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PASCAL AUSCHER AND MORITZ EGERT

Abstract. For elliptic systems with block structure in the upper half-space and t-independent coefficients, we settle the study of boundary value problems by proving compatible well-posedness of Dirichlet, regularity and Neumann problems in optimal ranges of exponents. Prior to this work, only the two-dimensional situation was fully understood. In higher dimensions, partial results for existence in smaller ranges of exponents and for a subclass of such systems had been established. The presented uniqueness results are completely new. We also elucidate optimal ranges for problems with fractional regularity data.

The first part of the monograph, which can be read independently, provides optimal ranges of exponents for functional calculus and adapted Hardy spaces for the associated boundary operator.

Methods use and improve, with new results, all the machinery developed over the last two decades to study such problems: the Kato square root estimates and Riesz transforms, Hardy spaces associated to operators, off-diagonal estimates, non-tangential estimates and square functions and abstract layer potentials to replace fundamental solutions in the absence of local regularity of solutions.

This mostly self-contained monograph provides a comprehensive overview on the field and unifies many earlier results that have been obtained by a variety of methods.

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Contents

1. Introduction and main results

1.1. **Objective of the monograph.** Consider the elliptic system of m equations in $(1+n)$ dimensions, $n \geq 1$, given by

$$\sum_{i,j=0}^{n} \sum_{\beta=1}^{m} \partial_i\big(A_{i,j}^{\alpha,\beta}(x)\partial_j u^\beta(t,x)\big) = 0 \quad (\alpha = 1,\ldots,m,\ t>0,\ x \in \mathbb{R}^n),$$

where $\partial_0 \coloneqq \frac{\partial}{\partial t}$ and $\partial_i \coloneqq \frac{\partial}{\partial x_i}$ if $i = 1,\ldots,n$. Note that the coefficients do not depend on the normal variable $t > 0$. Ellipticity will be described below, but when $m = 1$, the uniformly elliptic equations are included.

Boundary value problems for such systems have been extensively studied since the pioneering work of Dahlberg [38] in the late 1970s. The upper half-space situation is prototypical for Lipschitz graph domains. The case of t-independent coefficients is already challenging and meaningful since t-dependent coefficients are usually treated via perturbation techniques.[1] As usual in the harmonic analysis treatment of elliptic boundary value problems, solutions are taken in the weak sense, interior estimates involve non-tangential maximal functions and/or conical square functions and convergence at the boundary is to be understood in an appropriate non-tangential sense.

In this monograph, we consider the class of systems in *block form*, that is, when there are no mixed $\frac{\partial}{\partial t}\frac{\partial}{\partial x_i}$-derivatives. In short notation, the system can be written as

$$\partial_t(a\partial_t u) + \operatorname{div}_x(d\nabla_x u) = 0 \tag{1.1}$$

where the matrix $A = (A_{i,j}^{\alpha,\beta}(x))$ above is block diagonal with diagonal (matrix) entries $a = a(x)$ and $d = d(x)$, hence the name. These systems enjoy the additional feature that one can always produce strong solutions using the Poisson semigroup $\mathrm{e}^{-tL^{1/2}}$ associated with the sectorial operator $L \coloneqq -a^{-1} \operatorname{div}_x d\nabla_x$ on the boundary.[2] Existence and uniqueness of solutions to the boundary value problems are therefore inseparably tied to operator theoretic properties of L.

Our goal is to identify all spaces of boundary data of Hardy, Lebesgue and homogeneous Hölder-type, for which the Dirichlet and Neumann boundary value problems have weak solutions, and then prove uniqueness in these cases. Thus, we aim at proving well-posedness results for the largest possible class of boundary spaces.

To this end, we unify and improve, with several new results along the way, all the machinery developed over the last two decades to study such problems: the Kato square root estimates and Riesz transforms, Hardy spaces associated to operators, off-diagonal estimates,

[1] The reader can refer to Kenig's excellent survey [71] for background on these topics. They lie beyond the scope of our monograph.

[2] We identify the boundary of the upper half-space with $\mathbb{R}^n$.

non-tangential estimates and square functions, abstract layer potentials replacing fundamental solutions in the absence of local regularity of solutions, ...

Prior to this work, only the two-dimensional situation was fully understood for the boundary value problems. In higher dimensions, partial results for existence in smaller ranges of exponents and for a subclass of such systems had been established. The uniqueness results are completely new. We essentially close this topic by obtaining well-posedness in ranges of boundary spaces likely to be sharp in all dimensions.

For Dirichlet-type problems these ranges go beyond the semigroup theory for $\mathrm{e}^{-tL^{1/2}}$ on Lebesgue or Sobolev spaces. The global picture is that for the regularity problem, one can go one Sobolev exponent down from the semigroup range and for the Dirichlet problem, one can go one Sobolev exponent up. In particular, we exhibit for the first time the possibility of solving Dirichlet problems for Hölder and BMO-data without relying on any sort of duality with an adjoint problem with data in a Hardy space. For the Neumann problem, we shall provide a missing link to the existing literature, so that well-posedness in the optimal range of boundary spaces follows from earlier results. This range is the one provided by the semigroup theory.

Natural extensions of the results above are the Dirichlet and Neumann problems for data with fractional regularity between 0 and 1, for which we also provide well-posedness results. This concerns data in Besov and even Hardy–Sobolev spaces. We believe they are optimal in the formulation of the problem as well as in the ranges of spaces.

Most recent results in the field rely on one of two opposing strategies, sometimes referred to as *second*- and *first*-order approaches. None of these two approaches can be used 'off-the-shelf' in order to cover the full range of results that we are aiming at here. Indeed, in the former, the Poisson semigroup $\mathrm{e}^{-tL^{1/2}}$ is usually treated by comparison with the heat semigroup e^{-t^2L}, which offers better decay properties.[3] When $a \neq 1$, it may happen that L is sectorial of angle larger than $\pi/2$, and hence $-L$ does not generate a heat semigroup. This forces us to rely on resolvents $(1+t^2L)^{-1}$ instead, which offer sufficient off-diagonal decay but introduce new and partly unsuspected technicalities. In the first-order approach, the elliptic equation is rewritten as an equivalent first-order system of Cauchy–Riemann-type for the variable $F = [a\partial_t u, \nabla_x u]^\top$ called the *conormal gradient*.[4] The approach is genuinely built on the use of resolvents of a first order operator, but the range of admissible data spaces is limited since it treats the interior estimates for Dirichlet and Neumann problems simultaneously.

[3] References for these techniques are [13, 32, 33, 76].

[4] In this context the idea is pioneered in [7, 9].

Most of our arguments are carried out at the second-order level, but whenever convenient, we employ first-order methods to give more efficient proofs and novel results, even when $a = 1$. Readers, who are not familiar with the first-order approach, may find in this monograph a light introduction to some important features of the theory, while keeping technicalities at the absolute minimum. We also characterize all ranges of boundary spaces that have previously been obtained through first-order methods, using only the second-order operator L. We believe that this helps in rendering accessible the cornerstones of the first-order method to the broader audience that they deserve. At the same time, the block structure will reveal interesting new phenomena that could not be captured by the first-order method.

1.2. **The elliptic equation.** Consider again the elliptic equation (1.1). The value of m (the number of equations) is irrelevant to everything that follows and the reader may assume $m = 1$ when it comes to differential operators such as gradient and divergence.[5] We write (1.1) as

$$\mathcal{L}u := -\operatorname{div}(A\nabla u) = -\partial_t(a\partial_t u) - \operatorname{div}_x d\nabla_x u = 0,$$

where

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : \mathbb{R}^n \to \mathcal{L}(\mathbb{C}^m \times \mathbb{C}^{mn})$$

is the coefficient matrix of dimension $m(1+n)$ in block form. The equation is understood in the weak sense: By $\mathcal{L}u = 0$ we mean that $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^m)$ satisfies

$$\iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi} \, \mathrm{d}t\mathrm{d}x = 0 \quad (\phi \in \mathrm{C}^\infty_0(\mathbb{R}^{1+n}_+; \mathbb{C}^m)).$$

We assume that A is measurable and that there is a constant $\lambda \in (0, \infty)$ called *ellipticity constant*, such that the following hold. First, A is bounded from above:

$$\|A\|_\infty \leq \lambda^{-1}.$$

Second, A is bounded from below on the subspace $\mathcal{H}$ of vector fields $f = [f_0, \ldots f_n]^\top$ in $\mathrm{L}^2(\mathbb{R}^n; (\mathbb{C}^m)^{1+n})$ that satisfy the curl-free condition $\partial_j f_k = \partial_k f_j$ whenever $1 \leq j, k \leq n$:

$$\operatorname{Re}\langle Af, f\rangle \geq \lambda \|f\|_2^2 \quad (f \in \mathcal{H}), \tag{1.2}$$

where the angular brackets denote the inner product on L^2. Due to the block form, this lower bound can be written equivalently as two

[5] Notation in the case $m > 1$ looks exactly the same and is explained in Section 1.9.

separate conditions[6] : *Strict ellipticity*[7] of a,

$$\operatorname{Re}\langle a(x)\xi, \xi\rangle \geq \lambda|\xi|^2 \quad (x \in \mathbb{R}^n,\ \xi \in \mathbb{C}^m), \tag{1.3}$$

so that a is also invertible in $\mathrm{L}^\infty(\mathbb{R}^n; \mathbb{C}^m)$, and the *Gårding inequality* for d,

$$\operatorname{Re}\langle d\nabla_x v, \nabla_x v\rangle \geq \lambda\|\nabla_x v\|_2^2 \quad (v \in \mathrm{C}_0^\infty(\mathbb{R}^n; \mathbb{C}^m)), \tag{1.4}$$

which in general is weaker than strict ellipticity.

1.3. **The critical numbers.**

We use Hardy and homogeneous Hardy–Sobolev spaces H^p and $\dot{\mathrm{H}}^{1,p}$ in the range $p \in (1_*, \infty)$ with the convention that for $p \in (1, \infty)$ they coincide with Lebesgue and homogeneous Sobolev spaces L^p and $\dot{\mathrm{W}}^{1,p}$, respectively. We denote by p_* and p^* the lower and upper Sobolev conjugates of p. In particular, $1_* := n/(n+1)$.

We keep on denoting by

$$L = -a^{-1} \operatorname{div}_x d\nabla_x$$

the boundary operator associated with (1.1), defined as a sectorial operator in L^2 with maximal domain in $\mathrm{W}^{1,2}$.

The applications to boundary value problems require understanding the functional properties of the Poisson semigroup $(\mathrm{e}^{-tL^{1/2}})_{t>0}$, which comes as the natural solution operator, on Hardy and Hardy–Sobolev spaces. The existence of the Poisson semigroup operator $\mathrm{e}^{-tL^{1/2}}$ is granted from the functional calculus for L on L^2.[8]

Two intervals will rule our entire theory:

- $(p_-(L), p_+(L))$ is the maximal open set within $(1_*, \infty)$ for which the family $(a(1 + t^2 L)^{-1} a^{-1})_{t>0}$ is uniformly bounded on H^p.
- $(q_-(L), q_+(L))$ is the maximal open set within $(1_*, \infty)$ for which $(t\nabla_x(1 + t^2 L)^{-1} a^{-1})_{t>0}$ is uniformly bounded on H^p.

The endpoints $p_\pm(L), q_\pm(L)$ are called *critical numbers* associated with L.[9] They have various characterizations proved throughout the monograph. For example, replacing $(1 + t^2 L)^{-1}$ by $\mathrm{e}^{-tL^{1/2}}$ leads to the same intervals, which shows that the critical numbers capture sharp uniform boundedness properties of the Poisson semigroup for L in Hardy spaces.[10] We give a systematic study of these numbers, their inner

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[6]This follows since in the definition of $\mathcal{H}$ the first component f_0 is arbitrary and the curl-free condition is equivalent to $[f_1, \ldots, f_n]^\top = \nabla_x h$ for some distribution h, see [85, p. 59]. Then use that C_0^∞ is dense in the homogeneous Sobolev space $\dot{\mathrm{W}}^{1,2}$, see [87, Thm. 1].

[7]The term *strict accretivity* is also common.

[8]This is a classical construction. We give the necessary background in Section 3.

[9]The idea to use critical numbers for the sake of a flexible theory that applies to any given operator originates in [6]. Therein, they have been defined for $a = 1$ through L^p-boundedness of the heat semigroup. We shall prove that when $a = 1$ our intervals coincide with the ones of [6] in the range $(1, \infty)$, see Section 12.

[10]This is proved in Section 12.

relationship and their values depending on the dimension for the class of all L. In particular, we shall show that they are independent of a.[11] Of course, that does not mean that we can assume $a = 1$ in general.

For now, all one needs to know is that the best conclusion for the critical numbers for the class of all L is

$$(p_-(L), p_+(L)) \supseteq \begin{cases} (\frac{1}{2}, \infty) & \text{if } n = 1 \\ [1, \infty) & \text{if } n = 2 \\ [\frac{2n}{n+2}, \frac{2n}{n-2}] & \text{if } n \geq 3 \end{cases}$$

and

$$(q_-(L), q_+(L)) \supseteq \begin{cases} (\frac{1}{2}, \infty) & \text{if } n = 1 \\ [\frac{2n}{n+2}, 2] & \text{if } n \geq 2 \end{cases}$$

and that in general $p_-(L) = q_-(L)$ and $p_+(L) \geq (q_+(L))^*$. Including systematically exponents $p \in (1_*, 1]$ is a novelty of our approach for both the functional properties of L for its own sake[12] and the applications to boundary value problems.

1.4. **Square root problem and Hardy spaces.** One may wonder how we determine the spaces of data for the boundary value problems. Typically, they should include Lebesgue spaces, and Sobolev spaces in the range $p > 1$ and also Hardy and Hardy–Sobolev spaces in the range $p \leq 1$, as well as their intermediate fractional spaces. Indeed, it is natural from the point of view of regularity theory to incorporate the possibility of having estimates for $p \leq 1$, as is the case for instance for equations with real coefficients. The limitation to $p > 1_*$ can be understood from Sobolev embeddings and duality: The best one can hope for in absence of smoothness of the coefficients is regularity theory in Hölder spaces of exponents less than 1.

The whole theory is built from the case $p = 2$. For the regularity problem[13], it was Kenig[14] who observed that the required interior estimates are linked to the Kato conjecture for L, that is, the homogeneous estimate

$$\|aL^{1/2}f\|_2 \simeq \|\nabla_x f\|_2,$$

which identifies the domain of $L^{1/2}$ as the Sobolev space $\mathrm{W}^{1,2}$ since a is invertible in L^∞. This conjecture is now solved.[15]

[11] This is proved in Section 6.

[12] Section 10 is about consequences for the functional calculus and Section 14 provides a connection to kernel estimates.

[13] More precisely, the problem $(R)_2^{\mathcal{L}}$ defined in Section 1.5.

[14] See [71, Rem. 2.5.6].

[15] In the case $a = 1$, these are the results in [34] when $n = 1$, [60] when $n = 2$ and [12] in all dimensions. When $a \neq 1$, this is proved in [70] when $n = 1$ and then [25] in all dimensions.

The H^p-theory for the square root of L consists in comparing $aL^{1/2}$ and ∇_x in H^p. One estimate is the H^p-boundedness of the Riesz transform $\nabla_x L^{-1/2} a^{-1}$, namely $\|\nabla_x f\|_{\mathrm{H}^p} \lesssim \|aL^{1/2} f\|_{\mathrm{H}^p}$, and then there is the reverse estimate. Of course, the left multiplication with the strictly elliptic function a can be omitted when $p > 1$. The conclusion is[16]

$$\|\nabla_x f\|_{\mathrm{H}^p} \lesssim \|aL^{1/2} f\|_{\mathrm{H}^p} \quad \text{if and only if } q_-(L) < p < q_+(L)$$

for the Riesz transform and that the reverse estimate holds in a larger range, namely

$$\|aL^{1/2} f\|_{\mathrm{H}^p} \lesssim \|\nabla_x f\|_{\mathrm{H}^p} \quad \text{if } (q_-(L)_* \vee 1_*) < p < p_+(L).$$

What allows us to push the discussion to the range of exponents $1_* < p \leq 1$ is the systematic use of Hardy and Hardy–Sobolev spaces $\mathbb{H}^p_L$ and $\mathbb{H}^{1,p}_L$ associated with L that are defined using square functions involving the functional calculus of L.

This foreshadows the main operator theoretic result of the monograph. Indeed, our approach to obtaining square function bounds and non-tangential maximal function bounds as in Theorem 1.1 and Theorem 1.2 below is to determine the ranges of exponents for which abstract Hardy and Hardy–Sobolev spaces associated to L coincide with concrete spaces.[17] The upshot is that up to equivalent p-quasinorms, we are able to show

$$\mathbb{H}^p_L = \mathrm{H}^p \cap \mathrm{L}^2 \quad \text{if and only if } p_-(L) < p < p_+(L) \tag{1.5}$$

and

$$\mathbb{H}^{1,p}_L = \dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2 \quad \text{if } (q_-(L)_* \vee 1_*) < p < q_+(L), \tag{1.6}$$

where identification fails at the upper endpoint.[18] Even for the functional calculus *per se* these identifications yield interesting new results.[19] We now come to the boundary value problems.

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[16]This is proved in Section 11. In the Lebesgue range $(1, \infty)$ it was first done in [23] when $n = 1$ and $1 < p < \infty$, and reproved in [16]. For all dimensions, when $a = 1$, the optimal range of p within $(1, \infty)$ was settled in [6] after earlier works of [30, 56]. For discussions in the Hardy range $p \leq 1$ when $a = 1$, see [58]. Smaller intervals within the Lebesgue and Hardy range when $a \neq 1$ have been obtained in [22, 48, 63].

[17]This approach is of course not new and the very reason why these spaces have been introduced. The latest development and exposition can be found in [3]. Elaborations on Hardy–Sobolev spaces associated to L were previously considered in [58] when $a = 1$ and then in [22, 48] for general Dirac operators.

[18]This is proved in Section 9, except for the openness of $\mathcal{H}(L)$ and $\mathcal{H}^1(L)$ at the upper endpoint, which are obtained in Section 11 as a consequence of the results for the Riesz transform. When $a = 1$ and $m = 1$, results are obtained in [58] with a different definition for the Hardy–Sobolev space and $p \leq 2$, and limitations to $p > 1$ for the identification for the Hardy space.

[19]See Section 10.

1.5. **Main results on Dirichlet problems.** Since for general systems the solutions might not be regular, we use the Whitney average variants of the non-tangential maximal function in order to pose our boundary value problems. Also we formulate the approach to the boundary in a non-tangential fashion using Whitney averages. When we get back to systems where solutions have pointwise values, these variants turn out to be equivalent to the usual non-tangential pointwise control and limits. More precisely, we let

$$\widetilde{N}_*(F)(x) := \sup_{t>0} \bigg(⨌_{W(t,x)} |F(s,y)|^2 \, \mathrm{d}s\mathrm{d}y \bigg)^{1/2} \quad (x \in \mathbb{R}^n).$$

with $W(t,x) := (t/2, 2t) \times B(x,t)$.

For $1 < p < \infty$, the L^p *Dirichlet problem* with non-tangential maximal control and data $f \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{C}^m)$ consists in solving

$$(D)_p^{\mathcal{L}} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \widetilde{N}_*(u) \in \mathrm{L}^p(\mathbb{R}^n), \\ \lim_{t\to 0} ⨌_{W(t,x)} |u(s,y) - f(x)| \, \mathrm{d}s\mathrm{d}y = 0 & (\text{a.e. } x \in \mathbb{R}^n). \end{cases}$$

For the endpoint problem $(D)_1^{\mathcal{L}}$ the natural data class turns out to be a subspace of L^1, namely the image of H^1 under multiplication with the bounded function a^{-1}.

As usual, well-posedness means existence, uniqueness and continuous dependence on the data. Compatible well-posedness means well-posedness together with the fact that the solution agrees with the energy solution that can be constructed via the Lax–Milgram lemma if the data f also belongs to the boundary space $\dot{\mathrm{H}}^{1/2,2}(\mathbb{R}^n;\mathbb{C}^m)$ for energy solutions.

Let us formulate our principal result on the Dirichlet problem, where we denote by S the standard conical square function

$$S(F)(x) := \bigg(\iint_{|x-y|<s} |F(s,y)|^2 \, \frac{\mathrm{d}s\mathrm{d}y}{s^{1+n}} \bigg)^{\frac{1}{2}} \quad (x \in \mathbb{R}^n).$$

Theorem 1.1 (Dirichlet problem)**.** *Let $p \geq 1$ be such that $p_-(L) < p < p_+(L)^*$. Given $f \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{C}^m)$ when $p > 1$ and $f \in a^{-1}\mathrm{H}^1(\mathbb{R}^n;\mathbb{C}^m)$ when $p = 1$, the Dirichlet problem $(D)_p^{\mathcal{L}}$ is compatibly well-posed. The solution has the following additional properties.*

(i) *There is comparability*

$$\|\widetilde{N}_*(u)\|_p \simeq \|af\|_{\mathrm{H}^p} \simeq \|S(t\nabla u)\|_p.$$

(ii) *The non-tangential convergence improves to L^2-averages*

$$\lim_{t\to 0} ⨌_{W(t,x)} |u(s,y) - f(x)|^2 \, \mathrm{d}s\mathrm{d}y = 0 \quad (a.e.\ x \in \mathbb{R}^n).$$

(iii) *When $p < p_+(L)$, then au is of class*[20] $\mathrm{C}_0([0,\infty); \mathrm{H}^p(\mathbb{R}^n;\mathbb{C}^m)) \cap \mathrm{C}^\infty((0,\infty); \mathrm{H}^p(\mathbb{R}^n;\mathbb{C}^m))$ *with $u(0,\cdot) = f$ and*

$$\sup_{t>0} \|au(t,\cdot)\|_{\mathrm{H}^p} \simeq \|af\|_{\mathrm{H}^p}.$$

(iv) *When $p \geq p_+(L)$, then for all $T > 0$ and compact $K \subseteq \mathbb{R}^n$, u is of class $\mathrm{C}([0,T]; \mathrm{L}^2(K;\mathbb{C}^m))$ with $u(0,\cdot) = f$ and there is a constant $c = c(T,K)$ such that*

$$\sup_{0<t\leq T} \|u(t,\cdot)\|_{\mathrm{L}^2(K)} \lesssim c\|f\|_p.$$

As expected, the solution above is given by $u(t,x) = \mathrm{e}^{-tL^{1/2}} f(x)$ if in addition we have $f \in \mathrm{L}^2$ and by an extension by density of this expression for the respective topologies for general f. In the range $p < p_+(L)$ we can use the extension to a proper C_0-semigroup on the data space, which explains the regularity result (iii). However, and this was never observed before, the range of exponents in the statement exceeds by one Sobolev exponent the range provided by the semigroup theory.[21] This means that in this case u is understood as a function of both variables t and x simultaneously that does not come from a semigroup action.

Parts (i) and (iii) in the theorem remain true for the Poisson semigroup extension $u(t,x) = \mathrm{e}^{-tL^{1/2}} f(x)$ of data $f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$, even when $p_-(L) < p < 1$. This is why we have systematically incorporated multiplication by a in our estimates, although it can be omitted when $p > 1$.[22]

For $1_* < p < \infty$, the H^p *regularity problem* consists in solving, given $f \in \dot{\mathrm{H}}^{1,p}(\mathbb{R}^n;\mathbb{C}^m)$,

$$(R)_p^{\mathcal{L}} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \widetilde{N}_*(\nabla u) \in \mathrm{L}^p(\mathbb{R}^n), \\ \lim_{t\to 0} ⨏⨏_{W(t,x)} |u(s,y) - f(x)| \,\mathrm{d}s\mathrm{d}y = 0 & (\text{a.e. } x \in \mathbb{R}^n). \end{cases}$$

As a (quasi-)Banach space $\dot{\mathrm{H}}^{1,p}$ is a space of tempered distributions modulo constants but this point of view is not appropriate for the regularity problem. What we mean here is that the data f is a tempered distribution such that $\nabla_x f \in \mathrm{H}^p$. By Hardy–Sobolev embeddings any

[20] As usual, the notation $\mathrm{C}_0([0,\infty))$ means continuity and limit 0 at infinity.

[21] When $a = 1$, Mayboroda [76] dealt with variants where the L^2-averages in the maximal functions are replaced with L^p-averages. Her range of exponents is not the same and indeed, she shows that well-posedness is limited to the semigroup range.

[22] These estimates can be extended to f in a closure of the data class for the quasinorm $\|a \cdot\|_{\mathrm{H}^p}$. However, since H^p does not embed into $\mathrm{L}^1_{\mathrm{loc}}$ for $p < 1$ and a is not smooth, it is unclear whether this abstract extension has any reasonable (e.g. distributional) interpretation on the level of the boundary value problem. Even if $a = 1$, (ii) has no meaning for us.

such distribution is a locally integrable function and this gives a meaning to the boundary condition.[23]

Our principal result exhibits again an extended range of compatible well-posedness.[24] The solution is given by the Poisson semigroup if the data also belongs to L^2 and appropriate extensions thereof in the general case.

Theorem 1.2 (Regularity problem)**.** *Let $(q_-(L)_* \vee 1_*) < p < q_+(L)$. Then the regularity problem $(R)_p^{\mathcal{L}}$ is compatibly well-posed. Given $f \in \dot{\mathrm{H}}^{1,p}(\mathbb{R}^n;\mathbb{C}^m)$, the unique solution u has the following additional properties.*

(i) *There are estimates*

$$\|\widetilde{N}_*(\nabla u)\|_p \simeq \|S(t\nabla \partial_t u)\|_p \simeq \|\nabla_x f\|_{\mathrm{H}^p} \gtrsim \|g\|_{\mathrm{H}^p}$$

with $g = -aL^{1/2}f$ being the conormal derivative of u, where the square root extends from $\dot{\mathrm{H}}^{1,p}(\mathbb{R}^n;\mathbb{C}^m) \cap \mathrm{W}^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$ by density.

(ii) *For a.e. $x \in \mathbb{R}^n$ and all $t > 0$,*

$$\left(\fint\!\!\fint_{W(t,x)} |u(s,y) - f(x)|^2 \,\mathrm{d}s\mathrm{d}y \right)^{\frac{1}{2}} \lesssim t\widetilde{N}_*(\nabla u)(x).$$

In particular, the non-tangential convergence improves to L^2-averages. Moreover, $\lim_{t\to 0} u(t,\cdot) = f$ in $\mathcal{D}'(\mathbb{R}^n)$.

(iii) *If $p \geq 1$, then for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} \left| \begin{bmatrix} a\partial_t u \\ \nabla_x u \end{bmatrix} - \begin{bmatrix} g(x) \\ \nabla_x f(x) \end{bmatrix} \right|^2 \mathrm{d}s\mathrm{d}y = 0,$$

where g is as in (i).

(iv) *$\nabla_x u$ is of class $\mathrm{C}_0([0,\infty);\mathrm{H}^p(\mathbb{R}^n;\mathbb{C}^m)) \cap \mathrm{C}^\infty((0,\infty);\mathrm{H}^p(\mathbb{R}^n;\mathbb{C}^m))$ with $\nabla_x u(0,\cdot) = \nabla_x f$ and*

$$\|\nabla_x f\|_{\mathrm{H}^p} \simeq \sup_{t>0} \|\nabla_x u(t,\cdot)\|_{\mathrm{H}^p}.$$

If $p < n$, then up to a constant[25] $u \in \mathrm{C}_0([0,\infty);\mathrm{L}^{p^*}(\mathbb{R}^n;\mathbb{C}^m)) \cap \mathrm{C}^\infty((0,\infty);\mathrm{L}^{p^*}(\mathbb{R}^n;\mathbb{C}^m))$ with $u(0,\cdot) = f$ and

$$\|f\|_{p^*} \leq \sup_{t>0} \|u(t,\cdot)\|_{p^*} \lesssim \|\nabla_x f\|_{\mathrm{H}^p} + \|f\|_{p^*}.$$

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[23] In fact, the condition $\widetilde{N}_*(\nabla u) \in \mathrm{L}^p(\mathbb{R}^n)$ guarantees existence of a trace in $\dot{\mathrm{H}}^{1,p}$ in the sense of this limit at the boundary. See Appendix A.

[24] The fact that there is an extended range related to a Sobolev exponent down was observed by Mayboroda [76] when $a = 1$, who establishes $\|\widetilde{N}_*(\nabla u)\|_p \lesssim \|\nabla_x f\|_p$ for $p \in (p_-(L)_* \vee 1, 2+\varepsilon]$ inspired from the estimate $\|L^{1/2}f\|_{\mathrm{H}^p} \lesssim \|\nabla_x f\|_{\mathrm{H}^p}$ in a similar range from [6]. We point out that Step V in the proof of [76, Thm. 4.1] has a flaw that can be fixed (personal communication of S. Hofmann) or treated differently, see the argument in [32].

[25] The constant is chosen via Hardy–Sobolev embeddings such that $f \in \mathrm{L}^{p^*}$.

(v) *If $p > p_-(L)$, then $a\partial_t u$ is of class $\mathrm{C}_0([0,\infty);\mathrm{H}^p(\mathbb{R}^n;\mathbb{C}^m))$ and, with g as in (i),*

$$\|\widetilde{N}_*(\partial_t u)\|_p \simeq \sup_{t\geq 0}\|a\partial_t u(t,\cdot)\|_{\mathrm{H}^p} \simeq \|g\|_{\mathrm{H}^p} \simeq \|\nabla_x f\|_{\mathrm{H}^p}.$$

As mentioned earlier, prior to these two results the situation was fully understood only in the case of boundary dimension $n = 1$.[26]

One may wonder whether in the case $p_+(L) > n$ there are results for the Dirichlet problem with exponents 'beyond ∞', which, in view of Sobolev embeddings, we think of corresponding to the homogeneous Hölder spaces $\dot{\Lambda}^\alpha(\mathbb{R}^n;\mathbb{C}^m)$, $0 \leq \alpha < 1$, with the endpoint case $\dot{\Lambda}^0 :=$ BMO. We define the Carleson functional

$$C_\alpha(F)(x) := \sup_{t>0}\frac{1}{t^\alpha}\bigg(\frac{1}{t^n}\int_0^t\int_{B(x,t)}|F(s,y)|^2\frac{\mathrm{d}y\mathrm{d}s}{s}\bigg)^{1/2} \quad (x\in\mathbb{R}^n).$$

For $\alpha \in (0,1)$, one formulation of the Dirichlet problem with data $f \in \dot{\Lambda}^\alpha(\mathbb{R}^n;\mathbb{C}^m)$ consists in solving

$$(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+),\\ C_\alpha(t\nabla u)\in \mathrm{L}^\infty(\mathbb{R}^n), \\ \lim_{t\to 0}\fint\!\!\fint_{W(t,x)}|u(s,y)-f(x)|\,\mathrm{d}s\mathrm{d}y = 0 & (\text{a.e. } x\in\mathbb{R}^n).\end{cases}$$

The interior control from the Carleson functional alone implies existence of a non-tangential trace $f \in \dot{\Lambda}^\alpha(\mathbb{R}^n;\mathbb{C}^m)$ as in the third line[27], so that this is the weakest possible formulation of the boundary behavior. Again, we regard $\dot{\Lambda}^\alpha$ as a space of functions to make sense of the limit condition. This non-tangential trace also satisfies $\|\widetilde{N}_{\sharp,\alpha}(u-f)\|_\infty \lesssim \|C_\alpha(t\nabla u)\|_\infty$, where on the left-hand side we use the sharp functional on Whitney averages

$$\widetilde{N}_{\sharp,\alpha}(u-f)(x) := \sup_{t>0}\frac{1}{t^\alpha}\bigg(\fint\!\!\fint_{W(t,x)}|u(s,y)-f(y)|^2\,\mathrm{d}s\mathrm{d}y\bigg)^{1/2} \ (x\in\mathbb{R}^n).$$

Such a trace result is not available for $\alpha = 0$ and we formulate the boundary behavior for the endpoint problem differently, using convergence of Cesàro averages, which is natural from the point of view of

[26] This is due to [23], where existence and uniqueness are shown in the largest possible range $1 < p < \infty$ as well as existence for a Dirichlet problem in the Hardy range $1_* = 1/2 < p \leq 1$. When $n \geq 2$ and $a = 1$, non-tangential maximal functions estimates pertaining to the Dirichlet and regularity problems first appeared in [76] and some related square functions estimates are in [13]. Uniqueness has not been considered in general, except for systems having regular solutions [59,61]. A possible strategy for general elliptic systems has been developed in [11], but it only covers some smaller range of exponents when it comes to the block situation.

[27] We include a proof of the trace theorem in Appendix A.

both our construction and our approach to uniqueness theorems:

$$(D)^{\mathcal{L}}_{\dot{\Lambda}^0} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ C_0(t\nabla u) \in \mathrm{L}^\infty(\mathbb{R}^n), \\ \lim_{t\to 0} \fint_t^{2t} |u(s,\cdot) - f| \,\mathrm{d}s = 0 & (\text{in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^m)). \end{cases}$$

The discussion of non-tangential traces naturally leads us to formulating a modified $\dot{\Lambda}^\alpha$-Dirichlet problem

$$(\widetilde{D})^{\mathcal{L}}_{\dot{\Lambda}^\alpha} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \widetilde{N}_{\sharp,\alpha}(u-f) \in \mathrm{L}^\infty(\mathbb{R}^n), \\ \lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y) - f(x)| \,\mathrm{d}s\mathrm{d}y = 0 & (\text{a.e. } x \in \mathbb{R}^n). \end{cases}$$

As we have seen above, this second problem is *a priori* comparable to the first one when $\alpha > 0$.[28]

We obtain compatible well-posedness for both problems in the same range of exponents. In order to formulate the theorem, and systematically throughout this book, we denote by $L^\sharp$ the boundary operator for the adjoint equation $\mathcal{L}^*u = 0$, that is $L^\sharp = -(a^*)^{-1} \operatorname{div}_x d^* \nabla_x$.

Theorem 1.3 ($\dot{\Lambda}^\alpha$ Dirichlet problem)**.** *Suppose that $p_+(L) > n$ and that $0 \le \alpha < 1 - n/p_+(L)$. Then the Dirichlet problems $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ and $(\widetilde{D})^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ are compatibly well-posed. Given $f \in \dot{\Lambda}^\alpha(\mathbb{R}^n;\mathbb{C}^m)$, the unique solution u is the same for both problems and has the following additional properties.*

(i) *There is comparability*

$$\|C_\alpha(t\nabla u)\|_\infty \simeq \|f\|_{\dot{\Lambda}^\alpha}.$$

(ii) *One has the upper bound*

$$\|\widetilde{N}_{\sharp,\alpha}(u-f)\|_\infty \lesssim \|f\|_{\dot{\Lambda}^\alpha}$$

and convergence

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y) - f(x)|^2 \,\mathrm{d}s\mathrm{d}y = 0 \quad (a.e.\ x \in \mathbb{R}^n).$$

In addition, u is of class $\mathrm{C}([0,T];\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^m))$ with $u(0,\cdot) = f$ for every $T > 0$.

(iii) *If, moreover $p_-(L^\sharp) < 1$ and $\alpha < n(1/p_-(L^\sharp) - 1)$, then u is of class $\mathrm{C}_0([0,\infty);\dot{\Lambda}^\alpha_{weak^*}(\mathbb{R}^n;\mathbb{C}^m)) \cap \mathrm{C}^\infty((0,\infty);\dot{\Lambda}^\alpha_{weak^*}(\mathbb{R}^n;\mathbb{C}^m))$ and*

$$\sup_{t>0} \|u(t,\cdot)\|_{\dot{\Lambda}^\alpha} \simeq \|f\|_{\dot{\Lambda}^\alpha}.$$

[28] Uniqueness for the BMO-Dirichlet problem with interior Carleson control and Whitney average convergence at the boundary appears to be out of reach. See [74,75] for a very recent account on such Fatou-type theorems in the case of elliptic systems with *constant* coefficients.

In addition, u is of class $\dot{\Lambda}^\alpha(\overline{\mathbb{R}^{1+n}_+};\mathbb{C}^m)$, with

$$\|u\|_{\dot{\Lambda}^\alpha(\overline{\mathbb{R}^{1+n}_+})} \lesssim \|f\|_{\dot{\Lambda}^\alpha}.$$

Since $\dot{\Lambda}^\alpha \cap \mathrm{L}^2$ is *not* dense in $\dot{\Lambda}^\alpha$, we cannot extend the Poisson semigroup to the boundary space by density. In (iii), $\dot{\Lambda}^\alpha$ is considered as the dual space of H^p, where $\alpha = n(1/p - 1)$, with the weak* topology. The assumption in (iii) implies $p_+(L) = \infty$ and that the solution can be constructed by duality, using the extension of the Poisson semigroup for $L^* = a^* L^\sharp (a^*)^{-1}$ to H^p. Therefore the solution keeps the $\dot{\Lambda}^\alpha$-regularity in the interior. This construction has appeared earlier.[29]

The construction of the solution under the mere assumption that $p_+(L) > n$ is much more general and we have

$$u(t,x) = \lim_{j \to \infty} \mathrm{e}^{-tL^{1/2}} (\mathbf{1}_{\{|\cdot|<2^j\}} f)(x),$$

where $p_+(L) > n$ is used already to prove convergence of the right-hand side in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+;\mathbb{C}^m)$. This opens the possibility of uniquely solving Dirichlet problems for Hölder continuous (or BMO) data, while producing solutions that have no reason to be in the same class in the interior of the domain. To the best of our knowledge this phenomenon is observed for the first time. Note also that $p_+(L) > n$ always holds in dimension $n \leq 4$, so that in these dimensions both BMO-Dirichlet problems are compatibly well-posed.

1.6. **Dirichlet problems with fractional spaces of data.** If we think of the Dirichlet problem $(D)^{\mathcal{L}}_p$ as a boundary value problem with regularity $s = 0$ for the data and the regularity problem $(R)^{\mathcal{L}}_p$ as a Dirichlet problem with regularity $s = 1$, we can depict the exponents for both problems simultaneously in an $(1/p, s)$-diagram. There are two classical scales of data spaces to fill the intermediate area of points with $0 < s < 1$: The homogeneous Hardy–Sobolev spaces $\dot{\mathrm{H}}^{s,p}$ that can be obtained from the endpoints by complex interpolation and the homogeneous Besov spaces $\dot{\mathrm{B}}^{s,p}$ that result from real interpolation.[30]

[29] References are [19, 22, 61].

[30] We give a detailed account on all sorts of relevant function spaces in Section 2.

For $0 < p \leq \infty$ and $0 < s < 1$ satisfying ${}^1\!/_p < 1 + {}^s\!/_n$[31], the Dirichlet problem with data $f \in \dot{\mathrm{B}}^{s,p}(\mathbb{R}^n; \mathbb{C}^m)$ consists in solving

$$(D)^{\mathcal{L}}_{\dot{\mathrm{B}}^{s,p}} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ W(t^{1-s}\nabla u) \in \mathrm{L}^p(\mathbb{R}^{1+n}_+; \frac{\mathrm{d}t\mathrm{d}x}{t}), \\ \lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y) - f(x)| \,\mathrm{d}s\mathrm{d}y = 0 & (\text{a.e. } x \in \mathbb{R}^n), \end{cases}$$

where $W(F)$ is the Whitney average functional

$$W(F)(t,x) = \left(\fint\!\!\fint_{W(t,x)} |F(s,y)|^2 \,\mathrm{d}s\mathrm{d}y \right)^{\frac{1}{2}} \quad ((t,x) \in \mathbb{R}^{1+n}_+).$$

For $0 < p < \infty$ and $0 < s < 1$ satisfying ${}^1\!/_p < 1 + {}^s\!/_n$, the Dirichlet problem with data $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n; \mathbb{C}^m)$ consists in solving

$$(D)^{\mathcal{L}}_{\dot{\mathrm{H}}^{s,p}} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ S(t^{1-s}\nabla u) \in \mathrm{L}^p(\mathbb{R}^n), \\ \lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y) - f(x)| \,\mathrm{d}s\mathrm{d}y = 0 & (\text{a.e. } x \in \mathbb{R}^n) \end{cases}$$

where S is the same conical square function as before.[32]

For $p = \infty$ we can identify $\mathrm{B}^{s,\infty} = \dot{\Lambda}^s$, so that $(D)^{\mathcal{L}}_{\dot{\mathrm{B}}^{s,\infty}}$ is a third formulation of a Dirichlet problem for that space of data. The end-point problems for the Hardy–Sobolev scale are formulated for data in Strichartz' BMO-Sobolev spaces $\dot{\mathrm{H}}^{s,\infty} = \dot{\mathrm{BMO}}^s$ and consist in solving

$$(D)^{\mathcal{L}}_{\dot{\mathrm{H}}^{s,\infty}} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \mathrm{C}_0(t^{1-s}\nabla u) \in \mathrm{L}^\infty(\mathbb{R}^n), \\ \lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y) - f(x)| \,\mathrm{d}s\mathrm{d}y = 0 & (\text{a.e. } x \in \mathbb{R}^n). \end{cases}$$

We note that the approach to the boundary in these problems is not in the sense of the usual trace theory, that is by extension of the restriction map to the boundary defined on smooth functions. In fact, this approach would work for Besov spaces[33] but not for Hardy–Sobolev spaces which are not trace spaces in this sense. Our choice of a non-tangential

[31]When $0 < p < \infty$, this Sobolev-type condition characterizes the spaces that can be obtained by interpolation between data spaces for the Dirichlet problem (L^p with $p > 1$) and the regularity problem (H^p with $p > 1_*$), see Section 2.6. In particular, it is the natural restriction guaranteeing that all distributions in $\dot{\mathrm{X}}^{s,p}$ are locally integrable functions. The spaces $\mathrm{B}^{s,\infty}$ and $\mathrm{H}^{s,\infty}$ also have this property, see Section 2.5.

[32]Boundary value problems for general elliptic equations ($m = 1$) with data of fractional regularity have been pioneered by Barton–Mayboroda [28]. They treat $\dot{\mathrm{B}}^{s,p}$-data for equations with the de Giorgi–Nash–Moser property. This assumption was then removed in the first-order approach by Amenta along with the first author [3] and their approach includes the problems with $\dot{\mathrm{H}}^{s,p}$-data. Thanks to the block structure we do not have to include a limiting condition for u as $t \to \infty$ in the formulation of our fractional Dirichlet problems. Such a condition appears in the general framework of [3] but not in [28].

[33]This is the point of view taken in [28]. See also [3].

convergence of Whitney averages has one main advantage, valid for all situations: each interior control implies existence of a unique measurable function f, called non-tangential trace (in the sense of Whitney averages), such that the third condition holds, whether or not u is a weak solution to $\mathcal{L}u = 0$. In this sense, we prescribe the boundary limit in the weakest possible form. If, via a trace operator, $\lim_{t \to 0} u(t, \cdot)$ also exists in the sense of distributions (modulo constants), then the two notions of boundary trace coincide (modulo constants).[34] The same limit condition was taken in the boundary value problems from the previous section (except for one of the Dirichlet problems with BMO-data). We stress again that we consider the data spaces as classes of measurable functions and not as distributions (modulo constants) and that this is possible due to the assumption $1/p < 1 + s/n$.[31]

In the figures below, we collect compatible well-posedness results from the previous section on thick horizontal boundary segments at $s = 0$ and $s = 1$. For $p = \infty$, we can represent these results also on a thick vertical segment at $1/p = 0$. Empty circles indicate boundary points that are not contained in a segment of well-posedness. This allows us to create a map $f \mapsto u$ for different values of $(1/p, s)$ on these lines and, roughly speaking, we can interpolate to fill in a shaded region for compatible solvability of both fractional problems.[35]

Of course, interpolation does not preserve uniqueness. Still, we shall be able to show uniqueness (and hence compatible well-posedness) even in a possibly larger region than for existence of a solution.[36]

Theorem 1.4. *Let $0 < s < 1$ and $1_* < p \leq \infty$. If $(1/p, s)$ belongs to the region displayed in Figure 1, Figure 2 or Figure 3 (including the thick vertical segment), then $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ is compatibly well-posed.*

As customary, we obtain continuous dependence on the data: the interior control is bounded by the data in the boundary space. For the problems corresponding to all thick segments we have also seen the reverse estimates in the previous section. Various additional regularity properties in the spirit of Theorem 1.1 - 1.3 hold depending on the particular boundary space.[37]

In the following diagrams a color code allows us to distinguish different zones that explain the relation of the corresponding well-posedness results with the first- and second-order operator theory that we develop

[34] All this is shown in Appendix A. Similar trace theorems appear in [3, Sec. 6.6], where they are used to derive non-tangential convergence of the solution at the boundary *a posteriori*.

[35] The fact that not only the data spaces but also the interior control from the functionals S and W interpolate, shows again that these are natural classes of solutions from our perspective.

[36] The corresponding regions and the proof of the uniqueness theorems can all be found in Section 21.

[37] Precise results are stated and proved in Section 19.

in parallel. A reader who is not familiar with these tools (yet) might ignore the different colors for the time being and focus only on the shape of the regions.

- Gray corresponds to what can be obtained from the theory of first order DB-adapted Hardy spaces in [3].
- Blue shows extra information obtained from the theory of L-adapted Hardy–Sobolev spaces.
- Red indicates results outside of the theory of operator-adapted Hardy spaces.

All shaded regions in the strip $0 < s < 1$ capture a situation that is common to Hardy–Sobolev and Besov data and we set $\dot{\mathrm{X}}$ to designate $\dot{\mathrm{H}}$ or $\dot{\mathrm{B}}$. They depict three different cases: first $p_+(L) \leq n$, next $p_+(L) > n$ but $p_-(L^\sharp) \geq 1$ and eventually $p_-(L^\sharp) < 1$, which turns out to imply $p_+(L) = \infty$ by duality.[38]

We begin by illustrating the situation when $p_+(L) \leq n$. In this case we obtain the segment on the bottom line for $s = 0$ and the top line for $s = 1$ from Theorems 1.1 and 1.2, respectively. This leads to Figure 1. In all such figures we shall write p_+^L instead of $p_+(L)$ and so on for the sake of a clearer typeset.

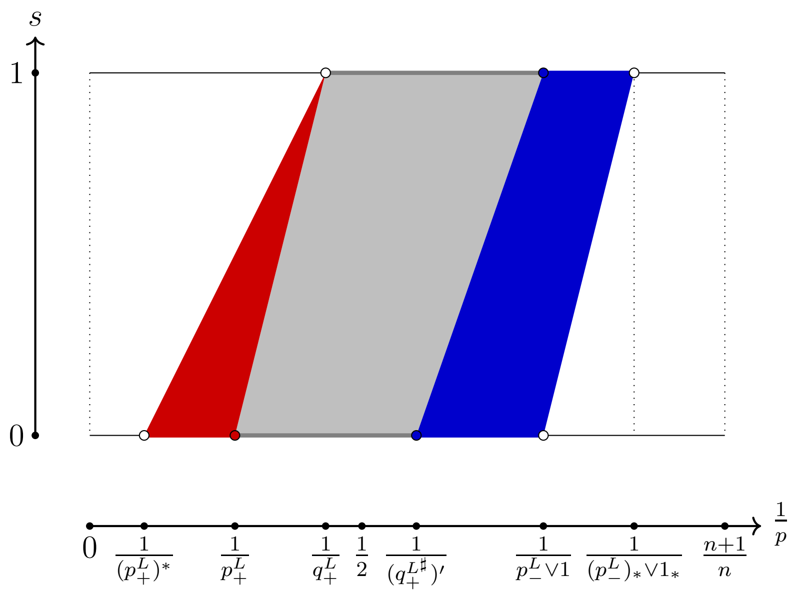

FIGURE 1. Compatible well-posedness region for Besov and Hardy–Sobolev data when $p_+(L) \leq n$.

