Fourier transform of the Lippmann-Schwinger equation for 3D Vectorial Electromagnetic Scattering : a direct relationship between fields and shape

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Abstract

In classical Physics, the Lippmann-Schwinger equation links the field scattered by an ensemble of particles – of arbitrary size, shape and material – to the incident field. This singular vectorial integral equation is generally formulated and solved in the direct space $\mathbb{R}^n$ (typically, $n = 2$ or $n = 3$), and often approximated by a scalar description that neglects polarization effects. Computing rigorously the Fourier transform of the fully vectorial Lippmann-Schwinger equation in $\mathscr{S}'(\mathbb{R}^3)$, we obtain a simple expression in the Fourier space. Besides, we can draw an explicit link between the shape of the scatterer and the scattered field. This expression gives a general, tridimensional, picture of the well known Rayleigh-Sommerfeld expression of bidimensional scattering through small apertures.

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I. INTRODUCTION

The Fourier Transform (FT) permits to turn Partial Differential Equations (PDE) into algebraic equations that can be handled or even solved more easily. Its use is popular to solve *eg*. Heat equation [1], or to treat problems of scalar scattering by a potential, described by a classical or quantum Lippmann-Schwinger Equation (LSE) [2–4]. However, in some class of problems that involve a finite size, solid scatterer – for example in Hydrodynamics [5] or in Optics [6–8] – or when vectorial scattering is considered – to model the field polarization – we note that the LSE is mostly (if not always) solved in real space. Its kernel has indeed both, a singularity at the source location, and a $L^2(\mathbb{R}^3)$ but not $L^1(\mathbb{R}^3)$ behaviour. If the first problem can be cured by proper regularization [9, chap. 15], the second one necessitates to use Generalized Functions when computing the FT, what is not done so often.

The first use of Fourier transformation to describe scattered light might date back to the early theories of Kirchhoff or Rayleigh and Sommerfeld [10], in which the far field scattered by an aperture in a screen was related to the FT of the hole shape. More recently, several attempts to solve Maxwell or Helmoltz equations in tridimensional Fourier space have been done in the literature [11, 12], but were limited either to simple shapes (*eg*. 1D multilayered material), to a far-field description, or a scalar description that does not take polarisation into account [4]. In some works, the calculation of the *free space* Green function or tensor has been done in Fourier space [13–16] – sometimes using generalized functions in an heuristic way [16] –, however, without considering the presence of scatterers. In other works, polarization of electromagnetic waves in complex systems was modeled, *eg*. solving the Bethe-Salpeter Equation in Fourier space [17], but this was limited to an ensemble of independant point scatterers. Budko and co-authors [6, 7] have studied the LSE for finite size scatterers, considering the full vectorial problem, but gave the FT of its singular part only, considering a scatterer described by Hölder continuous functions. Therefore, we believe it is interesting to make the FT of the Lippmann-Schwinger Equation in the general case, and to relate the scattered field to the Fourier transform of the object shape – described analytically by its indicatrix function [18] –, which is known to be a convenient descriptor of the geometrical properties of the scatterer [19].

In the present article, we aim at presenting in detail and rigorously, how to Fourier Transform the LSE for vectorial electromagnetic scattering in $\mathbb{R}^3$. At the difference with previous works, we shall consider a scatterer of arbitrary shape, material and size, placed in a uniform background of arbitrary permittivity, and non magnetic. In section Sec. (II), we make a rapid review of the regularization procedure that permits to define unambiguously the Integral equation, see [3, 8]. Then, we will detail the calculation of the FT of each part of the equation, see Sec. (III) and Sec. (IV). Our approach is based on the work by Grafakos and Teschl [20], regarding the FT of radial (generalized) functions. We shall gather the results in Sec. (V), to give the Fourier transform of electromagnetic LSE in $\mathbb{R}^3$. Note that the lengthy proofs and lemma have been gathered in [21].

II. BACKGROUNDS AND PURPOSE

A. The LS equation for Electromagnetic scattering

Stemming directly from Maxwell equations, the LS equation that describes the field scattered by an arbitrary object in a uniform background is [6, 8], in $\mathbb{R}^3$,

$$\mathbf{E}(\mathbf{x},\omega) = \mathbf{E}_{\text{inc}}(\mathbf{x},\omega) + \left[{k_b}^2 + \mathbf{grad}\,\text{div}\right] \int_{\mathbb{R}^3} \mathrm{G}\left(\|\mathbf{x}-\mathbf{x}'\|\right)\chi\left(\mathbf{x}',\omega\right)\mathbf{E}(\mathbf{x}',\omega)\,\mathrm{d}\mathbf{x}' \tag{II.1}$$

where

- $\mathbf{E}$ and $\mathbf{E}_{\text{inc}}$ are the total and incident electric field ;
- k_b is the wavenumber in the background medium, assumed to be real;
- χ is the relative difference between scatterer and background complex permittivities: $\chi(\mathbf{x}) := [\varepsilon(\mathbf{x}) - \varepsilon_b]/\varepsilon_b$. The scatterer medium is possibly dispersive – χ depends on ω–, dissipative –χ is complex–, and anisotropic – χ is a 3×3 tensor. In the case of an isotropic medium, χ boils down to a scalar. If $\varepsilon(\mathbf{x})$ is constant inside the scatterer, χ is proportional to the indicatrix.
- G is the Green function of Helmholtz equation in vacuum (with $\mathbf{l} := \mathbf{x}-\mathbf{x}'$ and $l := \|\mathbf{l}\|$). Note that, if one chooses the $\exp(-i\omega t)$ convention for harmonic time evolution (as we do), the outgoing Sommerfeld condition [22] imposes :

$$G(\mathbf{l}) := \frac{e^{+ik_b l}}{4\pi l}, \tag{II.2}$$

so that far enough from the scatterer, the wave behaves as an outgoing spherical wave, that carries energy outwards.

As shown in textbooks [8, p. 36], [9, chap. 15], the **grad** div operator in Eq. (II.1) cannot be put straightforwardly under the integral sign. However, upon a careful procedure, Eq. (II.1) can be recasted as the singular integral equation

$$\mathbf{E}_{\mathrm{inc}}(\mathbf{x},\omega) = \left[\mathbb{I}+\frac{\chi}{3}\right]\mathbf{E}(\mathbf{x},\omega) - \lim_{\varepsilon\to 0}\int_{\mathbf{x}'\in\mathbb{R}^3\setminus B(\mathbf{x},\varepsilon)} \mathbb{G}_0\left(\mathbf{x}-\mathbf{x}'\right)\chi\left(\mathbf{x}',\omega\right)\mathbf{E}(\mathbf{x}',\omega)\,\mathrm{d}\mathbf{x}' \\ - \int_{\mathbf{x}'\in\mathbb{R}^3} \mathbb{G}_1\left(\mathbf{x}-\mathbf{x}'\right)\chi\left(\mathbf{x}',\omega\right)\mathbf{E}(\mathbf{x}',\omega)\,\mathrm{d}\mathbf{x}' \tag{II.3}$$

where $\mathbb{I}$ is the unit tensor, $\mathbb{G}_0$ and $\mathbb{G}_1$ are respectively the "singular" part and the "regular" part of the Green tensors (with $\mathbf{l} := \mathbf{x}-\mathbf{x}'$ and $l := \|\mathbf{l}\|$) :

$$\mathbb{G}_0\left(\mathbf{l}\right) \quad := \tfrac{1}{4\pi l^3}\left(3\mathbb{Q}-\mathbb{I}\right), \tag{II.4}$$

$$\mathbb{G}_1\left(\mathbf{l}\right) := G_{1I}(l)\mathbb{I} + G_{1Q}(l)\mathbb{Q},$$

where

$$\mathbb{Q} := \frac{\mathbf{l}\,({}^t\mathbf{l})}{l^2} = \frac{1}{l^2}\begin{pmatrix} l_x l_x & l_x l_y & l_x l_z \\ l_y l_x & l_y l_y & l_y l_z \\ l_z l_x & l_z l_y & l_z l_z \end{pmatrix} \tag{II.5}$$

and

$$G_{1I}(l) := \frac{e^{ik_b l}}{4\pi l^3}\left(-1+ik_b l-(ik_b l)^2\right)+\frac{1}{4\pi l^3} \tag{II.6}$$

$$G_{1Q}(l) := \frac{e^{ik_b l}}{4\pi l^3}\left(3-3ik_b l+(ik_b l)^2\right)-\frac{3}{4\pi l^3} \tag{II.7}$$

Note that in the quasistatic limit, i.e. when $1/k_b$ is much larger than the size of the scatterer, the term with $\mathbb{G}_1$ cancels out and only the principal value that contains $\mathbb{G}_0$ contributes.

Eq. (II.3) defines a linear operator whose properties have been studied both in 2D [7] and 3D [6]. Our purpose is to give an explicit formula for the Fourier transform of Eq. (II.3). The key point being the computation of the Fourier transform of Green tensors $\mathbb{G}_0$ and $\mathbb{G}_1$. We will proceed in two steps. $\mathbb{G}_0$ will be Fourier transformed in a more general way than what done previously [6], and $\mathbb{G}_1$ will be handled using generalized functions theory.

III. FOURIER TRANSFORM OF THE SINGULAR PART

The definitions and properties of the Fourier Transform we use are given in Appendix A.

A. General expression as a convolution

In this section, we carry out the FT of the part of Eq. (II.3) that contains the Principal Value. Such result has been given previously in [6], a work that refers to [8] for the proof, based there on a decomposition of the characteristic – the function f in Eq. (III.8) – on a basis of spherical harmonics, see [8, *cf.* Appendix]. In these previous works [6, 8], the scatterer is supposed to be Hölder-continuous. On the contrary, in the present work, we use Generalized functions [23] and extend the result to a space of integrable functions for the scatterer profile : $\mathbf{x} \mapsto \chi(\mathbf{x})$.

Let f be an integrable function on the sphere $\mathbb{S}^{n-1}$ with mean 0. Then, the FT can be defined, possibly as a Generalized function.

