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Abstract. We study two families of probability measures on integer partitions, which are Schur measures with parameters tuned in such a way that the edge fluctuations are characterized by a critical exponent different from the generic 1/3. We find that the first part asymptotically follows a "higher-order analogue" of the Tracy–Widom GUE distribution, previously encountered by Le Doussal, Majumdar and Schehr in quantum statistical physics. We also compute limit shapes, and discuss an exact mapping between one of our families and the multicritical unitary matrix models introduced by Periwal and Shevitz.

Abstract. Nous considérons deux familles de mesures de Schur dont les fluctuations de bord sont caractérisées par un exposant différant de la valeur générique 1/3. Les distributions-limites, généralisant la loi de Tracy-Widom, ont été précédemment rencontrées par Le Doussal, Majumdar et Schehr. Nous calculons les formes-limites et discutons du lien avec les modèles de matrices unitaires de Periwal et Shevitz.

1 Introduction

Background. Schur measures, introduced by Okounkov [12], are probability measures on integer partitions λ of the form

$$\mathbb{P}(\lambda) = Z^{-1} s_\lambda[\theta_1, \theta_2, \dots] s_\lambda[\theta'_1, \theta'_2, \dots]. \tag{1.1}$$

Here, the θ_i, θ'_i are numbers such that $Z = \exp \sum_{i \geq 1} \frac{\theta_i \theta'_i}{i}$ is well-defined, and $s_\lambda[\theta_1, \theta_2, \dots]$ is the Schur symmetric function indexed by λ and evaluated at the specialization sending the i-th power sum p_i to the value θ_i, for all $i \geq 1$. A more concrete expression is given by the Jacobi–Trudi identity $s_\lambda[\theta_1, \theta_2, \dots] = \det_{i,j} h_{\lambda_i - i + j}[\theta_1, \theta_2, \dots]$, the entries of the

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determinant being given by the generating series $\sum_{k\geq 0} h_k[\theta_1, \theta_2, \dots] z^k = \exp \sum_{i\geq 1} \frac{\theta_i z^i}{i}$. See [11, 17] for background on symmetric functions and specializations.

Example 1. For $\theta_1 = \theta'_1 = \theta$, and all other θ_i, θ'_i set to zero, we obtain the *poissonized Plancherel measure* $\mathbb{P}(\lambda) = e^{-\theta^2} \left(\theta^{|\lambda|} \frac{f_\lambda}{|\lambda|!} \right)^2$, discussed below. Here, f_λ denotes the number of standard Young tableaux of shape λ.

Example 2. For $\theta_1 = \theta'_1$, $\theta_2 = \theta'_2$, and all other θ_i, θ'_i set to zero, we get

$$\mathbb{P}(\lambda) = e^{-\theta_1^2 - \theta_2^2/2} \sum_{\mu = 1^{a_1} 2^{a_2}} \sum_{\nu = 1^{b_1} 2^{b_2}} \frac{\chi^\lambda(\mu) \chi^\lambda(\nu) \theta_1^{a_1+b_1} \theta_2^{a_2+b_2}}{2^{a_2+b_2} a_1! a_2! b_1! b_2!} \tag{1.2}$$

where χ^λ is the irreducible character of the symmetric group $S_{|\lambda|}$ indexed by λ and μ, ν are two-column partitions with $|\lambda| = |\mu| = |\nu|$ using the notation of [11, Ch. 1].

Schur measures and their generalizations appear in several combinatorial, probabilistic, and statistical mechanical models of mathematical and physical interest. For a brief list, see [12, 13, 4] and references therein. One notable instance is the resolution of Ulam's problem on longest increasing subsequence of random permutations [1]. Namely, if we consider the poissonized Plancherel measure in Example 1, then the Baik–Deift–Johansson theorem [1] states that the first part λ_1 satisfies

$$\lim_{\theta \to \infty} \mathbb{P} \left(\frac{\lambda_1 - 2\theta}{\theta^{1/3}} < s \right) = F_{\mathrm{TW}}(s) \tag{1.3}$$

with $F_{\mathrm{TW}}(s)$ the Tracy–Widom GUE distribution [18] from random matrix theory. By Schensted's theorem [16], λ_1 is equal in distribution to the longest increasing subsequence of a random permutation on S_N, the symmetric group of N letters, where N in our case is a Poisson random variable $N \sim \mathrm{Poisson}(\theta^2)$. See [15] for more on this topic.

