Estimation of extremes for heavy-tailed and light-tailed distributions in the presence of random censoring

Julien Worms, Rym Worms

Estimation of extremes for heavy-tailed and light-tailed distributions in the presence of random censoring

Julien Worms (1) & Rym Worms[1] (2)

(1) Université Paris-Saclay/Université de Versailles-Saint-Quentin-En-Yvelines
Laboratoire de Mathématiques de Versailles (CNRS UMR 8100),
F-78035 Versailles Cedex, France,
e-mail : julien.worms@uvsq.fr

(2) Université Paris-Est
Laboratoire d'Analyse et de Mathématiques Appliquées
(CNRS UMR 8050),
UGE, UPEC, F-94010, Créteil, France,
e-mail : rym.worms@u-pec.fr

Abstract In this paper, we use the flexible semi-parametric model $A_1(\tau, \theta)$ introduced in Gardes et al. (2011) for estimating extremes of censored data. Both the censored and the censoring variables are supposed to belong to this family of distributions. Solutions for modeling the tail of censored data which are between Weibull-tail and Pareto-tail behavior are considered. Estimators of the parameters, as well as high-quantiles, are proposed and asymptotic normality results are proved. Various combinations of the tails of censored and censoring distributions are covered, ranging from rather light censoring to severe censoring in the tail.

AMS Classification. Primary 62G32 ; Secondary 62N02

Keywords and phrases. Log-Weibull tail. Tail inference. Random censoring. Asymptotic representation.

[1]Corresponding author

1. Introduction

This paper proposes new contributions to the topic of extreme value statistics for data which are randomly censored from the right.

Consider the classical random censoring setup, where one observes a sample from a couple $(Z, \delta) = (\min(X, C), \mathbb{I}_{X \leqslant C})$ with X denoting the variable of interest, and C a censoring variable (independent from X) which may prevent the user from observing the data X. The observed data is a sample $(Z_1, \delta_1), \ldots, (Z_n, \delta_n)$ where $(X_1, \ldots, X_n)$ and $(C_1, \ldots, C_n)$ are independent samples of i.i.d. copies of X and C. The topic of extreme value statistics for randomly censored data deals with the estimation of the tail of X (extreme quantiles, rare probabilities of exceeding a large value), while observing such an incomplete data sample.

This topic has benefited from a number of contributions in the recent years (see Worms and Worms (2019) and references therein), which were stimulated by applications in a variety of domains, mainly reliability analysis, survival/lifetime analysis and insurance. A characteristic of all of these papers is that X and C are always supposed to *share the same type of tail*, *i.e.* a heavy tail censored by a heavy tail, a light (Gumbel) tail censored by another light tail, or a finite tail censored by a finite tail. This is for instance very well described by the 3 cases exhibited in formula (7) of the insightful paper Einmahl et al. (2008).

The main and initial objective of this paper is to broaden the type of tails in the Gumbel domain that the user will be able to handle for estimating extremes for censored data. As a matter of fact, the lighter-than-Pareto-tails situation was slightly overlooked in censored extremes works, and this may be considered unfortunate since several applications of the censored extremes question do not necessarily exhibit a heavy tail behavior (particularly in survival/lifetimes analysis). Essentially only two research papers proposed so far solutions for dealing with light tails. Einmahl et al. (2008) considered the double Gumbel case but with an assumption on the ultimate probability of non-censoring in the tail, and without parametrization of the tail (only extreme quantiles were estimated, without further exploration of the tail). Worms & Worms (2019) considered the general two Weibull-tails case, a subset of the double Gumbel case, which allows for interesting configurations where the utimate probability of non-censoring in the tail can be zero (see its definition in next Section).

The basement of the present work is the flexible semi-parametric model proposed in Gardes et al. (2011) (model $A_1(\tau, \theta)$ described in the next section), which encompasses the Gumbel and the Fréchet maximum domain of attraction, and therefore provides a more flexible option for modeling various phenomena. In this paper, estimation of the parameters of this model will be made possible in the presence of censoring, with very simple expressions for the estimators. In addition, this setup will allow for a more diverse combination of tails (without prior knowledge of that combination) than the Fréchet versus Fréchet or the Weibull-tail versus Weibull-tail cases.

The paper is organised as follows. Section 2 formally settles the framework and describes how the parameters of the observed Z can be deduced from those of X and C, thus explaining what is statistically at stake. Section 3 explains how the parameters and extreme quantiles of X can be estimated from the observed censored data, while Section 4 states the main results of this paper, along with the required assumptions on the number k_n of order statistics retained for the estimation. Section 5 contains simulations to illustrate the performance of our estimators. Part A to D of the Appendix are devoted to the proofs of our asymptotic results, while part E contains important technical aspects.

2. Description of the framework and assumptions

Let us now describe more formally the setting. Defining for $\tau \in [0, 1]$ the Box-Cox function

$$K_\tau(x) = \int_1^x u^{\tau-1} du = \begin{cases} (x^\tau - 1)/\tau & \text{if } \tau \in]0, 1], \\ \log(x) & \text{if } \tau = 0, \end{cases}$$

we consider, for parameters $\tau \in [0, 1]$ and $\theta > 0$, that a distribution function F belongs to the semi-parametric family $A_1(\tau, \theta)$ if the following holds (see Gardes et al. (2011) where this model was first introduced):

$A_1(\tau, \theta)$: for some $x^* > 0$ and every $x \geqslant x_*$, we have

$$1 - F(x) = \exp(-K_\tau^-(\log(H(x)))),$$

where H is an increasing positive function such that H^- is regularly varying at infinity with index θ (which will be denoted by $H^- \in RV_\theta$).

Let us highlight that the tail heaviness of a distribution belonging to $A_1(\tau, \theta)$ is mainly driven by τ, although in practice both shape parameters τ and θ play an important role in the properties and shape of the upper tail. It is easy to see that $A_1(1, \theta)$ corresponds to distributions in the Fréchet domain of attraction with extreme value index θ, $A_1(0, \theta)$ corresponds to Weibull-tail distributions with Weibull-tail coefficient θ. The case $\tau \in]0, 1[$ corresponds to distributions in the Gumbel domain having tails heavier than Weibull-type ones : such distributions can be conveniently qualified as having log-Weibull-type tails, and log-normal distributions belong to this category with $\tau = 1/2$ (see Gardes et al. (2011) for more examples).

In this work, the main assumption is that both the censored and the censoring variables have their distribution belonging to the $A_1(\tau, \theta)$ family. This assumption covers a quite flexible setting. We thus assume the following :

Assumption (A1) : there exist $\tau_X \in [0, 1]$, $\tau_C \in [0, 1]$, $\theta_X > 0$, $\theta_C > 0$ such that

$$F_X \in A_1(\tau_X, \theta_X) \quad \text{and} \quad F_C \in A_1(\tau_C, \theta_C).$$

This means that there exists positive functions H_X and H_C such that

$$\bar{F}_X(x) = 1 - F_X(x) = \exp(-K^{-}_{\tau_X}(\log(H_X(x)))) \quad \text{and} \quad \bar{F}_C(x) = 1 - F_C(x) = \exp(-K^{-}_{\tau_C}(\log(H_C(x))))$$

and, for some slowly varying functions $\bar{l}_X$ and $\bar{l}_C$ at infinity,

$$H^{-}_X(x) = x^{\theta_X}\bar{l}_X(x) \quad \text{and} \quad H^{-}_C(x) = x^{\theta_C}\bar{l}_C(x).$$

It is clear that under this condition we also have $H_X(x) = x^{1/\theta_X} l_X(x)$ and $H_C(x) = x^{1/\theta_C} l_C(x)$ where both l_X and l_C are slowly varying functions at infinity.

The estimation of the parameters τ_X and θ_X is the main objective of this work. To do so, some relation must be found between the parameters of X and C, and those of the observed variable $Z = \min\{X, C\}$. Under assumption (A1), we can prove that the distribution of Z also belongs to the same family of distributions as those of X and C, for some parameters τ_Z and θ_Z precised below :

Proposition 1. *Under Assumption (A1), the distribution function of $Z = \min(X, C)$ satisfies condition $A_1(\tau_Z, \theta_Z)$, where*

$$\tau_Z = \min(\tau_X, \tau_C) \quad \text{and} \quad \theta_Z = \begin{cases} \theta_X & \text{if } 0 \leqslant \tau_X < \tau_C \leqslant 1 \\ \theta_C & \text{if } 0 \leqslant \tau_C < \tau_X \leqslant 1 \\ (\theta_X^{-1/\tau_Z} + \theta_C^{-1/\tau_Z})^{-\tau_Z} & \text{if } 0 < \tau_X = \tau_C \leqslant 1 \\ \min(\theta_X, \theta_C) & \text{if } \tau_X = \tau_C = 0 \end{cases}$$

Therefore, there exists $x_* > 0$ such that for any $x \geqslant x_*$, we have

$$\mathbb{P}(Z > x) = \exp(-K^{-}_{\tau_Z}(\log(H_Z(x)))),$$

where $H^{-}_Z \in RV_{\theta_Z}$. Consequently, if E denotes a standard exponential distribution, we have

$$Z = H^{-}_Z(\exp K_{\tau_Z}(E)).$$

The proof of this Proposition is not very difficult but tedious. It is therefore omitted for brevity.

Remark 1. *It is interesting to note that :*

- *in the two-heavy-tails case $\tau_X = \tau_C = 1$, we recover the well-known fact that $\theta_Z = \gamma_Z = (\gamma_X^{-1} + \gamma_C^{-1})^{-1}$ where γ_X and γ_C are the extreme value indices of X and C (see Beirlant et al. (2007)).*
- *in the two-Weibull-tails case $\tau_X = \tau_C = 0$, we recover the fact that the Weibull-tail parameter of Z is equal to the minimum of those of X and C (see Worms and Worms (2019)).*
- *when $\tau_X = \tau_C$, we have $\theta_Z \leqslant \min(\theta_X, \theta_C)$, but otherwise this is not necessarily the case.*
- *the expression of θ_Z in the fourth case is continuously coherent with the third one in the sense that $\min(\theta_X, \theta_C)$ is indeed the limit of $(\theta_X^{-1/\tau} + \theta_C^{-1/\tau})^{-\tau}$ as $\tau \to 0$.*

In this paper, we will exclude the first two situations evoked in Remark 1 above, which have already been explored in anterior works, and therefore suppose that $(\tau_X, \tau_C) \in [0, 1]^2 \backslash \{ (0, 0) , (1, 1) \}$.

Continuing with the probabilistic features of this model, let us now point out that, if $p(x) = \mathbb{P}(\delta = 1|Z = x)$ denotes the probability of being non-censored at level x, the following holds true (it is a copy of statement (i) of the much more complete Lemma 2 stated in the Appendix) :

$$\lim_{x\to+\infty} p(x) = p := \begin{cases} 1 & \text{if } 0 \leqslant \tau_X < \tau_C \leqslant 1, \\ 0 & \text{if } 0 \leqslant \tau_C < \tau_X \leqslant 1, \\ \theta_X^{1/\tau_X} / (\theta_X^{1/\tau_X} + \theta_C^{1/\tau_X}) & \text{if } 0 < \tau_X = \tau_C < 1, \end{cases}$$

In the first situation (the light censoring one), the fact that the ultimate probability p of non-censoring in the tail is 1 and that the parameters of X are the same as those of Z (see Proposition 1) would suggest that taking into account the censoring is useless. However, as Worms and Worms (2019) already put forward, this is not advisable because those settings produce finite size data where censoring is still present and needs to be taken into account. Similarly, the strong censoring situation where the ultimate probability p is 0 produces, in practice, data which are not completely censored in the tail, and thus the statistical problem of estimating the tail parameters and extreme quantiles of X should and can be addressed. Finally, one can note that the particular situation where tails of X and C have the same heaviness ($\tau_X = \tau_C$) is interesting on its own.

Note that in Einmahl et al. (2008) the double Gumbel case was considered with the assumption $p \in]0, 1[$, which is difficult to check in practice.

Let us close this section by now describing the more technical assumptions required for our results to hold. This part of the section may be skipped on first reading. In order to achieve asymptotic normality of the estimators defined in this paper, the slowly varying functions l_X and l_C associated to H_X and H_C are supposed to satisfy a classical second order condition (usually called the SR2 condition) :

Assumption (A2) : there exist some negative constants ρ_X and ρ_C, and some rate functions b_X and b_C having constant sign at $+\infty$ and satisfying $|b_X| \in RV_{\rho_X}$ and $|b_C| \in RV_{\rho_C}$, such that, as $t \to +\infty$,

$$\frac{l_X(tx)/l_X(t) - 1}{b_X(t)} \longrightarrow K_{\rho_X}(x), \quad \text{and} \quad \frac{l_C(tx)/l_C(t) - 1}{b_C(t)} \longrightarrow K_{\rho_C}(x), \forall x > 0. \tag{1}$$

According to the last statement of Proposition 1 and to the expression of our estimators (see next Section), it will be important in the sequel to consider the functions

$$H_Z^-(x) = x^{\theta_Z}\tilde{l}(x) \quad \text{and} \quad H_X \circ H_Z^-(x) = x^a l(x) \quad \text{with} \quad a := \frac{\theta_Z}{\theta_X}, \tag{2}$$

where both $\tilde{l}$ and l are slowly varying. The crucial parameter $a = \theta_Z/\theta_X$ is equal to 1 in "low censoring" situations (in particular when $\tau_X < \tau_C$).

Our important technical Lemma 1, stated in Appendix E.1, ensures that functions H_Z and H_Z^- also satisfy a second order condition SR2. For technical reasons though, we need to consider the following stronger conditions on $\tilde{l}$ and l, respectively noted $R_{\tilde{l}}(\tilde{b}, \tilde{\rho})$ and $R_l(b, \rho)$, and defined by :

Assumption $\boldsymbol{R_\ell(B, \rho)}$: for some constant $\rho \leqslant 0$ and a rate function B satisfying $\lim_{x\to+\infty} B(x) = 0$, such that for all $\epsilon > 0$, we have

$$\sup_{\lambda\geqslant 1}\left|\frac{\ell(\lambda x)/\ell(x) - 1}{B(x)K_\rho(\lambda)} - 1\right| \leqslant \epsilon, \text{ for } x \text{ sufficiently large .}$$

Note that, according to Lemma 1 in the Appendix, we have $\rho = \tilde{\rho}$, and that this parameter is negative when either $\tau_X = 0$ or $\tau_C = 0$, but otherwise (*i.e.* in most cases) it is zero, an unpleasant fact which often implies some challenge in the proofs.

3. Construction of the estimators

Let us denote by Λ_X and Λ_C the cumulative hazard functions associated to F_X and F_C, respectively

$$\Lambda_X(x) = -\log \bar{F}_X(x) \qquad \text{and} \qquad \Lambda_C(x) = -\log \bar{F}_C(x),$$

and let $\hat{\Lambda}_{nX}$ denote the Nelson-Aalen estimator of Λ_X defined as

$$\hat{\Lambda}_{nX}(x) = \sum_{Z_{i,n} \leqslant x} \frac{\delta_{i,n}}{n-i+1}, \tag{3}$$

where $Z_{1,n} \leqslant \ldots \leqslant Z_{n,n}$ are the order statistics of the sample (Z_i) and $\delta_{1,n}, \ldots, \delta_{n,n}$ are the corresponding indicators associated to these reordered Z values. Let $k_n = o(n)$ be an intermediate sequence of integers (which will often be simply denoted by k). The estimators of τ_X and θ_X that we propose are $\hat{\tau}_X$ and $\hat{\theta}_{X,\hat{\tau}_X}$ where we define

$$\hat{\tau}_X := \frac{HH_{k,n}}{D_{k,0}} \quad \text{and} \quad \hat{\theta}_{X,\tau_X} := \frac{H_{k,n}}{D_{k,\tau_X}} \tag{4}$$

with

$$\begin{aligned} H_{k,n} &:= \frac{1}{k_n}\sum_{j=1}^{k_n} \log(Z_{n-j+1,n}) - \log(Z_{n-k_n,n}), \\ HH_{k,n} &:= \frac{1}{k_n}\sum_{j=1}^{k_n} \log\log(Z_{n-j+1,n}) - \log\log(Z_{n-k_n,n}), \\ D_{k,\tau_X} &:= \frac{1}{k_n}\sum_{j=1}^{k_n} K_{\tau_X}(\hat{\Lambda}_{nX}(Z_{n-j+1,n})) - K_{\tau_X}(\hat{\Lambda}_{nX}(Z_{n-k_n,n})). \end{aligned}$$

Note that the expressions of the estimators defined in (4) do not depend on the relative positions of τ_X and τ_C (or of θ_X and θ_C). They can be calculated whatever the combinations of the tails of X and C are, with the same formulas. However, we will see in the next Section that the rates of convergence, performances, and assumptions of these estimators can differ depending on the strength of censoring.

Remark 2. *In the case $\tau_X = \tau_C = 0$, corresponding to the purely Weibull-tail framework, the estimator $\hat{\theta}_{X,0}$ corresponds to the one studied in Worms and Worms (2019), because $K_{\tau_Z}(x) = \log(x)$ in that case. In the case $\tau_X = \tau_C = 1$, corresponding to the purely heavy-tail framework, the estimator $\hat{\theta}_{X,1}$ corresponds to the adapted Hill estimator studied in Beirlant et al. (2007), because in that case $K_{\tau_Z}(x) = x - 1$ and thus we have exactly $D_{k,1} = \hat{p}_k$ (see formula (7) below). As said earlier, these two particular cases are excluded from the scope of the statements of this paper because properties of $\hat{\theta}_{X,0}$ and $\hat{\theta}_{X,1}$ are already known.*

In the following lines, we derive the approximations that inspired the definitions in (4). Under Assumption (A1), H_X is regularly varying with index $1/\theta_X$ and $K_{\tau_X}(\Lambda_X(x)) = \log(H_X(x))$, hence, for u large, we have

$$K_{\tau_X}(u) \approx \frac{1}{\theta_X}\log(\Lambda_X^{-}(u)).$$

Moreover, for s large and any $u > 1$

$$\log\left(\frac{K_{\tau_X}(su)}{K_{\tau_X}(s)}\right) = \log\left(\frac{(su)^{\tau_X} - 1}{(s)^{\tau_X} - 1}\right) \simeq \tau_X \log u.$$

Combining these two results, we obtain a first approximation, for u and s large, relating τ_X to Λ_X :

$$\tau_X \log u \approx \log\log(\Lambda_X^{-}(su)) - \log\log(\Lambda_X^{-}(s)). \tag{5}$$

The second approximation comes from the fact that, for t large and any given $x > 1$, we have

$$\frac{H_X(tx)}{H_X(t)} = \exp(K_{\tau_X}(\Lambda_X(tx)) - K_{\tau_X}(\Lambda_X(t))) \simeq x^{1/\theta_X},$$

hence θ_X is related to τ_X and Λ_X via the formula :

$$\frac{1}{\theta_X}\log x \approx K_{\tau_X}(\Lambda_X(tx)) - K_{\tau_X}(\Lambda_X(t)). \tag{6}$$

Therefore, the two definitions in relation (4) come by applying approximation (5) to $s = \Lambda_X(Z_{n-k_n,n})$ and $u = \Lambda_X(Z_{n-j+1,n})/\Lambda_X(Z_{n-k_n,n})$ on one hand, and approximation (6) to $t = Z_{n-k_n,n}$ and $x = Z_{n-j+1,n}/Z_{n-k_n,n}$ on the other hand, and then by plugging in the Nelson-Aalen estimator of Λ_X and summing for $1 \leqslant j \leqslant k$.

The two estimators above are thus ratios involving on one hand the mean of either the log-spacings (*i.e.*

the Hill statistic) or the log-log-spacings, and on the other hand a denominator involving the Nelson-Aalen estimator at the k upper values of the observed Z sequence.