We define

$$\langle W_f, \varphi \rangle := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{f(x/|x|)}{|x|^n} \varphi(x)\, \mathrm{d}x \tag{III.8}$$

for $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Note that $\mathbb{G}_0$ is obtained by choosing specifically $f(l) = \frac{1}{4\pi}(3\mathbb{Q} - \mathbb{1})$, $\mathbb{Q}$ being defined Eq. (II.5). It has been shown that W_f is a tempered distribution (see *eg*. [23, p. 267]), provided that the hypothesis:"f has a zero mean over the sphere $\mathbb{S}$" is true (see also [6]). Eventually, the Fourier transform can be found (*cf.* [23, p. 269]). It is the function (a.e. finite) given by:

$$\widehat{W_f}(\xi) = \int_{\mathbb{S}^{n-1}} f(\theta) \Big(\log \frac{1}{|\xi \cdot \theta|} - \frac{i\pi}{2} \mathrm{sgn}(\xi \cdot \theta) \Big)\, \mathrm{d}\theta, \tag{III.9}$$

where $\mathbb{S}^{n-1}$ is the unit sphere on $\mathbb{R}^n$, and $\theta \in \mathbb{S}^{n-1}$ is the direction of the vector $x/|x|$, for $x \in \mathbb{R}^n$.

Remark III.1 *With the notations of Eq. (II.3), and using Eq. (II.4), we obtain*

$$\lim_{\varepsilon \to 0} \int_{\mathbf{x}' \in \mathbb{R}^3 \setminus B(\mathbf{x},\varepsilon)} \mathbb{G}_0\big(\mathbf{x} - \mathbf{x}'\big) \chi(\mathbf{x}', \omega) \mathbf{E}(\mathbf{x}', \omega)\, \mathrm{d}\mathbf{x}' := \Big\langle W_{\frac{1}{4\pi}(3\mathbb{Q}-\mathbb{1})}, \tau_{-\mathbf{x}}\big(\chi(\,.\,,\omega)\mathbf{E}(\,.\,,\omega)\big) \Big\rangle, \tag{III.10}$$

where $\tau_{-\mathbf{x}}$ *is a translation operator, i.e.* $\tau_{-\mathbf{x}}(g)(\mathbf{x}') := g(\mathbf{x}' + \mathbf{x})$.

To carry on with the Fourier transform, we define the continuous operator H_f by

$$H_f(u) := W_{f,1} \star u + W_{f,2} \star u, \tag{III.11}$$

where $\star$ represents the convolution, with $u \in \mathrm{L}^2$, and

$$\langle W_{f,1}, u\rangle := \lim_{\varepsilon \to 0} \int_{B(0,1)\setminus B(0,\varepsilon)} \frac{f(x/|x|)}{|x|^n} u(x)\,\mathrm{d}x \quad \text{and} \quad W_{f,2} := \mathbb{1}_{B(0,1)^c} \frac{f(x/|x|)}{|x|^n}.$$

Besides, in all this work we will assume the following hypothesis :

Hypothesis 1 $\widehat{W_f}$ *is bounded.*

Remark III.2 *As we will see, that will be always the case in our framework.*

H_f is a continuous operator over the Hilbert space $\mathrm{L}^2(\mathbb{R}^n)$ with a Fourier transform verifying :

$$\widehat{H_f(u)} = \widehat{W_f}\hat{u} \in \mathrm{L}^2 \tag{III.12}$$

The operator H_f is well-defined since in one hand $W_{f,1}$ is a distribution with compact support, u can be assimilated to a tempered distribution ($\mathrm{L}^2 \subset \mathscr{S}'$) and, in the other hand $W_{f,2}$ and u are in L^2.

As well, the Fourier transform of H_f is well-defined and

$$\begin{aligned}\widehat{H_f(u)} &= \widehat{W_{f,1}}\hat{u} + \widehat{W_{f,2}}\hat{u} \\ &= \widehat{W_f}\hat{u} \in \mathrm{L}^2\end{aligned}$$

since $\widehat{W_f}$ is supposed to be bounded. Then $\overline{\mathscr{F}}(\widehat{H_f}) \in \mathrm{L}^2$.

The linear behaviour of H_f is clear and, for the continuity, we have

$$\|H_f(u)\|_2 = \|\widehat{H_f(u)}\|_2 = \|\widehat{W_f}\hat{u}\|_2 \leq C\|\hat{u}\|_2 = C\|u\|_2.$$

We now have to compute $\widehat{W_f}$.

B. Explicit calculation in the case of Electromagnetic scattering

Note that at this stage, Eq. (III.9) is more general than the singular part of LS equation for Electromagnetic Scattering in a uniform background, Eq. (II.3). Indeed, f can be any function integrable on $\mathbb{S}^{n-1}$ with mean 0. We can simply say that the part with the log function, in Eq. (III.9), is null if f is an odd function, and the part with the sgn function is null if f is an even function.

To be more specific we make a choice for f from now on, taking an homogeneous polynomial of degree α (even) and writing $P(\theta) = c\prod_{k=1}^{n}\theta_k^{\alpha_k}$ (where $\alpha_k = 0$ is allowed). Again, this choice for f is more general than what strictly needed to Fourier Transform the principal value part of Eq. (II.3), in which only $\alpha = 0$ and $\alpha = 2$ play a role.

Then, let fix $\xi \in \mathbb{S}^{n-1}$ and let A be a rotation such that $\xi = Ae_1$. We consequently have

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}} P(\theta)\log\frac{1}{|\xi\cdot\theta|}\,\mathrm{d}\theta &= -c\int_{\mathbb{S}^{n-1}} \log|\xi\cdot\theta|\prod_{k=1}^{n}\theta_k^{\alpha_k}\,\mathrm{d}\theta \\
&= -c\int_{\mathbb{S}^{n-1}} \log|Ae_1\cdot\theta|\prod_{k=1}^{n}\theta_k^{\alpha_k}\,\mathrm{d}\theta \\
&= -c\int_{\mathbb{S}^{n-1}} \log|e_1\cdot{}^tA\theta|\prod_{k=1}^{n}\left(A\,{}^tA\theta\right)_k^{\alpha_k}\,\mathrm{d}\theta \\
&= -c\int_{\mathbb{S}^{n-1}} \log|e_1\cdot\theta|\prod_{k=1}^{n}\left(A\theta\right)_k^{\alpha_k}\,\mathrm{d}\theta \qquad \text{(by Lemma III.2 in [21, p. 14])} \\
&= -c\int_{\mathbb{S}^{n-1}} \log|\theta_1|\prod_{k=1}^{n}\left(\xi_k\theta_1 + a_{k,2}\theta_2 + \cdots + a_{k,n}\theta_n\right)^{\alpha_k}\,\mathrm{d}\theta.
\end{aligned}
$$

Let us now limit the study to $\alpha \leq 2$.

The case $\alpha = 0$ is straightforward, and for $\alpha = 2$, writing $P(\theta) = \eta\ \theta_i\theta_j$, we have, according to Lemma III.4 in [21, p. 15],

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}} P(\theta)\log\frac{1}{|\xi\cdot\theta|}\,\mathrm{d}\theta &= -\eta\int_{\mathbb{S}^{n-1}} \log|\theta_1|\Big(\xi_i\xi_j\theta_1^2 + \sum_{k=2}^{n} a_{i,k}a_{j,k}\theta_k^2\Big)\,\mathrm{d}\theta \\
&= -P(\xi)\int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_1^2\,\mathrm{d}\theta \\
&\quad -(\eta\delta_{i,j} - P(\xi))\int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_2^2\,\mathrm{d}\theta,
\end{aligned} \tag{III.13}
$$

since [24] for every $k \geq 2$, $\int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_k^2\,\mathrm{d}\theta = \int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_2^2\,\mathrm{d}\theta$ and since A is a rotation, the rows are an orthonormal basis, $\xi_i\xi_j + \sum_{k=2}^{n} a_{i,k}a_{j,k} = \delta_{i,j}$.

So, to compute the Fourier transform when f is an homogeneous polynomial of degree $\alpha = 0$ or 2, one needs to compute

$$
\int_{\mathbb{S}^{n-1}} \log|\theta_1|\,\mathrm{d}\theta, \qquad \int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_1^2\,\mathrm{d}\theta, \qquad \int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_2^2\,\mathrm{d}\theta, \tag{III.14}
$$

where θ_i is expressed using the rotation angles of $\mathbb{R}^n$, see Eq. (III.39) in [21, p. 15]. To compute these integrals, we can use the following formula (*cf.* [23, p. 442]), with $n \geq 2$,

$$\int_{\mathbb{S}^{n-1}} K(x\cdot\theta)\,\mathrm{d}\theta = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} K(s|x|)\left(\sqrt{1-s^2}\right)^{n-3} \mathrm{d}s, \tag{III.15}$$

where $K : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}^n \setminus \{0\}$.

It follows (with $x = e_1$)

$$\int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_1^{\alpha}\,\mathrm{d}\theta = \frac{4\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{0}^{1} \log(s)s^{\alpha}\left(\sqrt{1-s^2}\right)^{n-3} \mathrm{d}s. \tag{III.16}$$

Besides, the integral $\int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_2^2\,\mathrm{d}\theta$ can be expressed as a linear combination of integrals of previous type, i.e. $\int_{\mathbb{S}^{n-1}} \log|\theta_1|\theta_1^{\alpha}\,\mathrm{d}\theta$, with $\alpha = 0$ or $\alpha = 2$. This method is detailed in Sec.(III C) of [21], where the three integrals Eq. (III.14) are computed in dimension n.