Main contribution. We consider *multicritical* Schur measures, having as their salient feature an "edge" behavior different from (1.3). More precisely, for every $n \geq 2$, we construct Schur measures for which the $1/3$ fluctuation exponent is replaced by $1/(2n+1)$ (we recover the poissonized Plancherel measure for $n = 1$). The limiting distribution then becomes a "higher-order analogue" of the Tracy–Widom distribution. It is a τ-function of a higher-order differential equation of the Painlevé II hierarchy [5] in the same way the Tracy–Widom distribution is for the "classical" Painlevé II equation [18].

Our inspiration comes from the work of Le Doussal, Majumdar and Schehr [10], who found the same limiting distributions in the momenta statistics of fermions in nonharmonic traps. They also noted a coincidental connection with the multicritical unitary matrix models of Periwal and Shevitz [14], which involve the Painlevé II hierarchy in their double scaling limit.

Our multicritical Schur measures explain the origin of this connection. On the one hand, as observed by Okounkov [12], Schur measures admit a convenient description in terms of free fermions. Simple scaling arguments show that they have the same asymptotic edge behavior as the models considered in [10]. On the other hand, through a chain of classical identities that we will review, the distribution of λ_1 in a Schur measure can be expressed as the partition function of a unitary matrix model. For multicritical measures, we recover *exactly* the models of [14]. Let us point out that there is a known connection between Ulam's problem and the Gross–Witten unitary matrix model, see [9] and references therein. We comment on the relation with our work in the conclusion.

Outline. In Section 2, we define the multicritical Schur measures and state our main theorems (Theorems 1 and 2) concerning their edge behavior. We compute limit shapes in Section 3. Section 4 reviews the connection between Schur measures and unitary matrix integrals. The proof of Theorems 1 and 2 is sketched in Section 5. Finally, concluding remarks are gathered in Section 6.

This is an extended abstract of the paper [2]. For brevity, we do not include a discussion of the physical interpretation in terms of fermions here, but we note that they manifest themselves via the determinantal point processes used in Section 5.

2 Multicritical Schur measures and their edge behavior

A partition λ may be characterized by the set $S(\lambda) = \{\lambda_i - i + \frac{1}{2} | i \geq 1\} \subset \mathbb{Z} + \frac{1}{2}$, see Figure 1 below. Note that the largest element of $S(\lambda)$ is $\lambda_1 - \frac{1}{2}$, and the smallest element of its complement is $-\ell(\lambda) + \frac{1}{2}$, where $\ell(\lambda)$ denotes the length of λ (number of nonzero parts).

When λ is a distributed according to a Schur measure (1.1), it was shown by Okounkov [12] that $S(\lambda)$ is a determinantal point process, whose kernel admits an explicit expression (given in Section 5) in terms of the θ_i, θ'_i parameters.

The study of the edge behavior—the statistics of the largest element(s) of $S(\lambda)$, or of the smallest element(s) of its complement—is most conveniently done via a saddle-point analysis [13]. For generic parameters θ_i, θ'_i (and, in particular, for the poissonized Plancherel measure), it is found that the edge behavior is characterized by the coalescence of two saddle points, which implies that the "action" has a double critical point, also known as "monkey saddle", explaining the $1/3$ fluctuation exponent. Multicritical Schur measures are obtained by tuning the parameters in such a way that the action has a critical point of higher order.

For simplicity, we restrict to the case where $\theta_i = \theta'_i$—ensuring that the probability (1.1) is indeed nonnegative—and where the set $\{i : \theta_i \neq 0\}$ is finite and of fixed cardinal $n \geq 1$. By symmetry reasons, the edge critical point is always of even order, and by tuning the

θ_i we expect $2n$ to be the maximal possible order. This is indeed the case.

Theorem 1 ("odd-even multicritical measure"). *Let $\mathbb{P}^{\mathrm{oe}}_{n,\theta}$ denote the Schur measure (1.1) where we set $\theta_i = \frac{(-1)^{i+1}(n-1)!(n+1)!}{(n-i)!(n+i)!}\theta$ for $i = 1, \dots, n$, and $\theta_i = 0$ for $i > n$. Then, we have:*

$$\lim_{\theta\to\infty} \mathbb{P}^{\mathrm{oe}}_{n,\theta}\left[\frac{\lambda_1 - b\theta}{(\theta/d)^{\frac{1}{2n+1}}} < s\right] = F(2n+1; s), \quad \lim_{\theta\to\infty} \mathbb{P}^{\mathrm{oe}}_{n,\theta}\left[\frac{\ell(\lambda) - \tilde{b}\theta}{(\theta/\tilde{d})^{\frac{1}{3}}} < s\right] = F(3; s) \tag{2.1}$$

with $b = \frac{n+1}{n}$, $d = \binom{2n}{n-1}$, $\tilde{b} = \frac{n+1}{n}\left(\frac{(2n)!!}{(2n-1)!!} - 1\right)$, $\tilde{d} = \frac{2^{2n-2}n}{\binom{2n}{n-1}}$, $F(3; s) = F_{\mathrm{TW}}(s)$ the Tracy-Widom GUE distribution and $F(2n+1, s)$ its higher-order analogue defined at (2.6) below.