[38]Let us mention that the diagrams are up to scale when $p_-(L) \geq 1$ but not when $p_-(L) < 1$. In this latter case, the top blue point is always situated at $(1/p_-(L), 1)$, while the bottom point would be $(1, 0)$.

In the case $p_+(L) > n$ we can extend the bottom line to exponents 'beyond infinity', using Theorem 1.3. The point corresponding to compatible well-posedness of $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ is $(-\alpha/n, 0)$. We shall see that this also leads to compatible well-posedness of $(D)^{\mathcal{L}}_{\dot{\mathrm{B}}^{\alpha,\infty}}$ at $(0,\alpha)$ as stated. A similar result holds for $(D)^{\mathcal{L}}_{\dot{\mathrm{H}}^{\alpha,\infty}}$ at $(0,\alpha)$.[39] Figure 2 illustrates this extension in the case that $p_+(L) > n$ but $p_-(L^\sharp) \geq 1$. This is the generic situation in dimensions $n = 3, 4$.

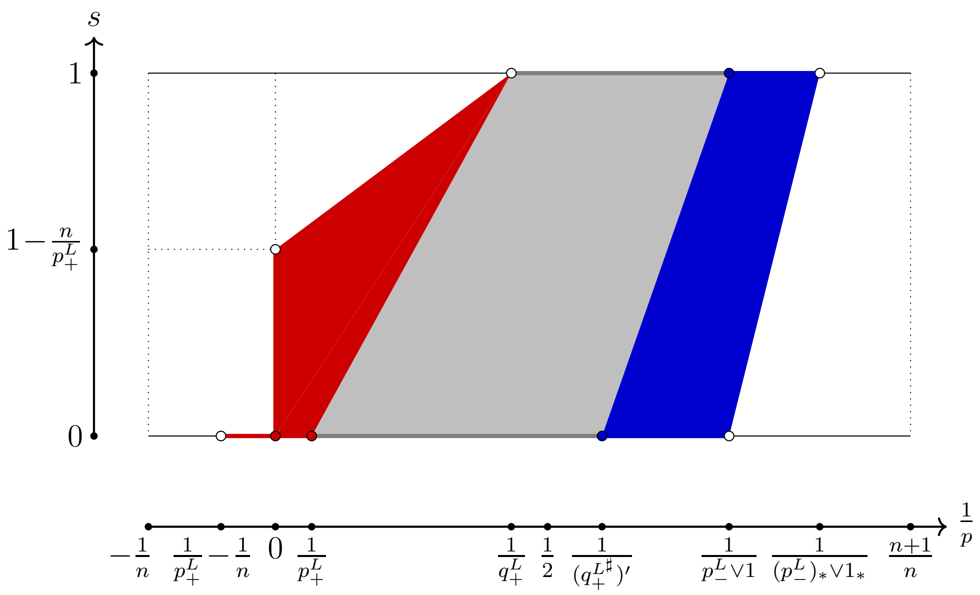

FIGURE 2. Compatible well-posedness region for Besov and Hardy–Sobolev data when $p_+(L) > n$ but $p_-(L^\sharp) \geq 1$.

Figure 3 describes the case when $p_-(L^\sharp) < 1$, which happens for instance when $n = 1, 2$ or for special classes of systems such as equations ($m = 1$) with real-valued coefficients d.[40]

[39] See Proposition 19.9.

[40] More examples are given in Section 14.3.

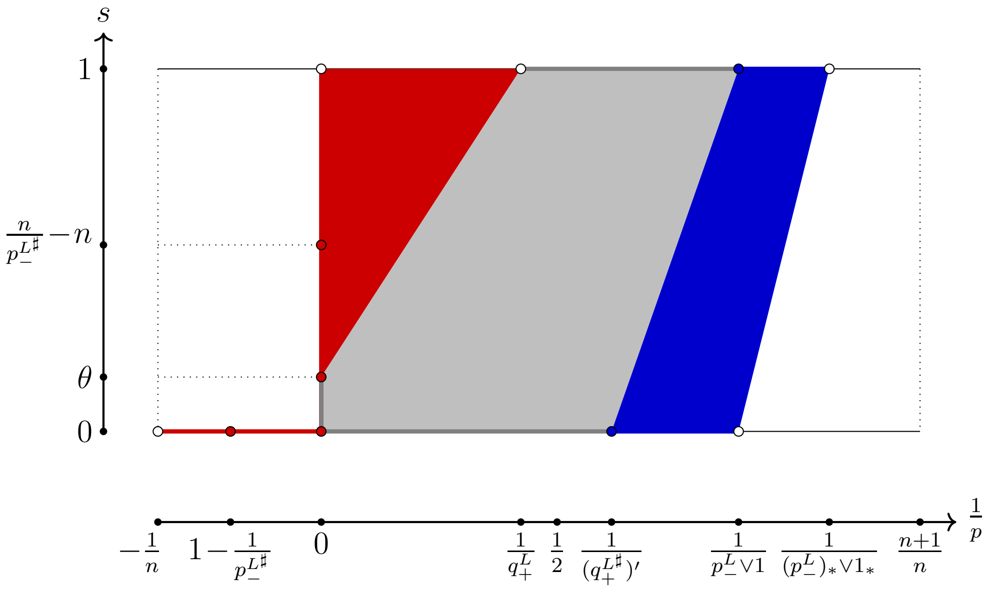

FIGURE 3. Compatible well-posedness region for Besov and Hardy–Sobolev data when $p_-(L^\sharp) < 1$. This implies $p_+(L) = \infty$ and hence there is no horizontal thick red line as in Figure 2. The number θ from [3] has a specific meaning, see Proposition 19.3, and is not larger than $n(1/p_-^{L^\sharp} - 1)$, which is the limitation of part (iii) in Theorem 1.3 for Besov-data.

1.7. **Neumann problems.** Although this is not central to our monograph, we complete the discussion with results on the Neumann problem. For $1_* < p < \infty$, the Neumann problem with data $g \in \mathrm{H}^p(\mathbb{R}^n; \mathbb{C}^m)$ consists in solving (modulo constants)

$$(N)_p^{\mathcal{L}} \qquad \begin{cases} \mathcal{L}u = 0 & \text{in } \mathbb{R}^{1+n}_+, \\ \widetilde{N}_*(\nabla u) \in \mathrm{L}^p(\mathbb{R}^n), \\ \lim_{t\to 0} a\partial_t u(t,\cdot) = g & (\text{in } \mathcal{D}'(\mathbb{R}^n; \mathbb{C}^m)). \end{cases}$$

Note that due to the block structure $a\partial_t u$ is indeed the conormal derivative $\partial_{\nu_A} u = e_0 \cdot A\nabla u$. Here, constants are solutions which do not change the Neumann data so we must argue modulo constants.

In order to understand how our results help in deducing a range of exponents for which the Neumann problem is compatibly well-posed from existing literature, we recall the first-order approach. For block systems it simply begins by writing (1.1) in the equivalent form

$$\partial_t \begin{bmatrix} a\partial_t u \\ \nabla_x u \end{bmatrix} + \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} a\partial_t u \\ \nabla_x u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{1.7}$$

where the second line is a dummy equation, or in short notation

$$\partial_t F + DBF = 0, \tag{1.8}$$

where $F = \nabla_A u := [a\partial_t u, \nabla_x u]^\top$ is the *conormal gradient* and DB is called *perturbed Dirac operator*. This operator is bisectorial and there are associated abstract Hardy spaces $\mathbb{H}^p_{DB}$. The idea then is to work backwards from that: first classify all weak solutions to (1.8) in the usual classes and then try to reconstruct u from its conormal gradient.

The principal thesis in the work of the first author with Stahlhut [22] and Mourgoglou [19] is that there is an open interval $I_L \subseteq (1_*, \infty)$ such that if $p \in I_L$, then

- the conormal gradient of every weak solution to $(N)^{\mathcal{L}}_p$ has an *a priori* representation via the semigroup associated with $|DB| := ((DB)^2)^{1/2}$,
- existence of a compatible solution for every $g \in \mathrm{H}^p(\mathbb{R}^n; \mathbb{C}^m)$ implies uniqueness and hence compatible well-posedness[41].

The interval I_L corresponds to identification $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ of abstract and concrete Hardy spaces up to equivalent p-quasinorms and a certain L^p-coercivity assumption of B when $p > 2$.

In the block case one can produce a formal solution to the Neumann problem by $u(t,x) := -L^{-1/2}\mathrm{e}^{-tL^{1/2}}(a^{-1}g)(x)$, so that once this is made rigorous, compatible well-posedness of $(N)^{\mathcal{L}}_p$ follows in the range $p \in I_L$. This being said, our main contribution for the Neumann problem lies in proving the equality[42]

$$I_L = (q_-(L), q_+(L)) \tag{1.9}$$

and then we conclude the following result.

Theorem 1.5 (Neumann problem)**.** *Let $q_-(L) < p < q_+(L)$. Then the Neumann problem $(N)^{\mathcal{L}}_p$ is compatibly well-posed (modulo constants).*

With the determination of I_L at hand, one can write down all further implications from [19] for solutions with the *a priori* representation of $\nabla_A u$. This would lead us too far from the objective of our monograph. Let us just mention that there are additional regularity properties for solutions to $(N)^{\mathcal{L}}_p$ in Theorem 1.5, similar to Theorem 1.2, and that well-posedness of an adjoint 'rough' Neumann problem follows by duality.[43] Finally, in the spirit of Section 1.6, there are fractional Neumann problems in between for which ranges of compatible well-posedness

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[41] This is Theorem 1.8 in [19]

[42] The proof is in Section 15, Corollary 15.2 and the principal issue is to *prove* the p-coercivity for $p > 2$. Before it was only known that when $a = 1$, I_L cannot be larger than $(q_-(L), q_+(L))$ and that its upper endpoint is $q_+(L)$ if in addition d is strictly elliptic, see [22, Sec. 12.4.1].

[43] For further regularity in the Neumann problem, see Corollary 1.2 in [19]. Therein, the Dirichlet data is given by $f = -L^{-1/2}(a^{-1}g) \in \dot{\mathrm{H}}^{1,p}$ using a suitable extension of the square root. For the duality with the rough Neumann problem see Theorem 1.6 and then Theorem 1.3 and Corollary 1.4 in [19] for the *a priori* representation and regularity for its solutions.

have also been described via I_L.[44] In fact, this is the gray region in the diagrams above.

1.8. **Synthesis.** We close the introduction with a comment further explaining the color code in the diagrams in Section 1.6. Heuristically, the H^p-theory for DB comprises the theory for L at both smoothness scales $s = 0$ and $s = 1$. On the level of Hardy spaces, this becomes apparent in the fact that the interval in (1.9) is the intersection of intervals of identification for $\mathbb{H}^p_L$ and $\mathbb{H}^{1,p}_L$, compare with (1.5) and (1.6). On the level of boundary value problems, the first-order approach via DB yields ranges of exponents in which problems with Neumann and Dirichlet data are simultaneously well-posed — this is the gray region. The L-adapted theory allows us to separate issues and obtain significantly larger ranges for the problems with Dirichlet data — gray and blue regions. Finally, there is a new phenomenon – solving Dirichlet problems for one Sobolev conjugate above the limitation of the Hardy space theory in the red region.

1.9. **Notation.** The following notation will be used throughout the monograph.

Geometry and measure. We let $B(x,r) \subseteq \mathbb{R}^n$ the open ball of radius $r > 0$ around $x \in \mathbb{R}^n$. Given a ball $B \subseteq \mathbb{R}^n$ of radius $r(B)$, we write cB for the concentric ball of radius $cr(B)$ and define the annular regions $C_j(B)$, $j \in \mathbb{N}$, by

$$C_1(B) \coloneqq 4B, \quad C_j(B) \coloneqq 2^{j+1}B \setminus 2^j B \quad (j \geq 2).$$

The same type of notation will be used for cubes instead of balls. In this case, $\ell(Q)$ denotes the sidelength of Q. In order to avoid even the slightest confusion, let us explicitly state that for us $\mathbb{N} \coloneqq \{1, 2, 3, \dots\}$.

We write the Euclidean distance on finite-dimensional vector spaces as $\mathrm{d}(x,y) \coloneqq |x-y|$ and extend the notation to sets $E, F \subseteq \mathbb{R}^n$ via

$$\mathrm{d}(E,F) \coloneqq \inf\{\mathrm{d}(x,y) : x \in E,\, y \in F\}.$$

In $\mathbb{R}^{1+n}$ we denote points by (t,x) and define the open upper halfspace

$$\mathbb{R}^{1+n}_+ \coloneqq \{(t,x) : t > 0,\, x \in \mathbb{R}^n\}.$$

We write $|\cdot|$ for the Lebesgue measure if the underlying Euclidean space is clear form the context. For integral averages we use $⨍$ and $⨌$ in $\mathbb{R}^n$ and $\mathbb{R}^{1+n}$, respectively, as well as the notation $(f)_E \coloneqq ⨍_E f$. We use the (uncentered) Hardy–Littlewood maximal operator defined for measurable functions on $\mathbb{R}^n$ via

$$\mathcal{M}(f)(x) \coloneqq \sup_{B \ni x} ⨍_B |f| \, \mathrm{d}y \quad (x \in \mathbb{R}^n),$$

[44] See [3] for an introduction to and results on these problems.

where the supremum runs over all balls B that contain x. Occasionally, we also use cubes instead of balls.

Gradient and divergence of vector-valued functions. Partial derivatives of $\mathbb{C}^m$-valued functions are taken componentwise. If f is a $\mathbb{C}^m$-valued function on a subset of $\mathbb{R}^n$ or $\mathbb{R}^{1+n}$, then

$$\nabla_x f := [\partial_{x_1} f, \ldots, \partial_{x_n} f]^\top$$

is a function valued in $\mathbb{C}^{mn} \cong (\mathbb{C}^m)^n$. In the opposite direction, if $F = [F_1, \ldots, F_n]^\top$ is $\mathbb{C}^{mn}$-valued, then we let

$$\operatorname{div}_x F = \partial_{x_1} F_1 + \ldots + \partial_{x_n} F_n.$$

Gradient and divergence with respect to all variables in $\mathbb{R}^{1+n}$ are defined as $\nabla f = [\partial_t f, \nabla_x f]^\top$ and $\operatorname{div} = \partial_t F_\perp + \operatorname{div}_x F_\parallel$ if $F = [F_\perp, F_\parallel]^\top$ is valued in $\mathbb{C}^m \times \mathbb{C}^{mn}$.

Exponents. We let

$$\begin{aligned}
\frac{1}{p'} &= 1 - \frac{1}{p} && (p \in [1,\infty], \text{Hölder conjugate}),\\
\frac{1}{p_*} &= \frac{1}{p} + \frac{1}{n} && (p \in (0,\infty], \text{lower Sobolev conjugate}),\\
\frac{1}{p^*} &= \frac{1}{p} - \frac{1}{n} && (p \in (0,n), \text{upper Sobolev conjugate}),\\
\frac{1}{[p_0,p_1]_\theta} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1} && (p_i \in (0,\infty],\ \theta \in [0,1], \text{interpolating index}).
\end{aligned}$$

The underlying dimensions for Sobolev conjugates is usually n and will always be clear from the context. We also agree on $p^* := \infty$ for $p \geq n$.

Constants. Given $a, b \in [0,\infty]$, we write $a \lesssim b$ to mean $a \leq Cb$ for some $C > 0$ (often times called 'implicit constant') that is independent of a and b. We write $a \simeq b$ to mean $a \lesssim b$ and $b \lesssim a$. In this case one of a, b is equal to ∞ (or 0) precisely when both are. Unless stated otherwise, estimates in this monograph are *quantitative* in the sense that constants in estimates depend only on constants quantified in the relevant hypotheses. Such dependence will usually be clear.

Index. This monograph has an index. For the sake of readability we shall occasionally refer to results by their name listed in the index instead of a number in the text.

2. Preliminaries on function spaces

Throughout, we consider $\mathbb{C}^k$-valued functions for some fixed $k \in \mathbb{N}$. For simplicity we often drop the dependence of k in the notation and write $\mathrm{L}^2(\mathbb{R}^n) = \mathrm{L}^2(\mathbb{R}^n; \mathbb{C}^k)$, and so on. On $\mathbb{R}^n$ we abbreviate further $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R}^n)$. Concerning the dilemma that parts of the literature only treat scalar-valued functions, we agree on using such results for $k > 1$ without further notice in the following cases:

- Splitting into components is immediately clear from the definition (e.g. $\mathrm{L}^2(\mathbb{R}^n;\mathbb{C}^k) \cong \bigotimes_{j=1}^k \mathrm{L}^2(\mathbb{R}^n;\mathbb{C})$),
- Proofs are exactly the same except for a systematic replacement of absolute values by Hermitian norms (e.g. Calderón–Zygmund decompositions or atomic decompositions).

The reader can consult this section to find all necessary background whenever new function spaces pop later on in the text.

2.1. **Lebesgue spaces and distributions.** On a (Lebesgue) measurable set $E \subseteq \mathbb{R}^n$ we let $\mathrm{L}^p(E)$, $p \in (0,\infty]$, be the (quasi-)Banach space of functions classes with finite (quasi)norm

$$\|f\|_{\mathrm{L}^p(E)} \coloneqq \bigg(\int_E |f|^p \,\mathrm{d}x \bigg)^{\frac{1}{p}}.$$

The right-hand side is interpreted as the essential supremum when $p = \infty$. We abbreviate $\|\cdot\|_p \coloneqq \|\cdot\|_{\mathrm{L}^p(\mathbb{R}^n)}$. The classes of functions that are p-integrable on compact subsets of E are denoted by $\mathrm{L}^p_{\mathrm{loc}}(E)$ and carry the natural Fréchet topology.

We write $\mathrm{C}_0^\infty(O)$, where $O \subseteq \mathbb{R}^n$ is open, and $\mathcal{S}(\mathbb{R}^n)$ for the test functions with compact support and of Schwartz-type, respectively. Their topological duals are the distribution spaces $\mathcal{D}'(O)$ and $\mathcal{S}'(\mathbb{R}^n)$. The subspace $\mathcal{Z}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ is the space of Schwartz functions f whose Fourier transform $\mathcal{F}f$ satisfies $D^\alpha \mathcal{F}f(0) = 0$ for all multi-indices $\alpha \in \mathbb{N}_0^n$. The dual $\mathcal{Z}'(\mathbb{R}^n)$ can be identified with the quotient $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ is the space of polynomials on $\mathbb{R}^n$, see [92, Sec. 5.2.1].

For $p \in [1,\infty]$ the Sobolev spaces $\mathrm{W}^{1,p}(O)$ is the collection of those $f \in \mathrm{L}^p(O)$ that satisfy $\nabla_x f \in \mathrm{L}^p(O)$ in the sense of distributions. Again, there are local versions denoted by $\mathrm{W}^{1,p}_{\mathrm{loc}}(O)$.

2.2. **Tent spaces.** Tent spaces have been introduced by Coifman–Meyer–Stein in [35]. Good sources for detailed proofs are [1, 2].

For $x \in \mathbb{R}^n$ we introduce the cone with vertex x,

$$\Gamma(x) \coloneqq \{(s,y) \in \mathbb{R}^{1+n}_+ : |x-y| < s\},$$

and define the corresponding (conical) *square function* for measurable functions $F : \mathbb{R}^{1+n}_+ \to \mathbb{C}^N$ by

$$(SF)(x) \coloneqq \bigg(\iint_{\Gamma(x)} |F(s,y)|^2 \frac{\mathrm{d}s\mathrm{d}y}{s^{1+n}} \bigg)^{\frac{1}{2}} \quad (x \in \mathbb{R}^n). \tag{2.1}$$

For $\alpha \geq 0$ the Carleson functional is defined as

$$C_\alpha F(x) \coloneqq \sup_{r>0} \frac{1}{r^\alpha} \bigg(\frac{1}{r^n} \int_0^r \int_{B(x,r)} |F(t,y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t} \bigg)^{\frac{1}{2}}. \tag{2.2}$$

With a slight abuse of notation, we denote by $t^{-s}F$ the function $(t,y) \mapsto t^{-s}F(t,y)$.

Definition 2.1. Let $s \in \mathbb{R}$, $\alpha \geq 0$ and $p \in (0,\infty]$. For finite p the *tent space* $\mathrm{T}^{s,p}$ consists of all functions $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ with finite quasi-norm

$$\|F\|_{\mathrm{T}^{s,p}} := \|S(t^{-s}F)\|_p.$$

For $p = \infty$ the tent space $\mathrm{T}^{s,\infty;\alpha}$ consists of all functions $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ with finite norm

$$\|F\|_{\mathrm{T}^{s,\infty;\alpha}} := \|C_\alpha(t^{-s}F)\|_\infty.$$

Remark 2.2. For brevity we set $\mathrm{T}^p := \mathrm{T}^{0,p}$ for finite p and we abbreviate and $\mathrm{T}^{s,\infty} = \mathrm{T}^{s,\infty;0}$ with the special case $\mathrm{T}^\infty := \mathrm{T}^{0,\infty;0}$. We also note that $F \mapsto t^sF$ is an isometric isomorphism from T^p onto $\mathrm{T}^{s,p}$ and from $\mathrm{T}^{\infty;\alpha}$ onto $\mathrm{T}^{s,\infty;\alpha}$.

All tent spaces are quasi-Banach spaces (Banach when $p \geq 1$) and their topology is finer than the one on $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$. Both statements follow directly from the bounds

$$SF(x) \geq t^{-\frac{1+n}{2}} \|F\|_{\mathrm{L}^2((t,2t)\times B(x,t))}$$
$$C_\alpha F(x) \geq t^{-\frac{1+n}{2}-\alpha} \|F\|_{\mathrm{L}^2((\frac{t}{2},t)\times B(x,t))}$$

for $t > 0$ and $x \in \mathbb{R}^n$ and Fatou's lemma. Moreover, for $p < \infty$ there is a *universal approximation technique* by functions in $\mathrm{L}^2(\mathbb{R}^{1+n}_+)$ with compact support [2, Prop. 1.4]:

$$\forall F \in \mathrm{T}^{s,p}: \quad \lim_{j\to\infty} \mathbf{1}_{(j^{-1},j)\times B(0,j)}F = F \quad (\text{in } \mathrm{T}^{s,p}).$$

'Universal' refers to the fact that the same approximating sequence can be used in all tent spaces that F belongs to. Results of this type will be important for us since we shall often work with intersections of spaces. We could also change the cones $\Gamma(x)$ to

$$\Gamma_\alpha(x) := \{(s,y) \in \mathbb{R}^{1+n}_+ : |x-y| < \alpha s\}$$

for any fixed $\alpha > 0$. This *change of angle* yields equivalent tent space norms [35, Prop. 4].

If $p \in (0,\infty)$, then the (anti-)dual space of $\mathrm{T}^{s,p}$ can be identified through the L^2 duality pairing

$$\langle F, G\rangle = \iint_{\mathbb{R}^{1+n}_+} F(s,y)\cdot\overline{G(s,y)}\,\frac{\mathrm{d}s\mathrm{d}y}{s}, \tag{2.3}$$

see [2, Prop. 1.9 & Thm. 1.11]. We have

$$(\mathrm{T}^{s,p})^* = \begin{cases} \mathrm{T}^{-s,p'} & \text{if } p > 1 \\ \mathrm{T}^{-s,\infty;n(\frac{1}{p}-1)} & \text{if } p \leq 1 \end{cases}.$$

In particular, $\mathrm{T}^2 = \mathrm{L}^2(\mathbb{R}^{1+n}_+, \frac{\mathrm{d}t\mathrm{d}x}{t})$ with equivalent norms, which can also be seen directly by Fubini's theorem:

$$\int_{\mathbb{R}^n} \iint_{|x-y|<s} |F(s,y)|^2 \frac{\mathrm{d}s\mathrm{d}y}{s^{1+n}} \mathrm{d}x = \omega_n \iint_{\mathbb{R}^{1+n}_+} |F(s,y)|^2 \frac{\mathrm{d}s\mathrm{d}y}{s},$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. This technique is called *averaging trick* in the following.

We shall need one more tent space that is related to the (modified) *non-tangential maximal function*

$$\widetilde{N}_* F(x) := \sup_{t>0} \left(\fint\!\!\fint_{W(t,x)} |F(s,y)|^2 \, \mathrm{d}s\mathrm{d}y \right)^{\frac{1}{2}}, \tag{2.4}$$

where $x \in \mathbb{R}^n$ and $W(t,x) := (t/2, 2t) \times B(x,t)$ is called *Whitney box*.

Definition 2.3. Let $p \in (0,\infty)$. The tent spaces $\mathrm{T}^{0,p}_\infty$ consists of all functions $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ with finite (quasi-)norm

$$\|F\|_{\mathrm{T}^{0,p}_\infty} := \|\widetilde{N}_* F\|_p.$$

As before, these are quasi-Banach (Banach when $p \geq 1$) spaces with a topology that is stronger than $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$. Moreover, a *change of Whitney parameters* to $W(t,x) = (c_0^{-1}t, c_0 t) \times B(x, c_1 t)$ with $c_0 > 1$ and $c_1 > 0$ leads to an equivalent $\mathrm{T}^{0,p}_\infty$-norm. For the reader's convenience we reprove this fact in Section A together with further auxiliary properties of non-tangential maximal functions.

2.3. **Z-spaces.** In the context of boundary value problems these spaces emerged from the work of Barton–Mayboroda [28] under a different name. Their relation to tent spaces has been noted by Amenta [2].

For measurable functions F on $\mathbb{R}^{1+n}_+$ we introduce the *Whitney average functional*

$$W(F)(t,x) = \left(\fint\!\!\fint_{W(t,x)} |F(s,y)|^2 \, \mathrm{d}s\mathrm{d}y \right)^{\frac{1}{2}} \quad ((t,x) \in \mathbb{R}^{1+n}_+).$$

Definition 2.4. Let $s \in \mathbb{R}$ and $p \in (0,\infty]$. The Z-*space* $\mathrm{Z}^{s,p}$ consists of all functions $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ with finite quasi-norm

$$\|F\|_{\mathrm{Z}^{s,p}} := \|W(t^{-s}F)\|_{\mathrm{L}^p(\mathbb{R}^{1+n}_+, \frac{\mathrm{d}t\mathrm{d}x}{t})}.$$

All Z-spaces are quasi-Banach spaces (Banach when $p \geq 1$), their topology is finer than the one on $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ and for $p < \infty$ they have the same *universal approximation technique* as the tent spaces. This can simply be checked by hand or deduced by real interpolation since Z-spaces are the real interpolants of tent spaces, see Section 2.6 below. Many properties of tent spaces have a Z-space analog: A change of Whitney parameters leads to equivalent quasi-norms (Remark A.2), the averaging trick reveals $\mathrm{Z}^{0,2} = \mathrm{L}^2(\mathbb{R}^{1+n}_+, \frac{\mathrm{d}t\mathrm{d}x}{t}) = \mathrm{T}^{0,2}$ and the L^2 duality pairing (2.3) gives rise to

$$(\mathrm{Z}^{s,p})^* = \begin{cases} \mathrm{Z}^{-s,p'} & \text{if } p > 1 \\ \mathrm{Z}^{-s+n(\frac{1}{p}-1),\infty} & \text{if } p \leq 1 \end{cases},$$

see [3, Prop. 2.22 & Thm. 2.28].

2.4. **Hardy spaces.** For $p > 1$ we set $\mathrm{H}^p := \mathrm{L}^p$ and for $p \leq 1$ we denote by H^p the real Hardy space of Fefferman–Stein [43]. For $p = 1$ we have the continuous inclusion $\mathrm{H}^1 \subseteq \mathrm{L}^1$.

We shall exclusively work in the range $p > 1_*$ and for most of our applications it will be convenient to think of H^p-spaces in terms of atoms.

Definition 2.5. Let $p \in (1_*, 1]$ and $q \in (1, \infty]$. An L^q*-atom for* H^p is a function a supported in a cube $Q \subseteq \mathbb{R}^n$ such that $\|a\|_q \leq \ell(Q)^{\frac{n}{q}-\frac{n}{p}}$ and $\int_{\mathbb{R}^n} a \,\mathrm{d}x = 0$.

Of course we could also use balls instead of cubes in the definition. The *atomic decomposition* [89, Sec. III.3.2] states that every $f \in \mathrm{H}^p$ can be written as $f = \sum_{i=1}^\infty \lambda_i a_i$, where the sum converges unconditionally in H^p, the a_i are L^∞-atoms for H^p and the scalars λ_i satisfy

$$\|(\lambda_i)\|_{\ell^p} \lesssim \|f\|_{\mathrm{H}^p}. \tag{2.5}$$

Moreover

$$\|f\|_{\mathrm{H}^p} \simeq \inf_{f=\sum_{i=1}^\infty \lambda_i a_i} \|(\lambda_i)\|_{\ell^p}.$$

When working with operators that are defined on some space L^s, $s \in (1, \infty)$, but not on distributions, the following compatibility property will be important: If $f \in \mathrm{H}^p \cap \mathrm{L}^s$, then the series that realizes (2.5) can be taken such that it also converges in L^s. In fact, the explicit construction in [89] has this property, as carefully verified in [82].

Occasionally, we shall need that for $p \in (1_*, 1]$ smooth functions with compact support and integral zero are dense in H^p. This follows, for example, by mollification of L^∞-atoms for H^p with a smooth kernel [44, Thm. 3.33].

2.5. **Homogeneous smoothness spaces.** Good textbooks for further background are [52, 81, 84, 92]. An operator-theoretic perspective on these spaces will emerge later on in Section 8.5. All function spaces will be on $\mathbb{R}^n$ and for the sake of a clear exposition we omit this from our notation.

Let $\psi \in \mathrm{C}_0^\infty$ be supported in the annulus $\frac{1}{2} \leq |\xi| \leq 2$ and normalized to

$$\sum_{j \in \mathbb{Z}} \psi(2^j \xi) = 1 \quad (\xi \in \mathbb{R}^n \setminus \{0\})$$

and introduce for $j \in \mathbb{Z}$ the associated Littlewood–Paley operators $\Delta_j f := \mathcal{F}^{-1}(\psi(2^j \cdot)\mathcal{F} f)$. Here $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^n$. Whenever $f \in \mathcal{Z}'$, then

$$\sum_{j \in \mathbb{Z}} \Delta_j f = f \quad (\text{in } \mathcal{Z}'), \tag{2.6}$$

see [84, Prop. 2.11]. The Paley–Wiener–Schwartz theorem [62, Thm. 1.7.7] asserts that the packets $\Delta_j f$ are smooth functions of moderate growth and the general idea behind the following homogeneous smoothness spaces is to measure them in Lebesgue-type norms.

Definition 2.6. Let $s \in \mathbb{R}$ and $p \in (0, \infty]$. The homogeneous *Hardy–Sobolev space* $\dot{\mathrm{H}}^{s,p}$ when $p < \infty$ is the set of those $f \in \mathcal{Z}'$ with finite (quasi)norm

$$\|f\|_{\dot{\mathrm{H}}^{s,p}} := \| \| j \mapsto 2^{js} \Delta_j f(\cdot) \|_{\ell^2(\mathbb{Z})} \|_p.$$

The endpoint space $\dot{\mathrm{H}}^{s,\infty}$ is determined by the norm

$$\|f\|_{\dot{\mathrm{H}}^{s,\infty}} := \inf_{f = \sum_{j \in \mathbb{Z}} \Delta_j f_j} \| \| j \mapsto 2^{js} |f_j(\cdot)| \|_{\ell^2(\mathbb{Z})} \|_\infty.$$

The homogeneous *Besov space* $\dot{\mathrm{B}}^{s,p}$ is the set of those $f \in \mathcal{Z}'$ with finite (quasi)norm

$$\|f\|_{\dot{\mathrm{B}}^{s,p}} := \| \| j \mapsto 2^{js} \|\Delta_j f\|_p \|_{\ell^p(\mathbb{Z})}.$$

Remark 2.7. Within the full scale of homogeneous Besov–Triebel–Lizorkin spaces the common notation for $\dot{\mathrm{H}}^{s,p}$ and $\dot{\mathrm{B}}^{s,p}$ is $\dot{\mathrm{F}}^s_{p,2}$ and $\dot{\mathrm{B}}^s_{p,p}$, respectively.

In the following let X denote either B or H. Then $\dot{\mathrm{X}}^{s,p}$ is a quasi-Banach space (Banach when $p \geq 1$), different choices of ψ lead to equivalent (quasi)norms and there are continuous inclusions

$$\mathcal{Z} \subseteq \dot{\mathrm{X}}^{s,p} \subseteq \mathcal{Z}'.$$

Moreover, $\mathcal{Z}$ is dense in $\dot{\mathrm{X}}^{s,p}$ when $p < \infty$ via a *universal approximation technique* [92, Sec. 5.1.5]: If $\varphi \in \mathcal{S}$ is such that $\varphi(0) = 1$ and $\mathcal{F}\varphi$ is supported in $|\xi| \leq 1$, then

$$\forall f \in \dot{\mathrm{X}}^{s,p} : \quad \lim_{N \to \infty} \lim_{\delta \to 0} \Big(\varphi(\delta \, \cdot) \sum_{|j| \leq N} \Delta_j f \Big) = f \quad (\text{in } \dot{\mathrm{X}}^{s,p}).$$

'Universal' has the same meaning and purpose as for the tent spaces and the approximants are in $\mathcal{Z}$ provided that $\delta < 2^{-N-1}$.

While the ambient space $\mathcal{Z}'$ is well-suited for general considerations, applications to boundary value problems require more concrete 'realizations' of $\dot{\mathrm{X}}^{s,p}$. This issue can be resolved thanks to an observation due to Peetre [81, p. 52-56], see also [84, Sec. 2.4.3]. Suppose that $L \in \mathbb{N}_0$ is such that $L > s - {}^n/_p$ and let $\mathcal{P}_{L-1}$ be the space of polynomials of degree at most $L - 1$. Then for any $f \in \dot{\mathrm{X}}^{s,p}$ the series in

(2.6) converges in $\mathcal{S}'/\mathcal{P}_{L-1}$ and identifying f with the limit yields an isometric copy of $\dot{\mathrm{X}}^{s,p}$ that is continuously embedded into the ambient space $\mathcal{S}'/\mathcal{P}_{L-1}$. In particular, the spaces of smoothness $s < 1$ can be viewed as subspaces of $\mathcal{S}'/\mathbb{C}$ and even of $\mathcal{S}'$ if $s \leq 0$ and $p < \infty$.

Within these smaller ambient spaces, $\dot{\mathrm{X}}^{s,p}$ can often be given an equivalent and more familiar quasinorm that does not make sense modulo *all* polynomials. For example, we have the Littlewood–Paley theorem

$$\dot{\mathrm{H}}^{0,p} = \mathrm{H}^p = \mathrm{L}^p \quad (1 < p < \infty),$$
$$\dot{\mathrm{H}}^{0,p} = \mathrm{H}^p \quad (p \leq 1),$$

see [52, Sec 6.2 & 6.4] and in accordance with the observation above L^p and H^p do not contain any polynomials besides 0. For $p = \infty$ we have

$$\dot{\mathrm{H}}^{0,\infty} = \mathrm{BMO} =: \dot{\Lambda}^0, \tag{2.7}$$

see [92, Sec. 5.2.4] and references therein. Here, BMO is the John-Nirenberg space of functions modulo constants with bounded mean oscillation

$$\|f\|_{\mathrm{BMO}} \coloneqq \sup_B \fint_B |f(x) - (f)_B| \,\mathrm{d}x,$$

where the supremum is taken over all balls in $\mathbb{R}^n$. For $0 < s < 1$ we denote by $\dot{\Lambda}^s$ the Hölder space of functions modulo constants with finite norm

$$\|f\|_{\dot{\Lambda}^s} \coloneqq \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^s},$$

which can be identified with

$$\dot{\Lambda}^s = \dot{\mathrm{B}}^{s,\infty} \quad (0 < s < 1),$$

see [92, Thm. 5.2.3.2].

Next, we recall relevant duality results in the case of finite exponents p. Since in this case $\mathcal{Z}$ is dense in $\dot{\mathrm{X}}^{s,p}$, we can view the (anti-)dual space $(\dot{\mathrm{X}}^{s,p})^*$ as a subspaces of $\mathcal{Z}'$ by restricting functionals to $\mathcal{Z}$. In this sense we have

$$(\dot{\mathrm{X}}^{s,p})^* = \dot{\mathrm{X}}^{-s,p'} \quad (1 \leq p < \infty).$$

A direct proof for inhomogeneous spaces that applies *mutatis mutandis* in our homogeneous setting is given in [92, Sec. 2.11.2], see also [92, Sec. 5.2.5]. For $s = 0$ and $p = 1$ this is the famous H^1 –BMO duality of Fefferman–Stein [43]. In the case $p < 1$ we shall only need the duality

$$(\dot{\mathrm{H}}^{0,p})^* = \dot{\Lambda}^{n(\frac{1}{p}-1)} \quad (1_* < p \leq 1), \tag{2.8}$$

see [67, Thm. 4.2] or again [92, Sec. 2.11.2]. An alternative proof is given in [45, Rem. 5.14].

Spaces for different smoothness parameters are related via a *lifting property*. The Riesz potential $I_\sigma \coloneqq \mathcal{F}^{-1}(|\cdot|^{-\sigma}\mathcal{F}f)$ is an isomorphism $\dot{\mathrm{X}}^{s,p} \to \dot{\mathrm{X}}^{s+\sigma,p}$. This is proved in [92, Sec. 5.2.3] for $p < \infty$ and follows by duality for $p = \infty$, see [92, Rem. 2.3.8.2]. On the basis of (2.7) we find that

$$\dot{\mathrm{H}}^{s,\infty} = I_s(\mathrm{BMO}) =: \mathrm{B\dot{M}O}^s \quad (0 < s < 1)$$

agree up to equivalent norms with Strichartz' $\mathrm{B\dot{M}O}^s$-spaces. We have $\dot{\mathrm{H}}^{s,\infty} \subseteq \dot{\mathrm{B}}^{s,\infty}$ with continuous inclusion as a mere consequence of the definitions and the inclusion $\ell^2(\mathbb{Z}) \subseteq \ell^\infty(\mathbb{Z})$. In particular, $\dot{\mathrm{H}}^{s,\infty}$ is a space of Hölder continuous functions of exponent s. An equivalent, more concrete norm is given by

$$\|f\|_{\mathrm{B\dot{M}O}^s} \coloneqq \sup_Q \left(\frac{1}{|Q|} \int_Q \int_Q \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}, \tag{2.9}$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$, see [90, Thm. 3.3].

Together with the Mihlin multiplier theorem [92, Sec. 5.2.2/3] the lifting property also yields

$$\dot{\mathrm{H}}^{1,p} = \{ f \in \mathcal{S}'/\mathbb{C} : \nabla_x f \in \dot{\mathrm{H}}^p \}$$
$$\|f\|_{\dot{\mathrm{H}}^{1,p}} \simeq \|\nabla_x f\|_{\dot{\mathrm{H}}^p}.$$

For $p > 1$ these are the more common homogeneous Sobolev spaces and we write

$$\dot{\mathrm{W}}^{1,p} \coloneqq \dot{\mathrm{H}}^{1,p} \quad \& \quad \dot{\mathrm{W}}^{-1,p} = (\dot{\mathrm{W}}^{1,p'})^* = \dot{\mathrm{H}}^{-1,p} \quad (p > 1).$$

In our usual range of exponents $p \in (1_*, \infty)$ any distribution $f \in \dot{\mathrm{H}}^{1,p}$ can be identified with a locally integrable function. This follows by density of $\mathcal{Z}$ in $\dot{\mathrm{H}}^{1,p}$ and the extended Sobolev embedding theorem that we recall for later reference.

Proposition 2.8. *There are continuous embeddings*

$$\dot{\mathrm{H}}^{1,p} \subseteq \dot{\mathrm{H}}^{p^*} \qquad (1_* < p < n),$$
$$\dot{\mathrm{H}}^{1,p} \subseteq \dot{\Lambda}^{1-\frac{n}{p}} \quad (n \leq p < \infty).$$

The second part is the classical Morrey inequality [50, Thm. 7.17]. The first part is a special case of the general embedding theorem

$$\dot{\mathrm{X}}^{s_0,p_0} \subseteq \dot{\mathrm{X}}^{s_1,p_1} \quad (0 < p_0 < p_1 < \infty,\ s_0 - {}^{n}/_{p_0} = s_1 - {}^{n}/_{p_1}),$$

see [67, Thm. 2.1].

2.6. **Interpolation functors.** Here, and throughout the monograph, 'complex interpolation' refers to the Kalton–Mitrea complex interpolation method [69, §3], which is well-defined for quasi-Banach spaces and agrees with the classical Calderón complex interpolation method on couples of Banach spaces. As usual, we write $[\cdot\,,\cdot]_\theta$, $\theta \in (0,1)$, for the complex interpolation bracket. 'Real interpolation' refers to the classical K-method [29, Sec. 3.10] and the corresponding interpolation bracket is denoted by $(\cdot\,,\cdot)_{\theta,p}$, $\theta \in (0,1)$, $p \in (0,\infty]$.

We gather the standard interpolation formulæ that will be needed in the further course. To this end we let $0 < p_0, p_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, $\theta \in (0,1)$ and set $p \coloneqq [p_0, p_1]_\theta$, $s \coloneqq (1-\theta)s_0 + \theta s_1$.

As for tent and Z spaces, we have up to equivalent quasi-norms,

$$\begin{aligned} \big[\, \mathrm{T}^{s_0,p_0}, \mathrm{T}^{s_1,p_1} \,\big]_\theta &= \mathrm{T}^{s,p} \quad (\text{one } p_i \text{ finite}), \\ \big(\, \mathrm{T}^{s_0,p_0}, \mathrm{T}^{s_1,p_1} \,\big)_{\theta,p} &= \mathrm{Z}^{s,p} \quad (s_0 \neq s_1), \\ \big(\, \mathrm{Z}^{s_0,p_0}, \mathrm{Z}^{s_1,p_1} \,\big)_{\theta,p} &= \mathrm{Z}^{s,p} \quad (s_0 \neq s_1), \end{aligned}$$

see [3, Thm. 2.12 & Thm. 2.30 & Prop. 2.31].

The required interpolation identities for $\dot{\mathrm{X}}^{s,p}$ have been proved in [3, Thm. 4.28 & Thm. 5.2] via an approach based on tent and Z spaces. Their proof uses the language of operator-adapted spaces that will be introduced in Section 8. We have up to equivalent quasinorms

$$\begin{aligned} \big[\dot{\mathrm{H}}^{s_0,p_0}, \dot{\mathrm{H}}^{s_1,p_1}\big]_\theta &= \dot{\mathrm{H}}^{s,p} \quad (\text{one } p_i \text{ finite}), \\ \big(\dot{\mathrm{X}}^{s_0,p_0}, \dot{\mathrm{X}}^{s_1,p_1}\big)_{\theta,p} &= \dot{\mathrm{B}}^{s,p} \quad (s_0 \neq s_1). \end{aligned}$$

Different proofs for some of the identities have been given in many earlier references including [45, 68, 92].

3. Preliminaries on operator theory

A particularly useful reference for our purpose is Haase's book [53] and the reader is advised to refer thereto whenever necessary.

3.1. **Definition of the elliptic operators.** We let a and d be the coefficients of $\mathcal{L}$ as in (1.1). The bounded multiplication operator B and the first-order *Dirac operator* D are defined with maximal domain in $\mathrm{L}^2(\mathbb{R}^n; \mathbb{C}^m \times \mathbb{C}^{mn})$ by

$$B \coloneqq \begin{bmatrix} a^{-1} & 0 \\ 0 & d \end{bmatrix}, \quad D \coloneqq \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix}.$$

We note that D is self-adjoint. Hence, it splits L^2 into an orthogonal sum $\mathsf{N}(D) \oplus \overline{\mathsf{R}(D)}$. The null space $\mathsf{N}(D)$ consists of all $f = [f_\perp, f_\parallel]^\top$ with $f_\perp = 0$ and $\operatorname{div}_x f_\parallel = 0$ and the closure of the range $\overline{\mathsf{R}(D)} = \mathcal{H}$ is the space in our ellipticity assumption (1.2). Consequently, (1.2) is equivalent to

$$\operatorname{Re} \int_{\mathbb{R}^n} Bf \cdot \overline{f} \,\mathrm{d}x \geq \lambda \int_{\mathbb{R}^n} |a^{-1} f_\perp|^2 + |f_\parallel|^2 \,\mathrm{d}x \quad (f \in \overline{\mathsf{R}(D)}),$$

or again, using angular brackets to denote inner products, equivalent to

$$\operatorname{Re}\langle BDu, Du \rangle \gtrsim \|Du\|_2^2 \quad (u \in \mathsf{D}(D)). \tag{3.1}$$

Because of this, we say that B is accretive (or elliptic) on the range of D.

The *perturbed Dirac operators*

$$BD := \begin{bmatrix} 0 & a^{-1}\operatorname{div}_x \\ -d\nabla_x & 0 \end{bmatrix}, \quad DB := \begin{bmatrix} 0 & \operatorname{div}_x d \\ -\nabla_x a^{-1} & 0 \end{bmatrix} \tag{3.2}$$

are again considered with maximal domain in L^2. Since B is bounded, DB is closed and as consequence of (3.1) also BD is closed. Their squares contain the following second-order operators:

$$\begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix} := \begin{bmatrix} -a^{-1}\operatorname{div}_x d\nabla_x & 0 \\ 0 & -d\nabla_x a^{-1}\operatorname{div}_x \end{bmatrix} = (BD)^2, \tag{3.3}$$

$$\begin{bmatrix} \widetilde{L} & 0 \\ 0 & \widetilde{M} \end{bmatrix} := \begin{bmatrix} -\operatorname{div}_x d\nabla_x a^{-1} & 0 \\ 0 & -\nabla_x a^{-1}\operatorname{div}_x d \end{bmatrix} = (DB)^2. \tag{3.4}$$

The definition of L coincides with the more traditional variational approach to defining second-order operators. Indeed, the Lax–Milgram lemma provides an isomorphism,

$$\Lambda : \dot{\mathrm{W}}^{1,2} \to \dot{\mathrm{W}}^{-1,2}, \quad \langle \Lambda u, v\rangle = \int_{\mathbb{R}^n} d\nabla_x u \cdot \overline{\nabla_x v}\,\mathrm{d}x. \tag{3.5}$$

We have $\Lambda u := -\operatorname{div}_x d\nabla_x u$ in the sense of distributions and one sees that $u \in \mathsf{D}(L)$ means that $u \in \mathrm{L}^2 \cap \dot{\mathrm{W}}^{1,2}$ with $\Lambda u \in \mathrm{L}^2$ and $Lu = a^{-1}\Lambda u$. Note that the domain of L does not depend on a. Occasionally, we will write

$$L_0 := -\operatorname{div}_x d\nabla_x,$$

for the divergence form operator L in the special case $a = 1$, that is to say, the maximal restriction of Λ to an operator in L^2.

3.2. **(Bi)sectorial operators.** Statements and proofs for sectorial and bisectorial operators usually go *mutadis mutandis*. Most authors have decided to showcase sectorial operators. In case of doubt the reader can consult [42, Ch. 3], which goes the other way round.