In the case $n = 3$,

we find in $\mathbb{R}^3$

$$\int_{\mathbb{S}^2} \log|\theta_1|\,\mathrm{d}\theta = -4\pi, \quad \int_{\mathbb{S}^2} \log|\theta_1|\theta_1^2\,\mathrm{d}\theta = -\frac{4}{9}\pi, \quad \int_{\mathbb{S}^2} \log|\theta_1|\theta_2^2\,\mathrm{d}\theta = -\frac{16}{9}\pi.$$

From Eq. (III.9) and Eq. (III.13), one obtains that, if $f = \eta$ is polynomial of degree 0,

$$\widehat{W_\eta}(\xi) = 4\pi\,\eta \tag{III.17}$$

if $f((x_1,x_2,x_3)) = \eta\; x_i x_j$, using Eq. (III.13),

$$\widehat{W_f}(\xi) = -\frac{4\pi}{3} f(\xi) + \frac{16\pi}{9}\eta\;\delta_{i,j} \tag{III.18}$$

So, if $f(l) = \frac{1}{4\pi}(3\mathbb{Q} - \mathbb{1})$,

$$\widehat{W_f}(\xi) = F(\tilde{\Theta}) = \frac{1}{3}\mathbb{1} - \tilde{\mathbb{Q}}, \tag{III.19}$$

where $\tilde{\Theta} := \frac{\xi}{\|\xi\|}$, and $\tilde{\mathbb{Q}} := \frac{\xi\,({}^t\xi)}{\|\xi\|^2}$. The notation $F(\tilde{\Theta})$ is introduced in agreement with [6], to emphasize that, due to the fact f is of zero mean on $\mathbb{S}^{n-1}$, the Fourier transform of the singular part only depends on $\frac{\xi}{\|\xi\|}$.

Remark III.3 *From Eq. (III.19), one immediatly notice that $\widehat{W_f}$ is a bounded operator, what validate our hypothesis.*

Finally, let us give the Fourier Transform in general notation. Since from Eq. (A.3), $\mathscr{F}^{a,b}h(k) = \left(\frac{|b|}{(2\pi)^{1-a}}\right)^{n/2} \mathscr{F}^{1,-1}h\big(-bk\big)$, we obtain :

$$\mathscr{F}^{a,b}\left(W_{\frac{1}{4\pi}(3\mathbb{Q}-\mathbb{1})}\right)(\xi) = c^3_{a,b}\left(\frac{1}{3}\mathbb{1} - \tilde{\mathbb{Q}}\right) \tag{III.20}$$

with

$$c^3_{a,b} = \left(\frac{|b|}{(2\pi)^{1-a}}\right)^{3/2} \tag{III.21}$$

Remark III.4 (Generalization and Perspectives) *Let us remark that the method developped above, e.g. Eq. (III.13) could be generalized to any polynomial expansion of higher degree for f. In such case, however, the integral expressions analoguous to Eq. (III.14) would be more tedious to compute, since we could not use the simplification induced by Lemma (III.4), see [21, p. 15].*

The Fourier Transform of the singular part of the LS equation has been computed and studied in [6], but on the subset of Hölder functions of L^2. In the present work, we have shown that the same expression, given in general Fourier notation by Eq. (III.20), is also valid for the FT of the singular part, even when functions $\mathbf{x} \mapsto \chi(\mathbf{x},\omega)\mathbf{E}(\mathbf{x},\omega)$ are not Hölder, but in the whole space $\mathrm{L}^2(\mathbb{R}^3)$.

IV. FOURIER TRANSFORM OF THE REGULAR PART

A. Radial distributions and Grafakos Theorem

As we will see below, in order to compute the FT of the regular part of Eq. (II.3), we shall compute FT of radial distributions, and use specific results on this topic, demonstrated in [20]. Let us first recall some definitions and properties.

1. Radial distributions

While $\mathscr{S}(\mathbb{R}^n)$ stands for the space of Schwartz functions on $\mathbb{R}^n$, we set:

$$\mathscr{S}_{\text{rad}}(\mathbb{R}^n) = \{\varphi \in \mathscr{S}(\mathbb{R}^n) \ : \ \varphi = \varphi \circ A, \ \forall A \in \mathscr{O}(n)\} \tag{IV.22}$$

$$\mathscr{S}_{\text{rad}}(\mathbb{R}) = \mathscr{S}_{\text{even}}(\mathbb{R}) = \{\varphi \in \mathscr{S}(\mathbb{R}) \ : \ \varphi(x) = \varphi(-x)\} \tag{IV.23}$$

where $\mathscr{O}(n)$ is the set of the orthogonal transformations of $\mathbb{R}^n$.

We define then the following functions :

$$\begin{cases} \mathscr{S}(\mathbb{R}^n) & \to & \mathscr{S}_{\text{rad}}(\mathbb{R}) \\ \varphi & \mapsto & \left(r \mapsto \varphi^o(r) := \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \varphi(r\theta)\,\mathrm{d}\theta\right) \end{cases} \tag{IV.24}$$

$$\begin{cases} \mathscr{S}_{\text{rad}}(\mathbb{R}) & \to & \mathscr{S}_{\text{rad}}(\mathbb{R}^n) \\ \varphi & \mapsto & \left(x \mapsto \varphi^O(x) := \varphi(|x|)\right) \end{cases} \tag{IV.25}$$

(where $\mathbb{S}^{n-1}$ is the unit sphere on $\mathbb{R}^n$ and ω_{n-1} its surface area ; with the convention $\omega_0 = 2$ and $\varphi^o(x) = \frac{1}{2}(\varphi(x) + \varphi(-x)$, for $\varphi \in \mathscr{S}(\mathbb{R})$).

Definition IV.1 *A distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ (with $\mathscr{S}'(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$) is called radial if for all $A \in \mathscr{O}(n)$,*

$$u = u \circ A,$$

that is,

$$\langle u, \varphi \rangle = \langle u, \varphi \circ A \rangle$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$. The set of all radial tempered distributions is denoted by $\mathscr{S}'_{\text{rad}}(\mathbb{R}^n)$.

Proposition IV.2 *For $u \in \mathscr{S}'_{\text{rad}}(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$,*

$$\langle u, \varphi \rangle = \langle u, \varphi^{\text{rad}} \rangle \tag{IV.26}$$

where $\varphi^{\text{rad}} := (\varphi^o)^O$ (i.e. $\varphi^{\text{rad}}(x) = \varphi^o(|x|)$).

Given a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and a fixed $x \in \mathbb{R}^n$, $\varphi^o(|x|)$ is then the mean over the sphere of radius $|x|$ of the function φ (so, if φ is already radial, φ will be of the form $f(|x|)$ with $f : \mathbb{R} \to \mathbb{R}$, and we will have directly $\varphi^o = f$).

Let us define the space

$$\mathscr{R}_n := r^{n-1}\mathscr{S}_{\text{rad}}(\mathbb{R}) = \left\{ \left(r \mapsto \psi(r) r^{n-1}\right),\ \psi \in \mathscr{S}_{\text{rad}}(\mathbb{R}) \right\}. \tag{IV.27}$$

Remark IV.3 *$\mathscr{R}_n$ is a subspace $\mathscr{S}(\mathbb{R})$ on which we can use the same topology ; we denote its dual (set of the linear continuous functions defined over $\mathscr{R}_n$) by $\mathscr{R}'_n$.*

We switch from $\mathscr{R}'_n$ to radials distributions of $\mathscr{S}'(\mathbb{R}^n)$ as follows :

- if u is radial distribution, we define $u_\diamond \in \mathscr{R}'_n$ by

$$\langle u_\diamond, \psi(r) r^{n-1}\rangle := \frac{2}{\omega_{n-1}} \langle u, \psi^O\rangle, \qquad \psi \in \mathscr{S}_{\mathrm{rad}}(\mathbb{R}) \tag{IV.28}$$

- if $u_\diamond \in \mathscr{R}'_n$, we define a radial distribution u by

$$\langle u, \varphi\rangle := \frac{\omega_{n-1}}{2} \langle u_\diamond, \varphi^o(r) r^{n-1}\rangle, \qquad \varphi \in \mathscr{S}(\mathbb{R}^n) \tag{IV.29}$$

2. Grafakos-Teschl theorem

Theorem IV.4 (Grafakos-Teschl (2013)) *Given v_1 in $\mathscr{S}'(\mathbb{R})$, we define a radial distribution v_k on $\mathbb{R}^k$ ($k \in \mathbb{N}^*$) by*

$$\langle v_k, \varphi\rangle := \frac{\omega_{k-1}}{2} \langle v_1, \varphi^o(r) r^{k-1}\rangle, \qquad \varphi \in \mathscr{S}_{\mathrm{rad}}(\mathbb{R}^k) \tag{IV.30}$$

(if $\varphi \in \mathscr{S}_{\mathrm{rad}}(\mathbb{R}^n)$, then $\varphi(x) = \varphi^o(|x|)$).

Let $u^k = \mathscr{F}_k^{a,b}(v_k)$. We have then

$$-\frac{(2\pi)^a}{|b| r} \frac{\mathrm{d}}{\mathrm{d}r} u^n_\diamond = u^{n+2}_\diamond \tag{IV.31}$$

In practice, we will not use directly this theorem, but the following corollary:

Corollary IV.5 *Given v_1 in $\mathscr{S}'(\mathbb{R})$, we define the radial distribution v_3 on $\mathbb{R}^3$ by*

$$\langle v_3, \varphi\rangle := \frac{\omega_2}{2} \langle v_1, \varphi^o(r) r^2\rangle, \qquad \varphi \in \mathscr{S}_{\mathrm{rad}}(\mathbb{R}^3) \tag{IV.32}$$

(if $\varphi \in \mathscr{S}_{\mathrm{rad}}(\mathbb{R}^n)$, then $\varphi(x) = \varphi^o(|x|)$). We have then, for all $\varphi \in \mathscr{S}_{\mathrm{rad}}(\mathbb{R}^3)$

$$\langle \mathscr{F}_3^{a,b}(v_3), \varphi\rangle = -\frac{(2\pi)^{a+1}}{|b|} \Big\langle r \frac{\mathrm{d}}{\mathrm{d}r} (\mathscr{F}_1^{a,b}(v_1)), \varphi^o(r) \Big\rangle \tag{IV.33}$$

With $u^3 := \mathscr{F}_3^{a,b}(v_3)$ and $u^1 := \mathscr{F}_1^{a,b}(v_3)$ we have

$$\begin{aligned}
\langle u^3, \varphi\rangle &= \frac{\omega_2}{2} \langle u^3_\diamond, \varphi^o(r) r^2\rangle && (\textit{cf.}\ \text{(IV.29)}) \\
&= \frac{\omega_2}{2} \Big\langle -\frac{(2\pi)^a}{|b| r} \frac{\mathrm{d}}{\mathrm{d}r} u^1, \varphi^o(r) r^2 \Big\rangle && (\textit{cf.}\ \text{(IV.31)}) \\
&= -\frac{(2\pi)^{a+1}}{|b|} \Big\langle r \frac{\mathrm{d}}{\mathrm{d}r}(u^1), \varphi^o(r) \Big\rangle
\end{aligned}$$

since $\omega_2 = 4\pi$.

