 \neq 0$ and $\gamma_0 > 0$. Let $f_1 \equiv 0$ in $(1.12)_1$ and $\mathcal{O}_2 = (0,L)$ in $(1.12)_2$. Then for every $T > 0$ and for any $(\rho^0, u^0) \in L^2(0,L) \times L^2(0,L)$, there exists a control $f_2 \in L^2(0,T;L^2(0,L))$ such that (ρ,u), the corresponding solution to (1.12) belongs to $C([0,T];L^2(0,L)) \times \left(L^2(0,T;H_0^1(0,L)) \cap C([0,T];L^2(0,L))\right)$ and satisfies

$$\rho(T,x) = u(T,x) = 0 \text{ for all } x \in (0,L).$$

Proof. Let (ρ,u) be the trajectory of (5.1) reaching to zero at time $T>0$ using the control g as constructed in Theorem 5.1. We define,

$$f_2 = g + \left(b_1 + \frac{2\gamma_0' b_0}{\gamma_0}\right)\partial_x u + \left(b_2 + \frac{\gamma_0'' b_0}{\gamma_0}\right)u + (d_2 - 1)\rho.$$

Then (ρ,u) satisfies the system (1.12) with f_2 defined above. Moreover, it satisfies

$$\rho(T,x) = u(T,x) = 0 \text{ for all } x \in (0,L).$$

$\square$

We now focus on the null controllability of the system (1.3)-(1.5). Let us recall the spaces from the introduction

$$L^2_{\mathrm{m}}(0,L) = \left\{ f \in L^2(0,L) \mid \int_0^L f\ \mathrm{d}x = 0 \right\}, \qquad H^1_{\mathrm{m}}(0,L) = H^1(0,L) \cap L^2_{\mathrm{m}}(0,L).$$

We prove the following result:

Theorem 5.5. *Assume (1.6), and*

$$a_0(x) = 0 = a_1(x), \quad c_1(x) \neq 0, \quad c_2(x) = c_1'(x) \text{ for all } x \in [0,L]. \tag{5.11}$$

Let $f_1 \equiv 0$ and $\mathcal{O}_2 = (0,L)$ in (1.3). The following results hold:

(i) *The system (1.3)-(1.5) with initial condition $(\rho^0,u^0) \in H^1(0,L) \times L^2(0,L)$ is not null controllable at any time $T>0$, if $\displaystyle\int_0^L \rho^0(x)\,\mathrm{d}x \neq 0$.*

(ii) *For every $T > 0$ and for any $(\rho^0, u^0) \in H^1_{\mathrm{m}}(0, L) \times L^2(0, L)$, there exists a control $f_2 \in L^2(0, T; L^2(0, L))$ such that (ρ, u), the corresponding solution to* (1.3)-(1.5)*, belongs to $C([0, T]; H^1_{\mathrm{m}}(0, L) \times L^2(0, L))$ and satisfies*

$$\rho(T, x) = u(T, x) = 0 \text{ for all } x \in (0, L). \tag{5.12}$$

Proof. (i) Integrating the first equation of (1.3) and using the above assumptions and boundary conditions, we deduce that

$$\int_0^L \rho(t, x)\,\mathrm{d}x = \int_0^L \rho^0(x)\,\mathrm{d}x \qquad \text{for all } t \in [0, T].$$

Therefore, for $(\rho_0, u_0) \in H^1(0, L) \times L^2(0, L)$, if (1.3) $-$ (1.5) is null controllable in $H^1(0, L) \times L^2(0, L)$ at time $T > 0$, then necessarily ρ^0 satisfies $\displaystyle\int_0^L \rho^0(x)\,\mathrm{d}x = 0$.

(ii) Let us assume that $(\rho^0, u^0) \in H^1_{\mathrm{m}}(0, L) \times L^2(0, L)$. Without loss of generality, we may assume that $c_1(x) > 0$, for all $x \in [0, L]$. Consider the following control system

$$\begin{cases} \partial_t \rho + \partial_x (c_1 u) = 0 & \text{in } (0, T) \times (0, L), \\ \partial_t u - \dfrac{b_0}{c_1} \partial_{xx}(c_1 u) + \partial_x \rho = g & \text{in } (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = 0 & \text{in } (0, T), \\ \rho(0) = \rho^0, \quad u(0) = u^0 & \text{in } (0, L), \end{cases} \tag{5.13}$$

where g is the control. Set $\eta = \partial_x \rho$. Then (η, u) satisfies

$$\begin{cases} \partial_t \eta + \partial_{xx} (c_1 u) = 0 & \text{in } (0, T) \times (0, L), \\ \partial_t u - \dfrac{b_0}{c_1} \partial_{xx}(c_1 u) + \eta = g & \text{in } (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = 0 & \text{in } (0, T), \\ \eta(0) = (\rho^0)', \quad u(0) = u^0 & \text{in } (0, L). \end{cases} \tag{5.14}$$

Note that, due to the average zero condition, the null controllability of (5.13) in $H^1_{\mathrm{m}}(0, L) \times L^2(0, L)$ is equivalent to the null controllability of the system (5.14) in $L^2(0, L) \times L^2(0, L)$. Therefore, by Theorem 5.1, for every $(\rho^0, u^0) \in H^1_{\mathrm{m}}(0, L) \times L^2(0, L)$ there exists a control $g \in L^2(0, T; L^2(0, L))$ such that (ρ, u), the corresponding solution to (5.13), belongs to $C([0, T]; H^1_{\mathrm{m}}(0, L) \times L^2(0, L))$ and satisfies

$$\rho(T, x) = u(T, x) = 0 \text{ for all } x \in (0, L).$$

Finally, by setting

$$f_2 = g + \left(b_1 + \frac{2c_1' b_0}{c_1}\right) \partial_x u + \left(b_2 + \frac{c_1'' b_0}{c_1}\right) u + (d_1 - 1)\partial_x \rho + d_2 \rho,$$

it is easy to verify that (ρ, u) solves the system (1.3)-(1.5) satisfying (5.12). $\square$