As we see, we obtain a nongeneric exponent $1/(2n+1)$ for the fluctuations of λ_1, but we still have the generic exponent $1/3$ for the fluctuations of $\ell(\lambda)$. It is actually possible to have a more symmetric situation if, rather than taking $\theta_1, \dots, \theta_n$ nonzero, we take $\theta_1, \theta_3, \dots, \theta_{2n-1}$ nonzero.

Theorem 2 ("odd multicritical measure"). *Let $\mathbb{P}^{\mathrm{o}}_{n,\theta}$ denote the Schur measure (1.1) where we set $\theta_{2i-1} = \frac{(-1)^{i+1}(n-1)!n!}{(2i-1)(n-i)!(n+i-1)!}\theta$ for $i = 1, \dots, n$, and all other θ_i to zero. Then, $\mathbb{P}^{\mathrm{o}}_{n,\theta}$ is invariant under the conjugation of partitions $\lambda \mapsto \lambda'$, and we have:*

$$\lim_{\theta\to\infty} \mathbb{P}^{\mathrm{o}}_{n,\theta}\left[\frac{\lambda_1 - b\theta}{(\theta/d)^{\frac{1}{2n+1}}} < s\right] = \lim_{\theta\to\infty} \mathbb{P}^{\mathrm{o}}_{n,\theta}\left[\frac{\ell(\lambda) - b\theta}{(\theta/d)^{\frac{1}{2n+1}}} < s\right] = F(2n+1; s) \tag{2.2}$$

with $b = \frac{2^{4n-1}}{n\binom{2n}{n}^2}$, $d = \frac{(2n-1)!!}{(2n-2)!!}$, and $F(2n+1; s)$ defined at (2.6) below.

Remark 3. For both measures, we have $\theta_1 = \theta$ and the parameters θ_i, b and d satisfy

$$\sum_i i^k \theta_i = \delta_{k,0}\, \tfrac{b\theta}{2} + (2n)!\delta_{k,2n} d\theta, \quad k = 0, 2, \dots, 2n-2, 2n. \tag{2.3}$$

When $n = 1$, both measures reduce to the poissonized Plancherel measure, and we recover the convergence in distribution (1.3). As soon as $n \geq 2$, they involve specializations which are not Schur positive, but the measures are nevertheless probability measures.

Example 4. For $n = 2$, $\mathbb{P}^{\mathrm{oe}}_{n,\theta}$ has the form given in Example 2 with $\theta_1 = \theta$, $\theta_2 = -\frac{\theta}{4}$, while $\mathbb{P}^{\mathrm{o}}_{n,\theta}$ has $\theta_1 = \theta$, $\theta_3 = -\frac{\theta}{9}$ as nonzero parameters.

The distributions $F(2n+1; s)$ appearing in Theorems 1 and 2 have been previously encountered in [10, 5], and we now give their definition in a self-contained way. First, we recall that, if K is an integral operator with kernel $K(x, y)$ acting on $L^2(X)$ (X is an open interval in what follows), it acts on functions $f \in L^2(X)$ via "matrix multiplication"

$(Kf)(x) = \int_X K(x,y)f(y)\mathrm{d}y$. For such operators which are trace-class one can define the *Fredholm determinant* of $1-K$ (1 the identity operator) on $L^2(X)$ by

$$\det(1-K)_{L^2(X)} = \sum_{m\geq 0} \frac{(-1)^m}{m!} \int_X \cdots \int_X \det_{1\leq i,j\leq m}[K(x_i,x_j)]\mathrm{d}x_1\cdots\mathrm{d}x_m \tag{2.4}$$

where there are m integrals in the m-th summand (and the case $m=0$ yields 1).

