Lyapunov-Krasovskii characterizations of stability notions for switching retarded systems

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December 7, 2020

Abstract

In this paper we characterize the global asymptotic and exponential stability of nonlinear switching retarded systems through direct and converse Lyapunov–Krasovskii theorems. Thanks to these theorems, a link between the exponential stability of an unforced switching retarded system and the input-to-state stability property is obtained. An example illustrating the applicability of our results is given.

Keywords: Converse theorems; Lyapunov–Krasovskii functionals; retarded functional differential equations; switching systems.

1 Introduction

A significative research has been devoted around the stability property of switching systems (see, e.g., [25]). Many sufficient conditions based on the existence of *common Lyapunov functions* [4] or *multiple Lyapunov functions* [2] are developed. Theses two approaches have been extended to switching retarded systems (see, e.g., [15, 26, 28] and references therein). Another interesting approach called *trajectory based approach* [17] is also investigated in the context of switching retarded systems (see, e.g., [1, 18]). In [15] the authors study the robustness of exponential stability of a class of nonlinear retarded switching systems to some specific class of perturbations. The key point is the existence of a common Lyapunov–Krasovskii functional for the nominal switching system, i.e. the non-perturbed one. In [18], based on a multiple Lyapunov functions approach, the authors study the asymptotic stability and robustness properties of nonlinear retarded switching systems. The key assumption in [18] is the existence of multiple Lyapunov functions with some weak properties.

The utility of converse Lyapunov theorems is well known in the literature of non-switching control systems. For example, converse Lyapunov theorems are the main tool for characterizing input-to-state stability and proving robust stability properties (see [14] for systems described by ordinary differential equations, and [22, 23, 27] for systems described by retarded and neutral functional differential equations). Converse theorems are also developed for switching systems. The existence of a common Lyapunov function is actually equivalent to the uniform global asymptotic stability of finite-dimensional switching systems (see, e.g., [4, 16]). Converse like results are also obtained for infinite-dimensional switching systems (see, e.g., [7, 11, 19]).

Converse theorems have been recently developed in [8] for linear retarded switching systems. Thanks to these theorems, sufficient conditions for the stability of interconnected linear uncertain delay systems are given in [9]. Converse like results are not yet developed for nonlinear retarded switching systems. In this paper we develop theorems characterizing the global asymptotic and exponential stability of such systems through the existence of a common Lyapunov–Krasovskii functional. These theorems are obtained

*This work is supported by a public grant overseen by the Research and Valorization Service (SRV) of ENSEA.

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for switching systems with piecewise constant switching signals. A theorem showing the link between input-to-state stability and global exponentially stability of an unforced switching retarded systems is given. An example highlighting the importance of such theorems is presented. Proofs are reported in the appendix.

2 Switching retarded systems

2.1 Notation

By $\mathbb{R}$ we denote the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers and $\overline{\mathbb{R}}$ the extended real line. Given $\Delta > 0$, $\mathcal{C} = (\mathcal{C}([-\Delta, 0], \mathbb{R}^n), \|\cdot\|_\infty)$ denotes the Banach space of continuous functions mapping $[-\Delta, 0]$ into $\mathbb{R}^n$, where $\|\cdot\|_\infty$ is the norm of uniform convergence. For a function $x : [-\Delta, b) \to \mathbb{R}^n$, with $0 < b \leq +\infty$, for $t \in [0, b)$, $x_t : [-\Delta, 0] \to \mathbb{R}^n$ denotes the history function defined by $x_t(\theta) = x(t + \theta)$, $-\Delta \leq \theta \leq 0$. For a positive real H and given $\phi \in \mathcal{C}$, $\mathcal{C}_H(\phi)$ denotes the subset $\{\psi \in \mathcal{C} : \|\phi - \psi\|_\infty \leq H\}$. We simply denote $\mathcal{C}_H(0)$ by $\mathcal{C}_H$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$; it is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and converges to zero as t tends to $+\infty$.

2.2 Definitions and assumptions

Let us consider the switching system described by the following retarded functional differential equation

$$\Sigma : \begin{array}{rcll} \dot{x}(t) & = & f_{\sigma(t)}(x_t), & a.e.\ t \geq 0, \\ x(\theta) & = & x_0(\theta), & \theta \in [-\Delta, 0], \end{array} \tag{1}$$

where: $x(t) \in \mathbb{R}^n$; n is a positive integer; Δ is a positive real (the maximum involved time-delay); $x_0 \in \mathcal{C}$ is the initial condition; the function $\sigma : \mathbb{R}_+ \to \mathrm{S}$ is the switching signal; S is a nonempty subset of $\mathbb{R}^D$; D is a positive integer. We introduce the following assumption.

Assumption 1. *For each $s \in \mathrm{S}$, $f_s(0) = 0$. Moreover, f_s is uniformly (with respect to $s \in \mathrm{S}$) Lipschitz on bounded subsets of $\mathcal{C}$, i.e., for any $H > 0$ there exists $L_H > 0$ such that*

$$|f_s(\phi) - f_s(\psi)| \leq L_H \|\phi - \psi\|_\infty, \quad \forall \phi, \psi \in \mathcal{C}_H, \forall s \in \mathrm{S}. \tag{2}$$

In addition, the switching signal $\sigma(\cdot)$ satisfies the following assumption:

Assumption 2. *The switching signal $\sigma : \mathbb{R}_+ \to \mathrm{S}$ is piecewise constant.*

Let us denote by $\mathcal{S}$ the set of all functions $\sigma : \mathbb{R}_+ \to \mathrm{S}$ satisfying Assumption 2. The following lemma discusses the existence and uniqueness of the solution of system Σ as well as its continuous dependence on the initial state.