Remark 5.6. *The purpose of Theorem 5.5 is to check if the anticipation in Remark 4.4 regarding the null controllability using everywhere control in the parabolic equation holds true. We observe that for the system* (1.3)-(1.5) *with $f_1 \equiv 0$ and with coefficients satisfying* (1.6) *along with*

$$a_0(x) = 0 = a_1(x), \quad c_1(x) \neq 0, \quad c_2(x) = c_1'(x), \quad d_1(x) \neq 0 \text{ for all } x \in [0, L],$$

both the lack of null controllability results Theorem 1.4(ii) and Theorem 4.3 and the null controllability result Theorem 5.5 hold. Theorem 1.4(ii) gives the lack of null controllability of the system in $L^2(0, L) \times L^2(0, L)$ at any time $T > 0$ using any control in the parabolic equation even acting everywhere in the domain. In Theorem 4.3, since $a_0 \equiv 0$ in $[0, L]$, $T_{\mathcal{O}_2} = \infty$, where $\mathcal{O}_2$ is a proper subset of $(0, L)$ and hence the system is not null controllable in $H^1(0, L) \times L^2(0, L)$ at any time $T > 0$ using any localized control in the parabolic equation. Then for the case of using everywhere control in the parabolic equation, Theorem 5.5 gives the null controllability of *the system in $H^1_{\mathrm{m}}(0,L)\times L^2(0,L)$, at any time $T>0$ and thus in this context, this is the best possible null controllability result expected for the system using a control only in the parabolic equation.*

It is expected to obtain an analogous result of Theorem 5.5 at any time $T>0$ even if a_0 is not identically zero in $(0,L)$, provided the coefficients in (1.3) satisfy suitable conditions.

6. Time dependent coefficients

In this section, we extend the above results to the case where the coefficients depend on both space and time.

6.1. **Extension of Theorem 1.2 and Theorem 1.4 to the time dependent case.** Let us assume that

$$\begin{aligned} a_i, b_j, c_i, d_i \in C^\infty([0,T]\times[0,L]), \quad &\text{for all } i=0,1, \quad \text{for all } j=0,1,2,\\ b_0(t,x)\geqslant \bar{b}>0 \quad &\text{for all } t\in[0,T], x\in[0,L]. \end{aligned} \tag{6.1}$$

Furthermore, we also assume one of the following three conditions is satisfied by $a_0(t,x)$ on the boundary:

$$a_0(t,0)>0, \quad a_0(t,L)\geqslant 0 \text{ for all } t\in[0,T], \tag{6.2}$$

or

$$a_0(t,0)\leqslant 0, \quad a_0(t,L)<0 \text{ for all } t\in[0,T], \tag{6.3}$$

or

$$a_0(t,0)\leqslant 0, \quad a_0(t,L)\geqslant 0 \text{ for all } t\in[0,T]. \tag{6.4}$$

Throughout this subsection we shall assume the coefficients satisfy (6.1), and one of (6.2) - (6.4), unless specified otherwise.

We consider the system (1.3) with the above hypothesis on coefficients. More precisely, we consider

$$\begin{cases} \partial_t\rho + a_0\partial_x\rho + a_1\rho + c_1\partial_x u + c_2 u = \mathbb{1}_{\mathcal{O}_1} f_1 & \text{in } (0,T)\times(0,L),\\ \partial_t u - b_0\partial_{xx}u + b_1\partial_x u + b_2 u + d_1\partial_x\rho + d_2\rho = \mathbb{1}_{\mathcal{O}_2} f_2 & \text{in } (0,T)\times(0,L),\\ u(t,0)=u(t,L)=0 & \text{in } (0,T),\\ \rho(t,0)=0 & \text{in } (0,T), \text{ if } a_0(t,0)>0,\\ \rho(t,L)=0 & \text{in } (0,T), \text{ if } a_0(t,L)<0,\\ \rho(0,\cdot)=\rho^0, \qquad u(0,\cdot)=u^0 \text{ in } (0,L). \end{cases} \tag{6.5}$$

Note that the conditions (6.2)-(6.4) ensure that the inflow boundary is time independent. We first study the well-posedness of the above system.

Theorem 6.1. (i) *Let $(\rho^0,u^0)\in \Big(L^2(0,L)\Big)^2$ and $f_i\in L^2(0,T;L^2(0,L))$, $i=1,2$. Then the system (6.5) admits an unique solution $(\rho,u)\in C\Big([0,T];\Big(L^2(0,L)\Big)^2\Big)$ together with the estimate*

$$\|(\rho,u)\|_{C\left([0,T];(L^2(0,L))^2\right)} \leqslant C\left(\|(\rho_0,u_0)\|_{L^2(0,L)\times L^2(0,L)} + \|(f_1,f_2)\|_{L^2(0,T;(L^2(0,L))^2)}\right), \tag{6.6}$$

where the positive constant C depends only on T, L and the coefficients of the system.

(ii) *Assume (6.1) and (6.4). Let $(\rho^0,u^0)\in H^1(0,L)\times L^2(0,L)$, $f_1=0$ and $f_2\in L^2(0,T;L^2(0,L))$. Then the system (6.5) admits an unique solution $(\rho,u)\in C([0,T];H^1(0,L)\times L^2(0,L))$ together with the estimate*

$$\|(\rho,u)\|_{C([0,T];H^1(0,L)\times L^2(0,L))} \leqslant C\left(\|(\rho_0,u_0)\|_{H^1(0,L)\times L^2(0,L)} + \|f_2\|_{L^2(0,T;L^2(0,L))}\right), \tag{6.7}$$

where the positive constant C depends only on T, L and the coefficients of the system.