The main issue in the proofs lies in the treatment of the denominators D_{k,τ_X} and $D_{k,0}$. In fact, the statistic D_{k,τ_X} defined below (4) turns out to be related to the proportion of uncensored data in the tail via the relation (see Lemma 3 in the Appendix for the details)

$$D_{k,\tau_X} \approx \left(\hat{\Lambda}_{nX}(Z_{n-k,n})\right)^{\tau_X - 1} \hat{p}_k \quad \text{where} \quad \hat{p}_k := \frac{1}{k}\sum_{j=1}^{k} \delta_{n-j+1,n}$$

because of the nature of the Box-Cox transformation K_{τ_X}, Taylor's formula, and of the fact that

$$\frac{1}{k}\sum_{j=1}^{k}\left(\hat{\Lambda}_{nX}(Z_{n-j+1,n}) - \hat{\Lambda}_{nX}(Z_{n-k,n})\right) = \frac{1}{k}\sum_{j=1}^{k}\sum_{l=j}^{k}\frac{\delta_{n-l+1,n}}{l} = \frac{1}{k}\sum_{j=1}^{k}\delta_{n-j+1,n}. \tag{7}$$

Therefore, the properties of our estimators will rely on a careful study of two sequences. The first one is $\hat{\Lambda}_{nX}(Z_{n-k,n})$ (in particular, how it can be approximated by $\Lambda_X(Z_{n-k,n})$ and written as an increasing function of $\log n/k$; see Lemma 4 in the Appendix). The second one is the sequence $\hat{p}_k$, which converges to 0, 1 or a value $p \in]0,1[$ depending on the position of τ_X with respect to τ_C (Proposition 2 provides the full details about this, and relies on sharp second order developments of the different regularly varying functions that appear in this framework, *cf* the important technical Lemmas 1 and 2 in the Appendix).

Finally, let us deal with the estimation of an extreme quantile $x_{p_n} := \bar{F}_X^{-}(p_n)$ of the distribution of X, with $p_n \to 0$, as $n \to +\infty$. Applying the approximation (6) now to $t = Z_{n-k,n}$ and $x = x_{p_n}/Z_{n-k,n}$, we can propose the following estimator of x_{p_n} (with both θ_X and τ_X being unknown) :

$$\hat{x}_{p_n} := Z_{n-k,n} \exp\left\{\hat{\theta}_{X,\hat{\tau}_X}\left(K_{\hat{\tau}_X}(-\log(p_n)) - K_{\hat{\tau}_X}(\hat{\Lambda}_{nX}(Z_{n-k,n}))\right)\right\}. \tag{8}$$

Note that if we know that $\tau_X = 0$ and set $\hat{\tau}_X = 0$, then this estimator is the same as the one proposed in Worms and Worms (2019).

4. Asymptotic results

The main assumptions on the model and the different notations have been stated in the previous sections. In order to obtain the asymptotic normality of our estimators, we naturally need the sequence (k_n) (number of top order statistics to use) to satisfy some conditions (we will note $k = k_n$ from now on). The first one is standard in the literature on Weibull-tail models :

$H_1 : k \to +\infty$, $\frac{k}{n} \to 0$, $\frac{\log k}{\log n} \to 0$, as $n \to +\infty$.

Moreover, introducing the important notation

$$L_{nk} = \log(n/k),$$

let v_n be a factor which will contribute to the rates of convergence of our estimators, it depends on the censoring strength in the tail :

$$v_n := \begin{cases} 1 & \text{if } 0 < \tau_X < \tau_C \leqslant 1 \text{ or } 0 < \tau_X = \tau_C < 1 \text{ or } 0 = \tau_X < \tau_C \leqslant 1, \\ L_{nk}^{\frac{1}{2}(\frac{\tau_C}{\tau_X} - 1)} & \text{if } 0 < \tau_C < \tau_X \leqslant 1, \\ L_{nk}^{-1/2}(\log L_{nk})^{\frac{1}{2}(\frac{1}{\tau_X} - 1)} & \text{if } 0 = \tau_C < \tau_X \leqslant 1. \end{cases}$$

We also consider the following conditions

$H_2 : 0 < \tau_X < \tau_C \leqslant 1$ and $\begin{cases} (i) \sqrt{k} L_{nk}^{\tau_X/\tau_C - 1} \to 0 \text{ if } \frac{1}{\tau_C} - \frac{1}{\tau_X} \geqslant -1 \\ (ii) \sqrt{k} L_{nk}^{-\tau_X} \to 0 \text{ if } \frac{1}{\tau_C} - \frac{1}{\tau_X} < -1 \end{cases}$

$H_3 : 0 < \tau_C < \tau_X \leqslant 1$ and $\begin{cases} (i) \sqrt{k} v_n \to +\infty \\ (ii) \sqrt{k} v_n L_{nk}^{\tau_C/\tau_X - 1} \to 0 \text{ if } \frac{1}{\tau_X} - \frac{1}{\tau_C} \geqslant -1 \\ (iii) \sqrt{k} v_n L_{nk}^{-\tau_C} \to 0 \text{ if } \frac{1}{\tau_X} - \frac{1}{\tau_C} < -1 \end{cases}$

$H_4 : 0 < \tau_X = \tau_C < 1$ and $\sqrt{k} L_{nk}^{-\tau_X} \to 0$

$H_5 : 0 = \tau_X < \tau_C \leqslant 1$ and $\exists \delta > 0, \sqrt{k} L_{nk}^{\tilde{\rho} + \delta} \to 0$

$$H_6 : 0 = \tau_C < \tau_X \leqslant 1 \text{ and } \begin{cases} (i)\sqrt{k}v_n \to +\infty \\ (ii)\sqrt{k}v_n(\log L_{nk})^{-1} \to 0 \end{cases}$$

(in assumption H_5 above, $\tilde{\rho}$ denotes the second order parameter associated to the slowly varying function $\tilde{l}$, which is negative in this case ; see formula (2) in Section 2 as well as Lemma 1 in Appendix E.1)

The following four theorems respectively state the convergence in distribution of the estimators $\hat{\tau}_X$, $\hat{\theta}_{X,\tau_X}$ (with τ_X known), $\hat{\theta}_{X,\hat{\tau}_X}$, and $\hat{x}_{p_n}$, all of them being defined in the previous section.

Theorem 1. *Let assumptions (A1) and (A2) hold, as well as $R_l(b,\rho)$ and $R_{\tilde{l}}(\tilde{b},\tilde{\rho})$. If (k_n) satisfies H_1 and one of the conditions $H_2,\ldots,H_6$, then we have, as $n \to \infty$,*

$$\text{if } \tau_X \neq 0, \quad \sqrt{k}v_n(\hat{\tau}_X - \tau_X) \xrightarrow{d} N\left(0, \tau_X^2\sigma^2\right),$$
$$\text{if } \tau_X = 0, \quad \hat{\tau}_X = O_{\mathbb{P}}(1/\log(L_{nk})) \xrightarrow{\mathbb{P}} 0,$$

where $a = \theta_Z/\theta_X$ and

$$\sigma^2 = \begin{cases} 1 & \text{if } 0 \leqslant \tau_X < \tau_C \leqslant 1, \\ a^{-1/\tau_X}\left(\frac{\tau_X}{\tau_C}\right)^{1-1/\tau_X} & \text{if } 0 < \tau_C < \tau_X \leqslant 1, \\ a^{-1/\tau_X} & \text{if } 0 < \tau_X = \tau_C < 1, \\ a^{-1/\tau_X}\tau_X^{1-1/\tau_X} & \text{if } 0 = \tau_C < \tau_X \leqslant 1. \end{cases}$$

Theorem 2. *Under the same assumptions as Theorem 1, we have, as $n \to \infty$,*

$$\sqrt{k}v_n(\hat{\theta}_{X,\tau_X} - \theta_X) \xrightarrow{d} N\left(0, \theta_X^2\sigma^2\right).$$

Remark 3. *When $\tau_X < \tau_C$, the ultimate probability p of non-censoring is 1, this is the light-censoring situation. When $\tau_X = \tau_C$, it is easy to see that $\theta_Z < \theta_X$ and thus the asymptotic variance is larger than in the case $\tau_X < \tau_C$ (i.e. we have $\sigma^2 > 1$). When $0 < \tau_C < \tau_X$ (strong censoring setting), the ultimate probability of non-censoring p is zero, and the factor σ^2 is < 1 when $\theta_C > \theta_X$, but otherwise this is not necessarily the case.*

Theorem 3. *Under the same assumptions as Theorem 1, if $\tau_X > 0$ and if we further assume that*

$$\frac{\sqrt{k}v_n}{\log L_{nk}} \to +\infty \quad (\text{if } \tau_C \neq 0) \qquad \text{or} \qquad \frac{\sqrt{k}v_n}{\log\log L_{nk}} \to +\infty \quad (\text{if } \tau_C = 0), \tag{9}$$

we then have, as $n \to \infty$,

$$\text{if } \tau_C \neq 0 \quad \frac{\sqrt{k}v_n}{\log L_{nk}}(\hat{\theta}_{X,\hat{\tau}_X} - \theta_X) \xrightarrow{d} N\left(0, \theta_X^2\sigma^2\tau_Z^2\right),$$
$$\text{if } \tau_C = 0 \quad \frac{\sqrt{k}v_n}{\log\log L_{nk}}(\hat{\theta}_{X,\hat{\tau}_X} - \theta_X) \xrightarrow{d} N\left(0, \theta_X^2\sigma^2\right).$$

Remark 4. *Note that the rate of convergence and asymptotic variance of $\hat{\theta}_{X,\hat{\tau}_X}$ are altered and different from that of $\hat{\theta}_{X,\tau_X}$ due to the plug-in of $\hat{\tau}_X$.*

Theorem 4. *Under the same assumptions as Theorem 3, if moreover*

$$\frac{\sqrt{k}v_n}{\log\log(1/p_n)(-\log(p_n))^{\tau_X}} \to +\infty \tag{10}$$

and

$$\frac{\log L_{nk}}{\log\log(1/p_n)} \to 0 \quad (\text{if } \tau_C \neq 0) \qquad \text{or} \qquad \frac{\log\log L_{nk}}{\log\log(1/p_n)} \to 0 \quad (\text{if } \tau_C = 0), \tag{11}$$

we then have, as $n \to \infty$,

$$\frac{\sqrt{k}v_n}{\log\log(1/p_n)(-\log(p_n))^{\tau_X}}\left(\frac{\hat{x}_{p_n}}{x_{p_n}} - 1\right) \xrightarrow{d} N\left(0, \theta_X^2\sigma^2\right).$$

Remark 5. *There is some sort of phase transition phenomenon in the above results. As a matter of fact, not only the rate of convergence of our estimators vary whether τ_X is $\leqslant \tau_C$ or not, but the closeness of the parameters τ_X and τ_C also play a role (see assumptions H_2 and H_3) : the assumptions vary whether τ_X is lower than τ_C but not too close to it (i.e. $1 < \frac{1}{\tau_X} - \frac{1}{\tau_C}$), lower than τ_C but close to it (i.e. $0 < \frac{1}{\tau_X} - \frac{1}{\tau_C} \leqslant 1$), equal to τ_C, larger than and close to τ_C (i.e. $0 < \frac{1}{\tau_C} - \frac{1}{\tau_X} \leqslant 1$), or sufficiently larger than τ_C (i.e. $1 < \frac{1}{\tau_C} - \frac{1}{\tau_X}$). However, in practice, for finite and moderate values of n, visualizing these findings on simulations is not easy, because other factors (than just the tail parameters) play a non-negligible role in the estimation quality.*

Let us finish this section by providing a hint of the consistency of our estimators. Let us note $\hat{\tau}_Z^{(c)}$ and $\hat{\theta}_Z^{(c)}$ the following estimators of τ_Z and θ_Z

$$\hat{\theta}_Z^{(c)} = \frac{H_{k,n}}{\mu_{1,\tau_Z}(L_{nk})} \quad \text{and} \quad \hat{\tau}_Z^{(c)} = \frac{HH_{k,n}}{\mu_{1,0}(L_{nk})} \quad \text{where} \quad \mu_{1,\tau}(t) = \int_0^\infty \left(K_\tau(x+t) - K_\tau(t)\right) e^{-x}\, dx. \tag{12}$$

The first one was introduced in Gardes et al. (2011). The second one is similar to the estimator proposed in Albert et al. (2020) (in a slightly different setting) ; by the way, note that $\hat{\tau}_Z^{(c)}$ is a new estimator of τ in the $A_1(\tau,\theta)$ model without censoring, and thus a competitor of the estimator which was proposed in El Methni et al. (2012) (which required the delicate choice of two intermediate sequences k_n and k'_n).

Using the material of Gardes et al. (2011) and Albert et al. (2020), one can prove that $\hat{\theta}_Z^{(c)}$ and $\hat{\tau}_Z^{(c)}$ are consistent estimators of θ_Z and τ_Z, and we have

$$\hat{\theta}_{X,\tau_X} = \hat{\theta}_Z^{(c)} \times \frac{\mu_{1,\tau_Z}(L_{nk})}{D_{k,\tau_X}} \quad \text{and} \quad \hat{\tau}_X = \hat{\tau}_Z^{(c)} \times \frac{\mu_{1,0}(L_{nk})}{l\mu_{1,\tau_Z}(L_{nk})} \times \frac{l\mu_{1,\tau_Z}(L_{nk})}{D_{k,0}} \tag{13}$$

where $l\mu_{1,\tau}(t) := \int_0^\infty \left(\log(K_\tau(x+t)) - \log(K_\tau(t))\right) e^{-x}\, dx$. The consistency of $\hat{\theta}_{X,\tau_X}$ will thus come from the convergence of the ratio $\mu_{1,\tau_Z}(L_{nk})/D_{k,\tau_X}$ to $1/a = \theta_X/\theta_Z$, which is deduced from Corollary 1 (stated in Appendix A of this paper). The consistency of $\hat{\tau}_X$ comes from the convergence of $l\mu_{1,\tau_Z}(L_{nk})/D_{k,0}$ to τ_X (which is deduced from Corollary 2 in Appendix B), and the fact that $\mu_{1,0}(t)/l\mu_{1,\tau_Z}(t)$ converges to $1/\tau_Z$ as $t \to \infty$ (which is deduced from relations (A.3) and (B.3) in the Appendix).

It is noteworthy that equation (13) describes a way of adapting to the censoring context any estimators of θ or τ which are known in the complete data setting, by simply dividing by the appropriate expression involving D_{k,τ_X} or $D_{k,0}$.

5. Finite sample comparisons

In this section, we illustrate, using few simulations, the finite sample performances of our estimators of τ_X, θ_X and x_{p_n} (for small p_n), in terms of observed bias and mean squared error (MSE). Note that numerous different situations could be considered with our flexible framework : a thorough and extensive simulation study is however not possible within the limits of the present paper. We generate $N = 1000$ samples of size $n = 500$.

We consider three classes of distributions of Log-Weibull-tail type, for the target variable X and the censoring variable C :

- Log-Weibull(θ) distribution such that its logarithm has c.d.f. $1 - \exp(-x^{1/\theta})$ ($x > 0$). It satisfies assumption $A_1(\theta,\theta)$.
- Log-Normal distribution $LN(\mu,\sigma^2)$, which satisfies assumption $A_1(\frac{1}{2}, \frac{\sigma\sqrt{2}}{2})$, according to Proposition 3 in Gardes et al. (2011).
- Model $\mathscr{F}$ with c.d.f. F_τ satisfying $A_1(\tau, 1/5)$, with $H^-(x) = x^{1/5}(1 + x^{-1/2})$ ($\forall x$).

We then consider three cases : a Log-Weibull(θ_X) distribution censored by the Log-Normal$(1, 1/2)$ distribution (Figure 1), the Log-Normal$(1, 1/2)$ distribution censored by a Log-Weibull(θ_C) distribution (Figure 2), and then a distribution in the $\mathscr{F}$ model censored by another distribution in the $\mathscr{F}$ model (Figure 3). In each case, we consider three situations with $\tau_X < \tau_C$, $\tau_X = \tau_C$ or $\tau_X > \tau_C$, corresponding to different (ultimate) intensities of censoring in the tail.

In parts (a),(b),(c) of Figures 1, 2 and 3, we present the bias and the MSE of our estimators $\hat{\tau}_X$ and $\hat{\theta}_{X,\hat{\tau}_X}$ as a function of k. In parts (d),(e),(f) of Figures 1, 2 and 3, we present the relative bias and the relative

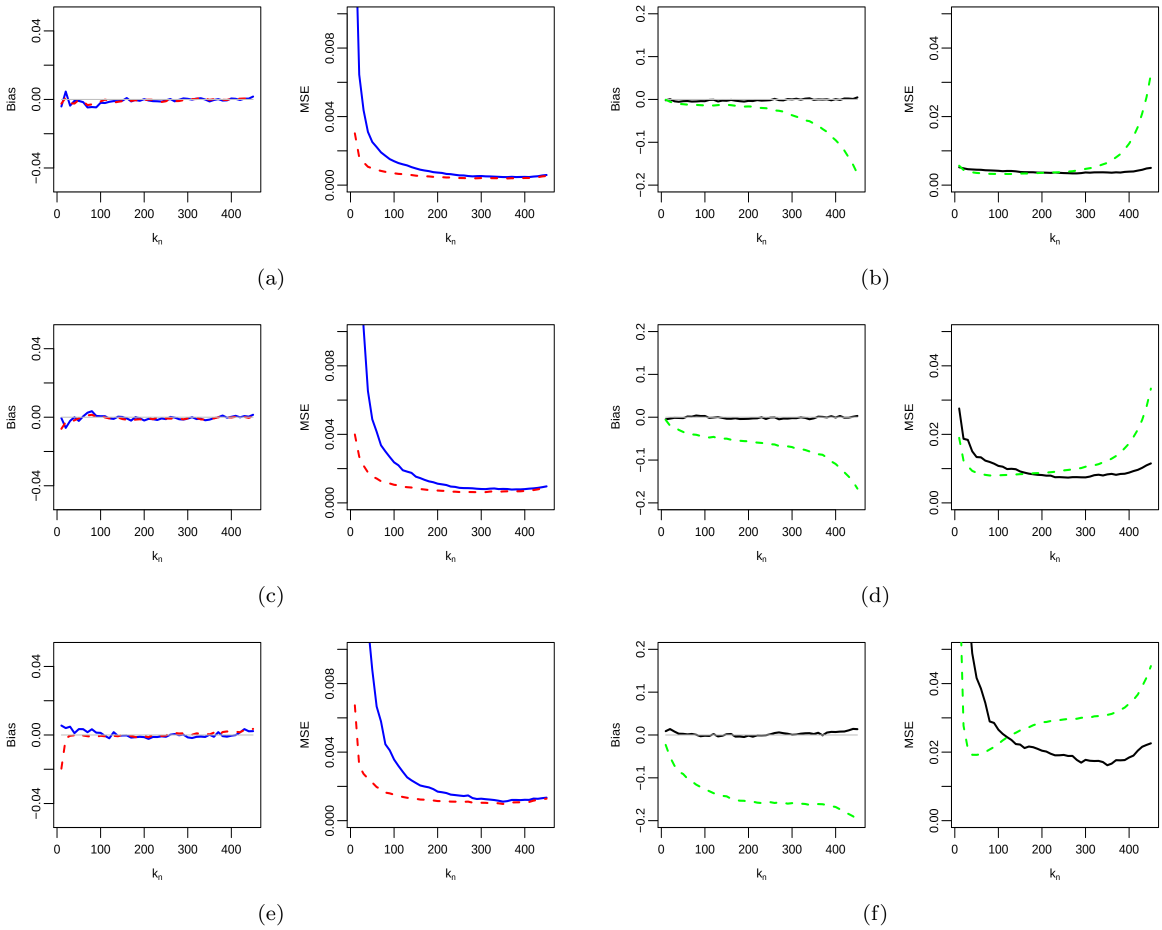

Figure 1: Simulation with X log-Weibull censored by C log-Normal, where $\tau_X = 0.4 < \tau_C = 0.5$ in first line (figures (a)-(b)), $\tau_X = 0.5 = \tau_C$ in second line (figures (c)-(d)), and $\tau_X = 0.6 > \tau_C = 0.5$ in third line (figures (e)-(f)). The graphs represent observed bias and MSE of estimators $\hat{\tau}_X$ (blue) and $\hat{\theta}_{X,\hat{\tau}_X}$ (dashed red) in figures (a)-(c)-(e), and relative bias and MSE of estimators $\hat{x}_{p_n}$ (black) and $\hat{x}_{p_n}^{EFG}$ (dashed green) in figures (b)-(d)-(f).