Lemma 1. *For any $x_0 \in \mathcal{C}$ and $\sigma \in \mathcal{S}$, there exists, uniquely, an absolutely continuous solution $x(t, x_0, \sigma)$ of Σ in a maximal time interval $[0, b)$, with $0 < b \leq +\infty$. If $b < +\infty$, then the solution is unbounded in $[0, b)$. Moreover, for any $\varepsilon > 0$, for any $c \in (0, b)$, there exists $\delta > 0$ such that, for any $\phi \in \mathcal{C}_\delta(x_0)$, the solution $x(t, \phi, \sigma)$ exists in $[0, c]$ and, furthermore, the following inequality holds*

$$|x(t, x_0, \sigma) - x(t, \phi, \sigma)| \leq \varepsilon, \quad \forall\ t \in [0, c]. \tag{3}$$

Proof. The proof is straightforward from the theory of systems described by RFDEs (see, e.g. [10]). $\square$

Definition 1. *We say that the system Σ is* PC-0-*GAS if there exist a function $\beta \in \mathcal{KL}$ such that, for any $x_0 \in \mathcal{C}$ and $\sigma \in \mathcal{S}$, the corresponding solution exists in $\mathbb{R}_+$ and, furthermore, satisfies the inequality*

$$|x(t, x_0, \sigma)| \leq \beta(\|x_0\|_\infty, t). \tag{4}$$

Definition 2. *We say that the system* Σ *is* PC-0-*GES if there exist positive reals* M *and* λ *such that, for any* $x_0 \in \mathcal{C}$ *and* $\sigma \in \mathcal{S}$*, the corresponding solution exists in* $\mathbb{R}_+$ *and, furthermore, satisfies the inequality*

$$|x(t, x_0, \sigma)| \leq M e^{-\lambda t} \|x_0\|_\infty. \tag{5}$$

Notice, from Definitions 1 and 2, that here we deal with stability properties which are uniform with respect to both the initial condition x_0 in compact sets and the switching signals $\sigma \in \mathcal{S}$.

Let us recall the following definition about Driver's form derivative of a continuous functional $V : \mathcal{C} \to \mathbb{R}_+$. This definition is a variation of the one given in [5, 20], for retarded functional differential equations without switching.

Definition 3. *For a continuous functional* $V : \mathcal{C} \to \mathbb{R}_+$*, its Driver's form derivative,* $D^+V : \mathcal{C} \to \overline{\mathbb{R}}$*, is defined, for the switching system* Σ*, for* $\phi \in \mathcal{C}$*, as follows,*

$$D^+V(\phi) = \sup_{s \in \mathrm{S}} \limsup_{h \to 0^+} \frac{V(\phi_h^{\Sigma,s}) - V(\phi)}{h}, \tag{6}$$

where $\phi_h^{\Sigma,s} \in \mathcal{C}$ *is defined, for* $h \in [0, \Delta)$ *and* $\theta \in [-\Delta, 0]$*, as follows*

$$\phi_h^{\Sigma,s}(\theta) = \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h) \\ \phi(0) + (\theta + h) f_s(\phi), & \theta \in [-h, 0]. \end{cases} \tag{7}$$

3 Converse theorems by common Lyapunov–Krasovskii functionals

The following theorems provide necessary and sufficient conditions for the PC-0-GAS and PC-0-GES properties of system Σ.

Theorem 1. *System* Σ *is* PC-0-*GAS if and only if there exist a functional* $V : \mathcal{C} \to \mathbb{R}_+$*, Lipschitz on bounded subsets of* $\mathcal{C}$*, and functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*,* $\alpha_3 \in \mathcal{K}$ *such that the following inequalities hold for any* $\phi \in \mathcal{C}$

(i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty)$,

(ii) $D^+V(\phi) \leq -\alpha_3(|\phi(0)|)$.

Theorem 2. *System* Σ *is* PC-0-*GES if and only if there exist a functional* $V : \mathcal{C} \to \mathbb{R}_+$*, Lipschitz on bounded subsets of* $\mathcal{C}$*, and positive reals* $\alpha_1, \alpha_2, \alpha_3$ *and* p*, such that the following inequalities hold for any* $\phi \in \mathcal{C}$

(i) $\alpha_1 \|\phi\|_\infty^p \leq V(\phi) \leq \alpha_2 \|\phi\|_\infty^p$,

(ii) $D^+V(\phi) \leq -\alpha_3 \|\phi\|_\infty^p$.

If, in addition, f_s*, for* $s \in \mathrm{S}$*, is uniformly globally Lipschitz then* V *is globally Lipschitz.*

4 Application to input-to-state stability of switching retarded systems

Consider the following switching retarded system

$$\begin{array}{rcll} \dot{x}(t) & = & f_{\sigma(t)}(x_t, u(t)), & a.e.\ t \geq 0, \\ x(\theta) & = & x_0(\theta), & \theta \in [-\Delta, 0], \end{array} \tag{8}$$

where $u : \mathbb{R}_+ \to \mathbb{R}^m$ is a piecewise continuous function which represents the input of the system and $f_s : \mathcal{C} \times \mathbb{R}^m \to \mathbb{R}^n$, for $s \in \mathrm{S}$, is uniformly Lipschitz on bounded subsets of $\mathcal{C} \times \mathbb{R}^m$, with $f(0, 0) = 0$. Thanks to Theorem 2, we give a link between the input-to-state stability of system (8) and the exponential stability of the unforced system. Firstly, let us recall the following standard definition (see [24]).

Definition 4. *We say that system* (8) *is input-to-state stable if there exist a function* $\beta \in \mathcal{KL}$ *and a function* $\gamma \in \mathcal{K}$ *such that, for any* $x_0 \in \mathcal{C}$*, any piecewise continuous input* u *and any* $\sigma \in \mathcal{S}$*, the corresponding solution exists in* $\mathbb{R}_+$ *and, furthermore, satisfies the inequality*

$$|x(t, x_0, u, \sigma)| \leq \beta(\|x_0\|_\infty, t) + \gamma(\|u_{[0,t)}\|_\infty). \tag{9}$$

We have the following theorem (inspired by [27]).