Proof. Without loss of generality, let us assume that (6.2) holds. We will show the existence and uniqueness of the solution of (6.5) by a fixed point argument. Let $0 < T_1 \leqslant T$, $\widehat{\rho} \in C([0,T]; L^2(0,L))$, and we consider the following system

$$\begin{cases} \partial_t \rho + a_0 \partial_x \rho + a_1 \rho + c_1 \partial_x u + c_2 u = \mathbb{1}_{\mathcal{O}1} f_1 & \text{in } (0,T)\times(0,L), \\ \partial_t u - b_0 \partial_{xx} u + b_1 \partial_x u + b_2 u = \mathbb{1}_{\mathcal{O}2} f_2 - d_1 \partial_x \widehat{\rho} - d_2 \widehat{\rho} & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = 0 & \text{in } (0,T), \\ \rho(t,0) = 0 & \text{in } (0,T), \\ \rho(0,\cdot) = \rho^0, \qquad u(0,\cdot) = u^0 \text{ in } (0,L). \end{cases} \tag{6.8}$$

Note that, the above system can be solved in cascades. Indeed, $\mathbb{1}_{\mathcal{O}2} f_2 - d_1 \partial_x \widehat{\rho} - d_2 \widehat{\rho} \in L^2(0,T; H^{-1}(0,L))$, and therefore using standard results for parabolic equation, we obtain $u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L))$ satisfying the estimate for all $t \in (0,T_1]$,

$$\|u(t)\|_{L^2(0,L)} + \|u\|_{L^2(0,T_1;H^1(0,L))} \leqslant M_1 \left(\|u_0\|_{L^2(0,L)} + \|f_2\|_{L^2(0,T_1;L^2(0,L))} + \|\widehat{\rho}\|_{L^2(0,T_1;L^2(0,L))} \right), \tag{6.9}$$

where M_1 depends on T, L and the coefficients, but independent of T_1. Next, solving the transport equation $(6.8)_1$ we have $\rho \in C([0,T_1]; L^2(0,L))$ and

$$\|\rho(t)\|_{L^2(0,L)} \leqslant M_2 \left(\left\|\rho^0\right\|_{L^2(0,L)} + \|f_1\|_{L^2(0,T_1;L^2(0,L))} + \|u\|_{L^2(0,T_1;H_0^1(0,L))} \right) \quad t \in (0,T_1], \tag{6.10}$$

where M_2 depends on T, L and the coefficients, but independent of T_1. This allows us to define a map $\mathcal{I}$ from $C([0,T_1]; L^2(0,L))$ into itself by $\mathcal{I}(\widehat{\rho}) = \rho$, where (ρ, u) solves (6.8). Let $\widehat{\rho}_1, \widehat{\rho}_2 \in C([0,T]; L^2(0,L))$, and (ρ_1, u_1), (ρ_2, u_2) be the corresponding solutions of (6.8) when $\widehat{\rho} = \widehat{\rho}_1$ and $\widehat{\rho} = \widehat{\rho}_2$ respectively. Note that $(\rho_1 - \rho_2, u_1 - u_2)$ satisfies (6.8) with initial condition $(0,0)$ and with $f_1 = 0 = f_2$ and it obeys the corresponding estimates (6.9)-(6.10). Combing the above estimates it is easy to see that

$$\begin{aligned} \|\mathcal{I}(\widehat{\rho}_1) - \mathcal{I}(\widehat{\rho}_2)\|_{C([0,T_1];L^2(0,L))} &\leqslant M_2 M_1 \|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^2(0,T_1;L^2(0,L))} \\ &\leqslant M_2 M_1 \sqrt{T_1}\, \|\widehat{\rho}_1 - \widehat{\rho}_2\|_{C([0,T_1];L^2(0,L))}. \end{aligned}$$

Let N be a natural number such that $N > T M_1^2 M_2^2$. We take $T_1 = T/N$. Then $\mathcal{I}$ is a contraction on $C([0,T_1]; L^2(0,L))$. It is standard to pass from local to global existence by subdividing $[0,T]$ for $T > T_1$, into N subintervals and getting the existence in each $[kT_1, (k+1)T_1]$ using above. This completes the proof of (i).

To prove (ii) under assumptions (6.1) and (6.4), let us define

$$\eta = \frac{c_1}{b_0} u + \partial_x \rho \quad \text{in } (0,T)\times(0,L).$$

Then (η, u) solves the system

$$\begin{cases} \partial_t \eta + a_0 \partial_x \eta + \tilde{a}_1 \eta + \tilde{c}_1 \partial_x u + \tilde{c}_2 u = \tilde{f}_1 & \text{in } (0,T)\times(0,L), \\ \partial_t u - b_0 \partial_{xx} u + b_1 \partial_x u + \tilde{b}_2 u + \tilde{d}_2 \eta = \tilde{f}_2 & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = 0 & \text{in } (0,T), \\ \eta(0,\cdot) = \dfrac{c_1(0,\cdot)}{b_0(0,\cdot)} u^0(\cdot) + \left(\rho^0\right)'(\cdot), \qquad u(0,\cdot) = u^0(\cdot) \text{ in } (0,L), \end{cases} \tag{6.11}$$

where

$$\begin{aligned}
\tilde{a}_1 =&\partial_x a_0 + a_1 + \frac{c_1 d_1}{b_0}, \ \tilde{c}_1 = \partial_x c_1 + c_2 + \frac{c_1 b_1 - a_0 c_1}{b_0} \\
\tilde{c}_2 =&\partial_x c_2 + \frac{c_1 b_2 - \partial_x a_0 c_1 - a_1 c_1}{b_0} - \frac{c_1^2 d_1}{b_0^2} - \partial_t\left(\frac{c_1}{b_0}\right) - a_0 \partial_x \left(\frac{c_1}{b_0}\right) \\
\tilde{f}_1 =&\frac{c_1}{b_0}\mathbb{1}_{\mathcal{O}_2} f_2 - \left(\partial_x a_1 + \frac{c_1 d_2}{b_0}\right)\rho, \ \tilde{b}_2 = b_2 - \frac{c_1 d_1}{b_0}, \ \tilde{d}_2 = d_1, \ \tilde{f}_2 = \mathbb{1}_{\mathcal{O}_2} f_2 - d_2 \rho.
\end{aligned}$$