MSE of our estimator $\hat{x}_{p_n}$ for the value $p_n = 0.01$, compared with those of the existing estimator defined, in a more general censored setting, by equation (8) in Einmahl et al. (2008) :

$$\hat{x}_{p_n}^{EFG} = Z_{n-k,n} + \hat{a}_k \frac{((1-\hat{F}_n(Z_{n-k}))/p_n)^{\hat{\gamma}^{c,Mom}} - 1}{\hat{\gamma}^{c,Mom}}, \tag{14}$$

where $\hat{\gamma}^{c,Mom}$ is the moment estimator of the extreme value index γ_X of F adapted to censoring and $\hat{F}_n$ stands for the Kaplan-Meier estimator of the c.d.f. F. We refer to Einmahl et al. (2008) for the expression of $\hat{a}_k$. Note that no formal asymptotic result is currently available for $\hat{x}_{p_n}^{EFG}$.

Concerning the performance of the estimators $\hat{\theta}_{X,\hat{\tau}_X}$ and $\hat{\tau}_X$, we observe that when X has a Log-Weibull tail, the bias and the MSE for both estimators are very small. When one deviates from this situation, though, they are not very satisfactory on the situations presented here. Note however that these estimators are the first to be proposed in this context, which is why no comparison to competitors is presented . Another remark is that the quality of the estimators do not systematically deteriorate when censoring gets stronger.

Concerning the performance of the high quantile estimator, the figures show very good performances when X has a Log-Weibull tail. When one deviates from this situation, things may become worse. It is particularly true here in the Log-Normal versus Log-Weibull case. However, our estimator remains competitive in terms of bias and MSE in a number of other situations, for instance in Figure 3.

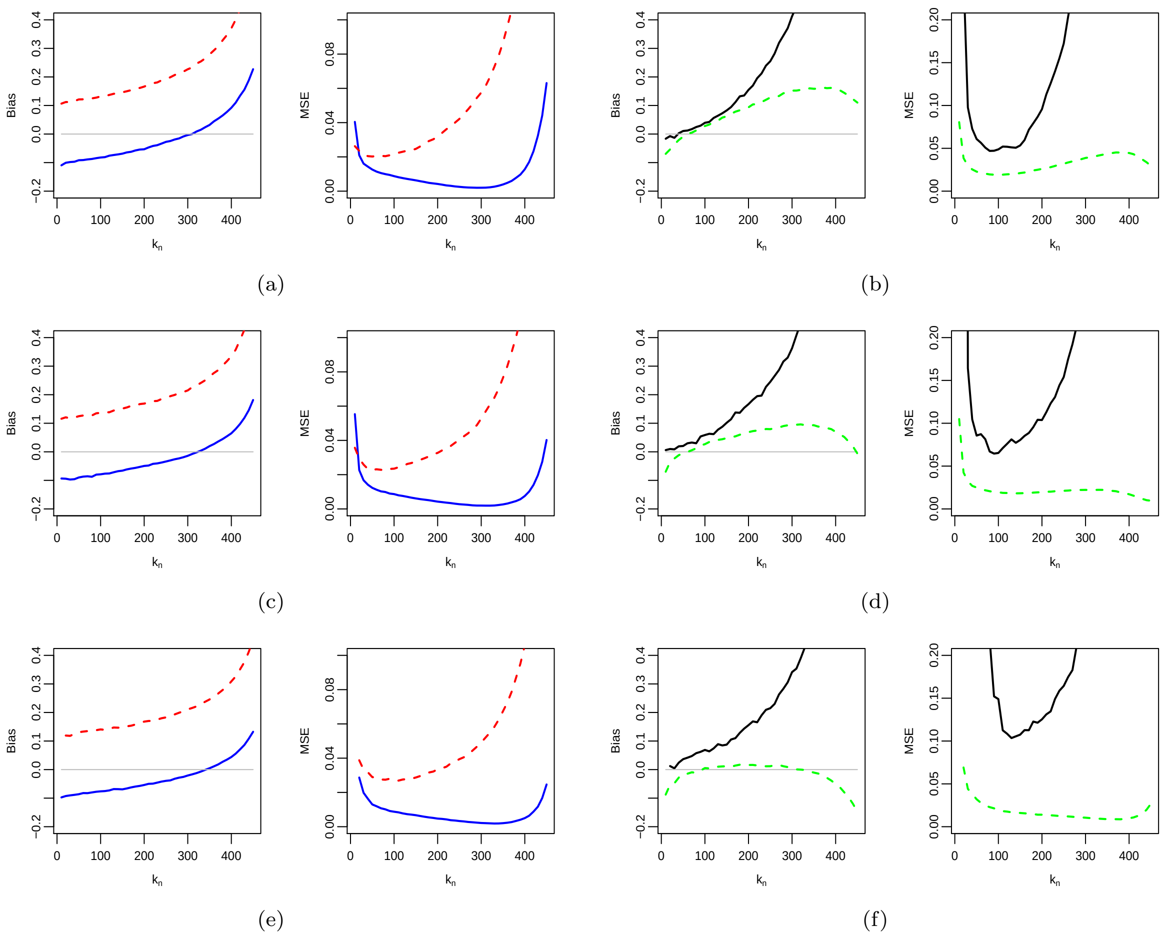

Figure 2: Simulation with X log-Normal censored by C log-Weibull, where $\tau_X = 0.5 < \tau_C = 0.6$ in first line (figures (a)-(b)), $\tau_X = 0.5 = \tau_C$ in second line (figures (c)-(d)), and $\tau_X = 0.5 > \tau_C = 0.4$ in third line (figures (e)-(f)). The graphs represent observed bias and MSE of estimators $\hat{\tau}_X$ (blue) and $\hat{\theta}_{X,\hat{\tau}_X}$ (dashed red) in figures (a)-(c)-(e), and relative bias and MSE of estimators $\hat{x}_{p_n}$ (black) and $\hat{x}_{p_n}^{EFG}$ (dashed green) in figures (b)-(d)-(f).

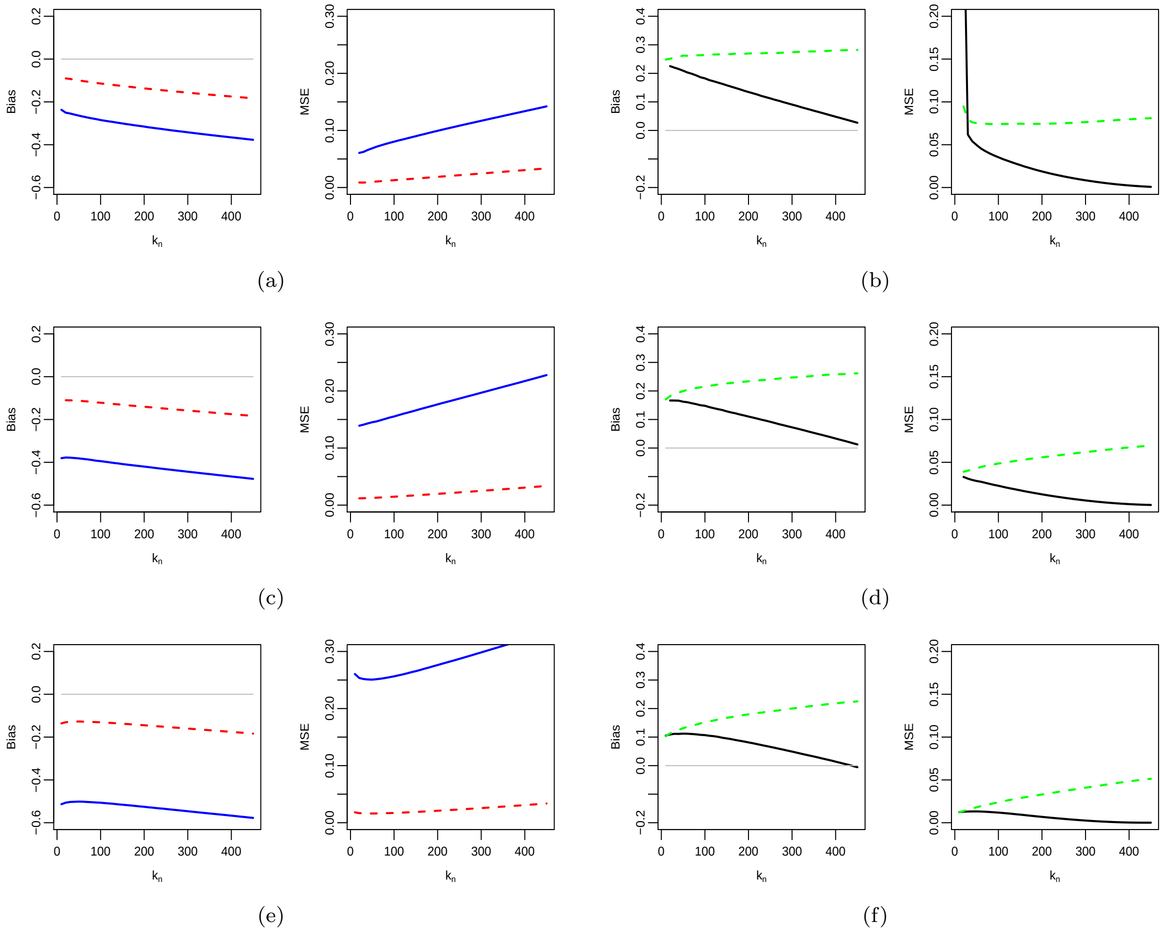

Figure 3: Simulation with X and C in the F model, where $\tau_X = 0.4 < \tau_C = 0.6$ in first line (figures (a)-(b)), $\tau_X = 0.5 = \tau_C$ in second line (figures (c)-(d)), and $\tau_X = 0.6 > \tau_C = 0.4$ in third line (figures (e)-(f)). The graphs represent observed bias and MSE of estimators $\hat{\tau}_X$ (blue) and $\hat{\theta}_{X,\hat{\tau}_X}$ (dashed red) in figures (a)-(c)-(e), and relative bias and MSE of estimators $\hat{x}_{p_n}$ (black) and $\hat{x}_{p_n}^{EFG}$ (dashed green) in figures (b)-(d)-(f).

6. Conclusion

In this paper we proposed a solution for dealing with tail and extreme quantile estimation of data which are randomly right censored, within a rather large family of distributions encompassing power tail distributions, Weibull-tail distributions, and intermediary situations such as (for instance) log-normal distributions. This family was first introduced in a complete data context in Gardes et al. (2011). Our asymptotic normality results support all possible amounts of censoring in the tail, even very strong ones where the ultimate probability of being censored in the tail is equal to one.

The main two contributions of this work are that very diverse combinations of tails of the censored and censoring distributions are dealt with (not just a combination of tails from the same category), and that tail estimation of log-Weibull-type distributions (not heavier than Pareto tails though) are dealt with as well. The fact that one can estimate the tail parameters of this flexible model, and not just the extreme quantiles, means that the user may consider estimating more elaborated parameters than the extreme quantiles (for instance, expected tail losses $\mathbb{E}(X|X > F_X^{-}(1-p))$ for small p).

Concerning the performances, the bias of our estimators of θ and τ remains a problem, as soon as one moves away from the pure log-Weibull situation. However our opinion is that this bias problem was already present for the original estimators of τ and θ (which inspired ours) in the non-censoring context. This topic of bias reduction still needs to be explored for this family of distributions, even in the non-censored situation. In this paper, we did not try to detail the asymptotic bias, mainly because of the great diversity of situations that our model handled, which already made the exposition quite complicated. This would require further work.

Finally, a continuation of this work could be to look for estimators of τ and θ which are weighted modifications of their non-censored versions (the estimators in equation (12)), but with varying weights, not the constant weights $D_{k,0}$ and D_{k,τ_X}, with in mind a possible improvement in terms of bias and mean-squared error.

References

C. Albert, A. Dutfoy, L. Gardes and S. Girard . An extreme quantile estimator for the log-generalized Weibull-tail model. In *Econometrics and Statistics* **13**, pages 137-174 (2020)

J. Beirlant, G. Dierckx, A. Fils-Villetard and A. Guillou . Estimation of the extreme value index and extreme quantiles under random censoring. In *Extremes* **10**, pages 151-174 (2007)

J. Beirlant, Y. Goegebeur, J. Segers and J. Teugels Statistics of extremes: Theory and applications Wiley (2004)

J. Einmahl, A. Fils-Villetard and A. Guillou . Statistics of extremes under random censoring. In *Bernoulli* **14**, pages 207-227 (2008)

J. El Methni, L. Gardes, S. Girard and A. Guillou . Estimation of extreme quantiles from heavy and light tailed distributions. In *J. Stat. Planning and Inference* **142**, pages 2735-2747 (2012)

L. Gardes, S. Girard and A. Guillou . Weibull tail-distributions revisited: A new look at some tail estimators. In *Journal of Statistical Planning and Inference* **141**, pages 429-444 (2011)

L. de Haan and A. Ferreira . Extreme Value Theory : an Introduction. *Springer Science + Business Media* (2006)

J. Worms and R. Worms. Estimation of extremes for Weibull-tail distributions in the presence of random censoring. In *Extremes* **22**, pages 667-704 (2019)

Appendix

Let us first summarize the contents of the Appendix. It is composed of 4 main parts.

Part A is devoted to the proof of Theorem 2.

Part B is devoted to the proof of Theorem 1.

Part C is devoted to the proof of Theorem 3

Part D is devoted to the proof of Theorem 4.

Part E contains different technical aspects.

Remind that L_{nk} is the notation for $\log(n/k)$. Let us introduce the following notations :

$$\Lambda_k = \Lambda_F(Z_{n-k,n}) \quad \text{and} \quad \hat{\Lambda}_k = \hat{\Lambda}_{nX}(Z_{n-k,n}).$$

Appendix A. Proof of Theorem 2

This section details how the asymptotic normality of $\hat{\theta}_{X,\tau_X}$ stems from the combination of properties of the Hill estimator $H_{k,n}$ (relations (A.1), (A.2) and (A.4) below) and of the proportion $\hat{p}_k$ of uncensored data in the tail (Proposition 2 stated next page), via the important decomposition (A.6). Some details are postponed to other sections, in particular the crucial technical Lemma 2 (stated in Appendix E.1) which states the second order properties of the function $p(x) = \mathbb{P}(\delta = 1|Z = x)$. The behavior of the (numerous) remainder terms is detailed in Proposition 3.

First, remind that $\hat{\theta}_{X,\tau_X} = \dfrac{H_{k,n}}{D_{k,\tau_X}}$, with

$$H_{k,n} = \frac{1}{k}\sum_{j=1}^{k} \log(Z_{n-j+1,n}) - \log(Z_{n-k,n}) \text{ and } D_{k,\tau_X} = \frac{1}{k}\sum_{j=1}^{k} K_{\tau_X}(\hat{\Lambda}_{nX}(Z_{n-j+1,n})) - K_{\tau_X}(\hat{\Lambda}_{nX}(Z_{n-k,n})).$$

According to Proposition 1, we have $Z_i = H_Z^{-}(\exp(K_{\tau_Z}(E_i)))$, where $E_1, \ldots, E_n$ are n independent standard exponential random variables and (see relation (2)) $H_Z^{-}(x) = x^{\theta_Z}\tilde{l}(x)$, $\tilde{l}$ being RV_0. Hence

$$H_{k,n} = \theta_Z M_n + R_{n,\tilde{l}} \tag{A.1}$$

where

$$M_n := \frac{1}{k}\sum_{j=1}^{k} K_{\tau_Z}(E_{n-i+1,n}) - K_{\tau_Z}(E_{n-k,n}) \quad \text{and} \quad R_{n,\tilde{l}} := \frac{1}{k}\sum_{j=1}^{k} \log\left(\frac{\tilde{l}(\exp(K_{\tau_Z}(E_{n-j+1,n})))}{\tilde{l}(\exp(K_{\tau_Z}(E_{n-k,n})))}\right).$$

By the Renyi representation, we have $E_{n-j+1,n} - E_{n-k} \overset{d}{=} F_{k-j+1,k}$, where $F_1, \ldots, F_k$ are k independent standard exponential random variables. As was done in Gardes et al. (2011) (and borrowing their notations), we have

$$M_n \overset{d}{=} \theta_{n,1}(E_{n-k}) \quad \text{where} \quad \theta_{n,1}(t) := \frac{1}{k}\sum_{j=1}^{k} K_{\tau_Z}(F_i + t) - K_{\tau_Z}(t). \tag{A.2}$$

Introducing, for $q \in \mathbb{N}^*$, (see Lemma 2 of Gardes et al. (2011))

$$\mu_{q,\tau_Z}(t) := \mathbb{E}(\theta_{n,q}(t)) = \int_0^\infty (K_{\tau_Z}(x+t) - K_{\tau_Z}(t))^q \, e^{-x} \, dx = (q!)\, t^{q(\tau_Z - 1)}(1 + o(1)) \quad (\text{as } t \to +\infty) \tag{A.3}$$

and $\sigma^2_{1,\tau_Z}(t) := \mu_{2,\tau_Z}(t) - \mu^2_{1,\tau_Z}(t)$, it is proved in Lemma 5 of Gardes et al. (2011) that

$$\sqrt{k}A_{1,n} \overset{d}{\longrightarrow} N(0,1) \quad \text{where} \quad A_{1,n} := \frac{\theta_{n,1}(E_{n-k}) - \mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})}. \tag{A.4}$$

Moreover, we prove in Lemma 3 (stated in Appendix E.4) that

$$D_{k,\tau_X} = \hat{\Lambda}_k^{\tau_X - 1}\hat{p}_k + R_{1,n} \tag{A.5}$$

where $\hat{p}_k$ denotes the proportion of uncensored data among the k upper data values (see Lemma 3 for the definition of the remainder term $R_{1,n}$). Formulas $(A.1)$ and $(A.5)$ thus easily entail the following important intermediate relation :

$$\hat{\theta}_{X,\tau_X} - \theta_X \overset{d}{=} \frac{\theta_Z M_n - \theta_X \Lambda_k^{\tau_X - 1}\hat{p}_k}{D_{k,\tau_X}} + \sum_{i=1}^{3} T_{i,n},$$

where

$$
\begin{aligned}
T_{1,n} &:= \frac{R_{n,\tilde{l}}}{D_{k,\tau_X}} \\
T_{2,n} &:= -\theta_X \frac{R_{1,n}}{D_{k,\tau_X}} \\
T_{3,n} &:= -\theta_X \frac{\hat{\Lambda}_k^{\tau_X-1} - \Lambda_k^{\tau_X-1}}{D_{k,\tau_X}} \hat{p}_k.
\end{aligned}
$$

Concerning now $\hat{p}_k$, reminding that $a := \theta_Z/\theta_X$, we prove in Lemma 5 (stated in Appendix E.4) that, when $\tau_X \geqslant 0$ and $\tau_C > 0$,

$$\Lambda_k^{\tau_X-1}\hat{p}_k = \left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X} E_{n-k}^{\tau_Z(1-1/\tau_X)}\hat{p}_k + R_{2,n}$$

