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QUANTUM NUMBERS AND q -DEFORMED CONWAY-COXETER FRIEZES

SOPHIE MORIER-GENOUD AND VALENTIN OVSIENKO

To the memory of John Conway

The notion of q -deformed real numbers was recently introduced in [15, 16]. The first steps of this theory were largely influenced by John Conway. In the fall of 2013, we had a chance to spend a week in his company. The occasion was a conference that we organized in Luminy, at which Conway gave two wonderful talks. We cannot forget the long conversations we had with John there; between desperate attempts to teach us his famous doomsday algorithm, he told us much about his work with Coxeter. Our understanding of Conway and Coxeter's theorem has transformed into a certain combinatorial viewpoint on continued fractions [14], that eventually led us to q -deformations.

One goal of this article is to explain the origins of q -numbers and sketch their main properties. We will also introduce q -deformed Conway-Coxeter friezes. We would have loved to have had the chance to talk all of this over with John.

1. q -DEFORMED CONWAY-COXETER FRIEZES

Friezes, or frieze patterns, are perhaps one of the most ingenious and the least known of Coxeter's inventions [6]. As he confessed, friezes “caused [him] many restless nights,” [5] but Conway [3] was able to put his mind somewhat to rest in answering his question about the classification of friezes of positive integers. As a result, these friezes are often called Conway-Coxeter friezes. Here, we introduce the notion of q -deformations of Conway-Coxeter friezes.

$$\begin{array}{cccccccccc}
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
\cdots & & 1 & & [4]_q & & [2]_q & & 1 & & [3]_q & & [2]_q & & [2]_q & & 1 \\
& 1 & & q[3]_q & & \{7\}_q & & 1 & & q[2]_q & & \{5\}_q & & [3]_q & & 1 \\
1 & & q^2[2]_q & & q\{5\}_q & & [3]_q & & q^2 & & q[3]_q & & \{7\}_q & & 1 & & \cdots \\
& q^3 & & q^2[3]_q & & q[2]_q & & q^2[2]_q & & q^3 & & q[4]_q & & [2]_q & & q^3 & & \cdots \\
\cdots & & q^4 & & q^2 & & q^3 & & q^3 & & q^4 & & q & & q^3 & & \cdots
\end{array}$$

FIGURE 1. The q -deformation of the frieze of Example 1.1.

Starting from the line $([c_i]_q)$ and using (3), one can calculate every next row of the q -frieze inductively. The q -deformation of the frieze in Example 1.1 is shown in Figure 1, where $\{7\}_q = 1 + 2q + 2q^2 + q^3 + q^4$ and $\{5\}_q = 1 + 2q + q^3 + q^4$, and $[c]_q$ is as in (1). For instance, $\{7\}_q$ is calculated twice, the first time as $\{7\}_q = [4]_q[2]_q - q^3$ and then as $\{7\}_q = \frac{\{5\}_q[3]_q - q^3}{[2]_q}$.

At first glance, definition 1.2 does not look natural. In particular, (3) is no longer local, but it leads to a frieze with nice properties. We will see in Section 2.3, that $C_{ij}(q)$ is a polynomial with positive integer coefficients. In particular, the last row of a q -frieze consists of powers of q . Quantum friezes deserve thorough study, and we are currently at work on this project.

2. INTRODUCING QUANTUM RATIONALS & IRRATIONALS

Like the integers, rational numbers $\frac{r}{s}$ cannot be q -deformed on their own, without including them in a sequence. Attempts to q -deform the numerator and denominator separately lead to notions that lack nice properties. A very naive formula $[\frac{r}{s}]_q = \frac{[r]_q}{[s]_q}$ and a more standard one $[x]_q = \frac{1-q^x}{1-q}$ (note that they coincide modulo a rescaling of the parameter q) are among them.