Theorem 3. *Let system* (8) *with* $u \equiv 0$ *be* PC-0-*GES. If there exist positive reals* L *and* l *and a nonnegative real* $p < 1$ *such that:*

1) $\forall \phi, \psi \in \mathcal{C}, \forall u \in \mathbb{R}^m, \forall s \in \mathrm{S}$*, the following holds*

$$|f_s(\phi, u) - f_s(\psi, u)| \leq L\|\phi - \psi\|_\infty; \tag{10}$$

2) $\forall \phi \in \mathcal{C}, \forall u \in \mathbb{R}^m, \forall s \in \mathrm{S}$*, the following holds*

$$|f_s(\phi, u) - f_s(\phi, 0)| \leq l \max\{\|\phi\|_\infty^p, 1\}|u|; \tag{11}$$

then system (8) *is input-to-state stable.*

5 Example

Consider the neural network with constant delays

$$\dot{x}(t) = Ax(t) + g(x_t), \tag{12}$$

where $A = \mathrm{diag}(a_1, \cdots, a_n)$, with $a_i < 0$ for $i \in \{1, \cdots, n\}$, and the nonlineartiy $g : \mathcal{C} \to \mathbb{R}^n$ is given by

$$g_i(\varphi) = \sum_{j=1}^{n} b_{ij} f_j(\varphi_j(-\tau_{ij})), \tag{13}$$

where $b_{ij} \in \mathbb{R}$, $\tau_{ij} \in [0, \Delta]$, for some $\Delta > 0$, for $i, j \in \{1, \cdots, n\}$.

Assumption 3. *For each* $i \in \{1, \cdots, n\}$ *the function* f_i *is globally Lipschitz with Lipschitz constant* $\mu_i > 0$.

If the following condition is satisfied

$$a_i + \sum_{j=1}^{n} \mu_j |b_{ij}| < 0, \quad \text{for } i = 1, \cdots, n, \tag{14}$$

it has been proven in [3] that system (12) is globally exponentially stable.
System (12) is useful in many applications such as signal processing, robotics and neurosciences. However, in general, the delays and even the vector field g may depend on different uncertain parameters and it is more suitable to consider a perturbed switching neural network with uncertain time-varying delays instead of constant ones. For this, let us consider the following modified perturbed system

$$\dot{x}(t) = Ax(t) + g_{\sigma(t)}(x_t, u(t)), \tag{15}$$

where $\sigma : \mathbb{R}_+ \to \mathrm{S}$ is a piecewise constant switching signal, $u : \mathbb{R}_+ \to \mathbb{R}^m$ is a piecewise continuous input and, for each fixed mode $s \in \mathrm{S}$, the nonlineartiy $g_s : \mathcal{C} \times \mathbb{R}^m \to \mathbb{R}^n$ is given, for $(\phi, u) \in \mathcal{C} \times \mathbb{R}^m$, by

$$g_{i,s}(\varphi, u) = \sum_{j=1}^{n} b_{ij} f_{j,s}(\varphi_j(-\tau_{ij,s})) + u_i, \tag{16}$$

where $f_{j,s} \in \{f_1, \cdots, f_n\}$ and $\tau_{ij,s} \in [0, \Delta]$, for all $i, j \in \{1, \cdots, n\}$.

As said previously, if condition (14) is satisfied then system (12) is globally exponentially stable. Therefore, by [22, Theorem 2.5], there exists a globally Lipschitz functional $V : \mathcal{C} \to \mathbb{R}_+$ with Lipschitz constant $\alpha_4 > 0$ such that the inequalities (i) and (ii) of Theorem 2 hold with $p = 1$. Computing the Driver's derivative of V along the trajectories of the switching system (15), in the case when $u \equiv 0$, gives

$$\begin{aligned} D^+V(\varphi) &= \sup_{s\in \mathrm{S}} \limsup_{h\to 0^+} \frac{V(\varphi_h^{(15),s}) - V(\varphi)}{h} \\ &\leq -\alpha_3\|\varphi\|_\infty + \sup_{s\in \mathrm{S}} \limsup_{h\to 0^+} \frac{V(\varphi_h^{(15),s}) - V(\varphi_h^{(12)})}{h} \\ &\leq -\alpha_3\|\varphi\|_\infty + \alpha_4 \sup_{s\in \mathrm{S}} \limsup_{h\to 0^+} \frac{\|\varphi_h^{(15),s} - \varphi_h^{(12)}\|_\infty}{h} \\ &\leq -\alpha_3\|\varphi\|_\infty + 2\bar{\mu}\alpha_4 \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n |b_{ij}|\right)^2} \|\varphi\|_\infty, \end{aligned}$$

where $\bar{\mu} = \max\{\mu_1, \cdots, \mu_n\}$. By consequence, by Theorem 2, in addition to condition (14), if the Lipschitz constants μ_i, for $i \in \{1, \cdots, n\}$, are sufficiently small, then the global exponential stability of the non-perturbed system (15), i.e. with $u \equiv 0$, is preserved. In particular, if α_4 is known, sufficient condition for the global exponential stability of the non-perturbed system (15) is given by condition (14) together with the following inequality

$$\bar{\mu} \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n |b_{ij}|\right)^2} < \frac{\alpha_3}{2\alpha_4}. \tag{17}$$

In this case, thanks to Theorem 3, we have that the perturbed switching uncertain delay system (15) is input-to-state stable.

6 Conclusions

In this paper we establish converse Lyapunov–Krasovskii theorems regarding global asymptotic and exponential stability of nonlinear retarded switching systems. These theorems are obtained for switching systems with piecewise constant switching signals. A link between input-to-state stability and exponential stability is given. An example discussing the applicability of our results is presented.

Acknowledgements

The authors are grateful to ENSEA and DISIM-Univaq for warm hospitality during the preparation of this work.

7 Proof of Theorem 1

7.1 Preliminary results

By exploiting the fact that f_s is uniformly (with respect to $s \in \mathrm{S}$) Lipschitz on bounded subsets of $\mathcal{C}$, the following two lemmas follows directly from [22, Lemma A.4] and [22, Lemma A.6], respectively.