B. Fourier transform of G_{1I}

1. Procedure for computing the FT in $\mathbb{R}^3$

We aim at computing the FT of a radial distribution, say v_3. We will proceed using Corollary (IV.5), in the following way:

1. We need to find a one-dimensional distribution, v_1, that verifies

$$\langle v_3, \varphi \rangle = \frac{\omega_2}{2} \langle v_1, \varphi^o(r) r^2 \rangle. \tag{IV.34}$$

($\omega_2 = 4\pi$ being the 1-sphere surface). Note that finding v_1 may be, generally speaking, a difficult task.

2. One computes $u_1 := \hat{v}_1$.

3. The FT we seek, $u_3 := \hat{v}_3$, is then defined by

$$\langle u_3, \varphi \rangle = -\left(\frac{\omega_2}{2}\right)^2 \langle r \tfrac{\mathrm{d}u_1}{\mathrm{d}r}, \varphi^o(r) \rangle. \tag{IV.35}$$

4. Note that if $\frac{\mathrm{d}u_1}{\mathrm{d}r} = f(r)$ is not a distribution, but a regular function, one directly writes

$$u_3 = -\pi \frac{f(\|l\|)}{\|l\|}. \tag{IV.36}$$

In practice, we want to compute the 3D Fourier transform of

$$4\pi G_{1I}(l) = -i\frac{\sin(k_b l)}{l^3} + \frac{1-\cos(k_b l)}{l^3} + ik_b\frac{\cos(k_b l)}{l^2} - k_b\frac{\sin(k_b l)}{l^2} + k_b^2\frac{\cos(k_b l)}{l} + ik_b^2\frac{\sin(k_b l)}{l}$$

The idea is thus to reduce the problem, to a one dimensional Fourier transform computation. To proceed, we will use Theorem IV.4. However this later is not sufficient. Indeed in this theorem, a one-dimensional distribution (denoted by v_1) is *given* and is used to compute the associated (Equation (IV.32)) multidimensional distribution v_3. In our work, on the contrary, we start from a multidimensional distribution v_3, and v_1 *is not known a priori*. To overcome this problem, we will first define (in an heuristic way) a one-dimensional distribution and use Theorem IV.4 to obtain an associated multidimensional distribution T (using equation (IV.32)) whose Fourier Transform is established by this very theorem. Finally, to conclude, it will remain necessary to prove that the Fourier transform of T is the same as the Fourier transform of v_3. The proof relies on a recent result [25]).

Let us start from the following even generalized function of $\mathscr{S}'(\mathbb{R})$:

$$v_1 = -i\sin(k_b r)T_{\frac{1}{r^3}} + (1-\cos(k_b r))T_{\frac{1}{|r|^3}} + ik_b\cos(k_b r)T_{\frac{1}{r^2}} - k_b\sin(k_b r)T_{\frac{\operatorname{sgn}(r)}{r^2}}$$
$$+ k_b^2\cos(k_b r)T_{\frac{1}{|r|}} + ik_b^2\sin(k_b r)T_{\frac{1}{r}}. \tag{IV.37}$$

The Theorem IV.4, then allows to compute the Fourier transform of the distribution $v_3 \in \mathscr{S}'_{\text{rad}}(\mathbb{R}^3)$, defined by

$$\langle v_3, \varphi\rangle := \frac{\omega_2}{2}\langle v_1, r^2\varphi^o\rangle.$$

We stress out, at this point, that it is not a trivial fact that v_3 will be equal to $G_{1I}(l)$, which will be assured by [25].

We group together the terms with a $T_{\frac{1}{r^m}}$ factor (which will give the imaginary part) versus the terms containing $T_{\frac{1}{|r|^m}}$ (which will give the real part). Using corollaries (II.5) and (II.10), see respectively Sec.(II C 4) and Sec.(II D 4) of [21], we obtain :

$$\mathfrak{I}(\widehat{v_1}) = \frac{\pi}{4}(k^2+k_b^2)(\operatorname{sgn}(k+k_b) - \operatorname{sgn}(k-k_b)) \tag{IV.38}$$

$$\mathfrak{Re}(\widehat{v_1}) = -k_b^2\gamma + k^2\ln(|k|) - \frac{1}{2}(k^2+k_b^2)\big(\ln(|k-k_b|) + \ln(|k+k_b|)\big) \tag{IV.39}$$

The detailed computation is provided in [21], see Sec.(IV A).

To compute the FT of the associated 3D distribution v_3, according to Corollary IV.5 (with $a = 1$, $b = -1$), we will have to compute the derivative of $\widehat{v_1}$:

$$\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}k}\mathfrak{I}(\widehat{v_1}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}k}\left((k^2+k_b^2)\big(\operatorname{sgn}(k+k_b) - \operatorname{sgn}(k-k_b)\big)\right)$$
$$= k\ \operatorname{sgn}(k+k_b) - k\ \operatorname{sgn}(k-k_b) + 2k_b^2(\delta_{-k_b} - \delta_{k_b}) \tag{IV.40}$$

and,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}\mathfrak{Re}(\widehat{v_1}) \;=\;& 2k\ln(|k|) + k + (-(k-k_b)-k_b)\ln(|k-k_b|) - \tfrac{1}{2}(k-k_b) - k_b kT_{\frac{1}{k-k_b}}\\ &+ (-(k+k_b)+k_b)\ln(|k+k_b|) - \tfrac{1}{2}(k+k_b) + k_b kT_{\frac{1}{k+k_b}}\\ =\;& 2k\ln(|k|) - k\ln(|k-k_b|) - k\ln(|k+k_b|) - k_b^2 T_{\frac{1}{k-k_b}} - k_b^2 T_{\frac{1}{k+k_b}},\end{aligned} \tag{IV.41}$$

since $k_b kT_{\frac{1}{k+k_b}} = (k_b(k+k_b) - k_b^2)T_{\frac{1}{k+k_b}} = k_b - k_b^2 T_{\frac{1}{k+k_b}}$.

Then, Corollary IV.5, with $a = 1$, $b = -1$, $u_3 = \widehat{v_3}$ and $u_1 = \widehat{v_1}$, gives

$$\langle u_3, \varphi\rangle = -\frac{\omega_2^2}{4}\left\langle k\frac{\mathrm{d}}{\mathrm{d}k}u_1, \varphi^o(k)\right\rangle. \tag{IV.42}$$

So, by linearity, we have to compute

$$-\frac{\omega_2^2}{4}\left\langle k\frac{\mathrm{d}}{\mathrm{d}k}\mathfrak{Re}(\widehat{v_1}),\varphi^o(k)\right\rangle \text{ and } -\frac{\omega_2^2}{4}\left\langle k\frac{\mathrm{d}}{\mathrm{d}k}\mathfrak{I}(\widehat{v_1}),\varphi^o(k)\right\rangle.$$

- $\frac{\omega_2}{2}\left\langle k\frac{\mathrm{d}}{\mathrm{d}k}\mathfrak{Re}(\widehat{v_1}),\varphi^o(k)\right\rangle$: from (IV.41), we can can break the computation in three pieces,

1. for $k^2\ln(|k|)$, it is straightforward that

$$\begin{aligned}\frac{\omega_2}{2}\langle k^2\ln(|k|),\varphi^o(k)\rangle &= \omega_2\int_0^{+\infty}k^2\ln(|k|)\varphi^o(k)\,\mathrm{d}k\\ &= \int_{\mathbb{R}^3}\ln(|x|)\varphi^o(|x|)\,\mathrm{d}x = \int_{\mathbb{R}^3}\ln(|x|)\varphi(x)\,\mathrm{d}x = \langle\ln(l),\varphi\rangle\end{aligned}$$

   (with $l=|x|$) ;

2. then, (since $E := k\mapsto -k^2(\ln(|k-k_b|)+\ln(|k+k_b|))$ is even)

$$\begin{aligned}\frac{\omega_2}{2}\big\langle E(k),\varphi^o(k)\big\rangle &= -\,\omega_2\int_0^{+\infty}k^2\Big(\ln(|k-k_b|)+\ln(|k+k_b|)\Big)\varphi^o(k)\,\mathrm{d}k\\ &= -\int_{\mathbb{R}^3}\Big(\ln(||x|-k_b|)+\ln(||x|+k_b|)\Big)\varphi(x)\,\mathrm{d}x\\ &= \Big\langle -\ln(|l-k_b|)-\ln(|l+k_b|),\varphi\Big\rangle\end{aligned}$$

3. Finally,

$$\begin{aligned}-\frac{\omega_2}{2}\langle kT_{\frac{1}{k-k_b}}+kT_{\frac{1}{k+k_b}},\varphi^o(k)\rangle = -\frac{\omega_2}{2}\lim_{\varepsilon\to0^+}\Bigg(&\int_{-\infty}^{k_b-\varepsilon}\frac{k}{k-k_b}\varphi^o(k)\,\mathrm{d}k+\int_{k_b+\varepsilon}^{+\infty}\frac{k}{k-k_b}\varphi^o(k)\,\mathrm{d}k\\ &+\int_{-\infty}^{-k_b-\varepsilon}\frac{k}{k+k_b}\varphi^o(k)\,\mathrm{d}k+\int_{-k_b+\varepsilon}^{+\infty}\frac{k}{k+k_b}\varphi^o(k)\,\mathrm{d}k\Bigg)\end{aligned}$$