Consider first the following *generalized* (order $2n+1$) *Airy function*:

$$\mathrm{Ai}_{2n+1}(x) = \int_{i\mathbb{R}+\delta} \exp\left(\frac{(-1)^{n-1}\zeta^{2n+1}}{2n+1} - x\zeta\right)\frac{\mathrm{d}\zeta}{2\pi i} \tag{2.5}$$

where $\delta > 0$ is small and the contour is up-oriented.[1] Notice they satisfy the generalized Airy differential equation $\left(\frac{d}{dx}\right)^{2n} A(x) = (-1)^{n-1}xA(x)$ and that Ai_3 is the usual Airy *Ai* function. Then $F(2n+1;s)$ is the following Fredholm determinant

$$F(2n+1;s) = \det(1-\mathcal{A}_{2n+1})_{L^2(s,\infty)} \tag{2.6}$$

where $\mathcal{A}_{2n+1}$ is the higher order Airy kernel given by

$$\begin{aligned}\mathcal{A}_{2n+1}(x,y) &= \int_{i\mathbb{R}-\delta} \frac{\mathrm{d}\omega}{2\pi i}\int_{i\mathbb{R}+\delta}\frac{\mathrm{d}\zeta}{2\pi i}\frac{\exp\left(\frac{(-1)^{n-1}\zeta^{2n+1}}{2n+1} - x\zeta\right)}{\exp\left(\frac{(-1)^{n-1}\omega^{2n+1}}{2n+1} - y\omega\right)}\frac{1}{\zeta-\omega} \\ &= \int_0^\infty \mathrm{Ai}_{2n+1}(x+t)\mathrm{Ai}_{2n+1}(y+t)\mathrm{d}t \\ &= (-1)^{n-1}\frac{\sum_{i=0}^{2n-1}(-1)^i \mathrm{Ai}_{2n+1}^{(i)}(x)\mathrm{Ai}_{2n+1}^{(2m-1-i)}(y)}{x-y}\end{aligned} \tag{2.7}$$

(both contours above are up-oriented). Let us note that $\mathcal{A}_3(x,y) = \frac{\mathrm{Ai}_3(x)\mathrm{Ai}_3'(y)-\mathrm{Ai}_3'(x)\mathrm{Ai}_3(y)}{x-y}$ is the usual Airy kernel and $F(3;s) = F_{\mathrm{TW}}(s)$ the Tracy–Widom GUE distribution [18]. In the $x=y$ case, the third equality should be taken in the l'Hôpital limit sense.

3 Limit shapes

In this section we describe limit shapes for the multicritical $\mathbb{P}^{\mathrm{o}}_{n,\theta}$- and $\mathbb{P}^{\mathrm{oe}}_{n,\theta}$-distributed random partitions of Theorems 1 and 2. This section is descriptive; stating precise results is harder not least because of topological and analytical considerations beyond the scope and space afforded by this note; see [15] for precise statements in the $n=1$ case.

[1] Comparing with [10, Eq. (5)], we chose different integration conventions for the same function. Their expression is different for n even and comes from the change of variables $z=-\zeta$. Otherwise said, the contours of [10, Eq. (5)] are such that $\Re(z^{2n+1}) < 0$ whereas ours have $\Re((-1)^{n-1}\zeta^{2n+1}) < 0$.

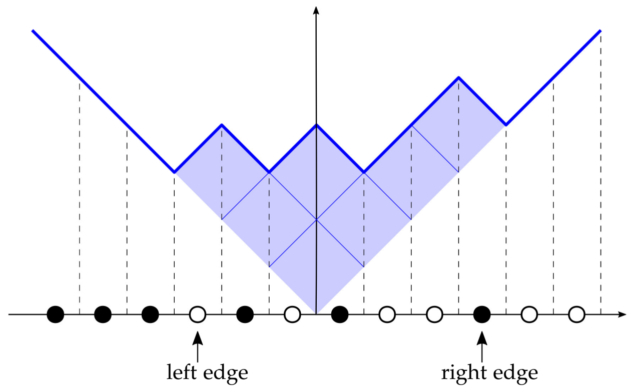

Figure 1: The profile (thick blue) and the set $S(\lambda)$ (dots) for $\lambda = (4, 2, 1)$.

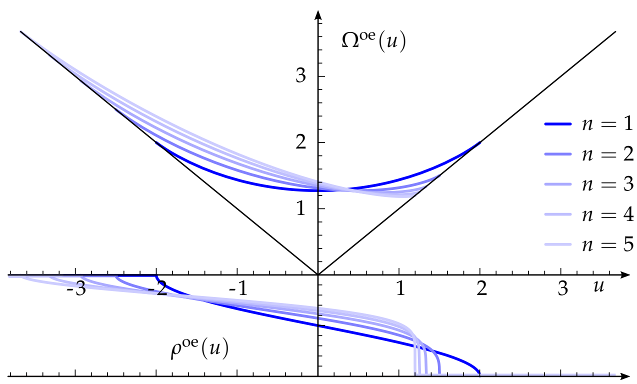

Figure 2: Limit shape and density profile of $\mathbb{P}^{\text{oe}}_{n,\theta}$-distributed random partitions.

To begin, recall that the Young diagram of a partition can be represented in "Russian convention" as the graph of a piecewise linear function composed of slope ± 1 segments, which we call its *profile*. See Figure 1.