Lemma 2. *There exists a continuous increasing function $\overline{L} : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $\phi, \psi \in \mathcal{C}$, the following inequality holds*

$$|f_s(\phi) - f_s(\psi)| \leq \overline{L}(\|\phi\|_\infty + \|\psi\|_\infty)\|\phi - \psi\|_\infty, \ \forall s \in \mathrm{S}. \tag{18}$$

Lemma 3. *Suppose that system Σ is* PC-0-*GAS. Let $\overline{L}$ be a continuous increasing function as depicted in Lemma 2. Let $L : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined as*

$$L(r) = \overline{L}(2\beta(r, 0)), \quad r \in \mathbb{R}_+, \tag{19}$$

where the function β is given in Definition 1. Then, for any $t \in \mathbb{R}_+$, for any $\sigma \in \mathcal{S}$, and for any $\phi, \psi \in C$, the following inequality holds

$$\|x_t(\phi, \sigma) - x_t(\psi, \sigma)\|_\infty \leq e^{L(\|\phi\|_\infty + \|\psi\|_\infty)t}\|\phi - \psi\|_\infty. \tag{20}$$

The following lemma is given in [24].

Lemma 4. *[24, Proposition 7] For a given function $\beta \in \mathcal{KL}$, there exist two functions $\overline{\alpha}_1, \overline{\alpha}_2 \in \mathcal{K}_\infty$, such that:*

$$\beta(r, t) \leq \overline{\alpha}_1^{-1}(e^{-2t}\overline{\alpha}_2(r)), \quad r \geq 0, t \geq 0. \tag{21}$$

The proof of the following lemma follows the same methodology as in [12, 22].

Lemma 5. *Suppose that system Σ is* PC-0-*GAS. Let, for any positive integer q, $U_q : \mathcal{C} \to \mathbb{R}_+$ be the functional defined, for all $\phi \in \mathcal{C}$, by*

$$U_q(\phi) = \sup_{t \geq 0, \sigma \in \mathcal{S}} \left(\max\left\{0, \overline{\alpha}_1\left(\|x_t(\phi, \sigma)\|_\infty\right) - \frac{1}{q}\right\} e^t \right), \tag{22}$$

where $\overline{\alpha}_1$ is defined in Lemma 4. Let $G : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ be the function defined as

$$G(r, q) = (1 + q\overline{\alpha}_2(r))^{(L(2r)+1)}, \quad r \in \mathbb{R}_+, q \in \mathbb{Z}_+, \tag{23}$$

where L is as depicted in Lemma 3 and $\overline{\alpha}_2$ is defined in Lemma 4. Then, for any positive integer q, we have:

1) $\max\left\{0, \overline{\alpha}_1\left(\|\phi\|_\infty\right) - \frac{1}{q}\right\} \leq U_q(\phi) \leq \overline{\alpha}_2(\|\phi\|_\infty)$, $\forall\ \phi \in \mathcal{C}$;

2) $U_q(x_\tau(\phi, \tilde{\sigma})) \leq e^{-\tau}U_q(\phi), \quad \forall\ \tau \geq 0,\ \tilde{\sigma} \in \mathcal{S},\ \phi \in \mathcal{C}$;

3) for any positive real H, for any $\phi, \psi \in \mathcal{C}_H$, the following inequality holds

$$|U_q(\phi) - U_q(\psi)| \leq G(H, q)\|\phi - \psi\|_\infty.$$

Proof. Let, for any positive integer q and any fixed $\sigma \in \mathcal{S}$, $U_q^\sigma : \mathcal{C} \to \mathbb{R}_+$ be the functional defined, for all $\phi \in \mathcal{C}$, by

$$U_q^\sigma(\phi) = \sup_{t \geq 0} \left(\max\left\{0, \overline{\alpha}_1\left(\|x_t(\phi, \sigma)\|_\infty\right) - \frac{1}{q}\right\} e^t \right). \tag{24}$$

Following the same steps as in the proof of [12, Theorem 2.9], we have

$$\max\left\{0, \overline{\alpha}_1\left(\|\phi\|_\infty\right) - \frac{1}{q}\right\} \leq U_q^\sigma(\phi) \leq \overline{\alpha}_2(\|\phi\|_\infty), \tag{25}$$

for any $\phi \in \mathcal{C}$ and any $\sigma \in \mathcal{S}$. Knowing that the left and right hand sides of inequalities (25) are independent from σ, the inequalities in 1) follow directly from (25) by applying the supremum over $\sigma \in \mathcal{S}$. Moreover, as far as the inequality in 2) is concerned, for any $\tau \geq 0$, any $\phi \in \mathcal{C}$, and any fixed $\sigma, \tilde{\sigma} \in \mathcal{S}$, we have,

$$\begin{aligned}
&U_q^\sigma(x_\tau(\phi,\tilde{\sigma})) = \sup_{t\geq 0}\left(\max\left\{0, \overline{\alpha}_1\left(\|x_t(x_\tau(\phi,\tilde{\sigma}),\sigma)\|_\infty\right) - \frac{1}{q}\right\} e^t\right) \\
&\leq \sup_{t\geq 0,\sigma\in\mathcal{S}}\left(\max\left\{0, \overline{\alpha}_1\left(\|x_{t+\tau}(\phi,\sigma)\|_\infty\right) - \frac{1}{q}\right\} e^t\right). \\
&\leq e^{-\tau}\sup_{t\geq 0,\sigma\in\mathcal{S}}\left(\max\left\{0, \overline{\alpha}_1\left(\|x_t(\phi,\sigma)\|_\infty\right) - \frac{1}{q}\right\} e^t\right). \\
&= e^{-\tau}U_q(\phi).
\end{aligned}$$

Inequality 2) follows from (26) by applying the supremum over $\sigma \in \mathcal{S}$. As far as the point 3) is concerned, let H be a positive real and let $\phi, \psi \in \mathcal{C}_H$. Following the same steps as in the proof of [12, Theorem 2.9], we have

$$|U_q^\sigma(\phi) - U_q^\sigma(\psi)| \leq G(H,q)\|\phi-\psi\|_\infty, \quad \forall\, \sigma \in \mathcal{S}. \tag{26}$$

By consequence, knowing that the right hand side of inequality (27) is independent from σ, we have

$$|U_q(\phi) - U_q(\psi)| = \left|\sup_{\sigma\in\mathcal{S}} U_q^\sigma(\phi) - \sup_{\sigma\in\mathcal{S}} U_q^\sigma(\psi)\right| \leq \sup_{\sigma\in\mathcal{S}}\left|U_q^\sigma(\phi) - U_q^\sigma(\psi)\right| \leq G(H,q)\|\phi-\psi\|_\infty. \tag{27}$$