Let $\omega \in (0, \pi)$. We define the sector $\mathrm{S}_\omega^+ := \{z \in \mathbb{C} : |\arg z| < \omega\}$ and agree on $\mathrm{S}_0^+ := (0, \infty)$. A linear operator T on a reflexive Banach space X is *sectorial* of angle $\omega \in [0, \pi)$ if its spectrum is contained in $\overline{\mathrm{S}_\omega^+}$ and if for every $\mu \in (\omega, \pi)$,

$$M_{T,\mu} := \sup_{z \in \mathbb{C}\setminus\overline{\mathrm{S}_\mu^+}} \|z(z-T)^{-1}\|_{X\to X} < \infty. \tag{3.6}$$

Usually, ω_T denotes the smallest angle ω with this property. A sectorial operator is densely defined, induces a topological kernel/range splitting

$$X = \mathsf{N}(T) \oplus \overline{\mathsf{R}(T)}, \tag{3.7}$$

and the restriction of T to $\overline{\mathsf{R}(T)}$ is sectorial, injective and has dense range [53, Prop. 2.1.1].

Bisectorial operators of angle $\omega \in [0, \pi/2)$ are defined analogously upon replacing sectors with bisectors $\mathrm{S}_\omega := \mathrm{S}_\omega^+ \cup (-\mathrm{S}_\omega^+)$ and share the same properties. If T is bisectorial of angle ω, then writing

$$(z^2 - T^2)^{-1} = -(z - T)^{-1}(-z - T)^{-1}$$

we see that T^2 is sectorial of angle 2ω. Moreover, $\mathsf{N}(T^2) = \mathsf{N}(T)$ and hence $\overline{\mathsf{R}(T^2)} = \overline{\mathsf{R}(T)}$, see [53, Prop. 2.1.1e)].

As prototypical examples, BD and DB are bisectorial of the same angle $\omega_{BD} = \omega_{DB}$ with

$$\mathsf{R}(BD) = B\,\mathsf{R}(D), \quad \mathsf{R}(DB) = \mathsf{R}(D), \tag{3.8}$$

see [9, Prop. 3.3]. From (3.3) and (3.4) we obtain that L, M, $\widetilde{L}$, $\widetilde{M}$ are sectorial of angle not larger than $2\omega_{BD}$, but possibly exceeding $\pi/2$, with

$$\overline{\mathsf{R}(L)} \times \overline{\mathsf{R}(M)} = (\mathrm{L}^2(\mathbb{R}^n; \mathbb{C}^m)) \times d\overline{\mathsf{R}(\nabla_x)} \tag{3.9}$$

$$\overline{\mathsf{R}(\widetilde{L})} \times \overline{\mathsf{R}(\widetilde{M})} = (\mathrm{L}^2(\mathbb{R}^n; \mathbb{C}^m)) \times \overline{\mathsf{R}(\nabla_x)}. \tag{3.10}$$

In particular, L and $\widetilde{L}$ have dense range and hence they are injective.

3.3. **Classes of holomorphic functions.** Let $\mu \in (0, \pi)$. The classes $\Psi_\sigma^\tau(\mathrm{S}_\mu^+)$, $\sigma, \tau \in \mathbb{R}$, consist of those holomorphic functions $\varphi : \mathrm{S}_\mu^+ \to \mathbb{C}$ that satisfy

$$|\varphi(z)| \lesssim |z|^\sigma \wedge |z|^{-\tau} \quad (z \in \mathrm{S}_\mu^+).$$

We write $\mathrm{H}^\infty(\mathrm{S}_\mu^+) := \Psi_0^0(\mathrm{S}_\mu^+)$ for the bounded holomorphic functions on S_μ^+. The classes of functions with some decay and arbitrarily large polynomial decay at 0 and ∞ are

$$\Psi_+^+(\mathrm{S}_\mu^+) := \bigcup_{\sigma,\tau>0} \Psi_\sigma^\tau(\mathrm{S}_\mu^+) \quad \text{and} \quad \Psi_\infty^\infty(\mathrm{S}_\mu^+) := \bigcap_{\sigma,\tau>0} \Psi_\sigma^\tau(\mathrm{S}_\mu^+),$$

respectively. We suppress reference to S_μ^+ in the notation when the relevant sector is clear from the context.

On bisectors we use the same notation and call a function *non-degenerate* if it does not identically vanish on one of the two connected components. An example of a degenerate function is $z + [z]$, where

$$[z] := \sqrt{z^2} \quad (z \in \mathbb{C} \setminus \mathrm{i}\mathbb{R})$$

is defined via the principal branch of the logarithm.

3.4. **Holomorphic functional calculi.** For the same reason as before, we can focus on the sectorial case. So, let T be sectorial and let $\mu \in (\omega_T, \pi)$. If ψ is of the form $\psi(z) = a + b(1+z)^{-1} + \varphi(z)$ for some $\alpha, \beta \in \mathbb{C}$ and $\varphi \in \Psi_+^+(\mathrm{S}_\mu^+)$, then $\psi(T)$ is defined as a bounded operator on X via

$$\psi(T) = \alpha + \beta(1+T)^{-1} + \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu^+} \varphi(z)(z - T)^{-1} \, \mathrm{d}z, \tag{3.11}$$

where $\nu \in (\omega_T, \mu)$, the choice of which does not matter in view of Cauchy's theorem, and $\partial \mathrm{S}_\nu^+$ is oriented such that it surrounds the spectrum of T counter-clockwise in the extended complex plane. The definition extends to larger classes of functions by *regularization*: If $e(T)$ and $(e\psi)(T)$ are already defined by the procedure above and if $e(T)$ is injective, then

$$\psi(T) := e(T)^{-1}(e\psi)(T)$$

is defined as a closed operator and can be shown not to depend on the choice of e. The expected relations

$$\psi(T) + \phi(T) \subseteq (\psi + \phi)(T)$$
$$\psi(T)\phi(T) \subseteq (\psi\phi)(T)$$

hold and there is equality if $\psi(T)$ is bounded.

Since the restriction of T to $\overline{\mathsf{R}(T)}$ is an injective sectorial operator, $e(z) = z(1+z)^{-2}$ regularizes any bounded holomorphic function in $\mathrm{H}^\infty(\mathrm{S}_\mu^+)$. The *convergence lemma* states that if $(\psi_j)_j$ is a bounded sequence in $\mathrm{H}^\infty(\mathrm{S}_\mu^+)$ that converges pointwise to ψ, then

$$\psi(T) = \lim_{j \to \infty} \psi_j(T) \tag{3.12}$$

in the sense of strong convergence on $\overline{\mathsf{R}(T)}$.

We say that T has a *bounded* H^∞*-calculus on* $\overline{\mathsf{R}(T)}$ (of angle $\mu \geq \omega_T$) if for all $\nu \in (\mu, \pi)$ there is a constant $M_{T,\nu}^\infty$ such that

$$\|\psi(T)\|_{\overline{\mathsf{R}(T)} \to \overline{\mathsf{R}(T)}} \leq M_{T,\nu}^\infty \|\psi\|_{\mathrm{L}^\infty(\mathrm{S}_\nu^+)} \quad (\psi \in \mathrm{H}^\infty(\mathrm{S}_\nu^+)). \tag{3.13}$$

In fact, by the convergence lemma, it suffices to have the bound for all $\psi \in \Psi_+^+(\mathrm{S}_\nu^+)$. In Hilbert spaces, these properties are independent of the angle μ. This is one of the statements of the following fundamental result due to McIntosh [77], see also [53, Thm. 7.3.1]. The dependence of the implicit constants easily follows from the proof and is also explicitly stated in [64, Thm. 10.4.16/19].

Theorem 3.1 (McIntosh)**.** *Let T be a (bi)sectorial operator in a Hilbert space H. Then T has a bounded H^∞-calculus of some angle on $\overline{\mathsf{R}(T)}$ (equivalently, of angle ω_T) if and only if the quadratic estimate*

$$\|f\|_H \simeq \left(\int_0^\infty \|\varphi(tT)f\|_H^2 \frac{\mathrm{d}t}{t} \right)^{1/2}$$

holds for all $f \in \overline{\mathsf{R}(T)}$ and some (equivalently, all) admissible and non-degenerate $\varphi \in \Psi_+^+$.

For fixed angle ν, the bound $M_{T,\nu}^\infty$ for the H^∞-calculus depends on $M_{T,\mu}$ for some $\mu \in (\omega_T, \nu)$ and implicit constants in the quadratic estimates, and vice versa.

We also recall the important reproducing formula for sectorial operators [53, Thm. 5.2.6] and remark that up to the usual modifications there is a bisectorial version [22, Prop. 4.2].

Lemma 3.2 (Calderón reproducing formula)**.** *Let T be a sectorial operator in a reflexive Banach space X and let $\varphi \in \Psi_+^+$ on a suitable sector be such that $\int_0^\infty \varphi(t) \frac{\mathrm{d}t}{t} = 1$. Then*

$$\int_0^\infty \varphi(tT) f \, \frac{\mathrm{d}t}{t} = f \quad (f \in \overline{\mathsf{R}(T)})$$

as an improper strong Riemann integral.

Remark 3.3. For any non-zero $\phi \in \mathrm{H}^\infty$ there is $\psi \in \Psi_\infty^\infty$ on the same sector such that $\varphi := \phi\psi$ satisfies the Calderón reproducing formula, for example $\psi(z) := c\overline{\phi(\overline{z})}\mathrm{e}^{-z-1/z}$, where $c^{-1} = \int_0^\infty |\phi(t)|^2 \mathrm{e}^{-t-1/t} \frac{\mathrm{d}t}{t}$.

Coming back to concrete operators, quadratic estimates (and hence bounded functional calculi) for BD and DB is a deep result due to Axelsson–Keith–McIntosh [25]. For a condensed proof, see also [8, Thm. 1.1]

Theorem 3.4 (Axelsson–Keith–McIntosh)**.** *The operators BD and DB have bounded H^∞-calculi on the closure of their ranges.*

Let now $\mu \in (2\omega_{BD}, \pi)$ and $\psi \in \Psi_+^+(\mathrm{S}_\mu^+)$. Then φ defined by $\varphi(z) := \psi(z^2)$ belongs to $\Psi_+^+(\mathrm{S}_{\mu/2})$. From (3.3) we obtain

$$\begin{bmatrix} \psi(L) & 0 \\ 0 & \psi(M) \end{bmatrix} = \psi((BD)^2) = \varphi(BD). \tag{3.14}$$

The same argument works for $\widetilde{L}$, $\widetilde{M}$ by referring to DB instead. McIntosh's theorem implies the following

Corollary 3.5. *The operators L and $\widetilde{L}$ have bounded H^∞-calculi on L^2. Likewise, M and $\widetilde{M}$ have bounded H^∞-calculi on the closure of their ranges.*

Since B is accretive on $\mathcal{H} = \overline{\mathsf{R}(DB)}$ and maps this space onto $B\mathcal{H} = \overline{\mathsf{R}(BD)}$, it follows that $B|_\mathcal{H} : \overline{\mathsf{R}(DB)} \to \overline{\mathsf{R}(BD)}$ is invertible and that the restrictions of BD and DB to the closure of their ranges are similar under conjugation with $B|_\mathcal{H}$. Therefore

$$\varphi(BD)B = B\varphi(DB) \tag{3.15}$$

holds as unbounded operators from $\overline{\mathsf{R}(DB)}$ into $\overline{\mathsf{R}(BD)}$, whenever one side is defined by the respective functional calculus. Elaborating further along these line, we obtain

Lemma 3.6 (Intertwining relations)**.** *Let $\varphi \in \mathrm{H}^\infty$ on a suitable bisector and $\psi \in \mathrm{H}^\infty$ on a suitable sector. Then*

$$D\varphi(BD)f = \varphi(DB)Df \quad (f \in \mathsf{D}(D))$$

and

$$\operatorname{div}_x \psi(M) f_\parallel = \psi(\widetilde{L}) \operatorname{div}_x f_\parallel, \quad (f_\parallel \in \mathsf{D}(\operatorname{div}_x)),$$
$$-\nabla_x \psi(L) f_\perp = \psi(\widetilde{M}) \nabla_x f_\perp, \quad (f_\perp \in \mathrm{W}^{1,2}).$$

Proof. For the first identity we note that $Df \in \mathsf{R}(D) = \mathsf{R}(DB)$ by (3.8). Hence, we can apply (3.15) to Df in order to obtain

$$BD\varphi(BD)f = B\varphi(DB)Df$$

and the claim follows since B is accretive on $\overline{\mathsf{R}(D)}$. By means of (3.14) and the analogous identity for DB the identities for L and M follow. □

3.5. **Adjoints.** We note that the adjoint of a (bi)sectorial operator in a Hilbert space is again bisectorial of the same angle [53, Prop. 2.1.1] and that B^* has the same properties as B. Since B is bounded, we have $(BD)^* = DB^*$ and likewise $(B^*D)^* = DB$, which yields $(DB)^* = B^*D$ because B^*D is closed. Since all these operators are bisectorial, we obtain $((BD)^2)^* = (DB^*)^2$, which in matrix form reads

$$\begin{bmatrix} L^* & 0 \\ 0 & M^* \end{bmatrix} = \begin{bmatrix} -\operatorname{div}_x d^* \nabla_x (a^*)^{-1} & 0 \\ 0 & -\nabla_x (a^*)^{-1} \operatorname{div}_x d^* \end{bmatrix}. \tag{3.16}$$

The Ψ_+^+-calculus of any (bi)sectorial operator dualizes in the expected manner $\psi(T)^* = \psi^*(T^*)$, where $\psi^*(z) = \overline{\psi(\overline{z})}$. If T has dense range, for example $T = L$, then this relation also holds for all $\psi \in \cup_{\sigma,\tau \in \mathbb{R}} \Psi_\sigma^\tau$, see [53, Prop. 7.0.1(d)]. When $a = 1$, the operator L^* is in the same class as L. When $a \neq 1$, the operator L^* is not in the same class as L but is similar to such an operator under conjugation with a^*. This is why instead of L^* we usually work with

$$L^\sharp := -(a^*)^{-1} \operatorname{div}_x d^* \nabla_x = (a^*)^{-1} L^* a^*$$

when it comes to duality arguments.

3.6. **Kato problem and Riesz transform.** Since $z \mapsto [z]/z$ and its inverse are bounded and holomorphic on any bisector, the bounded H^∞-calculus for BD entails that BD and $[BD]$ share the same domain along with comparability

$$\|BDf\|_2 \simeq \|[BD]f\|_2 \quad (f \in \mathsf{D}(BD)).$$

The left-hand side is also comparable to $\|Df\|_2$ by ellipticity. Looking at the first component and using the specific form of BD and its square, see (3.2) and (3.3), we obtain the resolution of the *Kato conjecture.*

Theorem 3.7 (Resolution of the Kato conjecture)**.** *It follows that* $\mathsf{D}(L^{1/2}) = \mathrm{W}^{1,2}$ *with the homogeneous estimate* $\|L^{1/2} f\|_2 \simeq \|\nabla_x f\|_2$.

As a consequence, we obtain a bounded extension $L^{1/2} : \dot{\mathrm{W}}^{1,2} \to \mathrm{L}^2$ by density that is injective with closed range. It is an isomorphism since its range contains $\mathsf{R}(L)$, which is dense in L^2 by (3.9). We denote its inverse by $L^{-1/2}$. In particular, the *Riesz transform* $\nabla_x L^{-1/2}$ is a bounded operator on L^2.

The domains of fractional powers of exponent $\alpha \in (0, 1/2)$ can be determined by complex interpolation.

Corollary 3.8. *If $\alpha \in (0, 1/2)$, then $\mathsf{D}(L^\alpha) = \dot{\mathrm{H}}^{2\alpha,2} \cap \mathrm{L}^2$ with the homogeneous estimate $\|L^\alpha f\|_2 \simeq \|f\|_{\dot{\mathrm{H}}^{2\alpha,2}}$.*

Proof. By [15, Thm. 5.1] we have $\mathsf{D}(L^\alpha) = [\mathrm{L}^2, \dot{\mathrm{W}}^{1,2}]_{2\alpha} \cap \mathrm{L}^2$ with the homogeneous estimate $\|L^\alpha f\|_2 \simeq \|f\|_{[\mathrm{L}^2,\dot{\mathrm{W}}^{1,2}]_{2\alpha}}$ and from Section 2.6 we know that $[\mathrm{L}^2, \dot{\mathrm{W}}^{1,2}]_{2\alpha} = \dot{\mathrm{H}}^{2\alpha,2}$. □

3.7. **Off-diagonal estimates.** We develop on these estimates in Section 4 below. Here we only gather the well-known L^2-bounds for our standard operators from Section 3.2.

Definition 3.9. Let $\Omega \subseteq \mathbb{C} \setminus \{0\}$ and let V_1, V_2 be finite-dimensional Hilbert spaces. A family $(T(z))_{z\in\Omega}$ of linear operators $\mathrm{L}^2(\mathbb{R}^n; V_1) \to \mathrm{L}^2(\mathbb{R}^n; V_2)$ satisfies L^2 *off-diagonal estimates of order* $\gamma > 0$ if there exists a constant C such that

$$\|\mathbf{1}_F T(z) \mathbf{1}_E f\|_2 \leq C\Big(1 + \frac{\mathrm{d}(E,F)}{|z|}\Big)^{-\gamma} \|\mathbf{1}_E f\|_2$$

holds for all measurable subsets $E, F \subseteq \mathbb{R}^n$, all $z \in \Omega$ and all $f \in \mathrm{L}^2(\mathbb{R}^n; V_1)$. It there are constants $C, c > 0$ such that the stronger estimate

$$\|\mathbf{1}_F T(z) \mathbf{1}_E f\|_2 \leq C\mathrm{e}^{-c\frac{\mathrm{d}(E,F)}{|z|}} \|\mathbf{1}_E f\|_2$$

holds, then the family is said to satisfies *off-diagonal estimates of exponential order*.

While decay of polynomial order is most suitable for the abstract theory that we develop in the upcoming sections, our prototypes actually satisfy the exponential estimate. For completeness, we include the argument from [8, Prop. 5.1].

Proposition 3.10. *The resolvent families $((1 + \mathrm{i}tBD)^{-1})_{t\in\mathbb{R}\setminus\{0\}}$ and $((1 + \mathrm{i}tDB)^{-1})_{t\in\mathbb{R}\setminus\{0\}}$ satisfy L^2 off-diagonal estimates of exponential order.*

Proof. We begin with the resolvents of $T(t) := ((1 + \mathrm{i}tBD)^{-1}$. Fix t, E, F and set $\mathrm{d} := \mathrm{d}(E,F)$. The family $(T(t))_{t\in\mathbb{R}\setminus\{0\}}$ is uniformly bounded in L^2 since BD is bisectorial. Hence, it suffices to obtain the exponential estimate for $|t| \leq \alpha\mathrm{d}$, where $\alpha > 0$ will be chosen later on in dependence of dimensions and ellipticity.

We introduce $G := \{x \in \mathbb{R}^n : \mathrm{d}(x, F) \leq {}^{\mathrm{d}}/_2\}$. As $\mathrm{d}(F, {}^cG) \geq {}^{\mathrm{d}}/_2$, we can pick a smooth function φ that satisfies $\mathbf{1}_F \leq \varphi \leq \mathbf{1}_G$ and $\|\nabla_x \varphi\|_\infty \leq {}^C/_{\mathrm{d}}$ for some dimensional constant C. Let $\eta := \mathrm{e}^{(\alpha \mathrm{d}/|t|)\varphi} - 1$ and observe that

$$\eta = 0 \quad (\text{on } E) \quad \text{and} \quad \eta = \mathrm{e}^{\frac{\alpha \mathrm{d}}{|t|}} - 1 \geq \frac{1}{2}\mathrm{e}^{\frac{\alpha \mathrm{d}}{|t|}} \quad (\text{on } F).$$

Thus, we obtain for all $f \in \mathrm{L}^2$ that

$$\frac{1}{2}\mathrm{e}^{\frac{\alpha \mathrm{d}}{|t|}} \|\mathbf{1}_F T(t) \mathbf{1}_E f\|_2 \leq \|\eta T(t) \mathbf{1}_E f\|_2 = \|[\eta, T(t)]\mathbf{1}_E f\|_2, \tag{3.17}$$

where $[\eta, T(t)] = \eta T(t) - T(t)(\eta\,\cdot)$ is the commutator between $T(t)$ and multiplication with η and we have used $\eta \mathbf{1}_E = 0$. Next, we expand

$$[\eta, T(t)] = T(t)[1 + \mathrm{i}tBD, \eta]T(t) = \mathrm{i}tT(t)B[D, \eta]T(t). \tag{3.18}$$

By the product rule we find that $[D, \eta]$ acts via multiplication by a function $\theta \mathrm{e}^{(\alpha \mathrm{d}/|t|)\varphi}$, where θ is supported in G and uniformly bounded by a dimensional multiple of ${}^{\alpha \mathrm{d}}/_{|t|}\|\nabla \varphi\|_\infty \leq {}^{C\alpha}/_{|t|}$. Since $T(t)$ and B are (uniformly) bounded on L^2, we conclude that

$$\begin{aligned}\|\eta T(t)\mathbf{1}_E f\|_2 &\leq C\alpha \big\|\mathrm{e}^{\frac{\alpha \mathrm{d}}{|t|}\varphi} T(t)\mathbf{1}_E f\big\|_2 \\ &\leq C\alpha\big(\|\eta T(t)\mathbf{1}_E f\|_2 + \|T(t)\mathbf{1}_E f\|_2\big),\end{aligned}$$

where C depends on ellipticity and dimension and the second step merely follows from $\eta = \mathrm{e}^{(\alpha \mathrm{d}/|t|)\varphi} - 1$. Setting $\alpha := {}^1/_{2C}$, we can absorb the first term on the right-hand side back into the left-hand side and we are left with

$$\|\eta T(t)\mathbf{1}_E f\|_2 \leq \frac{1}{2}\|T(t)\mathbf{1}_E f\|_2.$$

Using (3.17) on the left and uniform boundedness of $T(t)$ on the right completes the proof for the resolvents of BD.

For DB the only modification in the argument concerns (3.18), where B appears on the right of $[D, \eta]$. □

Remark 3.11. The off-diagonal estimates extend to complex parameters $t = z \in \mathrm{S}_\mu$ for any $\mu \in (0, {}^\pi/_2 - \omega_{BD})$. The proof is literally the same but it is also instructive to remark that one can use Stein interpolation against the uniform resolvent bounds. This argument appears in greater generality in Lemma 4.13 below.

Corollary 3.12. *The following families $(T(z))_{z \in \mathrm{S}_\mu^+}$ satisfy off-diagonal estimates of exponential order:*

(i) $T(z) = (1 + z^2 T)^{-1}$ *if* $\mu \in (0, {}^{(\pi - \omega_T)}/_2)$ *and* $T \in \{L, \widetilde{L}, M, \widetilde{M}\}$.
(ii) $T(z) = z\nabla_x (1 + z^2 L)^{-1}$ *if* $\mu \in (0, {}^{(\pi - \omega_L)}/_2)$.

In particular, these families satisfy L^2 *off-diagonal estimates of arbitrarily large order.*

Proof. By Stein interpolation, see the preceding remark, it suffices to argue for $z \in \mathbb{R} \setminus \{0\}$. Thanks to Proposition 3.10 we have off-diagonal estimates of exponential order for

$$\frac{1}{2}\Big((1+\mathrm{i}zBD)^{-1} + (1-\mathrm{i}zBD)^{-1}\Big) = (1+z^2(BD)^2)^{-1},$$

as well as for the corresponding family with DB replacing BD. Thus, (i) follows from (3.3) and (3.4). Similarly, we have

$$\frac{1}{2}\Big((1+\mathrm{i}zDB)^{-1} - (1-\mathrm{i}zDB)^{-1}\Big) = \begin{bmatrix} -\mathrm{i}z \operatorname{div}_x d(1+z^2\widetilde{M})^{-1} \\ \mathrm{i}z\nabla_x a^{-1}(1+z^2\widetilde{L})^{-1} \end{bmatrix}$$

and we obtain the required off-diagonal estimates for

$$z\nabla_x a^{-1}(1+z^2\widetilde{L})^{-1} = z\nabla_x(1+z^2L)^{-1}a^{-1}$$

as stated in (ii). □

4. $\mathrm{H}^p - \mathrm{H}^q$ BOUNDED FAMILIES

In this section we discuss general principles for $\mathrm{H}^p - \mathrm{H}^q$-bounded operator families. We provide a toolbox that will allow us to manipulate resolvent families associated with our first and second-order operators efficiently on an abstract level.

4.1. **Abstract principles.**

Throughout we work under the following assumption unless stated otherwise:

(4.1)
- $(T(z))_{z\in\Omega}$ is a family of bounded operators $\mathrm{L}^2(\mathbb{R}^n; V_1) \to \mathrm{L}^2(\mathbb{R}^n; V_2)$ indexed over some set $\Omega \subseteq \mathbb{C} \setminus \{0\}$, where the V_i are finite-dimensional Hilbert spaces,
- $a_i \in \mathrm{L}^\infty(\mathbb{R}^n; \mathcal{L}(V_i))$, $i = 1, 2$, are such that $a_i(x)$ is invertible for a.e. x and $a_i^{-1} \in \mathrm{L}^\infty(\mathbb{R}^n; \mathcal{L}(V_i))$.

Definition 4.1. Let $(T(z))_{z\in\Omega}$ be an operator family as in (4.1) and let $0 < p \le q < \infty$. This family is *$a_1\,\mathrm{H}^p - a_2\,\mathrm{H}^q$-bounded* if

$$\|a_2^{-1}T(z)a_1 f\|_{\mathrm{H}^q} \lesssim |z|^{\frac{n}{q}-\frac{n}{p}}\|f\|_{\mathrm{H}^p} \quad (z \in \Omega, f \in \mathrm{H}^p \cap \mathrm{L}^2). \tag{4.2}$$

Usually, Ω is a half-line, a sector or a bisector in our application, hence the follow-up on the scaling in (4.2).

Remark 4.2. (i) We omit Ω and simply write $(T(z))$ when the context is clear. We speak of *$a\,\mathrm{H}^p$-boundedness* when $a_1 = a_2 = a$ and $p = q$. If $q > 1$, then multiplication by a_2 is an automorphism of $\mathrm{H}^q = \mathrm{L}^q$ and hence a_2 may be dropped on the left-hand side of (4.2). We simply speak of *$a_1\,\mathrm{H}^p - \mathrm{L}^q$-boundedness*. If also $p > 1$, then a_1 may be dropped as well and we speak of *$\mathrm{L}^p - \mathrm{L}^q$-boundedness* (*$\mathrm{L}^p$-boundedness* if $p = q$).

(ii) Occasionally, we shall use the following extensions to the notions above. First, we can include endpoint Lebesgue spaces for $a_1 \mathrm{H}^p - \mathrm{L}^q$, $q \in \{1, \infty\}$, and $\mathrm{L}^p - \mathrm{L}^q$-boundedness, $p, q \in \{1, \infty\}$. Second, when $0 < p < \infty$ and $0 \leq \alpha < 1$, we speak of $a_1 \mathrm{H}^p - a_2 \dot{\Lambda}^\alpha$-boundedness if

$$\|a_2^{-1} T(z) a_1 f\|_{\dot{\Lambda}^\alpha} \lesssim |z|^{-\alpha - \frac{n}{p}} \|f\|_{\mathrm{H}^p} \quad (z \in \Omega, f \in \mathrm{H}^p \cap \mathrm{L}^2)$$

and make the same kind of notational abbreviations and extensions as before.

Since the Hardy spaces interpolate by the complex method and have a universal approximation technique, the notion of $a_1 \mathrm{H}^p - a_2 \mathrm{H}^q$-boundedness interpolates as well. Moreover, the notions 'dualize' in the expected way as the next lemma shows.

Lemma 4.3. *Let* $(T(z))$ *be as in* (4.1)*.*

(i) *If* $1 \leq p \leq q \leq \infty$*, then* $(T(z))$ *is* $\mathrm{L}^p - \mathrm{L}^q$*-bounded if and only if* $(T(z)^*)$ *is* $\mathrm{L}^{q'} - \mathrm{L}^{p'}$*-bounded.*

(ii) *If* $1_* < p \leq 1 \leq q \leq \infty$*, then* $(T(z))$ *is* $a_1 \mathrm{H}^p - \mathrm{L}^q$*-bounded if and only if* $(T(z)^*)$ *is* $\mathrm{L}^{q'} - (a_1^*)^{-1} \dot{\Lambda}^{n(\frac{1}{p} - 1)}$*-bounded.*

Proof. We can assume $a_1 = 1$ and $a_2 = 1$ — otherwise we replace $(T(z))$ by $(a_2^{-1} T(z) a_1)$. All of the claims take the abstract form that one of $(T(z))$ and $(T(z)^*)$ is $X_1 - X_2$-bounded and the other one should be $X_3 - X_4$-bounded. As

$$\langle T(z) f, g \rangle = \langle f, T(z)^* g \rangle \quad (z \in \Omega,\ f, g \in \mathrm{L}^2),$$

it suffices to know that the X_4-norm can be computed by testing against functions in $X_1 \cap \mathrm{L}^2$. Above, either X_1 is a Hardy or Lebesgue space and X_4 is its dual (so the claim follows since $X_1 \cap \mathrm{L}^2$ is dense in X_1) or $X_1 = \mathrm{L}^\infty$ and $X_4 = \mathrm{L}^1$ (and the claim follows by testing against characteristic functions of bounded sets). □

The next lemma provides us with a useful criterion for a family to map a given H^q-space back into $\mathrm{H}^2 = \mathrm{L}^2$.

Lemma 4.4. *Let* $(T(z))$ *be a family as in* (4.1) *with* $V_1 = V_2 =: V$ *and* $a_1 = a_2 =: a$*. Suppose that there exist* $p, \varrho \in (0, 2)$ *for which* $(T(z))$ *is* $a \mathrm{H}^p - a \mathrm{H}^p$ *and* $a \mathrm{H}^\varrho - \mathrm{L}^2$*-bounded. Then, for each* $q \in (p, 2)$*, there exists an integer* $\beta = \beta(p, q, \varrho)$ *such that* $(T^\beta(z))$ *is* $a \mathrm{H}^q - \mathrm{L}^2$*-bounded.*

Proof. If $\varrho \leq p$, then we can simply interpolate and take $\beta = 1$. Henceforth, we assume $p < \varrho$.

Consider a $(1/s, 1/t)$-plane as in Figure 4 where $(1/s, 1/t)$ is marked provided $(T(z))$ is $a \mathrm{H}^s - a \mathrm{H}^t$-bounded. The initial configuration are the vertices $A = (1/p, 1/p)$, $B = (1/2, 1/2)$ and $C = (1/\varrho, 1/2)$. By interpolation, we obtain their convex hull, that is to say, the closed triangle ABC.

Boundedness properties for $(T^2(z))$ are visualized in Figure 4 as follows: Take a point $X = (1/s, 1/t)$ on AC, move to AB on a horizontal line, then move to AC on a vertical line and call that point $X' = (1/t, 1/t')$. Then $(T^2(z))$ is $a\,\mathrm{H}^s - a\,\mathrm{H}^{t'}$-bounded.

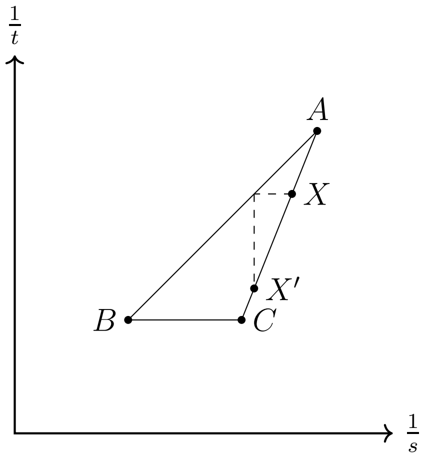

FIGURE 4. Visualization of the proof of Lemma 4.4.

If $1/q \leq 1/\varrho$, then ABC contains the point $(1/q, 1/2)$ and we can take $\beta = 1$. Otherwise, the segment AC contains at least one point X_0 with abscissa $1/q$. Starting from there, we construct $X_\beta \coloneqq (X_{\beta-1})'$ as above. After a finite number $\beta(p, q, \varrho)$ of steps X_β lies on the segment BC with constant ordinate $1/2$. Hence $(T^\beta(z))$ is $a\,\mathrm{H}^q - a\,\mathrm{H}^2$-bounded, that is, $a\,\mathrm{H}^q - \mathrm{L}^2$-bounded. □

4.2. **Off-diagonal estimates.** For Lebesgue spaces we shall make extensive use of off-diagonal estimates.

Definition 4.5. Let $1 \leq p \leq q \leq \infty$. An operator family $(T(z))_{z \in \Omega}$ as in (4.1) satisfies $\mathrm{L}^p - \mathrm{L}^q$ *off-diagonal estimates* of order $\gamma > 0$ if

$$\|\mathbf{1}_F T(z) \mathbf{1}_E f\|_q \lesssim |z|^{\frac{n}{q} - \frac{n}{p}} \left(1 + \frac{\mathrm{d}(E, F)}{|z|}\right)^{-\gamma} \|\mathbf{1}_E f\|_p$$

for all measurable subsets $E, F \subseteq \mathbb{R}^n$, all $z \in \Omega$ and all $f \in \mathrm{L}^p \cap \mathrm{L}^2$. If there are is a constant $c > 0$ such that the stronger estimate

$$\|\mathbf{1}_F T(z) \mathbf{1}_E f\|_q \lesssim |z|^{\frac{n}{q} - \frac{n}{p}} \mathrm{e}^{-c \frac{\mathrm{d}(E,F)}{|z|}} \|\mathbf{1}_E f\|_p$$

holds, then the family is said to satisfies *off-diagonal estimates of exponential order*.

As usual, we shall simply speak of L^p *off-diagonal estimates* when $p = q$. For $p = q = 2$ this notion is consistent with Definition 3.9. Duality for Lebesgue spaces yields the principle that $(T(z))$ satisfies $\mathrm{L}^p - \mathrm{L}^q$ off-diagonal estimates of order γ (resp. of exponential order) if and only if $(T(z)^*)$ satisfies $\mathrm{L}^{q'} - \mathrm{L}^{p'}$ off-diagonal estimates of order γ (resp. of exponential order). As for composition of off-diagonal estimates, we have the following rule.

Lemma 4.6. *Let $1 \leq p \leq q \leq \infty$. Let $(T(z))$ and $(S(z))$ be families as in (4.1) that are compatible in the sense that $(S(z)T(z))$ is defined. Suppose that they satisfy $\mathrm{L}^p - \mathrm{L}^q$ and $\mathrm{L}^q - \mathrm{L}^r$ off-diagonal estimates of orders γ_S and γ_T, respectively. Then $(S(z)T(z))$ satisfies $\mathrm{L}^p - \mathrm{L}^r$ off-diagonal estimates of order $\gamma_S \wedge \gamma_T$. If the order is exponential for both families, then the same is true for the composition.*

Proof. Given $E, F \subseteq \mathbb{R}^n$, we put $\mathrm{d} := \mathrm{d}(E,F)$ and define $G := \{x \in \mathbb{R}^n : \mathrm{d}(x,E) \leq \mathrm{d}/2\}$. Since we have $\mathrm{d}(E, {}^c G) \geq \mathrm{d}/2$ and $\mathrm{d}(F,G) \geq \mathrm{d}/2$, the claim follows on splitting

$$\mathbf{1}_F S(z) T(z) \mathbf{1}_E = \mathbf{1}_F S(z) \mathbf{1}_G T(z) \mathbf{1}_E + \mathbf{1}_F S(z) \mathbf{1}_{{}^c G} T(z) \mathbf{1}_E$$

and applying $\mathrm{L}^p - \mathrm{L}^q$ and $\mathrm{L}^q - \mathrm{L}^r$ off-diagonal estimates. □

Taking $E = F = \mathbb{R}^n$, we see that $\mathrm{L}^p - \mathrm{L}^q$ off-diagonal estimates are a stronger notion than $\mathrm{L}^p - \mathrm{L}^q$-boundedness, but more is true. This is well-known but we include a proof for convenience.

Lemma 4.7. *Let $1 \leq p \leq q \leq \infty$. If an operator family $(T(z))$ as in (4.1) satisfies $\mathrm{L}^p - \mathrm{L}^q$ off-diagonal estimates of order $\gamma > n$, then it is L^q-bounded and L^p-bounded.*

Proof. If $p = \infty$, then $q = \infty$, and L^∞ off-diagonal estimates imply L^∞-boundedness. From now on we may assume $p < \infty$.

Let $f \in \mathrm{L}^p$. For fixed z, we partition $\mathbb{R}^n$ into closed, axis-parallel cubes $\{Q_k\}_{k \in \mathbb{Z}^n}$ of sidelength $|z|$ with center $|z|k$. From Hölder's inequality and the assumption we obtain

$$\begin{aligned} \|T(z)f\|_p^p &= \sum_{k \in \mathbb{Z}^n} \|\mathbf{1}_{Q_k} T(z) f\|_p^p \\ &\leq |z|^{n - \frac{np}{q}} \sum_{k \in \mathbb{Z}^n} \|\mathbf{1}_{Q_k} T(z) f\|_q^p \\ &\leq |z|^{n - \frac{np}{q}} \sum_{k \in \mathbb{Z}^n} \bigg(\sum_{j \in \mathbb{Z}^n} \|\mathbf{1}_{Q_k} T(z) \mathbf{1}_{Q_j} f\|_q \bigg)^p \\ &\leq \sum_{k \in \mathbb{Z}^n} \bigg(\sum_{j \in \mathbb{Z}^n} C \Big(1 + \frac{\mathrm{d}(Q_j, Q_k)}{|z|} \Big)^{-\gamma} \|\mathbf{1}_{Q_j} f\|_p \bigg)^p. \end{aligned}$$

Let $|\cdot|_\infty$ be the ℓ^∞-norm on $\mathbb{R}^n$ and d_∞ the corresponding distance. Then

$$\mathrm{d}(Q_j, Q_k) \geq \mathrm{d}_\infty(Q_j, Q_k) = |z| \max\{|j - k|_\infty - 1, 0\}.$$

Young's convolution inequality yields

$$\begin{aligned}\|T(z)f\|_p^p &\leq \sum_{k\in\mathbb{Z}^n}\bigg(\sum_{j\in\mathbb{Z}^n} C\big(1+|j-k|_\infty\big)^{-\gamma}\|\mathbf{1}_{Q_j}f\|_p\bigg)^p \\ &\leq C\bigg(\sum_{k\in\mathbb{Z}^n}\big(1+|k|_\infty\big)^{-\gamma}\bigg)^p\bigg(\sum_{j\in\mathbb{Z}^n}\|\mathbf{1}_{Q_j}f\|_p^p\bigg).\end{aligned}$$

The sum in k converges since for fixed $m\in\mathbb{N}$ there are $(2m+1)^n - (2m-1)^n = \mathcal{O}(m^{n-1})$ points $k\in\mathbb{Z}^n$ with $|k|_\infty = m$. The sum in j equals $\|f\|_p^p$. This proves the L^p-boundedness of $(T(z))$.

The same argument applies to the dual family, which satisfies $\mathrm{L}^{q'} - \mathrm{L}^{p'}$ off-diagonal estimates of order γ. This yields L^q-boundedness of $(T(z))$. □

Remark 4.8. Re-examining the above proof reveals that $(T(z))$ even satisfies L^p and L^q off-diagonal estimates, both of order $\gamma - n$, and that the order is exponential provided that this is the case for the $\mathrm{L}^p - \mathrm{L}^q$ off-diagonal estimates.

Indeed, assume that f is supported in a set E and that the L^p-norm is taken on a set F with $\mathrm{d} := \mathrm{d}_\infty(E,F) \geq 4|z|$. All cubes Q_k and Q_j that are necessary to cover E and F, respectively, satisfy $2|k-j|_\infty|z| \geq \mathrm{d}$. Consequently, we only need to sum over $k\in\mathbb{Z}^n$ with $|k| \geq {}^{\mathrm{d}}/_{2|z|}$ in the final estimate. This sum is dominated by a multiple of $(1+{}^{\mathrm{d}}/_{|z|})^{-\gamma+n}$. If the order for the $\mathrm{L}^p - \mathrm{L}^q$ off-diagonal estimates is exponential, then we would sum over $\mathrm{e}^{-c|k|_\infty}$ and get control by $\mathrm{e}^{-\frac{c\mathrm{d}}{2|z|}}$. By duality, the same conclusions are true on L^q.

The previous lemma provides a means to obtain uniform boundedness in one space from sufficient decay between different spaces. We also need a result of this type for $p < 1$.

Lemma 4.9. *Let $(T(z))$ be an operator family as in (4.1). Suppose that $\varrho \in (1_*, 1)$ and $q \in (1,\infty)$ are such that $(T(z))$ is $a_1\,\mathrm{H}^\varrho - \mathrm{L}^q$-bounded and satisfies L^q off-diagonal estimates of arbitrarily large order. In addition assume $\int_{\mathbb{R}^n}((a_2)^{-1}T(z)a_1 f)(x)\,\mathrm{d}x = 0$ for all z and all compactly supported $f \in \mathrm{L}^2$ with integral zero. Then $(T(z))$ is $a_1\,\mathrm{H}^p - a_2\,\mathrm{H}^p$-bounded for every $p\in(\varrho,1]$.*

Proof. We can assume $a_1 = 1$ and $a_2 = 1$. Otherwise we replace $T(z)$ with $(a_2)^{-1}T(z)a_1$. Relying on the (L^2-convergent) atomic decomposition for $\mathrm{H}^p \cap \mathrm{L}^2$ (see Section 2.4) it suffices to show that there is a constant C such that $\|T(z)a\|_{\mathrm{H}^p} \leq C$ for all z and all L^∞-atoms a for H^p.

Step 1: Molecular decay. We show that there exist C, ε independently of a, z and $j \geq 1$ such that

$$\|T(z)a\|_{\mathrm{L}^q(C_j(Q))} \leq C(2^j\ell(Q))^{\frac{n}{q}-\frac{n}{p}}2^{-\varepsilon j}, \tag{4.3}$$

where Q is the cube associated with a. For $j = 1$ we can simply use L^q-boundedness and Hölder's inequality:

$$\|T(z)a\|_q \leq C\|a\|_q \leq C\ell(Q)^{\frac{n}{q}}\|a\|_\infty \leq C\ell(Q)^{\frac{n}{q}-\frac{n}{p}}.$$

For $j \geq 2$ the off-diagonal assumption yields

$$\begin{aligned}\|T(z)a\|_{\mathrm{L}^q(C_j(Q))} &\leq C_\gamma\left(\frac{2^j\ell(Q)}{|z|}\right)^{-\gamma}\|a\|_{\mathrm{L}^q(Q)} \\ &\leq C_\gamma\left(\frac{2^j\ell(Q)}{|z|}\right)^{-\gamma}\ell(Q)^{\frac{n}{q}-\frac{n}{p}}\end{aligned} \tag{4.4}$$

with $\gamma > 0$ at our disposal. Likewise, $\mathrm{H}^\varrho - \mathrm{L}^q$-boundedness yields

$$\|T(z)a\|_{\mathrm{L}^q(C_j(Q))} \leq C|z|^{\frac{n}{q}-\frac{n}{\varrho}}\|a\|_{\mathrm{H}^\varrho} \leq C|z|^{\frac{n}{q}-\frac{n}{\varrho}}\ell(Q)^{\frac{n}{\varrho}-\frac{n}{p}}, \tag{4.5}$$

where in the second step we have used that $\ell(Q)^{n/p-n/\varrho}a$ is an L^∞-atom for H^ϱ. Now, fix $\delta > 0$ such that $1/p - 1/q = (1-2\delta)(1/\varrho - 1/q)$. This is possible since we have $p > \varrho$. For $|z| \geq 2^{j(1-\delta)}\ell(Q)$ we use (4.5) to get

$$\begin{aligned}\|T(z)a\|_{\mathrm{L}^q(C_j(Q))} &\leq C2^{j(1-\delta)(\frac{n}{q}-\frac{n}{\varrho})}\ell(Q)^{\frac{n}{q}-\frac{n}{p}} \\ &= C(2^j\ell(Q))^{\frac{n}{q}-\frac{n}{p}}2^{-\delta(\frac{n}{\varrho}-\frac{n}{q})j},\end{aligned}$$

whereas for $|z| \leq 2^{j(1-\delta)}\ell(Q)$ we employ (4.4) and find

$$\|T(z)a\|_{\mathrm{L}^q(C_j(Q))} \leq C_\gamma 2^{-j\gamma\delta}\ell(Q)^{\frac{n}{q}-\frac{n}{p}}.$$

We take $\gamma > \delta^{-1}(n/p - n/q)$ to make sure that these bounds take the form (4.3).

Step 2: Conclusion. Since $f := T(z)a$ has integral zero by assumption, (4.3) implies $\|f\|_{\mathrm{H}^p} \leq C$ for some constant independent of f. Indeed, this is due to the molecular theory of Taibleson–Weiss [91, Thm. 2.9] but we include their argument in our special case in the subsequent lemma. □

Lemma 4.10. *Let $p \in (1_*, 1]$ and $q \in (1,\infty)$. Suppose $f \in \mathrm{L}^2$ has integral zero and satisfies for some $C, \varepsilon > 0$, some cube $Q \subseteq \mathbb{R}^n$ and all $j \geq 1$,*

$$\|f\|_{\mathrm{L}^q(C_j(Q))} \leq C(2^j\ell(Q))^{\frac{n}{q}-\frac{n}{p}}2^{-\varepsilon j}.$$

Then, there exists a constant C' depending on C, ε and dimensions, and L^q-atoms a_j for H^p with support in $C_{j+1}(Q) \cup C_j(Q)$, such that

$$f = \sum_{j=1}^\infty C' 2^{-\varepsilon j} a_j$$

with unconditional convergence in $\mathrm{L}^1_{\mathrm{loc}}$. In particular, the sum also converges in H^p and $\|f\|_{\mathrm{H}^p} \leq \frac{C'}{2^\varepsilon - 1}$.

Proof. The final statement follows from the atomic representation, using the maximal function characterization of H^p, see also [89, p.106]. To prove the rest, we set

$$f_j := \mathbf{1}_{C_j} f, \quad p_j := (f)_{C_j(Q)} \mathbf{1}_{C_j(Q)}.$$

Then $f_j - p_j$ has mean value zero and satisfies

$$\|f_j - p_j\|_q \leq 2\|f_j\|_q \leq 2C(2^j \ell(Q))^{\frac{n}{q}-\frac{n}{p}} 2^{-\varepsilon j}.$$

This means that $2^{\varepsilon j} 2^{n/q - n/p}(2C)^{-1}(f_j - p_j)$ is an L^q-atom for H^p. Next, letting $c_j := \sum_{k=j}^\infty \int_{C_k(Q)} f \,\mathrm{d}x$, summation by parts gives a pointwise identity

$$\sum_{j=1}^\infty p_j = \sum_{j=1}^\infty (c_j - c_{j+1}) \frac{\mathbf{1}_{C_j(Q)}}{|C_j(Q)|} = \sum_{j=1}^\infty b_j \tag{4.6}$$

with

$$b_j := c_{j+1} \bigg(\frac{\mathbf{1}_{C_{j+1}(Q)}}{|C_{j+1}(Q)|} - \frac{\mathbf{1}_{C_j(Q)}}{|C_j(Q)|} \bigg).$$

There are no boundary terms since we have $c_1 = 0$ and, from the assumption,

$$\begin{aligned} |c_j| \leq \sum_{k=j}^\infty |C_k(Q)|^{1-\frac{1}{q}} \|\mathbf{1}_{C_k(Q)} f\|_q &\lesssim \sum_{k=j}^\infty (2^k \ell(Q))^{n-\frac{n}{p}} 2^{-\varepsilon k} \\ &\simeq (2^j \ell(Q))^{n-\frac{n}{p}} 2^{-\varepsilon j}, \end{aligned}$$

so ${}^{|c_j|}/_{|C_j(Q)|}$ tends to 0 as $j \to \infty$. Identity (4.6) holds in $\mathrm{L}^1_{\mathrm{loc}}$ with unconditional convergence because the sums are locally finite since b_j has support in $C_{j+1}(Q) \cup C_j(Q)$. Moreover, b_j has mean value zero and satisfies

$$\|b_j\|_q \leq |c_{j+1}| \bigg(\frac{|C_{j+1}(Q)|^{\frac{1}{q}}}{|C_{j+1}(Q)|} + \frac{|C_j(Q)|^{\frac{1}{q}}}{|C_j(Q)|} \bigg) \leq C''(2^j \ell(Q))^{\frac{n}{q}-\frac{n}{p}} 2^{-\varepsilon j}.$$

Hence, $2^{\varepsilon j} 4^{n/q - n/p}(C'')^{-1} b_j$ is an L^q-atom for H^p and $f = \sum_{j=1}^\infty f_j - p_j + b_j$ is the representation we are looking for. □

4.3. **Interpolation principles.** We continue with interpolation properties. Our main tool will be the Stein interpolation theorem, which we state in an abstract version due to Voigt [93]. In the following we work on the strip $S := \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$.