   We eventually obtain (the proof is detailed Sec.(IV A 3) of [21]):

$$-\frac{\omega_2}{2}\left\langle kT_{\frac{1}{k-k_b}}+kT_{\frac{1}{k+k_b}},\varphi^o(k)\right\rangle = \frac{1}{k_b}\left(-\left\langle \mathrm{vp}\Big(\frac{1}{|x|-k_b}\Big),\varphi(x)\right\rangle+\left\langle\frac{1}{|x|+k_b},\varphi(x)\right\rangle\right)$$

with

$$\left\langle \mathrm{vp}\Big(\tfrac{1}{|x|-k_b}\Big),\varphi(x)\right\rangle := \lim_{\varepsilon\to0^+}\int_{B(0,k_b-\varepsilon)\cup B^c(0,k_b+\varepsilon)}\tfrac{\varphi(x)}{|x|-k_b}\,\mathrm{d}x.$$

Multiplying by $-\omega_2/2$ and using Eq. (IV.42) we can conclude that,

$$\widehat{\mathfrak{Re}(v_3)} = 2\pi\left[\ln\left(\frac{|k^2-k_b^2|}{k^2}\right)+k_b\ \mathrm{vp}\Big(\frac{1}{k-k_b}\Big)-k_b\frac{1}{k+k_b}\right] \qquad \text{(IV.43)}$$

Remark IV.6 *With the same type of definition, we can write*

$$\frac{1}{2k_b}\left(\mathrm{vp}\Big(\frac{1}{k-k_b}\Big)-\frac{1}{k+k_b}\right)=\mathrm{vp}\Big(\frac{1}{k^2-k_b^2}\Big). \tag{IV.44}$$

- $\frac{\omega_2}{2}\left\langle k\frac{\mathrm{d}}{\mathrm{d}k}\Im(\widehat{v_1}),\varphi^o(k)\right\rangle$: from (IV.40), we can can break the computation in two pieces,

1. for the term that contains $k\ \mathrm{sgn}(k+k_b)-k\ \mathrm{sgn}(k-k_b)$,

$$\begin{aligned}
&\frac{\pi\omega_2}{4}\langle k^2\mathrm{sgn}(k+k_b)-k^2\mathrm{sgn}(k-k_b),\varphi^o(k)\rangle\\
&\qquad=\frac{\pi\omega_2}{2}\left(\int_0^{+\infty}k^2\varphi^o(k)\,\mathrm{d}k-\int_0^{k_b}-k^2\varphi^o(k)\,\mathrm{d}k+\int_{k_b}^{\infty}-k^2\varphi^o(k)\,\mathrm{d}k\right)\\
&\qquad=\frac{\pi\omega_2}{2}\int_0^{k_b}2k^2\varphi^o(k)\,\mathrm{d}k\\
&\qquad=\pi\int_{B(0,k_b)}\varphi^o(|x|)\,\mathrm{d}x=\pi\langle\mathbb{1}_{B(0,k_b)}(l),\varphi\rangle
\end{aligned}$$

2. and for $2k_b^2(\delta_{-k_b}-\delta_{k_b})$,

$$\begin{aligned}
\frac{\pi\omega_2k_b^2}{4}\langle 2(k\delta_{-k_b}-k\delta_{k_b}),\varphi^o(k)\rangle&=-\frac{\pi\omega_2k_b^2}{2}k_b(\varphi^o(-k_b)+\varphi^o(k_b))\\
&=-\pi k_b^3\omega_2\varphi^o(k_b)\\
&=-\pi k_b^3\omega_2\frac{1}{\omega_2}\int_{S(0,1)}\varphi(k_b\sigma)\,\mathrm{d}\sigma,
\end{aligned}$$

where σ represents the solid angle.

Carrying on, we obtain:

$$\begin{aligned}
\frac{\pi\omega_2k_b^2}{4}\langle 2(k\delta_{-k_b}-k\delta_{k_b}),\varphi^o(k)\rangle&=-\pi k_b^3\int_0^{2\pi}\int_0^{\pi}\varphi(k_b\Theta(\theta,\phi))\sin(\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&=-\pi k_b\int_0^{2\pi}\int_0^{\pi}\varphi(k_b\Theta(\theta,\phi))k_b^2\sin(\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&=-\pi k_b\langle\delta_{S(0,k_b)},\varphi\rangle,
\end{aligned}$$

where the last integral defines a Surfacic Dirac Distribution :

$$\left\langle\delta_{S(0,k_b)},\varphi\right\rangle=\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\varphi(k_b\Theta(\theta,\phi))k_b^2\sin(\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Note that the latter computation permits to understand where the "surfacic Dirac", introduced in a more phenomenological way in [16], comes from. We now observe that it also appears in the FT of Maxwell Equation around a scatterer, what generalizes previous observations focused on Helmoltz equations in vacuum.

Multiplying by $-\omega_2/2$ and using Eq. (IV.42) we can conclude that,

$$\widehat{\Im(v_3)} = 2\pi^2\big(k_b\delta_{S(0,k_b)} - \mathbb{1}_{B(0,k_b)}(k)\big) \tag{IV.45}$$

Gathering (IV.45) and (IV.43), we obtain :

Proposition IV.7 *The Fourier Transform of*

$$l \mapsto G_{1I}(l) := l \mapsto \frac{e^{ik_bl}}{4\pi l^3}\big(-1+ik_bl-(ik_bl)^2\big)+\frac{1}{4\pi l^3}$$

is (with $k=\sqrt{k_x^2+k_y^2+k_z^2}$*)*

$$k \mapsto \frac{1}{2}\ln\left(\left|1-\frac{k_b^2}{k^2}\right|\right)+k_b^2\mathrm{vp}\left(\frac{1}{k^2-k_b^2}\right)+i\frac{\pi}{2}\big(k_b\delta_{S(0,k_b)}-\mathbb{1}_{B(0,k_b)}(k)\big). \tag{IV.46}$$

C. Fourier transform of $G_{1Q}(l)\underline{\underline{Q}}$

Contrary to $G_{1I}(l)$, the components of $G_{1Q}(l)\underline{\underline{Q}}$ do not possess radial symmetry, since they have the form $\frac{l_il_j}{l^2}G_{1Q}(l)$ $(\in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3))$: we cannot use directly Corollary IV.5. So, in order to proceed, we will first build a radial distribution T such that

$$l_il_jT = \frac{l_il_j}{l^2}G_{1Q}(l).$$

We will then have (denoting by D the differentiation in the sense of the distributions)

$$\mathscr{F}\left(\tfrac{l_il_j}{l^2}G_{1Q}(l)\right) = \mathscr{F}(l_il_jT) = -\mathrm{D}_{l_i}\mathrm{D}_{l_j}\mathscr{F}(T). \tag{IV.47}$$

Heuristically (since the following $\frac{1}{l^2}G_{1Q}(l)$ function is not definite at 0, *cf.* Remark IV.8), we have to compute the 3D Fourier transform of

$$\frac{4\pi}{l^2}G_{1Q}(l) = 3i\frac{\sin(k_bl)}{l^5}+3\frac{\cos(k_bl)-1}{l^5}-3ik_b\frac{\cos(k_bl)}{l^4}+3k_b\frac{\sin(k_bl)}{l^4}-k_b^2\frac{\cos(k_bl)}{l^3}-ik_b^2\frac{\sin(k_bl)}{l^3}$$

and, to use a unidimensional Fourier Transform computation (Corollary IV.5).

Let us set the following even distribution from $\mathscr{S}'(\mathbb{R})$:

$$\begin{aligned} v_1 := {} & 3i\sin(k_br)T_{\frac{1}{r^5}}+3(\cos(k_br)-1)T_{\frac{1}{|r|^5}}-3ik_b\cos(k_br)T_{\frac{1}{r^4}} \\ & +\ 3k_b\sin(k_br)T_{\frac{\mathrm{sgn}(r)}{r^4}}-k_b^2\cos(k_br)T_{\frac{1}{|r|^3}}-ik_b^2\sin(k_br)T_{\frac{1}{r^3}}. \end{aligned} \tag{IV.48}$$

Remark IV.8 *we define then,* from v_1 itself *the distribution* $v_3 \in \mathscr{S}'_{\text{rad}}(\mathbb{R}^3)$ *by*

$$\langle v_3, \varphi \rangle := \frac{\omega}{2} \langle v_1, r^2 \varphi^o \rangle.$$

As discussed in Sec. (IV B), a very important and non trivial point is to verify that

$$\tfrac{l_i l_j}{l^2} G_{1Q}(l) = \frac{1}{4\pi} l_i l_j v_3.$$

which is a consequence of [25]

Fourier transform of v_1

We group together the terms with a $T_{\frac{1}{r^m}}$ component versus the terms with a $T_{\frac{1}{|r|^m}}$ component. We eventually obtain :

$$\mathfrak{Re}(\hat{v_1}) = \frac{k^4}{4}(\gamma + \ln(|k|)) - \frac{1}{8}(k+k_b)^2(k-k_b)^2\big(2\gamma + \ln(|k^2 - k_b^2|)\big) \tag{IV.49}$$

$$\mathfrak{I}(\hat{v_1}) = \frac{\pi}{16}(k+k_b)^2(k-k_b)^2\big(\text{sgn}(k+k_b) - \text{sgn}(k-k_b)\big). \tag{IV.50}$$

The detailed computation is given in Sec.(IV B) of [21]. Then, we need to compute the derivative of these expressions (in the sense of the distributions) ; using the Lemma II.12, see [21, p. 13] , we eventually obtain :

$$\mathrm{D}_k \mathfrak{Re}(\hat{v_1}) = k k_b^2\Big(\gamma + \frac{1}{4}\Big) + k^3 \ln(|k|) - \frac{1}{2}k(k^2 - k_b^2)\ln(|k^2 - k_b^2|) \tag{IV.51}$$

$$\mathrm{D}_k \mathfrak{I}(\hat{v_1}) = \frac{\pi}{4}k(k^2 - k_b^2)\big(\text{sgn}(k+k_b) - \text{sgn}(k-k_b)\big) \tag{IV.52}$$

The computation is detailed in Sec.(IV B 3) of [21].