The proof of the lemma is complete. □

7.2 Proof of Theorem 1

Now we prove the sufficiency part of Theorem 1. Let $x_0 \in \mathcal{C}$ and let $\sigma \in \mathcal{S}$. Let $x(\cdot, x_0, \sigma)$ be the corresponding solution in a maximal interval $[0, b)$, $0 < b \leq +\infty$. Let $w : [0, b) \to \mathbb{R}_+$ be the function which is defined in $[0, b)$ as

$$w(t) = V(x_t(x_0,\sigma)), \quad t \in [0,b).$$

The function w is continuous in $[0, b)$. By [5, 20], we have, for $t \in [0, b)$,

$$\begin{aligned}
D^+w(t) &= \limsup_{h\to 0^+}\frac{w(t+h)-w(t)}{h} \\
&= \limsup_{h\to 0^+}\frac{V\left(x_{t+h}(x_0,\sigma)\right) - V(x_t(x_0,\sigma))}{h} \\
&= \limsup_{h\to 0^+}\frac{V\left((x_t(x_0,\sigma))_h^{\Sigma,\sigma(t)}\right) - V(x_t(x_0,\sigma))}{h} \\
&\leq \sup_{s\in \mathrm{S}}\limsup_{h\to 0^+}\frac{V\left((x_t(x_0,\sigma))_h^{\Sigma,s}\right) - V(x_t(x_0,\sigma))}{h} \\
&= D^+V(x_t(x_0,\sigma)).
\end{aligned} \tag{28}$$

By inequality (ii), it follows from (28) that

$$D^+w(t) \leq -\alpha_3(|x(t,x_0,\sigma)|) \leq 0, \quad \forall\, t \in [0,b). \tag{29}$$

Since w is continuous, it follows from [6] that

$$w(t) \leq w(0), \quad \forall\, t \in [0,b). \tag{30}$$

Finally, from (30) and the inequalities (i), we obtain

$$|x(t, x_0, \sigma)| \leq \alpha_1^{-1} \circ \alpha_2(\|x_0\|_\infty), \quad \forall\, t \in [0, b). \tag{31}$$

It follows, from Lemma 1, that $b = +\infty$. So, for any $\varepsilon > 0$, if $\delta = \alpha_2^{-1} \circ \alpha_1(\varepsilon)$, $\|x_0\|_\infty \leq \delta$ implies $|x(t, x_0, \sigma)| \leq \varepsilon$, for $t \geq 0$. Now, by analogous reasoning as in [10], let H be a positive real. Let ε be an arbitrary positive real. We wish to show that there exists a positive real T such that, for any $\sigma \in \mathcal{S}$, for any $x_0 \in \mathcal{C}_H$, the inequality $|x(t, x_0, \sigma)| \leq \varepsilon$ holds for all $t \geq T$. Notice that, for any $\sigma \in \mathcal{S}$ and for any $x_0 \in \mathcal{C}_H$, by (31), the following inequality holds

$$\|x_t(x_0, \sigma)\|_\infty \leq H + \alpha_1^{-1} \circ \alpha_2(H), \quad \forall\, t \geq 0. \tag{32}$$

Let $\delta = \alpha_2^{-1} \circ \alpha_1(\varepsilon)$. Let $\overline{k}$ be the smallest integer satisfying

$$\overline{k} \geq \frac{\Gamma \alpha_2(H)}{\delta \alpha_3(\frac{\delta}{2})} + 1,$$

where

$$\Gamma = \max \left\{ \sup_{s \in \mathrm{S},\ \phi \in \mathcal{C}_{H + \alpha_1^{-1} \circ \alpha_2(H)}} |f_s(\phi)|, \frac{2\delta}{\Delta} \right\}. \tag{33}$$

From (32) and (33), it follows that, for any $s \in \mathrm{S}$ and any $x_0 \in \mathcal{C}_H$, $|f_s(x_t(x_0, \sigma))| \leq \Gamma$, for any $t \geq 0$. Let $T = 2\overline{k}\Delta$. Let $x_0 \in \mathcal{C}_H$ and let $\sigma \in \mathcal{S}$. Then, if the corresponding solution satisfies $\|x_\tau(x_0, \sigma)\|_\infty \leq \delta$, for some $\tau \in [0, T]$, the inequality $|x(t, x_0, \sigma)| \leq \varepsilon$ holds, $\forall\, t \geq T$. Provided that σ is fixed in $\mathcal{S}$, the existence of such τ can be proved by following the same reasoning as in [10], and the proof of the sufficiency part is complete, since T does not depend on σ.

Let us prove now the necessity part. Let $V : \mathcal{C} \to \mathbb{R}_+$ be the functional defined, for $\phi \in \mathcal{C}$, as

$$V(\phi) = \sum_{q=1}^{+\infty} \frac{2^{-q}}{1 + G(q, q)} U_q(\phi), \tag{34}$$

where $G : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ is the function defined in (23) and $U_q : \mathcal{C} \to \mathbb{R}_+$ are the functionals defined in (22). Using the same reasoning as in the proof of [22, Lemma A.8] we have that the inequalities (i) hold true. Now, we prove that the inequality (ii) holds. Let, for $s \in \mathrm{S}$, $\sigma_s \in \mathcal{S}$ be defined, for $t \in \mathbb{R}_+$, as $\sigma_s(t) = s$. For any $\phi \in \mathcal{C}$, the following inequalities hold

$$\begin{aligned}
D^+V(\phi) &= \sup_{s \in \mathrm{S}} \limsup_{h \to 0^+} \frac{V(\phi_h^{\Sigma,s}) - V(\phi)}{h} = \sup_{s \in \mathrm{S}} \limsup_{h \to 0^+} \frac{V(x_h(\phi, \sigma_s)) - V(\phi)}{h} \\
&= \sup_{s \in \mathrm{S}} \limsup_{h \to 0^+} \frac{\sum_{q=1}^{+\infty} \frac{2^{-q}}{1 + G(q, q)} \left(U_q(x_h(\phi, \sigma_s) - U_q(\phi)\right)}{h} \\
&\leq \sup_{s \in \mathrm{S}} \limsup_{h \to 0^+} \frac{\sum_{q=1}^{+\infty} \frac{2^{-q}}{1 + G(q, q)} U_q(\phi) \left(e^{-h} - 1\right)}{h} = \\
&\sup_{s \in \mathrm{S}} \limsup_{h \to 0^+} \frac{e^{-h} - 1}{h} V(\phi) = -V(\phi) \leq -\alpha_1(\|\phi\|_\infty).
\end{aligned}$$

So the inequality (ii) is satisfied with the $\mathcal{K}_\infty$ function $\alpha_3(s) = \alpha_1(s)$. The fact that the functional V is Lipschitz on bounded subsets of $\mathcal{C}$ follows exactly the same lines of the proof of [22, Lemma A.8]. The proof of the necessity part is complete. The proof of Theorem 1 is complete.