Note that no boundary condition is needed for η because of (6.4). Then using (i) we complete the proof of (ii). $\square$

Next, we show that the controllability of the system (6.5) is equivalent to final-state observability of the adjoint system. In particular, we want to prove Proposition 4.2 when the coefficients depend both t and x variable. Consider the adjoint system (4.1) with the hypothesis (6.1) - (6.4) on the coefficients. Following the proof of Theorem 6.1, we can show that the adjoint system (4.1) well-posed, and Proposition 2.5 holds in this case also. We have the following identity equivalent to null controllability.

Proposition 6.2. *The system* (6.5) *is null controllable in $\mathcal{Z}$ at time $T > 0$ using two controls f_1 and f_2 in $L^2(0,T;L^2(0,L))$ with supports in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively if and only if*

$$\int_0^T \int_{\mathcal{O}_1} \sigma(T-t,x) f_1(t,x) \ \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} v(T-t,x) f_2(t,x) \ \mathrm{d}x\mathrm{d}t \\ + \int_0^L \rho^0(x)\sigma(T,x) \ \mathrm{d}x + \int_0^L u^0(x) v(T,x) \ \mathrm{d}x = 0, \quad (6.12)$$

for all $(\sigma^0, v^0) \in \mathcal{Z}$, and (σ, v) is the corresponding solution of (4.1) *with coefficients satisfying the hypothesis* (6.1) - (6.4).

Proof. Let (ρ, u) be the solution of the system (6.5) with initial condition $(\rho^0, u^0) \in \mathcal{Z}$ and two controls f_1 and f_2 in $L^2(0,T;L^2(0,L))$ with supports in $\mathcal{O}_1$ and $\mathcal{O}_2$. Let for any $(\sigma^0, v^0) \in \mathcal{Z}$, (σ, v) be the corresponding solution of (4.1).

Multiplying $(6.5)_1$ by $\sigma(T-t,x)$ and $(6.5)_2$ by $v(T-t,x)$ and using an integration by parts for continuous data and then using a density argument, for any $(\rho^0, u^0) \in \mathcal{Z}$ and f_1 and f_2 in $L^2(0,T;L^2(0,L))$ with supports in $\mathcal{O}_1$ and $\mathcal{O}_2$ and $(\sigma^0, v^0) \in \mathcal{Z}$, we obtain

$$\int_0^T \int_{\mathcal{O}_1} \sigma(T-t,x) f_1(t,x) \ \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} v(T-t,x) f_2(t,x) \ \mathrm{d}x\mathrm{d}t \\ + \int_0^L \rho^0(x)\sigma(T,x) \ \mathrm{d}x + \int_0^L u^0(x) v(T,x) \ \mathrm{d}x = \int_0^L \rho(T,x)\sigma^0(x) \ \mathrm{d}x + \int_0^L u(T,x) v^0(x) \ \mathrm{d}x.$$

From the above identity, it follows that $(\rho(T,\cdot), u(T,\cdot)) = (0,0)$ in $\mathcal{Z}$ for the controls f_1 and f_2 in $L^2(0,T;L^2(0,L))$ with supports in $\mathcal{O}_1$ and $\mathcal{O}_2$ if and only if (6.12) holds and hence the proposition follows.

$\square$

With the help of the above proposition, we now prove Proposition 4.2 in the case when coefficients are also time dependent.

Proof of Proposition 4.2 for non-autonomous case: Let us assume that the system (6.5) is null controllable in $\mathcal{Z}$ at time $T > 0$ using two controls f_1 and f_2 in $L^2(0,T;L^2(0,L))$ with supports in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. We want to show that (4.2) holds. We prove it using a contradiction argument. Assume that (4.2) is not true. Then there exists a sequence (σ_n^0, v_n^0) in $\mathcal{Z}$ such that, the corresponding solution (σ_n, v_n) to (4.1) with the hypothesis (6.1) - (6.4) on coefficients satisfies

$$\int_0^T \int_{\mathcal{O}_1} |\sigma_n(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} |v_n(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{n^2}\Big(\int_0^L |\sigma_n(T,x)|^2 \,\mathrm{d}x \;+\; \int_0^L |v_n(T,x)|^2 \,\mathrm{d}x\Big). \tag{6.13}$$

We set $(\tilde{\sigma}_n^0, \tilde{v}_n^0) = \frac{\sqrt{n}}{\|(\sigma_n(T), v_n(T))\|_{\mathcal{Z}}}(\sigma_n^0, v_n^0)$. Let $(\tilde{\sigma}_n, \tilde{v}_n)$ be the corresponding solution to (4.1). Then, for each $n \in \mathbb{N}$, it yields

$$\|(\tilde{\sigma}_n(T), \tilde{v}_n(T))\|_{\mathcal{Z}} = \sqrt{n}, \tag{6.14}$$

and (6.13) gives

$$\int_0^T \int_{\mathcal{O}_1} |\tilde{\sigma}_n(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} |\tilde{v}_n(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{n}. \tag{6.15}$$