If λ is distributed according to the measures $\mathbb{P}^{\text{oe}}_{n,\theta}$ or $\mathbb{P}^{\text{o}}_{n,\theta}$ of Theorems 1 and 2, and if we rescale by a factor $1/\sqrt{\theta}$ in both directions, then the profile converges as $\theta \to \infty$ to the graph of a deterministic 1-Lipschitz function, denoted Ω. We have $\Omega' = 1 - 2\rho$, where ρ is the limiting density profile of the set $S(\lambda)$.

The limiting density profiles may be computed exactly. Let us denote them as follows:

$$\rho^{\text{o/oe}}(u) = \lim_{\theta \to \infty} \sum_{\lambda : \theta u \in S(\lambda)} \mathbb{P}^{\text{o/oe}}_{n,\theta}(\lambda). \tag{3.1}$$

In the oe case we have, with $b = \frac{n+1}{n}, \tilde{b} = \frac{n+1}{n}\left(\frac{(2n)!!}{(2n-1)!!} - 1\right)$:

$$\rho^{\text{oe}}(u) = \frac{1}{\pi} \arccos\left(1 - \frac{1}{2}\binom{2n}{n-1}^{\frac{1}{n}} (b-u)^{\frac{1}{n}}\right), \quad u \in \left[-\tilde{b}, b\right] \tag{3.2}$$

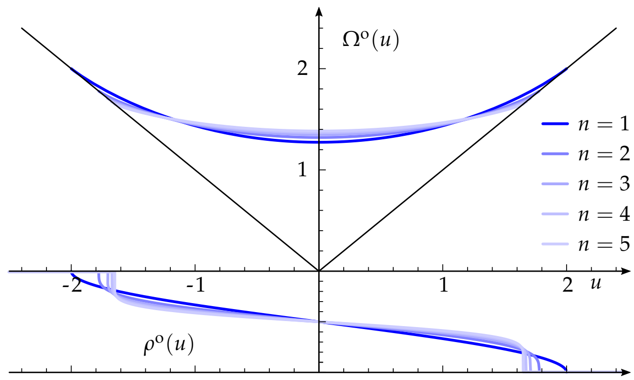

Figure 3: Limit shape and density profile of $\mathbb{P}^{\rm o}_{n,\theta}$-distributed random partitions, for $n = 1, \dots, 5$. Notice the symmetry with respect to the vertical axis.

and $\rho^{\rm oe}(u) = 1, u < -\tilde{b}$, $\rho^{\rm oe}(u) = 0, u > b$. The limit profile—depicted in Figure 2—is $\Omega^{\rm oe}(u) = \tilde{b} + \int_{-\tilde{b}}^{u} [1 - 2\rho(v)]\, \mathrm{d}v$. A similar profile, for n=2, was recently observed in tight-binding fermions [3].

In the o case and for $b = \frac{2^{4n-1}}{n\binom{2n}{n}^2}$ we have:

$$\rho^{\rm o}(u) = \frac{\chi(u)}{\pi}, \quad \int_0^{\chi(u)} (2\sin\phi)^{2n-1} \mathrm{d}\phi = (-1)^{n+1} \binom{2n-1}{n} u, \quad u \in [-b, b] \tag{3.3}$$

continued to $\rho^{\rm o}(u) = 1, u < -b$, $\rho^{\rm o}(u) = 0, u > b$. The limit shape, *symmetric under the vertical axis* and shown in Figure 3, is $\Omega^{\rm o}(u) = b + \int_{-b}^{u} [1 - 2\rho(v)]\, \mathrm{d}v$.

Both $\Omega^{\rm o}$ and $\Omega^{\rm oe}$ are extensions of the Vershik–Kerov–Logan–Shepp limit curve—see e.g. [15]—to multicritical random partitions; indeed they become the former if $n = 1$.

4 Toeplitz determinants and unitary matrix integrals

In this section we review the connection between Schur measures and unitary matrix integrals, and we relate our multicritical measures to the integrals studied in [14]. For simplicity, we assume that the parameters θ_i, θ_i' of (1.1) are such that $\theta_i = \theta_i'$ for all i, and $\theta_i = 0$ for i large enough. We introduce the polynomials V and $\tilde{V}$ defined by

$$V(z) = \sum_{i \geq 1} \theta_i \frac{z^i}{i}, \quad \tilde{V}(z + z^{-1}) = V(z) + V(z^{-1}). \tag{4.1}$$

In physical parlance $\tilde{V}$, modulo a multiplicative constant, is often called the *potential*.