Recalling (Corollary IV.5, with $a = 1$ and $b = -1$), with $u_3 := \widehat{v_3}$ and $u_1 := \widehat{v_1}$, that

$$-\frac{2}{\omega_2}\langle u_3, \varphi \rangle = \frac{\omega_2}{2}\left\langle r\frac{\mathrm{d}}{\mathrm{d}r}u_1, \varphi^o(r) \right\rangle.$$

We proceed in the same way we did in section Sec. (IV B). Gathering all the terms, which are computed in detail in Sec.(IV B 4) of [21], we obtain:

$$-\frac{2}{\omega_2}\hat{v_3} = k_b^2\Big(\gamma + \frac{1}{4}\Big) + k^2\ln(k) - \frac{1}{2}(k^2 - k_b^2)\ln(|k^2 - k_b^2|) + i\frac{\pi}{2}\mathbb{1}_{B(0,k_b)}(k)(k^2 - k_b^2) \tag{IV.53}$$

Which leads straightforwardly to :

$$\mathscr{F}(T) = \frac{-1}{2}\left[k_b^2\Big(\gamma + \frac{1}{4}\Big) + k^2\ln(k) - \frac{1}{2}(k^2 - k_b^2)\big(\ln(|k^2 - k_b^2|)\big) + i\frac{\pi}{2}\mathbb{1}_{B(0,k_b)}(k)(k^2 - k_b^2)\right].$$

Partial Derivatives of $\mathscr{F}(T)$

Now that $\mathscr{F}(T)$ has been computed, one need to compute its partial derivative in the sense of distribution, so as to use Eq. (IV.47) afterwards.

Real part

Let us first deal with the real part. From Lemma (II.13), see [21, p. 13], since on $\mathbb{R}^3 \setminus \{0\}$

$$\partial_i(k) = \frac{k_i}{k}, \qquad \partial_i(\ln(k)) = \frac{k_i}{k} \times \frac{1}{k} = \frac{k_i}{k^2}$$

and

$$\partial_i(k^2 \ln(k)) = 2k_i \ln(k) + k^2 \times \frac{k_i}{k^2} = 2k_i \ln(k) + k_i \in \mathscr{C}(\mathbb{R}^3)$$

We obtain:

$$\partial_j \partial_i(k^2 \ln(k)) = \delta_{ij}(2\ln(k)+1) + 2\frac{k_i k_j}{k^2} \tag{IV.54}$$

However, for $f(k) := (k^2 - k_b^2)\ln(|k^2 - k_b^2|)$ we cannot use Lemma II.13, since the derivative

$$\partial_i f(k) = 2k_i\big(\ln(|k+k_b|) + \ln(|k-k_b|)\big) + 2k_i$$

is not in L^1 anymore.
We provide a detailed calculation, see Sec.(IV B 6) of [21], in order to show that this problem can be overcomed.

To do so, we use the Green Formula (see Theorem IV.3) and split $\mathbb{R}^3$ in several domains, using $\Omega_{-\varepsilon} := B(0, k_b - \varepsilon)$ and $\Omega_\varepsilon := B^c(0, k_b + \varepsilon)$. This permits to isolate a crust of thickness 2ε (a region of $\mathbb{R}^3$ around $k = k_b$), where $k \mapsto \partial_i f(k)$ diverges, and to prove that:

$$\int_{\mathbb{R}^3} f(x)\partial_i\partial_j\varphi(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\Omega_{-\varepsilon}\cup\Omega_\varepsilon} \partial_j\partial_i f(x)\varphi(x)\,\mathrm{d}x,$$

where φ is a test function.

Eventually, we obtain:

$$\mathrm{D}_{k_i}\mathrm{D}_{k_j}(k^2 - k_b^2)\ln(|k^2 - k_b^2|) = 2(1 + \ln(|k^2 - k_b^2|)\delta_{i,j} + 2\frac{k_i k_j}{k}\left(\mathrm{v.p.}\frac{1}{k-k_b} + \frac{1}{k+k_b}\right),$$

and

$$\mathrm{D}_{k_i}\mathrm{D}_{k_j}\mathfrak{Re}(\mathscr{F}(T)) = -\frac{k_i k_j}{k^2} + \ln\left(\frac{|k^2 - k_b^2|}{k^2}\right)\frac{\delta_{i,j}}{2} + \left(\mathrm{v.p.}\frac{1}{k-k_b} + \frac{1}{k+k_b}\right)\frac{k_i k_j}{2k} \tag{IV.55}$$

Imaginary part

For $g(\vec{k}) = \mathbb{1}_{B(0,k_b)}(k)(k^2 - k_b^2)$, since it is a smooth function on $\Omega_{-\varepsilon}$, using Eq. (IV.52), in [21, Sec.(IV B 6)]

$$\int_{\Omega_{-\varepsilon}} g(x)\partial_i\partial_j\varphi(x)\,\mathrm{d}x = \int_{\Omega_{-\varepsilon}} \partial_j\partial_i g(x)\varphi(x)\,\mathrm{d}x - \int_{\partial\Omega_{-\varepsilon}} \varphi(x)\partial_i g(x) n_j^{-\varepsilon}(x)\,\mathrm{d}\sigma + \int_{\partial\Omega_{-\varepsilon}} \psi(x) g(x) n_i^{-\varepsilon}(x)\,\mathrm{d}\sigma$$

On $\Omega_{-\varepsilon}$,

$$\partial_i g = 2k_i, \qquad \partial_j\partial_i g = 2\delta_{i,j}$$

so

$$\int_{\Omega_{-\varepsilon}} g(x)\partial_i\partial_j\varphi(x)\,\mathrm{d}x = 2\delta_{i,j}\int_{\Omega_{-\varepsilon}} \varphi(x)\,\mathrm{d}x - \int_{\partial\Omega_{-\varepsilon}} \varphi(x) 2x_i n_j^{-\varepsilon}(x)\,\mathrm{d}\sigma + \underbrace{\int_{\partial\Omega_{-\varepsilon}} \psi(x) g(x) n_i^{-\varepsilon}(x)\,\mathrm{d}\sigma}_{\longrightarrow 0}$$

and with

$$\begin{aligned}
\int_{\partial\Omega_{-\varepsilon}} \varphi(x) 2x_i n_j^{-\varepsilon}(x)\,\mathrm{d}\sigma &= 2\int_0^{2\pi}\int_0^{\pi} \varphi((k_b-\varepsilon)\Theta)\Theta_i\Theta_j(k_b-\varepsilon)^3\sin(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\longrightarrow 2\int_0^{2\pi}\int_0^{\pi} \varphi(k_b\Theta)\Theta_i\Theta_j k_b^3\sin(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \int_{S(0,k_b)} \varphi(x) 2x_i x_j/k_b\,\mathrm{d}\sigma \\
&= \frac{2}{k_b}\langle k_j k_i \delta_{S(0,k_b)}\varphi\rangle.
\end{aligned}$$

So

$$\mathrm{D}_{k_i}\mathrm{D}_{k_j}\Im\mathscr{F}(T) = \frac{\pi}{2}\left(-\mathbb{1}_{B(0,k_b)}(k)\delta_{i,j} + \frac{k_j k_i}{k_b}\delta_{S(0,k_b)}\right) \tag{IV.56}$$

V. FOURIER TRANSFORM OF THE LS EQUATION AND GENERALIZATION

A. LS equation in Fourier space

Using Eqs. (IV.55, IV.56) and Eq. (IV.47), one obtains the Fourier Transform of $G_{1Q}(l)\underline{\underline{Q}}$. Then, using Eq. (II.5) and Eq. (IV.46), the result takes a remarkably simple form :

$$\mathscr{F}\left(\mathbb{G}_1\right) = \left(\mathrm{v.p.}\frac{{k_b}^2}{k^2 - {k_b}^2} + i\frac{\pi}{2}k_b\delta_{S(0,k_b)}(k)\right)\left[\mathbb{I} - \tilde{\mathbb{Q}}\right], \tag{V.57}$$

Using the general Fourier notation – see [21, Sec.(II D 1)] for the conversion between the $\mathscr{F}^{(1,-1)}$ and $\mathscr{F}^{(a,b)}$ notations –, we obtain:

$$\mathscr{F}\left(\mathbb{G}_1\right) = A_{a,b}(k)\left[\mathbb{I} - \tilde{\mathbb{Q}}\right], \tag{V.58}$$

$$A_{a,b}(k) = c_{a,b}^3\left(\text{v.p.}\frac{{k_b}^2}{b^2k^2 - {k_b}^2} + i\frac{\pi}{2}\frac{k_b}{|b|}\delta_{S(0,k_b/|b|)}(k)\right), \tag{V.59}$$

where $c_{a,b}^3$ is defined by Eq. (III.21).

Finally, we obtain the Fourier Transform of the Lippmann-Schwinger equations, using the general Fourier Transform notations defined Eq. (A.1).

$$\mathscr{F}\left(\mathbf{E}_{\text{inc}}\right)_{(\mathbf{k},\omega)} = \mathscr{F}\left(\mathbf{E}\right)_{(\mathbf{k},\omega)} + \left(\frac{|b|}{(2\pi)^{a+1}}\right)^{\frac{3}{2}}\left[A_{a,b}(k)\left(\tilde{\mathbb{Q}} - \mathbb{I}\right) + \tilde{\mathbb{Q}}\right]\left(\mathscr{F}\left(\chi\right) \star \mathscr{F}\left(\mathbf{E}\right)\right) \tag{V.60}$$

with $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$, and $\mathscr{F}\left(\chi\right)$ is, e.g. in the case of an isotropic dielectric, the (3D) Fourier Transform of $(x,y,z;\omega) \mapsto \varepsilon(x,y,z;\omega)/\varepsilon_b(\omega) - 1$, where $\varepsilon_b(\omega)$ is the background permittivity. In the anistropic case, both χ and $\mathscr{F}\left(\chi\right)$ are 3x3 matrix.

B. Comments and Discussion

Equations (V.59, V.60) thus give the FT of the initial LS equation (II.1), computed with the general notations, in $\mathbb{R}^3$. Obviously, we obtain in Fourier space a much more simple expression than in real space, *cf.* Eq. (II.3), as both operators $\mathbb{G}_o$ and $\mathbb{G}_1$ are not in the kernel anymore, and the limit $\varepsilon \to 0$ has been removed.

