Wave finite element method for waveguides and periodic structures subjected to arbitrary loads

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Abstract

The wave finite element method has been developed for waveguides and periodic structures with advantages in the calculation time. However, this method cannot be applied easily if the structure is subjected to complex or density loads and this is the aim of this article. Based on the finite element method, the dynamic equation of one period of the structure is rewritten to obtain a relation between the responses (DOF and nodal loads) on the left and right boundaries. This relation presents an additive term which links to the loads applying on the period. Then, by using WFE technique, we can compute the wave mode of the period and the wave decomposition. Because of the periodicity, we can also obtain a relation between the response and the left and right ends of the structure. Afterwards, the response of the structure is calculated by using the wave decomposition to apply in the dynamic stiffness matrix (DSM) approach or the wave analysis (WA). For the DMS approach, this technique shows that the external loads have no contribution to the global matrix but they lead to a equivalent force in the dynamic equation. Meanwhile, the external loads create waves propagating to the left and right of the structure in the WA approach.

Keywords: Wave Finite Element, Dynamic, Vibration, Periodic structure, Waveguide

1. Introduction

The dynamic response of a periodic structure is a subject of numerous publications. Beside the classical finite element method, some numerical methods has been developed in order to reduce the calculation time. One of them is the wave finite element method (WFE), which has been developed originally in the seventies in order to describe the wave propagation along one-dimensional periodic elastic structures [1]. Afterwards, this method has been developed for more complex periodic structures [2–9]. Based on the periodicity, the wave mode is calculated from the dynamic stiffness matrix of one period of the structure, provided that it is not submitted to any external load [2, 10]. Then, the response is decomposed in order to compute the wave amplitudes by the dynamic stiffness matrix (DMS) or the wave analysis (WA) approaches. For a structure subjected to external loads, this method can be applied by considering with different parts as superelements [11, 12].Another technique of WFE has been developed to deal with the loads by using the Fourier transform and the contour integration. Based on the solution of an infinite waveguide to a concentrated load, [13] the wave amplitudes of the response have been represented in terms of the wave amplitude of the loads yielded from the integration. This technique is efficient for waveguides and simple periodic structures [14–16]. However, it cannot be applied easily for periodic structures subjected to external loads at inner nodes of each period, which come from complex shapes. In addition, the integration step makes more calculations.

In this article, we present a new development of WFE for periodic structures of any shape and load. To do that, the dynamic equation of one period is rewritten by taking into account the loads at the inner and boundary loads. Then, by using the same technique of the classical WFE, we can obtain a recurrent relation between the responses (DOF and nodal loads) on the left and right boundary of the period. The external loads appear naturally as a additive term in this relation which is missing on the classical WFE. Then, we use the wave basis of WFE to decompose the vector of responses and the additive term. Finally, the responses of the global structure are assembled in the both approaches: DMS and WA. The results show that the external loads present global equivalent loads in the DMS approach or an addition wave amplitudes in the WA analysis.

2. Basic framework

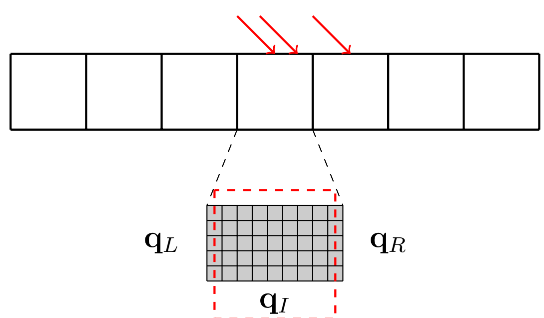

Figure 1: A periodic structure subjected to external loads

Let's consider a periodic structure as shown in Figure 1. Each period of the structure contains the left (L), right (R) boundaries and inner (I) nodes.

By using the finite element method, the dynamic equation of the period can be written in matrix form

$$\tilde{\mathbf{D}}\mathbf{q} = \mathbf{F} \tag{1}$$

with $\tilde{\mathbf{D}} = \mathbf{K} + j\omega\mathbf{C} - \omega^2\mathbf{M}$ is the dynamic stiffness matrix, and $\mathbf{q}$, $\mathbf{F}$ are the vectors of degrees of freedom (DOF) and nodal loads. We can rewrite the dynamic equation to separate the boundaries and inner DOF as follows

$$\begin{bmatrix} \tilde{\mathbf{D}}_{II} & \tilde{\mathbf{D}}_{IL} & \tilde{\mathbf{D}}_{IR} \\ \tilde{\mathbf{D}}_{LI} & \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR} \\ \tilde{\mathbf{D}}_{RI} & \tilde{\mathbf{D}}_{RL} & \tilde{\mathbf{D}}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_I \\ \mathbf{F}_L \\ \mathbf{F}_R \end{bmatrix} \tag{2}$$

We can rewrite the first row of equation (2) as follows

$$\mathbf{q}_I = \tilde{\mathbf{D}}_{II}^{-1} \left[\mathbf{F}_I - \tilde{\mathbf{D}}_{IL}\ \mathbf{q}_L - \tilde{\mathbf{D}}_{IR}\ \mathbf{q}_R \right] \tag{3}$$

Then, by substituting the aforementioned equation into the second and the third rows of equation (2) , we obtain

$$\begin{aligned} \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1} \left[\mathbf{F}_I - \tilde{\mathbf{D}}_{IL}\ \mathbf{q}_L - \tilde{\mathbf{D}}_{IR}\ \mathbf{q}_R \right] + \tilde{\mathbf{D}}_{LL}\ \mathbf{q}_L + \tilde{\mathbf{D}}_{LR}\ \mathbf{q}_R = \mathbf{F}_L \\ \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1} \left[\mathbf{F}_I - \tilde{\mathbf{D}}_{IL}\ \mathbf{q}_L - \tilde{\mathbf{D}}_{IR}\ \mathbf{q}_R \right] + \tilde{\mathbf{D}}_{RL}\ \mathbf{q}_L + \tilde{\mathbf{D}}_{RR}\ \mathbf{q}_R = \mathbf{F}_R \end{aligned} \tag{4}$$