Example 5. If $V(z) = \theta_1 z + \frac{\theta_2}{2} z^2 + \frac{\theta}{3} z^3$ we have $\tilde{V}(x) = -\theta_2 + (\theta_1 - \theta_3)x + \frac{\theta_2}{2} x^2 + \frac{\theta_3}{3} x^3$.

Proposition 6. *For λ distributed as above, we have:*

$$e^{\sum_i \theta_i^2/i} \cdot \mathbb{P}(\ell(\lambda) \leq \ell) = \det_{1 \leq i,j \leq \ell} [f_{j-i}] = \mathbb{E}_{U \in \mathcal{U}(\ell)} \left[\exp \operatorname{tr} \tilde{V}(U + U^*)\right] \tag{4.2}$$

where the middle Toeplitz determinant has symbol $\sum_{k \in \mathbb{Z}} f_k z^k = \exp \tilde{V}(z + z^{-1})$, and $\mathbb{E}_{U \in \mathcal{U}(\ell)}$ is the expectation with respect to the Haar measure over the unitary group $\mathcal{U}(\ell)$.

Proof. The left-hand side is equal to $\sum_{\ell(\lambda) \leq \ell} (s_\lambda[\theta_1, \theta_2, \ldots])^2$ which, by Gessel's identity [8, Thm. 16], is equal to the middle Toeplitz determinant. The second equality is Heine's identity—see e.g. [7]. □

We also have the following similar identity regarding λ_1.

Proposition 7. *For λ distributed as above, we have:*

$$e^{\sum_i \theta_i^2/i} \cdot \mathbb{P}(\lambda_1 \leq \ell) = \det_{1 \leq i,j \leq \ell} [g_{j-i}] = \mathbb{E}_{U \in \mathcal{U}(\ell)} \left[\exp \operatorname{tr} \left(-\tilde{V}(-U - U^*)\right)\right] \tag{4.3}$$

where the middle Toeplitz determinant has symbol $\sum_{k \in \mathbb{Z}} g_k z^k = \exp(-\tilde{V}(-z - z^{-1}))$.

It is a straightforward consequence of Proposition 6 and the following:

Lemma 8. *If λ is distributed according to the Schur measure (1.1), then the conjugate partition λ' is distributed according to the Schur measure of parameters $\tilde{\theta}_i = (-1)^{i-1}\theta_i$, $\tilde{\theta}_i{}' = (-1)^{i-1}\theta'_i$.*

Proof. This follows from the relation $s_\lambda[\theta_1, \theta_2, \ldots] = s_{\lambda'}[\tilde{\theta}_1, \tilde{\theta}_2, \ldots]$ that results from the classical involution ω on the algebra of symmetric function mapping the power sum p_i to $(-1)^{i-1} p_i$ and the Schur function s_λ to $s_{\lambda'}$. □

Another consequence of the above lemma is the fact, mentioned in Theorem 2, that $\mathbb{P}^{\mathrm{o}}$ is invariant under conjugation.

When we specialize Proposition 7 to the multicritical measures $\mathbb{P}^{\mathrm{oe}}_{n,\theta}$ of Theorem 1, then the right-hand side of (4.3) matches, up to a change of variable $U \to -U$, the multicritical unitary matrix integrals of Periwal and Shevitz [14]. Indeed, the derivative $V'_k(z)$ given on p. 737 of *op. cit.* is proportional to $V'(z)$ for $k = n$ in our present notations, and the proportionality constant can be reabsorbed in θ. It seems that the case of an odd multicritical potential was not considered in their paper.

5 Sketch of proof

Let us sketch the proof of Theorems 1 and 2. We present the argument for the $\mathbb{P}^{\mathrm{o}}_{n,\theta}$ measure as it is slightly simpler, and make comments at the end on the difference with the $\mathbb{P}^{\mathrm{oe}}_{n,\theta}$ measure.

We use the fact, already mentioned at the beginning of Section 2, that $S(\lambda)$ is a determinantal point process. This means that, fixing m and $k_1, \dots, k_m \in \mathbb{Z} + \frac{1}{2}$, we have

$$\mathbb{P}^{\mathrm{o}}(\{k_1, \dots, k_m\} \in S(\lambda)) = \det_{1 \le i,j \le m} K(k_i, k_j) \tag{5.1}$$

where, by [12], the discrete (ℓ^2 operator) kernel K equals (for some small $\epsilon > 0$)

$$K(k, \ell) = \frac{1}{(2\pi i)^2} \oint_{|w|=1-\epsilon} \oint_{|z|=1+\epsilon} \frac{e^{V(z)-V(z^{-1})}}{e^{V(w)-V(w^{-1})}} \frac{\mathrm{d}z\mathrm{d}w}{z^{k+1/2} w^{-\ell+1/2}(z-w)} \tag{5.2}$$