(note that the first term is equal to $E_{n-k}^{-1}\hat{p}_k$ when $0 = \tau_X < \tau_C \leqslant 1$, since then $\tau_Z = \tau_X$ and $a = 1$), and when $\tau_X > 0$ and $\tau_C = 0$,

$$\Lambda_k^{\tau_X-1}\hat{p}_k = (a\tau_X)^{1-1/\tau_X} (\log E_{n-k})^{1-1/\tau_X}\, \hat{p}_k + R_{2,n},$$

where the remainder term $R_{2,n}$ is detailed for each case in the statement of Lemma 5.
Consequently, defining $T_{4,n} := -\theta_X \frac{R_{2,n}}{D_{k,\tau_X}}$, we obtain the following decomposition : when $\tau_X \geqslant 0$ and $\tau_C > 0$

$$\hat{\theta}_{X,\tau_X} - \theta_X \stackrel{d}{=} \frac{\sigma_{1,\tau_Z}(E_{n-k})}{D_{k,\tau_X}} \left(\theta_Z A_{1,n} - \theta_X \frac{\mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})} \left(\left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X} \frac{E_{n-k}^{\tau_Z(1-1/\tau_X)}}{\mu_{1,\tau_Z}(E_{n-k})}\hat{p}_k - \frac{\theta_Z}{\theta_X}\right)\right) + \sum_{i=1}^{4} T_{i,n},$$

and, when $\tau_X > 0$ and $\tau_C = 0$,

$$\hat{\theta}_{X,\tau_X} - \theta_X \stackrel{d}{=} \frac{\sigma_{1,\tau_Z}(E_{n-k})}{D_{k,\tau_X}} \left(\theta_Z A_{1,n} - \theta_X \frac{\mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})} \left((a\tau_X)^{1-1/\tau_X} \frac{(\log E_{n-k})^{1-1/\tau_X}}{\mu_{1,\tau_Z}(E_{n-k})}\hat{p}_k - \frac{\theta_Z}{\theta_X}\right)\right) + \sum_{i=1}^{4} T_{i,n}.$$

Then, reminding that $\mu_{1,\tau_Z}(t) \sim t^{\tau_Z-1}$ as $t \to \infty$, we define the following remainder term as (note again that $a\tau_X/\tau_Z = 1$ and $\tau_Z(1-1/\tau_X) = -1$ when $\tau_X = 0 < \tau_C$)

$$R_{3,n} := \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X} \hat{p}_k \left(\frac{(E_{n-k})^{\tau_Z(1-1/\tau_X)}}{\mu_{1,\tau_Z}(E_{n-k})} - L_{nk}^{1-\tau_Z/\tau_X}\right) & \text{when } \tau_X \geqslant 0,\ \tau_C > 0, \\ (a\tau_X)^{1-1/\tau_X}\, \hat{p}_k \left(\frac{(\log E_{n-k})^{1-1/\tau_X}}{\mu_{1,0}(E_{n-k})} - L_{nk}(\log L_{nk})^{1-\tau_Z/\tau_X}\right) & \text{when } 0 = \tau_C < \tau_X. \end{cases}$$

Finally, using the additional fact that, thanks to $(A.3)$, $\frac{\mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} 1$, we can state the main relation of the proof of Theorem 2 :

$$\hat{\theta}_{X,\tau_X} - \theta_X \stackrel{d}{=} \frac{\mu_{1,\tau_Z}(E_{n-k})}{D_{k,\tau_X}} \left(\theta_Z A_{1,n} - \theta_X A_{2,n}(1 + o_{\mathbb{P}}(1))\right) + \sum_{i=1}^{5} T_{i,n}, \tag{A.6}$$

where the second important term $A_{2,n}$ is defined as

$$A_{2,n} := \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X} L_{nk}^{1-\tau_Z/\tau_X}\, \hat{p}_k - a & \text{if } \tau_X \geqslant 0 \text{ and } \tau_C > 0, \\ (a\tau_X)^{1-1/\tau_X} L_{nk}(\log L_{nk})^{1-\tau_Z/\tau_X}\, \hat{p}_k - a & \text{if } 0 = \tau_C < \tau_X, \end{cases}$$

and the last remainder term to be introduced is $T_{5,n} := \theta_Z R_{3,n}(1 + o_{\mathbb{P}}(1))$.

We deal with the asymptotic normality of $A_{2,n}$ and the reminder terms $T_{i,n}$ in the following two propositions, which are proved, respectively, in Appendix E.2 and Appendix E.3. Remind that the rate v_n is defined as

$$v_n := \begin{cases} 1 & \text{if } 0 < \tau_X < \tau_C \leqslant 1 \text{ or } 0 < \tau_X = \tau_C < 1 \text{ or } 0 = \tau_X < \tau_C < 1 \\ L_{nk}^{\frac{1}{2}(\frac{\tau_C}{\tau_X}-1)} & \text{if } 0 < \tau_C < \tau_X \leqslant 1 \\ L_{nk}^{-1/2}(\log L_{nk})^{\frac{1}{2}(\frac{1}{\tau_X}-1)} & \text{if } 0 = \tau_C < \tau_X < 1 \end{cases}$$

Proposition 2. *Under the conditions of Theorem 1,*

$$\begin{aligned}
&\text{if} \quad 0 \leqslant \tau_X < \tau_C \leqslant 1, \quad \sqrt{k} v_n A_{2,n} = \sqrt{k} v_n (\hat{p}_k - a) = \sqrt{k}(\hat{p}_k - 1) \xrightarrow{\mathbb{P}} 0,\\
&\text{if} \quad 0 < \tau_C < \tau_X \leqslant 1, \quad \sqrt{k} v_n A_{2,n} = \sqrt{k} v_n \left(\left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}} L_{nk}^{1-\tau_Z/\tau_X} \hat{p}_k - a \right) \xrightarrow{d} N\left(0, a^{2-1/\tau_X} \left(\frac{\tau_X}{\tau_C}\right)^{1-1/\tau_X}\right),\\
&\text{if} \quad 0 < \tau_X = \tau_C < 1, \quad \sqrt{k} v_n A_{2,n} = \sqrt{k}(a^{1-1/\tau_X} \hat{p}_k - a) \xrightarrow{d} N\left(0, a^{2-1/\tau_X}(1 - a^{1/\tau_X})\right),\\
&\text{if} \quad 0 = \tau_C < \tau_X < 1, \quad \sqrt{k} v_n A_{2,n} = \sqrt{k} v_n \left((a\tau_X)^{1-\frac{1}{\tau_X}} L_{nk} (\log L_{nk})^{1-\frac{1}{\tau_X}} \hat{p}_k - a \right) \xrightarrow{d} N\left(0, a^{2-1/\tau_X} \tau_X^{1-1/\tau_X}\right).
\end{aligned}$$

Proposition 3. *Under the conditions of Theorem 1, for all* $1 \leqslant i \leqslant 5$, $\sqrt{k} v_n T_{i,n} \xrightarrow{\mathbb{P}} 0$, *as* n *tends to infinity.*

Let us now explain how the combination of relations (A.6) and (A.4) and Propositions 2 and 3 imply that $\sqrt{k} v_n (\hat{\theta}_{X,\tau_X} - \theta_X) \xrightarrow{d} N(0, v)$ where $v = \theta_X^2 \sigma^2$ and ends the proof of Theorem 2.

When $0 \leqslant \tau_X < \tau_C \leqslant 1$, Proposition 2 states that $\sqrt{k} A_{2,n}$ converges to 0. Hence, the leading term in $(A.6)$ is $\sqrt{k} A_{1,n}$ which converges in distribution to $N(0,1)$ (see $(A.4)$), and we thus obtain as desired $v = (\frac{1}{a})^2 \theta_Z^2 = \theta_X^2$.

When $0 < \tau_X = \tau_C < 1$, Proposition 2 states that $\sqrt{k} A_{2,n} \xrightarrow{d} N\left(0, a^{2-1/\tau_X}(1 - a^{1/\tau_X})\right)$. Moreover $\sqrt{k} A_{1,n}$ converges in distribution to $N(0,1)$. Since $A_{1,n}$ and $A_{2,n}$ are independent, we obtain as desired

$$v = \frac{\theta_Z^2}{a^2} + \frac{\theta_X^2}{a^2} a^{2-1/\tau_X}(1 - a^{1/\tau_X}) = \theta_X^2 + \theta_X^2 (a^{-1/\tau_X} - 1) = \theta_X^2 a^{-1/\tau_X}.$$

In the other two cases, since $v_n \to 0$, $\sqrt{k} v_n A_{1,n}$ converges to 0, and on the other hand Proposition 2 states that $\sqrt{k} v_n A_{2,n}$ converges in distribution to $N(0, D)$ with variances described above, and it is easy to check that $(\frac{1}{a})^2 \theta_X^2 D$ equals to $\theta_X^2 \sigma^2$ as stated. This ends the proof of Theorem 2.

Let us end this section with the following corollary of Propositions 2 and 3.

Corollary 1. *Under the conditions of Theorem 1, we have* $\dfrac{D_{k,\tau_X}}{\mu_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} a$, *as* n *tends to infinity.*

Indeed, according to $(A.5)$, and since $\mu_{1,\tau_Z}(t) \sim t^{\tau_Z - 1}$ as $t \to \infty$ (see relation (A.3)),

$$\frac{D_{k,\tau_X}}{\mu_{1,\tau_Z}(E_{n-k})} = L_{nk}^{1-\tau_Z} \Lambda_k^{\tau_X - 1} \hat{p}_k (1 + o(1)) \stackrel{d}{=} (A_{2,n} + a)(1 + o(1)) \xrightarrow{\mathbb{P}} a.$$

Of course, the conditions of Theorem 1 are too strong for Corollary 1 to hold.

Appendix B. Proof of Theorem 1

The proof is very similar to the previous one. First, remind that $\hat{\tau}_X = \frac{HH_{k,n}}{D_{k,0}}$. Concerning the numerator, we have by Proposition 1 that $Z_i = H_Z^{-}(\exp(K_{\tau_Z}(E_i)))$, where $E_1, \ldots, E_n$ are standard exponential, and thus

$$HH_{k,n} := \frac{1}{k_n} \sum_{j=1}^{k_n} \log\log(Z_{n-j+1,n}) - \log\log(Z_{n-k_n,n}) = LM_n + RR_{n,\tilde{l}} \tag{B.1}$$

where

$$LM_n := \frac{1}{k} \sum_{j=1}^{k} \log(K_{\tau_Z}(E_{n-i+1,n})) - \log(K_{\tau_Z}(E_{n-k,n})) \quad \text{and} \quad RR_{n,\tilde{l}} := \frac{1}{k} \sum_{j=1}^{k} \log \left(\frac{1 + \frac{\log(\tilde{l}(\exp(K_{\tau_Z}(E_{n-j+1,n}))))}{\theta_Z K_{\tau_Z}(E_{n-j+1,n})}}{1 + \frac{\log(\tilde{l}(\exp(K_{\tau_Z}(E_{n-k,n}))))}{\theta_Z K_{\tau_Z}(E_{n-k,n})}} \right).$$

By the Renyi representation, for some independent standard exponential random variables $F_1, \ldots, F_k$ we have

$$LM_n \stackrel{d}{=} l\theta_{n,1}(E_{n-k}) \quad \text{where} \quad l\theta_{n,1}(t) := \frac{1}{k} \sum_{j=1}^{k} \log(K_{\tau_Z}(F_i + t)) - \log(K_{\tau_Z}(t)). \tag{B.2}$$

Introducing, for $q \in \mathbb{N}^*$,

$$l\mu_{q,\tau_Z}(t) := \mathbb{E}(l\theta_{n,q}(t)) = \int_0^\infty (\log(K_{\tau_Z}(x+t)) - \log(K_{\tau_Z}(t)))^q \, e^{-x} \, dx$$

and $l\sigma^2_{1,\tau_Z}(t) := l\mu_{2,\tau_Z}(t) - l\mu^2_{1,\tau_Z}(t)$, we have

$$l\mu_{q,\tau_Z}(t) = \begin{cases} (q!)\tau_Z^q t^{-q}(1+o(1)) & \text{if } \tau_Z \neq 0, \\ (q!)t^{-q}(\log(t))^{-q}(1+o(1)) & \text{if } \tau_Z = 0. \end{cases} \tag{B.3}$$

We can then prove that (the proof is similar to that of Lemma 5 in Gardes et al. (2011))

$$\sqrt{k} LA_{1,n} \xrightarrow{d} N(0,1) \quad \text{ where } \quad LA_{1,n} := \frac{l\theta_{n,1}(E_{n-k}) - l\mu_{1,\tau_Z}(E_{n-k})}{l\sigma_{1,\tau_Z}(E_{n-k})}. \tag{B.4}$$

Concerning now the denominator, we prove in Lemma 3 (stated in Appendix E.4) that

$$D_{k,0} := \frac{1}{k_n} \sum_{j=1}^{k_n} \log(\hat{\Lambda}_{nX}(Z_{n-j+1,n})) - \log(\hat{\Lambda}_{nX}(Z_{n-k_n,n})) = \hat{\Lambda}_k^{-1} \hat{p}_k + R_{1,n}, \tag{B.5}$$

where

$$R_{1,n} = \frac{1}{k} \sum_{j=1}^{k} \left(\log\left(1 + \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}\right) - \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k} \right)$$

and $\hat{p}_k$ denotes the proportion of uncensored data in the tail. From now on we consider that $\tau_X \neq 0$ (see Remark 6 below for the $\tau_X = 0$ case). Formulas $(B.1)$ and $(B.5)$ easily entail the following important intermediary relation :

$$\hat{\tau}_X - \tau_X \stackrel{d}{=} \frac{LM_n - \tau_X \Lambda_k^{-1} \hat{p}_k}{D_{k,0}} + \sum_{i=1}^{3} TT_{i,n},$$

where

$$\begin{aligned} TT_{1,n} &:= \frac{RR_{n,\tilde{l}}}{D_{k,0}} \\ TT_{2,n} &:= -\tau_X \frac{R_{1,n}}{D_{k,0}} \\ TT_{3,n} &:= -\tau_X (\hat{\Lambda}_k^{-1} - \Lambda_k^{-1})(D_{k,0})^{-1} \hat{p}_k. \end{aligned}$$

Moreover, we prove in Lemma 6 (stated in Appendix E.4) that, when $\tau_X > 0$ and $\tau_C > 0$ (the case $\tau_X > 0$ and $\tau_C = 0$ is omitted for brevity),

$$\Lambda_k^{-1} \hat{p}_k = \left(\frac{a\tau_X}{\tau_Z}\right)^{-1/\tau_X} E_{n-k}^{-\tau_Z/\tau_X} \hat{p}_k + RR_{2,n},$$

the expression for the remainder term $RR_{2,n}$ being detailed for each case in the statement of Lemma 6. Consequently, defining $TT_{4,n} := -\tau_X \frac{RR_{2,n}}{D_{k,0}}$, we obtain the following decomposition : when $\tau_X > 0$ and $\tau_C > 0$

$$\hat{\tau}_X - \tau_X \stackrel{d}{=} \frac{l\sigma_{1,\tau_Z}(E_{n-k})}{D_{k,0}} \left(LA_{1,n} - \tau_X \frac{l\mu_{1,\tau_Z}(E_{n-k})}{l\sigma_{1,\tau_Z}(E_{n-k})} \left(\left(\frac{a\tau_X}{\tau_Z}\right)^{-1/\tau_X} \frac{E_{n-k}^{-\tau_Z/\tau_X}}{l\mu_{1,\tau_Z}(E_{n-k})} \hat{p}_k - \frac{1}{\tau_X} \right) \right) + \sum_{i=1}^{4} T_{i,n}.$$

But $l\mu_{1,\tau_Z}(t) \sim \tau_Z t^{-1}$, so we define the following remainder term as

$$RR_{3,n} := \left(\frac{a\tau_X}{\tau_Z}\right)^{-1/\tau_X} \hat{p}_k \left(\frac{(E_{n-k})^{-\tau_Z/\tau_X}}{l\mu_{1,\tau_Z}(E_{n-k})} - \frac{1}{\tau_Z} L_{nk}^{1-\tau_Z/\tau_X} \right).$$

Finally, using the additional fact that $\frac{l\mu_{1,\tau_Z}(E_{n-k})}{l\sigma_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} 1$, we can state the main relation of the proof of Theorem 1 :

$$\hat{\tau}_X - \tau_X \stackrel{d}{=} \frac{l\mu_{1,\tau_Z}(E_{n-k})}{D_{k,0}} \left(LA_{1,n} - a^{-1} A_{2,n}(1 + o_{\mathbb{P}}(1)) \right) + \sum_{i=1}^{5} TT_{i,n}, \tag{B.6}$$

where $LA_{1,n}$ is defined in (B.4), the second main term $A_{2,n}$ is defined in section Appendix A and the last remainder term to be introduced is $TT_{5,n} := -\tau_X RR_{3,n}(1 + o_{\mathbb{P}}(1))$. The asymptotic normality of $A_{2,n}$ is dealt with in Proposition 2. Concerning the remainder terms $TT_{i,n}$, we prove the following proposition :

Proposition 4. *Under the conditions of Theorem 1, for all* $1 \leqslant i \leqslant 5$, $\sqrt{k} v_n TT_{i,n} \xrightarrow{\mathbb{P}} 0$, *as n tends to infinity.*

The proof of this Proposition is very similar to the proof of Proposition 3. It is omitted. The proof of Theorem 1 can be concluded in the same way as was that of Theorem 2 , details are also omitted. □

Finally the following statement is a Corollary of Propositions 2 and 4, in the same way that Corollary 1 was deduced from Propositions 2 and 3.

Corollary 2. *Under the conditions of Theorem 1, when* $\tau_X \neq 0$ *we have*

$$\frac{D_{k,0}}{l\mu_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} \frac{1}{\tau_X}$$

and, when $0 = \tau_X < \tau_C$, *we have as* $n \to \infty$

$$\frac{D_{k,0}}{l\mu_{1,0}(E_{n-k})} = (\log L_{nk})(1 + o_{\mathbb{P}}(1)).$$

Remark 6. *In the case* $0 = \tau_X < \tau_C$, *we have* $D_{k,0}/l\mu_{1,0}(E_{n-k,n}) \overset{\mathbb{P}}{\sim} \log L_{nk}$, *and thus the estimator* $\hat{\tau}_X \overset{d}{=} l\theta_{n,1}(E_{n-k}) \,/\, D_{k,0} + TT_{1,n}$ *is contiguous to* $l\mu_{1,0}(E_{n-k}) \,/\, D_{k,0} + TT_{1,n}$, *which is itself equivalent in probability to* $1/\log L_{nk}$. *Thus only the consistency and rate of convergence of* $\hat{\tau}_X$ *is obtained in this case.*

Appendix C. Proof of Theorem 3

Remind that $\hat{\theta}_{X,\hat{\tau}_X} = H_{k,n}/D_{k,\hat{\tau}_X}$ where

$$H_{k,n} = \frac{1}{k}\sum_{j=1}^{k} \log(Z_{n-j+1,n}) - \log(Z_{n-k,n}) \text{ and } D_{k,\hat{\tau}_X} = \frac{1}{k}\sum_{j=1}^{k} K_{\hat{\tau}_X}(\hat{\Lambda}_{nX}(Z_{n-j+1,n})) - K_{\hat{\tau}_X}(\hat{\Lambda}_{nX}(Z_{n-k,n})).$$

Moreover

$$\log\left(\frac{\hat{\theta}_{X,\hat{\tau}_X}}{\theta_X}\right) = \log\left(\frac{\hat{\theta}_{X,\hat{\tau}_X}}{\hat{\theta}_{X,\tau_X}}\right) + \log\left(\frac{\hat{\theta}_{X,\tau_X}}{\theta_X}\right). \tag{C.1}$$

Theorem 2 and the delta-method yields that the second term of the right-hand side in $(C.1)$ satisfies

$$\sqrt{k} v_n \log\left(\frac{\hat{\theta}_{X,\tau_X}}{\theta_X}\right) \xrightarrow{d} N\left(0, \sigma^2\right). \tag{C.2}$$

Now let us treat the first term. Since $D_{k,\tau_X} = (\hat{\Lambda}_k)^{\tau_X - 1}\hat{p}_k + R_{1,n}$ (see Lemma 3) and, similarly, $D_{k,\hat{\tau}_X} = (\hat{\Lambda}_k)^{\hat{\tau}_X - 1}\hat{p}_k + \hat{R}_{1,n}$, where $\hat{R}_{1,n}$ is obtained by replacing τ_X by $\hat{\tau}_X$ in the expression for $R_{1,n}$, we obtain

$$\log\left(\frac{\hat{\theta}_{X,\tau_X}}{\hat{\theta}_{X,\hat{\tau}_X}}\right) = (\hat{\tau}_X - \tau_X)\log(\hat{\Lambda}_k) - \log\left(1 + \frac{R_{1,n}}{\hat{\Lambda}_k^{\tau_X - 1}\hat{p}_k}\right) + \log\left(1 + \frac{\hat{R}_{1,n}}{\hat{\Lambda}_k^{\hat{\tau}_X - 1}\hat{p}_k}\right).$$

Let us study separately the first two terms in the expression above (the third one being similar to the second one). The starting point is

$$(\hat{\tau}_X - \tau_X)\log(\hat{\Lambda}_k) = (\hat{\tau}_X - \tau_X)\log(\Lambda_k) + (\hat{\tau}_X - \tau_X)\log\left(\frac{\hat{\Lambda}_k}{\Lambda_k}\right).$$

Let us continue with the case $\tau_X \neq 0$ and $\tau_C \neq 0$ (the case $0 = \tau_C < \tau_X$ being similar and the case $0 = \tau_X < \tau_C$ being excluded, see Remark 7 below).