Proposition 4.11 ([93]). *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of Banach spaces and let Z be a dense subspace of $X_0 \cap X_1$. Let $(T(z))_{z \in S}$ be a family of linear mappings $Z \to Y_0 + Y_1$ with the following properties for all $f \in Z$:*

(i) *The function $T(\cdot)f : S \to Y_0 + Y_1$ is continuous, bounded and holomorphic in the interior of S.*

(ii) *For $j = 0, 1$ the restriction $T(\cdot)f : j + \mathrm{i}\mathbb{R} \to Y_j$ is continuous and there is a constant M_j that does not depend on f such that*

$$\sup_{t \in \mathbb{R}} \|T(j + \mathrm{i}t)f\|_{Y_j} \leq M_j \|f\|_{X_j}.$$

Then for all $\theta \in (0, 1)$ and all $f \in Z$,

$$\|T(\theta)f\|_{[Y_0,Y_1]_\theta} \leq M_0^{1-\theta} M_1^\theta \|f\|_{[X_0,X_1]_\theta}.$$

Remark 4.12. The classical version of the theorem is when X_j and Y_j are L^p-spaces, $1 \leq p \leq \infty$, and Z is the space of step functions, [51, Thm. 1.3.4]. Then continuity is not required in (ii) and in (i) it suffices to assume that for all $f, g \in Z$ and all $z \in S$ the integral $\int_{\mathbb{R}^n} T(z)f \cdot \overline{g} \, \mathrm{d}x$ converges absolutely and defines a bounded and continuous function of z that is holomorphic in the interior of S. For example, it suffices that $T(\cdot) : S \to \mathrm{L}^2$ is bounded, continuous and holomorphic in the interior. Such weakening of assumptions is not possible for general interpolation couples [37].

As a first application we state the following

Lemma 4.13. *Let $p_0, q_0, p_1, q_1 \in [1, \infty]$, $p_i \leq q_i$ and $\omega \in (0, \pi)$. Let $(T(z))_{z \in \mathrm{S}^+_\omega}$ be a uniformly bounded family on L^2 as in (4.1) that depends holomorphically on z. Let $\theta \in (0, 1)$ and set*

$$p_\theta := [p_0, p_1]_\theta \quad \text{and} \quad q_\theta := [q_0, q_1]_\theta.$$

(i) *If $(T(z))_{z \in (0,\infty)}$ is $\mathrm{L}^{p_0} - \mathrm{L}^{q_0}$-bounded and $(T(z))_{z \in \mathrm{S}^+_\omega}$ is $\mathrm{L}^{p_1} - \mathrm{L}^{q_1}$-bounded, then $(T(z))_{z \in \mathrm{S}^+_{\theta\omega}}$ is $\mathrm{L}^{p_\theta} - \mathrm{L}^{q_\theta}$-bounded*

(ii) *If $(T(z))_{z \in (0,\infty)}$ is $\mathrm{L}^{p_0} - \mathrm{L}^{q_0}$-bounded and $(T(z))_{z \in \mathrm{S}^+_\omega}$ satisfies $\mathrm{L}^{p_1} - \mathrm{L}^{q_1}$ off-diagonal estimates of order γ, then $(T(z))_{z \in \mathrm{S}^+_{\theta\omega}}$ satisfies $\mathrm{L}^{p_\theta} - \mathrm{L}^{q_\theta}$ off-diagonal estimates of order $\theta\gamma$.*

(iii) *If $(T(z))_{z \in (0,\infty)}$ satisfies $\mathrm{L}^{p_0} - \mathrm{L}^{q_0}$ off-diagonal estimates of order γ and $(T(z))_{z \in \mathrm{S}^+_\omega}$ is $\mathrm{L}^{p_1} - \mathrm{L}^{q_1}$-bounded, then $(T(z))_{z \in \mathrm{S}^+_{\theta\omega}}$ satisfies $\mathrm{L}^{p_\theta} - \mathrm{L}^{q_\theta}$ off-diagonal estimates of order $(1 - \theta)\gamma$.*

Exponential order in the assumptions leads to exponential order in the conclusion with the decay parameter c changed accordingly.

Proof. We begin with part (ii). We fix $\nu \in (-\omega, \omega)$ and $r > 0$. Then we fix measurable sets $E, F \subseteq \mathbb{R}^n$ and consider the family $S(z) := \mathrm{e}^{(z-\theta)^2} \mathbf{1}_F T(r\mathrm{e}^{\mathrm{i}\nu z}) \mathbf{1}_E$. This family is uniformly bounded on L^2 and holomorphic in an open neighborhood of the strip $0 \leq \operatorname{Re} z \leq 1$. By assumption we have for all $t \in \mathbb{R}$ and all step functions f,

$$\begin{aligned} \|S(\mathrm{i}t)f\|_{q_0} &\leq C_0 |r\mathrm{e}^{-\nu t}|^{\frac{n}{q_0} - \frac{n}{p_0}} \mathrm{e}^{1-t^2} \|f\|_{p_0} \\ &\leq C_0 (r\mathrm{e}^{-\omega|t|})^{\frac{n}{q_0} - \frac{n}{p_0}} \mathrm{e}^{1-t^2} \|f\|_{p_0} \end{aligned}$$

and

$$\|S(1 + \mathrm{i}t)f\|_{q_1} \leq C_1 |r\mathrm{e}^{-\nu t}|^{\frac{n}{q_1} - \frac{n}{p_1}} \Big(1 + \frac{\mathrm{d}(E, F)}{|r\mathrm{e}^{-\nu t}|}\Big)^{-\gamma} \mathrm{e}^{1-t^2} \|f\|_{p_1}$$

$$\leq C_1(r\mathrm{e}^{-\omega|t|})^{\frac{n}{q_1}-\frac{n}{p_1}}\Big(1+\frac{\mathrm{d}(E,F)}{r\mathrm{e}^{\omega|t|}}\Big)^{-\gamma}\mathrm{e}^{1-t^2}\|f\|_{p_1}.$$

We use $1+{}^{\mathrm{d}(E,F)}/_{r\mathrm{e}^{\omega|t|}} \geq \mathrm{e}^{-\omega|t|}(1+{}^{\mathrm{d}(E,F)}/_{r})$ in the second line and that the additional factor e^{-t^2} acts in our favor, in order to give

$$\|S(\mathrm{i}t)f\|_{q_0} \leq M_0 r^{\frac{n}{q_0}-\frac{n}{p_0}}\|f\|_{p_0},$$
$$\|S(1+\mathrm{i}t)f\|_{q_1} \leq M_1 r^{\frac{n}{q_1}-\frac{n}{p_1}}\Big(1+\frac{\mathrm{d}(E,F)}{r}\Big)^{-\gamma}\|f\|_{p_1},$$

where the M_j are still also independent of r, ν and E, F. Stein interpolation yields

$$\|S(\theta)f\|_{[q_0,q_1]_\theta} \leq M_0^{1-\theta}M_1^{\theta} r^{\frac{n}{q_\theta}-\frac{n}{p_\theta}}\Big(1+\frac{\mathrm{d}(E,F)}{r}\Big)^{-\theta\gamma}\|f\|_{[p_0,p_1]_\theta}.$$

Since we have $S(\theta)f = \mathbf{1}_F T(r\mathrm{e}^{\mathrm{i}\nu\theta})\mathbf{1}_E f$, this estimates means that $T(z)$ satisfies $\mathrm{L}^{p_\theta}-\mathrm{L}^{q_\theta}$ off-diagonal estimates of order $\theta\gamma$ for $z \in \mathrm{S}^+_{\theta\omega}$.

The proof of part (iii) is exactly the same except that now the estimate for $S(\mathrm{i}t)$ comes with decay.

The proof of part (i) does not need the sets E, F and uses the same interpolation argument for $z \mapsto \mathrm{e}^{(z-\theta)^2}T(r\mathrm{e}^{\mathrm{i}\nu z})$.

Finally, the proof in case of exponential order in the assumptions follows *mutadis mutandis.* □

If we freeze z and view $T(z)$ as a constant family, then the same argument leads to

Lemma 4.14. *Let $p_0, q_0, p_1, q_1 \in [1,\infty]$ with $p_i \leq q_i$ and suppose that a family as in (4.1) is $\mathrm{L}^{p_0}-\mathrm{L}^{q_0}$-bounded and satisfies $\mathrm{L}^{p_1}-\mathrm{L}^{q_1}$ off-diagonal estimates of order γ (of exponential order). Then for each $\theta \in (0,1)$ it satisfies $\mathrm{L}^{[p_0,p_1]_\theta}-\mathrm{L}^{[q_0,q_1]_\theta}$ off-diagonal estimates of order $\theta\gamma$ (of exponential order).*

4.4. **Applications to the functional calculus.** We turn to the more specific setting that the family $(T(z))$ is modeled after the resolvents of a sectorial operator. In this section, we assume that

(4.7)
- T is a sectorial operator on $\mathrm{L}^2(\mathbb{R}^n;V)$ of some angle $\omega \in [0,\pi)$, where V is a finite-dimensional Hilbert space,
- $((1+t^2T)^{-1})_{t>0}$ satisfies L^2 off-diagonal estimates of arbitrarily large order.

Lemma 4.15. *Let $p \in (1,\infty)$ be such that $((1+t^2T)^{-1})_{t>0}$ is L^p-bounded. Let $\theta \in (0,1]$. Then for every $\mu \in (0, {}^{\theta(\pi-\omega)}/_2)$ the family $((1+z^2T)^{-1})_{z\in\mathrm{S}^+_\mu}$ satisfies $\mathrm{L}^{[p,2]_\theta}$ off-diagonal estimates of arbitrarily large order.*

Proof. The resolvent $z \mapsto (1+z^2T)^{-1}$ on L^2 is a bounded holomorphic function on S^+_μ for any $\mu \in (0, {}^{(\pi-\omega)}/_2)$. We apply Lemma 4.13 twice.

First, interpolation between the L^2-bounds on sectors and the L^2 off-diagonal estimates on the positive real axis yields L^2 off-diagonal estimates of arbitrarily large order on S^+_μ for any $\mu \in (0, {}^{(\pi-\omega)}/_2)$. Second, interpolation between the L^2 off-diagonal estimates on sectors and the L^p-bounds on the positive real axis yields the claim. $\square$

We obtain off-diagonal estimates for the functional calculus similar to [22, Part II]. In applications we usually work with holomorphic functions that are in the respective classes on any sector and the technical conditions on the angles can be ignored. On the other hand, the order of off-diagonal decay is of utmost importance: It is mainly the decay of ψ at $z = 0$, quantified by the classes Ψ^τ_σ from Section 3.4, that limits the available off-diagonal for $(\psi(t^2T))_{t>0}$.

Lemma 4.16. *Let $p \in (1, \infty)$ be such that $((1 + t^2T)^{-1})_{t>0}$ is L^p-bounded. Let $\theta \in (0, 1]$, put $q := [p, 2]_\theta$ and fix an angle $\mu \in (0, {}^{\theta(\pi-\omega)}/_2)$. Let $\sigma, \tau > 0$ and $\psi \in \Psi^\tau_\sigma(S^+_{\pi-2\mu})$. Then the following estimates hold.*

(i) *Let $(\eta(t))_{t>0}$ be a continuous and uniformly bounded family of functions in $H^\infty(S^+_{\pi-2\mu})$. Then for all measurable sets $E, F \subseteq \mathbb{R}^n$, all $t > 0$ and all $f \in L^q \cap L^2$,*

$$\|\mathbf{1}_F \eta(t)(T)\psi(t^2T)\mathbf{1}_E f\|_q \lesssim \|\eta\| \|\psi\|_{\sigma,\tau,\mu} \left(1 + \frac{d(E, F)}{t}\right)^{-2\sigma} \|f\|_q.$$

The norms are $\|\psi\|_{\sigma,\tau,\mu} := \sup_{z\in S^+_{\pi-2\mu}} {}^{|\psi(z)|}/_{(|z|^\sigma \wedge |z|^{-\tau})}$ and $\|\eta\| := \sup_{t>0} \|\eta(t)\|_\infty$.

(ii) *Furthermore, if $\eta(t)(z) = \varphi(t^2z)$ for some $\varphi \in \Psi^0_\sigma(S^+_{\pi-2\mu})$, then for all $0 < r \le t$ and with the same dependencies,*

$$\|\mathbf{1}_F \varphi(r^2T)\psi(t^2T)\mathbf{1}_E f\|_q \lesssim \|\psi\|_{\sigma,\tau,\mu} \|\varphi\|_{\sigma,0,\mu} \left(1 + \frac{d(E, F)}{r}\right)^{-2\sigma} \|f\|_q.$$

(iii) *Finally for each $\gamma \in [0, \sigma] \cap [0, \tau)$, it follows for all $r > 0$ with the same dependencies,*

$$\|\varphi(r^2T)\psi(t^2T)f\|_q \lesssim \|\psi\|_{\sigma,\tau,\mu} \|\varphi\|_{\sigma,0,\mu} \left(\frac{r^2}{t^2}\right)^\gamma \|f\|_q.$$

Proof. Throughout, let $\|f\|_q = 1$. We pick an angle $\nu \in (\mu, {}^{\theta(\pi-\omega)}/_2)$. By Lemma 4.15 we have L^q off-diagonal estimates of arbitrarily large order for the resolvents for $z \in \overline{S^+_\nu}$. Here, we use the order $2\sigma + 1$.

We begin with the first estimate and put $X := {}^{d(E,F)}/_t$. Since

$$\begin{aligned} \eta(t)(T)\psi(t^2T) &= \frac{1}{2\pi i} \int_{\partial S^+_{\pi-2\nu}} \eta(t)(z)\psi(t^2z)(z - T)^{-1} \, dz \\ &= \frac{1}{2\pi i} \int_{\partial S^+_{\pi-2\nu}} \eta(t)(zt^{-2})\psi(z)(1 - t^2z^{-1}T)^{-1} \, \frac{dz}{z}, \end{aligned} \tag{4.8}$$

where $(-t^2 z^{-1})^{1/2} \in \overline{\mathrm{S}^+_\nu}$, we obtain

$$\begin{aligned}
&\|\mathbf{1}_F \eta(t)(T)\psi(t^2T)\mathbf{1}_E f\|_q \\
&\qquad \lesssim \int_{\partial \mathrm{S}^+_{\pi-2\nu}} \|\eta\| \frac{|\psi(z)|}{(1+|z|^{1/2}X)^{2\sigma+1}} \frac{\mathrm{d}|z|}{|z|} \\
&\qquad \leq \|\eta\| \|\psi\|_{\sigma,\tau,\mu} \int_{\partial \mathrm{S}^+_{\pi-2\nu}} \frac{|z|^\sigma \wedge |z|^{-\tau}}{(1+|z|^{1/2}X)^{2\sigma+1}} \frac{\mathrm{d}|z|}{|z|}.
\end{aligned} \tag{4.9}$$

In the case $X \leq 1$, we minimize the denominator by 1 to derive the desirable bound

$$\|\mathbf{1}_F \eta(t)(T)\psi(t^2T)\mathbf{1}_E f\|_q \lesssim \|\eta\| \|\psi\|_{\sigma,\tau,\mu}.$$

In the case $X \geq 1$, we split the integral at $|z| = X^{-2}$ to give the desirable bound

$$\begin{aligned}
&\|\mathbf{1}_E \eta(t)(T)\psi(t^2T)\mathbf{1}_F f\|_q \\
&\qquad \lesssim \|\eta\| \|\psi\|_{\sigma,\tau,\mu} \bigg(\int_0^{X^{-2}} |z|^\sigma \frac{\mathrm{d}|z|}{|z|} + \int_{X^{-2}}^\infty \frac{|z|^\sigma}{(|z|^{1/2}X)^{2\sigma+1}} \frac{\mathrm{d}|z|}{|z|} \bigg) \\
&\qquad \lesssim \|\eta\| \|\psi\|_{\sigma,\tau,\mu} X^{-2\sigma}.
\end{aligned}$$

This completes the proof of (i).

Turning to the second estimate, we take $\eta(r)(z) = \varphi(r^2 z)$ in (4.8) and change variables to

$$\varphi(r^2T)\psi(t^2T) = \frac{1}{2\pi i} \int_{\partial \mathrm{S}^+_{\pi-2\nu}} \varphi(z)\psi(t^2 r^{-2} z)(1 - r^2 z^{-1} T)^{-1} \frac{\mathrm{d}z}{z}.$$

This time we set $X := {}^{\mathrm{d}(E,F)}/_r$ and obtain

$$\|\mathbf{1}_F \varphi(r^2T)\psi(t^2T)\mathbf{1}_E f\|_q \lesssim \int_{\partial \mathrm{S}^+_{\pi-2\nu}} \frac{|\varphi(z)||\psi(t^2r^{-2}z)|}{(1+|z|^{1/2}X)^{2\sigma+1}} \frac{\mathrm{d}|z|}{|z|}. \tag{4.10}$$

The important observation is that

$$|\varphi(z)| \leq \|\varphi\|_{\sigma,0,\mu} (|z|^\sigma \wedge 1) \tag{4.11}$$

and, since $r \leq t$,

$$|\psi(t^2r^{-2}z)| \leq \|\psi\|_\infty \wedge (\|\psi\|_{\sigma,\tau,\mu} |t^2r^{-2}z|^{-\tau}) \leq \|\psi\|_{\sigma,\tau,\mu}(1 \wedge |z|^{-\tau}),$$

so that

$$|\varphi(z)||\psi(t^2r^{-2}z)| \leq \|\psi\|_{\sigma,\tau,\mu} \|\varphi\|_{\sigma,0,\mu} (|z|^\sigma \wedge |z|^{-\tau}).$$

Thus, we can bound the right-hand side in (4.10) by the same parameter integral that already appeared on the far right in (4.9) and get the same bound $(1+X)^{-2\sigma}$ for the integral. Now, (ii) follows.

As for (iii), we first argue as in (ii) with $E = F = \mathbb{R}^n$ and $X = 0$ to obtain

$$\|\varphi(r^2T)\psi(t^2T)f\|_q$$

$$\lesssim \|\psi\|_{\sigma,\tau,\mu}\|\varphi\|_{\sigma,0,\mu}\int_0^\infty (1\wedge|z|^\sigma)\big(|t^2r^{-2}z)|^\sigma\wedge|t^2r^{-2}z)|^{-\tau}\big)\,\frac{\mathrm{d}|z|}{|z|}.$$

Using $(1\wedge|z|^\sigma)\leq|z|^\gamma$ in order to get a homogeneous estimate and changing variables, we conclude

$$\|\varphi(r^2T)\psi(t^2T)f\|_q\lesssim\|\psi\|_{\sigma,\tau,\mu}\|\varphi\|_{\sigma,0,\mu}\left(\frac{r^2}{t^2}\right)^\gamma\int_0^\infty|z|^\gamma\big(|z|^\sigma\wedge|z|^{-\tau}\big)\,\frac{\mathrm{d}|z|}{|z|}$$

and the remaining integral is finite since we assume $0\leq\gamma<\tau$. □

The decay of ψ at the origin can be replaced by the assumption that $\psi(z)$ has a limit as $|z|\to0$ with order of convergence $\mathcal{O}(|z|^\sigma)$ for some $\sigma>0$. The exemplary result of this type is as follows. The obtained order of decay is optimal and already attained when $T=-\Delta_x$.

Corollary 4.17. *Let $p\in(1,\infty)$ be such that $((1+t^2T)^{-1})_{t>0}$ is L^p-bounded and let $\theta\in(0,1)$. Then $(\mathrm{e}^{-tT^{1/2}})_{t>0}$ satisfies $\mathrm{L}^{[p,2]_\theta}$ off-diagonal estimates of order* 1.

Proof. This is a consequence of the preceding two lemmata since we can write $\mathrm{e}^{-z^{1/2}}=\psi(z)+(1+z)^{-1}$ with $\psi\in\Psi^1_{1/2}$ on any sector. □

5. Conservation properties

In order to extend the operator theory for L to Hardy spaces, we need to guarantee that certain operators $f(L)$ preserve vanishing zeroth moments or have the *conservation property* $f(L)c=c$ whenever c is a constant. In absence of integral kernels, the action of such operators on constants is explained via off-diagonal estimates as follows.

Proposition 5.1. *Let T be a bounded linear operator on $\mathrm{L}^2(\mathbb{R}^n;V)$, where V is a finite dimensional Hilbert space. If T satisfies L^p off-diagonal estimates of order $\gamma>n/p$ for some $p\in[2,\infty)$, then T can be extended to a bounded operator $\mathrm{L}^\infty(\mathbb{R}^n;V)\to\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;V)$ via*

$$Tf:=\sum_{j=1}^\infty T(\mathbf{1}_{C_j(B(0,1))}f). \tag{5.1}$$

Moreover, if $(\eta_j)\subseteq\mathrm{L}^\infty(\mathbb{R}^n;\mathbb{C})$ is a family such that

(5.2)
- $\sup_j\|\eta_j\|_\infty<\infty$,
- $\sum_{j=1}^\infty\eta_j(x)=1$ *for a.e.* $x\in\mathbb{R}^n$,
- η_j *has compact support, which for some C,c and all sufficiently large j is contained in $B(0,C2^j)\setminus B(0,c2^j)$,*

then

$$Tf=\sum_{j=1}^\infty T(\eta_jf),$$

where the right-hand side converges in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n; V)$ *and in particular in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n; V)$*.*

Remark 5.2. A particular example for a family with the required properties is $\eta_j = \mathbf{1}_{C_j(B)}$ for an arbitrary ball (or cube) $B \subseteq \mathbb{R}^n$.

Proof. We put $B := B(0,1)$ and fix any compact set $K \subseteq \mathbb{R}^n$. For all large enough j we have $\mathrm{d}(K, C_j(B)) \geq 2^{j-1}$ and therefore

$$\begin{aligned}\|T(\mathbf{1}_{C_j(B)}f)\|_{\mathrm{L}^p(K)} &\lesssim 2^{-j\gamma}\|f\|_{\mathrm{L}^p(C_j(B))}\\ &\lesssim 2^{j(\frac{n}{p}-\gamma)}\|f\|_\infty.\end{aligned}$$

Hence, the series on the right-hand side of (5.1) converges absolutely in $\mathrm{L}^p(K)$ and the limit satisfies $\|Tf\|_{\mathrm{L}^p(K)} \leq C_K\|f\|_\infty$ for a constant C_K that depends on K but not on f.

Next, we pick an integer $j_0 \geq 1$ such that $c2^{j_0} \geq 1$ and therefore $2^J B \subseteq B(0, c2^{J+j_0})$ for all $J \geq 1$. If J is large enough so that the annular support of η_j is granted, then $\sum_{j=1}^J \mathbf{1}_{C_j(B)} - \sum_{j=1}^{J+j_0} \eta_j$ vanishes on $2^{J+1}B$, has support in $C'2^J B$ for some C' that does not depend on J and is uniformly bounded. The off-diagonal bounds yield again

$$\Big\| \sum_{j=1}^J T(\mathbf{1}_{C_j(B)}f) - \sum_{j=1}^{J+j_0} T(\eta_j f)\Big\|_{\mathrm{L}^p(K)} \lesssim 2^{J(\frac{n}{p}-\gamma)}\|f\|_\infty,$$

which shows that $\sum_{j=1}^\infty T(\eta_j f)$ converges to Tf in $\mathrm{L}^p(K)$. □

We begin with the conservation property for the resolvents of the perturbed Dirac operator BD that has appeared implicitly in several earlier works [22, 83]. The proof relies on the cancellation property $Dc = 0$ for constants c (where D is understood in the sense of distributions).

Proposition 5.3. *If* $\alpha \in \mathbb{N}$ *and* $z \in \mathrm{S}_{\pi/2-\omega_{BD}}$*, then for all* $c \in \mathbb{C}^m \times \mathbb{C}^{mn}$*,*

$$(1 + \mathrm{i}zBD)^{-\alpha}c = c = (1 + z^2(BD)^2)^{-\alpha}c.$$

Proof. Let $R > 0$ and (η_j) be a smooth partition of unity on $\mathbb{R}^n$ subordinate to the sets

$$D_1 := B(0, 4R), \quad D_j := B(0, 2^{j+1}R) \setminus B(0, 2^{j-1}R) \quad (j \geq 2),$$

such that $\|\eta_j\|_\infty + 2^j R\|\nabla_x \eta_j\|_\infty \leq C$ for a dimensional constant C.

We begin with the resolvents of BD, which satisfy L^2 off-diagonal estimates of arbitrarily large order by Proposition 3.10 and composition. According to Proposition 5.1 we can write

$$(1 + \mathrm{i}zBD)^{-\alpha}c = \sum_{j=1}^\infty (1 + \mathrm{i}zBD)^{-\alpha}(\eta_j c),$$

so that

$$(1+\mathrm{i}zBD)^{-\alpha+1}c-(1+\mathrm{i}zBD)^{-\alpha}c=\sum_{j=1}^{\infty}\mathrm{i}z(1+\mathrm{i}zBD)^{-\alpha}BD(\eta_j c),$$

where we set $(1+\mathrm{i}zBD)^0c := c$ and used $\eta_j c \in \mathsf{D}(D) = \mathsf{D}(BD)$. Now, $BD(\eta_j c)$ has support in $B(0,2^{j+1}R)\setminus B(0,2^{j-1}R)$ also for $j=1$ and satisfies $\|BD(\eta_j c)\|_\infty \leq C|c|\|B\|_\infty R^{-1}$. The off-diagonal estimates yield

$$\|(1+\mathrm{i}zBD)^{-\alpha+1}c-(1+\mathrm{i}zBD)^{-\alpha}c\|_{\mathrm{L}^2(B(0,R/2))}\lesssim R^{\frac{n}{2}-\gamma-1}\sum_{j=1}^{\infty}2^{j(\frac{n}{2}-\gamma)}$$

with an implicit constant that is independent of R. Sending $R\to\infty$ gives $(1+\mathrm{i}zBD)^{-\alpha+1}c=(1+\mathrm{i}zBD)^{-\alpha}c$. Since $(1+\mathrm{i}zBD)^0c=c$, we conclude $(1+\mathrm{i}zBD)^{-\alpha}c=c$ for all α.

The argument for the resolvents of $(BD)^2$ is identical and draws upon the identity

$$\begin{aligned}(1+z^2(BD)^2)^{-\alpha+1}c&-(1+z^2(BD)^2)^{-\alpha}c\\&=\sum_{j=1}^{\infty}z^2BD(1+z^2(BD)^2)^{-\alpha}BD(\eta_j c).\end{aligned}$$

The off-diagonal decay for $z^2BD(1+z^2(BD)^2)^{-\alpha}$ follows again by composition since this operator can be written as

$$-\frac{\mathrm{i}z}{2}\Big((1-\mathrm{i}zBD)^{-1}-(1+\mathrm{i}zBD)^{-1}\Big)(1+z^2(BD)^2)^{-\alpha+1}. \qquad \square$$

As a corollary we obtain the conservation property for the second-order operator L. The reader can refer to [80, Sec. 4.4] and references therein for related conservation properties in the realm of semigroups.

Corollary 5.4. *Let $\alpha\in\mathbb{N}$ and $z\in \mathrm{S}^+_{(\pi-\omega_L)/2}$. Let $c\in\mathbb{C}^m$ and let $f\in\mathrm{L}^2$ have compact support. Then one has the conservation formula*

$$(1+z^2L)^{-\alpha}c=c$$

and its dual version

$$\int_{\mathbb{R}^n}a(1+z^2L)^{-\alpha}a^{-1}f\,\mathrm{d}x=\int_{\mathbb{R}^n}f\,\mathrm{d}x.$$

Proof. The left-hand sides are holomorphic functions of z (valued in $\mathrm{L}^2_{\mathrm{loc}}$ and $\mathbb{C}^m$, respectively). Hence, it suffices to argue for $z=t\in(0,\infty)$. We have

$$(1+t^2(BD)^2)^{-\alpha}=\begin{bmatrix}(1+t^2L)^{-\alpha}&0\\0&(1+t^2M)^{-\alpha}\end{bmatrix},$$

so the fist claim follows from the conservation property for BD. As $(a^*)^{-1}L^*a^*$ belongs to the same class as L, we also get

$$\begin{aligned}\int_{\mathbb{R}^n} a(1+t^2L)^{-\alpha}a^{-1}f\cdot\overline{c}\,\mathrm{d}x &= \int_{\mathbb{R}^n} f\cdot\overline{(1+t^2(a^*)^{-1}L^*a^*)^{-\alpha}c}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f\cdot\overline{c}\,\mathrm{d}x\end{aligned}$$

and since $c\in\mathbb{C}^m$ is arbitrary, the second claim follows. □

We turn to more general operators in the functional calculus. In view of Lemma 4.16 the decay of the auxiliary function at the origin limits the available off-diagonal decay and hence, in contrast with the case of resolvents, we have to use Proposition 5.1 for exponents $p\neq 2$.

Lemma 5.5. *Let $p\in[2,\infty)$ be such that $((1+t^2L)^{-1})_{t>0}$ is L^p-bounded. Suppose that ψ is of class Ψ_σ^τ on any sector, where $\tau>0$ and $\sigma>{}^n/_{(2p)}$. Then*

$$\psi(t^2L)c=0 \quad (c\in\mathbb{C}^m,\ t>0).$$

Proof. Let $\theta\in(0,1]$ be such that $q:=[p,2]_\theta$ satisfies $\sigma>{}^n/_{(2q)}$. According to Lemma 4.16 the family $(\psi(t^2L))_{t>0}$ satisfies L^q off-diagonal estimates of order $2\sigma>{}^n/_q$. Hence, $\psi(t^2L)c$ is defined via Proposition 5.1.

Lemma 4.13 provides L^q off-diagonal decay for the resolvents of L of arbitrarily large order on some sector S_μ^+. We pick $\nu\in(0,\mu)$ and write the definition of $\psi(t^2L)$ as

$$\psi(t^2L)=\frac{1}{2\pi i}\int_{\partial\mathrm{S}^+_{\pi-2\nu}}\psi(t^2z)(1-z^{-1}L)^{-1}\,\frac{\mathrm{d}z}{z}.$$

Setting $B:=B(0,1)\subseteq\mathbb{R}^n$, we formally have

$$\begin{aligned}\sum_{j\geq1}\psi(t^2L)(\mathbf{1}_{C_j(B)}c) &= \frac{1}{2\pi i}\int_{\partial\mathrm{S}^+_{\pi-2\nu}}\psi(t^2z)\sum_{j\geq1}(1-z^{-1}L)^{-1}(\mathbf{1}_{C_j(B)}c)\,\frac{\mathrm{d}z}{z}\\ &= \frac{1}{2\pi i}\int_{\partial\mathrm{S}^+_{\pi-2\nu}}\psi(t^2z)c\,\frac{\mathrm{d}z}{z}\\ &=0,\end{aligned}$$

where the second line uses the conservation property and the third one Cauchy's theorem. It remains to justify convergence and interchanging sum and integral sign in the first line.

To this end, fix any compact set $K\subseteq\mathbb{R}^n$. Using off-diagonal estimates, we obtain for all j large enough to grant for $\mathrm{d}(K,C_j(B))\geq 2^{j-1}$ that

$$\begin{aligned}&\|\psi(t^2z)(1-z^{-1}L)^{-1}(\mathbf{1}_{C_j(B)}c)\|_{\mathrm{L}^q(K)}\\ &\qquad\lesssim|\psi(t^2z)|(1+2^{j-1}|z|^{\frac12})^{-\gamma}\|c\|_{\mathrm{L}^q(C_j(B))}\end{aligned}$$

$$\lesssim 2^{j(\frac{n}{q}-\gamma)} \begin{cases} t^{-2\tau}|z|^{-\frac{\gamma}{2}-\tau} & \text{if } |z| \geq 1 \\ t^{2\sigma}|z|^{\sigma-\frac{\gamma}{2}} & \text{if } |z| \leq 1 \end{cases},$$

where $\gamma > 0$ is at our disposal. We take $n/q < \gamma < 2\sigma$, in which case the right-hand side takes the form $2^{-j\varepsilon} F_t(z)$ with $\varepsilon > 0$ and $F_t \in \mathrm{L}^1(\partial \mathrm{S}^+_{\pi-2\nu}, \mathrm{d}|z|/|z|)$, locally uniformly in t. This justifies at once convergence and interchanging sum and integral sign in $\mathrm{L}^q(K)$. □

Our third conservation property concerns the Poisson semigroup. In line with the previous result we need L^p-boundedness of the resolvents for large p to compensate for the poor decay of $\mathrm{e}^{-\sqrt{z}} - 1$ at the origin.

Proposition 5.6 (Conservation property for the Poisson semigroups)**.** *If $((1+t^2L)^{-1})_{t>0}$ is L^p-bounded for some $p > n$, then*

$$\mathrm{e}^{-tL^{1/2}} c = c \quad (c \in \mathbb{C}^m,\, t > 0).$$

Proof. We have $\mathrm{e}^{-\sqrt{z}} = (1+z)^{-1} + \psi(z)$ with $\psi \in \Psi^1_{1/2}$ on any sector and the claim follows from Corollary 5.4 and Lemma 5.5. □

6. The four critical numbers

We introduce the sets

$$\mathcal{J}(L) \coloneqq \big\{p \in (1_*, \infty) : ((1+t^2L)^{-1})_{t>0} \text{ is } a^{-1}\,\mathrm{H}^p\text{-bounded}\big\}$$

and

$$\mathcal{N}(L) \coloneqq \big\{p \in (1_*, \infty) : (t\nabla_x(1+t^2L)^{-1})_{t>0} \text{ is } a^{-1}\,\mathrm{H}^p - \mathrm{H}^p\text{-bounded}\big\},$$

where we recall that $1_* = n/(n+1)$. These sets contain $p = 2$ (Corollary 3.12) and since the notion of $a_1\,\mathrm{H}^p - a_2\,\mathrm{H}^p$-boundedness interpolates, they are in fact intervals.

Definition 6.1. The lower and upper endpoints of $\mathcal{J}(L)$ are denoted by $p_-(L)$ and $p_+(L)$, respectively. Similarly, the endpoints of $\mathcal{N}(L)$ are denoted by $q_-(L)$ and $q_+(L)$.

The exponents $p_\pm(L)$ and $q_\pm(L)$ are called *critical numbers* in the following. In this section we study intrinsic relations between these numbers, using the machinery developed in Section 4. For the various duality arguments in this section we recall that $L^\sharp = -(a^*)^{-1} \operatorname{div}_x d^* \nabla_x$ is in the same class as L and similar to L^* under conjugation with a^*. In particular, we have

$$\begin{aligned} 1 \vee p_-(L^\sharp) &= p_+(L)', \\ (1 \vee p_-(L))' &= p_+(L^\sharp). \end{aligned} \tag{6.1}$$

6.1. **General facts on critical numbers.** Here, we prove the following general relations between the four critical numbers. In fact, there are only three of them since $p_-(L)$ and $q_-(L)$ coincide. The two inequalities are best possible in the class of all operators L, see Remark 6.8 further below.

Theorem 6.2. *The critical numbers satisfy*

$$\begin{aligned} p_-(L) &= q_-(L), \\ p_+(L) &\geq q_+(L)^*, \\ p_-(L) &\leq (q_+(L^\sharp)')_*. \end{aligned}$$

We prepare the proof through a sequence of lemmata that are of independent interest.

Lemma 6.3. *Let $n \geq 2$. Then $(2_*, 2^*) \subseteq \mathcal{J}(L)$ and $((1+t^2L)^{-1})_{t>0}$ is $\mathrm{L}^2 - \mathrm{L}^q$-bounded and $\mathrm{L}^{q'} - \mathrm{L}^2$-bounded for every $q \in [2, 2^*] \cap [2, \infty)$.*

Proof. We have $2^* = \infty$ when $n = 2$ and $2^* < \infty$ when $n \geq 3$. The restriction on q is precisely such that we have the Gagliardo–Nirenberg inequality

$$\|u\|_q \lesssim \|\nabla_x u\|_2^\alpha \|u\|_2^{1-\alpha} \quad (u \in \mathrm{W}^{1,2}(\mathbb{R}^n)),$$

where $\alpha = {}^n/_2 - {}^n/_q$. We set $u \coloneqq (1+t^2L)^{-1}f$, $f \in \mathrm{L}^2$, $t > 0$, and use the L^2-boundedness of the resolvent and gradient families to give

$$\|(1+t^2L)^{-1}f\|_q \lesssim t^{-\alpha}\|f\|_2.$$

Hence, the resolvents are $\mathrm{L}^2 - \mathrm{L}^q$-bounded. Interpolation with the L^2 off-diagonal estimates by means of Lemma 4.14 leads to $\mathrm{L}^2 - \mathrm{L}^q$ off-diagonal estimates of arbitrarily large order for any $q \in [2, 2^*)$ and L^q-boundedness follows from Lemma 4.7.

The rest follows by duality and similarity by applying the above to $L^\sharp$ in place of L. □

In dimension $n = 1$ we have $2_* = {}^2/_3$ and by analogy with the previous lemma we expect that $({}^2/_3, \infty) \subseteq \mathcal{J}(L)$. However, in the one-dimensional situation we have $\mathrm{div}_x = \nabla_x$ and this allows us to improve the lower bound to the best possible value $1_* = {}^1/_2$.

Lemma 6.4. *Let $n = 1$. Then $({}^1/_2, \infty) \subseteq \mathcal{J}(L)$ and $(2, \infty) \subseteq \mathcal{N}(L)$. Moreover $((1+t^2L)^{-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^p - \mathrm{L}^2$-bounded for every $p \in ({}^1/_2, 2]$ and $((1+t^2L)^{-1})_{t>0}$ and $(t\frac{\mathrm{d}}{\mathrm{d}x}(1+t^2L)^{-1})_{t>0}$ are both $\mathrm{L}^2 - \mathrm{L}^q$-bounded for every $q \in [2, \infty)$.*

Proof. In the one-dimensional setting the operator L takes the form $L = -a^{-1}\frac{\mathrm{d}}{\mathrm{d}x}(d\frac{\mathrm{d}}{\mathrm{d}x})$ and the space $\mathcal{H}$ in (1.2) coincides with L^2. In particular, just as a, also d is strictly elliptic.

Step 1: $\mathrm{L}^2 - \mathrm{L}^q$-bound for the gradients. It suffices to obtain the bound for $t = 1$ with an implicit constant that depends on the coefficients only through ellipticity. Indeed, for $t \neq 1$ we can use the change of variable $f_t(x) := f(tx)$ in order to write

$$t\tfrac{\mathrm{d}}{\mathrm{d}x}(1+t^2L)^{-1}f(x) = (\tfrac{\mathrm{d}}{\mathrm{d}x}(1+L_t)^{-1}f_t)(t^{-1}x),$$

where $L_t := -a_t^{-1}\frac{\mathrm{d}}{\mathrm{d}x}(d_t\frac{\mathrm{d}}{\mathrm{d}x})$ has the same ellipticity constant as L.

Let now $f \in \mathrm{L}^2$ and set $u := (1+L)^{-1}f$, so that $\frac{\mathrm{d}}{\mathrm{d}x}(d\frac{\mathrm{d}}{\mathrm{d}x}u) = af - au$. In one dimension the Sobolev embedding $\mathrm{W}^{1,2} \subseteq \mathrm{L}^q$ holds for any $q \in [2,\infty)$. Thus, we have

$$\begin{aligned}\|\tfrac{\mathrm{d}}{\mathrm{d}x}u\|_q &\simeq \|d\tfrac{\mathrm{d}}{\mathrm{d}x}u\|_q \\ &\lesssim \|d\tfrac{\mathrm{d}}{\mathrm{d}x}u\|_{\mathrm{W}^{1,2}} \\ &\lesssim \|d\tfrac{\mathrm{d}}{\mathrm{d}x}u\|_2 + \|af\|_2 + \|au\|_2 \\ &\lesssim \|f\|_2,\end{aligned}$$

where in the final step we have used the L^2-boundedness of the resolvent and gradient families. This is the required $\mathrm{L}^2-\mathrm{L}^q$-bound.

Step 2: L^q-bound for the gradients. This follows from Lemma 4.14 and Lemma 4.7 as in the previous proof. Hence, we have $(2,\infty) \subseteq \mathcal{N}(L)$.

Step 3: Bounds for the resolvents. Let $q \in [2,\infty)$ and define $\varrho \in (1/2, 2/3]$ through $1 - 1/q = 1/\varrho - 1$.

For $f \in \mathrm{L}^2$ and $t > 0$ we use the Sobolev embedding $\dot{\mathrm{W}}^{1,q} \subseteq \dot{\Lambda}^{1-1/q}$ and the result of Step 1 for $L^\sharp$ to give

$$\begin{aligned}\|(1+t^2L^\sharp)^{-1}f\|_{\dot{\Lambda}^{1-1/q}} &\lesssim \|\tfrac{\mathrm{d}}{\mathrm{d}x}(1+t^2L^\sharp)^{-1}f\|_q \\ &\lesssim t^{-1+\frac{1}{q}-\frac{1}{2}}\|f\|_2.\end{aligned}$$

Hence, the resolvents of $L^\sharp$ are $\mathrm{L}^2-\dot{\Lambda}^{1-1/q}$-bounded. Since we have $L^\sharp = (a^*)^{-1}L^*a^*$, we obtain by duality that the resolvents of L are $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^2$-bounded, see Lemma 4.3. They also satisfy L^2 off-diagonal estimates of arbitrarily large order and have the cancellation property $\int_{\mathbb{R}^n} a(1+t^2L)^{-1}a^{-1}f\,\mathrm{d}x = 0$ if $f \in \mathrm{L}^2$ has compact support and integral zero, see Corollary 5.4. Hence, we are in a position to apply Lemma 4.9 and obtain $a^{-1}\,\mathrm{H}^p$-boundedness for $p \in (\varrho, 1]$.

Since $q \in [2,\infty)$ was arbitrary, the conclusion is that the resolvents are $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^2$-bounded and $a^{-1}\,\mathrm{H}^p$-bounded for all for all $\varrho \in (1/2, 2/3]$ and all $p \in (1/2, 1]$. By interpolation with the L^2-bound we can allow all $\varrho, p \in (1/2, 2]$. Finally, the $\mathrm{L}^2 - \mathrm{L}^q$ and L^q-bounds of the resolvents for all $q \in (2,\infty)$ follow again duality and similarity, by applying the results for $p \in (1,2]$ to $L^\sharp$. □

We also need a result that allows us to switch between powers of the resolvent in $\mathrm{H}^p - \mathrm{H}^q$-estimates.

Lemma 6.5. *Let $1_* < p \leq q < \infty$ with $q > 1$ and $n/p - n/q < 1$. Suppose that there exists an integer $\beta \geq 1$ such that $(t\nabla_x(1+t^2L)^{-\beta-1})_{t>0}$ is*

$a^{-1}\,\mathrm{H}^p - \mathrm{L}^q$-bounded. Then also $(t\nabla_x(1+t^2L)^{-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^p - \mathrm{L}^q$-bounded.

Proof. Let $t > 0$ and $f \in \mathrm{L}^2$. The Calderón reproducing formula for the injective sectorial operator $T = 1 + t^2 L$ and the auxiliary function $\varphi(z) = z(1+z)^{-\beta-1}$ reads

$$f = \beta \int_0^\infty (1+t^2L)(1+u+t^2uL)^{-\beta-1} f \,\mathrm{d}u. \tag{6.2}$$

Applying the bounded operator $t\nabla_x(1+t^2L)^{-1}$ and re-arranging terms gives

$$\begin{aligned} &t\nabla_x(1+t^2L)^{-1}f \\ &= \beta \int_0^\infty \frac{1}{(1+u)^{\beta+\frac{1}{2}}u^{\frac{1}{2}}} \Big(\frac{ut^2}{1+u}\Big)^{\frac{1}{2}} \nabla_x \Big(1 + \frac{ut^2}{1+u}L\Big)^{-\beta-1} f \mathrm{d}u. \end{aligned}$$

Now, we let $f \in \mathrm{H}^p \cap \mathrm{L}^2$, apply the formula to $a^{-1}f$, and take L^q norms on both sides, in order to give

$$\begin{aligned} &\|t\nabla_x(1+t^2L)^{-1}a^{-1}f\|_q \\ &\quad\lesssim \|f\|_{\mathrm{H}^p} \int_0^\infty \frac{1}{(1+u)^{\beta+\frac{1}{2}}u^{\frac{1}{2}}}\Big(\frac{ut^2}{1+u}\Big)^{\frac{1}{2}(\frac{n}{q}-\frac{n}{p})} \mathrm{d}u \\ &\quad\leq t^{\frac{n}{q}-\frac{n}{p}} \|f\|_{\mathrm{H}^p} \int_0^\infty \frac{u^{\frac{n}{2q}-\frac{n}{2p}-\frac{1}{2}}}{(1+u)^{\beta+\frac{1}{2}+\frac{n}{2q}-\frac{n}{2p}}} \mathrm{d}u. \end{aligned}$$

The numerical integral in u converges as we have $n/q - n/p > -1$ by assumption. □

Proof of Theorem 6.2. The argument is in two steps.

Step 1: Resolvent estimates from gradient bounds. Here, we show the upper bound in the first line and the second and third lines. In dimension $n = 1$ we have $p_-(L) = 1_*$ and $p_+(L) = \infty$ by Lemma 6.4 and there is nothing to prove. For the rest of the step we assume $n \geq 2$.

Let $\varrho \in \mathcal{N}(L)$. If $\varrho < n$, then a Sobolev embedding yields for all $f \in \mathrm{H}^\varrho \cap \mathrm{L}^2$ and all $t > 0$ that

$$\begin{aligned} \|(1+t^2L)^{-1}a^{-1}f\|_{\mathrm{L}^{\varrho^*}} &\lesssim t^{-1}\|t\nabla_x(1+t^2L)^{-1}a^{-1}f\|_{\mathrm{H}^\varrho} \\ &\lesssim t^{-1}\|f\|_{\mathrm{H}^\varrho}. \end{aligned}$$

Hence, the resolvents of L are $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^{\varrho^*}$-bounded. Likewise, if $\varrho > n$, then we obtain for all $f \in \mathrm{L}^\varrho \cap \mathrm{L}^2$ and all $t > 0$ that

$$\|(1+t^2L)^{-1}f\|_{\dot{\Lambda}^{1-n/\varrho}} \lesssim t^{-1}\|f\|_{\mathrm{L}^\varrho}.$$

By duality the resolvents of $L^\sharp$ are $(a^*)^{-1}\,\mathrm{H}^r - \mathrm{L}^{\varrho'}$-bounded. The exponent r is determined by $1 - n/\varrho = n(1/r - 1)$, that is, $r = (\varrho')_*$. From these observations, we can infer further mapping properties in each case.