Here we give several equivalent definitions of q -rationals and explain the connection to q -friezes. For more definitions, a combinatorial interpretation, and a connection to Jones polynomial and cluster algebras, see [15]. Surprisingly, the q -deformation $[x]_q$ of an irrational $x \in \mathbb{R}$ is a Laurent series in q , owing to an unexpected phenomenon of stabilization of Taylor series of sequences of q -deformed rationals converging to x .

2.1. Deformed continued fractions. Every rational number $\frac{r}{s} > 0$, where $r, s \in \mathbb{Z}_{>0}$ are coprime, has a standard finite continued fraction expansion $\frac{r}{s} = [a_1, a_2, \dots]$. Choosing an even number of coefficients (and removing the ambiguity $[a_1, \dots, a_n, 1] = [a_1, \dots, a_n + 1]$), we have the unique expansion $\frac{r}{s} = [a_1, \dots, a_{2m}]$. Similarly, there is also a unique expansion with minus signs called the Hirzebruch-Jung continued fraction:

$$\frac{r}{s} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{2m}}}} = c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_k}}},$$

where $a_i \geq 1$ and $c_j \geq 2$ (except for $a_1 \geq 0, c_1 \geq 1$). The notation used by Hirzebruch is $\frac{r}{s} = \llbracket c_1, \dots, c_k \rrbracket$; the coefficients a_i and c_j are connected by the Hirzebruch formula; see, e.g., [14].

FIGURE 2. The weighted Farey graph between $\frac{0}{1}$ and $\frac{1}{0}$; the colored area corresponds to the frieze of Example 1.1.

Definition 2.1. The q -deformed regular continued fraction is defined by

$$(4) \quad [a_1, \dots, a_{2m}]_q := [a_1]_q + \frac{q^{a_1}}{[a_2]_{q^{-1}} + \frac{q^{-a_2}}{[a_3]_q + \frac{q^{a_3}}{[a_4]_{q^{-1}} + \frac{q^{-a_4}}{\ddots} \frac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}$$

where $[a]_q$ is the Euler q -integer. The q -deformed Hirzebruch-Jung continued fraction is

$$(5) \quad \llbracket c_1, \dots, c_k \rrbracket_q := [c_1]_q - \frac{q^{c_1-1}}{[c_2]_q - \frac{q^{c_2-1}}{\ddots \frac{q^{c_{k-1}-1}}{[c_k]_q}}}$$

For $\frac{r}{s} = [a_1, \dots, a_{2m}] = \llbracket c_1, \dots, c_k \rrbracket$, the rational functions (4) and (5) coincide. This rational function is called the q -rational and denoted $\llbracket \frac{r}{s} \rrbracket_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$.

Both polynomials, $\mathcal{R}$ and $\mathcal{S}$, depend on r and s .

Example 2.2. For instance, we obtain $\llbracket \frac{5}{2} \rrbracket_q = \frac{1+2q+q^2+q^3}{1+q}$ and $\llbracket \frac{5}{3} \rrbracket_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$. Observe that “quantum 5” in the numerator depends on the denominator.

2.2. The weighted Farey graph and Stern-Brocot tree. Let us give a recursive definition.

The set of rational numbers $\mathbb{Q}$, completed by $\infty := \frac{1}{0}$, are vertices of a graph called the *Farey graph*. Two rationals $\frac{r}{s}$ and $\frac{r'}{s'}$ are connected by an edge if and only if $rs' - r's = \pm 1$. Edges of the Farey graph are often represented as geodesics of the hyperbolic plane which is triangulated.

Definition 2.3. The *weighted Farey graph*, see Figure 2, is just the classical Farey graph, in which the vertices are labeled by rational functions in q and the edges are weighted by powers of q . The weights and the labels are defined recursively via the following local rule:

from the initial triangle $(\frac{0}{1}, \frac{1}{1}, \frac{1}{0})$ that remains undeformed.