8 Proof of Theorem 2

We prove first the sufficiency part. Let $x_0 \in \mathcal{C}$ and let $\sigma \in \mathcal{S}$. Let $x(\cdot, x_0, \sigma)$ be the corresponding solution in a maximal interval $[0, b)$, $0 < b \leq +\infty$. Let $w : [0, b) \to \mathbb{R}_+$ be the function which is defined in $[0, b)$ as

$$w(t) = V(x_t(x_0, \sigma)), \quad t \in [0, b).$$

Following the same reasoning as in the proof of the sufficiency part of Theorem 1 and using inequalities (i) and (ii), we obtain

$$D^+ w(t) \leq -\alpha_3 \|x_t(x_0, \sigma)\|_\infty^p \leq -\frac{\alpha_3}{\alpha_2} w(t), \quad \forall\, t \in [0, b). \tag{35}$$

Since w is continuous, it follows from [6] that

$$w(t) \leq e^{-\frac{\alpha_3}{\alpha_2} t} w(0), \quad \forall\, t \in [0, b). \tag{36}$$

Finally, from (36) and the inequalities (i), we obtain

$$|x(t, x_0, \sigma)| \leq \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p}} e^{-\frac{\alpha_3}{p\alpha_2} t} \|x_0\|_\infty, \quad \forall\, t \in [0, b). \tag{37}$$

It follows, from Lemma 1, that $b = +\infty$, and, given the arbitrarity of $x_0 \in \mathcal{C}$ and $\sigma \in \mathcal{S}$, the PC-0-GES property of the system Σ is proved. The proof of the sufficiency part of the theorem is complete.

Let us prove now the necessity part. Let $M \geq 1$ and λ be the positive reals such that, for any $x_0 \in \mathcal{C}$ and $\sigma \in \mathcal{S}$, the following inequality holds

$$|x(t, x_0, \sigma)| \leq M e^{-\lambda t} \|x_0\|_\infty. \tag{38}$$

By analogous reasoning as in [13], let $T = \frac{1}{\lambda} \ln(2M)$. Let $V : \mathcal{C} \to \mathbb{R}_+$ be the functional defined, for $\phi \in \mathcal{C}$, as follows

$$V(\phi) = \sup_{\sigma \in \mathcal{S}} \int_0^T \|x_t(\phi, \sigma)\|_\infty dt + \sup_{\sigma \in \mathcal{S}} \sup_{t \in [0,T]} \|x_t(\phi, \sigma)\|_\infty. \tag{39}$$

Let us prove, first, that the functional V is Lipschitz on bounded subsets of $\mathcal{C}$. For this, let H be any positive real and let $\phi, \psi \in \mathcal{C}_H$. Let

$$A = \sup_{\sigma \in \mathcal{S}} \int_0^T \|x_t(\phi, \sigma)\|_\infty dt - \sup_{\sigma \in \mathcal{S}} \int_0^T \|x_t(\psi, \sigma)\|_\infty dt \tag{40}$$

and

$$B = \sup_{\sigma \in \mathcal{S}} \sup_{t \in [0,T]} \|x_t(\phi, \sigma)\|_\infty - \sup_{\sigma \in \mathcal{S}} \sup_{t \in [0,T]} \|x_t(\psi, \sigma)\|_\infty. \tag{41}$$

We have

$$|A| \quad \leq \quad \sup_{\sigma \in \mathcal{S}} \int_0^T |\|x_t(\phi, \sigma)\|_\infty - \|x_t(\psi, \sigma)\|_\infty| d \leq \sup_{\sigma \in \mathcal{S}} \int_0^T \|x_t(\phi, \sigma) - x_t(\psi, \sigma)\|_\infty dt \tag{42}$$

From Lemma 3 and inequality (42) it follows that

$$|A| \quad \leq \quad \sup_{\sigma \in \mathcal{S}} \int_0^T e^{L(\|\phi\|_\infty + \|\psi\|_\infty)T} \|\phi - \psi\|_\infty dt \leq T e^{\overline{L}(4MH)T} \|\phi - \psi\|_\infty.$$

Moreover, as far as the term B is concerned, we have, taking into account (20),

$$\begin{aligned}
|B| &\leq \sup_{\sigma\in\mathcal{S}}\left|\sup_{t\in[0,T]}\|x_t(\phi,\sigma)\|_\infty - \sup_{t\in[0,T]}\|x_t(\psi,\sigma)\|_\infty\right| \\
&\leq \sup_{\sigma\in\mathcal{S}}\sup_{t\in[0,T]}|\|x_t(\phi,\sigma)\|_\infty - \|x_t(\psi,\sigma)\|_\infty| \\
&\leq \sup_{\sigma\in\mathcal{S}}\sup_{t\in[0,T]}\|x_t(\phi,\sigma) - x_t(\psi,\sigma)\|_\infty \\
&\leq e^{\overline{L}(4MH)T}\|\phi-\psi\|_\infty.
\end{aligned}$$

Finally, we have the following inequalities

$$|V(\phi)-V(\psi)| \leq |A|+|B| \leq (1+T)e^{\overline{L}(4MH)T}\|\phi-\psi\|_\infty.$$