Since the system (6.5) is assumed to be null controllable in $\mathcal{Z}$, by Proposition 6.2, we have

$$\int_0^T \int_{\mathcal{O}_1} \tilde{\sigma}_n(T-t,x) f_1(t,x) \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} \tilde{v}_n(T-t,x) f_2(t,x) \,\mathrm{d}x\mathrm{d}t + \int_0^L \rho^0(x)\tilde{\sigma}_n(T,x) \,\mathrm{d}x + \int_0^L u^0(x)\tilde{v}_n(T,x) \,\mathrm{d}x = 0. \tag{6.16}$$

From the above identity and (6.15), it is easy to see that $(\tilde{\sigma}_n(T), \tilde{v}_n(T))$ converges weakly to 0 in $\mathcal{Z}$, and hence the sequence $\{(\tilde{\sigma}_n(T), \tilde{v}_n(T))\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{Z}$. This is a contradiction to (6.14).

Conversely, let us assume that the observability inequality (4.2) holds. We want to show that the system (6.5) is null controllable in $\mathcal{Z}$. Denoting, $\widetilde{\mathcal{U}} = L^2(\mathcal{O}_1) \times L^2(\mathcal{O}_2)$, consider the subspace $\mathcal{X}$ of $L^2(0,T;\widetilde{\mathcal{U}})$ defined by

$$\mathcal{X} = \left\{ (\mathbb{1}_{\mathcal{O}_1}\sigma, \mathbb{1}_{\mathcal{O}_2} v) \mid (\sigma, v) \text{ solves (4.1) for some } (\sigma^0, v^0) \in \mathcal{Z} \right\}.$$

Given $(\rho^0, u^0) \in \mathcal{Z}$, consider the linear functional $\mathcal{F}$ on $\mathcal{X}$ defined by

$$\mathcal{F}(\mathbb{1}_{\mathcal{O}_1}\sigma, \mathbb{1}_{\mathcal{O}_2} v) = -\left(\int_0^L \rho^0(x)\sigma(T,x) \,\mathrm{d}x + \int_0^L u^0(x) v(T,x) \,\mathrm{d}x \right).$$

By the observability inequality (4.2), $\mathcal{F}$ is well-defined and bounded linear functional on $\mathcal{X}$. Thus by Hahn-Banach theorem, we can extend the linear functional $\mathcal{F}$ to a bounded linear functional, still denoted by $\mathcal{F}$, on $L^2(0,T;\widetilde{\mathcal{U}})$. By the Riesz representation theorem, there exists $(\tilde{f}_1, \tilde{f}_2) \in L^2(0,T;\widetilde{\mathcal{U}})$ such that

$$\int_0^T \int_{\mathcal{O}_1} \sigma(t,x)\tilde{f}_1(t,x) \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathcal{O}_2} v(t,x)\tilde{f}_2(t,x) \,\mathrm{d}x\mathrm{d}t + \int_0^L \rho^0(x)\sigma(T,x) \,\mathrm{d}x + \int_0^L u^0(x) v(T,x) \,\mathrm{d}x = 0. \tag{6.17}$$

By setting $f_1(t,x) = \tilde{f}_1(T-t,x)$, for all $x \in \mathcal{O}_1$ and $t \in (0,T)$ and $f_2(t,x) = \tilde{f}_2(T-t,x)$, for all $x \in \mathcal{O}_2$ and $t \in (0,T)$, and using Proposition 6.2, it is easy to see that, (f_1, f_2) is the desired control to obtain the null controllability of the system (6.5) in $\mathcal{Z}$. □

Now we can follow exactly same steps used in the proofs of Theorem 1.2 and Theorem 1.4 in Section 4, to conclude that Theorem 1.2 and Theorem 1.4 also hold for time dependent coefficients. Obviously, we just need to modify the definition of X given in (1.7) in the following way

$$\begin{cases} \partial_t X(t,x) = -a_0\left(t, X(t,x)\right) & (t \geqslant 0), \\ X(0,x) = x \qquad x \in \mathbb{R}. \end{cases} \tag{6.18}$$

6.2. **Extension of Theorem 1.6 and Theorem 4.3 to the time dependent case.** Throughout this subsection we assume (6.1)-(6.4) and

$$c_0(t,x) \neq 0 \text{ for all } t \in [0,T], x \in [0,L]. \tag{6.19}$$

We consider the system (1.12) with time dependent coefficients:

$$\begin{cases} \partial_t \rho + a_0 \partial_x \rho + a_1 \rho + c_0 \partial_{xx} u + c_1 \partial_x u + c_2 u = \mathbb{1}_{\mathcal{O}1} f_1 & \text{in } (0,T)\times(0,L), \\ \partial_t u - b_0 \partial_{xx} u + b_1 \partial_x u + b_2 u + d_2 \rho = \mathbb{1}_{\mathcal{O}2} f_2 & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = 0 & \text{in } (0,T), \\ \rho(t,0) = 0 & \text{in } (0,T), \text{ if } a_0(t,0) > 0, \\ \rho(t,L) = 0 & \text{in } (0,T), \text{ if } a_0(t,L) < 0, \\ \rho(0,x) = \rho^0(x), \qquad u(0,x) = u^0(x) & \text{in } (0,L). \end{cases} \tag{6.20}$$

Regarding existence and uniqueness of the solution of the above system we have the following result.