Theorem 2 ([15]). *The vertices of the weighted Farey graph are, indeed, labeled by q -rationals.*

Remark 2.4. The Farey graph contains $\mathbb{Z}$ as a subgraph (that forms a sequence of triangles, $(\frac{n}{1}, \frac{n+1}{1}, \frac{1}{0})$). Our q -deformation restricted to $\mathbb{Z}$ leads to Euler's formula (1).

Alternatively, one can use the Stern-Brocot tree instead of the Farey graph.

local rule

The weight of every edge of the weighted Stern-Brocot tree is determined by the local rule along the tree: the weight of the right branch is multiplied by q , the weight of the left branch is 1. This allows us to calculate q -rationals inductively. For instance, $[\frac{2}{1}]_q = \frac{q+1}{1} = [2]_q$, $[\frac{1}{2}]_q = \frac{q}{q+1}$. The left branch of the tree consists of the classical q -integers (1).

2.3. q -deformed $\text{SL}(2, \mathbb{Z})$. The *modular group* $\text{SL}(2, \mathbb{Z})$ is useful as a tool in working with continued fractions. In our case, we need to q -deform $\text{SL}(2, \mathbb{Z})$, so we consider the following three matrices:

$$(6) \quad R_q := \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad L_q := \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}, \quad S_q := \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}.$$

For $q = 1$, we obtain the standard matrices, any two of which can be chosen as generators of $\text{SL}(2, \mathbb{Z})$.

Proposition 2.5 ([15]). *Given a rational, $\frac{r}{s} = [a_1, \dots, a_{2m}] = \llbracket c_1, \dots, c_k \rrbracket$, the polynomials $\mathcal{R}(q)$ and $\mathcal{S}(q)$ of $[\frac{r}{s}]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ can be calculated as the entries of the first column of the following matrix products*

$$R_q^{a_1} L_q^{a_2} \dots R_q^{a_{2m-1}} L_q^{a_{2m}} = \begin{pmatrix} q\mathcal{R} & \mathcal{R}' \\ q\mathcal{S} & \mathcal{S}' \end{pmatrix}, \quad R_q^{c_1} S_q R_q^{c_2} S_q \dots R_q^{c_k} S_q = \begin{pmatrix} \mathcal{R} & -q^{c_k-1} \mathcal{R}'' \\ \mathcal{S} & -q^{c_k-1} \mathcal{S}'' \end{pmatrix},$$

where $\mathcal{R}', \mathcal{S}'$ and $\mathcal{R}'', \mathcal{S}''$ are lower degree polynomials corresponding to the previous convergents of the continued fractions.

This way of understanding q -rationals clarifies Definition 1.2. Indeed, (3) is the determinant of the second matrix. Note that the matrices (6) are proportional to the matrices arising in quantum Teichmüller theory; see [7]. This interesting connection has yet to be investigated.

2.4. Connection to q -friezes. It turns out that q -rationals appear everywhere in q -deformed Conway-Coxeter friezes, as quotients of the neighboring entries, so that we can consider q -friezes as yet another way to describe q -deformations of rationals.

Let us first recall the connection between the Conway-Coxeter friezes and continued fractions.

Proposition 2.6 ([6]). *If (c_{ij}) are the entries of a frieze, then $\frac{c_{ij}}{c_{i+1,j}} = \llbracket c_i, \dots, c_j \rrbracket$.*

This statement has a straightforward q -analogue.

Proposition 2.7. *If (c_{ij}) are the entries of a frieze and $(C_{ij}(q))$ the entries of the q -deformed frieze, then $\frac{C_{ij}(q)}{C_{i+1,j}(q)} = \left[\frac{c_{ij}}{c_{i+1,j}} \right]_q = \llbracket c_i, \dots, c_j \rrbracket_q$.*

The q -frieze of Figure 1 contains many examples of q -rationals: $\frac{\{5\}}{\{2\}} = \left[\frac{5}{2} \right]_q$, $\frac{\{7\}}{\{3\}} = \left[\frac{7}{3} \right]_q$, etc.