In the other hand, we can write

$$\begin{bmatrix} \mathbf{D}_{LI}\mathbf{F}_I \\ \mathbf{D}_{RI}\mathbf{F}_I \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_L \\ \mathbf{F}_R \end{bmatrix} \tag{5}$$

where

$$\begin{aligned} &\mathbf{D}_{LL} = \tilde{\mathbf{D}}_{LL} - \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IL} && \mathbf{D}_{LR} = \tilde{\mathbf{D}}_{LR} - \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IR} \\ &\mathbf{D}_{RL} = \tilde{\mathbf{D}}_{RL} - \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IL} && \mathbf{D}_{RR} = \tilde{\mathbf{D}}_{RR} - \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IR} \\ &\mathbf{D}_{LI} = \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1} && \mathbf{D}_{RI} = \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1} \end{aligned} \tag{6}$$

We see that equation (5) presents a relation between the responses (DOF and nodal loads) at the left and right boundaries of a period. It contains a term of $\mathbf{F}_I$ which is zero when the period is free. For two consecutive connected periods (n) and $(n+1)$, the right boundary of (n) is also the left boundary of $(n+1)$. Thus, we have

$$\begin{aligned} \mathbf{q}_R^{(n)} &= \mathbf{q}_L^{(n+1)} \\ \mathbf{F}_R^{(n)} + \mathbf{F}_L^{(n+1)} &= -\mathbf{F}_{\partial R}^{(n)} \end{aligned} \tag{7}$$

where $\mathbf{F}_{\partial R}^{(n)}$ is the external nodal load at the right boundary R of the period (n). By combining equations (5) and (7), we obtain

$$\begin{bmatrix} \mathbf{q}_L^{(n+1)} \\ -\mathbf{F}_L^{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{qI}\mathbf{F}_I^{(n)} \\ \mathbf{D}_{fI}\mathbf{F}_I^{(n)} + \mathbf{F}_{\partial R}^{(n)} \end{bmatrix} + \mathbf{S} \begin{bmatrix} \mathbf{q}_L^{(n)} \\ -\mathbf{F}_L^{(n)} \end{bmatrix} \tag{8}$$

where

$$\mathbf{S} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{LR}^{-1} \\ \mathbf{D}_{RL} - \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1} \end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix} \mathbf{D}_{qI} \\ \mathbf{D}_{fI} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} \\ \mathbf{D}_{RI} - \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} \end{bmatrix} \tag{10}$$

We can rewrite equation (8) as follows

$$\mathbf{u}^{(n+1)} = \mathbf{S}\mathbf{u}^{(n)} + \mathbf{b}^{(n)} \tag{11}$$

where

$$\mathbf{u}^{(n)} = \begin{bmatrix} \mathbf{q}_L^{(n)} \\ -\mathbf{F}_L^{(n)} \end{bmatrix}, \quad \mathbf{b}^{(n)} = \begin{bmatrix} \mathbf{D}_{qI}\mathbf{F}_I^{(n)} \\ \mathbf{D}_{fI}\mathbf{F}_I^{(n)} + \mathbf{F}_{\partial R}^{(n)} \end{bmatrix} \tag{12}$$

Equation (11) presents a relation between the periods (n) and $(n+1)$. Here $\mathbf{b}^{(n)}$ presents the external loads at the inner nodes $\mathbf{F}_I$ and at the right boundary $\mathbf{F}_{\partial R}$ of the period (n). When the period is free, $\mathbf{b}^{(n)} = 0$ and we meet the expression of $\mathbf{S}$ of the classical WFE.

In the other hand, equation (11) presents also a recurrent relation with regard to n. Therefore, we can develop a relation between $\mathbf{u}^{(n)}$ ans $\mathbf{u}^{(1)}$ as follows

$$\mathbf{u}^{(n)} = \mathbf{S}^{n-1}\mathbf{u}^{(1)} + \sum_{k=1}^{n-1} \mathbf{S}^{n-k-1}\mathbf{b}^{(k)} \tag{13}$$

Similarly, we can get the relation between $\mathbf{u}^{(N+1)}$ and $\mathbf{u}^{(n)}$

$$\mathbf{u}^{(N+1)} = \mathbf{S}^{N-n+1}\mathbf{u}^{(n)} + \sum_{k=n}^{N} \mathbf{S}^{N-k}\mathbf{b}^{(k)} \tag{14}$$

Equations (13) and (14) are the relations between the period (n) and the left and the right ends of a structure of N periods. Next, we will develop these expressions with the help of the wave decomposition.

Remark: $\mathbf{F}_{\partial R}^{(n)}$ in equation (7) is the external nodal load at the right boundary but it is not included on the left end of the period. This will explain the different expressions for the left and right boundary in the DSM and WA approaches presented in section 3.

3. Wave decomposition

We are looking for wave modes $\{(\mu_j, \phi_j)\}$ which are the eigenvalues and eigenvectors of the matrix $\mathbf{S}$ such that $\mathbf{S}\phi_j = \mu_j\phi_j$. Due to the symplectic nature [11] of the matrix $\mathbf{S}$, its eigenvalues come in pairs as $(\mu_j, 1/\mu_j)$. Then, the eigenvectors are separated to ϕ_i for the eigenvalue less than 1 and $\phi_i^\star$ for the rest. We call ϕ_i the left (to right) wave and $\phi_i^\star$ the right (to left) wave. If we note $\mathbf{\Phi} = [\phi_1 \cdots \phi_n]$ and $\mathbf{\Phi}^\star = [\phi_1^\star \cdots \phi_n^\star]$, we call $\{\mathbf{\Phi}, \mathbf{\Phi}^\star\}$ the wave mode of the transformation $\mathbf{S}$. We can also separate the components of the wave mode corresponding to $\mathbf{q}$ and $\mathbf{F}$ as follows

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_q \\ \mathbf{\Phi}_F \end{bmatrix} \quad \mathbf{\Phi}^\star = \begin{bmatrix} \mathbf{\Phi}_q^\star \\ \mathbf{\Phi}_F^\star \end{bmatrix} \tag{15}$$

It is important to note that the wave mode is symplectic orthogonal basis. We can normalize this basis by considering the weighting matrix given by

$$\begin{aligned} \mathbf{\Psi} &= \mathbf{\Phi}^{\star T}\mathbf{J}\mathbf{\Phi} = \mathbf{\Phi}_q^{\star T}\mathbf{\Phi}_F - \mathbf{\Phi}_F^{\star T}\mathbf{\Phi}_q \\ \mathbf{\Psi}^\star &= \mathbf{\Phi}^{T}\mathbf{J}\mathbf{\Phi}^\star = \mathbf{\Phi}_q^{T}\mathbf{\Phi}_F^\star - \mathbf{\Phi}_F^{T}\mathbf{\Phi}_q^\star \end{aligned} \tag{16}$$

where

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \tag{17}$$

We can verify that $\mathbf{\Psi}^\star = -\mathbf{\Psi}^T$ and they are diagonal matrices. Therefore, we can define a normalized basis by multiplying on the right side with $\mathbf{\Psi}^{-1/2}$ (e.i. the basis is given by $\mathbf{\Phi}\mathbf{\Psi}^{-1/2}$ and $\mathbf{\Phi}^\star\mathbf{\Psi}^{-1/2}$).