Since $\sqrt{k} v_n(\hat{\tau}_X - \tau_X) \xrightarrow{d} N\left(0, \sigma^2 \tau_X^2\right)$ (Theorem 1), and, according to Lemma 7, $\log(\Lambda_k) = \frac{\tau_Z}{\tau_X}(\log L_{nk})(1 + o_{\mathbb{P}}(1))$, we obtain that

$$\frac{\sqrt{k} v_n}{\log L_{nk}}(\hat{\tau}_X - \tau_X)\log(\Lambda_k) \xrightarrow{d} N\left(0, \sigma^2 \tau_Z^2\right)$$

and $\frac{\sqrt{k} v_n}{\log L_{nk}}(\hat{\tau}_X - \tau_X)\log\left(\frac{\hat{\Lambda}_k}{\Lambda_k}\right) = o_{\mathbb{P}}(1)$ (because $\frac{\hat{\Lambda}_k}{\Lambda_k} = O_{\mathbb{P}}(1)$).

Now, $\log\left(1+\frac{R_{1,n}}{(\hat{\Lambda}_k)^{\tau_X-1}\hat{p}_k}\right) = \frac{R_{1,n}}{(\hat{\Lambda}_k)^{\tau_X-1}\hat{p}_k}(1+o_{\mathbb{P}}(1))$, and we prove in Proposition 3 that $\sqrt{k}v_n\frac{R_{1,n}}{(\hat{\Lambda}_k)^{\tau_X-1}\hat{p}_k} = o_{\mathbb{P}}(1)$. Hence $\frac{\sqrt{k}v_n}{\log L_{nk}}\log\left(1+\frac{R_{1,n}}{(\hat{\Lambda}_k)^{\tau_X-1}\hat{p}_k}\right) = o_{\mathbb{P}}(1)$. This ensures that

$$\frac{\sqrt{k}v_n}{\log L_{nk}}\log\left(\frac{\hat{\theta}_{X,\tau_X}}{\hat{\theta}_{X,\hat{\tau}_X}}\right)\xrightarrow{d} N\left(0,\sigma^2\tau_Z^2\right).$$

Finally, $(C.1)$ and $(C.2)$ yield

$$\frac{\sqrt{k}v_n}{\log L_{nk}}\log\left(\frac{\hat{\theta}_{X,\hat{\tau}_X}}{\theta_X}\right)\xrightarrow{d} N\left(0,\sigma^2\tau_Z^2\right).$$

This entails the announced asymptotic normality, via the delta-method. □

Remark 7. *In the case $\tau_X = 0$, $\log(\Lambda_k) = a(\log L_{nk})(1+o_{\mathbb{P}}(1))$, according to Lemma 7. Hence, $\hat{\tau}_X\log(\Lambda_k)$ does not converge to 0, in this case. This is why $\tau_X = 0$ is excluded from the asymptotic result of $\hat{\theta}_{X,\hat{\tau}_X}$.*

Appendix D. Proof of Theorem 4

Remind that $x_{p_n} = \bar{F}_X^{-}(p_n) = H_X^{-}(\exp(K_{\tau_X}(-\log p_n)))$ and

$$\hat{x}_{p_n} = Z_{n-k,n}\exp\left(\hat{\theta}_{X,\hat{\tau}_X}\left(K_{\hat{\tau}_X}(-\log(p_n)) - K_{\hat{\tau}_X}(\hat{\Lambda}_k)\right)\right)$$

where $H_X^{-}(x) = x^{\theta_X}\bar{l}_X(x)$, and $\bar{l}_X$ is slowly varying at infinity. Moreover, since $Z_{n-k,n} = \bar{F}_X^{-}(\exp(-\Lambda_k))$, it is easy to prove that

$$\begin{aligned}
\log\left(\frac{\hat{x}_{p_n}}{x_{p_n}}\right) = \; & \hat{\theta}_{X,\hat{\tau}_X}\left\{(K_{\hat{\tau}_X}(-\log(p_n)) - K_{\hat{\tau}_X}(\Lambda_k)) - (K_{\tau_X}(-\log(p_n)) - K_{\tau_X}(\Lambda_k))\right\} \\
& +(\hat{\theta}_{X,\hat{\tau}_X}-\theta_X)K_{\tau_X}(-\log(p_n)) + \hat{\theta}_{X,\hat{\tau}_X}\left(K_{\hat{\tau}_X}(\Lambda_k) - K_{\hat{\tau}_X}(\hat{\Lambda}_k)\right) \\
& -(\hat{\theta}_{X,\hat{\tau}_X}-\theta_X)K_{\tau_X}(\Lambda_k) + \log\left(\frac{\bar{l}_X(\exp(K_{\tau_X}(\Lambda_k)))}{\bar{l}_X(\exp(K_{\tau_X}(-\log(p_n))))}\right) \\
& =: Q_1 + Q_2 + Q_3 + Q_4 + Q_5.
\end{aligned}$$

Let us treat separately these five terms, in the case $\tau_X \neq 0$ and $\tau_C \neq 0$, the case $0 = \tau_X < \tau_C$ being similar. Remind that

$$L_k := \begin{cases} (a\tau_X/\tau_Z)^{1/\tau_X}\,(L_{nk})^{\tau_Z/\tau_X} & \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0, \\ (a\tau_X)^{1/\tau_X}\,(\log L_{nk})^{1/\tau_X} & \text{if } \tau_X \neq 0 \text{ and } \tau_C = 0. \end{cases}$$

Consider the temporary notations

$$\sigma_n := \left(\sqrt{k}v_n\right)^{-1} \quad \text{and} \quad w_n := \int_{L_k}^{-\log(p_n)} u^{\tau_x-1}\log u\,du.$$

By integration by parts, and under assumption (11) (which implies that $L_k = o(-\log(p_n))$), we can prove that

$$w_n = \frac{1}{\tau_X}\log(\log(1/p_n))\,(-\log(p_n))^{\tau_X}\,(1+o(1)), \tag{D.1}$$

and similarly $\tilde{w}_n := \int_{L_k}^{-\log(p_n)} u^{\tau_x-1}\log^2 u\,du = \frac{1}{\tau_X}(\log(\log(1/p_n)))^2\,(-\log(p_n))^{\tau_X}\,(1+o(1))$.

- Let us prove that $\sigma_n^{-1}w_n^{-1}Q_1$ converges in distribution to $\mathscr{N}(0,\theta_X^2\tau_X^2\sigma^2)$, which (via (D.1)) will imply that

$$\frac{\sqrt{k}v_n}{\log\log(1/p_n)(-\log p_n)^{\tau_X}}Q_1 \xrightarrow{d} \mathscr{N}(0,\theta_X^2\sigma^2). \tag{D.2}$$

According to Theorem 1, $\hat{\tau}_X = \tau_X + \sigma_n\xi_n$, where ξ_n converges in distribution to $\mathscr{N}(0,\tau_X^2\sigma^2)$. Hence,

$$\begin{aligned}
Q_1 &= \hat{\theta}_{X,\hat{\tau}_X}\left(\int_{\Lambda_k}^{-\log p_n} u^{\tau_X+\sigma_n\xi_n-1}du - \int_{\Lambda_k}^{-\log p_n} u^{\tau_X-1}du\right) \\
&= \hat{\theta}_{X,\hat{\tau}_X}\left(\int_{L_k}^{-\log p_n} u^{\tau_X-1}(u^{\sigma_n\xi_n}-1)du - \int_{L_k}^{\Lambda_k} u^{\tau_X-1}(u^{\sigma_n\xi_n}-1)du\right).
\end{aligned}$$

Let us introduce $\phi(x) = e^x - 1 - x$. Consequently,

$$Q_1 = \sum_{i=1}^{4} Q_1^{(i)},$$

where

$$\begin{array}{rcl} Q_1^{(1)} &=& \hat\theta_{X,\hat\tau_X} \int_{L_k}^{-\log p_n} u^{\tau_X - 1}\phi(\sigma_n \xi_n \log u) du \\ Q_1^{(2)} &=& \hat\theta_{X,\hat\tau_X} \sigma_n \xi_n \int_{L_k}^{-\log p_n} u^{\tau_X - 1} \log u \; du \\ Q_1^{(3)} &=& -\hat\theta_{X,\hat\tau_X} \int_{L_k}^{\Lambda_k} u^{\tau_X - 1}\phi(\sigma_n \xi_n \log u) du \\ Q_1^{(4)} &=& -\hat\theta_{X,\hat\tau_X} \sigma_n \xi_n \int_{L_k}^{\Lambda_k} u^{\tau_X - 1} \log u \; du \end{array}$$

Now, there exists $\eta > 0$, such that $x < \log \eta$ implies that $|\phi(x)| < (\eta/2)x^2$. As a consequence, since $\sigma_n \log\log(1/p_n) \to 0$ and $\sigma_n \log L_k \to 0$ (according to (10) and (11)),

$$|Q_1^{(1)}| < \hat\theta_{X,\hat\tau_X} \frac{\eta}{2} \sigma_n^2 \xi_n^2 \int_{L_k}^{-\log p_n} u^{\tau_X - 1} (\log u)^2 \; du = \eta \, O_{\mathbb{P}}(1) \, \sigma_n^2 \tilde w_n.$$

Hence, via (10) and the previous approximations of w_n and $\tilde w_n$,

$$\sigma_n^{-1} w_n^{-1} |Q_1^{(1)}| < \eta \, O_{\mathbb{P}}(1) \, \sigma_n \tilde w_n / w_n = \eta \, O_{\mathbb{P}}(1) \, \sigma_n \log\log(1/p_n) \xrightarrow{\mathbb{P}} 0.$$

Concerning $Q_1^{(2)}$, we have

$$\sigma_n^{-1} w_n^{-1} Q_1^{(2)} = \hat\theta_{X,\hat\tau_X} \xi_n \xrightarrow{d} \mathcal{N}(0, \theta_X^2 \tau_X^2 \sigma^2).$$

Let us now consider $Q_1^{(3)}$. We proceed as for $Q_1^{(1)}$ to obtain

$$\begin{array}{rcl} \sigma_n^{-1} w_n^{-1} |Q_1^{(3)}| &<& \hat\theta_{X,\hat\tau_X} \frac{\eta}{2} \sigma_n \xi_n^2 \frac{\int_{L_k}^{\Lambda_k} u^{\tau_X - 1} (\log u)^2 \; du}{\int_{L_k}^{-\log p_n} u^{\tau_X - 1} \log u \; du} \\ &<& \hat\theta_{X,\hat\tau_X} \frac{\eta}{2} \sigma_n \max(\log \Lambda_k, \log L_k) \xi_n^2 \frac{\int_{L_k}^{\Lambda_k} u^{\tau_X - 1} \log u \; du}{\int_{L_k}^{-\log p_n} u^{\tau_X - 1} \log u \; du}. \end{array}$$

Since $\sigma_n \log \Lambda_k \xrightarrow{\mathbb{P}} 0$ (this is an easy consequence of assumption (11) and Lemma 7), the right hand-side tends to 0, according to Lemma 8 and assumption (11).

Concerning $Q_1^{(4)}$, Lemma 8 and assumption (11) entails that $\sigma_n^{-1} w_n^{-1} Q_1^{(4)}$ tends to 0. This completes the proof of (D.2).

- Let us prove that $\sigma_n^{-1} w_n^{-1} Q_2 = o_{\mathbb{P}}(1)$: according to Theorem 3,

$$Q_2 = \sigma_n (\log L_{nk}) K_{\tau_X}(-\log(p_n)) \delta_n,$$

where δ_n converges to a gaussian distribution. Hence,

$$\sigma_n^{-1} w_n^{-1} Q_2 = \frac{(\log L_{nk}) \; K_{\tau_X}(-\log(p_n))}{\int_{L_k}^{-\log(p_n)} u^{\tau_x - 1} \log u \; du} \delta_n,$$

and assumption (11) yields the result.

- In order to prove that $\sigma_n^{-1} w_n^{-1} Q_3 = o_{\mathbb{P}}(1)$, we obtain via a Taylor expansion that

$$\sigma_n^{-1} w_n^{-1} |Q_3| = \hat\theta_{X,\hat\tau_X} \sqrt{k} |\Lambda_k - \hat\Lambda_k| \left| K'_{\hat\tau_X}(\hat T_k) \frac{v_n}{w_n} \right|$$

where $\hat T_k$ is a value between Λ_k and $\hat\Lambda_k$. The fact that $\sqrt{k}|\Lambda_k - \hat\Lambda_k| = O_{\mathbb{P}}(1)$ (see Lemma 7 in Worms and Worms (2019)) and assumption (11) yields the result.

- Let us prove that $\sigma_n^{-1} w_n^{-1} Q_4 = o_{\mathbb{P}}(1)$: as above (see treatment of Q_2)

$$Q_4 = \sigma_n \log L_{nk} K_{\tau_X}(\Lambda_k) \delta_n,$$

where δ_n converges to a gaussian distribution. Moreover $K_{\tau_X}(\Lambda_k) \stackrel{d}{=} a K_{\tau_Z}(L_{nk})(1 + o_{\mathbb{P}}(1))$ (see Lemma

4 (i)). Hence

$$\sigma_n^{-1} w_n^{-1} Q_4 \stackrel{d}{=} a \frac{K_{\tau_Z}(L_{nk}) \ \log L_{nk}}{\int_{L_k}^{-\log(p_n)} u^{\tau_x - 1} \log u \ du} \ \delta_n(1 + o_{\mathbb{P}}(1)).$$

Assumption (11) yields the result.

- Let us finally prove that $\sigma_n^{-1} w_n^{-1} Q_5 = o_{\mathbb{P}}(1)$: remind that

$$\begin{aligned} Q_5 &= \log\left(\frac{\bar{l}_X(\exp(K_{\tau_X}(\Lambda_k)))}{\bar{l}_X(\exp(K_{\tau_X}(-\log(p_n))))}\right) \\ &= \log\left(\frac{\bar{l}_X(\exp(K_{\tau_X}(L_k)))}{\bar{l}_X(\exp(K_{\tau_X}(-\log(p_n))))}\right) + \log\left(\frac{\bar{l}_X(\exp(K_{\tau_X}(\Lambda_k)))}{\bar{l}_X(\exp(K_{\tau_X}(L_k)))}\right) \\ &= Q_5^{(1)} + Q_5^{(2)}. \end{aligned}$$

Concerning $Q_5^{(1)}$, we know that $\bar{l}_X$ satisfies the SR2 condition (see Remark 8). Hence

$$\begin{aligned} -Q_5^{(1)} &= \log\left(\frac{\bar{l}_X(\lambda_n x_n)}{\bar{l}_X(x_n)}\right) \\ &= \bar{b}_X(x_n) K_{\theta_X \rho_X}(\lambda_n)(1 + o_{\mathbb{P}}(1)), \end{aligned}$$

where $|\bar{b}_X| \in RV_{\theta_X \rho_X}$, $\lambda_n = \frac{\exp(K_{\tau_X}(-\log(p_n)))}{\exp(K_{\tau_X}(L_k))}$ and $x_n = \exp(K_{\tau_X}(L_k))$. Moreover, since λ_n tends to $+\infty$, as n tends to infinity (because $\frac{K_{\tau_X}(L_k)}{K_{\tau_X}(-\log(p_n))}$ tends to 0 under assumption (11)), we obtain that $K_{\theta_X \rho_X}(\lambda_n)$ tends to $-1/(\theta_X \rho_X)$. Moreover, $\sqrt{k} v_n \bar{b}_X(\exp(K_{\tau_X}(L_k)))$ tends to 0 under the appropriate assumption among $H_2, \ldots, H_5$. Hence,

$$\sqrt{k} v_n \frac{Q_5^{(1)}}{K_{\tau_X}(-\log(p_n))} = \sqrt{k} v_n \bar{b}_X(\exp(K_{\tau_X}(L_k))) \frac{K_{\rho_X}(\lambda_n)}{K_{\tau_X}(-\log(p_n))},$$

tends to 0. Then,

$$\sigma_n^{-1} w_n^{-1} Q_5^{(1)} = \sqrt{k} v_n \frac{Q_5^{(1)}}{K_{\tau_X}(-\log(p_n))} \frac{K_{\tau_X}(-\log p_n)}{\int_{L_k}^{-\log(p_n)} u^{\tau_x - 1} \log u \ du},$$

which tends to 0 thanks to $(D.1)$.

Similarly, we have

$$\begin{aligned} Q_5^{(2)} &= \log\left(\frac{\bar{l}_X(\lambda_n x_n)}{\bar{l}_X(x_n)}\right) \\ &= \bar{b}_X(x_n) K_{\theta_X \rho_X}(\lambda_n)(1 + o_{\mathbb{P}}(1)), \end{aligned}$$

where $x_n = \exp(K_{\tau_X}(L_k))$ and

$$\lambda_n = \frac{\exp(K_{\tau_X}(\Lambda_k))}{\exp(K_{\tau_X}(L_k))} = \exp(\tau_X^{-1}(\Lambda_k^{\tau_X} - L_k^{\tau_X})) = \exp(cst.L_{nk}^{\tau_Z - \alpha}(1 + o(1))),$$

where, according to Lemma 4, the constant above is negative and

$$\alpha = \begin{cases} \tau_Z & \text{when either } \tau_X = \tau_C, \text{ or } \tau_X \neq \tau_C \text{ and } r \leqslant 0, \\ \tau_Z(1-r) & \text{when } \tau_X \neq \tau_C \text{ and } r \in]0,1[. \end{cases}$$

In the case where $\alpha = \tau_Z$, $K_{\theta_X \rho_X}(\lambda_n)$ converges to a constant. Hence we obtain, for the term $Q_5^{(1)}$, that

$$\sqrt{k} v_n \frac{Q_5^{(2)}}{K_{\tau_X}(-\log(p_n))} \xrightarrow{\mathbb{P}} 0.$$

Therefore $\sigma_n^{-1} w_n^{-1} Q_5^{(2)} \xrightarrow{\mathbb{P}} 0$, thanks to $(D.1)$.