So, the Lipschitz property of the functional V, on bounded subsets of $\mathcal{C}$, is proved. Observe that if f_s, $s\in\mathrm{S}$, is uniformly globally Lipschitz then V is globally Lipschitz.
Now, we prove that the inequalities (i) and (ii) hold. The first inequality in (i) is satisfied with $\alpha_1 = 1$. As far as the second inequality in (i) is concerned, the following inequalities hold, for any $\phi\in\mathcal{C}$,

$$V(\phi) \leq \sup_{\sigma\in\mathcal{S}}\int_0^T Me^{-\lambda t}\|\phi\|_\infty dt + \sup_{\sigma\in\mathcal{S}}\sup_{t\in[0,T]} Me^{-\lambda t}\|\phi\|_\infty \leq \left(\int_0^T Me^{-\lambda t}dt + M\right)\|\phi\|_\infty. \tag{43}$$

So, the second inequality in (i) is satisfied with $\alpha_2 = \int_0^T Me^{-\lambda t}dt + M$. As far as the inequality (ii) is concerned, let $V_1:\mathcal{C}\to\mathbb{R}_+$ and $V_2:\mathcal{C}\to\mathbb{R}_+$ be defined, for $\phi\in\mathcal{C}$, as

$$V_1(\phi) = \sup_{\sigma\in\mathcal{S}}\int_0^T \|x_t(\phi,\sigma)\|_\infty dt \tag{44}$$

$$V_2(\phi) = \sup_{\sigma\in\mathcal{S}}\sup_{t\in[0,T]}\|x_t(\phi,\sigma)\|_\infty. \tag{45}$$

Let, for $s\in\mathrm{S}$, $\sigma_s\in\mathcal{S}$ be defined, for $t\in\mathbb{R}_+$, as $\sigma_s(t)=s$. The following inequality holds, for any $\phi\in\mathcal{C}$

$$D^+V(\phi) \leq \sup_{s\in\mathrm{S}}\left(\limsup_{h\to0^+}\frac{V_1(x_h(\phi,\sigma_s))-V_1(\phi)}{h} + \limsup_{h\to0^+}\frac{V_2(x_h(\phi,\sigma_s))-V_2(\phi)}{h}\right). \tag{46}$$

As far as the first limit in the right-hand side of the inequality in (46) is concerned, let us define, for given $s\in\mathrm{S}$, and for given positive real h, the set $M_{\mathrm{S}}^{h,s}$ as follows (see, [11])

$$M_{\mathrm{S}}^{h,s} = \{\sigma\in\mathcal{S} : \sigma(t)=s,\ \forall\, t\in[0,h)\}. \tag{47}$$

So, we have the following equalities/inequality,

$$\begin{aligned}
&V_1(\phi) = \sup_{\sigma\in\mathcal{S}}\int_0^T \|x_t(\phi,\sigma)\|_\infty dt \qquad (48)\\
&= \sup_{\sigma\in\mathcal{S}}\left(\int_0^h \|x_t(\phi,\sigma)\|_\infty dt + \int_h^T \|x_t(\phi,\sigma)\|_\infty dt\right) \\
&\geq \sup_{\sigma\in M_{\mathrm{S}}^{h,s}}\left(\int_0^h \|x_t(\phi,\sigma)\|_\infty dt + \int_h^T \|x_t(\phi,\sigma)\|_\infty dt\right) \\
&= \int_0^h \|x_t(\phi,\sigma_s)\|_\infty dt + \sup_{\sigma\in M_{\mathrm{S}}^{h,s}}\int_h^T \|x_t(\phi,\sigma)\|_\infty dt.
\end{aligned}$$

Furthermore, we have the following equalities/inequality

$$\begin{aligned}V_1(x_h(\phi,\sigma_s)) &= \sup_{\sigma\in\mathcal{S}}\int_0^T \|x_t(x_h(\phi,\sigma_s),\sigma)\|_\infty dt = \sup_{\sigma\in M_{\mathrm{S}}^{h,s}}\int_h^{T+h}\|x_t(\phi,\sigma)\|_\infty dt \\ &\leq \sup_{\sigma\in M_{\mathrm{S}}^{h,s}}\int_h^T\|x_t(\phi,\sigma)\|_\infty dt + \sup_{\sigma\in M_{\mathrm{S}}^{h,s}}\int_T^{T+h}\|x_t(\phi,\sigma)\|_\infty dt.\end{aligned}\tag{49}$$

So, from (48) and (49), it follows that

$$\begin{aligned}V_1(\phi) &\geq \int_0^h\|x_t(\phi,\sigma_s)\|_\infty dt + \sup_{\sigma\in M_{\mathrm{S}}^{h,s}}\int_h^T\|x_t(\phi,\sigma)\|_\infty dt \\ &\geq \int_0^h\|x_t(\phi,\sigma_s)\|_\infty dt + V_1(x_h(\phi,\sigma_s)) - \sup_{\sigma\in M_{\mathrm{S}}^{h,s}}\int_T^{T+h}\|x_t(\phi,\sigma)\|_\infty dt.\end{aligned}\tag{50}$$

From, (50), the following inequality holds

$$V_1(x_h(\phi,\sigma_s)) - V_1(\phi) \leq \sup_{\sigma\in M_{\mathrm{S}}^{h,s}}\int_T^{T+h}\|x_t(\phi,\sigma)\|_\infty dt - \int_0^h\|x_t(\phi,\sigma_s)\|_\infty dt.\tag{51}$$

Finally, from (51), the following equalities/inequalities hold for the first limit in the right-hand side of (46)

$$\begin{aligned}\limsup_{h\to0^+}\frac{V_1(x_h(\phi,\sigma_s))-V_1(\phi)}{h} &\leq \limsup_{h\to0^+}\frac{1}{h}\left(\int_T^{T+h}Me^{-\lambda t}\|\phi\|_\infty dt - \int_0^h\|x_t(\phi,\sigma_s)\|_\infty dt\right) \\ &= -\left(1-Me^{-\lambda T}\right)\|\phi\|_\infty = -\frac{1}{2}\|\phi\|_\infty.\end{aligned}\tag{52}$$