Step 1a: The Lebesgue case $1 < \varrho < n$. The resolvents of L are $\mathrm{L}^\varrho - \mathrm{L}^{\varrho^*}$-bounded. Lemma 4.14 yields $\mathrm{L}^{[\varrho,2]_\theta} - \mathrm{L}^{[\varrho^*,2]_\theta}$ off-diagonal estimates of arbitrarily large order, where $\theta \in (0,1)$ is arbitrary. Lemma 4.7 yields both $\mathrm{L}^{[\varrho,2]_\theta}$ and $\mathrm{L}^{[\varrho^*,2]_\theta}$-boundedness. Consequently, we must have $p_-(L) \leq \varrho$ and $p_+(L) \geq \varrho^*$.

Step 1b: The Hardy case $\varrho \leq 1$. Since $\mathcal{N}(L)$ is an interval, we have $(\varrho, 2) \subseteq \mathcal{N}(L)$. The first part applies to all exponents in $(1,2)$ instead of ϱ and we first get L^q-boundedness of the resolvents of L for all $q \in (1, 2^*)$ and then L^q off-diagonal estimates of arbitrarily large order by interpolation.

If $\rho = 1$, then $p_-(L) \leq \varrho$ follows directly.

Now, assume $\varrho < 1$. As $\varrho > 1_*$, we can take $q := \varrho^*$ and have L^q off-diagonal estimates of arbitrarily large order and $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^q$-boundedness. For compactly supported $f \in \mathrm{L}^2$, Corollary 5.4 yields $\int_{\mathbb{R}^n} a(1+t^2L)^{-1}a^{-1} f \,\mathrm{d}x = 0$. We have verified the assumptions of Lemma 4.9 and obtain that the resolvents of L are $a^{-1}\,\mathrm{H}^p$-bounded for every $p \in (\varrho, 1]$. Therefore, we have again $p_-(L) \leq \varrho$.

Step 1c: The Hölder case $\varrho > n$. From the preliminary discussion we know that the resolvents of $L^\sharp$ are $(a^*)^{-1}\,\mathrm{H}^{(\varrho')_*} - \mathrm{L}^{\varrho'}$-bounded. We claim that they satisfy $\mathrm{L}^{\varrho'}$ off-diagonal estimates of arbitrarily large order. Taking the claim for granted, $p_-(L^\sharp) \leq (\varrho')_*$ follows as in the previous step.

For the claim we first prove $(1,2) \subseteq \mathcal{J}(L^\sharp)$. In dimension $n = 2$ this is due to Lemma 6.3. In dimension $n \geq 3$ we have $(2,n) \subseteq \mathcal{N}(L)$ since the latter is an interval that contains 2 and ϱ. Step 1a applies to all exponents in $(2,n)$ in place of ϱ and yields $(2,\infty) \subseteq \mathcal{J}(L)$. By duality, we get again $(1,2) \subseteq \mathcal{J}(L^\sharp)$. As we have $\varrho' \in (1,2)$, the $\mathrm{L}^{\varrho'}$ off-diagonal estimates for the resolvents of $L^\sharp$ follow by interpolation with the L^2-result.

Let us conclude Step 1. In dimension $n \geq 3$ the set $\mathcal{N}(L) \cap (1_*, n)$ is non-empty because it contains 2. Letting ϱ vary over $\mathcal{N}(L) \cap (1_*, n)$, we conclude $p_-(L) \leq q_-(L)$ and $p_+(L) \geq q_+(L)^*$ from Steps 1a & 1b. In dimension $n = 2$ the same argument applies unless $q_-(L) = 2$. But in this case[45] the inequalities in question trivially hold because we have $p_-(L) \leq 1$ and $p_+(L) = \infty$ by Lemma 6.3.

As for the third line in the theorem, if $q_+(L^\sharp) \leq n$, then

$$p_-(L) \leq (p_+(L^\sharp))' \leq (q_+(L^\sharp)^*)' = (q_+(L^\sharp)')_*$$

follows from (6.1) and the second line. If $q_+(L^\sharp) > n$, then the inequality $p_-(L) \leq (q_+(L^\sharp)')_*$ follows from Step 1c with the roles of L and $L^\sharp$ switched.

[45]In fact this case never occurs as we shall see later on.

Step 2: Gradient bounds from resolvent estimates. Let $p \in \mathcal{J}(L)$ with $p < 2$. Hence, $((1+t^2L)^{-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^p$-bounded. Lemmata 6.3 and 6.4 guarantee that this family is $\mathrm{L}^\varrho - \mathrm{L}^2$-bounded for some $\varrho \in (1,2)$. According to Lemma 4.4, we find for every $q \in (p,2)$ an integer $\beta(q) \geq 1$ such that $((1+t^2L)^{-\beta(q)})_{t>0}$ is $a^{-1}\,\mathrm{H}^q - \mathrm{L}^2$-bounded. By composition with the L^2-bounded gradient family, $(t\nabla_x(1+t^2L)^{-\beta(q)-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^q - \mathrm{L}^2$-bounded.

Step 2a: The Lebesgue case $p \geq 1$. We know that $(t\nabla_x(1+t^2L)^{-\beta(q)-1})_{t>0}$ is $\mathrm{L}^q - \mathrm{L}^2$-bounded. By composition, this family also satisfies L^2 off-diagonal estimates of arbitrarily large order. Since this holds for every $q \in (p,2)$, we can run the usual argument: $\mathrm{L}^q - \mathrm{L}^2$ off-diagonal estimates of arbitrarily larger order follow by interpolation and this implies L^q-boundedness. Thanks to Lemma 6.5 we get L^q-boundedness also for $(t\nabla_x(1+t^2L)^{-1})_{t>0}$. Since $q \in (p,2)$ was arbitrary, we have $q_-(L) \leq p$.

Step 2b: The Hardy case $p < 1$. We slightly refine the argument in the Lebesgue case by appealing to Lemma 4.9. In the following let $q \in (p,1)$ and $s \in (1,2)$ such that $1/q - 1/s < 1/n$. Such s exists since we have $p > 1_*$.

First, consider the family $(t\nabla_x(1+t^2L)^{-\beta(q)-1})_{t>0}$. It is $a^{-1}\,\mathrm{H}^q - \mathrm{L}^2$-bounded and satisfies L^2 off-diagonal estimates of arbitrarily large order. For compactly supported $f \in \mathrm{L}^2$ we get that $(1+t^2L)^{-\beta(q)}a^{-1}f$ and $\nabla_x(1+t^2L)^{-\beta(q)}a^{-1}f$ are in L^1 from the L^2 off-diagonal decay of order $\gamma > n/2$. The integral of the gradient of a $\mathrm{W}^{1,1}$-function vanishes, so $\int_{\mathbb{R}^n} t\nabla_x(1+t^2L)^{-\beta(q)}a^{-1}f\,\mathrm{d}x = 0$. We have checked the assumptions of Lemma 4.9 and obtain $a^{-1}\,\mathrm{H}^q - \mathrm{H}^q$-boundedness for every q. This interpolates with the original $a^{-1}\,\mathrm{H}^q - \mathrm{L}^2$-boundedness, so that the conclusion is $a^{-1}\,\mathrm{H}^q - \mathrm{L}^s$-boundedness for all q and s.

Now, we consider $(t\nabla_x(1+t^2L)^{-1})_{t>0}$. Lemma 6.5 yields $a^{-1}\,\mathrm{H}^q - \mathrm{L}^s$-boundedness for all q and s. Step 2a applies to s and yields L^s-boundedness, which implies L^s off-diagonal decay of arbitrarily large order for every s by interpolation with the L^2-result. As before, we also have $\int_{\mathbb{R}^n} t\nabla_x(1+t^2L)^{-1}a^{-1}f\,\mathrm{d}x = 0$ for compactly supported $f \in \mathrm{L}^2$. We have again verified the assumptions of Lemma 4.9 and conclude for $a^{-1}\,\mathrm{H}^q - \mathrm{H}^q$-boundedness for every q. Thus, we have $q_-(L) \leq p$.

As $p \in \mathcal{J}(L) \cap (1_*, 2)$ was arbitrary, Steps 2a & 2b yield the missing inequality $q_-(L) \leq p_-(L)$ that completes the proof of the first line in the theorem. □

6.2. **Worst-case estimates for the critical numbers.** The following extrapolation from the L^2-theory has been proved by an application of Šneĭberg's stability theorem [10, 86].

Proposition 6.6 ([21, Prop. 4.5])**.** *There exists $\varepsilon > 0$, depending on ellipticity and dimensions, such that whenever $p \in [2-\varepsilon, 2+\varepsilon]$, then $((1+\mathrm{i}tDB)^{-1})_{t\in\mathbb{R}}$ is L^p-bounded.*

We use this result to give the following global picture for the critical numbers for the class of all L in all dimensions.

Proposition 6.7. *The following relations hold.*

(i) *In dimension $n = 1$,*

$$p_-(L) = q_-(L) = \frac{1}{2} \quad \& \quad p_+(L) = q_+(L) = \infty.$$

(ii) *In dimension $n \geq 2$ there exists $\varepsilon > 0$, depending on ellipticity and dimensions, such that*

$$p_-(L) = q_-(L) \leq 2_* - \varepsilon \quad \& \quad p_+(L) \geq 2^* + \varepsilon \quad \& \quad q_+(L) \geq 2 + \varepsilon.$$

Proof. The identification $q_-(L) = p_-(L)$ in any dimension is due to Theorem 6.2. In dimension $n = 1$, Lemma 6.4 shows that $p_-(L) = 1/2$ and $q_+(L) = \infty = p_+(L)$ take the best possible values. Hence, (i) follows.

As for (ii), we use (3.4) to write, whenever $t > 0$,

$$\frac{1}{2}\Big((1 + \mathrm{i}tDB)^{-1} - (1 - \mathrm{i}tDB)^{-1}\Big) = -\mathrm{i}tDB(1 + t^2(DB)^2)^{-1}$$
$$= \begin{bmatrix} -\mathrm{i}t \operatorname{div}_x d(1 + t^2\widetilde{M})^{-1} \\ \mathrm{i}t\nabla_x a^{-1}(1 + t^2\widetilde{L})^{-1} \end{bmatrix}.$$

This family is L^p-bounded for $p \in [2-\varepsilon, 2+\varepsilon]$ due to Proposition 6.6. In particular, the second component is L^p-bounded and since a is strictly elliptic, the same is true for $t\nabla_x a^{-1}(1 + t^2\widetilde{L})^{-1} a = t\nabla_x(1 + t^2L)^{-1}$. Hence, for a possibly different choice of ε we have $q_+(L) \geq 2 + \varepsilon$ and $q_-(L) \leq 2 - \varepsilon$. The same thing for $L^\sharp$. Now, the claim follows from Theorem 6.2. □

Remark 6.8. (i) In the one-dimensional setting the identification of the critical numbers could also be obtained from the kernel estimates in [18]. They are only stated for $m = 1$ but the argument literally applies to systems ($m > 1$) under our ellipticity assumption. In fact, the proof of Lemma 6.4 mimics some intermediate steps in [18]. The value $p_-(L) = 1/2$ has appeared in a related context in [23].

(ii) In higher dimensions the bounds above cannot be improved in general, even when $a = 1$ and $m = 1$. More precisely, given $\varepsilon > 0$, any of $p_-(L) < 2_* - \varepsilon$, $p_+(L) > 2^* + \varepsilon$, $q_+(L) > 2 + \varepsilon$ can fail for some L.

Indeed, for $p_\pm$ in dimensions $n \geq 3$, counterexamples rely on Frehse's irregular solution [47] and can be found in [58, Prop. 2.10]. In view of Theorem 6.2 such counterexamples satisfy $2^* + \varepsilon \geq p_+(L) \geq q_+(L)^* \geq 2^*$. Hence, they also serve as counterexamples to the general improvement of q_+ and show that the inequalities in Theorem 6.2 are best possible in the class of all operators L.

When $n = 2$, the counterexample for q_+ due to Kenig comes with d real symmetric [24, Sec. 4.2.2]. The same operator is a counterexample for the general improvement on p_-, that is, $p_-(L)$ can be as close to $2_* = 1$ as one wants. Hence, the final inequality in Theorem 6.2 is again best possible.

6.3. ***a*-independence of critical numbers.** It is tempting to compare the critical numbers for L with those for

$$L_0 = -\operatorname{div}_x d\nabla_x,$$

seeing $L = a^{-1}L_0$ as a multiplicative perturbation of L_0. Let us prove that the critical numbers for both operators are indeed the same.

Theorem 6.9. *The critical numbers for L and L_0 coincide, that is,*

$$p_\pm(L) = p_\pm(L_0) \quad \& \quad q_\pm(L) = q_\pm(L_0).$$

Proof. The claim in dimension $n = 1$ is an immediate consequence of Proposition 6.7. The proof in dimensions $n \geq 2$ is divided into six steps.

Step 1: $p_-(L) \leq p_-(L_0)$. Let $p \in \mathcal{J}(L_0) \cap (1_*, 2_*]$. This interval is non-empty thanks to Proposition 6.7. We set $p_0 := p$, define iteratively $p_k := p_{k-1}^*$ and stop at the first exponent $k^+ \geq 0$ with $p_{k^+} \in (2_*, 2]$. We shall prove by backward induction that $(p_k, 2] \subseteq \mathcal{J}(L)$ for all k. Hence, we eventually find $(p, 2] \subseteq \mathcal{J}(L)$ and taking the infimum over all p yields $p_-(L) \leq p_-(L_0)$.

Once again by Proposition 6.7, we have $(p_{k^+}, 2] \subseteq \mathcal{J}(L)$. For the inductive step we assume $(p_k, 2] \subseteq \mathcal{J}(L)$ and pick any $q \in (p_{k-1}, 2_*]$. For all $t > 0$ we have

$$1 = (a + t^2 L_0)(1 + t^2 L_0)^{-1} + (1 - a)(1 + t^2 L_0)^{-1}$$

as operators on L^2. Multiplication by $(1 + t^2 L)^{-1} a^{-1} = (a + t^2 L_0)^{-1}$ from the left yields the key identity

$$\begin{aligned} &(1 + t^2 L)^{-1} a^{-1} \\ &\quad = (1 + t^2 L_0)^{-1} + (1 + t^2 L)^{-1} a^{-1} (1 - a)(1 + t^2 L_0)^{-1}. \end{aligned} \tag{6.3}$$

On the right-hand side $((1+t^2 L)^{-1})_{t>0}$ is L^{q^*}-bounded by the induction hypothesis. By Theorem 6.2 we have $q_-(L_0) = p_-(L_0)$ so that $((t\nabla_x(1+t^2 L_0)^{-1})_{t>0}$ is H^q-bounded. By a Sobolev embedding we have

$$\|(1 + t^2 L_0)^{-1} f\|_{q^*} \lesssim \|\nabla_x (1 + t^2 L_0)^{-1} f\|_{\mathrm{H}^q} \lesssim t^{-1} \|f\|_{\mathrm{H}^q},$$

whenever $f \in \mathrm{H}^q \cap \mathrm{L}^2$ and $t > 0$. Hence, $((1 + t^2 L_0)^{-1})_{t>0}$ is $\mathrm{H}^q - \mathrm{L}^{q^*}$-bounded. Now, it follows from (6.3) that $((1+t^2 L)^{-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^q - \mathrm{L}^{q^*}$-bounded. This was the key step.

If $q > 1$, then we have $\mathrm{L}^q - \mathrm{L}^{q^*}$-boundedness for the resolvents of L. Interpolation with the L^2 off-diagonal estimates (Lemma 4.14) followed by Lemma 4.7 yields $(q, 2] \subseteq \mathcal{J}(L)$.

If $q = 1$, then $(p_{k-1}, 2_*]$ also contains exponents that are strictly smaller than 1 and we can jump right into the following case.

In the remaining case $q < 1$ we have $a^{-1}\,\mathrm{H}^q - \mathrm{L}^{q^*}$-boundedness for the resolvents of L. As q^* is an interior point of $\mathcal{J}(L)$ by the induction hypothesis, we get again L^{q^*} off-diagonal estimates of arbitrarily large order from the ones on L^2 by interpolation. For compactly supported $f \in \mathrm{L}^2$, Corollary 5.4 yields $\int_{\mathbb{R}^n} a(1+t^2L)^{-1}a^{-1}f \,\mathrm{d}x = 0$. This means that we have verified the assumptions of Lemma 4.9 and $(q, 2] \subseteq \mathcal{J}(L)$ follows.

Step 2: $p_-(L_0) \leq p_-(L)$. We only need a key identity replacing (6.3) and allowing us to deduce $\mathrm{H}^q - \mathrm{L}^{q^*}$-boundedness of $((1+t^2L_0)^{-1})_{t>0}$ from L^{q^*}-boundedness of $((1+t^2L_0)^{-1})_{t>0}$ and $a^{-1}\,\mathrm{H}^q - \mathrm{L}^{q^*}$-boundedness of $((1+t^2L)^{-1})_{t>0}$. The rest of the proof for $p_-(L) \leq p_-(L_0)$ was symmetric in L and L_0.

For the new key identity we split

$$1 = (1+t^2L_0)(a+t^2L_0)^{-1} + (a-1)(a+t^2L_0)^{-1}$$

and multiply by $(1+t^2L_0)^{-1}$ from the left in order to get the desirable decomposition

$$(1+t^2L_0)^{-1} = (1+t^2L)^{-1}a^{-1} + (1+t^2L_0)^{-1}(a-1)(1+t^2L)^{-1}a^{-1}.$$

Step 3: $q_-(L) = q_-(L_0)$. It follows from the first two steps and Theorem 6.2.

Step 4: $p_+(L) = p_+(L_0)$. Simply note that by duality relations (6.1) and the first two steps we have

$$p_+(L) = (1 \vee p_-(L^\sharp))' = (1 \vee p_-(L_0^\sharp))' = p_+(L_0).$$

Step 5: $q_+(L_0) \leq q_+(L)$. Let $2 \leq q < q_+(L_0)$. For $t > 0$ we use a new decomposition, namely

$$\begin{aligned} &t\nabla_x(1+t^2L)^{-1} \\ &\quad= t\nabla_x(1+t^2L_0)^{-1}(a+t^2L_0+1-a)(1+t^2L)^{-1} \\ &\quad= t\nabla_x(1+t^2L_0)^{-1}a + t\nabla_x(1+t^2L_0)^{-1}(1-a)(1+t^2L)^{-1}. \end{aligned} \tag{6.4}$$

On the right-hand side $(t\nabla_x(1+t^2L_0)^{-1})_{t>0}$ is L^q-bounded by assumption and $((1+t^2L)^{-1})_{t>0}$ is L^q-bounded since we have $q_+(L_0) \leq p_+(L_0) = p_+(L)$ by Theorem 6.2 and Step 4. Thus, $(t\nabla_x(1+t^2L)^{-1})_{t>0}$ is L^q-bounded. Taking the supremum over all q, we obtain $q_+(L_0) \leq q_+(L)$.

Step 6: $q_+(L) \leq q_+(L_0)$. The argument follows by reversing the roles of L and L_0 in Step 4 and using the identity

$$t\nabla_x(1+t^2L_0)^{-1}$$

$$= t\nabla_x(a+t^2L_0)^{-1}(1+t^2L_0+a-1)(1+t^2L_0)^{-1}$$
$$= t\nabla_x(1+t^2L)^{-1}a^{-1} + t\nabla_x(1+t^2L)^{-1}a^{-1}(a-1)(1+t^2L_0)^{-1}$$

instead of (6.4). □

As an application of Theorem 6.9 we determine the critical numbers of multiplicative perturbations of the (coordinatewise acting) Laplacian.

Corollary 6.10. *In any dimension it follows that*

$$p_-(-a^{-1}\Delta_x) = q_-(-a^{-1}\Delta_x) = 1_*,$$
$$p_+(-a^{-1}\Delta_x) = q_+(-a^{-1}\Delta_x) = \infty.$$

This result is originally due to McIntosh–Nahmod, see Theorem 3.3 and §5.(v) in [78]. Here, we have used a rather different and simpler method. In Section 14, we shall discuss kernel estimates

Proof of Corollary 6.10. In view of Theorem 6.2 we only have to prove that $q_+(-a^{-1}\Delta_x) = \infty$. By Theorem 6.9 we have $q_+(a^{-1}\Delta_x) = q_+(-\Delta_x)$ and there are many ways to see that $q_+(-\Delta_x) = \infty$. One is to note that $t\nabla_x(1+t^2(-\Delta_x))^{-1}$ corresponds to the Fourier multiplier $\xi \mapsto \mathrm{i}t(1+t^2|\xi|^2)^{-1}\xi$, which falls under the scope of the Mihlin multiplier theorem. □

7. Riesz transform estimates: Part I

We introduce the set

$$\mathcal{I}(L) := \big\{p \in (1_*, \infty) : \nabla_x L^{-1/2} \text{ is } a^{-1}\,\mathrm{H}^p - \mathrm{H}^p\text{-bounded}\big\}. \tag{7.1}$$

Some clarification on the meaning of $\nabla_x L^{-1/2}$ being $a^{-1}\,\mathrm{H}^p - \mathrm{H}^p$-bounded is necessary since there are two possible interpretations:

- As we have seen in Section 3.6, $L^{1/2} : \mathrm{W}^{1,2} \to \mathrm{L}^2$ extends to an isomorphism $\dot{\mathrm{W}}^{1,2} \to \mathrm{L}^2$ that we denote again by $L^{1/2}$. In this sense $R_L := \nabla_x L^{-1/2}$ is defined as a bounded operator on L^2. The question of $a^{-1}\,\mathrm{H}^p - \mathrm{H}^p$-boundedness for R_L fits into the abstract framework of Section 4 and means that
$$\|R_L a^{-1} f\|_{\mathrm{H}^p} \lesssim \|f\|_{\mathrm{H}^p} \quad (f \in \mathrm{H}^p \cap \mathrm{L}^2)$$
and when $p > 1$ equivalently that
$$\|R_L f\|_p \lesssim \|f\|_p \quad (f \in \mathrm{L}^p \cap \mathrm{L}^2).$$
- We could also avoid the extension, work directly with $\nabla_x L^{-1/2}$ defined on $\mathsf{R}(L^{1/2})$ and ask for $\|\nabla_x L^{-1/2} a^{-1} f\|_{\mathrm{H}^p} \lesssim \|f\|_{\mathrm{H}^p}$ for all $f \in \mathrm{H}^p \cap \mathsf{R}(aL^{1/2})$.

We opt for the first interpretation, which is stronger. Then, by interpolation, $\mathcal{I}(L)$ is an interval and we make the following

Definition 7.1. The lower and upper endpoint of $\mathcal{I}(L)$ are denoted by $r_-(L)$ and $r_+(L)$, respectively.

The two interpretations above agree if $\mathrm{H}^p \cap \mathsf{R}(aL^{1/2})$ is dense in $\mathrm{H}^p \cap \mathrm{L}^2$, but *a priori* this information might not be available. It happens for $p \in \mathcal{J}(L) \cap (1,\infty)$ though, as the following more general lemma shows.

Lemma 7.2. *If $p \in \mathcal{J}(L) \cap (1,\infty)$, then the spaces $\mathrm{L}^p \cap \mathsf{D}(L^k) \cap \mathsf{R}(L^k)$, $k \in \mathbb{N}$, are all dense in $\mathrm{L}^p \cap \mathrm{L}^2$.*

Proof. By the Hahn–Banach theorem it suffices to check density for the weak topology. Given $f \in \mathrm{L}^p \cap \mathrm{L}^2$, we consider approximants in $\mathsf{D}(L^k) \cap \mathsf{R}(L^k)$ defined by

$$\begin{aligned} f_j &:= (jL)^k (1+jL)^{-k} (1+j^{-1}L)^{-k} \\ &= (1-(1+jL)^{-1})^k (1+j^{-1}L)^{-k} \quad (j \in \mathbb{N}). \end{aligned}$$

By the convergence lemma we have $f_j \to f$ in L^2 as $j \to \infty$. On the other hand, (f_j) is bounded in $\mathrm{L}^p \cap \mathrm{L}^2$ and this space is reflexive since it is isomorphic to a closed subspace of a reflexive space, namely the diagonal in $\mathrm{L}^p \times \mathrm{L}^2$. Hence, it has a weak accumulation point in $\mathrm{L}^p \cap \mathrm{L}^2$, which by L^2-convergence has to be f. □

In this section we shall identify $(r_-(L), r_+(L)) \cap (1,\infty)$. Hence, we are studying L^p-boundedness of $\nabla_x L^{-1/2}$. Later on, in Section 11, we will complete the results on the Riesz transform by identifying $\mathcal{I}(L)$ in the full range of exponents. This will require different methods.

Here is our main result on the Riesz transform in the L^p-scale.

Theorem 7.3. *The endpoints of $\mathcal{I}(L) \cap (1,\infty)$ can be characterized as follows:*

$$r_-(L) \vee 1 = p_-(L) \vee 1 \quad \& \quad r_+(L) = q_+(L).$$

Theorem 7.3 requires establishing four implications that we shall present in a separate section each. The outline follows [6, Ch. 5]. To begin with, we need suitable singular integral representations for R_L. Let $\alpha \in \mathbb{N}$. Writing out the Calderón reproducing formula for the auxiliary function $z^{3\alpha+1/2}(1+z)^{-9\alpha}$ and applying $R_L = \nabla_x L^{-1/2}$ on both sides, we have for all $f \in \mathrm{L}^2$ the representation via an improper Riemann integral

$$R_L f = \frac{1}{c_\alpha} \int_0^\infty t \nabla_x (1+t^2L)^{-3\alpha} (t^2L)^{3\alpha} (1+t^2L)^{-6\alpha} f \, \frac{\mathrm{d}t}{t}, \tag{7.2}$$

where c_α is a constant depending on α. We note that on the right-hand side we do not have to deal with the extension of the square root. More precisely, the truncated Riesz transforms defined for $\varepsilon \in (0,1)$ via

$$R_L^\varepsilon f := \frac{1}{c_\alpha} \nabla_x L^{-1/2} \int_\varepsilon^{1/\varepsilon} (1+t^2L)^{-3\alpha} (t^2L)^{3\alpha+1/2} (1+t^2L)^{-6\alpha} f \, \frac{\mathrm{d}t}{t} \tag{7.3}$$

converge strongly on L^2 towards R_L as $\varepsilon \to 0$. The way to treat the kernel in (7.2) or (7.3) will be through $\mathrm{L}^p - \mathrm{L}^2$ and $\mathrm{L}^2 - \mathrm{L}^p$ off-diagonal bounds that we record in the next lemma.

Lemma 7.4. *Let $p \in \mathcal{J}(L)$ and let q be between p and 2. There exists an integer $\beta = \beta(p,q,n)$ with the following property.*

(i) *If $p < 2$, then $((1+t^2L)^{-\beta})_{t>0}$ and $(t\nabla_x(1+t^2L)^{-\beta-1})_{t>0}$ satisfy $\mathrm{L}^q - \mathrm{L}^2$ off-diagonal estimates of arbitrarily large order.*

(ii) *If $p > 2$, then $((1+t^2L)^{-\beta})_{t>0}$ satisfies $\mathrm{L}^2 - \mathrm{L}^q$ off-diagonal estimates of arbitrarily large order.*

Proof. We begin with (i). The resolvents are L^p-bounded by assumption and $\mathrm{L}^\varrho - \mathrm{L}^2$-bounded for some $\varrho = \varrho(n) \in (1,2)$ due to Lemmata 6.3 and 6.4. Lemma 4.4 furnishes an integer $\beta = \beta(p,q,n)$ such that $((1+t^2L)^{-\beta})_{t>0}$ is $\mathrm{L}^q - \mathrm{L}^2$-bounded. This holds for all such exponents q, so the off-diagonal estimates follow by interpolation with the L^2-result. The claim for the gradients follows by composition since $(t\nabla_x(1+t^2L)^{-1})$ satisfies L^2 off-diagonal estimates of arbitrarily large order.

As for (ii), we can argue by duality and similarity. Indeed, (i) applies to $L^\sharp := (a^*)^{-1}L^*a^*$ and we have $(2, (p_-(L^\sharp) \vee 1)') = (2, p_+(L))$. □

7.1. **Sufficient condition for $1 < p < 2$.** We prove $(p_-(L) \vee 1, 2) \subseteq \mathcal{I}(L)$. Due to the L^2-bound and the Marcinkiewicz interpolation theorem it suffices to show that R_L is of weak type (p,p) for every $p \in (p_-(L) \vee 1, 2)$. We fix such p and use Blunck–Kunstmann's criterion [30] in its simplified version as stated in [6, Thm. 1.1]:

Proposition 7.5. *Let $p \in [1,2)$. Suppose that T is a sublinear operator of strong type $(2,2)$ and let A_r, $r > 0$, be a family of bounded linear operators on $\mathrm{L}^2(\mathbb{R}^n)$. Assume for $j \geq 2$ that*

$$\bigg(\frac{1}{|B|} \int_{C_j(B)} |T(1-A_{r(B)})f|^2\bigg)^{\frac{1}{2}} \leq g(j) \bigg(\frac{1}{|B|} \int_B |f|^p\bigg)^{\frac{1}{p}} \tag{7.4}$$

and for $j \geq 1$ that

$$\bigg(\frac{1}{|B|} \int_{C_j(B)} |A_{r(B)}f|^2\bigg)^{\frac{1}{2}} \leq g(j) \bigg(\frac{1}{|B|} \int_B |f|^p\bigg)^{\frac{1}{p}}, \tag{7.5}$$

for all balls B and all $f \in \mathrm{L}^2$ with support in B. If $\Sigma = \sum_j g(j) 2^{jn/2}$ is finite, then T is of weak type (p,p) with a bound depending on p_0, p, Σ and the strong type $(2,2)$-bound.

We check (7.4) and (7.5) for $T = R_L$ the Riesz transform and

$$A_r \coloneqq 1 - \varphi(r^2 L), \tag{7.6}$$

where

$$\varphi(z) \coloneqq (1 - (1+z)^{-\beta})^{3\alpha}. \tag{7.7}$$

Here, $\alpha \in \mathbb{N}$ is as in (7.2). It will be chosen larger in the further course. Since p is not the lower endpoint of $\mathcal{J}(L) \cap (1, 2]$, we can pick $\beta \in \mathbb{N}$ sufficiently large according to Lemma 7.4 to have $\mathrm{L}^p - \mathrm{L}^2$ off-diagonal estimates of arbitrarily large order for $((1 + t^2 L)^{-\beta})_{t>0}$ at our disposal.

Step 1: Verification of (7.5) with $g(j) = c2^{-\gamma j}$ and arbitrary $\gamma > 0$. Expanding

$$A_r = -\sum_{k=1}^{3\alpha} \binom{3\alpha}{k} (-1)^k (1 + r^2 L)^{-\beta k} \tag{7.8}$$

and using the $\mathrm{L}^p - \mathrm{L}^2$ off-diagonal decay, we immediately get (7.5) with $g(j) = c2^{-j\gamma}$ with $\gamma > 0$ as large as we want and c depending on α, β, γ. We take $\gamma > n/2$ to meet the summing condition in Proposition 7.5.

Step 2: Verification of (7.4) with $g(j) = c2^{j(n/2 - n/p - \alpha)}$. Let B be a ball of radius $r > 0$ and let f be supported in B. We abbreviate $C_j(B)$ by C_j and for $j \geq 2$ we introduce $D_j \coloneqq 2^{j-1} B$. Then

$$\mathrm{d}(C_j, D_j) \simeq 2^j r \simeq \mathrm{d}(B, {}^c D_j).$$

The representation (7.2) yields

$$\begin{aligned} &\|R_L(1 - A_r) f\|_{\mathrm{L}^2(C_j)} \\ &\leq \int_0^\infty \|t \nabla_x (1 + t^2 L)^{-3\alpha} \mathbf{1}_{D_j} \psi(t^2 L) \varphi(r^2 L) f\|_{\mathrm{L}^2(C_j)} \frac{\mathrm{d}t}{t} \\ &\quad + \int_0^\infty \|t \nabla_x (1 + t^2 L)^{-3\alpha} \mathbf{1}_{{}^c D_j} \psi(t^2 L) \varphi(r^2 L) f\|_{\mathrm{L}^2(C_j)} \frac{\mathrm{d}t}{t}, \end{aligned} \tag{7.9}$$

with an auxiliary function

$$\psi(z) \coloneqq c_\alpha z^{3\alpha} (1 + z)^{-6\alpha}. \tag{7.10}$$

From now on we require $3\alpha \geq \beta + 1$. Composing L^2 off-diagonal estimates for the resolvents and their gradients and $\mathrm{L}^p - \mathrm{L}^2$ off-diagonal estimates for the β-th powers of the resolvents, we find that

$$t \nabla_x (1 + t^2 L)^{-3\alpha} = t \nabla_x (1 + t^2 L)^{-3\alpha + \beta} (1 + t^2 L)^{-\beta}$$

satisfies $\mathrm{L}^p - \mathrm{L}^2$ off-diagonal estimates of arbitrarily large order. Thus,

$$\begin{aligned} &\|t \nabla_x (1 + t^2 L)^{-3\alpha} \mathbf{1}_{D_j} \psi(t^2 L) \varphi(r^2 L) f\|_{\mathrm{L}^2(C_j)} \\ &\lesssim t^{\frac{n}{2} - \frac{n}{p}} \left(1 + \frac{2^j r}{t}\right)^{-\gamma} \|\psi(t^2 L) \varphi(r^2 L) f\|_p, \end{aligned} \tag{7.11}$$

with $\gamma > 0$ at our disposal. From (7.7) and (7.10) we can read off the decay properties $\varphi \in \Psi_{3\alpha}^0$ and $\psi \in \Psi_{3\alpha}^{3\alpha}$. Thus we find by the third part of Lemma 4.16 that

$$\|\psi(t^2L)\varphi(r^2L)f\|_p \lesssim \left(\frac{r}{t}\right)^{2\alpha}\|f\|_p. \tag{7.12}$$

We remark that in applying Lemma 4.16 we do not need to switch to an exponent $q \in (p,2]$ since p is not the lower endpoint of $\mathcal{J}(L) \cap (1,2]$. The combination of the previous two estimates is

$$\begin{aligned}\|t\nabla_x(1+t^2L)^{-3\alpha}\mathbf{1}_{D_j}\psi(t^2L)\varphi(r^2L)f\|_{\mathrm{L}^2(C_j)}\\ \lesssim \left(\frac{r}{t}\right)^{2\alpha-\frac{n}{2}+\frac{n}{p}}\left(1+\frac{2^jr}{t}\right)^{-\gamma} r^{\frac{n}{2}-\frac{n}{p}}\|f\|_p\end{aligned}$$

and integrating the resulting bound with respect to $\mathrm{d}t/t$ and changing variables to $s = 2^jr/t$ leads us to

$$\int_0^\infty \|t\nabla_x(1+t^2L)^{-3\alpha}\mathbf{1}_{D_j}\psi(t^2L)\varphi(r^2L)f\|_{\mathrm{L}^2(C_j)}\frac{\mathrm{d}t}{t} \le g(j)r^{\frac{n}{2}-\frac{n}{p}}\|f\|_p,$$

where

$$g(j) := 2^{j(\frac{n}{2}-\frac{n}{p}-2\alpha)}\int_0^\infty \frac{s^{2\alpha-\frac{n}{2}+\frac{n}{p}}}{(1+s)^\gamma}\frac{\mathrm{d}s}{s}.$$

We take $\gamma > 2\alpha - n/2 + n/p$ to have a finite integral in s and $2\alpha > n - n/p$ to take care of the summing condition in Proposition 7.5. This completes the treatment of the first integral on the right of (7.9).

For the second integral the roles of uniform boundedness and off-diagonal estimates are reversed. Indeed, as ${}^cD_j$ and C_j intersect, our replacement for (7.11) becomes

$$\begin{aligned}\|t\nabla_x(1+t^2L)^{-3\alpha}\mathbf{1}_{{}^cD_j}\psi(t^2L)\varphi(r^2L)f\|_{\mathrm{L}^2(C_j)}\\ \lesssim t^{\frac{n}{2}-\frac{n}{p}}\|\psi(t^2L)\varphi(r^2L)f\|_{\mathrm{L}^p({}^cD_j)}\end{aligned} \tag{7.13}$$

and from the first and second part of Lemma 4.16 we obtain the bound

$$\|\psi(t^2L)\varphi(r^2L)f\|_{\mathrm{L}^p({}^cD_j)} \lesssim \begin{cases}\left(1+\frac{2^{j-1}r}{t}\right)^{-6\alpha}\|f\|_p & \text{if } t \le r\\ (1+2^{j-1})^{-6\alpha}\|f\|_p & \text{if } t \ge r.\end{cases}$$

In addition we still have the uniform bound (7.12) and thus, using both estimates raised to the power $1/2$, we have

$$\|\psi(t^2L)\varphi(r^2L)f\|_{\mathrm{L}^p({}^cD_j)} \lesssim \left(\frac{r}{t}\right)^{\alpha}\left(1+\frac{2^jr}{t}\right)^{-3\alpha}\|f\|_p. \tag{7.14}$$

We combine the latter estimate with (7.11), integrate in t and change variables to $s = 2^jr/t$ as before in order to obtain

$$\int_0^\infty \|t\nabla_x(1+t^2L)^{-3\alpha}\mathbf{1}_{{}^cD_j}\psi(t^2L)\varphi(r^2L)f\|_{\mathrm{L}^2(C_j)}\frac{\mathrm{d}t}{t} \lesssim g(j)r^{\frac{n}{2}-\frac{n}{p}}\|f\|_p$$

where this time

$$g(j) := 2^{j(\frac{n}{2}-\frac{n}{p}-\alpha)} \int_0^\infty \frac{s^{\alpha-\frac{n}{2}+\frac{n}{p}}}{(1+s)^{3\alpha}} \frac{\mathrm{d}s}{s}.$$

We take $\alpha > n/p - n/2$ to have a finite integral in s and $\alpha > n - n/p$ to take care of the summing condition in Proposition 7.5. This completes the treatment of the second integral on the right of (7.9) and also the proof of the weak (p,p)-bound for R_L is complete.

7.2. **Sufficient condition for $p > 2$.** We prove $(2, q_+(L)) \subseteq \mathcal{I}(L)$. We let $p \in (2, q_+(L))$ and prove that the Riesz transform R_L is L^p-bounded. We use again the singular integral representation (7.2) with a parameter $\alpha \in \mathbb{N}$ to be chosen large in the further course of the proof.

The kernel of the truncated Riesz transforms R_L^ε in (7.3) given by

$$\begin{aligned} &t\nabla_x(1+t^2L)^{-3\alpha}(t^2L)^{3\alpha}(1+t^2L)^{-6\alpha} \\ &\quad = t\nabla_x(1+t^2L)^{-1}\,(1-(1+t^2L)^{-1})^{3\alpha}(1+t^2L)^{-6\alpha} \end{aligned} \tag{7.15}$$

is L^p-bounded since we have $p < p_+(L)$ by Theorem 6.2. Consequently, each R_L^ε is L^p-bounded with a bound depending on ε and it suffices to establish a uniform L^p-bound in order to conclude for L^p-boundedness of R_L. To this end we ultimately fix some $p_0 \in (p, q_+(L))$ and employ the following criterion.

Proposition 7.6 ([6, Thm. 1.2])**.** *Let $p_0 \in (2, \infty]$. Suppose that T is a sublinear operator acting on $\mathrm{L}^2(\mathbb{R}^n)$ and let A_r, $r > 0$, a family of linear operators acting on $\mathrm{L}^2(\mathbb{R}^n)$. Assume*

$$\left(\frac{1}{|B|}\int_B |T(1-A_{r(B)})f|^2\right)^{\frac{1}{2}} \leq C(\mathcal{M}(|f|^2))^{\frac{1}{2}}(y) \tag{7.16}$$

and

$$\left(\frac{1}{|B|}\int_B |TA_{r(B)}f|^{p_0}\right)^{\frac{1}{p_0}} \leq C(\mathcal{M}(|Tf|^2))^{\frac{1}{2}}(y) \tag{7.17}$$

for all $f \in \mathrm{L}^2$, all balls B and all $y \in B$. If $2 < p < p_0$ and $Tf \in \mathrm{L}^p$ whenever $f \in \mathrm{L}^p \cap \mathrm{L}^2$, then

$$\|Tf\|_p \leq c\|f\|_p,$$

where c depends only on n, p, p_0, C.

As $p_0 < p_+(L)$, we can use again Lemma 7.4 to find some large $\beta \in \mathbb{N}$ for which $((1+t^2L)^{-\beta+1})_{t>0}$ satisfies $\mathrm{L}^2 - \mathrm{L}^{p_0}$ off-diagonal estimates of arbitrarily large order. Then we define the same approximating family A_r as in (7.6) and our task is to verify (7.16) and (7.17) for $T = R_L^\varepsilon$ and a constant C that does not depend on ε.

We assume right away that $6\alpha \geq \beta - 1$. By composition, this guarantees that the kernel in (7.15) is $\mathrm{L}^2 - \mathrm{L}^{p_0}$-bounded and hence that R_L^ε also maps L^2 into L^{p_0}.

Step 1: Verification of (7.16)*.* Let $f \in \mathrm{L}^2$ and B a ball of radius r. We claim that

$$\|R_L^\varepsilon(1 - A_r)f\|_{\mathrm{L}^2(B)} \lesssim r^{\frac{n}{2}} \sum_{j=1}^\infty g(j) \bigg(\fint_{2^{j+1}B} |f|^2 \bigg)^{\frac{1}{2}} \tag{7.18}$$

with $g(j) = C2^{-j(\alpha-n)}$ and C a constant that does not depend on ε. Since each integral on the right-hand side is bounded by $\mathcal{M}(|f|^2)(y)$ for every $y \in B$, this bound yields (7.16) provided that we take $\alpha > n$.

For the claim we write $f = \sum_{j=1}^\infty f_j$, where $f_j := \mathbf{1}_{C_j} f$ and $C_j := C_j(B)$, and obtain

$$\|R_L^\varepsilon(1 - A_r)f\|_{\mathrm{L}^2(B)} \leq \sum_{j=1}^\infty \|R_L^\varepsilon(1 - A_r)f_j\|_{\mathrm{L}^2(B)}.$$

The term for $j = 1$ is readily handled by L^2-boundedness of $R_L^\varepsilon(1-A_r)$:

$$\|R_L^\varepsilon(1 - A_r)f_1\|_{\mathrm{L}^2(B)} \lesssim \|f\|_{\mathrm{L}^2(4B)} \simeq r^{\frac{n}{2}} \bigg(\fint_{4B} |f|^2 \bigg)^{\frac{1}{2}}.$$

Note that the L^2-bound is independent of ε, r and depends only on dimensions and ellipticity. This follows from writing $R_L^\varepsilon(1 - A_r) = R_L F_{\varepsilon,r}(L)$ as in (7.3) and using the functional calculus on L^2. For $j \geq 2$ we re-introduce the auxiliary function ψ from (7.10) and the sets $D_j := 2^{j-1}B$. In analogy with (7.9) we write

$$\begin{aligned} &\|R_L^\varepsilon(1 - A_r)f_j\|_{\mathrm{L}^2(B)} \\ &\quad \leq \int_0^\infty \|t\nabla_x(1 + t^2L)^{-3\alpha}(\mathbf{1}_{{}^cD_j} + \mathbf{1}_{D_j})\psi(t^2L)\varphi(r^2L)f_j\|_{\mathrm{L}^2(B)} \frac{\mathrm{d}t}{t}. \end{aligned}$$

By composition, $t\nabla_x(1 + t^2L)^{-3\alpha}$ satisfies L^2 off-diagonal estimates of arbitrarily large order when $t > 0$. Therefore, we continue by

$$\begin{aligned} &\lesssim \int_0^\infty \bigg(1 + \frac{2^j r}{t}\bigg)^{-3\alpha} \|\psi(t^2L)\varphi(r^2L)f_j\|_{\mathrm{L}^2({}^cD_j)} \frac{\mathrm{d}t}{t} \\ &\quad + \int_0^\infty \|\psi(t^2L)\varphi(r^2L)f_j\|_{\mathrm{L}^2(D_j)} \frac{\mathrm{d}t}{t}. \end{aligned}$$

We can re-use (7.12) with $p = 2$ and likewise (7.14) if we replace $({}^cD_j, f)$ with (D_j, f_j) due to the different support properties in the ongoing argument. Indeed, these bounds have been obtained assuming only $p \in (p_-(L), 2]$. Altogether, we obtain a bound by

$$\begin{aligned} &\lesssim \int_0^\infty \Big(\frac{r}{t}\Big)^{2\alpha} \bigg(1 + \frac{2^j r}{t}\bigg)^{-3\alpha} \|f_j\|_2 + \Big(\frac{r}{t}\Big)^{\alpha} \bigg(1 + \frac{2^j r}{t}\bigg)^{-3\alpha} \|f_j\|_2 \frac{\mathrm{d}t}{t} \\ &\leq 2^{-j\alpha} \|f\|_{\mathrm{L}^2(2^{j+1}B)} \int_0^\infty \frac{s^{2\alpha} + s^\alpha}{1 + s^{3\alpha}} \frac{\mathrm{d}s}{s}, \end{aligned}$$

where the integral in s is finite. The claim (7.18) follows.

Step 2: Verification of (7.17)*.* Let $g \in \dot{\mathrm{W}}^{1,p_0} \cap \mathrm{W}^{1,2}$ and B a ball of radius r. We claim that

$$\Big(\fint_B |\nabla_x A_r g|^{p_0}\Big)^{\frac{1}{p_0}} \leq C \sum_{j=1}^{\infty} g(j) \Big(\fint_{2^{j+1}} |\nabla_x g|^2\Big)^{\frac{1}{2}} \tag{7.19}$$

holds with a summable sequence $g(j)$ that does not depend on ε. Taking this for granted, the right-hand side is bounded by $\mathcal{M}(|\nabla g|^2)(y)^{1/2}$ for every $y \in B$ and, given $f \in \mathrm{L}^2$, the function

$$g := \frac{1}{c_\alpha} \int_\varepsilon^{1/\varepsilon} t(1+t^2L)^{-3\alpha}(t^2L)^{3\alpha}(1+t^2L)^{-6\alpha} f \,\frac{\mathrm{d}t}{t}$$

verifies $\nabla_x A_r g = R_L^\varepsilon A_r f$ and $\nabla_x g = R_L^\varepsilon f$. At the beginning of the proof we have seen that R_L^ε maps L^2 into L^{p_0}. Therefore $g \in \dot{\mathrm{W}}^{1,p_0}$ and we obtain (7.17).

In order to prove (7.19), we perform two more reduction steps. Expanding A_r as in (7.8), we see that it suffices to establish (7.19) with A_r replaced by $(1+r^2L)^{-\beta k}$, $k \geq 1$. Moreover, thanks to the conservation property in Corollary 5.4 we can replace g by $g - (g)_B$.

