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Computing highest-order divisors for a class of quasi-linear partial differential equations

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Abstract

A differential polynomial G is called a divisor of a differential polynomial F if any solution of the differential equation $G = 0$ is a solution of the equation $F = 0$. We design an algorithm which for a class of quasi-linear partial differential polynomials of order $k + 1$ finds its quasi-linear divisors of order k .

Keywords: quasi-linear differential polynomial, divisor, algorithm

Introduction

The problem of factoring linear ordinary differential operators $L = T \circ Q$ was studied in [15]. Algorithms for this problem were designed in [8], [16] (in [8] a complexity bound better than for the algorithm from [15] was established). An algorithm is exhibited in [10] for factoring a partial linear differential operator in two variables with a separable symbol. In [9], an algorithm is constructed for finding all first-order factors of a partial linear differential operator in two variables. A generalization of factoring for D -modules (in other words, for systems of linear partial differential operators) was considered in [11, 17]. A particular case of factoring for D -modules is the Laplace problem [6, 19] (one can find a short exposition of the Laplace problem in [12]).

The meaning of factoring for search of solutions is that any solution of operator Q is a solution of operator L , thus, factoring allows one to diminish the order of operators.

Much less is known for factoring non-linear (even ordinary) differential equations.

We note that our definition of divisors is in the frame of differential ideals [14], rather than the definition of factorization from [18, 4] being in terms of a composition of nonlinear ordinary differential polynomials. In [4], a decomposition algorithm is designed.

We consider partial differential polynomials viewing them as polynomials in independent variables $x_1, \dots, x_n$ and in derivatives

$$\frac{d^{i_1+\dots+i_n}u}{dx_1^{i_1}\dots dx_n^{i_n}}$$

[14]. We study a class of *quasi-linear* differential polynomials in which the coefficients at all its highest derivatives, i. e., with the biggest value of the order $i_1 + \dots + i_n$, are constants.

We design an algorithm which for a given quasi-linear differential polynomial F of order $k + 1$ finds the algebraic variety of all its quasi-linear divisors G of order k . Moreover, we show that in this case, $\deg G \leq \deg F$ (treating F and G as algebraic polynomials). This result generalizes [13] where an algorithm was designed for finding quasi-linear divisors for quasi-linear *ordinary* differential polynomials F of order $k = 2$.

In Section 1, we bound the degree of a divisor, and in Section 2, we describe the algorithm.

It would be interesting to find divisors of F of arbitrary orders (rather than just of k) even in the case of ordinary differential equations. Also an extension to arbitrary differential polynomials (rather than quasi-linear) looks as a challenge.

Another issue to be studied is constructing a common multiple of a pair of partial differential polynomials, i. e., a differential polynomial whose solutions contain the solutions of both differential polynomials; for the case of quasi-linear ordinary differential polynomials, an algorithm was designed in [13].

1 Bound on a Degree of a Divisor

We study partial differential polynomials, i. e., polynomials of the form

$$F(\dots, \frac{d^{i_1+\dots+i_n}u}{dx_1^{i_1}\dots dx_n^{i_n}}, \dots, x_1, \dots, x_n)$$

with coefficients over $\overline{\mathbb{Q}}$ where the maximal value of $i_1 + \dots + i_n$ is denoted by $\text{ord } F$ (the order of F) [14]. We denote the differential ring of all partial differential polynomials by D .

Definition 1.1 *A differential polynomial G is a divisor of F if any solution u from the universal extension (see, e. g., p. 133 [14]) of the field of rational functions $\overline{\mathbb{Q}}(x_1, \dots, x_n)$ of equation $G = 0$ is a solution of $F = 0$ as well.*

Due to the differential Nullstellensatz (see, e. g., Corollary 1 p. 148 [14]) a differential polynomial G is a divisor of F iff F belongs to the radical differential ideal generated by G . We mention that a bound being in general not primitive-recursive, for the differential Nullstellensatz was established in [5].

We say that F of order $k + 1$ is *quasi-linear* if

$$F = \sum_{i_1+\dots+i_n=k+1} a_{i_1,\dots,i_n} \cdot \frac{d^{k+1}u}{dx_1^{i_1}\dots dx_n^{i_n}} + f$$

where coefficients $a_{i_1, \dots, i_n} \in \overline{\mathbb{Q}}$ and $\text{ord } f \leq k$.

In the present section we provide an algebraic criterion for a quasi-linear G of order k to be a divisor of F and bound the degree of G .