Theorem 6.3. *Let* $(\rho^0, u^0) \in \Big(L^2(0,L)\Big)^2$ *and* $f_i \in L^2(0,T;L^2(0,L))$, $i=1,2$. *Then the system* (6.20) *admits an unique solution* $(\rho,u) \in C\Big([0,T];\Big(L^2(0,L)\Big)^2\Big)$ *together with the estimate*

$$\|(\rho,u)\|_{C([0,T];L^2(0,L))^2} \leqslant C\left(\|(\rho_0,u_0)\|_{L^2(0,L)\times L^2(0,L)} + \|(f_1,f_2)\|_{L^2(0,T;L^2(0,L))^2}\right), \tag{6.21}$$

where the positive constant C depends only on T, L and the coefficients of the system.

Proof. Let us set $\eta = b_0\rho + c_0 u$. Then (η,u) solves the following system

$$\begin{cases} \partial_t \eta + a_0 \partial_x \eta + \tilde{a}_1 \eta + \tilde{c}_1 \partial_x u + \tilde{c}_2 u = \tilde{f}_1 & \text{in } (0,T)\times(0,L), \\ \partial_t u - b_0 \partial_{xx} u + b_1 \partial_x u + \tilde{b}_2 u + \tilde{d}_2 \eta = \tilde{f}_2 & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = 0 & \text{in } (0,T), \\ \eta(t,0) = 0 & \text{in } (0,T), \text{ if } a_0(t,0) > 0, \\ \eta(t,L) = 0 & \text{in } (0,T), \text{ if } a_0(t,L) < 0, \\ \eta(0,x) = \eta^0(x) := b_0(0,x)\rho^0(x) + c_0(0,x)u^0(x), \quad u(0,x) = u^0(x) & \text{in } (0,L), \end{cases} \tag{6.22}$$

where

$$\begin{aligned} \tilde{a}_1 =& \frac{1}{b_0}\left(a_1 b_0 + c_0 d_2 - \partial_t b_0 - a_0 \partial_x b_0\right), \ \tilde{c}_1 = b_0 c_1 + c_0 b_1 - a_0 c_0, \\ \tilde{c}_2 =& \frac{1}{b_0}\Big(b_0^2 c_2 + b_0 c_0 b_2 - b_0 c_o a_1 - c_0^2 d_2 + c_0\left(\partial_t b_0 + a_0 \partial_x b_0\right) - b_0\left(\partial_t c_0 + a_0 \partial_x c_0\right)\Big), \\ \tilde{f}_1 =& b_0 \mathbb{1}_{\mathcal{O}1} f_1 + c_0 \mathbb{1}_{\mathcal{O}2} f_2, \ \tilde{b}_2 = b_2 - \frac{d_2 c_0}{b_0}, \ \tilde{d}_2 = \frac{d_2}{b_0}, \ \tilde{f}_2 = \mathbb{1}_{\mathcal{O}2} f_2. \end{aligned}$$

The system (6.22) is similar to the system (6.5). Thus by proceeding similarly to the proof of Theorem 6.1 we can prove this theorem. $\square$

We now derive a suitable observability inequality, which is equivalent to the null controllability of the system (6.20). To this aim, we consider the adjoint system of (6.20) :

$$\begin{cases} \partial_t \sigma - \partial_x(a_0\sigma) + a_1\sigma + d_2 v = 0 & \text{in } (0,T)\times(0,L),\\ \partial_t v - \partial_{xx}(b_0 v - c_0\sigma) - \partial_x(b_1 v + c_1\sigma) + (b_2 v + c_2\sigma) = 0 & \text{in } (0,T)\times(0,L),\\ (b_0 v - c_0\sigma)(t,0) = (b_0 v - c_0\sigma)(t,L) = 0 & \text{in } (0,T),\\ \sigma(t,L) = 0 \text{ if } a_0(t,0) > 0, \qquad \sigma(t,0) = 0 \text{ if } a_0(t,L) < 0 & \text{in } (0,T),\\ \sigma(0,x) = \sigma^0(x), \quad v(0,x) = v^0(x) & \text{in } (0,L). \end{cases} \tag{6.23}$$

Following the similar arguments of the proof of Theorem 6.3, we can show that the adjoint system (6.23) is well-posed in $\mathcal{Z}$. Moreover, using the similar argument used in the preceding subsection, we can easily prove the following equivalence between the null controllability of the system (6.20) and the following observability inequality:

Proposition 6.4. *The system (6.20) is null controllable in $\mathcal{Z}$ at time $T>0$ using two controls f_1 and f_2 in $L^2(0,T;L^2(0,L))$ with supports in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, if and only if, for $T>0$, there exists a positive constant $C_T>0$ such that for any $(\sigma^0,v^0)\in\mathcal{Z}$, (σ,v), the solution of (6.23), satisfies the following observability inequality:*

$$\int_0^L |\sigma(T,x)|^2\,\mathrm{d}x + \int_0^L |v(T,x)|^2\,\mathrm{d}x \leqslant C_T\Big(\int_0^T\int_{\mathcal{O}_1}|\sigma(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathcal{O}_2}|v(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\Big). \tag{6.24}$$

Now using the above observability inequality and the Gaussian beam solutions constructed in Theorem 3.8, we can easily proof Theorem 1.6 for the case with time dependent coefficients. Moreover, Theorem 4.3 also holds provided the coefficients satisfy (6.1), (6.4) and

$$d_2(d_1\partial_x a_0 - a_0\partial_x d_1) = d_1(d_1\partial_x a_1 - a_0\partial_x d_2), \quad c_1\neq 0, \quad d_1\neq 0, \quad \text{in} \quad [0,\mathrm{T}]\times[0,\mathrm{L}].$$

6.3. **Extension of Theorem 5.1, Theorem 5.4 and Theorem 5.5 to the time dependent case.** The proofs given in Section 5 also hold for time dependent coefficients. Thus Theorem 5.1, Theorem 5.4 and Theorem 5.5 also hold for time dependent coefficients with suitable assumptions. In particular