Concerning off-diagonal estimates of arbitrarily large order, we obtain type $\mathrm{L}^2 - \mathrm{L}^{p_0}$ for

$$r\nabla_x(1+r^2L)^{-\beta k} = r\nabla_x(1+r^2L)^{-1}(1+r^2L)^{-\beta k+1}$$

by composition: Indeed, for the gradient family we have $\mathrm{L}^{p_0} - \mathrm{L}^{p_0}$ by interpolation of the L^2-result with L^q-boundedness for some $q \in (p_0, q_+(L))$, and β was chosen such that already the $(\beta-1)$-th powers of resolvents have $\mathrm{L}^2 - \mathrm{L}^{p_0}$. As usual, we split $g - g_B = \sum_{j \geq 1}(g-g_B)\mathbf{1}_{C_j(B)}$ and obtain

$$\Big(\fint_B |\nabla_x(1+r^2L)^{-\beta}(g-g_B)|^{p_0}\Big)^{\frac{1}{p_0}} \lesssim r^{-1-\frac{n}{2}} \sum_{j\geq 1} 2^{-j\gamma} \|g-g_B\|_{\mathrm{L}^2(C_j(B))},$$

where $\gamma > 0$ is at our disposal. Poincaré's inequality [50, Prop. 7.45] provides the bound

$$\|g-g_B\|_{\mathrm{L}^2(C_j(B))} \leq \|g-g_B\|_{\mathrm{L}^2(2^{j+1}B)} \lesssim 2^{jn} r \|\nabla_x g\|_{\mathrm{L}^2(2^{j+1}B)}.$$

We conclude that

$$\Big(\fint_B |\nabla_x(1+r^2L)^{-\beta}(g-g_B)|^{p_0}\Big)^{1/p_0} \lesssim \sum_{j\geq 1} 2^{j(\frac{3n}{2}-\gamma)} \Big(\fint_{2^{j+1}} |\nabla_x g|^2\Big)^{1/2}.$$

We take $\gamma > {}^{3n}/_{2}$ to grant summability of $g(j) := 2^{j(3n/2-\gamma)}$ and the proof of (7.19) is complete.

7.3. **Necessary condition for $1 < p < 2$.** We suppose that the Riesz transform is L^p-bounded for some $p \in (1,2)$ and prove that $p \geq p_-(L)$. In dimension $n \leq 2$ we have $p_-(L) \leq 1$, see Proposition 6.7. Hence, we can restrict ourselves to dimensions $n \geq 3$.

We set $p_0 := p$, define iteratively $p_k := p_{k-1}^*$ and stop at the first exponent $k^+ \geq 0$ with $p_{k^+} \in (2_*, 2]$. We shall prove by backward induction that $(p_k, 2] \subseteq \mathcal{J}(L)$ for all k. Hence, we eventually find $(p, 2] \subseteq \mathcal{J}(L)$, that is to say, $p \geq p_-(L)$.

We have $(p_{k^+}, 2] \subseteq (2_*, 2] \subseteq \mathcal{J}(L)$ by Proposition 6.7. For the inductive step we assume $(p_k, 2] \subseteq \mathcal{J}(L)$ and pick any $q \in (p_{k-1}, 2_*]$. Then q^* is an interior point of $\mathcal{J}(L)$ and hence $(tL^{1/2}(1+t^2L)^{-1})_{t>0}$ is L^{q^*}-bounded by Lemma 4.16. For $f \in \mathrm{L}^q \cap \mathsf{R}(L^{1/2})$ we can therefore estimate

$$\begin{aligned}\|(1+t^2L)^{-1}f\|_{q^*} &\lesssim t^{-1}\|L^{-1/2}f\|_{q^*} \\ &\lesssim t^{-1}\|\nabla_x L^{-1/2}f\|_q \\ &\lesssim t^{-1}\|f\|_q,\end{aligned}$$

where the final step uses $q \in (p, 2] \subseteq \mathcal{I}(L)$. We need to make sure that this estimates applies to sufficiently many functions f. We stress that Lemma 7.2 is useless in this regard since $q \in \mathcal{J}(L)$ is precisely what we are trying to prove.

Lemma 7.7. *In any dimension n, it follows that if $q \in \mathcal{I}(L)$ satisfies $q < 2_*$, then $\mathrm{H}^q \cap \mathrm{L}^2 \subseteq \mathsf{R}(aL^{1/2})$.*

Momentarily, let us take the lemma for granted. If $q > 1$, then multiplication by a is an automorphism of $\mathrm{L}^q \cap \mathrm{L}^2$. Hence, we have $\mathrm{L}^q \cap \mathrm{L}^2 \subseteq \mathsf{R}(L^{1/2})$ and the previous bound implies $\mathrm{L}^q - \mathrm{L}^{q^*}$-boundedness of the resolvents. As usual, we can interpolate with the L^2 off-diagonal estimates and then use Lemma 4.7 to obtain $(q, 2] \subseteq \mathcal{J}(L)$. Since $q \in (p_{k-1}, 2_*]$ was arbitrary, $(p_{k-1}, 2] \subseteq \mathcal{J}(L)$ follows.

This completes the proof modulo the

Proof of Lemma 7.7. For clarity we denote by T_L the extension of the bijection $L^{1/2} : \mathrm{W}^{1,2} \to \mathsf{R}(L^{1/2})$ to an isomorphism $\dot{\mathrm{W}}^{1,2} \to \mathrm{L}^2$, so that $R_L = \nabla_x T_L^{-1}$.

Let $f \in \mathrm{H}^q \cap \mathrm{L}^2$. Interpolation yields $f \in \mathrm{H}^{2_*} \cap \mathrm{L}^2$ and $2_* \in \mathcal{I}(L)$. Hence, $\nabla_x T_L^{-1} a^{-1} f \in \mathrm{H}^{2_*}$. Modulo constants we obtain $T_L^{-1} a^{-1} f \in \mathrm{L}^2$ by the Hardy–Sobolev embedding and consequently $T_L^{-1} a^{-1} f \in \mathrm{W}^{1,2}$. By definition of T_L this means that $a^{-1} f \in \mathsf{R}(L^{1/2})$. □

7.4. **Necessary condition for $p > 2$.** We let $p \in (2, r_+(L))$ and prove that $p \leq q_+(L)$. In fact, it suffices to prove $[2, p) \subseteq \mathcal{J}(L)$: For $q \in (2, p)$ we then obtain L^q-boundedness of

$$t\nabla_x(1+t^2L)^{-1} = (\nabla_x L^{-1/2})((t^2L)^{1/2}(1+t^2L)^{-1})$$

by composition, applying Lemma 4.16 to the second factor.

The argument is similar to the previous section. We set $p_0 := p$, define iteratively $p_k := (p_{k-1})_*$ and stop at the first exponent $k^- \geq 0$ with $p_{k^-} \in [2, 2^*)$. Then $[2, p_{k^-}) \subseteq \mathcal{J}(L)$ by Proposition 6.7. Now, assume $[2, p_k) \subseteq \mathcal{J}(L)$ and pick any $q \in [2^*, p_{k-1})$. Since q_* is an interior point of $\mathcal{J}(L)$, the family $(tL^{1/2}(1+t^2L)^{-1})_{t>0}$ is L^{q_*}-bounded by Lemma 4.16. Moreover, $q_* \in [2, p) \subseteq \mathcal{I}(L)$, so for all $f \in \mathrm{L}^q \cap \mathrm{L}^2$, we get

$$\begin{aligned}\|(1+t^2L)^{-1}f\|_q &\lesssim \|\nabla_x(1+t^2L)^{-1}f\|_{q_*} \\ &\lesssim \|L^{1/2}(1+t^2L)^{-1}f\|_{q_*} \\ &\lesssim t^{-1}\|f\|_{q_*},\end{aligned}$$

which shows that $((1+t^2L)^{-1})_{t>0}$ is $\mathrm{L}^{q_*} - \mathrm{L}^q$-bounded. Interpolation with the L^2 off-diagonal estimates and then Lemma 4.7 yield $[2, q) \subseteq \mathcal{J}(L)$. Since $q \in [2, p_{k-1})$ was arbitrary, $[2, p_{k-1}) \subseteq \mathcal{J}(L)$ follows. By backward induction we eventually arrive at the desired conclusion $[2, p) \subseteq \mathcal{J}(L)$.

8. Operator-adapted spaces

Operator-adapted Hardy–Sobolev spaces are our main tool in this monograph and will be essential for understanding most of the following sections. They have been developed in various references starting with semigroup generators in [17, 40, 57, 58] up to the recent monographs focusing on bisectorial operators [3, 22]. Still we need some unrevealed features and we take this opportunity to correct some inexact arguments from the literature.

For general properties of adapted Hardy spaces we closely follow [3, Sec. 3], where the authors develop an abstract framework of two-parameter operator families that provides a unified approach to sectorial and bisectorial operators. The application to bisectorial operators with first-order scaling has been detailed in [3, Sec. 4] and we review their results in Section 8.1. Section 8.2 provides all necessary details in order to apply the framework to sectorial operators with second-order scaling and we summarize the results that are relevant to us. This will justify using parts of [3] for sectorial operators in the further course.

The abstract framework allows us to treat operator-adapted Besov spaces simultaneously without any additional effort. These spaces will only be needed in the final Section 19 and the reader might ignore them till then.

8.1. **Bisectorial operators with first-order scaling.** To set the stage, we assume that

(8.1)
- T is a bisectorial operator in $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R}^n; V)$ of some angle $\omega \in [0, \frac{\pi}{2})$, where V is a finite-dimensional Hilbert space,
- T has a bounded H^∞-calculus on $\overline{\mathsf{R}(T)}$,
- $((1 + \mathrm{i}tT)^{-1})_{t \in \mathbb{R}\setminus\{0\}}$ satisfies L^2 off-diagonal estimates of arbitrarily large order.

These are called *Standard Assumptions* in [3, Ch. 4]. In fact, [3] requires for all $\nu \in (0, \pi/2 - \omega)$ that the family $((1+\mathrm{i}zT)^{-1})_{z \in \mathrm{S}_\nu}$ satisfies L^2 off-diagonal estimates of arbitrarily large order but this follows already from the first and third assumption in (8.1) by interpolation, see Lemma 4.13. The reader may recall from Sections 3.5 and 4 that T^* satisfies the standard assumptions as well.

In the following we suppress the reference to bisectors from notation of classes of holomorphic functions since we allow any bisector of angle larger than ω. We mimic the extension to the upper half-space by convolutions in the definition of the classical Hardy spaces by associating with each $\psi \in \mathrm{H}^\infty$ on a bisector the *extension operator*

$$\mathbb{Q}_{\psi,T} : \overline{\mathsf{R}(T)} \to \mathrm{L}^\infty(0,\infty; \mathrm{L}^2), \quad (\mathbb{Q}_{\psi,T} f)(t) = \psi(tT)f. \tag{8.2}$$

If in addition $\psi \in \Psi_+^+$, then $\mathbb{Q}_{\psi,T}$ is defined on all of L^2 and by McIntosh's theorem it maps L^2 boundedly into $\mathrm{L}^2(0,\infty, \frac{\mathrm{d}t}{t}; \mathrm{L}^2) = \mathrm{T}^{0,2} = \mathrm{Z}^{0,2}$. Hence, we can look at the bounded dual operator

$$\mathbb{C}_{\psi,T} \coloneqq (\mathbb{Q}_{\psi^*,T^*})^* : \mathrm{L}^2(0,\infty, \tfrac{\mathrm{d}t}{t}; \mathrm{L}^2) \to \mathrm{L}^2,$$

where $\psi^*(z) = \overline{\psi(\overline{z})}$, which is given by the weakly convergent integral

$$\mathbb{C}_{\psi,T} F = \int_0^\infty \psi(tT) F(t) \, \frac{\mathrm{d}t}{t}. \tag{8.3}$$

Of course, the integral converges strongly in L^2 if F has compact support in $(0,\infty)$. We call $\mathbb{C}_{\psi,T}$ a *contraction operator*. It is denoted by $\mathbb{S}_{\psi,T}$ in [3] and we change notation in order to distinguish it from conical square functions.

Definition 8.1. Let $\psi \in \mathrm{H}^\infty$, $s \in \mathbb{R}$ and $p \in (0,\infty)$. The sets

$$\mathbb{H}^{s,p}_{\psi,T} \coloneqq \{f \in \overline{\mathsf{R}(T)} : \mathbb{Q}_{\psi,T} f \in \mathrm{T}^{s,p} \cap \mathrm{T}^{0,2}\},$$
$$\mathbb{B}^{s,p}_{\psi,T} \coloneqq \{f \in \overline{\mathsf{R}(T)} : \mathbb{Q}_{\psi,T} f \in \mathrm{Z}^{s,p} \cap \mathrm{Z}^{0,2}\},$$

equipped with what will be shown to be quasinorms

$$\|f\|_{\mathbb{H}^{s,p}_{\psi,T}} \coloneqq \|\mathbb{Q}_{\psi,T} f\|_{\mathrm{T}^{s,p}}, \quad \|f\|_{\mathbb{B}^{s,p}_{\psi,T}} \coloneqq \|\mathbb{Q}_{\psi,T} f\|_{\mathrm{Z}^{s,p}},$$

are called *pre-Hardy–Sobolev* and *pre-Besov space* space of smoothness s and integrability p adapted to T, respectively. The function ψ is called *auxiliary function*.

In order to treat pre-Hardy–Sobolev and pre-Besov spaces simultaneously, we introduce the concise notation

$$\mathbb{X}^{s,p}_{\psi,T} \coloneqq \{f \in \overline{\mathsf{R}(T)} : \mathbb{Q}_{\psi,T} f \in \mathrm{Y}^{s,p} \cap \mathrm{Y}^{0,2}\},$$

where the pair $(\mathrm{Y}, \mathbb{X})$ is either $(\mathrm{T}, \mathbb{H})$ or $(\mathrm{Z}, \mathbb{B})$. These pairs are called $(\mathrm{X}, \mathbb{X})$ in [3] but it will be convenient to keep the symbol X for a different purpose. For $\psi \in \Psi^+_+$ the condition $\mathbb{Q}_{\psi,T} f \in \mathrm{Y}^{0,2}$ is redundant and if in addition ψ is non-degenerate, then by McIntosh's theorem we have up to equivalent norms

$$\mathbb{X}^{0,2}_{\psi,T} = \overline{\mathsf{R}(T)}. \tag{8.4}$$

For general values of s and p and auxiliary functions $\psi \in \mathrm{H}^\infty = \Psi^0_0$ we still have that $\mathbb{X}^{s,p}_{\psi,T}$ is quasinormed [3, Prop. 4.3] and, up to equivalent quasinorms, independent of the auxiliary function in the following classes.

Proposition 8.2 ([3, Prop. 4.4])**.** *Let $s \in \mathbb{R}$ and $p \in (0, \infty)$. Up to equivalent norms, $\mathbb{X}^{s,p}_{\psi,T}$ does not depend on the choice of $\psi \in \mathrm{H}^\infty$ as long as it is non-degenerate and of class Ψ^τ_σ with the following technical conditions on the decay parameters:*

- *$\tau > -s + |{}^n/_2 - {}^n/_p|$ and $\sigma > s$ if $p \le 2$,*
- *$\tau > -s$ and $\sigma > s + |{}^n/_2 - {}^n/_p|$ if $p \ge 2$.*

This allows us to drop the dependence on ψ.

Definition 8.3. Let $s \in \mathbb{R}$ and $p \in (0, \infty)$. Denote by $\mathbb{X}^{s,p}_T$ the quasinormed space $\mathbb{X}^{s,p}_{\psi,T}$ for any $\psi \in \Psi^\tau_\sigma$ as in Proposition 8.2. When $s = 0$, simply write $\mathbb{X}^p_T \coloneqq \mathbb{X}^{0,p}_T$.

Usually, we take ψ with sufficiently large decay to describe these spaces.

Proposition 8.4 ([3, Prop. 4.7])**.** *Let $s \in \mathbb{R}$ and $p \in (0, \infty)$ and suppose that $\psi \in \Psi^\tau_\sigma$ is non-degenerate, where*

- *$\tau > s$ and $\sigma > -s + |{}^n/_2 - {}^n/_p|$ if $p \le 2$,*
- *$\tau > s + |{}^n/_2 - {}^n/_p|$ and $\sigma > -s$ if $p \ge 2$.*

Then $\mathbb{X}^{s,p}_T = \mathbb{C}_{\psi,T}(\mathrm{Y}^{s,p} \cap \mathrm{Y}^{0,2})$ and

$$f \mapsto \inf\big\{\|F\|_{\mathrm{Y}^{s,p}} : F \in \mathrm{Y}^{s,p} \cap \mathrm{Y}^{0,2} \ \&\ \mathbb{C}_{\psi,T} F = f\big\}.$$

is an equivalent quasinorm.

The spaces $\mathbb{X}^{s,p}_T$ are not complete in general unless $p = 2$. This is why we use the subscript 'pre' and remove it when taking completions. As usual, a *completion* of a quasinormed space Q is an isometric map $\iota : Q \to \hat{Q}$, where $\hat{Q}$ is a complete quasinormed space and $\iota(Q)$ is dense in $\hat{Q}$. For $Q \coloneqq \mathbb{X}^{s,p}_T$, there are compatible completions of these spaces within the same ambient space $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$: the construction in [3, Prop. 4.20], called *canonical completion*, is to take

$$\iota := \mathbb{Q}_{\psi,T} \text{ with } \psi \in \Psi^\infty_\infty \quad \& \quad \hat{Q} := \overline{\mathbb{Q}_{\psi,T}(\mathbb{X}^{s,p}_T)} \subseteq \mathrm{Y}^{s,p}.$$

Definition 8.5. Let $\psi \in \Psi^\infty_\infty$ be non-degenerate. For $s \in \mathbb{R}$ and $p \in (0,\infty)$ denote by $\psi\mathbb{X}^{s,p}_T$ the canonical completion of the quasinormed space $\mathbb{X}^{s,p}_T$.

By the Calderón reproducing formula (here for bisectorial operators, see [22, Prop. 4.2]) the function ψ has a non-degenerate sibling $\varphi \in \Psi^\infty_\infty$ such that $\mathbb{C}_{\varphi,T}\mathbb{Q}_{\psi,T} = 1$ on $\overline{\mathsf{R}(T)}$. This allows us to summarize the full construction of operator adapted Hardy spaces in one commutative diagram, see Figure 5.

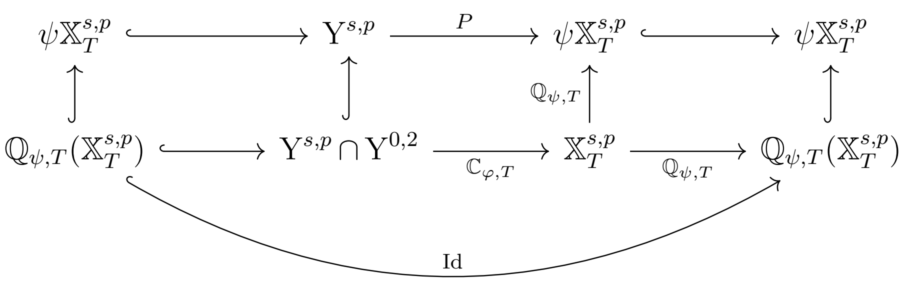

FIGURE 5. Canonical completion: $\varphi, \psi \in \Psi^\infty_\infty$ are siblings and P is the unique bounded linear map for which the diagram commutes. It follows that P is a projection from $\mathrm{Y}^{s,p}$ onto $\psi\mathbb{X}^{s,p}$. By the universal approximation technique for Y-spaces, projections for different choices of admissible spaces are compatible. The bottom part of the diagram also identifies $\psi\mathbb{X}^{s,p} \cap \mathbb{Q}_\psi(\mathbb{X}^{0,2}_T) = \mathbb{Q}_\psi(\mathbb{X}^{s,p}_T)$.

The canonical completions inherit many properties tent and Z-spaces via Figure 5. Two important examples are the following approximation results that have been tacitly used in [3]. By a slight abuse of notation we allow $\mathrm{X} \in \{\mathrm{B},\mathrm{H}\}$ to be different in the assumption and the conclusion.

Lemma 8.6. *Let $\psi \in \Psi^\infty_\infty$ be non-degenerate. If $F \in \psi\mathbb{X}^{s_0,p_0}_T$ for some $s_0 \in \mathbb{R}$, $p_0 \in (0,\infty)$, then there exists $(F_j)_j \subseteq \psi\mathbb{X}^{0,2}_T$ with $F_j \to F$ in every space of type $\psi\mathbb{X}^{s,p}_T$ that F belongs to.*

Proof. This is an immediate consequence of Figure 5. Indeed, since $\mathbf{1}_{(j^{-1},j)\times B(0,j)}F \in \mathrm{Y}^{0,2}$ is a universal approximation of F with respect to tent and Z-spaces, see Sections 2.2 and 2.3, we can take $F_j := P(\mathbf{1}_{(j^{-1},j)\times B(0,j)}F)$. □

Lemma 8.7. *Let $s_0 \in \mathbb{R}$ and $p_0 \in (0,\infty)$. Given $f \in \mathbb{X}_T^{s_0,p_0}$, there is a sequence $(f_j)_j$ in $\bigcap_{k\in\mathbb{Z}} \mathsf{R}(T^k)$ with $f_j \to f$ in every space of type $\mathbb{X}_T^{s,p}$ that f belongs to. In particular, convergence holds in $\mathbb{X}_T^{0,2} \subseteq \mathrm{L}^2$.*

Proof. Again by Figure 5 we have $f = \mathbb{C}_{\varphi,T}F$ with $F := \mathbb{Q}_{\psi,T}f$ and therefore $f_j := \mathbb{C}_{\varphi,T}(\mathbf{1}_{(j^{-1},j)\times B(0,j)}F)$ have the required universal approximation property. Thanks to $\varphi \in \Psi_\infty^\infty$ we also obtain that

$$f_j = T^k \int_{j^{-1}}^{j} (tT)^{-k}\varphi(tT)(\mathbf{1}_{B(0,j)}F(t))\,\frac{\mathrm{d}t}{t^{k+1}} \in \mathsf{R}(T^k) \quad (k \in \mathbb{Z}). \qquad \square$$

One necessity for the canonical completions is the following interpolation result.

Proposition 8.8 ([3, Thm. 4.28])**.** *Let $\psi \in \Psi_\infty^\infty$ be non-degenerate. Let $0 < p_0, p_1 < \infty$, $s_0, s_1 \in \mathbb{R}$, $\theta \in (0,1)$ and set $p := [p_0,p_1]_\theta$, $s := (1-\theta)s_0 + \theta s_1$. Up to equivalent quasinorms it follows that*

$$[\psi\mathbb{H}_T^{s_0,p_0}, \psi\mathbb{H}_T^{s_1,p_1}]_\theta = \psi\mathbb{H}_T^{s,p}$$

and if $s_0 \neq s_1$,

$$(\psi\mathbb{X}_T^{s_0,p_0}, \psi\mathbb{X}_T^{s_1,p_1})_{\theta,p} = \psi\mathbb{B}_T^{s,p}.$$

When $p \in (1,\infty)$, the spaces $\psi\mathbb{X}_T^p$ and $\varphi\mathbb{X}_{T^*}^{p'}$ are in natural duality with each other as described in [3, Prop. 4.23] provided that $\varphi, \psi \in \Psi_\infty^\infty$ are siblings. Since by definition the pre-Hardy–Sobolev and pre-Besov spaces are dense in their completions, we can equivalently state this result as follows.

Proposition 8.9. *Let $p \in (1,\infty)$. Then, whenever $f \in \mathrm{L}^2$,*

$$\sup_{g \in \mathbb{X}_{T^*}^{p'}} \frac{|\langle f, g\rangle|}{\|g\|_{\mathbb{X}_{T^*}^{p'}}} \simeq \|f\|_{\mathbb{X}_T^p},$$

where $\langle \cdot\,, \cdot \rangle$ is the inner product on L^2.

The 'raison d'être' of these spaces is that the H^∞-calculus of T extends to them in the best possible way.

Proposition 8.10 ([3, Thm. 4.14].)**.** *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $\nu \in (\omega, \pi/2)$. Then for all $\eta \in \mathrm{H}^\infty(\mathrm{S}_\nu)$,*

$$\|\eta(T)f\|_{\mathbb{X}_T^{s,p}} \lesssim \|\eta\|_\infty \|f\|_{\mathbb{X}_T^{s,p}} \quad (f \in \mathbb{X}_T^{s,p}).$$

Moreover, if $\varphi \in \Psi_{-1}^1(\mathrm{S}_\nu)$ and $\psi \in \Psi_1^{-1}(\mathrm{S}_\nu)$, then

$$\|\varphi(T)f\|_{\mathbb{X}_T^{s+1,p}} \lesssim \|f\|_{\mathbb{X}_T^{s,p}} \quad (f \in \mathsf{D}(\varphi(T)) \cap \mathbb{X}_T^{s,p})$$

and

$$\|\psi(T)f\|_{\mathbb{X}_T^{s-1,p}} \lesssim \|f\|_{\mathbb{X}_T^{s,p}} \quad (f \in \mathsf{D}(\psi(T)) \cap \mathbb{X}_T^{s,p}),$$

where the implicit constants also depend on φ and ψ.

The second part indicates that the spaces for different smoothness parameters are related through a *lifting property*. Indeed, recall that $(z/[z])(T)$ and its inverse are bounded operators on $\overline{\mathsf{R}(T)}$ since T has a bounded H^∞-calculus and that therefore T and $[T]$ share the same domain and range. Thus, using $(\varphi, \psi) = (1/\psi, \psi)$ with either $\psi(z) = z$ or $\psi(z) = [z]$ in the proposition above, we obtain

Corollary 8.11. *The operators T and $(T^2)^{1/2}$ are bijections $\mathbb{X}_T^{s+1,p} \cap \mathsf{D}(T) \to \mathbb{X}_T^{s,p} \cap \mathsf{R}(T)$ that satisfy*

$$\|Tf\|_{\mathbb{X}_T^{s,p}} \simeq \|f\|_{\mathbb{X}_T^{s+1,p}} \simeq \|[T]f\|_{\mathbb{X}_T^{s,p}}.$$

From the H^∞-calculus we immediately obtain that $(\mathrm{e}^{-t[T]})_{t \geq 0}$ is a bounded semigroup on $\mathbb{X}_T^{s,p}$. In fact, we also have strong continuity and stability.

Proposition 8.12 ([3, Prop. 4.33])**.** *Let $s \in \mathbb{R}$ and $p \in (0, \infty)$. For all $f \in \mathbb{X}_T^{s,p}$ the following limits hold in $\mathbb{X}_T^{s,p}$:*

$$\lim_{t \to 0} \mathrm{e}^{-t[T]} f = f \quad \text{and} \quad \lim_{t \to \infty} \mathrm{e}^{-t[T]} f = 0.$$

8.2. **Sectorial operators with second-order scaling.**

In this case our standard assumptions are that

(8.5)
- T is a sectorial operator on $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R}^n; V)$ of some angle $\omega \in [0, \pi)$, where V is a finite-dimensional Hilbert space,
- T has a bounded H^∞-calculus on $\overline{\mathsf{R}(T)}$,
- $((1 + t^2 T)^{-1})_{t>0}$ satisfies L^2 off-diagonal estimates of arbitrarily large order,

and we allow holomorphic functions on any sector of angle larger than ω in the following considerations.

We define the extension for $\psi \in \mathrm{H}^\infty$ with second-order scaling

$$\mathbb{Q}_{\psi,T} : \overline{\mathsf{R}(T)} \to \mathrm{L}^\infty(0, \infty; \mathrm{L}^2), \quad (\mathbb{Q}_{\psi,T} f)(t) = \psi(t^2 T) f$$

and if in addition Ψ_+^+, then $\mathbb{Q}_{\psi,T}$ is again defined on all of L^2, maps into $L^2(0, \infty, \frac{\mathrm{d}t}{t}; \mathrm{L}^2)$ and we have the dual operator

$$\mathbb{C}_{\psi,T} \coloneqq (\mathbb{Q}_{\psi^*,T^*})^*, \quad \mathbb{C}_{\psi,T} F = \int_0^\infty \psi(t^2 T) F(t) \,\frac{\mathrm{d}t}{t},$$

where the integral converges weakly in L^2.

Most of the theory in [3, Sec. 3 & 4] has been written for abstract continuous two-parameter families $(S_{t,\tau})_{t,\tau>0}$ on L^2 and hence applies *in extenso* to families

$$(\psi(t^2 T)\eta(T)\varphi(\tau^2 T)) \tag{8.6}$$

with a sectorial operator as above, instead of

$$(\psi(tT)\eta(T)\varphi(\tau T)) \tag{8.7}$$

with a bisectorial operator. Here, $\psi \in \Psi_{\sigma_1}^{\tau_1}$, $\varphi \in \Psi_{\sigma_2}^{\tau_2}$, $\eta \in \Psi_{\sigma_3}^{\tau_3}$ are auxiliary functions with $\sigma_j, \tau_j \in \mathbb{R}$. The only difference with the results of bisectorial operators lies in how large these parameters have to be in order to arrive at the desired conclusion.

The three fundamental mapping properties for families of type (8.7) in [3] – Lemma 3.17, Lemma 3.18 and Theorem 3.19 – remain to hold for families of type (8.6) and then the same conclusion holds already if one replaces σ_j, τ_j by ${}^{\sigma_j}/_2, {}^{\tau_j}/_2$ in the assumptions. Indeed, following the self-contained proofs in [3], one readily sees that the assumptions on the auxiliary functions are exclusively determined by [3, Thm. 3.8], which in turn provides the order of L^2 off-diagonal decay that one can get for families of the form $(\eta(t)\psi(tT))_{t>0}$ if $(\eta(t))_{t>0}$ is a continuous bounded family of functions in H^∞ and $\psi \in \Psi_\sigma^\tau$. Precisely, [3, Thm. 3.8] allows any order up to $\gamma = \sigma$. On the other hand, in Lemma 4.16 we have proved the same conclusion for $(\eta(t)\psi(t^2T))_{t>0}$ under the mere assumption $\psi \in \Psi_{\sigma/2}^{\tau/2}$.

From this discussion we conclude that qualitatively the results of Section 8.1 that build on [3] remain valid for sectorial operators with second-order scaling but there are the following quantitative changes. The technical conditions of Proposition 8.2 become

- $\tau > -{}^s/_2 + |{}^n/_4 - {}^n/_{(2p)}|$ and $\sigma > {}^s/_2$ if $p \leq 2$,
- $\tau > -{}^s/_2$ and $\sigma > {}^s/_2 + |{}^n/_4 - {}^n/_{(2p)}|$ if $p \geq 2$,

with the same type of modification in Proposition 8.4. In Proposition 8.10 the assumption on the angle is again best possible, that is $\nu \in (\omega, \pi)$ and $\eta \in \mathrm{H}^\infty(\mathrm{S}_\nu^+)$, and the second part of holds for $\varphi \in \Psi_{-1/2}^{1/2}(\mathrm{S}_\nu^+)$ and $\psi \in \Psi_{1/2}^{-1/2}(\mathrm{S}_\nu^+)$. As a consequence, the lifting property of Corollary 8.11 uses $\sqrt{T}$.

Performing only the purely symbolic replacement of $\sqrt{z^2}$ by $\sqrt{z}$ at all occasions in the statement and proof of Proposition 8.12, we immediately obtain the following version for sectorial operators.

Proposition 8.13. *Let $s \in \mathbb{R}$ and $p \in (0, \infty)$. For all $f \in \mathbb{X}_T^{s,p}$ the following limits hold in $\mathbb{X}_T^{s,p}$:*

$$\lim_{t\to 0} \mathrm{e}^{-t\sqrt{T}} f = f \quad \textit{and} \quad \lim_{t\to\infty} \mathrm{e}^{-t\sqrt{T}} f = 0.$$

8.3. **Molecular decomposition for adapted Hardy spaces.** Molecular decompositions for $\mathbb{H}_T^p$ with $p \in (0,1]$ have been pioneered in [40, 57, 58] for divergence form operators $T = -\operatorname{div}_x d\nabla_x$. For (bi)sectorial operators satisfying our standard assumptions, the same kind of decomposition has been used in many references including [3, 22] but a proof seems to be missing in the literature. We take the opportunity to close this gap. The construction closely follows [58] but heat semigroup bounds have to be replaced with more technical resolvent bounds.

Throughout this section T is again a (bi)sectorial operator that satisfies the standard assumptions of Section 8.1 or Section 8.2 and we define $\mathbb{H}^p_T$ by the abstract theory for first or second-order scaling, respectively.

Definition 8.14. Let $p \in (0,1]$, $\varepsilon > 0$ and $M \in \mathbb{N}$. A function $m \in \mathrm{L}^2$ is called $(\mathbb{H}^p_T, \varepsilon, M)$-*molecule* if there exists a cube $Q \subseteq \mathbb{R}^n$ and a function $b \in \mathsf{D}(T^M)$ that satisfies $T^M b = m$ and the following estimates for $j = 1, 2, \dots$ and $k = 0, 1, \dots, M$:

(i) If T is bisectorial with first-order scaling

$$\|((\ell(Q)T)^{-k} m\|_{\mathrm{L}^2(C_j(Q))} \leq (2^j \ell(Q))^{\frac{n}{2} - \frac{n}{p}} 2^{-j\varepsilon}.$$

(ii) If T is sectorial with second-order scaling

$$\|((\ell(Q)^2 T)^{-k} m\|_{\mathrm{L}^2(C_j(Q))} \leq (2^j \ell(Q))^{\frac{n}{2} - \frac{n}{p}} 2^{-j\varepsilon}.$$

Remark 8.15. Summing up the bounds in j gives the global L^2-bound $\|((\ell(Q)^\varrho T)^{-k} m\|_2 \leq c\ell(Q)^{n/2 - n/p}$, where $\varrho \in \{1, 2\}$ is the order of scaling and c depends on p, ε, M. If $\varepsilon > n/2$, then we can use Hölder's inequality before summing and obtain $\|((\ell(Q)^\varrho T)^{-k} m\|_1 \leq c\ell(Q)^{n - n/p}$.

Definition 8.16. Let $p \in (0,1]$, $\varepsilon > 0$ and $M \in \mathbb{N}$. A *molecular* $(\mathbb{H}^p_T, \varepsilon, M)$-*representation* of $f \in \overline{\mathsf{R}(T)}$ is a series $\sum_{i=0}^\infty \lambda_i m_i$ that converges towards f unconditionally in L^2 such that $(\lambda_i) \in \ell^p$ and each m_i is a $(\mathbb{H}^p_T, \varepsilon, M)$-molecule. The *molecular Hardy space*

$$\mathbb{H}^p_{T,\mathrm{mol},\varepsilon,M} := \Big\{ f \in \overline{\mathsf{R}(T)} : f \text{ has a molecular } (\mathbb{H}^p_T, \varepsilon, M)\text{-representation} \Big\}$$

is equipped with the quasi norm

$$\|f\|_{\mathbb{H}^p_{T,\mathrm{mol},M}} := \inf \|(\lambda_i)\|_{\ell^p},$$

where the infimum is taken over all admissible representations.

With these definitions at hand, we establish the following

Theorem 8.17. *Let $p \in (0,1]$, $\varepsilon > 0$ and $M \in \mathbb{N}$ with $M > n/p - n/2$ if T is bisectorial with first-order scaling or $M > n/(2p) - n/4$ if T is sectorial with second-order scaling. Then*

$$\mathbb{H}^p_{T,\mathrm{mol},\varepsilon,M} = \mathbb{H}^p_T$$

with equivalent quasinorms and the equivalence constants depend on T only through the bounds that are quantified in the standard assumptions.

As in many earlier references, the proof relies on the atomic decomposition for tent spaces that we recall beforehand.

Definition 8.18. Let $p \in (0,1]$. A T^p-*atom* associated with a cube $Q \subseteq \mathbb{R}^n$ is a measurable function $A : \mathbb{R}^{1+n}_+ \to \mathbb{C}^N$ with support in

$Q \times (0, \ell(Q))$ such that

$$\Big(\int_0^{\ell(Q)} \int_Q |A(s,y)|^2 \, \frac{\mathrm{d}s\mathrm{d}y}{s} \Big)^{\frac{1}{2}} \leq \ell(Q)^{\frac{n}{2} - \frac{n}{p}}.$$

Proposition 8.19 ([35, Prop. 5]). *Let $p \in (0,1]$. There is a constant C such that every $F \in \mathrm{T}^p$ can be written as $f = \sum_{i=0}^\infty \lambda_i A_i$ with unconditional convergence in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$, where each A_i is a T^p-atom and $\|(\lambda_i)\|_{\ell^p} \leq C\|F\|_{\mathrm{T}^p}$.*

Remark 8.20. The unconditional convergence is not stated explicitly but is immediate from the construction, see [35, (4.5)]. Indeed, we have $\lambda_i A_i = F\mathbf{1}_{\Delta_i}$, where $(\Delta_i)_i$ is a collection of pairwise disjoint subsets of $\mathbb{R}^{1+n}_+$. This also implies that for $f \in \mathrm{T}^p \cap \mathrm{T}^2$ the atomic decomposition converges in $\mathrm{T}^2 = \mathrm{L}^2(\mathbb{R}^{1+n}_+, \frac{\mathrm{d}t\mathrm{d}x}{t})$.

The proof of Theorem 8.17 relies on two lemmata.

Lemma 8.21. *Let $p \in (0,1]$ and $\varepsilon > 0$. Let $M \in \mathbb{N}$ and $\psi \in \Psi_+^+$ as follows:*

- *$M > {}^n/_p - {}^n/_2$ and $\psi(z) = z^{2M}(1+\mathrm{i}z)^{-4M}$ if T is bisectorial with first-order scaling,*
- *$M > {}^n/_{(2p)} - {}^n/_4$ and $\psi(z) = z^{2M}(1+z)^{-4M}$ if T is sectorial with second-order scaling.*

Then there exists a constant C depending on these parameters and the bounds that are quantified in the standard assumptions such that

$$\|\mathbb{Q}_{\psi,T} m\|_{\mathrm{T}^p} \leq C$$

holds for every $(\mathbb{H}^p_T, \varepsilon, M)$-molecule m.

Proof. We give the proof for bisectorial T with first-order scaling. Up to consistently changing the scaling, the argument for sectorial operators is identical. Since

$$\psi(z) = (-\mathrm{i})^{2M}((1+\mathrm{i}z)^{-1} - (1+\mathrm{i}z)^{-2})^{2M}$$

we obtain by composition that $(\psi(tT))_{t>0}$ satisfies L^2 off-diagonal estimates of arbitrarily large order.

Let m be a $(\mathbb{H}^p_T, \varepsilon, M)$-molecule associated with a cube Q of side-length ℓ. We need a uniform L^p-bound for the square function

$$S_{\psi,T} m(x) := \Big(\iint_{|x-y|<t} |\psi(tT)m(y)|^2 \, \frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}} \Big)^{1/2}.$$

Since $\mathbb{H}^2_T = \overline{\mathsf{R}(T)}$, we have that $\|S_{\psi,T} f\|_2 \lesssim \|f\|_2$ for all $f \in \overline{\mathsf{R}(T)}$. In particular, we obtain from Hölder's inequality and the molecular decay the local bound

$$\|S_{\psi,T} m\|_{\mathrm{L}^p(16Q)} \leq |16\ell|^{\frac{n}{p} - \frac{n}{2}} \|Sm\|_{\mathrm{L}^2(16Q)} \leq C.$$

It remains to prove that there is $\alpha > 0$ depending only on ε, M, p such that for all $j \geq 4$ we have a uniform bound

$$\|S_{\psi,T} m\|_{\mathrm{L}^2(C_j(Q))} \leq C 2^{-j\alpha} (2^j \ell)^{\frac{n}{2}-\frac{n}{p}}. \tag{8.8}$$

Indeed, this implies $\|S_{\psi,T} m\|_{\mathrm{L}^p(C_j(Q))} \leq C 2^{-j\alpha}$ as before and the global L^p-bound for $S_{\psi,T} m$ follows by summing up the p-th powers of these estimates.

In order to establish (8.8), we split the integral in t at height $2^{\theta(j-1)}\ell$, where $\theta \in (0,1)$ will be fixed later:

$$\begin{aligned}
\|S_{\psi,T} m\|_{\mathrm{L}^2(C_j(Q))} &\simeq \bigg(\int_{C_j(Q)} \iint_{|x-y|<t} |\psi(tT)m(y)|^2 \frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}} \,\mathrm{d}x \bigg)^{1/2} \\
&\lesssim \bigg(\int_0^{2^{\theta j(j-1)}\ell} \int_{D_j(Q)} |\psi(tT)m(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t} \bigg)^{1/2} \\
&\quad + \bigg(\int_{2^{\theta(j-1)}\ell}^\infty \int_{\mathbb{R}^n} |\psi(tT)m(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t} \bigg)^{1/2} \\
&=: \mathrm{I} + \mathrm{II},
\end{aligned}$$

where $D_j(Q) := 2^{j+2}Q \setminus 2^{j-1}Q$ and we have used Tonelli's theorem to bound the integrals in x. By the molecular properties, we can write $m = T^M b$. Since $\psi \in \Psi_{2M}^{2M}$, we have a uniform L^2-bound for $(tT)^M \psi(tT)$, which together with Remark 8.15 leads us to

$$\begin{aligned}
\mathrm{II} &= \bigg(\int_{2^{\theta(j-1)}\ell}^\infty \int_{\mathbb{R}^n} |(tT)^M \psi(tT) b(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{2M+1}} \bigg)^{1/2} \\
&\lesssim (2^{\theta j}\ell)^{-M} \|b\|_2 \\
&\leq C 2^{-j(\theta M + \frac{n}{2} - \frac{n}{p})} (2^j \ell)^{\frac{n}{2}-\frac{n}{p}}
\end{aligned}$$

and we can achieve $\alpha := \theta M + {}^n/_p - {}^n/_2 > 0$ by taking θ sufficiently close to 1. This completes the treatment of II.

As for I, we decompose further $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$, where I_k corresponds to replacing m with m_k defined as

$$m_1 := \mathbf{1}_{2^{j+3}Q \setminus 2^{j-2}Q} m, \quad m_2 := \mathbf{1}_{{}^c(2^{j+3}Q \setminus 2^{j-2})} m.$$

The L^2-bound for $S_{\psi,T}$ and the molecular estimates yield

$$\mathrm{I}_1 \lesssim \|m_1\|_2 \leq \sum_{k=j-2}^{j+2} \|m\|_{\mathrm{L}^2(C_k(Q))}^2 \leq C 2^{-2j\varepsilon} (2^j \ell)^{\frac{n}{2}-\frac{n}{p}}.$$

Since the support of m_2 is at distance at least $2^{j-2}\ell$ from $D_j(Q)$, we get can infer from the off-diagonal decay for $\psi(tT)$ that

$$\mathrm{I}_2 \lesssim \|m\|_2 \bigg(\int_0^{2^{\theta(j-1)}\ell} \Big(1 + \frac{2^{j-2}\ell}{t}\Big)^{-2\gamma} \frac{\mathrm{d}t}{t} \bigg)^{1/2}$$

$$\lesssim \|m\|_2 (2^j \ell)^{-2\gamma} \bigg(\int_0^{2^{\theta(j-1)}\ell} t^{2\gamma} \, \frac{\mathrm{d}t}{t} \bigg)^{1/2}$$
$$\leq C (2^j \ell)^{\frac{n}{2}-\frac{n}{p}} 2^{-j((1-\theta)\gamma + \frac{n}{2} - \frac{n}{p})},$$

where we have used again Remark 8.15 in the final step and γ is still at our disposal. We have already fixed $\theta \in (0,1)$ and it suffices to take γ large enough so that $\alpha := (1-\theta)\gamma + n/2 - n/p > 0$. This completes the treatment of I and hence we have established our goal (8.8). □

Lemma 8.22. *Let $p \in (0,1]$. Let $\varepsilon > 0$ and $M \in \mathbb{N}$. Let $\psi(z) = z^{2M}(1+\mathrm{i}z)^{-4M}$ if T is bisectorial with first-order scaling and $\psi(z) = z^{2M}(1+z)^{-4M}$ is T is sectorial with second-order scaling. There exists a constant c depending on these parameters and the bounds that are quantified in the standard assumptions, such that $c^{-1}\mathbb{C}_{\psi,T}A$ is a $(\mathbb{H}^p_T, \varepsilon, M)$-molecule, whenever A is a T^p-atom.*

Proof. Again we only do the proof in the bisectorial case and the sectorial case follows line by line up to the usual modifications.

Let A be a T^p-atom associated with a cube Q of sidelength ℓ and set

$$m := \mathbb{C}_{\psi,T} A = \int_0^\ell (tT)^{2M}(1+\mathrm{i}tT)^{-4M} A(t) \, \frac{\mathrm{d}t}{t},$$

where we have used the support property of A. The integral converges weakly in L^2 but as $M \geq 1$, the integral

$$b := \int_0^\ell t^M (tT)^M (1+\mathrm{i}tT)^{-4M} A(t) \, \frac{\mathrm{d}t}{t}$$

converges strongly and we have $T^M b = m$. We establish the molecular bounds for m up to a generic renorming factor c.

In preparation of the argument, let $g \in \mathrm{L}^2$. For $k = 0, \ldots, M$ we bound the L^2 inner product

$$\begin{aligned} |\langle (\ell T)^{-k} m, g \rangle| &\leq \ell^{-k} \int_0^\ell |\langle t^k (tT)^{2M-k}(1+\mathrm{i}tT)^{-4M} A(t), g\rangle| \, \frac{\mathrm{d}t}{t} \\ &= \ell^{-k} \int_0^\ell t^k |\langle A(t), \varphi(tT^*) g\rangle| \, \frac{\mathrm{d}t}{t} \\ &\leq \bigg(\int_0^\ell \|A(t)\|^2_{\mathrm{L}^2(Q)} \, \frac{\mathrm{d}t}{t} \bigg)^{1/2} \bigg(\int_0^\ell \|\varphi(tT^*) g\|^2_{\mathrm{L}^2(Q)} \, \frac{\mathrm{d}t}{t} \bigg)^{1/2} \\ &\leq \ell^{\frac{n}{2}-\frac{n}{p}} \bigg(\int_0^\ell \|\varphi(tT^*) g\|^2_{\mathrm{L}^2(Q)} \, \frac{\mathrm{d}t}{t} \bigg)^{1/2}, \end{aligned}$$

where $\varphi \in \Psi^M_M$ is given by $\varphi(z) := z^{2M-k}(1-\mathrm{i}z)^{-4M}$ and we have used the support and the molecular bound of A. Taking the supremum over all g with support in $4Q$ normalized to $\|g\|_2 = 1$ and controlling the square function via McIntosh's theorem, we obtain

$$\|(\ell T)^{-k} m\|_{\mathrm{L}^2(4Q)} \leq c \ell^{\frac{n}{2}-\frac{n}{p}},$$

which is the required molecular bound for $j = 1$. The family $(\varphi(tT))_{t>0}$ satisfies L^2 off-diagonal estimates of arbitrarily large order by decomposition since we can expand

$$\varphi(z) = (-\mathrm{i})^{2M-k}(1-(1+\mathrm{i}z)^{-1})^{2M-k}(1+\mathrm{i}z)^{-2M-k}.$$

For $j \geq 2$ we take the supremum over all normalized g in L^2 with support in $\mathrm{L}^2(C_j(Q))$ and obtain

$$\begin{aligned}\|(\ell T)^{-k} m\|_{\mathrm{L}^2(C_j(Q))} &\lesssim \ell^{\frac{n}{2}-\frac{n}{p}} \bigg(\int_0^\ell \Big(\frac{2^{j-1}\ell}{t}\Big)^{-2\gamma} \frac{\mathrm{d}t}{t} \bigg)^{1/2} \\ &\leq c\ell^{\frac{n}{2}-\frac{n}{p}} 2^{-(j-1)\gamma},\end{aligned}$$

with $\gamma > 0$ at our disposal. We take $\gamma > {}^n/_p - {}^n/_2 + \varepsilon$ to obtain the required molecular decay. □

Putting it all together, we give the

Proof of Theorem 8.17. Let $f \in \mathbb{H}^p_{T,\mathrm{mol},\varepsilon,M}$ and let $f = \sum_{i=0}^\infty \lambda_i m_i$ be an L^2 convergent molecular representation. We define $\mathbb{H}^p_T$ via the admissible auxiliary function ψ from Lemma 8.21. Let $\varrho \in \{1,2\}$ be the scaling order. We have

(8.9)

$$\begin{aligned}\|\mathbb{Q}_{\psi,T} f\|_{\mathrm{T}^p}^p &= \int_{x\in\mathbb{R}^n} \bigg(\iint_{|x-y|<t} |\psi(t^\varrho T) f(y)|^2 \frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}} \bigg)^{p/2} \mathrm{d}x \\ &\leq \int_{\mathbb{R}^n} \bigg(\sum_{i=0}^\infty |\lambda_i| \bigg(\iint_{|x-y|<t} |\psi(t^\varrho T) m_i(y)|^2 \frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}} \bigg)^{1/2} \bigg)^p \mathrm{d}x \\ &\leq \sum_{i=0}^\infty |\lambda_i|^p \|\mathbb{Q}_{\psi,T} m_i\|_{\mathrm{T}^p}^p \\ &\leq C^p \sum_{i=0}^\infty |\lambda_i|^p,\end{aligned}$$

where the first step uses L^2-convergence, the second step is due to $p \leq 1$ and monotone convergence and the third step is by Lemma 8.21. Taking the infimum over all representations yields $\|f\|_{\mathbb{H}^p_T} \leq C\|f\|_{\mathbb{H}_{T,\mathrm{mol},\varepsilon,M}}$.