For the sake of simplifying notations we will assume that there are just two independent variables x, y , i. e., $n = 2$. Denote a quasi-linear differential polynomial

$$F = \sum_{0 \leq i \leq k+1} a_i \cdot \frac{d^{k+1}u}{dx^i dy^{k+1-i}} + f. \quad (1)$$

Let a quasi-linear differential polynomial

$$G = \sum_{0 \leq i \leq k} b_i \cdot \frac{d^k u}{dx^i dy^{k-i}} + g \quad (2)$$

be a divisor of F where $\text{ord } g \leq k-1$ and $b_0, \dots, b_k \in \overline{\mathbb{Q}}$. Making a $\overline{\mathbb{Q}}$ -linear transformation of the independent variables x, y one can assume w.l.o.g. that $b_0 = 1$.

Theorem 1.2 *i) A quasi-linear differential polynomial G of order k is a divisor of a quasi-linear differential polynomial F of order $k+1$ (with $\deg F = d$) iff G divides (as polynomials)*

$$\left(F - c_1 \cdot \frac{dG}{dx} - c_2 \cdot \frac{dG}{dy} \right)^d$$

where

$$\text{ord} \left(F - c_1 \cdot \frac{dG}{dx} - c_2 \cdot \frac{dG}{dy} \right) \leq k$$

for suitable (unique) $c_1, c_2 \in \overline{\mathbb{Q}}$.

ii) In this case, $\deg G \leq \deg F$.

Introduce the *highest order derivatives forms* being homogeneous polynomials

$$A := \sum_{0 \leq i \leq k+1} a_i \cdot v^i \cdot w^{k+1-i}, \quad B := \sum_{0 \leq i \leq k} b_i \cdot v^i \cdot w^{k-i} \in \overline{\mathbb{Q}}[v, w]$$

of the differential polynomials F and G , respectively.

Lemma 1.3 *If a quasi-linear differential polynomial G with $\text{ord } G = k$ is a divisor of a quasi-linear differential polynomial F with $\text{ord } F = k+1$ then there exist unique $c_1, c_2 \in \overline{\mathbb{Q}}$ such that $(c_1 \cdot v + c_2 \cdot w) \cdot B = A$, in other words $B|A$. Moreover, in this case $\text{ord}(F - c_1 \cdot \frac{dG}{dx} - c_2 \cdot \frac{dG}{dy}) \leq k$.*

Proof of Lemma. Due to the differential Nullstellensatz we have for suitable integer m

$$F^m = \sum_q H_q \cdot G_q \quad (3)$$

where G_q are certain partial derivatives of G and $H_q \in D$. Introduce variables $u_{i,j}$ for $\frac{d^{i+j}u}{dx^i dy^j}$ and making use repeatedly of relations $\frac{du_{i,j}}{dx} = u_{i+1,j}$, $\frac{du_{i,j}}{dy} = u_{i,j+1}$ we can consider (3) as an equality of polynomials in the variables $\{u_{i,j}\}_{i,j}$, x , y . Let a derivative of G of an order higher than 1 occur in (3) and denote by $s \geq 2$ the highest order of derivatives of G occurring in (3).

Taking appropriate $\overline{\mathbb{Q}}$ -linear combinations of the equations

$$\frac{d^s G}{dx^i dy^{s-i}} = 0, \quad 0 \leq i \leq s,$$

and considering their highest order derivatives one can express the variables

$$u_{j,s+k-j} = \sum_{s < l \leq s+k} c_l \cdot u_{l,s+k-l} + g_j, \quad 0 \leq j \leq s \quad (4)$$

for suitable coefficients $c_l \in \overline{\mathbb{Q}}$ and differential polynomials g_j with $\text{ord } g_j < s + k$. Substituting expressions (4) into (3) we get rid of all the derivatives G_q of G of order s . Observe that these substitutions do not change the left-hand side of (3). After that substitute 0 in all H_q for variables $u_{l,s+k-l}$, $s < l \leq s + k$ and for all variables $u_{i,j}$ with $i + j > s + k$, we obtain a formula similar to (3) with orders of derivatives G_q of G less than s and with variables $u_{i,j}$ occurring in G_q and H_q satisfying $i + j < s + k$.

Continuing in this way, we get rid of all the variables $u_{i,j}$ in the right-hand side of (3) with $i + j > k + 1$.

After that we employ formulae (4) with $s = 1$ to achieve that the differential polynomial $F_0 := F - c_1 \cdot \frac{dG}{dx} - c_2 \cdot \frac{dG}{dy}$ does not contain derivatives $u_{0,k+1}$, $u_{1,k}$ for suitable $c_1, c_2 \in \overline{\mathbb{Q}}$. Then (3) implies that

$$F_0^m = H^{(1)} \cdot \frac{dG}{dx} + H^{(2)} \cdot \frac{dG}{dy} + H^{(0)} \cdot G \quad (5)$$

for some differential polynomials $H^{(1)}$, $H^{(2)}$, $H^{(0)}$ of orders at most $k + 1$. Now substitute formulae (4) with $s = 1$ in formula (5), this results in

$$F_0^m = H \cdot G \quad (6)$$

for appropriate $H \in D$. Therefore, since F_0 contains derivatives of order $k + 1$ with constant coefficients, all these coefficients vanish, thus, $\text{ord } F_0 \leq k$, hence $\text{ord } H \leq k$. Consequently,

$$F_0 = f - c_1 \cdot \frac{dg}{dx} - c_2 \cdot \frac{dg}{dy} \quad (7)$$

(see (1), (2)) and $(c_1 \cdot v + c_2 \cdot w) \cdot B = A$. The Lemma is proved. $\square$