Now we can decompose each vector of equation (11) in this normalized basis as follows

$$\begin{aligned} \mathbf{u}^{(n)} &= \mathbf{\Phi}\mathbf{Q}^{(n)} - \mathbf{\Phi}^\star\mathbf{Q}^{\star(n)} \\ \mathbf{b}^{(n)} &= \mathbf{\Phi}\mathbf{Q}_B^{(n)} - \mathbf{\Phi}^\star\mathbf{Q}_B^{\star(n)} \end{aligned} \tag{18}$$

It is remarkable that the sign minus in these decompositions yields a more convenient expression of the wave amplitudes as follows

$$\mathbf{Q}^{(n)} = \mathbf{\Phi}^{\star T}\mathbf{J}\mathbf{u}^{(n)}, \quad \mathbf{Q}^{\star(n)} = \mathbf{\Phi}^{T}\mathbf{J}\mathbf{u}^{(n)} \tag{19}$$

$$\mathbf{Q}_B^{(n)} = \mathbf{\Phi}^{\star T}\mathbf{J}\mathbf{b}^{(n)}, \quad \mathbf{Q}_B^{\star(n)} = \mathbf{\Phi}^{T}\mathbf{J}\mathbf{b}^{(n)} \tag{20}$$

In the other hand, from equation (12) we obtain

$$\begin{aligned}\mathbf{\Phi}^{\star T}\mathbf{J}\mathbf{b}^{(n)} &= \left(\mathbf{\Phi}_q^{\star T}\mathbf{D}_{fI} - \mathbf{\Phi}_F^{\star T}\mathbf{D}_{qI}\right)\mathbf{F}_I^{(n)} + \mathbf{\Phi}_q^{\star T}\mathbf{F}_{\partial R}^{(n)} \\ \mathbf{\Phi}^{T}\mathbf{J}\mathbf{b}^{(n)} &= \left(\mathbf{\Phi}_q^{T}\mathbf{D}_{fI} - \mathbf{\Phi}_F^{T}\mathbf{D}_{qI}\right)\mathbf{F}_I^{(n)} + \mathbf{\Phi}_q^{T}\mathbf{F}_{\partial R}^{(n)}\end{aligned} \tag{21}$$

Thereafter, by substituting $\mathbf{D}_{fI}$ and $\mathbf{D}_{qI}$ in equation (10) into the aforementioned equation and then combining the results with equation (20), we obtain

$$\begin{aligned}\mathbf{Q}_B^{(n)} &= \left[\left(\mathbf{\Phi}_F^{\star T} - \mathbf{\Phi}_q^{\star T}\mathbf{D}_{RR}\right)\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} + \mathbf{\Phi}_q^{\star T}\mathbf{D}_{RI}\right]\mathbf{F}_I^{(n)} + \mathbf{\Phi}_q^{\star T}\mathbf{F}_{\partial R}^{(n)} \\ \mathbf{Q}_B^{\star(n)} &= \left[\left(\mathbf{\Phi}_F^{T} - \mathbf{\Phi}_q^{T}\mathbf{D}_{RR}\right)\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} + \mathbf{\Phi}_q^{T}\mathbf{D}_{RI}\right]\mathbf{F}_I^{(n)} + \mathbf{\Phi}_q^{T}\mathbf{F}_{\partial R}^{(n)}\end{aligned} \tag{22}$$

We have also a relation between the components of the wave basis (see [10])

$$\begin{aligned}\mathbf{\Phi}_F &= \mathbf{D}_{RR}\mathbf{\Phi}_q + \mathbf{D}_{RL}\mathbf{\Phi}_q\boldsymbol{\mu}^\star \\ \mathbf{\Phi}_F^\star &= \mathbf{D}_{RR}\mathbf{\Phi}_q^\star + \mathbf{D}_{RL}\mathbf{\Phi}_q^\star\boldsymbol{\mu}\end{aligned} \tag{23}$$

By substituting the aforementioned equation into equation (22), we obtain

$$\begin{aligned}\mathbf{Q}_B^{(n)} &= \left(\boldsymbol{\mu}\mathbf{\Phi}_q^{\star T}\mathbf{D}_{LI} + \mathbf{\Phi}_q^{\star T}\mathbf{D}_{RI}\right)\mathbf{F}_I^{(n)} + \mathbf{\Phi}_q^{\star T}\mathbf{F}_{\partial R}^{(n)} \\ \mathbf{Q}_B^{\star(n)} &= \left(\boldsymbol{\mu}^\star\mathbf{\Phi}_q^{T}\mathbf{D}_{LI} + \mathbf{\Phi}_q^{T}\mathbf{D}_{RI}\right)\mathbf{F}_I^{(n)} + \mathbf{\Phi}_q^{T}\mathbf{F}_{\partial R}^{(n)}\end{aligned} \tag{24}$$

We see that the term $\mathbf{b}^n$ in equation (11), which links to the external loads on each period can be decomposed in the wave basis with the amplitudes calculated by equation (24). Now we will build the relation of these wave amplitudes with the amplitude of the response $\mathbf{u}^{(n)}$.

By substituting equation (18) into equations (13), we obtain

$$\mathbf{\Phi}\mathbf{Q}^{(n)} - \mathbf{\Phi}^\star\mathbf{Q}^{\star(n)} = \mathbf{S}^{n-1}\left(\mathbf{\Phi}\mathbf{Q}^{(1)} - \mathbf{\Phi}^\star\mathbf{Q}^{\star(1)}\right) + \sum_{k=1}^{n-1}\mathbf{S}^{n-k-1}\left(\mathbf{\Phi}\mathbf{Q}_B^{(n)} - \mathbf{\Phi}^\star\mathbf{Q}_B^{\star(n)}\right) \tag{25}$$

In addition, by definition we have $\mathbf{S}^k\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\mu}^k$ and $\mathbf{S}^k\boldsymbol{\Phi}^\star = \boldsymbol{\Phi}^\star\boldsymbol{\mu}^{\star k}$. Thus, by multiplying the aforementioned equation with $\boldsymbol{\Phi}^{\star T}\mathbf{J}$ on the left side and then combining with equation (19), we obtain

$$\mathbf{Q}^{(n)} = \boldsymbol{\mu}^{n-1}\mathbf{Q} + \sum_{k=1}^{n-1}\boldsymbol{\mu}^{n-k-1}\mathbf{Q}_B^{(k)} \tag{26}$$

where $\mathbf{Q} = \mathbf{Q}^{(1)}$.