Conversely, let $f \in \mathbb{H}^p_T$ and let ψ be the auxiliary function from Lemma 8.22. According to Proposition 8.4, we can write $f = \mathbb{C}_{\psi,T} F$ with $F \in \mathrm{T}^p \cap \mathrm{T}^2$ and $\|F\|_{\mathrm{T}^p} \leq 2\|f\|_{\mathbb{H}^p_T}$. According to Proposition 8.19 and the subsequent remark, we can write $F = \sum_{i=0}^\infty \lambda_i A_i$, where the sum converges unconditionally in T^2, each A_i is a T^p-atom and we have $\|(\lambda)_i\|_{\ell^p} \leq C\|F\|_{\mathrm{T}^p}$. Since $\mathbb{C}_{\psi,T} : \mathrm{T}^2 \to \mathrm{L}^2$ is bounded, we get an unconditionally L^2-convergent representation

$$f = \mathbb{C}_{\psi,T} F = \sum_{i=0}^\infty \lambda_i \mathbb{C}_{\psi,T} A_i = \sum_{i=0}^\infty (c\lambda_i) c^{-1} m_i,$$

where c is the constant from Lemma 8.22 and the $m_i := \mathbb{C}_{\psi,T} A_i$ are $(\mathbb{H}^p_T, \varepsilon, M)$-molecules. This proves $\|f\|_{\mathbb{H}_{T,\mathrm{mol},\varepsilon,M}} \leq 2Cc\|f\|_{\mathbb{H}^p_T}$. □

8.4. **Connection with the non-tangential maximal function.** We recall the non-tangential maximal function

$$\widetilde{N}_* F(x) := \sup_{t>0} \left(\fint\!\!\fint_{W(t,x)} |F(s,y)|^2 \, \mathrm{d}s\mathrm{d}y \right)^{1/2},$$

where $W(t,x) := (t/2, 2t) \times B(x,t)$. At this level of generality we do not know whether $\mathbb{H}^p_T$ could be characterized via $\widetilde{N}_*$ as in [40, 58] but, using the molecular decompositions, we can give upper bounds for the non-tangential maximal function of resolvent families and Poisson-type semigroups acting on $\mathbb{H}^p_T$ if $p \leq 1$. Such result can be extend to $p \leq 2$ by interpolation provided the result for $p = 2$ holds, which might be a concern in itself.

We begin with a simple comparison of the non-tangential maximal function and the uncentered Hardy–Littlewood maximal operator $\mathcal{M}$ in $\mathbb{R}^n$.

Lemma 8.23. *Let $\psi : (0,\infty) \to \mathcal{L}(\mathrm{L}^2)$ be a strongly measurable family that satisfies L^2 off-diagonal estimates of order $\gamma > n/2$. Then there is a constant C depending on dimensions and the off-diagonal bounds, such that*

$$\fint\!\!\fint_{W(t,x)} |\psi(s)f(y)|^2 \, \mathrm{d}s\mathrm{d}y \leq C \, \mathcal{M}(|f|^2)(x)$$

for all $f \in \mathrm{L}^2$ and all $(t,x) \in \mathbb{R}^{1+n}_+$.

Proof. Set $B := B(x,t)$ and split $f = \sum_{j \geq 0} f_j$, where $f_j := \mathbf{1}_{C_j(B)} f$. For $t/2 < s < 2t$ we have by assumption

$$\begin{aligned} \int_B |\psi(s)f_j(y)|^2 \, \mathrm{d}y &\lesssim \left(1 + \frac{(2^j - 1)t}{s}\right)^{-2\gamma} \|f\|^2_{\mathrm{L}^2(C_j(B))} \\ &\lesssim 2^{-2\gamma j} \int_{2^j B} |f|^2 \\ &\lesssim t^n 2^{-j(2\gamma - n)} \, \mathcal{M}(|f|^2)(x). \end{aligned}$$

The claim follows by summing in j and averaging in s. □

We also recall Kolmogorov's lemma for bounding the maximal operator on L^θ for $\theta < 1$, see for instance [39, Lem. 5.16].

Lemma 8.24 (Kolmogorov)**.** *Let $\theta \in (0,1)$ and $E \subseteq \mathbb{R}^n$ a set of finite measure. There is a constant $C = C(\theta, n)$ such that*

$$\int_E |\, \mathcal{M} f(y)|^\theta \, \mathrm{d}y \leq C|E|^{1-\theta} \|f\|_1^\theta \quad (f \in \mathrm{L}^1).$$

With these tools at hand, we establish a first non-tangential maximal bound on $\mathbb{H}^p_T$.

Proposition 8.25. *Let $p \in (0,1]$ and $\varepsilon > 0$. Let $M \in \mathbb{N}$ and $\psi \in \mathrm{H}^\infty$ as follows:*

- *$M > n/p - n/2$ and $\psi(z) = (1+\mathrm{i}z)^{-2M}$ if T is bisectorial with first-order scaling,*
- *$M > n/(2p) - n/4$ and $\psi(z) = (1+z)^{-2M}$ if T is sectorial with second-order scaling.*

Then there exists a constant C depending on these parameters and the bounds that are quantified in the standard assumptions such that

$$\|\widetilde{N}_*(\mathbb{Q}_{\psi,T} f)\|_p \leq C\|f\|_{\mathbb{H}^p_T} \quad (f \in \mathbb{H}^p_T).$$

Proof. Let $f \in \mathbb{H}^p_T$ and $f = \sum_{i=0}^\infty \lambda_i m_i$ be an L^2-convergent molecular representation as in Theorem 8.17. Then $\mathbb{Q}_{\psi,T} f = \sum_{i=0}^\infty \lambda_i \mathbb{Q}_{\psi,T} m_i$ in $\mathrm{L}^\infty(0,\infty;\mathrm{L}^2)$ and by sublinearity of the maximal function we find

$$\|\widetilde{N}_*(\mathbb{Q}_{\psi,T} f)\|_p^p \leq \left\| \sum_{i=0}^\infty |\lambda_i| \widetilde{N}_*(\mathbb{Q}_{\psi,T} m_i) \right\|_p^p \leq \sum_{i=0}^\infty |\lambda_i|^p \|\widetilde{N}_*(\mathbb{Q}_{\psi,T} m_i)\|_p^p.$$

Consequently, it suffices to treat the case that $f = m$ is an $(\mathbb{H}^p_T, \varepsilon, M)$-molecule (associated with a cube Q of sidelength ℓ). We only write out the argument in the bisectorial case. As usual, the proof is identical in the sectorial case upon changing the scaling.

Step 1: Local bound for $\widetilde{N}_*$. By composition, the family $(\psi(tT))_{t>0}$ satisfies L^2 off-diagonal estimates of arbitrarily large order. Therefore, Lemma 8.23 yields $\widetilde{N}_*(\mathbb{Q}_{\psi,T} m) \leq C(\mathcal{M}(|m|^2))^{1/2}$ a.e. on $\mathbb{R}^n$ and by means of Kolmogorov's lemma and Remark 8.15 we get

$$\|\widetilde{N}_*(\mathbb{Q}_{\psi,T} m)\|_{\mathrm{L}^p(16Q)}^p \lesssim \int_{16Q} |\mathcal{M}(|m|^2)(y)|^{\frac{p}{2}} \,\mathrm{d}y \leq |16Q|^{1-\frac{p}{2}} \|m\|_2^p \leq C^p.$$

Step 2: Decomposition of $\widetilde{N}_*$ on annuli. It remains to show that there is $\alpha > 0$ depending only on ε, M, p such that for all $j \geq 4$ we have a uniform bound

$$\|\widetilde{N}_*(\mathbb{Q}_{\psi,T} m)\|_{\mathrm{L}^p(C_j(Q))}^p \leq 2^{-j\alpha} C^p. \tag{8.10}$$

The claim then follows by summing up in j. To this end, we fix $j \geq 4$ and split

$$\widetilde{N}_*(\mathbb{Q}_{\psi,T} m) \leq \widetilde{N}_*^{\mathrm{loc}}(\mathbb{Q}_{\psi,T} m) + \widetilde{N}_*^{\mathrm{glob}}(\mathbb{Q}_{\psi,T} m),$$

where the local and global parts correspond to restricting the size of Whitney boxes in the definition of $\widetilde{N}_*$ to $t \leq \ell$ and $t \geq \ell$, respectively.

Step 3: Bound for $\widetilde{N}_*^{\mathrm{loc}}$ on $C_j(Q)$. Let $0 < t < \ell$ and $x \in C_j(Q)$. Splitting $m = \sum_{i \geq 0} m_i$, where $m_i := \mathbf{1}_{C_i(Q)} m$, we get

$$\left(⨌_{W(t,x)} |(1+\mathrm{i}sT)^{-2M} m|^2 \,\mathrm{d}s\mathrm{d}y \right)^{1/2}$$

$$\lesssim \sum_{|i-j|\geq 2} t^{-\frac{n}{2}}\Big(1+\frac{\mathrm{d}(B(x,t),C_i(Q))}{t}\Big)^{-\gamma}\|m_i\|_2 + \sum_{|i-j|\leq 1} \mathcal{M}(|m_i|^2)(x)^{\frac{1}{2}},$$

where we have used L^2 off-diagonal decay of the resolvents whenever $|i-j| \geq 2$ and Lemma 8.23 whenever $|i-j| \leq 1$. The order $\gamma > 0$ is at our disposal. For any set $E \subseteq \mathbb{R}^n$ we have

$$1+\frac{\mathrm{d}(B(x,t),E)}{t} \geq \frac{1}{2}+\frac{\mathrm{d}(x,E)}{4t}$$

as follows by distinguishing whether or not $t \geq \mathrm{d}(x,E)/2$. Specializing to $E = C_i(Q)$ with $|i-j| \geq 2$, we get

$$1+\frac{\mathrm{d}(B(x,t),C_i(Q))}{t} \geq \frac{1}{2}+\frac{\mathrm{d}(x,C_i(Q))}{4t} \gtrsim \frac{2^{i\vee j}\ell}{4t}.$$

We also have

$$\|m_i\|_2 \leq |2^{i\vee j}Q|^{\frac{1}{2}}\bigg(\fint_{2^{i\vee j}Q} |m_i|^2 \,\mathrm{d}y\bigg)^{\frac{1}{2}} \lesssim (2^{i\vee j}\ell)^{\frac{n}{2}}\,\mathcal{M}(|m_i|^2)(x)^{\frac{1}{2}}.$$

Applying these bounds on the right-hand side of our estimate leads us to

$$\begin{aligned}
&\bigg(\iint_{W(t,x)} |(1+\mathrm{i}sT)^{-2M}m|^2 \,\mathrm{d}s\mathrm{d}y\bigg)^{1/2} \\
&\quad\lesssim \sum_{i\leq j-2} (2^j\ell)^{\frac{n}{2}-\gamma}t^{\gamma-\frac{n}{2}}\,\mathcal{M}(|m_i|^2)(x)^{\frac{1}{2}} \\
&\qquad + \sum_{|i-j|\leq 1} \mathcal{M}(|m_i|^2)(x)^{\frac{1}{2}} \\
&\qquad + \sum_{i\geq j+2} (2^i\ell)^{\frac{n}{2}-\gamma}t^{\gamma-\frac{n}{2}}\,\mathcal{M}(|m_i|^2)(x)^{\frac{1}{2}}.
\end{aligned} \tag{8.11}$$

From now on we require $\gamma > n/2$. On the right-hand side t appears with positive exponent and hence the supremum over $0 < t \leq \ell$ is attained for $t = \ell$. We conclude that

$$\begin{aligned}
\widetilde{N}_*^{\mathrm{loc}}(\mathbb{Q}_{\psi,T}m)(x) &\lesssim \sum_{i\leq j-2} 2^{j(\frac{n}{2}-\gamma)}\,\mathcal{M}(|m_i|^2)(x)^{\frac{1}{2}} \\
&\quad + \sum_{|i-j|\leq 1} \mathcal{M}(|m_i|^2)(x)^{\frac{1}{2}} \\
&\quad + \sum_{i\geq j+2} 2^{i(\frac{n}{2}-\gamma)}\,\mathcal{M}(|m_i|^2)(x)^{\frac{1}{2}}.
\end{aligned} \tag{8.12}$$

Kolmogorov's lemma and the molecular bounds for m imply

$$\begin{aligned}
\int_{C_j(Q)} |\,\mathcal{M}(|m_i|^2)(x)|^{\frac{p}{2}} \,\mathrm{d}x &\leq |C_j(Q)|^{1-\frac{p}{2}}\|m_i\|_2^p \\
&\leq C2^{j(\frac{n}{p}-\frac{n}{2})p}2^{i(\frac{n}{2}-\frac{n}{p}-\varepsilon)p},
\end{aligned}$$

so that integrating the p-th power of (8.12) in $x \in C_j(Q)$ yields

$$\begin{aligned}\|\widetilde{N}_*^{\mathrm{loc}}(\mathbb{Q}_{\psi,T}m)\|_{\mathrm{L}^p(C_j(Q))}^p &\lesssim \sum_{i \le j-2} 2^{j(\frac{n}{p}-\gamma)p} 2^{i(\frac{n}{2}-\frac{n}{p}-\varepsilon)p} \\ &\quad + \sum_{|i-j|\le 1} 2^{j(\frac{n}{p}-\frac{n}{2})p} 2^{i(\frac{n}{2}-\frac{n}{p}-\varepsilon)p} \\ &\quad + \sum_{i \ge j-2} 2^{j(\frac{n}{p}-\frac{n}{2})p} 2^{i(n-\frac{n}{p}-\varepsilon-\gamma)p} \\ &\simeq 2^{j(\frac{n}{p}-\gamma)p} + 2^{-j\varepsilon} + 2^{j(\frac{n}{2}-\varepsilon-\gamma)p}.\end{aligned}$$

This establishes (8.10) for $\widetilde{N}_*^{\mathrm{loc}}$ provided that eventually we take $\gamma > {}^n/_p$ (which implies $\gamma > {}^n/_2$).

Step 4: Bound for $\widetilde{N}_*^{\mathrm{glob}}$ *on* $C_j(Q)$. We write $m = T^M b$ as in Definition 8.14. We have

$$\begin{aligned}(1+\mathrm{i}tT)^{-2M}m &= (\mathrm{i}t)^{-M}(it)^M(1+\mathrm{i}tT)^{-2M}m \\ &= (\mathrm{i}t)^{-M}((1+\mathrm{i}tT)^{-1} - (1+\mathrm{i}tT)^{-2})^M b \\ &=: (\mathrm{i}t)^{-M}\varphi(tT)b,\end{aligned}$$

where $(\varphi(tT))_{t>0}$ satisfies L^2 off-diagonal estimates of arbitrarily large order. Hence, we can repeat the first part of Step 3 with φ, b replacing ψ, m and due to the additional factor $(\mathrm{i}t)^{-M}$ our substitute for (8.11) becomes

$$\begin{aligned}&\left(\fint\!\!\!\!\fint_{W(t,x)} |(1+\mathrm{i}sT)^{-2M}m|^2 \,\mathrm{d}s\mathrm{d}y\right)^{1/2} \\ &\quad\lesssim \sum_{i\le j-2} (2^j\ell)^{\frac{n}{2}-\gamma} t^{\gamma-\frac{n}{2}-M}\, \mathcal{M}(|b_i|^2)(x)^{\frac{1}{2}} \\ &\qquad + \sum_{|i-j|\le 1} t^{-M}\, \mathcal{M}(|b_i|^2)(x)^{\frac{1}{2}} \\ &\qquad + \sum_{i\ge j+2} (2^i\ell)^{\frac{n}{2}-\gamma} t^{\gamma-\frac{n}{2}-M}\, \mathcal{M}(|b_i|^2)(x)^{\frac{1}{2}}\end{aligned}$$

with $\gamma > 0$ at our disposal and $b_i := \mathbf{1}_{C_i(Q)}b$. We require $\gamma < {}^n/_2 + M$. Then t appears with negative exponent on the right-hand side and passing to the supremum for all $t \ge \ell$, we get

$$\begin{aligned}\widetilde{N}_*^{\mathrm{glob}}(\mathbb{Q}_{\psi,T}m)(x) &\lesssim \sum_{i\le j-2} 2^{j(\frac{n}{2}-\gamma)}\, \mathcal{M}(|\ell^{-M}b_i|^2)(x)^{\frac{1}{2}} \\ &\quad + \sum_{|i-j|\le 1} \mathcal{M}(|\ell^{-M}b_i|^2)(x)^{\frac{1}{2}} \\ &\quad + \sum_{i\ge j+2} 2^{i(\frac{n}{2}-\gamma)}\, \mathcal{M}(|\ell^{-M}b_i|^2)(x)^{\frac{1}{2}}.\end{aligned}$$

Now, $\ell^{-M} b = (\ell T)^{-M} m$ satisfies the same L^2-bounds on annuli as m and we can repeat the arguments in Step 3 in order to conclude (8.10) for $\widetilde{N}_*^{\mathrm{glob}}$ provided that at the end of the proof we take again $\gamma > n/p$. This requirement is compatible with $\gamma < n/2 + M$ since we have $M > n/p - n/2$ by assumption. □

In the context of boundary value problems it will be important to have a statement as above with a Poisson-like semigroup replacing the resolvents. To this end we need the following fact.

Lemma 8.26. *Let $p \in (0, \infty)$. There is a constant $C = C(n, p)$ such that*

$$\|\widetilde{N}_*(F)\|_p \le C \|F\|_{\mathrm{T}^p} \quad (F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)).$$

We add a proof for convenience.

Proof. Let $(t, x) \in \mathbb{R}^{1+n}_+$. Since $(s, y) \in W(t, x)$ implies $|x - y| < t \le 2s$ and $t \ge s/2$, we have that

$$\left(\fint\!\!\fint_{W(t,x)} |F(s, y)|^2 \,\mathrm{d}s\mathrm{d}y \right)^{1/2} \le C \left(\iint_{|x-y|<2s} |F(s, y)|^2 \,\frac{\mathrm{d}s\mathrm{d}y}{s^{1+n}} \right)^{1/2}.$$

The right-hand side does not depend on t and its L^p-quasinorm in x is equivalent to $\|F\|_{\mathrm{T}^p}$ by a change of aperture. The claim follows by taking the supremum in t and integrating the p-th powers in x. □

Proposition 8.27. *Let $p \in (0, 1]$. Let $\psi(z) = \mathrm{e}^{-\sqrt{z^2}}$ if T is bisectorial with first-order scaling and $\psi(z) = \mathrm{e}^{-\sqrt{z}}$ if T is sectorial with second-order scaling. Then there exists a constant C depending on the bounds that are quantified in the standard assumptions, such that*

$$\|\widetilde{N}_*(\mathbb{Q}_{\psi,T} f)\|_p \le C \|f\|_{\mathbb{H}^p_T} \quad (f \in \mathbb{H}^p_T).$$

Moreover, the bound continues to hold for $p \in (1, 2]$ if it holds for $p = 2$.

Proof. First, let $p \in (0, 1]$ and define an auxiliary function φ as follows:

- If T is bisectorial with first-order scaling, let $M > n/p - n/2$ and $\varphi(z) \coloneqq \psi(z) - (1 + \mathrm{i}z)^{-2M}$. Then $\varphi \in \Psi_1^{2M}$, so that the technical condition in Proposition 8.2 holds.
- If T is sectorial with second-order scaling, let $M > n/(2p) - n/4$ and $\varphi(z) \coloneqq \psi(z) - (1 + z)^{-2M}$. Then $\varphi \in \Psi_{1/2}^{2M}$ and the corresponding technical condition for sectorial operators (Section 8.2) holds.

We find for all $f \in \mathbb{H}^p_T$ that

$$\begin{aligned} \|\widetilde{N}_*(\mathbb{Q}_{\psi,T} f)\|_p &\le \|\widetilde{N}_*(\mathbb{Q}_{\varphi,T} f)\|_p + \|\widetilde{N}_*(\mathbb{Q}_{\psi-\varphi,T} f)\|_p \\ &\lesssim \|\mathbb{Q}_{\varphi,T} f\|_{\mathrm{T}^p} + \|\widetilde{N}_*(\mathbb{Q}_{\psi-\varphi,T} f)\|_p \\ &\lesssim \|f\|_{\mathbb{H}^p_T}, \end{aligned}$$

where the second step is due Lemma 8.26 and the third step uses the definition of the $\mathbb{H}^p_T$-norm and Proposition 8.25.

Suppose in addition that this bound holds for $p = 2$. Let $\phi \in \Psi^\infty_\infty$ and recall the definition of $\mathbb{H}^p_T$ via the contraction mapping $\mathbb{C}_\phi$ (Proposition 8.4). The claim is then equivalent to $F \mapsto \widetilde{N}_*(\mathbb{Q}_{\psi,T}\mathbb{C}_\phi F)$ being bounded $\mathrm{T}^p \cap \mathrm{T}^2 \to \mathrm{L}^p$ for the respective p-norms. By assumption this holds for $p = 2$ and from the first part of the proof it follows $p = 1$, so the claim follows by complex interpolation for positive sublinear operators [66]. □

8.5. ***D*-adapted spaces.** The unperturbed Dirac operator D satisfies the standard assumptions of Section 8.1. In order to fully understand the associated Hardy–Sobolev and Besov spaces, we need the orthogonal projection $\mathbb{P}_D : \mathrm{L}^2 \to \overline{\mathsf{R}(D)} \subseteq \mathrm{L}^2$. From the specific form of D^2 in (3.3) we see that

$$D^2 f = -\Delta_x f \quad (f \in \mathsf{D}(D^2) \cap \mathsf{R}(D))$$

and hence that $\mathbb{P}_D = -\Delta_x^{-1} D^2$ holds on the dense subspace $\mathsf{D}(D^2)$ of L^2. Now, $-\Delta_x^{-1} D^2$ can also be viewed as a Fourier multiplier with symbol

$$\begin{bmatrix} 1_{\mathbb{C}^m} & 0 \\ 0 & (|\xi|^{-2}\xi \otimes \xi) \otimes 1_{\mathbb{C}^m} \end{bmatrix}, \tag{8.13}$$

where $\xi \in \mathbb{R}^n$ is the Fourier variable and we think of $\mathbb{C}^{mn} \simeq (\mathbb{C}^m)^n$ as n-vectors of elements in $\mathbb{C}^m$ just as in the definition of vector-valued gradient and divergence. This symbol is homogeneous of degree zero and smooth outside of 0 and hence falls in the scope of the Mihlin multiplier theorem [92, Thm. 5.2.2]. Therefore $-\Delta_x D^2$ extends boundedly to $\dot{\mathrm{X}}^{s,p}$, where $\mathrm{X} \in \{\mathrm{B}, \mathrm{H}\}$, for all $s \in \mathbb{R}$ and $p \in (0, \infty)$. The extension to L^2 is precisely $\mathbb{P}_D$ and we keep on denoting the extensions to other spaces by the same symbol. From (8.13) we also obtain the block structure

$$\mathbb{P}_D =: \begin{bmatrix} 1 & 0 \\ 0 & \mathbb{P}_{\mathrm{curl}_x} \end{bmatrix}. \tag{8.14}$$

Since $\overline{\mathsf{R}(D)}$ coincides with the space $\mathcal{H}$ in the ellipticity condition (1.2), we get that $\mathbb{P}_{\mathrm{curl}_x}$ is the projection onto the curl-free L^2 vector fields. By [3, Thm. 5.3] we have for $s \in \mathbb{R}$ and $p \in (0, \infty)$ that

$$\mathbb{X}^{s,p}_D = \mathbb{P}_D(\dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2)$$

with equivalence of p-quasinorms. In particular, $\mathbb{P}_D(\dot{\mathrm{X}}^{s,p})$ equipped with the norm of $\dot{\mathrm{X}}^{s,p}$ is a completion of $\mathbb{X}^{s,p}_D$ in $\mathcal{Z}'$. Let now $\psi \in \mathrm{H}^\infty$ for the sectorial functional calculus and put $\varphi(z) := \psi(z^2)$. Then (3.14)

$$\mathbb{X}^{s,p}_{-\Delta_x} \oplus \mathbb{X}^{s,p}_{-\nabla_x \operatorname{div}_x} = \mathbb{X}^{s,p}_{D} = \dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2 \oplus \mathbb{P}_{\operatorname{curl}_x}(\dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2).$$

$$\mathbb{X}^{s,p}_{-\Delta_x} = \dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2, \qquad \mathbb{X}^{s,p}_{-\nabla_x \operatorname{div}_x} = \mathbb{P}_{\operatorname{curl}_x}(\dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2)$$

FIGURE 6. Identification of Hardy–Sobolev and Besov spaces up to equivalent quasinorms in the unperturbed case $B = 1$.

with $B = 1$ yields for all $t > 0$ that

$$\varphi(tD) = \begin{bmatrix} \psi(-t^2\Delta_x) & 0 \\ 0 & \psi(-t^2\nabla_x \operatorname{div}_x) \end{bmatrix},$$

that is to say

$$\mathbb{Q}_{\varphi,D} = \begin{bmatrix} \mathbb{Q}_{\psi,-\Delta_x} & 0 \\ 0 & \mathbb{Q}_{\psi,-\nabla_x \operatorname{div}_x} \end{bmatrix}. \tag{8.15}$$

On taking ψ with sufficient decay at 0 and ∞, we conclude $\mathbb{X}^{s,p}_D = \mathbb{X}^{s,p}_{-\Delta_x} \oplus \mathbb{X}^{s,p}_{-\nabla_x \operatorname{div}_x}$. Along with (8.14) we can characterize the D-adapted spaces as in Figure 6.

As a matter of fact, Theorem 8.17 for D comprises a molecular decomposition for $\dot{\mathrm{H}}^{0,p} \cap \mathrm{L}^2 = \mathrm{H}^p \cap \mathrm{L}^2$ when $p \in (0, 1]$. In order to illustrate how operator-adapted and standard theory interact for a specific differential operator, we recover an atomic decomposition for H^p from the molecular decomposition of $\mathbb{H}^p_D$.

Proposition 8.28. *Let $p \in (1_*, 1]$. Every $f \in \mathbb{H}^p_D$ can be written as $f = \sum_{i=0}^\infty \lambda_i a_i$ with unconditional convergence in L^2, where each a_i is an L^2-atom for H^p. Moreover, $\|f\|_{\mathrm{H}^p} \simeq \inf \|(\lambda_i)\|_{\ell^p}$, where the infimum is taken over all such representations.*

Proof. Let C be such that $\|a\|_{\mathrm{H}^p} \le C$ for all L^2-atoms a for H^p. By the same argument as in (8.9) we get for any L^2-convergent atomic representation $f := \sum_{i=0}^\infty \lambda_i a_i$ that

$$\|f\|^p_{\mathrm{H}^p} \simeq \|f\|^p_{\mathbb{H}^p_D} \le C^p \|(\lambda)_i\|^p_{\ell^p}.$$

Conversely, let $f \in \mathbb{H}^p_D$. Due to Theorem 8.17 we have $f = \sum_{i=0}^\infty \lambda_i m_i$, where each m_i is an $(\mathbb{H}^p_D, 1, 1)$-molecule and $\|(\lambda_i)_i\|_{\ell^p} \le 2\|f\|_{\mathrm{H}^p}$. Consequently, it suffices to find atomic decompositions for each m_i.

Let Q_i be the associated cube and write $m_i = Db_i$ as in Definition 8.14. Let $(\chi_i^j)_{j=1}^\infty$ be a smooth partition of unity on $\mathbb{R}^n$ such that

$$0 \le \chi_i^j \le \mathbf{1}_{C_j(Q_i)}, \quad \|\nabla_x \chi_i^j\|_\infty \le c(n)(2^j \ell(Q_i))^{-1}. \tag{8.16}$$

Then $b_i = \sum_{j=1}^{\infty} \chi_i^j b_i$ unconditionally in L^2. Since D is a first-order differential operator, each $D(\chi_i^j b_i)$ is supported in $2^{j+1}Q_i$, has mean value zero and satisfies

$$\begin{aligned}\|D(\chi_i^j b_i)\|_2 &\leq \|m_i\|_{\mathrm{L}^2(C_j(Q_i))} + 2^{-j}c\|\ell(Q_i)^{-1}b_i\|_{\mathrm{L}^2(C_j(Q_i))} \\ &\leq c(2^j\ell(Q_i))^{\frac{n}{2}-\frac{n}{p}}2^{-j},\end{aligned}$$

where c only depends on dimensions. This means that $a_i^j := c^{-1}2^j D(\chi_i^j b_i)$ is an H^p-atom. Since D is closed, we obtain the atomic decomposition

$$m_i = Db_i = \sum_{j=1}^{\infty} D(\chi_i^j b_i) = \sum_{j=1}^{\infty} c2^{-j}a_i^j. \qquad \square$$

The proof above showed more.

Corollary 8.29. *Let $p \in (1_*, 1]$. There is a constant C that depend on dimensions and p such that every $(\mathbb{H}_D^p, 1, 1)$-molecule m satisfies $\|m\|_{\mathrm{H}^p} \leq C$.*

Note that Lemma 8.21 gives the same result provided that $n/p < 1 + n/2$. We have used the specific structure for D to get the conclusion without this restriction.

We shall also need atomic decomposition of $\dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$ as in [45], but with $\dot{\mathrm{W}}^{1,2}$-convergence rather than convergence in $\dot{\mathrm{H}}^{1,p}$. While this can certainly be inferred from inspection of the proof in [45], we prefer to give a direct and more transparent argument that relies on the lifting property from Corollary 8.11.

Definition 8.30. Let $p \in (1_*, 1]$. An L^2*-atom for* $\dot{\mathrm{H}}^{1,p}$ is a function a supported in a cube $Q \subseteq \mathbb{R}^n$ such that $\|\nabla_x a\|_2 \leq \ell(Q)^{\frac{n}{2}-\frac{n}{p}}$.

Proposition 8.31. *Let $p \in (1_*, 1]$. Every $f \in \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$ can be written as $f = \sum_{i=0}^{\infty} \lambda_i a_i$ with unconditional convergence in $\dot{\mathrm{W}}^{1,2}$, where each a_i is an L^2-atom for $\dot{\mathrm{H}}^{1,p}$. Moreover, $\|f\|_{\dot{\mathrm{H}}^{1,p}} \simeq \inf \|(\lambda_i)_i\|_{\ell^p}$, where the infimum is taken over all such representations.*

Proof. Corollary 8.11 and Figure 6 tell us that

$$D : \mathbb{H}_D^{1,p} \cap \mathsf{D}(D) \to \mathbb{H}_D^p \cap \mathsf{R}(D)$$

is bijective and satisfies $\|Dg\|_{\mathrm{H}^p} \simeq \|g\|_{\dot{\mathrm{H}}^{1,p}}$ and $\|Dg\|_{\mathrm{L}^2} \simeq \|g\|_{\dot{\mathrm{H}}^{1,2}}$ for all g. Also, if f is an L^2-atom for $\dot{\mathrm{H}}^{1,p}$, then $D([f,0]^\top) = [0, -\nabla_x f]$ is an L^2-atom for H^p.

If $f = \sum_{i=0}^{\infty} \lambda_i a_i$ is a $\dot{\mathrm{H}}^{1,p}$ atomic decomposition as above, then

$$D\begin{bmatrix} f \\ 0 \end{bmatrix} = \sum_{i=0}^{\infty} \lambda_i \begin{bmatrix} 0 \\ -\nabla_x a_i \end{bmatrix}$$

is a H^p atomic decomposition and $\|f\|_{\dot{\mathrm{H}}^{1,p}} \lesssim \|(\lambda_i)_i\|_{\ell^p}$ follows. Conversely, let $f \in \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$. Then $D([f,0]^\top) \in \mathbb{H}^p_D$ and the atomic decomposition obtained in the proof of Proposition 8.28 takes the form

$$D\begin{bmatrix} f \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\nabla_x f \end{bmatrix} = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c\lambda_i 2^{-j} a_i^j, \quad a_i^j = c^{-1}2^j D(\chi_i^j b_i),$$

where each a_i^j is an L^2-atom for H^p and the χ_i^j are smooth functions satisfying (8.16). The function $(c^{-1}2^j\chi_i^j b_i)_\perp$ has support in $2^{j+1}Q_i$ and satisfies $-\nabla_x(c^{-1}2^j\chi_i^j b_i)_\perp = (a_i^j)_\parallel$. Hence, it is an L^2-atom for $\dot{\mathrm{H}}^{1,p}$ and the decomposition we are looking for is

$$f = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c\lambda_i 2^{-j}(c^{-1}2^j\chi_i^j b_i)_\perp. \qquad \square$$

8.6. **Spaces adapted to perturbed Dirac operators.** Now, we apply the abstract theory with first-order scaling to the bisectorial operators BD and DB and relate the operator-adapted spaces to those obtained for the sectorial operators $L, M, \widetilde{L}, \widetilde{M}$ with second-order scaling. Thanks to the different orders of scaling, the meaning of s as a smoothness parameter is the same for all adapted spaces.

In analogy with (8.15) we have that whenever ψ is an admissible auxiliary function on a sector for the definition of $\mathbb{X}^{s,p}_L$ and $\mathbb{X}^{s,p}_M$, then $\varphi(z) := \psi(z^2)$ is admissible for $\mathbb{X}^{s,p}_{BD}$ and

$$\mathbb{Q}_{\varphi,BD} = \begin{bmatrix} \mathbb{Q}_{\psi,L} & 0 \\ 0 & \mathbb{Q}_{\psi,M} \end{bmatrix}. \tag{8.17}$$

This is again a consequence of (3.14). The same kind of relation holds with DB on the left and $\widetilde{L}$, $\widetilde{M}$ on the right and follows from (3.4). Merely by definition we obtain

$$\begin{aligned} \mathbb{X}^{s,p}_{BD} &= \mathbb{X}^{s,p}_L \oplus \mathbb{X}^{s,p}_M, \\ \mathbb{X}^{s,p}_{DB} &= \mathbb{X}^{s,p}_{\widetilde{L}} \oplus \mathbb{X}^{s,p}_{\widetilde{M}}. \end{aligned} \tag{8.18}$$

In this sense the theory for the perturbed Dirac operators encompasses the theory of all four second-order operators. Figure 7 summarizes their various relations.

As for the mapping between the second and third row in Figure 7, we first cite the following *regularity shift* from [3, Prop. 5.6]: we have that

$$D : \mathbb{X}^{s+1,p}_{BD} \cap \mathsf{D}(D) \to \mathbb{X}^{s,p}_{DB} \cap \mathsf{R}(D) \tag{8.19}$$

is bijective and bounded from below and above for the $\mathbb{X}$-quasinorms. In particular,

$$\|Df\|_{\mathbb{X}^{s,p}_{DB}} \simeq \|f\|_{\mathbb{X}^{s+1,p}_{BD}} \quad (f \in \mathbb{X}^{s+1,p}_{BD} \cap \mathsf{D}(D)).$$

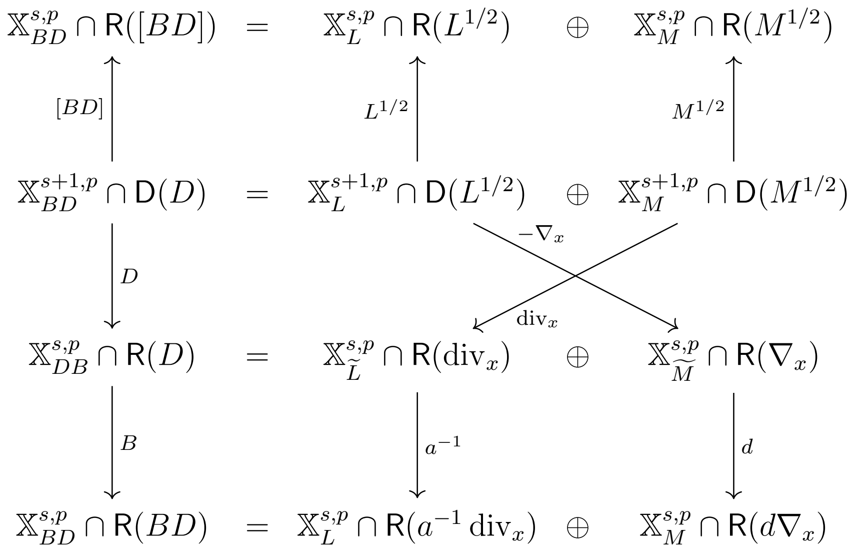

FIGURE 7. Splittings and identifications of pre-Hardy–Sobolev and pre-Besov spaces. Each arrow indicates a bijection that is bounded from below and above for the respective $\mathbb{X}$-quasinorms. Domains and ranges are taken for the corresponding operators on L^2 with maximal domain. Each appearing space is the intersection of an adapted space $\mathbb{X}^{s,p}_T$ with one of its dense subsets, where density is with respect to the norm $\|\cdot\|_{\mathbb{X}^{s,p}_T} + \|\cdot\|_2$, see Lemma 8.7.

This takes care of the left-hand side. The two ingredients for the proof in [3] are the intertwining property from Lemma 3.6 and the following

Lemma 8.32 (Local coercivity inequality, [22, Lem. 5.14])**.** *For any $u \in \mathrm{L}^2_{\mathrm{loc}}$ with $Du \in \mathrm{L}^2_{\mathrm{loc}}$ and any ball $B(x,t) \in \mathbb{R}^n$ it follows that*

$$\int_{B(x,t)} |Du|^2 \lesssim \int_{B(x,2t)} |BDu|^2 + t^{-2} \int_{B(x,2t)} |u|^2.$$

Remark 8.33. In Lemma 8.32 we understand $Du = [\operatorname{div}_x u_\parallel, -\nabla_x u_\perp]^\top$ in the sense of distributions. In particular, we can take $u \in \mathsf{D}(D)$.

On recalling $\mathsf{D}(D) = \mathsf{D}(BD) = \mathsf{D}([BD])$ from Section 3.6 and $\mathsf{R}(D) = \mathsf{R}(DB)$ from (3.8), we can split the regularity shift (8.19) in the spirit of (8.18) and obtain the right-hand side between the second and third row.

Similarly, the mappings between the first and second row in Figure 7 are due to (8.18) and Corollary 8.11.

Finally, the mapping from the third to the fourth line follows from the block diagonal structure of B and the following

Lemma 8.34. *Let $s \in \mathbb{R}$ and $p \in (0,\infty)$. The map*

$$B : \mathbb{X}^{s,p}_{DB} \cap \mathsf{R}(D) \to \mathbb{X}^{s,p}_{BD} \cap \mathsf{R}(BD)$$

is bijective and bounded from below and above for the respective $\mathbb{X}^{s,p}$-quasinorms.

Proof. Let $f \in \mathbb{X}^{s,p}_{DB} \cap \mathsf{R}(D)$. We have $Bf \in \mathsf{R}(BD)$ and for any $\psi \in \Psi^+_+$ we obtain $\mathbb{Q}_{\psi,BD}Bf = B\mathbb{Q}_{\psi,DB}f$ from (3.15). Since B is a bounded multiplication operator, we conclude $\|Bf\|_{\mathbb{X}^{s,p}_{BD}} \leq \|B\|_\infty \|f\|_{\mathbb{X}^{s,p}_{DB}}$.

Conversely, let $g \in \mathbb{X}^{s,p}_{BD} \cap \mathsf{R}(BD)$ and write $g = Bf$ with $f \in \mathsf{R}(D)$. In order to bound f in $\mathbb{X}^{s,p}_{DB}$, we take an auxiliary function $\psi \in \Psi^\infty_\infty$ and define $\varphi \in \Psi^\infty_\infty$ by $\varphi(z) = z\psi(z)$. For fixed $t > 0$ we have again the intertwining relation $D\psi(tBD)g = DB\psi(tDB)f$. The local coercivity inequality applied to $u := t\psi(tBD)g$ can therefore be rewritten as

$$\int_{B(x,t)} |\varphi(tDB)f|^2 \lesssim \int_{B(x,2t)} |\varphi(tBD)g|^2 + \int_{B(x,2t)} |\psi(tBD)g|^2.$$

Consequently,

$$\|\mathbb{Q}_{\varphi,DB}f\|_{\mathrm{Y}^{s,p}} \lesssim \|\mathbb{Q}_{\varphi,BD}g\|_{\mathrm{Y}^{s,p}} + \|\mathbb{Q}_{\psi,BD}g\|_{\mathrm{Y}^{s,p}},$$

where in the case $(\mathbb{X},\mathrm{Y}) = (\mathbb{H},\mathrm{T})$ we also used a change of angle in the tent space norms. The left-hand side compares to $\|f\|_{\mathbb{X}^{s,p}_{DB}}$ whereas both terms on the right compare to $\|g\|_{\mathbb{X}^{s,p}_{BD}}$. □

We could also write down a 'completed' version of Figure 7 in which all pre-Hardy–Sobolev and pre-Besov spaces are replaced by their canonical completions and all intersections vanish. While conceptually this might seem more satisfactory, the possibility of working with invertible maps in L^2 will have significant advantages for many of our proofs.

9. Identification of adapted Hardy spaces

This section is concerned with identifying three pre-Hardy spaces, $\mathbb{H}^p_L$, $\mathbb{H}^{1,p}_L$ and $\mathbb{H}^p_{DB}$, that play a crucial role for Dirichlet and regularity problems with classical smoothness spaces. To this end it will be convenient to have a version of Figure 7 around these particular spaces at hand:

As for the second and third row 'identifying' means determining whether the spaces remain the same as sets and with equivalent p-quasinorms when B is replaced by the identity matrix. In the fourth row for $\mathbb{H}^p_L$, we can then expect it is the image of $\mathrm{H}^p \cap \mathrm{L}^2$ under multiplication with a^{-1}. If $p > 1$, then multiplication by a^{-1} is invertible on $\mathrm{H}^p = \mathrm{L}^p$ and hence the image the same as $\mathrm{L}^p \cap \mathrm{L}^2$.

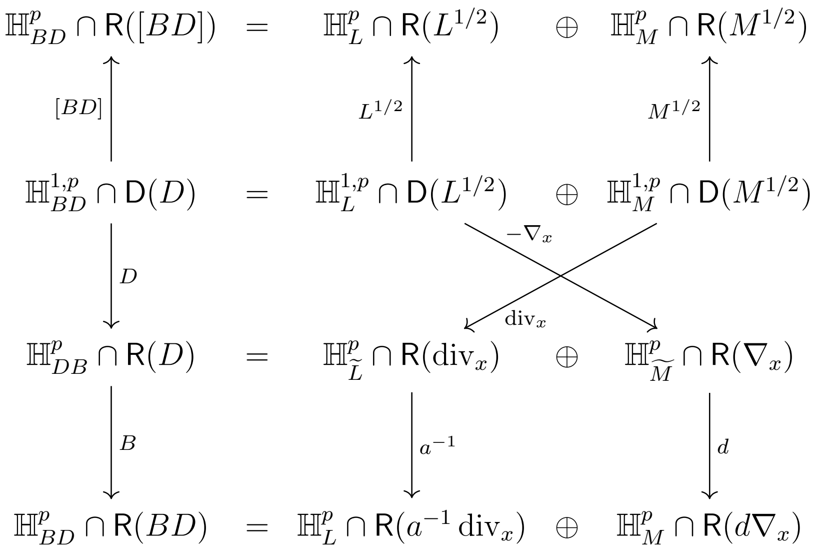

FIGURE 8. Figure 7 for $s = 0$ and $\mathbb{X} = \mathbb{H}$. Each appearing space is the intersection of an adapted space with one of its dense subsets.

9.1. **Identification regions.** We introduce three such sets of exponents:

$$\mathcal{H}(DB) \coloneqq \big\{p \in (1_*, \infty) : \|f\|_{\mathbb{H}^p_{DB}} \simeq \|f\|_{\mathbb{H}^p_D} \text{ for all } f \in \overline{\mathsf{R}(D)}\big\} \tag{9.1}$$

and

$$\mathcal{H}(L) \coloneqq \big\{p \in (1_*, \infty) : \|f\|_{\mathbb{H}^p_L} \simeq \|af\|_{\mathrm{H}^p} \text{ for all } f \in \mathrm{L}^2\big\},$$
$$\mathcal{H}^1(L) \coloneqq \big\{p \in (1_*, \infty) : \|f\|_{\mathbb{H}^{1,p}_L} \simeq \|f\|_{\dot{\mathrm{H}}^{1,p}} \text{ for all } f \in \mathrm{L}^2\big\}.$$

The identification region for DB turns out to be the intersection of the two regions associated with L. This has nothing to do with the particular Hardy spaces above and follows from Figure 8 for all sorts of adapted spaces. Identification regions for other DB- and L-adapted spaces will appear much later in the text in Section 19.

Lemma 9.1. *Let $s \in \mathbb{R}$ and $p \in (0, \infty)$. The following are equivalent :*
(i) $\mathbb{X}^{s,p}_{DB} = \mathbb{X}^{s,p}_D$ *with equivalent p-quasinorms.*
(ii) $\mathbb{X}^{s,p}_L = a^{-1}(\dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2)$ *and* $\mathbb{X}^{s+1}_L = \dot{\mathrm{X}}^{s+1,p} \cap \mathrm{L}^2$*, both with equivalent p-quasinorms.*

Specializing to $\mathbb{X} = \mathbb{H}$ and $s = 0$ in Lemma 9.1, we obtain

Corollary 9.2. *It follows that $\mathcal{H}(DB) = \mathcal{H}(L) \cap \mathcal{H}^1(L)$. In particular (by (8.4)) all three sets contain $p = 2$.*

Proof of Lemma 9.1. Throughout, equalities of spaces are up to comparable pre-Hardy quasinorms and spaces that arise from multiplication with a^{-1} carry the image topology.