(i) Theorem 5.1 holds for the system (5.1) with coefficients satisfying

$$\alpha_0,\gamma_0,b_0\in C^\infty([0,T]\times[0,L]), \qquad \min_{[0,T]\times[0,L]} b_0>0, \quad \alpha_0\neq 0, \quad \min_{[0,T]\times[0,L]}\gamma_0>0. \tag{6.25}$$

(ii) Theorem 5.4 holds for the system (6.20) with $f_1=0$ and $\mathcal{O}_2=(0,L)$ and with coefficients satisfying $a_0=a_1=0$ on $[0,T]\times[0,L]$ and

$$\begin{aligned} &b_j\in C^\infty([0,T]\times[0,L]), \quad j=0,1,2, \quad b_0(t,x)\geqslant\bar b>0 \quad \text{for all } t\in[0,T], x\in[0,L],\\ &\alpha_0,\gamma_0\in C^\infty([0,T]\times[0,L]), \quad \alpha_0(t,x)\neq 0, \quad \gamma_0(t,x)>0 \quad \text{for all } t\in[0,T], x\in[0,L],\\ &c_0=\alpha_0\gamma_0, \qquad c_1=2\alpha_0\partial_x\gamma_0, \qquad c_2=\alpha_0\partial_{xx}\gamma_0, \quad \text{on} \quad [0,\mathrm{T}]\times[0,\mathrm{L}]. \end{aligned}$$

(iii) Theorem 5.5 holds for the system (6.5) with $f_1=0$ and $\mathcal{O}_2=(0,L)$ and with the coefficients satisfying $a_0=0=a_1$ on $[0,T]\times[0,L]$ and

$$\begin{aligned} &b_j,c_i,d_i\in C^\infty([0,T]\times[0,L]), \quad i=0,1, \quad j=0,1,2,\\ &b_0(t,x)\geqslant\bar b>0, \quad c_1(t,x)\neq 0, \quad c_2(t,x)=\partial_x c_1(t,x), \quad \text{for all } t\in[0,T], x\in[0,L]. \end{aligned}$$

7. Extensions and comments

In this section, we give some possible extension of our results and formulate some open problems.

7.1. **Periodic boundary conditions.** Let us set $\mathcal{S} = \mathbb{R}/L\mathbb{Z}$. We consider the control system (1.3)-(1.4) in $(0,T)\times\mathcal{S}$. By working on the torus $\mathcal{S}$, we assume that all the coefficients and the quantities at stake are L-periodic with respect to x.

We obtain the following well-posedness result :

Theorem 7.1. *Assume* (1.6)*. Let us consider the system* (1.3)-(1.4) *in* $(0,T)\times\mathcal{S}$*. Let us denote by* $\mathcal{A}_\sharp$ *the associated linear operator. Then the operator* $\mathcal{A}_\sharp$ *generates a* C^0*-semigroup on* $L^2(\mathcal{S})\times L^2(\mathcal{S})$ *as well as on* $H^1(\mathcal{S})\times L^2(\mathcal{S})$*.*

The proof of this theorem is similar to the proof of Proposition 2.2 and Theorem 2.6. Regarding null controllability we have the following results.

Theorem 7.2. *Assume* (1.6)*. The system* (1.3)-(1.4) *with periodic boundary conditions is not null controllable at time* T *using interior controls* $f_1 \in L^2(0,T;L^2(\mathcal{S}))$ *supported in* $\mathcal{O}_1$ *and* $f_2 \in L^2(0,T;L^2(\mathcal{S}))$ *with support in* $\mathcal{O}_2$*, in the following scenarios:*

(i) *if* $0<T<T_{\mathcal{O}_1}$*;* $(\rho^0,u^0)\in L^2(\mathcal{S})\times L^2(\mathcal{S})$*;* $\mathcal{O}_1\subset\mathcal{S}$ *such that* $\mathcal{S}\setminus\overline{\mathcal{O}}_1$ *is a nonempty open subset of* $\mathcal{S}$*;* $\mathcal{O}_2\subseteq\mathcal{S}$*.*

(ii) *if* $0<T<T_{\mathcal{O}_2}$*;* $(\rho^0,u^0)\in H^1(\mathcal{S})\times L^2(\mathcal{S})$*;* $f_1\equiv 0$*;* $\mathcal{O}_2\subset\mathcal{S}$ *such that* $\mathcal{S}\setminus\overline{\mathcal{O}}_2$ *is a nonempty open subset of* $\mathcal{S}$*; and along with* (1.6)*, the coefficients in* (1.3) *satisfy*
$$d_2(a_0'd_1 - a_0d_1') = d_1(d_1a_1' - a_0d_2'), \quad c_1\neq 0, \quad d_1\neq 0, \quad \text{on} \quad [0,L].$$

Furthermore, assume (5.11)*,* $f_1\equiv 0$ *and* $\mathcal{O}_2=\mathcal{S}$*. Then the system* (1.3)-(1.4) *with periodic boundary conditions is null controllable in* $H^1(\mathcal{S})\cap L^2_{\mathrm{m}}(\mathcal{S})\times L^2(\mathcal{S})$*, at any time* $T>0$*, by control acting everywhere in the parabolic component.*

Remark 7.3. *In a similar fashion, we can also consider the system* (1.12) *with periodic boundary conditions. Both Theorem 1.6 and Theorem 5.4 hold in this case also. In fact, Theorem 1.2 and Theorem 1.6 can be extended to the coupled system with any suitable boundary conditions, where the corresponding linear operator is well-posed.*