In a similar way for equation (14), we obtain the following result

$$\mathbf{Q}^{(N+1)} = \boldsymbol{\mu}^{\star N-n+1}\mathbf{Q}^{\star(n)} + \sum_{k=n}^{N}\boldsymbol{\mu}^{\star N-k}\mathbf{Q}_B^{\star(k)} \tag{27}$$

In addition, $\boldsymbol{\mu}^\star = \boldsymbol{\mu}^{-1}$. Thus, we have

$$\mathbf{Q}^{\star(n)} = \boldsymbol{\mu}^{N+1-n}\mathbf{Q}^\star - \sum_{k=n}^{N}\boldsymbol{\mu}^{k+1-n}\mathbf{Q}_B^{\star(k)} \tag{28}$$

where $\mathbf{Q}^\star = \mathbf{Q}^{\star(N+1)}$.

We see that the two wave amplitudes $\mathbf{Q}^{(n)}$ and $\mathbf{Q}^{\star(n)}$ depend on the wave amplitudes of the first and the last periods ($\mathbf{Q}$ and $\mathbf{Q}^\star$) and the wave amplitudes of the external loads $\mathbf{Q}_B^{(k)}$ and $\mathbf{Q}_B^{\star(k)}$ which are calculated by equation (24). Next, we will use these relations to obtain the responses by using two different approaches.

4. Analysis of a complete structure

4.1. DSM approach

The principle of the DSM approach is to establish the relation between the DOFs and nodal loads of the left end $\mathbf{u}^{(1)}$ and the right ends $\mathbf{u}^{(N+1)}$ by using the wave decomposition. This relation is also yielded from the stiffness matrix of the whole structure with the help of FEM and this is the reason of the name of the approach.

For the first and the last period of the structure, we obtain the following results from equations (26) and (28)

$$
\begin{aligned}
\mathbf{u}^{(N+1)} &= \mathbf{\Phi}\boldsymbol{\mu}^N\mathbf{Q} - \mathbf{\Phi}^\star\mathbf{Q}^\star + \mathbf{\Phi}\sum_{k=1}^{N}\boldsymbol{\mu}^{N-k}\mathbf{Q}_B^{(k)} \\
\mathbf{u}^{(1)} &= \mathbf{\Phi}\mathbf{Q} - \mathbf{\Phi}^\star\boldsymbol{\mu}^N\mathbf{Q}^\star + \mathbf{\Phi}^\star\sum_{k=1}^{N}\boldsymbol{\mu}^{k}\mathbf{Q}_B^{\star(k)}
\end{aligned}
\tag{29}
$$

Then by multiplying the aforementioned equation with $\mathbf{\Phi}^{*T}\mathbf{J}$ for the first line and $\mathbf{\Phi}^T\mathbf{J}$ for the second line, we obtain

$$
\begin{aligned}
\mathbf{\Phi}^{*T}\mathbf{J}\mathbf{u}^{(N+1)} &= \boldsymbol{\mu}^N\mathbf{Q} + \sum_{k=1}^{N}\boldsymbol{\mu}^{N-k}\mathbf{Q}_B^{(k)} \\
\mathbf{\Phi}^{T}\mathbf{J}\mathbf{u}^{(1)} &= \boldsymbol{\mu}^N\mathbf{Q}^\star - \sum_{k=1}^{N}\boldsymbol{\mu}^{k}\mathbf{Q}_B^{\star(k)}
\end{aligned}
\tag{30}
$$

In addition, we have $\mathbf{Q} = \mathbf{\Phi}^{*T}\mathbf{J}\mathbf{u}^{(1)}$ and $\mathbf{Q}^\star = \mathbf{\Phi}^{T}\mathbf{J}\mathbf{u}^{(N+1)}$. Thus, by using the definition of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(N+1)}$ in equation (12), we obtain

$$
\begin{aligned}
\boldsymbol{\mu}^N\mathbf{\Phi}_q^{\star T}\mathbf{F}_L^{(1)} - \mathbf{\Phi}_q^{*T}\mathbf{F}_L^{(N+1)} &= -\boldsymbol{\mu}^N\mathbf{\Phi}_F^{*T}\mathbf{q}_L^{(1)} + \mathbf{\Phi}_F^{*T}\mathbf{q}_L^{(N+1)} + \sum_{k=1}^{N}\boldsymbol{\mu}^{N-k}\mathbf{Q}_B^{(k)} \\
-\mathbf{\Phi}_q^{T}\mathbf{F}_L^{(1)} + \boldsymbol{\mu}^N\mathbf{\Phi}_q^{T}\mathbf{F}_L^{(N+1)} &= \mathbf{\Phi}_F^{T}\mathbf{q}_L^{(1)} - \boldsymbol{\mu}^N\mathbf{\Phi}_F^{T}\mathbf{q}_L^{(N+1)} - \sum_{k=1}^{N}\boldsymbol{\mu}^{k}\mathbf{Q}_B^{\star(k)}
\end{aligned}
\tag{31}
$$

Otherwise, we have $\mathbf{F}_L^{(N+1)} = -\mathbf{F}_R^{(N)}$ and $\mathbf{q}_L^{(N+1)} = \mathbf{q}_R^{(N)}$. Thereforce, we can rewrite the aforementioned equation to obtain the following result (see Appendix A).

$$
\begin{bmatrix} \mathbf{F}_L^{(1)} \\ \mathbf{F}_R^{(N)} \end{bmatrix} = \mathbf{D}_T \begin{bmatrix} \mathbf{q}_L^{(1)} \\ \mathbf{q}_R^{(N)} \end{bmatrix} + \mathbf{F}_T \tag{32}
$$

where $\mathbf{D}_T, \mathbf{F}_T$ are the reduced dynamic stiffness and the external loads, which are calculated by equations (A.4) and (A.5).

Equation (32) is the relation between the nodal loads and DOF at the left and the right ends of the structure. When $\mathbf{F}_T = 0$ (which means $\mathbf{F}_I^{(k)} = \mathbf{F}_B^{(k)} = 0$), we obtain the same relation for the free loaded structure [2]. We can combine this relation with the DSM of the rest of the structure to get the reduced DSM of the whole structure or apply the boundary conditions on the left and the right ends of the structure to calculate the response.