The characteristic (or indicatrix) function of the scatterer in Fourier space, $\mathscr{F}\left(\chi\right)$, is now the kernel of the integral equation Eq.(V.60), what shows a direct link between the field and the shape of the scatterer.

An hypothesis we had to take is that $x \mapsto \chi(x,\omega) \cdot \mathbf{E}(x,\omega)$ is sufficiently smooth to be considered as a test function of $\mathscr{S}(\mathbb{R}^3)$. Although at first sight this may seem restrictive (scatterers are generally modeled with discontinuous functions of space), we can always consider that the scatterer is smooth enough provided its shape is convoluted by a mollifier. In this case, $\mathbf{x} \mapsto \chi(\mathbf{x},\omega)$ becomes a $\mathscr{C}^\infty$ function that approximates the scattering problem. Its solution, $\mathbf{x} \mapsto E(\mathbf{x},\omega)$ would then also be $\mathscr{C}^\infty$ (there is no field discontinuity at the scatterer boundary, as the transition between the outside and the inside part of the scatterer becomes smooth; i.e. $\mathscr{C}^\infty$).

In the definition of $A_{a,b}(k)$, a surfacic Dirac distribution appears. This is in agreement with the observations made concerning free space propagation (i.e. Helmholtz equation), where such distribution appeared either explicitely [16], either through the related concept of Ewald or Mc Cutchen Sphere [15], which is simply the ensemble of points $\mathbf{k} \in \mathbb{R}^3$ such that $\|\mathbf{k}\| = k_b$. We now understand that this concept, introduced for scalar diffraction in far field or light propagation in vacuum, is in fact present at all length scales – and also in the near field of a scatterer, for which the LS equation (II.1) is valid –but not the Helmholtz equation.

1. Solution for a low index contrast

To give an example, we solve analytically Equations (V.59, V.60) in the case of a low index contrast. In this case, $\mathscr{F}\Big(\chi\Big)_{\mathbf{k}} \ll 1$ in $\mathbb{R}^3$, and Eq.(V.60) can be inverted easily to give:

$$\mathscr{F}\Big(\mathbf{E}_{\text{inc}}\Big) = \mathscr{F}\Big(\mathbf{E}\Big) + \left(\frac{|b|}{(2\pi)^{a+1}}\right)^{\frac{3}{2}}\left[A_{a,b}(k)\Big(\tilde{\mathbb{Q}} - \mathbb{I}\Big) + \tilde{\mathbb{Q}}\right]\Big(\mathscr{F}\Big(\chi\Big) \star \mathscr{F}\Big(\mathbf{E}_{\text{inc}}\Big)\Big) \tag{V.61}$$

The solution is even more simple considering a plane wave excitation. In this case, using the simplified FT notation, see Appendix A, and denoting $\widehat{f}$ instead of $\mathscr{F}\Big(f\Big)$, we are led to impose $\widehat{\mathbf{E}_{\text{inc}}} = (2\pi)^3\mathbf{E_o}\delta(\mathbf{k}-\mathbf{k_b})$. Then, $\widehat{\chi} \star \widehat{\mathbf{E}_{\text{inc}}} = (2\pi)^3\mathbf{E_o}\widehat{\chi}(\mathbf{k}-\mathbf{k_b})$, and the scattered field, defined as $\mathbf{E_s} \equiv \mathbf{E} - \mathbf{E}_{\text{inc}}$ is given analytically by:

$$\widehat{\mathbf{E}_{s(\mathbf{k},\omega)}} = -\left[\frac{k_b^2\,\widehat{\chi}(\mathbf{k}-\mathbf{k_b})}{k^2-k_b^2} + i\frac{\pi}{2}k_b\delta(k-k_b)\widehat{\chi}(\mathbf{k}-\mathbf{k_b})\right]\Big(\tilde{\mathbb{Q}} - \mathbb{I}\Big)\mathbf{E_o} - \widehat{\chi}(\mathbf{k}-\mathbf{k_b})\tilde{\mathbb{Q}}\mathbf{E_o} \tag{V.62}$$

This equation gives the analytic solution, albeit in Fourier space for the (vectorial) field scattered by an arbitrarily shaped scatterer of low index. Thanks to, e.g. numerical inverse Fourier Transform, one can obtain the 3D field plots in real space, see Figs. 1 - 2, that compares our result and Mie theory for a dielectric sphere of radius ρ in water, in which case,

$$\widehat{\chi}(\mathbf{k}) \equiv \widehat{\chi}(\|\mathbf{k}\|) = \frac{\varepsilon - \varepsilon_b}{\varepsilon_b} j_1(\|\mathbf{k}\|\rho)/(\|\mathbf{k}\|\rho),$$

where j_1 is the spherical Bessel function of order 1, and we assume $\varepsilon \approx \varepsilon_b$. We observe on Figs. 1 - 2, that the agreement with Mie theory is very satisfying. Note the injected field has no **z** component, and this latter appears in the scattering process, see Fig. 1 (c,d). The change of light polarization, though tenuous it is, is well described by our vectorial approach.

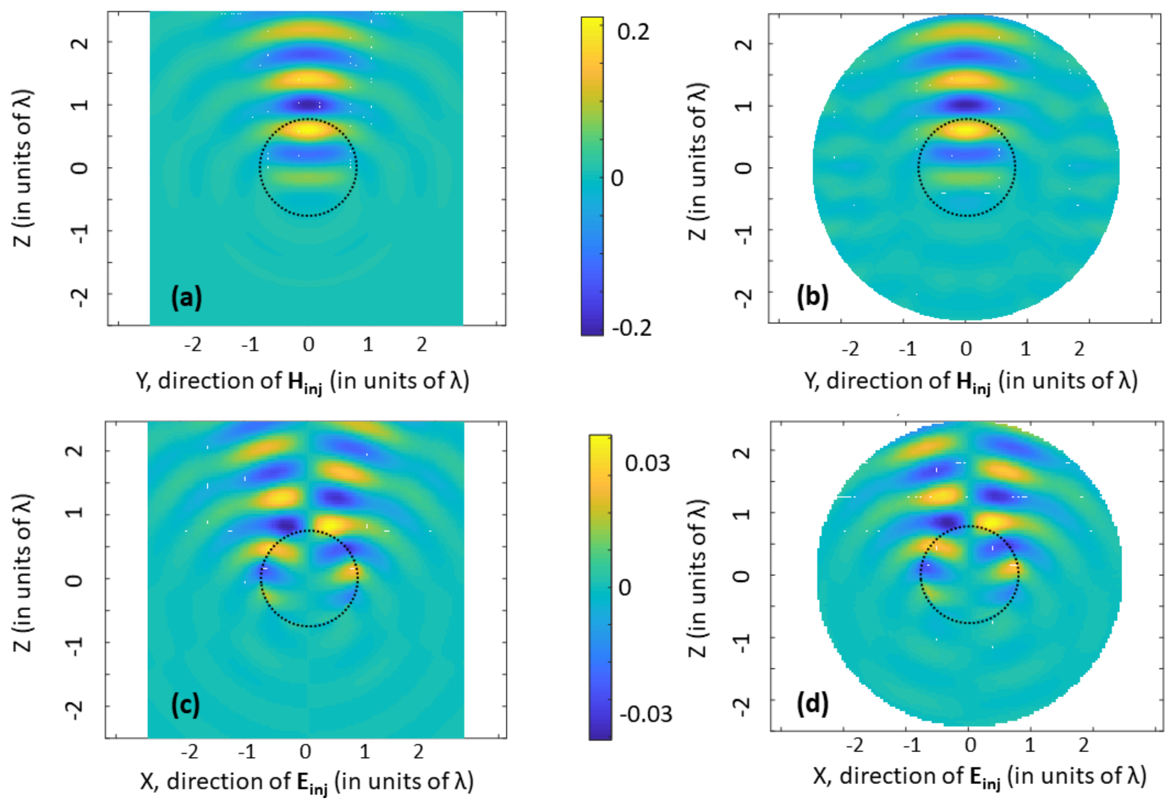

FIG. 1. Maps of the real part of the scattered field, from Mie theory – panel **(a)** and **(c)** – and inverse FT of Eq. (V.62) – panel **(b)** and **(d)** –. A plane wave polarized linearly along **x**, and propagating along **z**, towards $z > 0$, is impinging on a sphere of diameter $d = 1.5\lambda$, where λ is the vacuum wavelength. The sphere has an index $n_{int} = 1.35$, and is placed in water $n_{ext} = 1.33$. panel **(a)** and **(b)** display the **x**-component, panel **(c)** and **(d)** display the **z**-component. The black dotted circle figures the sphere boundary.

Note also that the far field / near field decomposition is readily visible in Eq. (V.62), as the transverse part in Fourier Space – that corresponds to the far field in real space – is given bythe first term, proportional to $\left(\tilde{\mathbb{Q}} - \mathbb{I}\right)\mathbf{E_0}$. The last term, proportional to $\tilde{\mathbb{Q}}\mathbf{E_0}$ is purely longitudinal and do not contribute to the far - field.