Proof of Theorem. Substitute formulae

$$\frac{dg}{dx} = \sum_{i+j \leq k-1} \frac{\partial g}{\partial u_{i,j}} \cdot u_{i+1,j} + \frac{\partial g}{\partial x}; \quad \frac{dg}{dy} = \sum_{i+j \leq k-1} \frac{\partial g}{\partial u_{i,j}} \cdot u_{i,j+1} + \frac{\partial g}{\partial y} \quad (8)$$

in (7), and we substitute the obtained expression for F_0 in the left-hand side of (6), then we substitute in the resulting formula the expression for $u_{0,k} = -\sum_{1 \leq i \leq k} b_i \cdot u_{i,k-i} - g$ from (2). After the latter substitution, the right-hand side of (6) vanishes, and we deduce (taking into account (2)) the equality

$$0 = f|_{(u_{0,k} = -\sum_{1 \leq i \leq k} b_i \cdot u_{i,k-i} - g)} - c_1 \cdot \left(\sum_{i+j \leq k-1} \frac{\partial g}{\partial u_{i,j}} \cdot u_{i+1,j} + \frac{\partial g}{\partial x} \right) + \quad (9)$$

$$c_2 \cdot \left(\frac{\partial g}{\partial u_{0,k-1}} \left(\sum_{1 \leq i \leq k} b_i u_{i,k-i} + g \right) - \sum_{i+j \leq k-1, (i,j) \neq (0,k-1)} \frac{\partial g}{\partial u_{i,j}} u_{i,j+1} - \frac{\partial g}{\partial y} \right) \quad (10)$$

One can rewrite

$$f|_{(u_{0,k} = -\sum_{1 \leq i \leq k} b_i \cdot u_{i,k-i} - g)} = f|_{(u_{0,k} = -\sum_{1 \leq i \leq k} b_i \cdot u_{i,k-i})} + h \cdot g$$

for suitable $h \in D$. Therefore, (9) and (10) imply the following divisibility relation

$$g|(f|_{(u_{0,k} = -\sum_{1 \leq i \leq k} b_i \cdot u_{i,k-i})} - c_1 \cdot \left(\sum_{i+j \leq k-1} \frac{\partial g}{\partial u_{i,j}} \cdot u_{i+1,j} + \frac{\partial g}{\partial x} \right) + \quad (11)$$

$$c_2 \cdot \left(\frac{\partial g}{\partial u_{0,k-1}} \cdot \sum_{1 \leq i \leq k} b_i \cdot u_{i,k-i} - \sum_{i+j \leq k-1, (i,j) \neq (0,k-1)} \frac{\partial g}{\partial u_{i,j}} \cdot u_{i,j+1} - \frac{\partial g}{\partial y} \right)) \quad (12)$$

Denote the polynomial in the variables $\{u_{i,j}\}_{i,j}$, x , y in the right-hand side of (11), (12) by P .

Our goal is to prove that $\deg g \leq \deg f$. Suppose the contrary. Then (11), (12) entail that $\deg P \leq \deg g$ (taking into account that $\deg f \leq \deg g$ by the supposition) and whence $P = c \cdot g$ for appropriate $c \in \overline{\mathbb{Q}}$. Consider a linear deglex ordering $\prec$ of monomials in $\{u_{i,j}\}_{i+j \leq k-1}$, x , y in which $u_{i,j} \prec u_{l,s}$ when $i+j > l+s$ (the remaining requirements on the ordering do not matter). We observe that the highest (w.r.t. $\prec$) monomial in g cannot occur in P since $\deg f < \deg g$ by the supposition. This leads to a contradiction with the equality $P = c \cdot g$ which proves inequality $\deg g \leq \deg f$. Summarizing, we conclude Theorem 1.2 ii).

To prove Theorem 1.2 i) in the direction when G is a divisor of F we apply Lemma 1.3 and note that one can take $m = d$ in (6) owing to Theorem 1.2 ii) because if $G|F_0^m$ for some m then $G|F_0^{\deg G}$. To prove the converse we observe that $G|(F_0 - c_1 \cdot \frac{dg}{dx} - c_2 \cdot \frac{dg}{dy})^d$ implies (3) (with $m = d$). $\square$

We present the following simple example just to illustrate the notations.