4.2. WA approach

The WA approach seeks to calculate the response via the wave amplitudes from the boundary condition at the right and left ends of the structure. Indeed, by combining equation (18) and (26), we obtain

$$\begin{aligned}
\mathbf{q}_L^{(n)} &= \mathbf{\Phi}_q \boldsymbol{\mu}^{n-1}\mathbf{Q} - \mathbf{\Phi}_q^\star \boldsymbol{\mu}^{N+1-n}\mathbf{Q}^\star + \mathbf{\Phi}_q \sum_{k=1}^{n-1} \boldsymbol{\mu}^{n-k-1}\mathbf{Q}_B^{(k)} + \mathbf{\Phi}_q^\star \sum_{k=n}^{N} \boldsymbol{\mu}^{k+1-n}\mathbf{Q}_B^{\star(k)} \\
-\mathbf{F}_L^{(n)} &= \mathbf{\Phi}_F \boldsymbol{\mu}^{n-1}\mathbf{Q} - \mathbf{\Phi}_F^\star \boldsymbol{\mu}^{N+1-n}\mathbf{Q}^\star + \mathbf{\Phi}_F \sum_{k=1}^{n-1} \boldsymbol{\mu}^{n-k-1}\mathbf{Q}_B^{(k)} + \mathbf{\Phi}_F^\star \sum_{k=n}^{N} \boldsymbol{\mu}^{k+1-n}\mathbf{Q}_B^{\star(k)}
\end{aligned} \tag{33}$$

The aforementioned equation presents the responses in terms of wave amplitudes $\{\mathbf{Q}, \mathbf{Q}^\star\}$. We note that $\mathbf{Q}_B^{(k)}, \mathbf{Q}_B^{\star(k)}$ are calculated from the external loads by equation (23). Therefore, if the boundary conditions are given on $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(N+1)}$, we can compute directly $\{\mathbf{Q}, \mathbf{Q}^\star\}$ from these conditions.

Otherwise, if the periodic structure is linked to the other parts of the structure, we can combine this wave decomposition with the dynamic equation of these parts. From the dynamic stiffness matrix at the left and right ends, we can write (see Appendix B)

$$
\begin{aligned}
-\mathbf{F}_L^{(1)} &= \mathbb{D}\mathbf{q}_L^{(1)} + \mathbb{D}_q\mathbf{q}_0 + \mathbb{D}_F\mathbf{F}_0 + \mathbf{F}_{\partial R}^{(0)} && (34)\\
\mathbf{F}_L^{(N+1)} &= \mathbb{D}^\star\mathbf{q}_L^{(N+1)} + \mathbb{D}_q^\star\mathbf{q}_0^\star + \mathbb{D}_F^\star\mathbf{F}_0^\star && (35)
\end{aligned}
$$

where $\mathbf{F}_{\partial R}^{(0)}$ is the external loads at the left end of the periodic structure. We note that the second line of equation (35) does not have the equivalent term because it is already taken into account at $\mathbf{b}^{(N)}$ defined in equation (12). By substituting equation (33) with $n = 1$ and $n = N+1$ into the aforementioned equations, we obtain

$$
\begin{aligned}
\left(\boldsymbol{\Phi}_F - \mathbb{D}\boldsymbol{\Phi}_q\right)\mathbf{Q} &= \left(\boldsymbol{\Phi}_F^\star - \mathbb{D}\boldsymbol{\Phi}_q^\star\right)\left(\boldsymbol{\mu}^N\mathbf{Q}^\star - \sum_{k=1}^{N}\boldsymbol{\Phi}_q^\star\boldsymbol{\mu}^k\mathbf{Q}_B^{\star(k)}\right) + \mathbb{D}_q\mathbf{q}_0 + \mathbb{D}_F\mathbf{F}_0 + \mathbf{F}_{\partial R}^{(0)}\\
\left(\boldsymbol{\Phi}_F^\star + \mathbb{D}^\star\boldsymbol{\Phi}_q^\star\right)\mathbf{Q}^\star &= \left(\boldsymbol{\Phi}_F + \mathbb{D}^\star\boldsymbol{\Phi}_q\right)\left(\boldsymbol{\mu}^N\mathbf{Q} + \sum_{k=1}^{N}\boldsymbol{\Phi}_q\boldsymbol{\mu}^k\mathbf{Q}_B^{(k)}\right) + \mathbb{D}_q^\star\mathbf{q}_0^\star + \mathbb{D}_F^\star\mathbf{F}_0^\star
\end{aligned}
\tag{36}
$$

Then we can rewrite equation (36) as follows

$$
\begin{aligned}
\mathbf{Q} &= \mathbb{C}\left[\boldsymbol{\mu}^N\mathbf{Q}^\star - \sum_{k=1}^{N}\boldsymbol{\mu}^k\mathbf{Q}_B^{\star(k)}\right] + \mathbb{F},\\
\mathbf{Q}^\star &= \mathbb{C}^\star\left[\boldsymbol{\mu}^N\mathbf{Q} + \sum_{k=1}^{N}\boldsymbol{\mu}^{N-k}\mathbf{Q}_B^{(k)}\right] + \mathbb{F}^\star
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
\mathbb{C} &= -\left[\mathbb{D}\boldsymbol{\Phi}_q - \boldsymbol{\Phi}_F\right]^{-1}\left[\mathbb{D}\boldsymbol{\Phi}_q^\star - \boldsymbol{\Phi}_F^\star\right]\\
\mathbb{F} &= -\left[\mathbb{D}\boldsymbol{\Phi}_q - \boldsymbol{\Phi}_F\right]^{-1}\left[\mathbb{D}_q\mathbf{q}_0 + \mathbb{D}_F\mathbf{F}_0 + \mathbf{F}_{\partial R}^{(0)}\right]\\
\mathbb{C}^\star &= -\left[\mathbb{D}^\star\boldsymbol{\Phi}_q^\star + \boldsymbol{\Phi}_F^\star\right]^{-1}\left[\mathbb{D}^\star\boldsymbol{\Phi}_q + \boldsymbol{\Phi}_F\right]\\
\mathbb{F}^\star &= -\left[\mathbb{D}^\star\boldsymbol{\Phi}_q + \boldsymbol{\Phi}_F\right]^{-1}\left[\mathbb{D}_q^\star\mathbf{q}_0^\star + \mathbb{D}_F^\star\mathbf{F}_0^\star\right]
\end{aligned}
\tag{38}
$$