with V as in (4.1). Combinatorially, the above integral is just coefficient extraction: we look at the coefficient of z^k/w^ℓ in the respective generating series (since $\epsilon > 0$, $\frac{1}{z-w}$ should be expanded as $\sum_{i \ge 0} \frac{w^i}{z^{i+1}}$). Moreover, inclusion-exclusion gives that the *gap* probability $\mathbb{P}^{\mathrm{o}}(\lambda_1 \le l)$ is equal to the discrete Fredholm determinant $\det(1-K)_{\ell^2\{l+1/2, l+3/2, \dots\}}$.

In the multicritical regime we look for numbers β and $\theta_1, \theta_3, \dots, \theta_{2n-1}$ satisfying

$$\sum_{i=1,3,\dots,2n-1} i^k \theta_i = -\delta_{k,0} \frac{\beta}{2}, \quad k = 0, 2, \dots, 2n-2 \tag{5.3}$$

and solve for each of them in terms of $\theta_1 = \theta$. The solution is, up to an overall factor, given in the statement of the theorem:

$$\theta_1 = \theta, \quad \beta = \frac{2^{4n-1}}{n\binom{2n}{n}^2}\theta, \quad \theta_{2i-1} = \frac{(-1)^{i+1}(n-1)!n!}{(2i-1)(n-i)!(n+i-1)!}\theta, \quad i = 2, 3, \dots, n. \tag{5.4}$$

We call $b = 2^{4n-1} n^{-1} \binom{2n}{n}^{-2} = \beta/\theta$. The correlation kernel becomes

$$K(k, \ell) = \frac{1}{(2\pi i)^2} \oint_{|w|=1-\epsilon} \oint_{|z|=1+\epsilon} \frac{e^{\theta[S_0(z)-S_0(w)]}\mathrm{d}z\mathrm{d}w}{z^{k+1/2} w^{-\ell+1/2}(z-w)} \tag{5.5}$$

with $S_0(z) = \sum_{i=1}^{n} \frac{(-1)^{i+1}(n-1)!n!}{(2i-1)(n-i)!(n+i-1)!} \frac{(z^{2i-1} - z^{1-2i})}{2i-1}$. The equations (5.3) ensure that

$$(z\partial_z)^i [S_0(z) - b \log z]\Big|_{z=1} = 0, \quad 1 \le i \le 2n \tag{5.6}$$

meaning $z = 1$ is a critical point of order $2n$. The same is true for $z = -1$. Notice that the relation (5.6) is automatically satisfied for even i by the symmetry relation $S_0(z) + S_0(z^{-1}) = 0$; the specific choice of coefficients ensures that it also holds for odd i between 1 and $2n-1$.

We now analyze the scaling regime

$$\theta \to \infty, \qquad k = \lfloor b\theta + x(\theta d)^{\frac{1}{2n+1}} \rfloor, \qquad \ell = \lfloor b\theta + y(\theta d)^{\frac{1}{2n+1}} \rfloor \tag{5.7}$$

with $d = \frac{(2n-1)!!}{(2n-2)!!}$. In this regime, the integral (5.5) will be dominated by the vicinity of the critical point $z = w = 1$ (if we considered instead the regime $k, \ell \approx -b\theta$, then the critical point $z = w = -1$ would dominate). We perform the change of variable

$$z = 1 + \zeta(d\theta^{-1})^{\frac{1}{2n+1}}, \quad w = 1 + \omega(d\theta^{-1})^{\frac{1}{2n+1}} \tag{5.8}$$

where ζ and ω are to be integrated over $i\mathbb{R} + \delta$ and $i\mathbb{R} - \delta$ respectively. The quantity $\theta S_0(z) - k \ln z$ which appears exponentiated in the integral may be approximated as

$$\frac{S^{(2n+1)}(1)}{(2n+1)!}\frac{\zeta^{2n+1}}{d} - x\zeta + O\left(\frac{1}{\theta^{1/(2n+1)}}\right) = (-1)^{n+1}\frac{\zeta^{2n+1}}{2n+1} - x\zeta + O\left(\frac{1}{\theta^{1/(2n+1)}}\right) \tag{5.9}$$

and we estimate $-\theta S_0(w) + \ell \ln w$ similarly. Plugging these estimates into (5.5), we recognize the double integral representation (2.7) for $\mathcal{A}_{2n+1}(x, y)$. Modulo standard arguments (dominated convergence, tail bounds...) to justify the approximation, we obtain

$$(d^{-1}\theta)^{\frac{1}{2n+1}} K\left(b\theta + x(d\theta)^{\frac{1}{2n+1}},\ b\theta + y(d\theta)^{\frac{1}{2n+1}}\right) \to \mathcal{A}_{2n+1}(x, y) \quad \text{as } \theta \to \infty. \tag{5.10}$$