Example 1 Here we use the notations $u_x = \frac{\partial u}{\partial x}$ and so on.

$$G = u_x + u_y + g(x, y);$$

$$F = u_{xx} + 5u_{xy} + 6u_{yy} + \frac{\partial g}{\partial x} + 3\frac{\partial g}{\partial y} + H(x, y, u, u_x, u_y) \cdot (u_x + 2u_y + g);$$

$$c_1 = 1, c_2 = 3, A = v^2 + 5vw + 6w^2, B = v + 2w;$$

$$F_0 = F - u_{xx} - 2u_{xy} - \frac{\partial g}{\partial x} - 3(u_{xy} + 2u_{yy} + \frac{\partial g}{\partial y}) = H \cdot G.$$

2 Algorithm to Find the Algebraic Variety of All the Divisors

Now we proceed to an algorithm which for a quasi-linear $F \in D$ with $\text{ord } F = k+1$, $\deg F = d$ yields the algebraic variety of all its divisors of order k (let $k \geq 1$). Making a $\overline{\mathbb{Q}}$ -linear transformation of independent variables x, y one can assume w.l.o.g. that the coefficient $a_0 = 1$ (see (1)), this is compatible with the assumption $b_0 = 1$ due to Lemma 1.3.

First the algorithm factorizes the highest order derivatives form $A = \sum_{0 \leq i \leq k+1} a_i \cdot v^i \cdot w^{k+1-i} \in \overline{\mathbb{Q}}[v, w]$ (see Lemma 1.3), say with the help of [2], [7]. Pick one of its at most of $k+1$ factors with degree k as a candidate for the highest order derivatives form $B = \sum_{0 \leq i \leq k} b_i \cdot v^i \cdot w^{k-i} \in \overline{\mathbb{Q}}[v, w]$ of a divisor G of F . One can assume w.l.o.g. that $b_0 = 1$ (if $b_0 = 0$ we discard this candidate). Hence $(c_1 \cdot v + c_2 \cdot w) \cdot B = A$ for some $c_1, c_2 \in \overline{\mathbb{Q}}$ (actually, $c_2 = 1$ since $a_0 = b_0 = 1$).

Due to Theorem 1.2 ii) $\deg G \leq \deg F$, and we write a candidate for G as a polynomial with indeterminate coefficients over $\overline{\mathbb{Q}}$. In view of Theorem 1.2 i) one has to verify whether G divides $(F - c_1 \cdot \frac{dG}{dx} - c_2 \cdot \frac{dG}{dy})^d$ employing (8) (cf. also (9), (10), (11), and (12)). For this goal we introduce H (see (6)) with indeterminate coefficients over $\overline{\mathbb{Q}}$ and verify the condition

$$\left(F - c_1 \cdot \frac{dG}{dx} - c_2 \cdot \frac{dG}{dy} \right)^d = H \cdot G \quad (13)$$

as a system of polynomial equations invoking the quantifier elimination algorithm from [3] (eliminating the indeterminate coefficients of H). The latter algorithm finds the irreducible components of the algebraic variety of all divisors G .

To estimate the complexity of the designed algorithm one has to specify how does the algorithm represent the coefficients of F from $\overline{\mathbb{Q}}$. A customary way to this end is to represent them as elements from an appropriate finite extension of $\mathbb{Q}$ (see e. g. [1, 2, 7, 3]). Denote by L a bound on the bit-size of such a representation (say, in a particular case of rational numbers p/q its bit-size is defined as $\lceil \log_2(p+1)(q+1) \rceil$).

Denote

$$N_0 := \binom{k+n}{n} + n, N := \binom{N_0 + d^2}{d^2}.$$

The complexity of the designed algorithm is majorated by the complexity of solving (13) which leads to the quantifier elimination for a system of polynomials in at most of N indeterminates being the coefficients at the monomials of degrees d for polynomial G and of degrees $d^2 - d$ for polynomial H in N_0 variables $\{u_{i_1, \dots, i_n} : i_1 + \dots + i_n \leq k\} \cup \{x_1, \dots, x_n\}$. The degrees of these polynomials do not exceed d , and their number is bounded by N . The bit-sizes of the coefficients of these polynomials are less than $L + O(\log N)$. The complexity of the quantifier elimination algorithm [3] applied to this system does not exceed a polynomial in L , d^{N^2} . Summarizing and utilizing the notations introduced above, we conclude with

Theorem 2.1 *There is an algorithm which for a given quasi-linear differential polynomial of an order $k+1$ produces the irreducible components of the algebraic variety of all its quasi-linear divisors of order k . The complexity of the algorithm can be bounded by a polynomial in L , d^{N^2} .*

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