As far as the second limit in the right-hand side of the inequality in (46) is concerned, we have

$$\begin{aligned}\limsup_{h\to0^+}\frac{V_2(x_h(\phi,\sigma_s))-V_2(\phi)}{h} &\leq \limsup_{h\to0^+}\frac{1}{h}\left(\sup_{\sigma\in\mathcal{S}}\sup_{t\in[0,T]}\|x_{t+h}(\phi,\sigma)\|_\infty - \sup_{\sigma\in\mathcal{S}}\sup_{t\in[0,T]}\|x_t(\phi,\sigma)\|_\infty\right) \\ &\leq \limsup_{h\to0^+}\frac{1}{h}\sup_{\sigma\in\mathcal{S}}\left(\sup_{t\in[0,T+h]}\|x_t(\phi,\sigma)\|_\infty - \sup_{t\in[0,T]}\|x_t(\phi,\sigma)\|_\infty\right).\end{aligned}\tag{53}$$

Since, for any $\sigma\in\mathcal{S}$, the following inequalities hold

$$\sup_{t\in[T,T+h]}\|x_t(\phi,\sigma)\|_\infty \leq Me^{-\lambda T}\|\phi\|_\infty = \frac{1}{2}\|\phi\|_\infty,\tag{54}$$

$$\sup_{t\in[0,T]}\|x_t(\phi,\sigma)\|_\infty \geq \|\phi\|_\infty,\tag{55}$$

it follows that, for any $\sigma\in\mathcal{S}$, the following inequality holds

$$\sup_{t\in[h,T+h]}\|x_t(\phi,\sigma)\|_\infty \leq \sup_{t\in[0,T]}\|x_t(\phi,\sigma)\|_\infty.\tag{56}$$

Therefore, from (53) and (56), it follows that

$$\limsup_{h\to0^+}\frac{V_2(x_h(\phi,\sigma_s))-V_2(\phi)}{h} \leq 0.\tag{57}$$

Finally, from (46), (52) and (57), it follows that

$$D^+V(\phi) \leq -\frac{1}{2}\|\phi\|_\infty.\tag{58}$$

So the inequality (ii) is satisfied with $\alpha_3 = \frac{1}{2}$. The proof of the necessity part is complete. The proof of Theorem 2 is complete.

9 Proof of Theorem 3

Proof. We prove first that the theorem holds with bounded piecewise continuous inputs. For this, let $x_0 \in \mathcal{C}$, $\sigma \in \mathcal{S}$ be fixed and let u be a piecewise continuous function such that $\|u\|_\infty \leq v$, for a suitable $v \geq 0$. Let $x(\cdot, x_0, u, \sigma)$ be the corresponding solution in a maximal interval $[0, b)$, $0 < b \leq +\infty$ (this is guaranteed using the same arguments as in Lemma 1). Knowing that system (8), with $u \equiv 0$, is PC-0-GES, then, thanks to Theorem 2, there exist a globally Lipschitz functional $V : \mathcal{C} \to \mathbb{R}_+$, with Lipschitz constant $\alpha_4 > 0$, such that inequalities (i) and (ii) of Theorem 2 hold. Let $w : [0, b) \to \mathbb{R}_+$ be the function which is defined in $[0, b)$ as

$$w(t) = V(x_t(x_0, u, \sigma)), \quad t \in [0, b).$$

The function w is continuous in $[0, b)$. In addition, for $t \in [0, b)$, we have

$$\begin{aligned} D^+ w(t) &= \limsup_{h\to 0^+} \frac{w(t+h) - w(t)}{h} \\ &= \limsup_{h\to 0^+} \frac{V(x_{t+h}(x_0, u, \sigma)) - V(x_t(x_0, u, \sigma))}{h} \\ &\leq \limsup_{h\to 0^+} \frac{V(x_h(x_t(x_0, u, \sigma), 0, \sigma(t+\cdot))) - V(x_t(x_0, u, \sigma))}{h} \\ &+ \limsup_{h\to 0^+} \frac{V(x_{t+h}(x_0, u, \sigma)) - V(x_h(x_t(x_0, u, \sigma), 0, \sigma(t+\cdot)))}{h} \\ &\leq -\alpha_3 \|x_t(x_0, u, \sigma)\|_\infty + \alpha_4 l \max\{\|x_t(x_0, u, \sigma)\|_\infty^p, 1\} v. \end{aligned}$$

Let $\omega \in \mathcal{K}_\infty$ defined by $\omega(s) = \frac{\alpha_3}{2l\alpha_4} \min\{s, s^{1-p}\}$. Let us introduce the set $\mathcal{I} = \{\varphi \text{ is continuous} : V(\varphi) \leq \alpha_2 \omega^{-1}(v)\}$. Following the same reasoning as in [21], we obtain that the set $\mathcal{I}$ is forward invariant. Now, suppose that $x_0 \notin \mathcal{I}$ and let $c \geq 0$ be defined as follows

$$c = \begin{cases} b & \text{if } x_t \notin \mathcal{I}, \forall t \in [0, b), \\ \bar{t} & \text{if } x_t \notin \mathcal{I}, \forall t \in [0, \bar{t}) \text{ and } x_{\bar{t}} \in \mathcal{I}. \end{cases} \tag{59}$$

On one hand, we have

$$|x(t, x_0, u, \sigma)| \leq \frac{\alpha_2}{\alpha_1} e^{-\frac{\alpha_3}{2\alpha_2} t} \|x_0\|_\infty, \quad \forall t \in [0, c). \tag{60}$$

On the other hand, we have

$$|x(t, x_0, u, \sigma)| \leq \frac{\alpha_2}{\alpha_1} \omega^{-1}(v), \quad \forall t \in [c, b). \tag{61}$$

Therefore, the following inequality holds

$$|x(t, x_0, u, \sigma)| \leq \beta(\|x_0\|_\infty, t) + \gamma(v), \quad \forall t \in [0, b), \tag{62}$$

with $\beta(s, t) = \frac{\alpha_2}{\alpha_1} e^{-\frac{\alpha_3}{2\alpha_2} t} s$ and $\gamma(s) = \frac{\alpha_2}{\alpha_1} \omega^{-1}(s)$. Using the same arguments as in the proof of Theorem (2), it follows that $b = +\infty$. By causality arguments, it follows that, for any $x_0 \in \mathcal{C}$, $\sigma \in \mathcal{S}$ and any piecewise continuous function u inequality (9) holds with the previously defined functions β and γ. $\square$

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