Finally, we deduce from equation (37) to obtain

$$\begin{bmatrix} \mathbf{I}_n & -\mathbb{C}\boldsymbol{\mu}^N \\ -\mathbb{C}^\star\boldsymbol{\mu}^N & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{Q} \\ \mathbf{Q}^\star \end{bmatrix} = \begin{bmatrix} \mathbb{F} \\ \mathbb{F}^\star \end{bmatrix} + \sum_{k=1}^{N} \begin{bmatrix} \mathbb{C}\boldsymbol{\mu}^k \mathbf{Q}_B^{\star(k)} \\ \mathbb{C}^\star\boldsymbol{\mu}^{N-k}\mathbf{Q}_B^{(k)} \end{bmatrix} \tag{39}$$

Equation (39) permits to calculate the wave amplitudes $\{\mathbf{Q}, \mathbf{Q}^\star\}$ from the boundary conditions and external loads of the whole structure. Then the response is calculated by using the wave decomposition in equation (33).

5. Examples

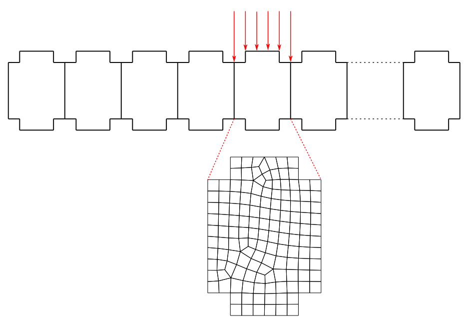

Figure 2: Fixed-free periodic beam subjected to dynamic loads

Let's consider a beam of width 0.1m, thickness varied between 0.1-0.14m and length of 2m as shown in Figure 2. The material parameter is given by the Young's modulus of 30GPa and the mass density of 2200kg/m^3. The beam is fixed at the left boundary and it is subjected to a pressure in an interval between 0.3m and 0.4m . In this structure, the boundary conditions are the following

$$\mathbf{q}^{(1)} = 0, \quad \mathbf{F}^{(N+1)} = 0 \tag{40}$$

We use the DMS approach presented in section 3 to calculate the beam responses. The beam is made of 20 substructures which are squares of dimension $a = 0.1$m and the same thickness. With the external load at the third substructure, we have $\mathbf{Q}_B^{(k)} = \mathbf{Q}_B^{\star(k)} = 0$, $\forall k \neq 4$. By using the finite element method, we obtain the DMS of the substructure with the mesh as shown in Figure 2. Then, by using the WFE, we can combine equations (32) and (40), we obtain

$$\begin{bmatrix} \mathbf{F}_L^{(1)} \\ \mathbf{0} \end{bmatrix} = \mathbf{D}_T \begin{bmatrix} \mathbf{0} \\ \mathbf{q}_R^{(N+1)} \end{bmatrix} + \mathbf{F}_T \tag{41}$$

Thus, we can calculate $\mathbf{q}_R^{(N+1)}$ from the second line of the aforementioned equation and then the force $\mathbf{F}_L^{(1)}$ from the first row.

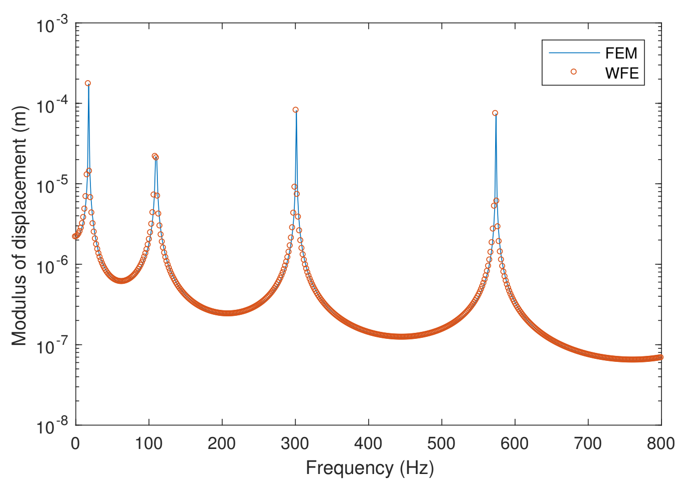

Figure 3: Response of the beam at the upper corner

Figure 3 shows the results obtained by WFE and the classical FEM for the displacement in the frequency domain at the upper right corner of the beam. We see that the WFE has the same quality as the FEM. Moreover, the calculation time for WFE is 14.0s while 120.2s for FEM, equivalent to 88% of time reduction.

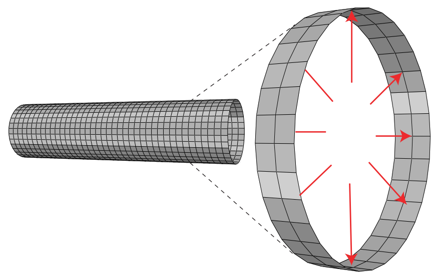

Figure 4: Fixed-fixed pipeline subjected to dynamic pressures

We take another example of a pipeline under pressure as shown in Figure 4. The pipeline is a cylinder of radius 1m and thickness 5mm, made of steel with Young modulus 210GPa and Poisson coefficient of 0.3. The cylinder has two fixed ends and it is subjected to a dynamic pressure. We can use the wave analysis by substituting the boundary condition into equation (38) and obtain

$$\begin{aligned} \mathbf{\Phi}_q \mathbf{Q} - \mathbf{\Phi}_q^\star \boldsymbol{\mu}^N \mathbf{Q}^\star + \mathbf{\Phi}_q^\star \sum_{k=1}^{N} \boldsymbol{\mu}^k \mathbf{Q}_B^{\star(k)} = 0 \\ \mathbf{\Phi}_q \boldsymbol{\mu}^N \mathbf{Q} - \mathbf{\Phi}_q^\star \mathbf{Q}^\star + \mathbf{\Phi}_q \sum_{k=1}^{N} \boldsymbol{\mu}^{N-k} \mathbf{Q}_B^{(k)} = 0 \end{aligned} \tag{42}$$

Then we can calculate $\mathbf{Q}$ and $\mathbf{Q}^\star$ from the aforementioned equation. The response is calculated then by equation (33).

Figure 5 presents a comparison of the results between the finite element method and the wave finite element method. The calculation time is reduced from 1277,7s with FEM to 332,8s with WFE, that means a reduction of 76% of calculation time.