In the case where $\alpha = \tau_Z(1-r)$, we have $K_{\theta_X \rho_X}(\lambda_n) = O(1)\exp(cst.L_{nk}^{r\tau_Z}(1 + o(1)))$, where here the constant is positive. Moreover, for some small $\delta > 0$,

$$\bar{b}_X(x_n) = \exp((\theta_X \rho_X + \delta) K_{\tau_X}(L_k)) o(1) = \exp\left((\theta_X \rho_X + \delta).cst.L_{nk}^{\tau_Z}(1 + o(1))\right) o(1),$$

where the constant above is positive. Consequently, $\sqrt{k} v_n \bar{b}_X(x_n) K_{\theta_X \rho_X}(\lambda_n)$ tends to 0 according to the appropriate assumption among $H_2, \ldots, H_5$. To conclude, we proceed as in the previous case. □

Appendix E. Technical aspects

Appendix E.1. Details about second order conditions and censoring probabilities

Remind that

$$\bar{F}_X(x) = \exp(-K^-_{\tau_X}(\log(H_X(x)))) \quad \text{and} \quad \bar{F}_C(x) = \exp(-K^-_{\tau_C}(\log(H_C(x))))$$

where

$$H^-_X(x) = x^{\theta_X}\bar{l}_X(x)\ ,\ H^-_C(x) = x^{\theta_C}\bar{l}_C(x)\ ,\ H_X(x) = x^{1/\theta_X}l_X(x)\ ,\ H_C(x) = x^{1/\theta_C}l_C(x).$$

Moreover (see Proposition 1),

$$\mathbb{P}(Z > x) = \exp(-K^-_{\tau_Z}(\log(H_Z(x)))),$$

where $H^-_Z(x) = x^{\theta_Z}\tilde{l}(x)$ and $\tilde{l}$ is slowly varying. This implies that $H_X \circ H^-_Z(x) = x^a l(x)$, with l a slowly varying function and $a = \theta_Z/\theta_X$.

Lemma 1 stated below provides details about the second order properties of the functions H^-_Z and $H_X \circ H^-_Z$ (and therefore, on the behavior of the variables Z_i and $\Lambda_X(Z_i)$). These properties not only depend on the position of the parameters τ_X and τ_C with respect to each other, but on their proximity through the parameter r defined by

$$r := 1 - \left|\frac{1}{\tau_C} - \frac{1}{\tau_X}\right| \in [-\infty, 1]$$

(if either $\tau_X = 0$ or $\tau_C = 0$, indeed consider that $r = -\infty$). The proof of the lemma is based on Theorem B.2.2 in de Haan and Ferreira (2006) as well as the concept of de Bruyn conjugate (see Proposition 2.5 in Beirlant et al. (2004)). Its demonstration is tedious, details are omitted for brevity.

Lemma 1. *Let conditions (A_1) and (A_2) hold.*

(i) For different slowly varying functions generically noted v, we have

$$\begin{aligned} l_X(x) &= c_X(1 - x^{\rho_X}v(x)) &\text{and}\quad l_C(x) &= c_C(1 - x^{\rho_C}v(x)) \\ \bar{l}_X(x) &= c_X^{-\theta_X}(1 - x^{\theta_X\rho_X}v(x)) &\text{and}\quad \bar{l}_C(x) &= c_C^{-\theta_C}(1 - x^{\theta_C\rho_C}v(x)). \end{aligned}$$

(ii) The slowly varying functions $\tilde{l}$ and l associated to H^-_Z and $H_X \circ H^-_Z$ satisfy a second order condition SR2 : as $t \to +\infty$,

$$\frac{\frac{\tilde{l}(tx)}{\tilde{l}(t)} - 1}{\tilde{b}(t)} \longrightarrow K_{\tilde{\rho}}(x) \quad \text{and} \quad \frac{\frac{l(tx)}{l(t)} - 1}{b(t)} \longrightarrow K_\rho(x)$$

where

$$\tilde{\rho} = \rho = \begin{cases} \max(\theta_X\rho_X, -1) & \text{if } 0 = \tau_X < \tau_C < 1 \\ \max(\theta_C\rho_C, -1) & \text{if } 0 = \tau_C < \tau_X < 1 \\ 0 & \text{in the other cases ,} \end{cases}$$

and $|\tilde{b}| \in RV_{\tilde{\rho}}$ and $|b| \in RV_\rho$. When $\rho = 0$, both $b(t)$ and $\tilde{b}(t)$ are (as $t \to +\infty$) of the order $O((\log t)^{r-1})$ when $r \neq 0$, and of the order $O((\log t)^{-2})$ when $r = 0$.

(iii) The function H_Z satisfies

$$\lim_{x\to+\infty} H_Z(x) = c_Z \begin{cases} \in]0, +\infty[& \text{if } \tau_X = \tau_C \text{ or } r \leqslant 0, \\ = +\infty & \text{if } \tau_X \neq \tau_C \text{ and } r \in]0, 1[\end{cases}$$

where in particular $c_Z = c_X$ if $\tau_X < \tau_C$ and $r < 0$, and $c_Z = c_C$ if $\tau_C < \tau_X$ and $r < 0$. Moreover we have (with the convention $(+\infty)^{-\theta} = 0$ when $\theta > 0$)

$$\tilde{l}(t) \to \tilde{c} := c_Z^{-\theta_Z} \quad \text{and} \quad l(t) \to c := c_X\tilde{c}^{1/\theta_X}, \quad \text{as } t \to +\infty.$$

When $\tau_X = \tau_C$ or $r \leqslant 0$, both c and $\tilde{c}$ are positive and the following relations hold:

$$\tilde{l}(t) = \tilde{c}(1 - x^{\tilde{\rho}}v(x)) \quad \text{and} \quad l(t) = c(1 - x^\rho v(x)),$$

where v is a generic slowly varying function.
When $\tau_X \neq \tau_C$ and $r \in]0, 1[$, both $\tilde{c}$ and c are zero and the following relation holds for some $\nu > 0$, as $x \to \infty$

$$\frac{\log l(\exp x)}{x} = -\nu.x^{r-1}(1 + o(1)) \longrightarrow 0 \quad \text{and} \quad \frac{\log \tilde{l}(\exp x)}{x} = -\theta_X\nu.x^{r-1}(1 + o(1)) \longrightarrow 0 \qquad \text{(E.1)}$$

Remark 8. *A consequence of this Lemma is that* $\bar{l}_X$ *and* $\bar{l}_C$ *also satisfy the SR2 condition with rate functions* $|\bar{b}_X| \in RV_{\theta_X \rho_X}$ *and* $|\bar{b}_C| \in RV_{\theta_C \rho_C}$ *respectively.*

Remind now that the function $p(\cdot)$ is defined by

$$p(x) = \mathbb{P}(\delta = 1 | Z = x).$$

The following lemma provides useful developments of functions $p(\cdot)$ and $r(\cdot)$

$$r(t) = p \circ H_Z^{-}(\exp(K_{\tau_Z}(-\log t))),$$

which are crucial to derive the properties of the random proportion $\hat{p}_k$ (and therefore the statements of Proposition 2). Its proof is based on the fact that

$$p(x) = \frac{\bar{F}_C(x) f_X(x)}{\bar{F}_C(x) f_C(x) + \bar{F}_X(x) G(x)} = \left(1 + \frac{(K_{\tau_C}^{-})'(\log H_C(x))}{(K_{\tau_X}^{-})'(\log H_X(x))} \frac{H_C'(x)/H_C(x)}{H_X'(x)/H_X(x)}\right)^{-1}$$

(where f_X and f_C are the respective probability density functions of X and C), as well as on the results of Lemma 1. It is omitted for brevity.

Lemma 2. *Let us define the constants*

$$A_X = \theta_X(\tau_X^{-1} - 1)(\tau_X^{-1} + \log c_X)\ ,\ A_C = \theta_C(\tau_C^{-1} - 1)(\tau_C^{-1} + \log c_C)$$

and

$$A = A_C - A_X \quad \text{and} \quad B = \frac{\theta_X}{\theta_C}\left(\frac{\tau_X}{\theta_X}\right)^{1-1/\tau_X}\left(\frac{\tau_C}{\theta_C}\right)^{1/\tau_C - 1}.$$

Let assumptions (A_1) *and* (A_2) *hold (the asymptotics below are* $x \to +\infty$ *and* $t \downarrow 0$*).*

(i) We have

$$p(x) \to p := \begin{cases} 1 & \text{if } 0 \leqslant \tau_X < \tau_C \leqslant 1, \\ 0 & \text{if } 0 \leqslant \tau_C < \tau_X \leqslant 1, \\ \dfrac{\theta_X^{1/\tau_X}}{(\theta_X^{1/\tau_X} + \theta_C^{1/\tau_X})} = a^{1/\tau_X} & \text{if } 0 < \tau_X = \tau_C < 1, \end{cases}$$

and, more precisely,

$$p(x) - p = \begin{cases} D\,(\log x)^{r-1}\left[1 + g(r)(\log x)^{\max(-1,r-1)}(1+o(1))\right] & \text{if } 0 < \tau_X \neq \tau_C \leqslant 1, \\ D\,x^{-1/\theta_X}(\log x)^{\tau_C^{-1}-1}\left[1 + A_C(\log x)^{-1}(1+o(1))\right] & \text{if } 0 = \tau_X < \tau_C \leqslant 1, \\ D\,x^{-1/\theta_C}(\log x)^{\tau_X^{-1}-1}\left[1 + A_X(\log x)^{-1}(1+o(1))\right] & \text{if } 0 = \tau_C < \tau_X \leqslant 1, \\ D\,(\log x)^{-1}(1 + O(1/\log x)) & \text{if } 0 < \tau_C = \tau_X < 1, \end{cases}$$

where

$$D = \begin{cases} -B & \text{if } 0 < \tau_X < \tau_C \leqslant 1, \\ B^{-1} & \text{if } 0 < \tau_C < \tau_X \leqslant 1, \\ -(\tau_C/\theta_C)^{\tau_C^{-1}-1}(\theta_X/\theta_C c_X) & \text{if } 0 = \tau_X < \tau_C \leqslant 1, \\ (\tau_X/\theta_X)^{\tau_X^{-1}-1}(\theta_C/\theta_X c_C) & \text{if } 0 = \tau_C < \tau_X \leqslant 1, \\ -AB(1+B)^{-2} & \text{if } 0 < \tau_C = \tau_X < 1, \end{cases}$$

and

$$g(r) = \begin{cases} A\mathbb{I}_{r<0} + (A - B)\mathbb{I}_{r=0} + (-B)\mathbb{I}_{r\in]0,1[} & \text{if } 0 < \tau_X < \tau_C \leqslant 1, \\ (-A)\mathbb{I}_{r<0} + (-A - B^{-1})\mathbb{I}_{r=0} + (-B^{-1})\mathbb{I}_{r\in]0,1[} & \text{if } 0 < \tau_C < \tau_X \leqslant 1. \end{cases}$$

(ii) When $\tau_Z > 0$ *and* $\tau_X \neq \tau_C$*, as* $t \downarrow 0$ *we have*

$$r(t) - p = D(\theta_Z/\tau_Z)^{r-1}(-\log t)^{-\tau_Z(1-r)}\left(1 + O\left((-\log t)^{-\tau_Z \min\{1,1-r\}}\right)\right),$$

in particular, when $0 < \tau_C < \tau_X \leqslant 1$*,*

$$r(t) = a^{1/\tau_X}(\tau_X/\tau_C)^{\tau_X^{-1}-1}(-\log t)^{\frac{\tau_C}{\tau_X}-1}\left(1 + O\left((-\log t)^{\max\{-\tau_C,\tau_C/\tau_X - 1\}}\right)\right).$$

When $\tau_Z > 0$ *and* $\tau_X = \tau_C$*, we have*

$$r(t) - p = -AB\left[(1+B)^2(\theta_Z/\tau_Z)\right]^{-1}(-\log t)^{-\tau_Z}\left(1 + O\left((-\log t)^{-\tau_Z}\right)\right).$$

When $\tau_Z = 0$, if $\tau_+ = \max(\tau_X, \tau_C)$ we have

$$r(t) - p = cst(-\log t)^{-1}(\log\log(1/t))^{\frac{1}{\tau_+}-1}\left(1 + O\left((\log\log(1/t))^{-1}\right)\right).$$

with the constant being equal to $\tau_X^{\frac{1}{\tau_X}-1} a^{1/\tau_X}$ when $0 = \tau_C < \tau_X \leqslant 1$.

Appendix E.2. Proof of Proposition 2

The function $p(\cdot)$ being defined in the previous subsection, and proceeding as in Einmahl et al. (2008), we carry on the proof by considering now that δ_i is related to Z_i by

$$\delta_i = \mathbb{I}_{U_i \leqslant p(Z_i)},$$

where $(U_i)_{i\leqslant n}$ denotes an independent sequence of standard uniform variables, independent of the sequence $(Z_i)_{i\leqslant n}$. We denote by $U_{[1,n]}, \ldots, U_{[n,n]}$ the (unordered) values of the uniform sample pertaining to the order statistics $Z_{1,n} \leqslant \ldots \leqslant Z_{n,n}$ of the observed sample $Z_1, \ldots, Z_n$.

Remind that $Z_i = H_Z^-(\exp(K_{\tau_Z}(E_i)))$, where $E_1, \ldots, E_n$ are independent standard exponential random variables (Proposition 1). We introduce, for every $1 \leqslant i \leqslant n$, the standard uniform random variables $V_i = 1 - \exp(-E_i)$ such that

$$Z_i = H_Z^-(\exp(K_{\tau_Z}(-\log(1 - V_i)))) = r(1 - V_i)$$

where the function $r(\cdot)$ was defined before the statement of Lemma 2, which provides valuable information about it. Let us provide a detailed proof of Proposition 2 in the case $0 < \tau_C < \tau_X \leqslant 1$ (the non-Weibull-tail strong censoring case) ; all the other cases are treated similarly. We start by writing

$$\begin{aligned}
\sqrt{k}v_n A_{2,n} &= \sqrt{k}v_n \left(\left(\tfrac{a\tau_X}{\tau_C}\right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X}\,\hat{p}_k - a\right)\\
&= \sqrt{k}v_n \left(\frac{a\tau_X}{\tau_C}\right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X}\,\frac{1}{k}\sum_{j=1}^{k}\left(\mathbb{I}_{U_{[n-j+1,n]}\leqslant r(1-V_{n-j+1,n})} - \mathbb{I}_{U_{[n-j+1,n]}\leqslant r(j/n)}\right)\\
&\quad + \sqrt{k}v_n \frac{1}{k}\sum_{j=1}^{k}\left(\left(\frac{a\tau_X}{\tau_C}\right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X}\,\mathbb{I}_{U_{[n-j+1,n]}\leqslant r(j/n)} - a\right)\\
&=: T_{1,k} + T_{2,k}.
\end{aligned}$$

We will prove below that the term $T_{1,k}$ above converges to 0 in probability. Let us, first, treat the term $T_{2,k}$. We write

$$\begin{aligned}
T_{2,k} &= \tfrac{1}{\sqrt{k}}v_n \left(\tfrac{a\tau_X}{\tau_C}\right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X}\sum_{j=1}^{k}\left(\mathbb{I}_{U_{n-j+1,n}\leqslant r(j/n)} - r(j/n)\right)\\
&\quad + \tfrac{1}{\sqrt{k}}v_n\sum_{j=1}^{k}\left(\left(\tfrac{a\tau_X}{\tau_C}\right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X}\,r(j/n) - a\right)\\
&=: T'_{2,k} + T''_{2,k},
\end{aligned}$$

Let us prove that $T'_{2,k} \xrightarrow{d} N(0, D)$ where $D = a^{2-1/\tau_X}\left(\frac{\tau_X}{\tau_C}\right)^{1-1/\tau_X}$, while $T''_{2,k} \xrightarrow{\mathbb{P}} 0$.

We deduce from Lemma 2 that

$$r(t) = a^{1/\tau_X}\left(\frac{\tau_X}{\tau_C}\right)^{1/\tau_X-1}(-\log t)^{\tau_C/\tau_X-1}(1 + o(1)) \to 0.$$

Hence,

$$\begin{aligned}
\mathbb{V}(T'_{2,k}) &= v_n^2\left(\tfrac{a\tau_X}{\tau_C}\right)^{2-2/\tau_X}(L_{nk})^{2-2\tau_C/\tau_X}\,\tfrac{1}{k}\sum_{j=1}^{k} r(j/n)(1 - r(j/n))\\
&= v_n^2 D\,(L_{nk})^{1-\tau_C/\tau_X}(1 + o(1))\tfrac{1}{k}\sum_{j=1}^{k}\left(\tfrac{L_{nj}}{L_{nk}}\right)^{\tau_C/\tau_X-1},
\end{aligned}$$

denoting $L_{nj} = \log(n/j)$. We have $\frac{1}{k}\sum_{j=1}^{k}\left(\frac{L_{nj}}{L_{nk}}\right)^{\tau_C/\tau_X-1}$ converges to 1, because $\frac{L_{nj}}{L_{nk}}$ converges uniformly to 1. Consequently,

$$\mathbb{V}(T'_{2,k}) = Dv_n^2\,(L_{nk})^{1-\tau_C/\tau_X}(1 + o(1)) \to D.$$

We conclude, for this term, using Lyapunov's Theorem (details are omitted).

Concerning $T''_{2,k}$, we see that $\left(\frac{a\tau_X}{\tau_C}\right)^{1-1/\tau_X}(L_{nk})^{1-\tau_C/\tau_X}\, r(j/n) = a+o(1)$. Hence, we need a second order development for $r(j/n)$. According to Lemma 2 (part (ii)), we have

$$\left(\frac{a\tau_X}{\tau_C}\right)^{1-\frac{1}{\tau_X}} L_{nk}^{1-\tau_C/\tau_X} r(j/n) - a = a\left(\left(\frac{L_{nj}}{L_{nk}}\right)^{\frac{\tau_C}{\tau_X}-1} - 1\right) + O(1)L_{nk}^{-\alpha}\left(\frac{L_{nj}}{L_{nk}}\right)^{-\alpha}.$$

where $\alpha = \max\{-\tau_C, \tau_C/\tau_X - 1\}$. Hence,

$$T''_{2,k} = a\sqrt{k}v_n\left(\frac{\tau_C}{\tau_X} - 1\right)L_{nk}^{-1}(1+o(1))\frac{1}{k}\sum_{j=1}^{k}\log(k/j) + O(1)\sqrt{k}v_n L_{nk}^{-\alpha}(1+o(1))\frac{1}{k}\sum_{j=1}^{k}\left(\frac{L_{nj}}{L_{nk}}\right)^{-\alpha}.$$

But $\frac{1}{k}\sum_{j=1}^{k}\log(k/j)$ and $\frac{1}{k}\sum_{j=1}^{k}\left(\frac{L_{nj}}{L_{nk}}\right)^{-\alpha}$ both tend to 1 . Hence, according to assumption H_3 ((ii) or (iii), depending on the closeness of τ_X w.r.t. τ_C), $T''_{2,k}$ indeed tends to 0. This concludes the proof for $T_{2,k}$.