To finish the proof, we show that $K(k, \ell)$ has exponential decay which then shows the discrete Fredholm determinant $\mathbb{P}^{\mathrm{o}}_{n,\theta}(\lambda_1 \leq l) = \det(1 - K)_{\ell^2\{l+1/2, l+3/2, \ldots\}}$ converges to the continuous one $\det(1 - \mathcal{A}_{2n+1})_{L^2(s,\infty)} = F(2n+1; s)$ when $l = b\theta + s(d\theta)^{\frac{1}{2n+1}}$.

In the odd+even multicritical case, the analysis of the scaling regime (5.7) is the same. However, we lose symmetry under conjugation. This means that $V(z)$ and hence the function $S_0(z)$ appearing in (2.7) are not odd functions of z anymore. At the point $z = -1$, which is relevant for studying the asymptotics of $\ell(\lambda)$, we find that $S_0(z) + \tilde{b} \ln z$ has a *generic* double critical point which, by standard calculations, leads to the second equality in (2.1).

6 Concluding remarks

In this paper we have introduced Schur measures displaying the same multicritical edge behavior as the fermionic models considered by Le Doussal, Majumdar and Schehr [10]. We also computed limit shapes and explained how our measures map exactly to the Periwal–Shevitz multicritical unitary matrix models [14]. This gives a combinatorial explanation to the coincidence noted in [10].

The approach of Periwal and Shevitz relies on the method of orthogonal polynomials. Through this approach, one obtains a different expression for the higher order distributions $F(2n+1; s)$ in terms of solutions of the Painlevé II hierarchy. It is shown in [5]—see also Appendix G of the arXiv version of [10]—that it is indeed equal to the

Fredholm determinant (2.6). Multicriticality of a similar flavor was also observed at the spectrum edge of *Hermitian* random matrix ensembles by Claeys, Its and Krasovsky [6].

For $n = 1$, our measures reduce to the poissonized Plancherel measure, while on the unitary random matrix side we obtain a model first studied by Gross and Witten, see e.g. [9] and references therein. Our work shows that the connection observed by Johansson in [9] extends to higher orders $n \geq 2$ of multicriticality, even though the formulation in terms of longest increasing subsequences seems more elusive.

Main result of this note (odd case). Let us summarize, in one place, the results of Theorem 2 on one hand and of Propositions 6 and 7 on the other.[2]

Fix $n \geq 1$ and let λ be $\mathbb{P}^{\mathrm{o}}_{n,\theta}$-distributed (1.1) with $\theta_{2i-1} = \frac{(-1)^{i+1}(n-1)!n!}{(2i-1)(n-i)!(n+i-1)!}\theta$ for $i = 1, \dots, n$ and $\theta > 0$. Let $b = \frac{2^{4n-1}}{n\binom{2n}{n}^2}$; $d = \frac{(2n-1)!!}{(2n-2)!!}$; $V(z) = \sum_{i=1}^n \frac{\theta_{2i-1} z^{2i-1}}{2i-1}$; $\tilde{V}(z + z^{-1}) = V(z) + V(z^{-1})$; and $\sum_{k \in \mathbb{Z}} f_k z^k = \exp[V(z) + V(z^{-1})]$. Then the quantities

$$\mathbb{P}^{\mathrm{o}}_{n,\theta}(\lambda_1 \leq \ell), \quad \mathbb{P}^{\mathrm{o}}_{n,\theta}(\ell(\lambda) \leq \ell), \quad \frac{\det_{1 \leq i,j \leq \ell}[f_{j-i}]}{e^{\sum_{i=1}^n \theta_{2i-1}^2/(2i-1)}}, \quad \frac{\mathbb{E}_{U \in \mathcal{U}(\ell)}\left[\exp \operatorname{tr} \tilde{V}(U + U^*)\right]}{e^{\sum_{i=1}^n \theta_{2i-1}^2/(2i-1)}} \tag{6.1}$$

are all equal, and equal to the Fredholm determinant $\det(1 - K)$ (K as in (5.2)) on $\{\ell + 1/2, \ell + 3/2, \dots\}$. Asymptotically, they all equal the distribution $F(2n+1; s)$ in (2.6) when $\ell = b\theta + s(\theta d)^{\frac{1}{2n+1}}$ and $\theta \to \infty$.

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[2] An analogous result could be stated for Theorem 1; we omit it for brevity.

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