VI. CONCLUSION

We have carried out all the more rigorously the Fourier Transform of the Lippmann-Schwinger equation that describes electromagnetic scattering in $\mathbb{R}^3$. We tried to be as general as possible (e.g. working as long as possible in $\mathbb{R}^n$). Therefore, using the work in [21], and the method described above, the reader can generalize this work to the FT of other convolution equations of the LSE

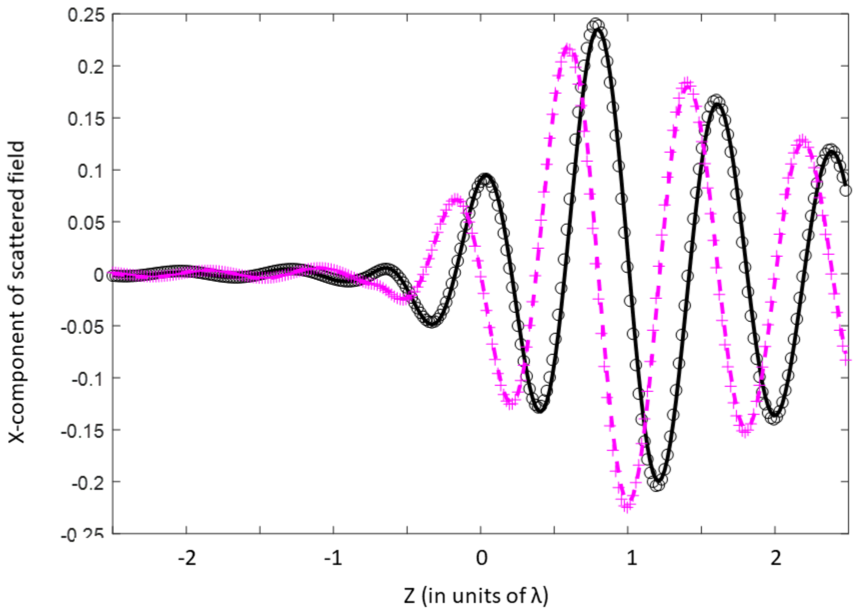

FIG. 2. Real (black) and Imaginary (magenta) part of the **x** component of the scattered field, along the direction of propagation. Parameters are those of Fig. 1. Symbols show the results computed with Mie theory, lines are for the inverse FT of Eq. (V.62).

type, with a scalar kernel such as: $x \mapsto \frac{R(\cos(\|x\|,\sin(\|x\|)}{\|x\|^m} P(x)$, where $R \in \mathbb{C}[X,Y]$, $P \in \mathbb{C}[X,Y,Z]$, and $m \in \mathbb{N}^\star$. The final FT result is surprisingly simple, and could be used in further work, to compute in a novel way the E-field, *eg*. so as to find the eigenmodes of optical resonators [26–28], or to study coherence of light in disordered systems, with correlated disorder [17].

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Appendix A: Fourier transform and notations

Fourier Transform of a function

There are a lot of definitions of the Fourier transform. Let us present a convenient notation to encompass all the cases once for all (with $f, g : \mathbb{R}^n \to \mathbb{R}$, $\mathscr{F}$ representing the direct Fourier transform). Let us denote $< f,g >$ the canonical scalar product, either on $\mathbb{R}^n$, or (unambiguously) on the function space $\mathbb{R}^n \to \mathbb{R}$. In the following, the fact that the electromagnetic field is a complex valued function will be handled by splitting real and imaginary part, in the calculations :

$$\mathscr{F}^{a,b} f(k) \ := \left(\frac{|b|}{(2\pi)^{1-a}}\right)^{n/2} \int_{\mathbb{R}^n} f(t) e^{ib\langle k,t\rangle}\, \mathrm{d}t, \tag{A.1}$$

$$\left(\mathscr{F}^{a,b}\right)^{-1} g(t) := \left(\frac{|b|}{(2\pi)^{1+a}}\right)^{n/2} \int_{\mathbb{R}^n} g(k) e^{-ib\langle t,k\rangle}\, \mathrm{d}k, \tag{A.2}$$

Note that detailed proof are given in the supplementary information, see Sec. (II) in [21].

To jump from a convention to another, we have the simple following corresponding formula :

$$\mathscr{F}^{a,b} f(k) = \left(\frac{|b/b'|}{(2\pi)^{a'-a}}\right)^{n/2} \mathscr{F}^{a',b'} f\left(\tfrac{b}{b'}k\right). \tag{A.3}$$

Fourier Transform of a tempered distribution

Let us recall that if $T \in \mathscr{S}'(\mathbb{R}^n)$ is a tempered distribution, we define the Fourier transform $\mathscr{F}^{a,b}T$ by

$$\langle \mathscr{F}^{a,b} T, \varphi\rangle := \langle T, \mathscr{F}^{a,b}\varphi\rangle, \tag{A.4}$$

where,

$$\langle f,g\rangle := \int_{\mathbb{R}^n} f(t) g(t)\, \mathrm{d}t. \tag{A.5}$$

One can generalize the "jump" formula Eq.(A.3) to distribution, by :

$$\mathscr{F}^{a,b} T = \left(\frac{|b/b'|}{(2\pi)^{a'-a}}\right)^{n/2} m_{\frac{b}{b'}} \mathscr{F}^{a',b'} T, \tag{A.6}$$

where, for all $\alpha \neq 0$,

$$\langle m_\alpha T, \varphi\rangle := \frac{1}{|\alpha|^n} \langle T, m_{\frac{1}{\alpha}} \varphi\rangle \tag{A.7}$$

The proof is given in Sec. (II B) in [21].

Fourier Transform of a convolution

Symmetrically, the FT of convolution can be expressed as :

$$\mathscr{F}^{a,b}(f \star g) = \left(\frac{(2\pi)^{1-a}}{|b|}\right)^{\frac{n}{2}} \left(\mathscr{F}^{a,b} f\right) \cdot \left(\mathscr{F}^{a,b} g\right), \tag{A.8}$$

The proof is given in Sec. (II C 2) in [21].

Fourier Transform of a product

The well known formula that links the FT of a product and the convolution of two FTs can be casted under the general notation, see Eq. (A.1, A.2). The result reads:

$$\mathscr{F}^{a,b}(f\cdot g) = \left(\frac{|b|}{(2\pi)^{a+1}}\right)^{\frac{n}{2}} \left(\mathscr{F}^{a,b} f\right) \star \left(\mathscr{F}^{a,b} g\right), \tag{A.9}$$

The proof is given in Sec. (II C 3) in [21].

Alternative notation used in this article

These general notations, in spite of being really useful to bring altogether the different Fourier transform conventions, can be a bit heavy while conducting computations. We will often use, for a function (or a distribution) u the simplified notation $\mathscr{F}(u)$, or even $\hat{u}$, to symbolize $\mathscr{F}^{1,-1}(u)$. However, the main results, as well as the FT of important (generalized) functions in the general notation are documented in appendix, see Sec. (II D 1) in [21].

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[1] J.-R. Li and L. Greengard, Journal of Computational Physics **226**, 1891 (2007).

[2] K. Takayanagi and M. Oishi, Journal of Mathematical Physics **56**, 022101 (2015).

[3] M. Reed and B. Simon, *Methods of modern mathematical Physics*, Scattering theory, Vol. 3 (Academic Press, 1979).

[4] A. C. Maioli and A. G. M. Schmidt, Journal of Mathematical Physics **59**, 122102 (2018).

[5] B. U. Felderhof and R. B. Jones, Physica A **136**, 77 (1986).

[6] N. V. Budko and A. B. Samokhin, SIAM Journal on Scientific Computing **28**, 682 (2006).

[7] G. P. Zouros and N. V. Budko, SIAM Journal on Scientific Computing **34**, B226 (2012).

[8] A. Samokhin, Y. Shestopalov, and I. Shestopalov, *Integral Equations and Iteration Methods in Electromagnetic Scattering* (VSP, 2001).

[9] L. Novotny and B. Hecht, *Principles of Nano-Optics* (Cambridge, 2006).

[10] J. Goodman, *Introduction to Fourier Optics* (Mc Graw-Hill, 1996).

[11] M. Paulus, P. Gay-Balmaz, and O. J. F. Martin, Phys. Rev. E **62**, 5797 (2000).

[12] C. Acquista, Appl. Opt. **15**, 2932 (1976).

[13] H. F. Arnoldus, J. Opt. Soc. Am. B **18**, 547 (2001).

[14] C. J. R. Sheppard, J. Lin, and S. S. Kou, J. Opt. Soc. Am. A **30**, 1180 (2013).

[15] C. J. R. Sheppard, S. S. Kou, and J. Lin, Frontiers in Physics **2**, 67 (2014).

[16] J. Schmalz, G. Schmalz, T. Gureyev, and K. Pavlov, American Journal of Physics **78**, 181 (2010).

[17] K. Vynck, R. Pierrat, and R. Carminati, Phys. Rev. A **89**, 013842 (2014).

[18] The indicatrix is a function whose value is 1 inside the scatterer, 0 outside.

[19] G. M. Gallatin, Journal of Mathematical Physics **53**, 013509 (2012).

[20] L. Grafakos and G. Teschl, Journal on Fourier Analysis and Applications **19**, 167 (2013).

[21] F. Gruy, M. Perrin, and V. Rabiet, SUPPLEMENTARY MATERIAL.

[22] S. H. Schot, Historia Mathematica **19**, 385 (1992).

[23] L. Grafakos, *Classical Fourier Analysis*, Graduate Texts in Mathematics, second edition (Springer, 2008).

[24] Indeed if we have a rotation B such that $Be_2 = e_k$, $Be_k = -e_2$, $Be_j = e_j$ elsewhere ; then

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} h(\theta_1)\theta_k^\alpha \,\mathrm{d}\theta &= \int_{\mathbb{S}^{n-1}} h(e_1\cdot\theta)(e_k\cdot\theta)^\alpha \,\mathrm{d}\theta \\ &= \int_{\mathbb{S}^{n-1}} h(e_1\cdot\theta)(Be_2\cdot\theta)^\alpha \,\mathrm{d}\theta \\ &= \int_{\mathbb{S}^{n-1}} h(B{}^tBe_1\cdot\theta)(e_2\cdot{}^tB\theta)^\alpha \,\mathrm{d}\theta \\ &= \int_{\mathbb{S}^{n-1}} h(\underbrace{{}^tBe_1}_{=e_1}\cdot{}^tB\theta)(e_2\cdot{}^tB\theta)^\alpha \,\mathrm{d}\theta \\ &= \int_{\mathbb{S}^{n-1}} h(e_1\cdot\theta)(e_2\cdot\theta)^\alpha \,\mathrm{d}\theta = \int_{\mathbb{S}^{n-1}} h(\theta_1)\theta_2^\alpha \,\mathrm{d}\theta.\end{aligned}$$

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[25] See https://hal.archives-ouvertes.fr/hal-02494837.

[26] M. Perrin, Optics Express **24**, 27137 (2016).

[27] P. Lalanne, W. Yan, K. Vynck, C. Sauvan, and J.-P. Hugonin, Laser Photonics Review **12**, 1700113 (2018).

[28] T. Meklachi, S. Moskow, and J. C. Schotland, Journal of Mathematial Physics **59**, 083502 (2018).