It remains to prove that $T_{1,k}$ above converges to 0 in probability. Following the same lines as in the proof of Lemma 2 (Subsection C.3) in Worms and Worms (2019), it turns out that this amounts to proving that, for some positive sequence $s_n = k^{-\delta}/n$ ($\delta > 0$) and some constant $c > 0$,

$$\sqrt{k}v_n S_{n,k} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{where} \quad S_{n,k} := \sup\left\{ |r(s) - r(t)| \; ; \; \frac{1}{n} \leqslant t \leqslant \frac{k}{n} \; , \; |s-t| \leqslant c\sqrt{k}/n \; , \; s \geqslant s_n \right\}. \tag{E.2}$$

In the case considered here, $0 < \tau_C < \tau_X \leqslant 1$, $r(t) = cst(-\log t)^{\tau_C/\tau_X - 1}v(-\log t)$, where v is a slowly varying function such that $v(-\log t)$ tends to 1 when $t \to 0$. Let $h(t) = (-\log t)^{\tau_C/\tau_X-1}$. Applying the mean value theorem, we obtain

$$\begin{aligned} |r(t) - r(s)| &\leqslant cst|t-s|\sup_{u\in[s,t]}\left|h'(u)v(-\log u)\left(1 + \tfrac{(-\log u)v'(-\log u)}{v(-\log u)}\right)\right| \\ &\leqslant cst|t-s|\sup_{u\in[s,t]}|h'(u)|, \end{aligned}$$

since $\frac{tv'(t)}{v(t)}$ tends to 1, as t tends to infinity. This entails that

$$S_{n,k} \leqslant cst\ k^{1/2+\delta}L_{nk}^{\tau_C/\tau_X-2}.$$

Remind that in this case $v_n = L_{nk}^{\frac{1}{2}(\tau_C/\tau_X-1)}$. Hence

$$\sqrt{k}v_n S_{n,k} \leqslant cst\left(\sqrt{k}L_{nk}^{\alpha+\delta'}\right)^{2(1+\delta)},$$

for some $\delta' > 0$ and $\alpha = \frac{3}{4}(\tau_C/\tau_X - 1) - \frac{1}{2}$. We easily prove that, if we choose $0 < \delta' < \frac{1}{2}$, $\sqrt{k}L_{nk}^{\alpha+\delta'} \to 0$, under assumption $H_3(ii)$ or $H_3(iii)$. □

Appendix E.3. Proof of Proposition 3

The proofs for the terms $T_{1,n}, \dots, T_{5,n}$ are respectively detailed in parts (1),…,(5) of this Section.

(1) Remind that $T_{1,n} = \frac{R_{n,\tilde{l}}}{D_{k,\tau_X}}$, where $R_{n,\tilde{l}} = \frac{1}{k}\sum_{j=1}^{k}\log\left(\frac{\tilde{l}(\exp(K_{\tau_Z}(E_{n-i+1,n})))}{\tilde{l}(\exp(K_{\tau_Z}(E_{n-k,n})))}\right)$. According to assumption $R_{\tilde{l}}(\tilde{b},\tilde{\rho})$, we have $\log\left(\frac{\tilde{l}(tx)}{\tilde{l}(t)}\right) \sim \tilde{b}(t)K_{\tilde{\rho}}(x)$, uniformly for $x \geqslant 1$, as $t \to +\infty$. The Renyi representation yields that $E_{n-i+1,n} - E_{n-k} \overset{d}{=} F_{k-i+1,k}$, where $F_1, \dots, F_k$ are k independent standard exponential random variables. Consequently, taking $t = \exp(K_{\tau_Z}(E_{n-k,n})) \to +\infty$ and $x = \exp(K_{\tau_Z}(E_{n-i+1,n}) - K_{\tau_Z}(E_{n-k,n})) \geqslant 1$, we obtain

$$R_{n,\tilde{l}} \overset{d}{=} \tilde{b}(\exp(K_{\tau_Z}(E_{n-k,n})))(1+o_{\mathbb{P}}(1))\frac{1}{k}\sum_{j=1}^{k}K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_i + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n}))).$$

But $\sqrt{k}v_n\tilde{b}(\exp(K_{\tau_Z}(E_{n-k,n})))$ tends to 0, under conditions H_2-H_6. Since $\frac{\mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})}$ tends to 1 (thanks to $(A.3)$), Corollary 1 yields that $\sigma_{1,\tau_Z}(E_{n-k})/D_{k,\tau_X} \overset{\mathbb{P}}{\longrightarrow} 1/a$. It thus remains to prove that

$$\frac{\frac{1}{k}\sum_{j=1}^{k}K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_i + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n})))}{\sigma_{1,\tau_Z}(E_{n-k})}$$

is bounded in probability. In the cases where $\tilde{\rho}$ is equal to 0,

$$\frac{1}{k}\sum_{j=1}^{k} K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_i+E_{n-k,n})-K_{\tau_Z}(E_{n-k,n}))) = \frac{1}{k}\sum_{j=1}^{k}(K_{\tau_Z}(F_i+E_{n-k,n})-K_{\tau_Z}(E_{n-k,n})) = \theta_{n,1}(E_{n-k}),$$

and $\frac{\theta_{n,1}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} 1$ (see $(A.4)$). In the cases where $\tilde{\rho} < 0$, we use the fact that $|K_{\tilde{\rho}}(e^u) - u| \leqslant |\tilde{\rho}|\frac{u^2}{2}$, and we easily prove that

$$\frac{\frac{1}{k}\sum_{j=1}^{k}\left(K_{\tau_Z}(F_i + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n})\right)^2}{\sigma_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} 0.$$

This concludes the proof for $T_{1,n}$.

(2) Remind that $T_{2,n} = -\theta_X \frac{R_{1,n}}{D_{k,\tau_X}}$, where $R_{1,n}$ is defined in Lemma 3 and we have (also in Lemma 3)

$$D_{k,\tau_X} = \hat{\Lambda}_k^{\tau_X - 1}\hat{p}_k + R_{1,n}.$$

It suffices to prove that $\sqrt{k}v_n \frac{R_{1,n}}{\hat{\Lambda}_k^{\tau_X-1}\hat{p}_k} \xrightarrow{\mathbb{P}} 0$. Let us consider the case where $\tau_X \neq 0$ and $\tau_C \neq 0$, and introduce the notations

$$\Lambda_j := \Lambda_X(Z_{n-j+1,n}) \quad \text{and} \quad \hat{\Lambda}_j := \hat{\Lambda}_{nX}(Z_{n-j+1,n}).$$

In this case (except when $\tau_X = 1$, since in that case $R_{1,n} = 0$),

$$R_{1,n} = \frac{\tau_X - 1}{2}\hat{\Lambda}_k^{\tau_X}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}\right)^2(1 + T_{j,k})^{\tau_X - 2},$$

with $\hat{\Delta}_{j,k} = \hat{\Lambda}_j - \hat{\Lambda}_k$ and $T_{j,k} \in]0, \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}[$. Since $\tau_X - 2 < 0$, we are led to prove that

$$\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}\right)^2 \xrightarrow{\mathbb{P}} 0,$$

and, introducing

$$\xi_{j,k} := \frac{\hat{\Lambda}_j}{\Lambda_j}\frac{\Lambda_k}{\hat{\Lambda}_k} - 1 \quad \text{and} \quad d_{j,k} := \frac{\Lambda_j}{\Lambda_k} - 1,$$

we have $(\hat{\Delta}_{j,k}/\hat{\Lambda}_k)^2 = (\frac{\Lambda_j}{\Lambda_k}\xi_{j,k} + d_{j,k})^2 \leqslant 2((\Lambda_j/\Lambda_k)^2\,\xi_{j,k}^2 + d_{j,k}^2)$. We thus need to prove that

$$\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\Lambda_j}{\Lambda_k}\right)^2\xi_{j,k}^2 \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^{k}d_{j,k}^2 \xrightarrow{\mathbb{P}} 0. \tag{E.3}$$

Let $E_1, \ldots E_n$ be i.i.d. standard exponential random variables. We have (see Lemma 4 (i))

$$\frac{\Lambda_j}{\Lambda_k} - 1 \stackrel{d}{=} (1 + x_{j,k})^{1/\tau_X} - 1,$$

where

$$\begin{aligned} x_{j,k} &= \frac{\tau_X aK_{\tau_Z}(E_{n-j+1,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-j+1,n}))) + 1}{\tau_X aK_{\tau_Z}(E_{n-k,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n}))) + 1} - 1 \\ &= (1 + o_{\mathbb{P}}(1))(A_{j,k} + B_{j,k}), \end{aligned}$$

with

$$A_{j,k} = 1 - \frac{K_{\tau_Z}(E_{n-j+1,n})}{K_{\tau_Z}(E_{n-k,n})} \text{ and } B_{j,k} = \frac{1}{aK_{\tau_Z}(E_{n-j+1,n})}\log\left(\frac{l(\exp(K_{\tau_Z}(E_{n-j+1,n})))}{l(\exp(K_{\tau_Z}(E_{n-j+1,n})))}\right).$$

Hence, $d_{j,k} = \tau_X^{-1}(A_{j,k} + B_{j,k})(1 + o_{\mathbb{P}}(1))$. Moreover, the Renyi representation yields that $E_{n-i+1,n} - E_{n-k,n} \stackrel{d}{=} F_{k-i+1,k}$, where $F_1, \ldots, F_k$ are k independent standard exponential random variables. Consequently,

$$\begin{aligned} A_{j,k} &= 1 - \frac{E_{n-j+1,n}^{\tau_Z} - 1}{E_{n-k,n}^{\tau_Z} - 1} \\ &\stackrel{d}{=} -\tau_Z\frac{F_{k-j+1,k}}{E_{n-k,n}}(1 + o_{\mathbb{P}}(1)). \end{aligned}$$

Concerning $B_{j,k}$, we use the second order condition $R_l(b,\rho)$ for l to write

$$B_{j,k} = \frac{b(\exp(K_{\tau_Z}(E_{n-k,n})))}{aK_{\tau_Z}(E_{n-k,n}))} K_\rho\left(\exp(K_{\tau_Z}(E_{n-j+1,n}) - K_{\tau_Z}(E_{n-k,n}))\right)(1+o_{\mathbb{P}}(1)).$$

Since $(A_{j,k}+B_{j,k})^2 \leqslant 2(A_{j,k}^2+B_{j,k}^2)$, we only have to prove that $\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k A_{j,k}^2 \xrightarrow{\mathbb{P}} 0$ and $\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k B_{j,k}^2 \xrightarrow{\mathbb{P}} 0$. Moreover $\Lambda_k \overset{d}{=} \left(\frac{a\tau_X}{\tau_Z}\right)^{1/\tau_X}(E_{n-k,n})^{\tau_Z/\tau_X}(1+o_{\mathbb{P}}(1))$, where $\frac{E_{n-k,n}}{L_{nk}} \xrightarrow{\mathbb{P}} 1$ and $\frac{\hat{\Lambda}_k}{\Lambda_k} \xrightarrow{\mathbb{P}} 1$. Hence

$$\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k A_{j,k}^2 \quad\overset{d}{=}\quad cste(1+o_{\mathbb{P}}(1))\sqrt{k}v_n\frac{L_{nk}^{\tau_Z/\tau_X-2}}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k F_j^2.$$

But $\left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}}(L_{nk})^{1-\tau_Z/\tau_X}\hat{p}_k \xrightarrow{\mathbb{P}} a$, according to Proposition 2. Consequently, $\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k A_{j,k}^2 \overset{d}{=} O_{\mathbb{P}}(1)\sqrt{k}v_nL_{nk}^{-1}$, which, using assumptions $H_2,\dots H_4$, goes to 0 in probability.
Now, according to Lemma 5 in Gardes et al. (2011), we have

$$\frac{1}{\mu_{2,\tau_Z}(E_{n-k})}\frac{1}{k}\sum_{j=1}^k K_\rho^2\left(\exp(K_{\tau_Z}(E_{n-j+1,n}) - K_{\tau_Z}(E_{n-k,n}))\right) \xrightarrow{\mathbb{P}} cst.$$

Hence,

$$\begin{aligned}\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k B_{j,k}^2 \quad&\overset{d}{=}\quad cst(1+o_{\mathbb{P}}(1))\sqrt{k}v_n\frac{L_{nk}^{\tau_Z/\tau_X}}{\hat{p}_k}\left(\frac{b(\exp(K_{\tau_Z}(E_{n-k,n})))}{aK_{\tau_Z}(E_{n-k,n}))}\right)^2\mu_{2,\tau_Z}(E_{n-k})\\ &\overset{d}{=}\quad cst(1+o_{\mathbb{P}}(1))\sqrt{k}v_nL_{nk}^{-1}b^2(\exp(K_{\tau_Z}(E_{n-k,n}))),\end{aligned}$$

since $\mu_{2,\tau_Z}(E_{n-k}) \sim 2L_{nk}^{2(\tau_Z-1)}$, according to Lemma 2 in Gardes et al. (2011). The second part of relation (E.3) is thus proved.

Let us now deal with the first part of relation (E.3). We have

$$\xi_{j,k} = \frac{\hat{\Lambda}_j}{\Lambda_j}\frac{\Lambda_k}{\hat{\Lambda}_k} - 1 = \left(\frac{\Lambda_k}{\hat{\Lambda}_k}\right)\left(\Delta_j\frac{\Lambda_k}{\Lambda_j} - \Delta_{k+1}\right)\Lambda_k^{-1},$$

where $\Delta_j := \hat{\Lambda}_j - \Lambda_j$ and $\Delta_{k+1} := \hat{\Lambda}_k - \Lambda_k$. Lemmas 6 and 7 in Worms and Worms (2019) ensure that $|\Delta_j| = O_{\mathbb{P}}(1/\sqrt{j-1})$ for all $j = 2,\dots,k+1$, $|\Delta_1| = O_{\mathbb{P}}(1)$ and $\frac{E_{n-k,n}}{L_{nk}} \xrightarrow{\mathbb{P}} 1$. Since in addition both $\frac{\Lambda_k}{\hat{\Lambda}_k}$ and $\frac{\Lambda_k}{\Lambda_j}$ tend to 1 in probability, and the latter is $\leqslant 1$, we thus obtain $|\xi_{1,n}| \leqslant (1+o_{\mathbb{P}}(1))\left(O_{\mathbb{P}}(1)+O_{\mathbb{P}}(1/\sqrt{k})\right)\Lambda_k^{-1}$ and

$$|\xi_{j,n}| \leqslant (1+o_{\mathbb{P}}(1))\left(O_{\mathbb{P}}(1/\sqrt{j-1})+O_{\mathbb{P}}(1/\sqrt{k})\right)\Lambda_k^{-1},\ \text{for } j=2,\dots,k.$$

Therefore,

$$\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k\left(\frac{\Lambda_j}{\Lambda_k}\right)^2\xi_{j,k}^2 \leqslant (1+o_{\mathbb{P}}(1))\frac{v_n}{\sqrt{k}}(\Lambda_k\hat{p}_k)^{-1}\left(O_{\mathbb{P}}(1)+\sum_{j=2}^k O_{\mathbb{P}}(1/(j-1))\right).$$

But $\Lambda_k \overset{d}{=} cst(1+o_{\mathbb{P}}(1))L_{nk}^{\tau_Z/\tau_X}$ and, according to Proposition 2, $L_{nk}^{1-\tau_Z/\tau_X}\hat{p}_k = cst(1+o_{\mathbb{P}}(1))$. Consequently

$$\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k\left(\frac{\Lambda_j}{\Lambda_k}\right)^2\xi_{j,k}^2 \leqslant O_{\mathbb{P}}(1)\sqrt{k}v_nL_{nk}^{1-2\tau_Z/\tau_X}\frac{\log k}{k},$$

due to $\frac{1}{k}\sum_{j=1}^k\frac{1}{j} \sim \frac{\log k}{k}$. If $\tau_Z = \tau_X$ (thus $v_n = 1$), then the right-hand side above becomes $O_{\mathbb{P}}(1)\sqrt{k}L_{nk}^{-1}\frac{\log k}{k}$, which tends to 0 in probability, under assumption H_2 or H_4. If $\tau_Z = \tau_C < \tau_X$ (thus $v_n = L_{nk}^{(\tau_C/\tau_X-1)/2}$), let $0<\epsilon<\frac{1}{2}$ and write

$$\sqrt{k}v_nL_{nk}^{1-2\tau_Z/\tau_X}\frac{\log k}{k} = \sqrt{k}v_nL_{nk}^{1-2\tau_C/\tau_X}k^{\epsilon-1}o(1) = L_{nk}^{\frac{3}{2}\frac{\tau_C}{\tau_X}-\frac{1}{2}}k^{\epsilon-1/2}o(1) = (\sqrt{k}L_{nk}^{-b})^{2\epsilon-1}o(1),$$

where $-b > \frac{3}{2}\frac{\tau_C}{\tau_X}-\frac{1}{2}$ It remains to ensure that $\sqrt{k}L_{nk}^{\frac{3}{2}\frac{\tau_C}{\tau_X}-\frac{1}{2}}$ tends to infinity : this is the case under assumption $H_3(i)$.

(3) Remind that $T_{3,n} = -\theta_X(\hat{\Lambda}_k^{\tau_X-1} - \Lambda_k^{\tau_X-1})(D_{k,\tau_X})^{-1}\hat{p}_k$. Since $D_{k,\tau_X} = \hat{\Lambda}_k^{\tau_X-1}\hat{p}_k + R_{1,n}$, according to Lemma 3 (stated in Appendix E.4 below) and $R_{1,n}/D_{k,\tau_X} = o_{\mathbb{P}}(1)$ (term $T_{2,n}$ in Proposition 3), we obtain that

$$T_{3,n} = O_{\mathbb{P}}(1)\left(1 - \left(\frac{\Lambda_k}{\hat{\Lambda}_k}\right)^{\tau_X-1}\right).$$

But $|\Lambda_k - \hat{\Lambda}_k| = O_{\mathbb{P}}(k^{-1/2})$ (see Lemma 7 in Worms and Worms (2019)). Hence

$$|T_{3,n}| \leqslant O_{\mathbb{P}}(k^{-1/2})\Lambda_k^{-1}.$$

But $\Lambda_k = K_{\tau_X}^{-}(aK_{\tau_Z}(E_{n-k}) + \log l(\exp(K_{\tau_Z}(E_{n-k}))))$ (see statement (i) of Lemma 4). In the case where both τ_X and τ_C are not equal to 0 (the other cases are treated similarly), this yields that $\Lambda_k = O_{\mathbb{P}}(1)L_{nk}^{\tau_Z/\tau_X}$. Since $v_n L_{nk}^{-\tau_Z/\tau_X} = o_{\mathbb{P}}(1)$, this concludes the proof for $T_{3,n}$.

(4) Remind that $T_{4,n} = -\theta_X\frac{R_{2,n}}{D_{k,\tau_X}}$, where $R_{2,n}$ is defined in Lemma 5.

Let us consider the case where $\tau_X > 0$ and $\tau_C > 0$. If $\tau_X = 1$, then $R_{2,n} = 0$ and there is nothing to prove, so we suppose $\tau_X \in]0,1[$. We then have

$$R_{2,n} = \left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}}(E_{n-k,n})^{\tau_Z(1-\frac{1}{\tau_X})}\hat{p}_k\left((1-E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}}\left(1+\frac{1+\tau_X\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} - 1\right)$$

According to Lemma 3 (stated in Appendix E.4 below) and the fact that $\frac{\hat{\Lambda}_k}{\Lambda_k} \xrightarrow{\mathbb{P}} 1$, since

$$\Lambda_k = \left(\frac{a\tau_X}{\tau_Z}\right)^{1/\tau_X}(E_{n-k,n})^{\tau_Z/\tau_X}(1 + o_{\mathbb{P}}(1)),$$

it remains to prove that $\sqrt{k}v_n R_n$, where

$$R_n := \left((1-E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}}\left(1+\frac{1+\tau_X\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} - 1\right).$$

But $l(x)$ tends to a constant c that can be 0, as x tends to $+\infty$. Hence,

$$R_n = bE_{n-k,n}^{-\tau_Z}(1 + o_{\mathbb{P}}(1)) \quad \text{if} \quad c \neq 0 \quad \text{and} \quad R_n = cst\frac{\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{K_{\tau_Z}(E_{n-k,n})}(1 + o_{\mathbb{P}}(1)) \quad \text{if} \quad c = 0,$$

where $b = (1/\tau_X - 1)(1 - a^{-1}\tau_Z/\tau_X - \tau_Z/a\log c)$. According to Lemma 1 (part (iii)), in the cases when $c = 0$, we have $\frac{\log l(e^x)}{x} = cst.x^{r-1}(1 + o(1))$ as $x \to +\infty$. Consequently,

$$R_n = cst.L_{nk}^{\tau_Z(r-1)}(1 + o_{\mathbb{P}}(1)).$$

Hence, $\sqrt{k}v_n R_n \xrightarrow{\mathbb{P}} 0$, under assumption H_2 or H_3. The cases when $c \neq 0$ are treated similarly. This concludes the proof for $T_{4,n}$ when $\tau_X > 0$ and $\tau_C > 0$. The other cases ($\tau_X = 0$ or $\tau_C = 0$) can be treated similarly, details are ommited.