7.2. **Boundary control.** Assume (1.6), $a_0(0)>0$ and $a_0(L)\geqslant 0$. We consider the following boundary control system:
$$\begin{cases} \partial_t\rho + a_0\partial_x\rho + a_1\rho + c_1\partial_x u + c_2 u = 0 & \text{in } (0,T)\times(0,L),\\ \partial_t u - b_0\partial_{xx}u + b_1\partial_x u + b_2 u + d_1\partial_x\rho + d_2\rho = 0 & \text{in } (0,T)\times(0,L),\\ u(t,0) = h_0(t), \quad u(t,L) = h_L(t) & t\in(0,T),\\ \rho(t,0) = g_0(t) & t\in(0,T), \end{cases} \tag{7.1}$$
where controls g_0, h_0, h_L belong to $L^2(0,T)$. In this case, the system (7.1) is null controllable in $\mathcal{Z}$ at time T by controls $(g_0,h_0,h_L)\in(L^2(0,T))^3$ if and only if, for $T>0$, there exists a positive constant $C_T>0$ such that for any $(\sigma^0,v^0)\in\mathcal{D}(\mathcal{A}^*;\mathcal{Z})$, defined in (2.15), (σ,v), the solution of (4.1), satisfies the following observability inequality:
$$\begin{aligned} &\int_0^L |\sigma(T,x)|^2\,\mathrm{d}x + \int_0^L |v(T,x)|^2\,\mathrm{d}x \\ &\leqslant C_T\Big(\int_0^T |a_0(0)\sigma(t,0)|^2\,\mathrm{d}t + \int_0^T |c_1(0)\sigma(t,0)+\partial_x(b_0v)(t,0)|^2\mathrm{d}t + \int_0^T |\partial_x(b_0v)(t,L)|^2\mathrm{d}t\Big). \end{aligned} \tag{7.2}$$

Recall the definitions of $T_\emptyset$ from (1.10). By proceeding similarly as the proof of Theorem 1.2, we obtain the following results:

Theorem 7.4. *Let* $T_\emptyset$ *be as defined in* (1.10)*. The system* (7.1) *is not null controllable at any time* $0<T<T_\emptyset$ *using controls* g_0,h_0,h_L *in* $L^2(0,T)$ *acting on the boundary.*

Remark 7.5. *In a similar manner, we can also prove that the system* (1.12) *is not null controllable at any time* $0<T<T_\emptyset$ *by the boundary controls.*

7.3. **Concluding remarks and open problems.** The main results in this article concern the lack of null controllability of coupled transport-parabolic systems with variable coefficients. These results are generalizations of the results available for the coupled systems with constant coefficients. Moreover, when the transport velocity $a_0 = 0$, these systems are null controllable at any time by the control acting everywhere in the parabolic equation, under suitable assumptions on the initial data and the coefficients. In view of our results, several open questions seem natural and are under investigation currently.

Null controllability: In [1, 5, 6], the null controllability results are proved for the coupled system with constant coefficients within the periodic setup. In view of these articles, it is reasonable to expect that, the systems in consideration are null controllable, under suitable geometric assumptions. Perhaps, one could follow the arguments of [11] to conclude null controllability of (1.3) with boundary controls, under suitable geometric assumptions. However, as far as we know, if a_0 is not identically equal to zero, there are no controllability results available for the systems (1.3) and (1.12), with Dirichlet boundary conditions and localized interior controls, even in the constant coefficient case. Furthermore, it would also be interesting to see whether the positive results in Section 5 can also be obtained by a moving control instead of control acting everywhere.

Lack of null controllability in $H^s \times L^2, 0 \leqslant s < 1$, by parabolic control: If the coefficients are constant, the system (1.3)-(1.5) is not null controllable in $H^s \times L^2, 0 \leqslant s < 1$, at any time $T > 0$, by a parabolic control acting everywhere in the domain, see for instance [7, Theorem 5.1] (if $a_0 = 0$) and [6, Theorem 1.3] (if $a_0 \neq 0$ and the system with periodic boundary conditions). In Theorem 1.4, we prove this result only for $s = 0$ under the assumption (1.11). It would be interesting to see whether Theorem 1.4 holds in $H^s \times L^2$, $0 < s < 1$.

Weaker type controllability - possibility of controlling only one component: In view of the lack of null controllability of the system (1.3)-(1.5), it may be natural to ask a weaker controllability result i.e., if only one component of the solution of the system can be brought at rest at time $T > 0$ and in that case if the minimal time is needed. We observe that the system (1.3) with periodic boundary conditions, $a_0 = c_1 = b_0 = b_1 = d_1 = 1$, and $a_1 = c_2 = b_2 = d_2 = 0$, can be reduced to an equation in u-variable as

$$\partial_{txx} u + \partial_{xxx} u - \partial_{tt} u - 2\partial_{tx} u = F,$$

with periodic boundary conditions and control F. This system exhibits hyperbolic nature, and therefore the minimal time is required to control such system. This indicates that we need the minimal time to obtain the controllability even for one component u. However, this reduction of the system to a single equation for general coefficients with Dirichlet boundary conditions is not so obvious at all. Another possibility could be to derive suitable observability inequality which is equivalent to the controllability for one component. Having suitable observability inequality, perhaps the Gaussian beam construction can be used to show the existence of the minimal time.

Degenerate coefficients: In this article, we have always assumed that, the "viscosity" coefficient b_0 is strictly positive. However, one can ask if the results hold, in the case when b_0 is degenerate either at boundary or at an interior point. But this require new techniques.

Coupling of several transport and parabolic systems: In the spirit of [1], it would also be interesting to consider coupling of several transport and parabolic equations with variable coefficients.

Multi-dimension: The techniques used in this article can be extended to the coupled system posed in higher space dimensions. Thus analogous results can be anticipated for the systems in higher dimensions.

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