2.5. q -deformed irrational numbers. Let $x \geq 0$ be an irrational number, and let $(x_n)_{n \geq 1}$ be a sequence of rationals converging to x . Our definition is the following.

Take the sequence of rational functions $[x_1]_q, [x_2]_q, \dots$. Consider their Taylor expansions at $q = 0$, for which we will use the notation $[x_n]_q = \sum_{k \geq 0} \varkappa_{n,k} q^k$.

Theorem 3 ([16]). (i) *For every $k \geq 0$, the coefficients of the Taylor series of $[x_n]_q$ stabilize as n grows.*
(ii) *The limit coefficients $\varkappa_k = \lim_{n \rightarrow \infty} \varkappa_{n,k}$ do not depend on the sequence $(x_n)_{n \geq 1}$, but only on x .*

This stabilization phenomenon allows us to define the q -deformation of $x \geq 0$ as a power series in q :

$$[x]_q = \varkappa_0 + \varkappa_1 q + \varkappa_2 q^2 + \varkappa_3 q^3 + \dots$$

In the case of $x < 0$, the definition of q -deformation is based on the recurrence $[x-1]_q := q^{-1} [x]_q - q^{-1}$; cf. Section 4.1. It turns out that the resulting series in q is a *Laurent series* (with integer coefficients):

$$[x]_q = -q^{-N} + \varkappa_{1-N} q^{1-N} + \varkappa_{2-N} q^{2-N} + \dots,$$

where $N \in \mathbb{Z}_{>0}$ such that $-N \leq x < 1 - N$.

3. EXAMPLES: q -FIBONACCI AND q -PELL NUMBERS, q -GOLDEN RATIO AND $\left[\sqrt{2} \right]_q$

Let us consider two remarkable sequences of rationals,

$$\frac{F_{n+1}}{F_n} = \underbrace{[1, 1, \dots, 1]}_n \quad \text{and} \quad \frac{P_{n+1}}{P_n} = \underbrace{[2, 2, \dots, 2]}_n,$$

where F_n is the n^{th} Fibonacci number, and P_n is the n^{th} Pell number. Quantizing them, we obtain sequences of polynomials with a very particular “swivel” property: the polynomials corresponding to F_n (resp. P_n) in the numerator and denominator are mirrors of each other. The stabilized Taylor series of $\left[\frac{F_{n+1}}{F_n} \right]_q$ and $\left[\frac{P_{n+1}}{P_n} \right]_q$ give rise of q -analogues of $\frac{1+\sqrt{5}}{2}$ and $1 + \sqrt{2}$, respectively.

3.1. q -Fibonacci numbers. Let $\mathcal{F}_n(q)$ be a sequence of polynomials defined by the recurrence

$$\mathcal{F}_{n+2} = [3]_q \mathcal{F}_n - q^2 \mathcal{F}_{n-2},$$

where $[3]_q = 1 + q + q^2$ is Euler’s quantum 3, and where the initial conditions are $(\mathcal{F}_0(q) = 0, \mathcal{F}_2(q) = 1)$ and $(\mathcal{F}_1(q) = 1, \mathcal{F}_3(q) = 1 + q)$. The sequence of polynomials $\mathcal{F}_n(q)$ is a q -deformation of the Fibonacci

sequence, i.e., $\mathcal{F}_n(1) = F_n$. Consider also the *mirror polynomials* $\tilde{\mathcal{F}}_n(q) := q^{n-2}\mathcal{F}_n(\frac{1}{q})$ ($n \geq 2$). The triangles of their coefficients

$$\begin{array}{cccccccc}
 & & & & & & & 1 \\
 & & & & & & & 1 \\
 & & & & & & 1 & 1 \\
 & & & & & 1 & 1 & 1 \\
 & & & 1 & 2 & 1 & 1 & \\
 & & 1 & 2 & 2 & 2 & 1 & \\
 & 1 & 3 & 3 & 3 & 2 & 1 & \\
 & 1 & 3 & 4 & 5 & 4 & 3 & 1 \\
 \dots & & & & & & & \dots
 \end{array}$$

are the well-studied sequences A079487 and A123245 of OEIS [17].