(5) Remind that $T_{5,n} = \theta_Z(1 + o_{\mathbb{P}}(1))R_{3,n}$, and that, in the case $\tau_X \neq 0$ and $\tau_C \neq 0$,

$$R_{3,n} = \left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X}\hat{p}_k\left(\frac{(E_{n-k})^{\tau_Z(1-1/\tau_X)}}{\mu_{1,\tau_Z}(E_{n-k})} - (L_{nk})^{1-\tau_Z/\tau_X}\right).$$

But, according to Proposition 2, $R_{3,n} = a(1 + o_{\mathbb{P}}(1))R_n$, where

$$R_n := \frac{L_{nk}^{\frac{\tau_Z}{\tau_X}-1}(E_{n-k})^{\tau_Z(1-\frac{1}{\tau_X})}}{\mu_{1,\tau_Z}(E_{n-k})} - 1 = R_n^{(1)} + R_n^{(2)} + R_n^{(3)},$$

and

$$R_n^{(1)} := \frac{L_{nk}^{\frac{\tau_Z}{\tau_X}-1}}{\mu_{1,\tau_Z}(E_{n-k})}\left((E_{n-k})^{\tau_Z(1-\frac{1}{\tau_X})} - L_{nk}^{\tau_Z(1-\frac{1}{\tau_X})}\right),$$
$$R_n^{(2)} := L_{nk}^{\tau_Z-1}\left(\frac{1}{\mu_{1,\tau_Z}(E_{n-k})} - \frac{1}{\mu_{1,\tau_Z}(L_{nk})}\right)$$
$$R_n^{(3)} := \frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(L_{nk})} - 1.$$

Let us prove that $\sqrt{k}v_n R_n^{(i)}$ tend to 0 , for $i=1,2,3$.

Concerning $R_n^{(1)}$, we use Lemma 4 of Gardes et al. (2011) to prove that $\sqrt{k}$ times the large brackets in the definition of $R_n^{(1)}$ is $O_{\mathbb{P}}(1)L_{nk}^{\tau_Z(1-\frac{1}{\tau_X})-1}$. Moreover, $\frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(E_{n-k})}$ tends to 1, in probability, according to see (A.3). Consequently, $\sqrt{k}v_n R_n^{(1)}=O_{\mathbb{P}}(1)v_n L_{nk}^{-1}$, which tends to 0.

Concerning $R_n^{(2)}$, we also use Lemma 4 of Gardes et al. (2011) to prove that $\sqrt{k}$ times the large brackets in the definition of $R_n^{(2)}$ is $O_{\mathbb{P}}(1)\frac{\mu'_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}{\mu^2_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}$. Since $\frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(L_{nk})}$ tends to 1, we obtain that

$$\sqrt{k}v_n R_n^{(2)}=O_{\mathbb{P}}(1)v_n\frac{\mu'_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}{\mu_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}\frac{\mu_{1,\tau_Z}(L_{nk})}{\mu_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))},$$

which tends to 0, according to Lemma 2 (iii) of Gardes et al. (2011).

Concerning $R_n^{(3)}$, remind that, if $\tau\neq 0$, $\mu_{1,\tau}(t)=\int_0^{+\infty}(K_\tau(x+t)-K_\tau(t))e^{-x}\,dx$ and $t^{\tau-1}=K'_\tau(t)$. This entails that

$$\begin{aligned}\frac{\mu_{1,\tau}(t)}{t^{\tau-1}} &= \int_0^{+\infty}x\frac{K_\tau(x+t)-K_\tau(t)}{xK'_\tau(t)}e^{-x}\,dx-\int_0^{+\infty}xe^{-x}\,dx\\ &= \int_0^{+\infty}\frac{x}{2}\frac{K''_\tau(t+\alpha)}{K'_\tau(t)}xe^{-x}\,dx\ (\alpha\in]0,x[)\\ &= \int_0^{+\infty}\frac{\tau-1}{2}\frac{x^2}{t}(1+\eta\frac{x}{t})^{\tau-2}e^{-x}\,dx\ (\eta\in]0,1[)\end{aligned}$$

Hence $R_n^{(3)}=\frac{1-\tau_Z}{2}L_{nk}^{-1}(1+o_{\mathbb{P}}(1))$ and $\sqrt{k}v_n R_n^{(3)}=O_{\mathbb{P}}(1)v_n L_{nk}^{-1}$, which tends to 0 under assumptions H_2,H_3,H_4. □

Appendix E.4. Technical Lemmas

Lemma 3. *The denominator of the estimator $\hat\theta_{X,\tau_X}$ satisfies the relation*

$$D_{k,\tau_X}=\frac{1}{k}\sum_{j=1}^k K_{\tau_X}(\hat\Lambda_{nX}(Z_{n-j+1,n}))-K_{\tau_X}(\hat\Lambda_{nX}(Z_{n-k,n}))=\hat\Lambda_k^{\tau_X-1}\hat p_k+R_{1,n},$$

where

$$R_{1,n}=\begin{cases}\frac{\tau_X-1}{2}\hat\Lambda_k^{\tau_X}\frac{1}{k}\sum_{j=1}^k\left(\frac{\hat\Delta_{j,k}}{\hat\Lambda_k}\right)^2(1+T_{j,k})^{\tau_X-2}, & \text{if } 0<\tau_X<1,\\ \frac{1}{k}\sum_{j=1}^k\left(\log\left(1+\frac{\hat\Delta_{j,k}}{\hat\Lambda_k}\right)-\frac{\hat\Delta_{j,k}}{\hat\Lambda_k}\right) & \text{if } \tau_X=0,\\ 0 & \text{if } \tau_X=1\end{cases}$$

with, for each $j=1,\ldots,k$, $\hat\Delta_{j,k}:=\hat\Lambda_{nX}(Z_{n-j+1,n})-\hat\Lambda_{nX}(Z_{n-k,n})$ and the random variable $T_{j,k}$ lies between 0 and $\frac{\hat\Delta_{j,k}}{\hat\Lambda_k}$.

Proof : straightforward via Taylor's formula and the definition of function K_{τ_Z} (the negligibility of $R_{1,n}$ is another story, it is dealt with in Appendix E.3, part (2)). □

For the following lemma, remind that (E_i) denote the i.i.d. standard exponential variable (E_i) satisfying $Z_i=H_Z^{-}(\exp(K_{\tau_Z}(E_i)))$, and that $l(\cdot)$ denotes the slowly varying function which properties are described in Lemma 1 and which is such that $H_X\circ H_Z^{-}(x)=x^a l(x)$. Note that in part (ii) of this lemma, the results also hold when one replaces $E_{n-k,n}$ by L_{nk}, or replaces $Z_{n-k,n}$ and $E_{n-k,n}$ by $Z_{n-j+1,n}$ and $E_{n-j+1,n}$ (this will occasionally prove useful).

Lemma 4. (i) *For every $i=1,\ldots,n$, and whether $\tau_Z>0$ or is equal to 0, we have*

$$\Lambda_X(Z_i)=K_{\tau_X}^{-}\left(aK_{\tau_Z}(E_i)+\log l(\exp K_{\tau_Z}(E_i))\right).$$

(ii) *When $\tau_Z>0$, we have*

$$\Lambda_X(Z_{n-k,n})=\left(a\frac{\tau_X}{\tau_Z}\right)^{1/\tau_X}E_{n-k,n}^{\tau_Z/\tau_X}(1+o_{\mathbb{P}}(1))=\left(a\frac{\tau_X}{\tau_Z}\right)^{1/\tau_X}E_{n-k,n}^{\tau_Z/\tau_X}\left(1+\beta E_{n-k,n}^{-\alpha}(1+o_{\mathbb{P}}(1))\right)\quad\text{(E.4)}$$

for some constant β and exponent $\alpha=\begin{cases}\tau_Z & \textit{when either } \tau_X=\tau_C\textit{, or } \tau_X\neq\tau_C \textit{ and } r\leqslant 0,\\ \tau_Z(1-r) & \textit{when } \tau_X\neq\tau_C \textit{ and } r\in]0,1[.\end{cases}$

When $0 = \tau_X < \tau_C$*, we have* $\Lambda_X(Z_{n-k,n}) = E_{n-k,n} l(E_{n-k,n}) = E_{n-k,n}(1 + o_{\mathbb{P}}(1))$.

When $0 = \tau_C < \tau_X$*, we have*

$$\Lambda_X(Z_{n-k,n}) = (a\tau_X)^{1/\tau_X} (\log E_{n-k,n})^{1/\tau_X} \left(1 + \beta(\log E_{n-k,n})^{-1}(1 + o_{\mathbb{P}}(1))\right).$$

Note that the constant β is negative in the case $\tau_X \neq \tau_C$ and $r \in]0, 1[$.

Proof of Lemma 4 :

The first statement (i) holds because on one hand, since $\bar{F}_X \in A_1(\tau_X, \theta_X)$, we have $\Lambda_X(x) = K^{-}_{\tau_X}(\log H_X(x))$, and on the other hand, $Z_i = H^{-}_Z(\exp(K_{\tau_Z}(E_i)))$ where $H_X \circ H^{-}_Z(x) = x^a l(x)$ (see beginning of Appendix E.1).

The second statement is essentially a consequence of the first one and of some of the second order results contained in Lemma 1. Suppose for the moment that $\tau_Z > 0$, *i.e.* $\tau_X > 0$ and $\tau_C > 0$. We thus have $K^{-}_{\tau_X}(x) = (\tau_X x + 1)^{1/\tau_X}$. Hence, noting temporarily $\phi(x) = \log l(\exp x)/x$, it is easy to see that (i) implies

$$\begin{aligned} \Lambda_X(Z_{n-k,n}) &= \{(a\tau_X K_{\tau_Z}(E_{n-k,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n}))) + 1\}^{1/\tau_X} \\ &= (a\tau_X)^{1/\tau_X} (K_{\tau_Z}(E_{n-k,n}))^{1/\tau_X} \left\{1 + (a\tau_X K_{\tau_Z}(E_{n-k,n}))^{-1} + a^{-1}\phi(K_{\tau_Z}(E_{n-k,n}))\right\}^{1/\tau_X} \end{aligned}$$

But $K_{\tau_Z}(E_{n-k,n}) = E^{\tau_Z}_{n-k,n}(1 - E^{-\tau_Z}_{n-k,n})/\tau_Z = E^{\tau_Z}_{n-k,n}(1 + o_{\mathbb{P}}(1))$, so

$$\Lambda_X(Z_{n-k,n}) = (a\tau_X/\tau_Z)^{1/\tau_X} E^{\tau_Z/\tau_X}_{n-k,n} \left(1 - \frac{1}{\tau_X} E^{-\tau_Z}_{n-k,n}(1 + o_{\mathbb{P}}(1))\right) \times B_n$$

where B_n denotes the quantity in curly brackets above. Thanks to part (iii) of Lemma 1, we have

$$B_n = 1 + \frac{\tau_Z}{a\tau_X} E^{-\tau_Z}_{n-k,n}(1 + o_{\mathbb{P}}(1)) + cst.E^{-\alpha}_{n-k,n}(1 + o_{\mathbb{P}}(1))$$

where either $\alpha = \tau_Z$ and $cst = (\log c)\tau_Z/a$ (when $\tau_X = \tau_C$ or $\tau_X \neq \tau_C$ and $r \leqslant 0$) or $\alpha = \tau_Z$ and $cst = -\nu a^{-1}\tau_Z^{1-r} < 0$ (when $\tau_X \neq \tau_C$ and $r \in]0, 1[$). The proof is thus over when $\tau_Z > 0$.

The cases $\tau_X = 0$ and $\tau_C > 0$, or $\tau_C = 0$ and $\tau_X > 0$, can be proved similarly. When $0 = \tau_X < \tau_C$, we have $\tau_Z = 0$ and $a = 1$ so it immediately comes $\Lambda_X(Z_{n-k,n}) = E_{n-k,n} l(E_{n-k,n}) = E_{n-k,n}(1 + o_{\mathbb{P}}(1))$ (because $c = 1$ in that case, see Lemma 1). When $0 = \tau_C < \tau_X$, we have $\tau_Z = 0$ and thus

$$\Lambda_X(Z_{n-k,n}) = \{a\tau_X \log(E_{n-k,n}) + \tau_X \log l(E_{n-k,n}) + 1\}^{1/\tau_X}$$

The end of the proof is then very similar to the first case covered in details above.

The fact that relation E.4 also holds when $E_{n-k,n}$ is replaced by L_{nk} is due to Lemma 4 in Gardes et al. (2011), which states that $\sqrt{k}(E_{n-k,n} - L_{nk})$ converges in distribution to a standard normal variable. □

Lemma 5. *Let* $E_1, \ldots, E_n$ *be i.i.d. standard exponential random variables.*

$$\Lambda_k^{\tau_X - 1}\hat{p}_k = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X} E^{\tau_Z(1-1/\tau_X)}_{n-k,n}\hat{p}_k + R_{2,n}, & \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} + R_{2,n}, & \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{1-1/\tau_X} (\log(E_{n-k,n}))^{1-1/\tau_X}\hat{p}_k + R_{2,n} & \text{if } 0 = \tau_C < \tau_X < 1, \end{cases}$$

where

$$R_{2,n} = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}} E^{\tau_Z(1-\frac{1}{\tau_X})}_{n-k,n}\hat{p}_k \left((1 - E^{-\tau_Z}_{n-k,n})^{1-\frac{1}{\tau_X}}\left(1 + \frac{1+\tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} - 1\right), \\ \text{if } 0 < \tau_X < 1 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}}\left(\frac{1}{l(E_{n-k,n})} - 1\right), \ \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{1-\frac{1}{\tau_X}} (\log(E_{n-k,n}))^{1-\frac{1}{\tau_X}}\hat{p}_k \left(\left(1 + \frac{1+\tau_X \log l(E_{n-k,n})}{a\tau_X \log(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} - 1\right), \ \text{if } 0 = \tau_C < \tau_X < 1 \\ 0, \ \text{if } \tau_X = 1 \end{cases}$$

Proof : Using part (i) of Lemma 4, we have

$$\Lambda_k = K^{-}_{\tau_X}\left(aK_{\tau_Z}(E_{n-k,n}) + \log l(\exp(K_{\tau_Z}(E_{n-k,n})))\right),$$

which yields, in the case $\tau_X \neq 0$ and $\tau_C \neq 0$,

$$\Lambda_k^{\tau_X-1} = \left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}} E_{n-k,n}^{\tau_Z(1-\frac{1}{\tau_X})}(1-E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}}\left(1+\frac{1+\tau_X\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}}.$$

The expression of $R_{2,n}$ follows in this case. The other cases are similar. □

Lemma 6. *Let $E_1,\dots,E_n$ be i.i.d. standard exponential random variables.*

$$\Lambda_k^{-1}\hat{p}_k = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{-1/\tau_X} E_{n-k,n}^{-\tau_Z/\tau_X}\hat{p}_k + RR_{2,n}, & \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} + RR_{2,n}, & \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{-1/\tau_X}(\log(E_{n-k,n}))^{-1/\tau_X}\hat{p}_k + RR_{2,n} & \text{if } 0 = \tau_C < \tau_X < 1, \end{cases}$$

where

$$RR_{2,n} = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{-\frac{1}{\tau_X}} E_{n-k,n}^{-\frac{\tau_Z}{\tau_X}}\hat{p}_k\left((1-E_{n-k,n}^{-\tau_Z})^{-\frac{1}{\tau_X}}\left(1+\frac{1+\tau_X\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{-\frac{1}{\tau_X}}-1\right), \\ \text{if } 0<\tau_X<1 \text{ and } \tau_C\neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}}\left(\frac{1}{l(E_{n-k,n})}-1\right), \ \text{if } 0=\tau_X<\tau_C<1 \\ (a\tau_X)^{-\frac{1}{\tau_X}}(\log(E_{n-k,n}))^{-\frac{1}{\tau_X}}\hat{p}_k\left(\left(1+\frac{1+\tau_X\log l(E_{n-k,n})}{a\tau_X\log(E_{n-k,n})}\right)^{-\frac{1}{\tau_X}}-1\right), \ \text{if } 0=\tau_C<\tau_X<1 \\ 0, \quad \text{if } \tau_X = 1 \end{cases}$$

The proof of the previous lemma is very similar to the one of Lemma 5, it is therefore omitted. The following one is an easy consequence of Lemma 4.

Lemma 7. *Under the same assumptions in Theorem 1, we have, as $n\to\infty$,*

$$\begin{array}{ll} \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0, & \log(\Lambda_k) = \frac{\tau_Z}{\tau_X}\log L_{nk}(1+o_{\mathbb{P}}(1)) \\ \text{if } \tau_X = 0, & \log(\Lambda_k) = a\log L_{nk}(1+o_{\mathbb{P}}(1)) \\ \text{if } \tau_X \neq 0, \text{ and } \tau_C = 0 & \log(\Lambda_k) = \frac{1}{\tau_X}\log\log L_{nk}(1+o_{\mathbb{P}}(1)) \end{array}$$

Lemma 8. *Under the assumptions of Theorem 4, we have, as n tends to infinity,*

$$\int_{L_k}^{\Lambda_k} u^{\tau_X-1}\log u\,du = \begin{cases} O_{\mathbb{P}}(\log L_{nk}) & \text{if } \tau_X\neq 0, \tau_C\neq 0 \text{ and } (\tau_X=\tau_C \text{ or } r\leqslant 0), \\ O_{\mathbb{P}}(L_{nk}^{r\tau_Z}\log L_{nk}) & \text{if } \tau_X\neq 0,\tau_C\neq 0,\tau_X\neq\tau_C \text{ and } r\in]0,1[, \\ O_{\mathbb{P}}(\log\log L_{nk}) & \text{if } \tau_X\neq 0 \text{ and } \tau_C = 0, \\ o_{\mathbb{P}}(\log L_{nk}) & \text{if } \tau_X = 0. \end{cases}$$

Proof : We only treat the case where both τ_X and τ_C are positive. In this case, remind that $L_k = (a\tau_X/\tau_Z)^{1/\tau_X}(L_{nk})^{\tau_Z/\tau_X}$ and, according to Lemma 4, $\frac{\Lambda_k}{L_k}\xrightarrow{\mathbb{P}} 1$. We have (with $v = u/L_k$)

$$\begin{aligned} \int_{L_k}^{\Lambda_k} u^{\tau_X-1}\log u\,du &= L_k^{\tau_X}\int_1^{\Lambda_k/L_k} v^{\tau_X-1}(\log v+\log L_k)\,dv \\ &= \frac{L_k^{\tau_X}}{\tau_X}\log\left(\frac{\Lambda_k}{L_k}\right)\left(\frac{\Lambda_k}{L_k}\right)^{\tau_X} - \frac{L_k^{\tau_X}}{\tau_X^2}\left(\left(\frac{\Lambda_k}{L_k}\right)^{\tau_X}-1\right) + \log L_k\frac{L_k^{\tau_X}}{\tau_X}\left(\left(\frac{\Lambda_k}{L_k}\right)^{\tau_X}-1\right). \end{aligned}$$

An immediate consequence of Lemma 4 is that both $\log\left(\frac{\Lambda_k}{L_k}\right)$ and $\left(\frac{\Lambda_k}{L_k}\right)^{\tau_X}-1$ are $O_{\mathbb{P}}\left((L_{nk})^{-\tau_Z}\right)$ if $\tau_X=\tau_C$ or $r\leqslant 0$, and are $O_{\mathbb{P}}\left((L_{nk})^{-\tau_Z(r-1)}\right)$ if $\tau_X\neq\tau_C$ and $r\in]0,1[$. The result follows easily. □