6. Conclusions

This article presents a new technique of WFE for periodic structures

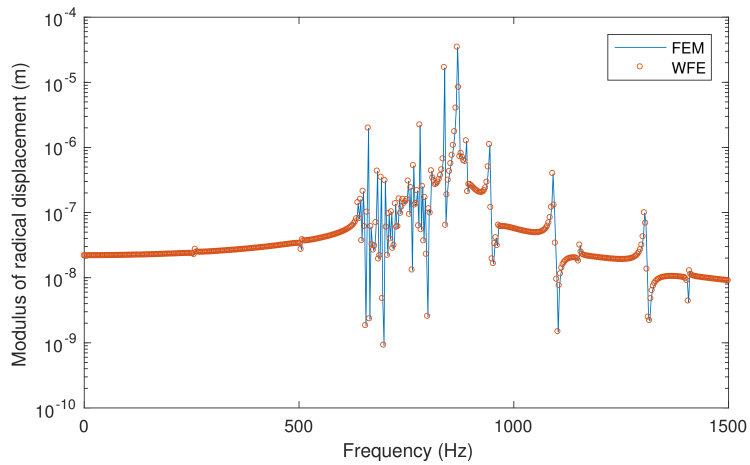

Figure 5: Response of the pipeline

subjected to external loads. By using the recurrent relation between the left and right boundaries of one period, we obtain a general formulation of the response in terms of loads. For the DMS approach, the external loads is represented by a global force in the reduced dynamic equation. For the WA approach, the loads on each period create the waves to the left and right sides of the structures. Moreover, the numerical application proves the efficiency of WFE in term of calculation time in comparing with classical FEM.

Appendix A. Calculation of DSM approach

We can rewrite equation (37) in the matrix form as follows

$$\begin{aligned}\begin{bmatrix} \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^{\star T} & \boldsymbol{\Phi}_q^{\star T} \\ \boldsymbol{\Phi}_q^T & \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^T \end{bmatrix} \begin{bmatrix} \mathbf{F}_L^{(1)} \\ \mathbf{F}_R^{(N)} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\mu}^N \boldsymbol{\Phi}_F^{\star T} & \boldsymbol{\Phi}_F^{\star T} \\ \boldsymbol{\Phi}_F^T & -\boldsymbol{\mu}^N \boldsymbol{\Phi}_F^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_L^{(1)} \\ \mathbf{q}_R^{(N)} \end{bmatrix} \\ + \sum_{k=1}^{N} \begin{bmatrix} \boldsymbol{\mu}^{N-k} \mathbf{Q}_B^{(k)} \\ \boldsymbol{\mu}^k \mathbf{Q}_B^{\star(k)} \end{bmatrix}\end{aligned} \tag{A.1}$$

Moreover, by substituting equation (24) into the aforementioned equation, we obtain

$$\sum_{k=1}^{N}\begin{bmatrix}\boldsymbol{\mu}^{N-k}\mathbf{Q}_B^{(k)}\\ \boldsymbol{\mu}^{k}\mathbf{Q}_B^{\star(k)}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\mu}^{N-k+1}\boldsymbol{\Phi}_q^{\star T} & \boldsymbol{\mu}^{N-k}\boldsymbol{\Phi}_q^{\star T}\\ \boldsymbol{\mu}^{k-1}\boldsymbol{\Phi}_q^{T} & \boldsymbol{\mu}^{k}\boldsymbol{\Phi}_q^{T}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{LI} & \mathbf{0}\\ \mathbf{D}_{RI} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{F}_I^{(k)}\\ \mathbf{F}_{\partial R}^{(k)}\end{bmatrix} \tag{A.2}$$

In the other hand, by using equation (23) we obtain the following result (see [2])

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\mu}^{N}\boldsymbol{\Phi}_F^{\star T} & \boldsymbol{\Phi}_F^{\star T}\\ \boldsymbol{\Phi}_F^{T} & \boldsymbol{\mu}^{N}\boldsymbol{\Phi}_F^{T}\end{bmatrix}&=\begin{bmatrix}\boldsymbol{\mu}^{N}\boldsymbol{\Phi}_q^{\star T} & \boldsymbol{\Phi}_q^{\star T}\\ \boldsymbol{\Phi}_q^{T} & \boldsymbol{\mu}^{N}\boldsymbol{\Phi}_q^{T}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{LL} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}_{RR}\end{bmatrix}\\ &\quad+\begin{bmatrix}\boldsymbol{\mu}^{N-1}\boldsymbol{\Phi}_q^{\star T} & \boldsymbol{\mu}\boldsymbol{\Phi}_q^{\star T}\\ \boldsymbol{\mu}\boldsymbol{\Phi}_q^{T} & \boldsymbol{\mu}^{N-1}\boldsymbol{\Phi}_q^{T}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{RL} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}_{LR}\end{bmatrix}\end{aligned} \tag{A.3}$$

By substituting equations (A.2), (A.3) into equation (A.1), we obtain the result in equation (32) with the global stiffness matrix $\mathbf{D}_T$ and global external force $\mathbf{F}_T$ calculated by

$$\begin{aligned}\mathbf{D}_T &= \begin{bmatrix}\mathbf{D}_{LL} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}_{RR}\end{bmatrix}+\begin{bmatrix}\boldsymbol{\Phi}_q^{\star -T}\boldsymbol{\mu}^{N}\boldsymbol{\Phi}_q^{\star T} & \mathbf{I}\\ \mathbf{I} & \boldsymbol{\Phi}_q^{-T}\boldsymbol{\mu}^{N}\boldsymbol{\Phi}_q^{T}\end{bmatrix}^{-1}\times\\ &\quad\begin{bmatrix}\boldsymbol{\Phi}_q^{\star -T}\boldsymbol{\mu}^{N-1}\boldsymbol{\Phi}_q^{\star T} & \boldsymbol{\Phi}_q^{\star -T}\boldsymbol{\mu}\boldsymbol{\Phi}_q^{\star T}\\ \boldsymbol{\Phi}_q^{-T}\boldsymbol{\mu}\boldsymbol{\Phi}_q^{T} & \boldsymbol{\Phi}_q^{-T}\boldsymbol{\mu}^{N-1}\boldsymbol{\Phi}_q^{T}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{RL} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}_{LR}\end{bmatrix}\end{aligned} \tag{A.4}$$