Proposition 3.1. *One has $\left[\frac{F_{n+1}}{F_n}\right]_q = \frac{\tilde{\mathcal{F}}_{n+1}(q)}{\tilde{\mathcal{F}}_n(q)}$.*

Example 3.2. The case $\left[\frac{5}{3}\right]_q$ has already been considered in Example 2.2. We then have

$$\begin{aligned}
 \left[\frac{8}{5}\right]_q &= \frac{1 + 2q + 2q^2 + 2q^3 + q^4}{1 + 2q + q^2 + q^3}, \\
 \left[\frac{13}{8}\right]_q &= \frac{1 + 2q + 3q^2 + 3q^3 + 3q^4 + q^5}{1 + 2q + 2q^2 + 2q^3 + q^4}, \\
 \left[\frac{21}{13}\right]_q &= \frac{1 + 3q + 4q^2 + 5q^3 + 4q^4 + 3q^5 + q^6}{1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5}, \\
 \dots & \quad \dots \quad \dots
 \end{aligned}$$

The coefficients of these rational functions grow at every fixed power of q , and there is no stabilization of rational functions.

3.2. q -Pell numbers. The sequence of polynomials $\mathcal{P}_n(q)$ satisfying the recursion

$$\mathcal{P}_{n+2} = \binom{4}{2}_q \mathcal{P}_n - q^4 \mathcal{P}_{n-2},$$

where $\binom{4}{2}_q$ is the Gaussian q -binomial and with the initial conditions ($\mathcal{P}_0(q) = 0, \mathcal{P}_2(q) = 1 + q$) and ($\mathcal{P}_1(q) = 1, \mathcal{P}_3(q) = 1 + q + 2q^2 + q^3$), are q -analogues of the classical *Pell numbers*. The mirror polynomials are defined by $\tilde{\mathcal{P}}_n := q^{2n-3}\mathcal{P}_n(\frac{1}{q})$ ($n \geq 2$). The coefficients of $\mathcal{P}_n(q)$

$$\begin{array}{cccccccc}
 & & & & & & & 1 \\
 & & & & & & & 1 \\
 & & & & & & 1 & 1 \\
 & & & 1 & 2 & 1 & 1 & \\
 & & 1 & 2 & 3 & 3 & 2 & 1 \\
 & 1 & 3 & 5 & 6 & 6 & 5 & 2 & 1 \\
 & 1 & 3 & 7 & 11 & 13 & 13 & 11 & 7 & 3 & 1 \\
 \dots & & & & & & & & & & \dots
 \end{array}$$

form a triangular sequence that was recently added to the OEIS (sequence A323670).

Proposition 3.3. *One has $\left[\frac{P_{n+1}}{P_n}\right]_q = \frac{\tilde{\mathcal{P}}_{n+1}(q)}{\tilde{\mathcal{P}}_n(q)}$.*

3.3. q -deformed golden ratio. To illustrate the stabilization phenomenon, we take the Taylor series of the Fibonacci quotients; see Example 3.2. For instance,

$$\begin{aligned} \left[\frac{8}{5}\right]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 7q^6 - 12q^7 + 21q^8 - 37q^9 + 65q^{10} - 114q^{11} + 200q^{12} \dots \\ \left[\frac{21}{13}\right]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 36q^8 - 75q^9 + 156q^{10} - 325q^{11} + 677q^{12} \dots \\ \left[\frac{55}{34}\right]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 - 82q^9 + 184q^{10} - 414q^{11} + 932q^{12} \dots \\ \left[\frac{144}{89}\right]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 - 82q^9 + 185q^{10} - 423q^{11} + 978q^{12} \dots \end{aligned}$$

More and more coefficients repeat as the series proceed with the series eventually stabilizing to

$$\begin{aligned} [\varphi]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 - 82q^9 + 185q^{10} - 423q^{11} + 978q^{12} - 2283q^{13} \\ &\quad + 5373q^{14} - 12735q^{15} + 30372q^{16} - 72832q^{17} + 175502q^{18} - 424748q^{19} + 1032004q^{20} \dots \end{aligned}$$

This power series is our quantized golden ratio $\left[\frac{1+\sqrt{5}}{2}\right]_q := [\varphi]_q$.

The series $[\varphi]_q$ satisfies the equation $q[\varphi]_q^2 - (q^2 + q - 1)[\varphi]_q - 1 = 0$, which is a q -analogue of $x^2 - x + 1 = 0$. Therefore, the generating function of the series $[\varphi]_q$ is

$$(7) \quad [\varphi]_q = \frac{q^2 + q - 1 + \sqrt{(q^2 + 3q + 1)(q^2 - q + 1)}}{2q}.$$

Let us mention that the coefficients of the series $[\varphi]_q$ remarkably coincide with Sequence A004148 of [17] (with alternating signs), called the Generalized Catalan numbers.

3.4. q -deformed $\sqrt{2}$. Quotients of the Pell polynomials stabilize to the series $[1 + \sqrt{2}]_q$, from which we deduce

$$\begin{aligned} [\sqrt{2}]_q &= 1 + q^3 - 2q^5 + q^6 + 4q^7 - 5q^8 - 7q^9 + 18q^{10} + 7q^{11} - 55q^{12} + 18q^{13} \\ &\quad + 146q^{14} - 155q^{15} - 322q^{16} + 692q^{17} + 476q^{18} - 2446q^{19} + 307q^{20} \dots \end{aligned}$$

This series is a solution of $q^2 [\sqrt{2}]_q^2 - (q^3 - 1)[\sqrt{2}]_q = q^2 + 1$, which is our version of the q -analogue of $x^2 = 2$. The generating function is then equal to

$$(8) \quad [\sqrt{2}]_q = \frac{q^3 - 1 + \sqrt{(q^4 + q^3 + 4q^2 + q + 1)(q^2 - q + 1)}}{2q^2}.$$

Note that the coefficients of $[\sqrt{2}]_q$ grow much more slowly than those of $[\varphi]_q$ and fail to match any known sequence.

3.5. Quadratic q -irrationals. Several observations can be made by analyzing (7) and (8). The polynomial under the radical of these q -numbers is a *palindrome*. This remarkable property remains true for arbitrary *quadratic irrationals*, i.e., numbers of the form $x = \frac{a+\sqrt{b}}{c}$, where $a, b > 0, c$ are integers.

Theorem 4 ([9]). *For every quadratic irrational, $\left[\frac{a+\sqrt{b}}{c}\right]_q = \frac{A(q)+\sqrt{B(q)}}{C(q)}$, where A, B and C are polynomials in q . Furthermore, B is a monic polynomial whose coefficients form a palindrome.*

A quadratic irrational can also be characterized as a fixed point of an element of $\text{PSL}(2, \mathbb{Z})$. A q -analogue of this property will be provided in Theorem 5 in the next section.