$$\begin{aligned}\mathbf{F}_T &= \begin{bmatrix}\boldsymbol{\Phi}_q^{\star -T}\boldsymbol{\mu}^{N}\boldsymbol{\Phi}_q^{\star T} & \mathbf{I}\\ \mathbf{I} & \boldsymbol{\Phi}_q^{-T}\boldsymbol{\mu}^{N}\boldsymbol{\Phi}_q^{T}\end{bmatrix}^{-1}\times\\ &\quad\sum_{k=1}^{N}\begin{bmatrix}\boldsymbol{\Phi}_q^{\star -T}\boldsymbol{\mu}^{N-k-1}\boldsymbol{\Phi}_q^{\star T} & \boldsymbol{\Phi}_q^{\star -T}\boldsymbol{\mu}^{N-k}\boldsymbol{\Phi}_q^{\star T}\\ \boldsymbol{\Phi}_q^{-T}\boldsymbol{\mu}^{k+1}\boldsymbol{\Phi}_q^{T} & \boldsymbol{\Phi}_q^{-T}\boldsymbol{\mu}^{k}\boldsymbol{\Phi}_q^{T}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{LI} & \mathbf{0}\\ \mathbf{D}_{RI} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{F}_I^{(k)}\\ \mathbf{F}_{\partial R}^{(k)}\end{bmatrix}\end{aligned} \tag{A.5}$$

with $[\cdot]^{-T}$ denotes for the inverse of the transpose matrix.

Appendix B. Calculation of WA approach

For the left end of the structure, we can write the dynamic equation as follows

$$\begin{bmatrix} (\mathbf{D}_1)_{LL} & (\mathbf{D}_1)_{LI} & (\mathbf{D}_1)_{LB} \\ (\mathbf{D}_1)_{IL} & (\mathbf{D}_1)_{II} & (\mathbf{D}_1)_{IB} \\ (\mathbf{D}_1)_{BL} & (\mathbf{D}_1)_{BI} & (\mathbf{D}_1)_{BB} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L^{(1)} \\ \mathbf{q}_{1I} \\ \mathbf{q}_0 \end{bmatrix} = \begin{bmatrix} -\mathbf{F}_{\partial R}^{(0)} - \mathbf{F}_L^{(1)} \\ \mathbf{F}_0 \\ \mathbf{F}_{1B} \end{bmatrix} \tag{B.1}$$

$$\begin{bmatrix} (\mathbf{D}_2)_{RR} & (\mathbf{D}_2)_{RI} & (\mathbf{D}_2)_{RB} \\ (\mathbf{D}_2)_{IR} & (\mathbf{D}_2)_{II} & (\mathbf{D}_2)_{IB} \\ (\mathbf{D}_2)_{BR} & (\mathbf{D}_2)_{BI} & (\mathbf{D}_2)_{BB} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L^{(N+1)} \\ \mathbf{q}_{2I} \\ \mathbf{q}_0^\star \end{bmatrix} = \begin{bmatrix} \mathbf{F}_L^{(N+1)} \\ \mathbf{F}_0^\star \\ \mathbf{F}_{2B} \end{bmatrix} \tag{B.2}$$

From the first line of the aforementioned equations, we can write

$$-\mathbf{F}_{\partial R}^{(0)} - \mathbf{F}_L^{(1)} = (\mathbf{D}_1)_{LL}\mathbf{q}_L^{(1)} + (\mathbf{D}_1)_{LI}\mathbf{q}_{1I} + (\mathbf{D}_1)_{LB}\mathbf{q}_0 \tag{B.3}$$

$$\mathbf{F}_L^{(N+1)} = (\mathbf{D}_2)_{RR}\mathbf{q}_L^{(N+1)} + (\mathbf{D}_2)_{RI}\mathbf{q}_{2I} + (\mathbf{D}_2)_{LB}\mathbf{q}_0^\star \tag{B.4}$$

In addition, from the second line of equations (B.1) and (B.2), we have

$$\mathbf{q}_{1I} = (\mathbf{D}_1)_{LI}^{-1}\left[\mathbf{F}_0 - (\mathbf{D}_1)_{IL}\mathbf{q}_L^{(1)} - (\mathbf{D}_1)_{IB}\mathbf{q}_0\right] \tag{B.5}$$

$$\mathbf{q}_{2I} = (\mathbf{D}_2)_{RI}^{-1}\left[\mathbf{F}_0^\star - (\mathbf{D}_2)_{IR}\mathbf{q}_L^{(N+1)} - (\mathbf{D}_2)_{IB}\mathbf{q}_0^\star\right] \tag{B.6}$$

Then, by combining equations (B.5) and (B.6), we obtain

$$-\mathbf{F}_L^{(1)} = \mathbb{D}\mathbf{q}_L^{(1)} + \mathbb{D}_q\mathbf{q}_0 + \mathbb{D}_F\mathbf{F}_0 + \mathbf{F}_{\partial R}^{(0)} \tag{B.7}$$

$$\mathbf{F}_L^{(N+1)} = \mathbb{D}^\star\mathbf{q}_R^{(N+1)} + \mathbb{D}_q^\star\mathbf{q}_0^\star + \mathbb{D}_F^\star\mathbf{F}_0^\star \tag{B.8}$$

where

$$\mathbb{D} = (\mathbf{D}_1)_{LL} - (\mathbf{D}_1)_{LI}(\mathbf{D}_1)_{II}^{-1}(\mathbf{D}_1)_{IL} \tag{B.9}$$

$$\mathbb{D}_q = (\mathbf{D}_1)_{LB} - (\mathbf{D}_1)_{LI}(\mathbf{D}_1)_{II}^{-1}(\mathbf{D}_1)_{IL} \tag{B.10}$$

$$\mathbb{D}_F = (\mathbf{D}_1)_{LI}(\mathbf{D}_1)_{II}^{-1} \tag{B.11}$$

and

$$\mathbb{D}^\star = (\mathbf{D}_2)_{RR} - (\mathbf{D}_2)_{RI}(\mathbf{D}_2)_{II}^{-1}(\mathbf{D}_2)_{IR} \tag{B.12}$$

$$\mathbb{D}_q^\star = (\mathbf{D}_2)_{RB} - (\mathbf{D}_2)_{RI}(\mathbf{D}_2)_{II}^{-1}(\mathbf{D}_1)_{IR} \tag{B.13}$$

$$\mathbb{D}_F^\star = (\mathbf{D}_2)_{RI}(\mathbf{D}_2)_{II}^{-1} \tag{B.14}$$

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