3.6. Radius of convergence. The modulus of the smallest (i.e., closest to 0) root of the polynomials under the radical in (7) and (8) is equal to

$$R_\varphi^- = \frac{3 - \sqrt{5}}{2} \quad \text{and} \quad R_{\sqrt{2}}^- = \frac{1 + \sqrt{2} - \sqrt{2\sqrt{2} - 1}}{2},$$

respectively. This are the *radii of convergence* of the Taylor series (7) and (8), resp. The modulus of the largest roots are $R_\varphi^+ = 1/R_\varphi^-$ and $R_{\sqrt{2}}^+ = 1/R_{\sqrt{2}}^-$, viz.,

$$R_\varphi^+ = \frac{3 + \sqrt{5}}{2} \quad \text{and} \quad R_{\sqrt{2}}^+ = \frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2}.$$

Aesthetic aspects of these formulas motivated us to analyze several more examples, one of which is another remarkable number, $\frac{9 + \sqrt{221}}{14}$, sometimes called the “bronze ratio.” This is the third, after φ and $\sqrt{2}$, badly approximated number in Markov theory. The modulus of the minimal and maximal roots of the polynomial under the radical in $\left[\frac{9 + \sqrt{221}}{14}\right]_q$ are

$$R^- = \frac{1 + \sqrt{13} - \sqrt{2(\sqrt{13} - 1)}}{4} \quad \text{and} \quad R^+ = \frac{1 + \sqrt{13} + \sqrt{2(\sqrt{13} - 1)}}{4}.$$

Note that $221 = 13 \cdot 17$. We do not know if the striking resemblance to the case of $\sqrt{2}$ is a coincidence. Work on the analytic properties of q -numbers is in progress [10].

4. SOME PROPERTIES OF q -RATIONALS

The notion of q -rationals arose from an attempt to understand the connection between several different theories, such as continued fractions, Jones polynomials of (rational) knots, quantum Teichmüller theory, and cluster algebras. These connections were discovered by many authors; see [7, 11]. Our definition is a specialization of such notions as F -polynomials, quantum geodesic length, snake graphs. We present some of those concrete properties of q -rationals that we consider most important; many reflect this deep connection. For more details, see [15, 16].

4.1. $\text{PSL}(2, \mathbb{Z})$ -action. Our first important property is the following.

Theorem 5 ([9]). *The procedure of q -deformation commutes with the $\text{PSL}(2, \mathbb{Z})$ -action.*

Indeed, the first recurrence (2) remains true for q -rationals. For every $x \in \mathbb{Q}$, we have

$$(9) \quad [x + 1]_q = q[x]_q + 1.$$

Recurrence (9) readily follows from (4) and is very useful as it allows us to define q -deformations of $x < 0$. Furthermore, we have

$$(10) \quad \left[-\frac{1}{x}\right]_q = -\frac{1}{q[x]_q}, \quad [-x]_q = -q^{-1}[x]_{q^{-1}}.$$

Together, (9) and (10) define an action of $\text{PSL}(2, \mathbb{Z})$ on q -rationals generated by the matrices R_q and qS_q in (6). This was (implicitly) checked in [15] (see Lemma 4.6) and will be further developed in [9].

Remark 4.1. Recurrence (9) appears, e.g., for q -integers and plays a crucial role in quantum algebra. For instance, this recurrence is necessary for the quantum binomial formula. Identities (10) look more intriguing and need to be better understood.

4.2. Total positivity. The polynomials in the numerator and denominator of a positive q -rational have positive integer coefficients, with 1 at lowest and higher orders. Moreover, the set of all q -rationals has a much stronger property of *total positivity*. Consider two q -rationals, $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ and $\left[\frac{r'}{s'}\right]_q = \frac{\mathcal{R}'(q)}{\mathcal{S}'(q)}$.

Theorem 6 ([15]). *If $\frac{r}{s} > \frac{r'}{s'} > 0$, then the polynomial*

$$(11) \quad \mathcal{X}_{\frac{r}{s}, \frac{r'}{s'}}(q) := \mathcal{R}(q)\mathcal{S}'(q) - \mathcal{S}(q)\mathcal{R}'(q)$$

has positive integer coefficients.

The main ingredient of the proof of this theorem is the fact that (11) is a monomial, i.e., proportional to a power of q , if and only if $\frac{r}{s}$ and $\frac{r'}{s'}$ are connected in the Farey graph.

Theorem 6 means that the “quantization preserves the order,” in the sense that it is a homeomorphism of $\mathbb{Q}$ into an ordered subset of a partially ordered set of rational functions. The notion of total positivity has a long history in mathematics and manifests itself in every area of it.

4.3. Relation to the Jones polynomial. One of the ingenious inventions of Conway [2] was to encode a certain large class of knots, called “rational” or “two-bridge” knots, by continued fractions. Every notion of knot theory, such as knot invariants, can then be directly associated to continued fractions.

The Jones polynomial is a powerful invariant in knot theory. It turns out that one can express the Jones polynomial $J_{\frac{r}{s}}(q)$ of the two-bridge knot associated to a rational $\frac{r}{s}$ as a combination of the polynomials $\mathcal{R}$ and $\mathcal{S}$ of the q -rational $\left[\frac{r}{s}\right]_q$.

Theorem 7 ([15]). *The Jones polynomial of a two-bridge knot is $J_{\frac{r}{s}}(q) = q\mathcal{R}(q) + (1 - q)\mathcal{S}(q)$.*

The proof is based on the connection to cluster algebras; see [11].

4.4. Unimodality conjecture. Long computer experimentation and evidence in the simplest cases convinced us of yet another property of q -rationals.

Conjecture 1 ([15]). *For every $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$, the coefficients of the polynomials $\mathcal{R}(q)$ and $\mathcal{S}(q)$ form unimodal sequences.*

This means that the coefficients of the polynomials increase from 1 to the maximal value (that can be taken by one or more consecutive coefficients) and then decrease to 1. Unimodal sequences appear in mathematics, and this property is interesting because it often hides some combinatorial or geometric structure.

A proof of several particular cases of the conjecture, as well a connection to old problems of combinatorics, was obtained in [12]. However, the general problem is open.

5. DISCUSSION ACTUALLY

To our great regret, we cannot discuss all of this with John Conway (at least not yet!), but we can easily imagine such a discussion.

Perhaps, John would ask about *number walls*, as he did when we talked about friezes. This notion, among hundreds of other surprising notions, can be found on leafing through John’s co-authored (with Richard Guy) book [4]. A number wall is a $\mathbb{Z}^2$ -lattice filled with integers satisfying a local rule different from the frieze rule:

$$\begin{array}{ccccccc} \dots & \bullet & N & \bullet & \dots & & \\ \dots & W & X & E & \dots & \text{local rule} & \\ \dots & \bullet & S & \bullet & \dots & X^2 = NS + EW. & \end{array}$$

As in the case of friezes, the array is bounded by lines (or zig-zags) of 1’s. “Can you q -deform this?,” John would likely ask. Our answer would be: “We don’t know!” The question, however, is interesting since number walls produce many interesting (known and unknown) sequences of integers.

He might also have wondered about *surreal numbers*, which he and Guy also discussed in [4]. Surreal numbers can be constructed using the same binary tree as in Section 2.2. Vertices of the tree are labeled

by dyadic rationals (rationals whose denominator is a power of 2):

The completion of this picture, according to Conway, contains not only all real numbers, but much more. Conway called surreal numbers an “enormous new world of numbers” and valued them as his greatest discovery; see [18, p. 568]. Once again, though, we do not know what q -deformed surreal numbers are, but it is tempting to apply the same quantization procedure to the binary tree. We are not aware, however, of any stabilization phenomenon.

At this point, John would understand that any sophisticated question he might ask us would result in the same answer He would probably look at us with indulgence and ask something simple like: “But how about quantum complex numbers?! What is the q -deformation of i , or of $\frac{1+i\sqrt{3}}{2}$?!” “Yes,” we would reply, “we *do* know the answer to that!:

$$[i]_q = \frac{i}{\sqrt{q}},$$

while $\frac{1+i\sqrt{3}}{2}$ remains undeformed, like 0 and 1,” and we would add, “we’ll will tell you more about it very very soon!”

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