We start by noting that (i) is equivalent to $\mathbb{X}^{s,p}_{DB} \cap \mathsf{R}(D) = \mathbb{X}^{s,p}_{D} \cap \mathsf{R}(D)$ since $\mathsf{R}(D) = \mathsf{R}(DB)$ is dense in both adapted spaces. The third row of Figure 7 yields equivalence to

$$\begin{aligned} \mathbb{X}^{s,p}_{\widetilde{L}} \cap \mathsf{R}(\operatorname{div}_x) &= \mathbb{X}^{s,p}_{-\Delta_x} \cap \mathsf{R}(\operatorname{div}_x), \\ \mathbb{X}^{s,p}_{\widetilde{M}} \cap \mathsf{R}(\nabla_x) &= \mathbb{X}^{s,p}_{-\nabla_x \operatorname{div}_x} \cap \mathsf{R}(\nabla_x). \end{aligned}$$

By moving to the second and fourth row, this is the same as having

$$\begin{aligned} \mathbb{X}^{s,p}_{L} \cap \mathsf{R}(a^{-1} \operatorname{div}_x) &= a^{-1}(\mathbb{X}^{p}_{-\Delta_x} \cap \mathsf{R}(\operatorname{div}_x)), \\ \mathbb{X}^{s+1,p}_{L} \cap \mathsf{D}(\nabla_x) &= \mathbb{X}^{s+1,p}_{-\Delta_x} \cap \mathsf{D}(\nabla_x), \end{aligned}$$

which, by density, is equivalent to having

$$\begin{aligned} \mathbb{X}^{s,p}_{L} &= a^{-1}\mathbb{X}^{s,p}_{-\Delta_x} \\ \mathbb{X}^{s+1,p}_{L} &= \mathbb{X}^{s+1,p}_{-\Delta_x}. \end{aligned}$$

The spaces associated with the Laplacian have been identified in Figure 6 and equivalence to (ii) follows. □

Remark 9.3. The argument above proves slightly more: it says that we have, all in the sense of continuous inclusions, $\mathbb{X}^{s,p}_{DB} \subseteq \mathbb{X}^{s,p}_{D}$ if and only if we have both $\mathbb{X}^{s,p}_{L} \subseteq a^{-1}(\dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2)$ and $\mathbb{X}^{s+1,p}_{L} \subseteq \dot{\mathrm{X}}^{s+1,p} \cap \mathrm{L}^2$, and that the same result holds upon reversing all inclusions.

In order to show that the identification regions are intervals, we borrow an interpolation argument from [3, Thm. 4.32] that uses the canonical completions of adapted Hardy spaces. In fact, for $\mathcal{H}(DB)$ and $\mathcal{H}^1(L)$ the result in [3] would apply 'off-the-shelf' but a slight variant is needed for $\mathcal{H}(L)$ because of the multiplication by a.

Lemma 9.4. *The sets $\mathcal{H}(DB)$, $\mathcal{H}(L)$ and $\mathcal{H}^1(L)$ are intervals.*

Proof. We begin with the proof for $\mathcal{H}(L)$. By definition, we have $p \in \mathcal{H}(L)$ if and only if the multiplication operators $a : \mathbb{H}^p_L \to \mathrm{H}^p \cap \mathrm{L}^2$ and $b := a^{-1} : \mathrm{H}^p \cap \mathrm{L}^2 \to \mathbb{H}^p_L$ are well-defined and bounded for the p-quasinorms. This is equivalent to saying that these operators have bounded extensions $\hat{a} : \psi\mathbb{H}^p_L \to \mathrm{H}^p$ and $\hat{b} : \mathrm{H}^p \to \psi\mathbb{H}^p_L$ to canonical completions in the sense that the following diagrams commute:

$$\begin{array}{ccc} \psi\mathbb{H}^p_L & \xrightarrow{\hat{a}} & \mathrm{H}^p \\ \uparrow & & \uparrow \\ \mathbb{H}^p_L & \xrightarrow{a} & \mathrm{H}^p \cap \mathrm{L}^2 \end{array} \qquad \begin{array}{ccc} \mathrm{H}^p & \xrightarrow{\hat{b}} & \psi\mathbb{H}^p_L \\ \uparrow & & \uparrow \\ \mathrm{H}^p \cap \mathrm{L}^2 & \xrightarrow{b} & \mathbb{H}^p_L \end{array}$$

Let now $p_0, p_1 \in \mathcal{H}(L)$. Since the spaces H^p and $\psi\mathbb{H}^p_L$ have universal approximation techniques, the extensions in the respective diagrams for p_0 and p_1 are compatible and we can use complex interpolation (Section 2.6 and Proposition 8.8) to obtain the same diagrams for all p between p_0 and p_1. Hence, these exponents are all in $\mathcal{H}(L)$.

The arguments for $\mathcal{H}^1(L)$ is identical except that we extend the identity operator. The same for $\mathcal{H}(DB)$ but instead of H^p we use a canonical completion $\psi\mathbb{H}^p_D$. □

Definition 9.5. (i) The upper and lower endpoints of $\mathcal{H}(DB)$ are denoted by $h_-(DB)$ and $h_+(DB)$.
(ii) The upper and lower endpoints of $\mathcal{H}(L)$ are denoted by $h_-(L)$ and $h_+(L)$. Likewise $h^1_\pm(L)$ are the endpoints of $\mathcal{H}^1(L)$.

9.2. **The identification theorem.** We come to the characterization of the identification region's endpoints through the critical numbers $p_-(L)$ and $q_\pm(L)$.

Theorem 9.6 (Identification Theorem)**.** *The endpoints of $\mathcal{H}(L)$ and $\mathcal{H}^1(L)$ can be characterized and controlled as follows:*

$$h_\pm(L) = p_\pm(L),$$
$$h^1_-(L) \le (p_-(L)_* \vee 1_*),$$
$$h^1_+(L) = q_+(L).$$

As a consequence, the endpoints of $\mathcal{H}(DB)$ are $h_-(DB) = p_-(L)$ and $h_+(DB) = q_+(L)$.

The relations for L imply those for DB since $\mathcal{H}(DB) = \mathcal{H}(L) \cap \mathcal{H}^1(L)$ and $q_+(L) \le p_+(L)$ by Theorem 6.2. We later precise this result by showing that these intervals are open at their ends except may be at the lower endpoint of $\mathcal{H}^1(L)$ for which we cannot even say whether the bound is sharp.

The proof of Theorem 9.6 is spread over 10 parts, using different methods for different regimes of parameters. Upper bounds on the size of $\mathcal{H}(L)$ are easy to obtain (Part 1), whereas lower bounds require establishing two continuous inclusions. Parts 2 - 5 focus on different inclusions of classical and L-adapted spaces. Parts 6 - 10 contain the synthesis of these preparatory steps.

Many arguments are known when $a = 1$. However, there are still some new difficulties when $a \ne 1$ that need to be taken care of and for some other parts we can simplify known arguments through the full strength of Figure 8 even when $a = 1$.

Part 1: $p_-(L) \le h_-(L)$ and $p_+(L) \ge h_+(L)$. Being slightly more precise, we show the inclusion $\mathcal{H}(L) \subseteq \mathcal{J}(L)$. Given $p \in (1_*, \infty)$, Proposition 8.10 yields

$$\|(1 + t^2 L)^{-1} f\|_{\mathbb{H}^p_L} \lesssim \|f\|_{\mathbb{H}^p_L}$$

uniformly for all $f \in \mathbb{H}^p_L$ and all $t > 0$. If now $p \in \mathcal{H}(L)$, then $\mathbb{H}^p_L = a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$ holds with equivalent Hardy norms and $p \in \mathcal{J}(L)$ follows.

Part 2: $\mathrm{L}^p \cap \mathrm{L}^2 \subseteq \mathrm{H}^p_L$ for $2 \leq p < \infty$. We are going to prove the continuous inclusion $\mathrm{L}^q \cap \mathrm{L}^2 \subseteq \mathbb{H}^q_L$ for $q \in [2, \infty)$.

We define $\mathbb{H}^q_L$ via the auxiliary function $\psi(z) := z^\alpha (1+z)^{-2\alpha}$ with an integer $\alpha > n/4$, so that this choice is admissible for all q, see Section 8.2. We have to establish the bound

$$\|\mathbb{Q}_{\psi,L} f\|_{\mathrm{T}^q} \lesssim \|f\|_q \quad (f \in \mathrm{L}^q \cap \mathrm{L}^2).$$

For a later purpose, we prove a more general statement. This uses the standard assumptions from Section 8.2. For $T = L$ the bound required here follows by simply taking the auxiliary parameters $\theta = 1$ and $p = 2$. The further interest in the lemma lies in picking p as large and θ as small as possible in order to allow for weaker decay assumption of ψ at the origin.

Lemma 9.7. *Let T be a sectorial operator that satisfies the standard assumptions (8.5). Fix $\mu \in (0, (\pi-\omega)/2)$ and $\sigma, \tau > 0$. Let $\psi \in \Psi^\tau_\sigma(\mathrm{S}^+_{\pi-2\mu})$ and consider the square function bound*

$$\|\mathbb{Q}_{\psi,T} f\|_{\mathrm{T}^q} \lesssim \|\psi\|_{\sigma,\tau,\mu} \|f\|_q \quad (f \in \mathrm{L}^q \cap \mathrm{L}^2),$$

where the implicit constant does not depend on ψ. Then this bound is valid for every $q \geq 2$ provided that one can find $p \in [2, \infty)$ and $\theta \in (0, 1]$ such that $((1 + t^2 T)^{-1})_{t>0}$ is L^p-bounded and

$$\mu \in \Big(0, \frac{\theta(\pi - \omega)}{2}\Big) \quad \& \quad \sigma > \frac{n}{2[p,2]_\theta}.$$

Proof. In the following implicit constants are allowed to depend on the fixed parameters but not on ψ itself. Via McIntosh's theorem the boundedness for $q = 2$ is equivalent to the bounded H^∞-calculus on $\overline{\mathsf{R}(T)}$. Hence, we can state

$$\|\mathbb{Q}_{\psi,T} f\|_{\mathrm{T}^2} \lesssim \|\psi\|_{\mathrm{L}^\infty(\mathrm{S}^+_{\pi-2\mu})} \|f\|_2 \quad (f \in \overline{\mathsf{R}(T)}).$$

Cauchy's theorem yields $\psi(t^2 T) f = 0$ for all $t > 0$ if $f \in \mathsf{N}(T)$. Hence, we can state same bound for all $f \in \mathrm{L}^2$. By complex interpolation it remains to treat the case $q = \infty$, that is to say, to prove for all balls $B \subseteq \mathbb{R}^n$ of radius $r > 0$ and all $f \in \mathrm{L}^\infty \cap \mathrm{L}^2$ that

$$\Big(\frac{1}{|B|} \int_0^r \int_B |\psi(t^2 T) f|^2 \, \frac{\mathrm{d}x \mathrm{d}t}{t}\Big)^{1/2} \lesssim \|\psi\|_{\tau,\sigma,\mu} \|f\|_\infty. \tag{9.2}$$

Having fixed B, we write $f = \sum_{j \geq 1} f_j$ with $f_j := \mathbf{1}_{C_j(B)} f$. For $j = 1$ we use that $\mathrm{T}^2 = \mathrm{L}^2(\mathbb{R}^{1+n}_+; \frac{\mathrm{d}t \mathrm{d}x}{t})$ and again the L^2-bound to give

$$\begin{aligned} &\frac{1}{|B|} \int_0^r \int_B |\psi(t^2 T) f_1(x)|^2 \, \frac{\mathrm{d}x \mathrm{d}t}{t} \\ &\lesssim \frac{1}{|B|} \|\mathbb{Q}_{\psi,L} f_1\|^2_{\mathrm{T}^2} \end{aligned}$$

$$\begin{aligned}&\lesssim \frac{1}{|B|}\|\psi\|^2_{\mathrm{L}^\infty(\mathrm{S}^+_{\pi-2\mu})}\|f_1\|_2^2\\ &\leq 4^n\|\psi\|^2_{\mathrm{L}^\infty(\mathrm{S}^+_{\pi-2\mu})}\|f\|_\infty^2.\end{aligned}$$

Next, we let $\varrho := [p,2]_\theta$ and obtain from Lemma 4.16.(i) the off-diagonal estimate

$$\begin{aligned}\|\psi(t^2L)f_j\|_{\mathrm{L}^\varrho(B)} &\lesssim \|\psi\|_{\tau,\sigma,\mu}\bigg(1+\frac{2^j r}{t}\bigg)^{-2\sigma}\|f\|_{\mathrm{L}^\varrho(C_j(B))}\\ &\leq \|\psi\|_{\tau,\sigma,\mu}\frac{t^{2\sigma}}{r^{2\sigma}}2^{-j(2\sigma-\frac{n}{\varrho})}|B|^{1/\varrho}\|f\|_\infty.\end{aligned}$$

Since $\varrho \geq 2$, we obtain from Hölder's inequality that

$$\|\psi(t^2L)f_j\|_{\mathrm{L}^2(B)} \lesssim \|\psi\|_{\tau,\sigma,\mu}\frac{t^{2\sigma}}{r^{2\sigma}}2^{-j(2\sigma-\frac{n}{\varrho})}|B|^{1/2}\|f\|_\infty$$

and taking L^2-norms with respect to ${}^{\mathrm{d}t}/_{t}$, we are led to

$$\begin{aligned}&\bigg(\frac{1}{|B|}\int_0^r\int_B|\psi(t^2L)f_j(x)|^2\,\frac{\mathrm{d}x\mathrm{d}t}{t}\bigg)^{1/2}\\ &\quad\lesssim \|\psi\|_{\tau,\sigma,\mu}2^{-j(2\sigma-\frac{n}{\varrho})}\|f\|_\infty\bigg(\int_0^r\frac{t^{4\sigma}}{r^{4\sigma}}\,\frac{\mathrm{d}t}{t}\bigg)^{1/2}\\ &\quad= 2^{-j(2\sigma-\frac{n}{\varrho})}\|\psi\|_{\tau,\sigma,\mu}(4\sigma)^{-\frac{1}{2}}\|f\|_\infty.\end{aligned}$$

By assumption, we have $2\sigma > {}^{n}/_{\varrho}$. Summing up in j yields (9.2). □

Remark 9.8. It becomes clear from the proof above that Lemma 9.7 has very little to do with sectorial operators and could be extended to more general extensions

$$(\mathbb{Q}_\psi f)(t,x) := (\psi(t)f)(x)$$

where $\psi : (0,\infty) \to \mathcal{L}(\mathrm{L}^2)$ is a strongly measurable family of operators. For example, with $p=2$ and $\theta=1$ the only properties of $(\psi(t))_{t>0}$ that we have used to get for every $q \geq 2$ a bound

$$\|\mathbb{Q}_\psi f\|_{\mathrm{T}^q} \lesssim \|f\|_q \quad (f \in \mathrm{L}^q \cap \mathrm{L}^2)$$

is the corresponding L^2-bound and L^2 off-diagonal estimates of order $\gamma > {}^{n}/_{2}$.

Part 3: Injection of classical spaces into L-adapted spaces for $p \in (1,2)$. For this part we work with the auxiliary function ψ defined by

$$\psi_\alpha(z) := z^{\alpha-1/2}(1+z)^{-3\alpha}, \tag{9.3}$$

where $\alpha \in \mathbb{N}$ will be chosen sufficiently large depending on exponents and dimensions. Throughout this part it will be convenient to write

$$S_{\psi_\alpha,L}f(x) := \bigg(\iint_{|x-y|<t}|\psi_\alpha(t^2L)f(y)|^2\,\frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}}\bigg)^{1/2}, \tag{9.4}$$

so that $\|S_{\psi,L}\cdot\|_p$ becomes an equivalent norm on $\mathbb{H}^p_L$ provided that $\alpha > {}^{n}/_{(2p)} - {}^{n}/_{4}$, compare with Section 8.2.

Our main objective is to establish the following extrapolation result for square functions.

Lemma 9.9. *Suppose for some $q \in (p_-(L) \vee 1, 2]$ and all sufficiently large α (depending on $q, p_-(L), n$) that*

$$\|S_{\psi_\alpha,L}(L^{1/2}u)\|_q \lesssim \|\nabla_x u\|_q \quad (u \in \dot{\mathrm{W}}^{1,q} \cap \mathrm{W}^{1,2}). \tag{9.5}$$

Then for all $p \in (q_* \vee 1, q)$ and all sufficiently large α (depending on $p, q, p_-(L), n$) it follows that

$$\|S_{\psi_\alpha,L}(L^{1/2}u)\|_p \lesssim \|\nabla_x u\|_p \quad (u \in \dot{\mathrm{W}}^{1,p} \cap \mathrm{W}^{1,2}).$$

Remark 9.10. Assumption (9.5) holds for $q = 2$ and any $\alpha \in \mathbb{N}$. Indeed, this follows from $\mathbb{H}^2_L = \overline{\mathsf{R}(L)} = \mathrm{L}^2$ and the solution of the Kato problem. Starting from there, we can iterate Lemma 9.9 in order to conclude that for every $q \in (p_-(L)_* \vee 1, 2]$ the bound (9.5) holds for all sufficiently large α.

Before giving the proof of Lemma 9.9, let us state the more important consequences of this lemma for the identification of L-adapted Hardy spaces.

Proposition 9.11. *If $p \in (p_-(L) \vee 1, 2]$, then $\mathrm{L}^p \cap \mathrm{L}^2 \subseteq \mathbb{H}^p_L$ with continuous inclusion for the p-norms.*

Proof. First let us assume $f \in \mathrm{L}^p \cap \mathsf{R}(L^{1/2})$. By Lemma 7.2 this is a dense subspace of $\mathrm{L}^p \cap \mathrm{L}^2$. We put $u := L^{-1/2}f$. Since the Riesz transform is L^p-bounded (Theorem 7.3), we have $u \in \dot{\mathrm{W}}^{1,p} \cap \mathrm{W}^{1,2}$ with $\|\nabla_x u\|_p \lesssim \|f\|_p$. Remark 9.10 yields

$$\|S_{\psi_\alpha,L} f\|_p \lesssim \|f\|_p$$

if α is sufficiently large. If in addition $\alpha > {}^{n}/_{(2p)} - {}^{n}/_{4}$, then ψ_α is admissible as auxiliary function for $\mathbb{H}^p_L$ and we obtain

$$\|f\|_{\mathbb{H}^p_L} \lesssim \|f\|_p$$

with an implicit constant independent of f. A general $f \in \mathrm{L}^p \cap \mathrm{L}^2$ can be approximated by $(f_j) \subseteq \mathrm{L}^p \cap \mathsf{R}(L^{1/2})$ in $\mathrm{L}^p \cap \mathrm{L}^2$. By L^2-convergence

$$\int_{B(x,t)} |\psi_\alpha(t^2L)f(y)|^2 \,\mathrm{d}y = \lim_{j\to\infty} \int_{B(x,t)} |\psi_\alpha(t^2L)f_j(y)|^2 \,\mathrm{d}y$$

holds for all $(t,x) \in \mathbb{R}^{1+n}$ and we invoke Fatou's lemma to give

$$\begin{aligned}\|f\|^p_{\mathbb{H}^p_L} &\leq \liminf_{j\to\infty} \int_{\mathbb{R}^n} \bigg(\iint_{|x-y|<t} |\psi_\alpha(t^2L)f_j(y)|^2 \,\frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}} \bigg)^{p/2} \mathrm{d}x \\ &= \liminf_{j\to\infty} \|f_j\|^p_{\mathbb{H}^p_L}.\end{aligned}$$

On the right-hand side $\|f_j\|_{\mathbb{H}^p_L}^p$ is under control by $\|f_j\|_p$ thanks to the first part of the proof and L^p-convergence of (f_j) gives the required bound by $\|f\|_p$. □

Proposition 9.12. *If $p \in (p_-(L)_* \vee 1, 2]$, then $\dot{\mathrm{W}}^{1,p} \cap \mathrm{L}^2 \subseteq \mathbb{H}^{1,p}_L$ with continuous inclusion for the p-norms.*

Proof. By the universal approximation technique even $\mathcal{Z}$ is dense in $\dot{\mathrm{W}}^{1,p} \cap \mathrm{L}^2$. Hence, the same approximation argument as in the previous proof shows that it suffices to check

$$\|u\|_{\mathbb{H}^{1,p}_L} \lesssim \|\nabla_x u\|_p \quad (u \in \dot{\mathrm{W}}^{1,p} \cap \mathrm{W}^{1,2}).$$

We take α large enough so that (9.5) holds at exponent $q = p$ and $\alpha > {}^{n}/_{(2p)} - {}^{n}/_{4}$ to make sure that $\mathbb{H}^{1,p}_L$ can be defined through the auxiliary function $\varphi_\alpha(z) := \sqrt{z}\psi_\alpha(z)$. We have $\psi_\alpha(t^2L)L^{1/2}u = t^{-1}\varphi_\alpha(t^2L)u$ for $t > 0$ and therefore

$$\begin{aligned} \|u\|_{\mathbb{H}^{1,p}_L} = \|\mathbb{Q}_{\varphi_\alpha,L} u\|_{\mathrm{T}^{1,p}} &= \|\mathbb{Q}_{\psi_\alpha,L}(L^{1/2}u)\|_{\mathrm{T}^{0,p}} \\ &= \|S_{\psi_\alpha,L}(L^{1/2}u)\|_p \\ &\lesssim \|\nabla_x u\|_p. \end{aligned}$$ □

We come to the proof of Lemma 9.9. We modify the strategy of [22, pp.42-45]. Henceforth we fix p, q as in the statement and we write $\psi = \psi_\alpha$, where α will be chosen larger from step to step in dependence of $p, q, p_-(L), n$.

Let $u \in \dot{\mathrm{W}}^{1,p} \cap \mathrm{W}^{1,2}$ and $\lambda > 0$. It will be enough to obtain the weak-type estimate

$$\Big|\Big\{x \in \mathbb{R}^n : S_{\psi,L}(L^{1/2}u)(x) > 3\lambda\Big\}\Big| \lesssim \frac{1}{\lambda^p}\|\nabla_x u\|_p^p \tag{9.6}$$

with implicit constant independent of u and λ. Indeed, consider the positive sublinear operator

$$T : \mathcal{Z} \to \mathrm{L}^2, \quad Tu := S_{\psi,L}(L^{1/2}(-\Delta_x)^{-1/2}u)$$

and recall that $\mathcal{Z}$ is dense in all (intersections of) L^r-spaces with $r > 1$. Now, T is of strong type (q, q) by (9.5) and of weak type (p, p) by (9.6). Hence, it is of strong type (r, r) for every $r \in (p, q]$ by the Marcinkiewicz interpolation theorem. As $(-\Delta_x)^{-1/2}$ is invertible on $\mathcal{Z}$, this means that we have

$$\|S_{\psi,L}(L^{1/2}u)\|_r \lesssim \|\nabla_x u\|_r \quad (u \in \mathcal{Z}).$$

This bound extends to $u \in \dot{\mathrm{W}}^{1,r} \cap \mathrm{W}^{1,2}$ by density as before. Since $p \in (q_* \vee 1, q)$ and $r \in (p, q]$ were arbitrary, the claim follows.

The proof of (9.6) itself comes in 8 steps.

Step 1: Calderón–Zygmund decomposition. We use the decomposition for Sobolev functions that was introduced in [6, Lem. 4.12], see [5] for the correction of an inaccuracy in the original proof.

Since $u \in \dot{\mathrm{W}}^{1,p}$, according to this decomposition, there is a countable collection of cubes $(Q_j)_{j\in J}$, measurable functions g and b_j and constants C and N that depend only on dimensions and p, such that

(i) $u = g + \sum_{j\in J} b_j$ pointwise almost everywhere,

(ii) $\|\nabla_x g\|_\infty \le C\lambda$,

(iii) b_j has support in Q_j and $\|\nabla_x b_j\|_p^p \,\mathrm{d}x \le C\lambda^p |Q_j|$,

(iv) $\sum_{j\in J} |Q_j| \le C\lambda^{-p} \|\nabla_x u\|_p^p$,

(v) $\sum_{j\in J} \mathbf{1}_{Q_j} \le N$.

More precisely, setting $\Omega \coloneqq \{\mathcal{M}(|\nabla_x u|^p) > \lambda^p\} \subseteq \mathbb{R}^n$, the b_j take the form $b_j = (u - u(x_j))\chi_j$ with $x_j \in 2Q_j \cap {}^c\Omega$ and $\chi_j \in \mathrm{C}_0^\infty(Q_j)$ such that $\|\chi_j\|_\infty + \ell(Q_j)\|\nabla_x \chi_j\|_\infty \le C$. The function u has a representative on ${}^c\Omega$ that satisfies $|u(x) - (u)_Q| \le C\lambda\ell(Q)$ whenever Q is a cube centered at $x \in {}^c\Omega$ and this is how we understand $u(x_j)$.

We recall these details on the construction because we need two additional properties in the proof of (9.6):

(i') If $u \in \dot{\mathrm{W}}^{1,r}$ for some $r \in (1,\infty)$, then $b_j \in \mathrm{W}^{1,r}$ for all j and $\sum_{j\in J} b_j$ converges unconditionally in $\dot{\mathrm{W}}^{1,r}$,

(ii') If $r \in (p, p^*)$, then $\|\nabla_x g\|_r^r \le C'\lambda^{r-p}\|\nabla_x g\|_p^p$ and $\|b_j\|_r^r \le C'\lambda^r |Q_j|^{1+\frac{r}{n}}$ for all j, where C' also depends on r.

To see property (i'), we let Q_j' be the cube centered at x_j with side-length $3\ell(Q_j)$ and write

$$b_j = (u - (u)_{Q_j})\chi_j + ((u)_{Q_j} - (u)_{Q_j'})\chi_j + ((u)_{Q_j'} - u(x_j))\chi_j. \tag{9.7}$$

The special property of u on ${}^c\Omega$ yields $|((u)_{Q_j'} - u(x_j))\nabla_x \chi_j| \le C\lambda$ on $\mathbb{R}^n$. Next, since $Q_j \subseteq Q_j'$, we obtain from Poincaré's inequality that

$$|(u)_{Q_j} - (u)_{Q_j'}| \lesssim \fint_{Q_j'} |u - (u)_{Q_j'}| \,\mathrm{d}x \lesssim \ell(Q_j) \fint_{Q_j'} |\nabla_x u| \,\mathrm{d}x.$$

The right-hand side is bounded by λ since $x_j \in {}^c\Omega$ and we obtain $|((u)_{Q_j} - (u)_{Q_j'})\nabla_x \chi_j| \le C\lambda$ on $\mathbb{R}^n$. Once again by Poincaré's inequality we have

$$\int_{\mathbb{R}^n} |\nabla_x((u - (u)_{Q_j})\chi_j)|^r \,\mathrm{d}x \lesssim \int_{Q_j} |\nabla_x u|^r \,\mathrm{d}x,$$

so that altogether we obtain from (9.7) the estimate

$$\int_{\mathbb{R}^n} |\nabla_x b_j|^r \,\mathrm{d}x \lesssim \lambda^r |Q_j| + \int_{Q_j} |\nabla_x u|^r \,\mathrm{d}x. \tag{9.8}$$

Since b_j has compact support, we have $b_j \in \mathrm{W}^{1,r}$ qualitatively. For any partial sum of j's we obtain from (v) and Hölder's inequality that

$$\int_{\mathbb{R}^n} \Big|\nabla_x \sum_j b_j\Big|^r \, \mathrm{d}x \leq N^{r-1} \int_{\mathbb{R}^n} \sum_j |\nabla_x b_j|^r \, \mathrm{d}x. \tag{9.9}$$

Properties (iv) and (v) justify using the dominated convergence theorem to conclude that $\sum_{j \in J} \nabla_x b_j$ converges in L^r. The limit is independent of the order of summation since the sum contains at most N non-zero terms at each $x \in \mathbb{R}^n$.

As for (ii'), the L^r-bound for b_j immediately follows from the Sobolev–Poincaré inequality [94, Cor. 4.2.3] and (iii). From (9.8) and (9.9) with $r = p$ and then (iv), we obtain $\|\nabla_x \sum_{j \in J} b_j\|_p \leq C\|\nabla_x u\|_p$. We conclude $\|\nabla_x g\|_p \leq C\|\nabla_x u\|_p$ from (i) and the required L^r-bound follows from (ii).

Step 2: Decomposition of the level set. For the same α as is the definition of $\psi = \psi_\alpha$ in (9.3) we introduce a function $\varphi \in \mathrm{H}^\infty$ through

$$\varphi(z) := z^\alpha (1+z)^{-\alpha} \tag{9.10}$$

and we decompose $u = g + \widetilde{g} + b$, using the series

$$\widetilde{g} := \sum_{j \in J} (1 - \varphi(\ell_j^2 L)) b_j,$$
$$b := \sum_{j \in J} \varphi(\ell_j^2 L) b_j.$$

In Step 4 we shall check that the series $\widetilde{g}$ converges in $\dot{\mathrm{W}}^{1,q}$, so that by (i') with $r = q$ the same is true for b.

Anticipating the convergence of $\widetilde{g}$, we obtain that the set on the left-hand side of (9.6) is contained in the union of

$$A_1 := \Big\{x \in \mathbb{R}^n : S_{\psi,L}(L^{1/2} g)(x) > \lambda\Big\},$$
$$A_2 := \Big\{x \in \mathbb{R}^n : S_{\psi,L}(L^{1/2} \widetilde{g})(x) > \lambda\Big\},$$
$$A_3 := \Big\{x \in \mathbb{R}^n : S_{\psi,L}(L^{1/2} b)(x) > \lambda\Big\},$$

where we do not make a notational distinction between $v \mapsto S_{\psi,L}(L^{1/2} v)$ and its bounded extension from $\dot{\mathrm{W}}^{1,q}$ into L^q. It suffices to bound the measure of each of the three sets by a generic multiple of $\lambda^{-p}\|\nabla_x u\|_p$.

Step 3: Bound of A_1. We use the Markov inequality, the assumption and (ii') to give

$$|A_1| \leq \lambda^{-q} \|S_{\psi,L}(L^{1/2} g)\|_q^q \lesssim \lambda^{-q} \|\nabla_x g\|_q^q \lesssim \lambda^{-p} \|\nabla_x u\|_p^p.$$

Step 4: Convergence and estimate of $\widetilde{g}$. For the time being, let j run only through a finite set of J. Consider the partial sum of $\widetilde{g}$ given by

$$\sum_j (1-\varphi(\ell_j^2 L))b_j = \sum_{\beta=1}^{\alpha} \binom{\alpha}{\beta}(-1)^{\beta-1}\Big(\sum_j (1+\ell_j^2 L)^{-\beta} b_j\Big), \tag{9.11}$$

where we have expanded φ from (9.10). We fix β and introduce

$$f_\beta := \sum_j (1+\ell_j^2 L)^{-\beta} b_j. \tag{9.12}$$

Since we have $b_j \in \mathrm{W}^{1,2} = \mathsf{D}(L^{1/2})$ by (i'), the same is true for f_β. We calculate its norm in $\dot{\mathrm{W}}^{1,q}$ by dualizing $\nabla_x f$ against $h \in \mathrm{C}_0^\infty$, normalized to $\|h\|_{q'} = 1$:

$$\langle \nabla_x f_\beta, h\rangle = \sum_j \sum_{k=1}^{\infty} \langle \nabla_x (1+\ell_j^2 L)^{-\beta} b_j, h_{j,k}\rangle,$$

where $h_{j,k} := \mathbf{1}_{C_k(Q_j)} h$. We take adjoints, use the support of b_j and then Hölder's inequality to give

$$|\langle \nabla_x f_\beta, h\rangle| \leq \sum_j \sum_{k=1} \|b_j\|_{\mathrm{L}^q(Q_j)} \|(\nabla_x (1+\ell_j^2 L)^{-\beta})^* h_{k,j}\|_{\mathrm{L}^{q'}(Q_j)}.$$

By (ii') we get

$$\begin{aligned} &|\langle \nabla_x f_\beta, h\rangle| \\ &\quad\leq \sum_j \sum_{k=1}^{\infty} \lambda |Q_j|^{\frac{1}{q}} \|(\ell_j \nabla_x (1+\ell_j^2 L)^{-\beta})^* h_{k,j}\|_{\mathrm{L}^{q'}(Q_j)}. \end{aligned} \tag{9.13}$$

For $t > 0$ the families $((1+t^2L)^{-1})$ and $(t\nabla_x(1+t^2L)^{-1})$ satisfy L^2 off-diagonal estimates of arbitrarily large order. Now q is an inner point of the interval of resolvent bounds $(p_-(L) \vee 1, 2)$, which by Theorem 6.2 is the same as $(q_-(L) \vee 1, 2)$ for gradient bounds. By interpolation (Lemma 4.14) both families have L^q off-diagonal bounds of arbitrarily large order. Composition and duality yield $\mathrm{L}^{q'}$ off-diagonal bounds of arbitrarily large order $\gamma > 0$ for $((t\nabla_x(1+t^2L)^{-\beta})^*)$. Consequently, we have

$$\begin{aligned} \|(\ell_j \nabla_x (1+\ell_j^2 L)^{-\beta})^* h_{k,j}\|_{\mathrm{L}^{q'}(Q_j)} &\lesssim 2^{-k\gamma} \|h\|_{\mathrm{L}^{q'}(C_k(Q_j))} \\ &\lesssim 2^{-k\gamma} |2^k Q_j|^{\frac{1}{q'}} (\mathcal{M}(|h|^{q'})(x))^{\frac{1}{q'}}, \end{aligned}$$

where $x \in Q_j$ is arbitrary. We take $\gamma > n/q'$ so that when substituting this estimate back into (9.13), we obtain a finite sum in k:

$$|\langle \nabla_x f_\beta, h\rangle| \lesssim \lambda \sum_j |Q_j| (\mathcal{M}(|h|^{q'})(x))^{\frac{1}{q'}}.$$

We average in $x \in Q_j$, take into account the finite overlap of the Q_j and apply Kolmogorov's Lemma, in order to conclude that

$$
\begin{aligned}
|\langle \nabla_x f_\beta, h\rangle| &\lesssim \lambda \sum_j \int_{Q_j} (\mathcal{M}(|h|^{q'})(x))^{\frac{1}{q'}} \, \mathrm{d}x \\
&\lesssim \lambda \int_{\cup_j Q_j} (\mathcal{M}(|h|^{q'})(x))^{\frac{1}{q'}} \, \mathrm{d}x \qquad (9.14)\\
&\lesssim \lambda \Big| \bigcup_j Q_j \Big|^{\frac{1}{q}} \|h\|_{q'}^{\frac{1}{q'}}.
\end{aligned}
$$

We recall the definition of f_β from (9.12) and that h was normalized in $\mathrm{L}^{q'}$. Hence we have shown the estimate

$$\Big\| \nabla_x \sum_j (1 + \ell_j^2 L)^{-\beta} b_j \Big\|_q \lesssim \lambda \Big(\sum_j |Q_j| \Big)^{\frac{1}{q}},$$

where j runs over a finite subset of J. Property (iv) of the Calderón–Zygmund decomposition implies that $\sum_{j \in J} (1 + \ell_j^2 L)^{-\beta} b_j$ converges in $\dot{\mathrm{W}}^{1,q}$ and that its norm is under control by $\lambda^{1-p/q} \|\nabla_x u\|_p^{p/q}$. By definition in (9.11), the series $\widetilde{g}$ is a finite sum in β over series of this type. Hence, it converges in $\dot{\mathrm{W}}^{1,q}$ as required and is bounded by

$$\|\nabla_x \widetilde{g}\|_q \lesssim \lambda^{1-\frac{p}{q}} \|\nabla_x u\|_p^{\frac{p}{q}}. \qquad (9.15)$$

Step 5: Bound of A_2. We argue as in Step 3 and use (9.15) instead of (ii') to give

$$|A_2| \lesssim \lambda^{-q} \|\nabla_x \widetilde{g}\|_q^q \lesssim \lambda^{-p} \|\nabla_x u\|_p^p.$$

Step 6: Preparation of the bound for A_3. By Markov's inequality and the boundedness of $v \mapsto \mathrm{S}_{\psi,L}(L^{1/2} v)$ from $\dot{\mathrm{W}}^{1,q}$ into L^q, we have

$$\Big| \Big\{ x \in \mathbb{R}^n : S_{\psi,L}(L^{1/2} v)(x) > \lambda \Big\} \Big| \lesssim \lambda^{-q} \|\nabla_x v\|_q \quad (v \in \dot{\mathrm{W}}^{1,q}).$$

In particular, the measure of the set on the left tends to 0 as v tends to 0 in $\dot{\mathrm{W}}^{1,q}$. Since the series b converges in $\dot{\mathrm{W}}^{1,q}$, this argument shows that it suffices to derive the desirable bound $\lambda^{-p} \|\nabla_x u\|_p$ for the measure of

$$\widetilde{A}_3 := \Big\{ x \in \mathbb{R}^n : S_{\psi,L}(L^{1/2} \widetilde{b})(x) > \frac{\lambda}{2} \Big\},$$

where $\widetilde{b} := \sum_j \varphi(\ell_j^2 L) b_j$ and j runs over a finite subset of J. Again, this reduction bears the advantage that $\widetilde{b}$ is contained in $\mathrm{W}^{1,2} = \mathsf{D}(L^{1/2})$ and hence we can properly work with the functional calculus of L. In fact, such type of reduction is necessary since p may lie outside of $\mathcal{J}(L)$ and therefore there is no hope for reasonable functional calculus bounds for L on L^p.

First, we can split off $E \coloneqq \bigcup_{j \in J} 6Q_j$ since its measure is under control by property (iii) of the Calderón–Zygmund decomposition. Next, by Markov's inequality and the definition of $S_{\psi,L}$, the measure of the remaining set is at most

$$\begin{aligned} |\widetilde{A}_3 \setminus E| &\leq 4\lambda^{-2} \int_{\widetilde{A}_3 \setminus E} |(S_{\psi,L}(L^{1/2}\widetilde{b}))(x)|^2 \,\mathrm{d}x \\ &\leq 4\lambda^{-2} \iint_{\mathbb{R}^{1+n}_+} |(\psi(t^2L)L^{1/2}\widetilde{b})(y)|^2 \frac{|B(y,t) \setminus E|}{t^n} \frac{\mathrm{d}t\mathrm{d}y}{t}. \end{aligned}$$

The set $B(y,t) \setminus E$ has of course measure controlled by t^n but if y is contained in the cube $4Q_j$, then this set is empty for all $t < \ell_j$. Hence, introducing the 'local' and 'global' parts

$$\begin{aligned} f_{\mathrm{loc}}(t,y) &\coloneqq \sum_j \mathbf{1}_{4Q_j}(y)\mathbf{1}_{(\ell_j,\infty)}(t)\Big(\psi(t^2L)L^{1/2}\varphi(\ell_j^2L)b_j\Big)(y), \\ f_{\mathrm{glob}}(t,y) &\coloneqq \sum_j \mathbf{1}_{^c(4Q_j)}(y)\Big(\psi(t^2L)L^{1/2}\varphi(\ell_j^2L)b_j\Big)(y), \end{aligned} \tag{9.16}$$

we obtain

$$|\widetilde{A}_3 \setminus E| \lesssim \lambda^{-2} \iint_{\mathbb{R}^{1+n}} |f_{\mathrm{loc}}(t,y)|^2 + |f_{\mathrm{glob}}(t,y)|^2 \frac{\mathrm{d}t\mathrm{d}y}{t}.$$

and we are left with bounding the two integrals on the right by generic multiples of $\lambda^{2-p}\|\nabla_x u\|_p^p$.

Step 7: The local part. Let $h \in \mathrm{L}^2(\mathbb{R}^{1+n}_+, \frac{\mathrm{d}t\mathrm{d}x}{t})$ and let $\langle \cdot\,, \cdot \rangle$ be the duality pairing on that space. By the Cauchy–Schwarz inequality we first find

$$|\langle f_{\mathrm{loc}}, h\rangle| \leq I_j \bigg(\int_{4Q_j} \int_{\ell_j}^\infty |h(t,y)|^2 \frac{\mathrm{d}t\mathrm{d}y}{t} \bigg)^{1/2}$$

and then, generously bounding the second integral by a maximal function in x, that

$$|\langle f_{\mathrm{loc}}, h\rangle| \leq \sum_j I_j |4Q_j|^{1/2} \inf_{x \in Q_j} (\mathcal{M}(H^2)(x))^{1/2}, \tag{9.17}$$

where

$$\begin{aligned} I_j &\coloneqq \bigg(\int_{\ell_j}^\infty \int_{4Q_j} |L^{1/2}\psi(t^2L)\varphi(\ell_j^2L)b_j(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t} \bigg)^{1/2}, \\ H(y) &\coloneqq \bigg(\int_0^\infty |h(t,y)|^2 \frac{\mathrm{d}t}{t} \bigg)^{1/2}. \end{aligned} \tag{9.18}$$

At this stage of the proof we introduce a fixed exponent $\varrho \in (p_-(L) \vee 1, q)$ and take the parameter α in (9.3) large enough to grant that

$(tL^{1/2}\psi(t^2L))_{t>0}$ is $\mathrm{L}^\varrho - \mathrm{L}^2$-bounded. This is possible by Lemma 7.4,(ii) since ϱ is not the lower endpoint of $\mathcal{J}(L)$ and we can expand

$$tL^{1/2}\psi(t^2L) = ((1+t^2L)^{-2} - (1+t^2L)^{-3})^\alpha$$

in terms of resolvents of power at least 2α. By interpolation with the L^2-bound we then have of course $\mathrm{L}^r - \mathrm{L}^2$-boundedness for all $r \in [\varrho, 2]$. Since φ from (9.10) is bounded, we obtain from the functional calculus on L^2 that

$$\|L^{1/2}\psi(t^2L)\varphi(\ell_j^2 L)f\|_2 \lesssim t^{-1}t^{\frac{n}{2}-\frac{n}{r}}\|f\|_r \quad (f \in \mathrm{L}^r \cap \mathrm{L}^2). \tag{9.19}$$

In this step we use the above estimate with $r = q$ and $f = b_j$ to bound I_j. As we have $n - 2n/q \leq 0$, integration in t leads us to

$$I_j \lesssim \|b_j\|_q \bigg(\int_{\ell_j}^\infty t^{n-\frac{2n}{q}-2}\,\frac{\mathrm{d}t}{t}\bigg)^{\frac{1}{2}} \lesssim \ell_j^{\frac{n}{2}-\frac{n}{q}-1}\|b_j\|_q \lesssim \lambda |Q_j|^{\frac{1}{2}},$$

where the final step uses (ii'). Going back to (9.17), we have established the bound

$$|\langle f_{\mathrm{loc}}, h\rangle| \lesssim \lambda \sum_j |Q_j| \inf_{x\in Q_j} (\mathcal{M}(H^2)(x))^{\frac{1}{2}}, \tag{9.20}$$

so that we can bring into play Kolmogorov's lemma as in (9.14) and then use property (iv) to conclude

$$|\langle f_{\mathrm{loc}}, h\rangle| \leq \lambda \bigg|\bigcup_j Q_j\bigg|^{\frac{1}{2}} \|H^2\|_1^{\frac{1}{2}} \lesssim \lambda^{1-\frac{p}{2}} \|\nabla_x u\|_p^{\frac{1}{2}} \|h\|_{\mathrm{L}^2(\frac{\mathrm{d}t\mathrm{d}x}{t})}.$$

Since h was arbitrary, we have proved the bound that was required at the end of Step 6:

$$\iint_{\mathbb{R}^{1+n}_+} |f_{\mathrm{loc}}(t,y)|^2 \,\frac{\mathrm{d}t\mathrm{d}y}{t} \lesssim \lambda^{2-p}\|\nabla_x u\|_p.$$

Step 8: The global part. We use the same duality argument as in Step 7 except that for f_{glob} we will have to work on the ${}^c(4Q_j)$, which we split into annuli $C_k(Q_j)$, $k \geq 2$. In this manner, our substitute for (9.17) becomes

$$|\langle f_{\mathrm{glob}}, h\rangle| \leq \sum_j \sum_{k\geq 2} I_{j,k} |2^{k+1}Q_j|^{\frac{1}{2}} \inf_{x\in Q_j} (\mathcal{M}(H^2)(x))^{\frac{1}{2}}, \tag{9.21}$$

where H is still as in (9.18) and

$$I_{j,k} := \bigg(\int_0^\infty \int_{C_k(Q_j)} |L^{1/2}\psi(t^2L)\varphi(\ell_j^2L)b_j(y)|^2 \,\frac{\mathrm{d}y\mathrm{d}t}{t}\bigg)^{\frac{1}{2}}.$$

From the definitions in (9.3) and (9.10) we see that $z \mapsto \sqrt{z}\psi(z)$ and φ are of class $\Psi_\alpha^{2\alpha}$ and Ψ_α^0, respectively. Lemma 4.16.(i) yields for all

$f \in \mathrm{L}^2$ with support in Q_j that

$$\|L^{1/2}\psi(t^2L)\varphi(\ell_j^2L)f\|_{\mathrm{L}^2(C_k(Q_j))} \lesssim t^{-1}\left(\frac{2^k\ell_j}{t}\right)^{-2\alpha}\|f\|_{\mathrm{L}^2(Q_j)}.$$

For fixed j, k, t, we interpolate this bound with (9.19) for $r = \varrho$ by means of the Riesz–Thorin theorem. This results in

$$\begin{aligned}\|L^{1/2}\psi(t^2L)\varphi(\ell_j^2L)f\|_{\mathrm{L}^2(C_k(Q_j))}\\ \lesssim t^{-1+\frac{n}{2}-\frac{n}{q}}\left(\frac{2^k\ell_j}{t}\right)^{-2\theta\alpha}\|f\|_{\mathrm{L}^q(Q_j)},\end{aligned} \tag{9.22}$$

where $\theta \in (0,1)$ is such that $q = [\varrho, 2]_\theta$. In exactly the same manner we can interpolate the assertion of Lemma 4.16.(ii) with (9.19) in order to obtain

$$\|L^{1/2}\psi(t^2L)\varphi(\ell_j^2L)f\|_{\mathrm{L}^2(C_k(Q_j))} \lesssim t^{-1+\frac{n}{2}-\frac{n}{q}}2^{-2\theta\alpha k}\|f\|_{\mathrm{L}^q(Q_j)}, \tag{9.23}$$

provided that $t \geq \ell_j$.

Now we come back to $I_{j,k}$, split the outer integral at $t = \ell_j$ and use (9.22) and (9.23) with $f = b_j$ to give

$$\begin{aligned}I_{j,k}^2 \lesssim 2^{-4\theta\alpha k}\ell_j^{-4\theta\alpha}\|b_j\|_q^2\int_0^{\ell_j} t^{-2+n-\frac{2n}{q}+4\theta\alpha}\,\frac{\mathrm{d}t}{t}\\ + 2^{-4\theta\alpha k}\|b_j\|_q^2\int_{\ell_j}^\infty t^{-2+n-\frac{2n}{q}}\,\frac{\mathrm{d}t}{t}.\end{aligned}$$

There is no issue with convergence of the second integral since we have $q \leq 2$. We pick α large in dependence of n, q, θ, in order to grant convergence of the first integral and get

$$I_{j,k}^2 \leq 2^{-4\theta\alpha k}\ell_j^{-2+n-\frac{2n}{q}}\|b_j\|_q^2 \lesssim \lambda^2 2^{-4\theta\alpha k}|Q_j|,$$

where the final step follows from (ii'). We pick $\alpha \geq n/(4\theta)$ so that when finally going back to (9.21), we find a convergent geometric series in k and obtain

$$|\langle f_{\mathrm{glob}}, h\rangle| \leq \lambda\sum_j |Q_j|^{\frac{1}{2}}\inf_{x\in Q_j}(\mathcal{M}(H^2)(x))^{\frac{1}{2}}.$$

At this point, the right-hand side is the same as in the treatment of the local part. We obtain the required bound for the global part by repeating the argument following (9.20). This concludes the proof of Lemma 9.9.

Part 4: Injection of L-adapted spaces into classical spaces for $p \leq 2$. In this section we establish the continuous inclusions

$$\mathbb{H}_L^p \subseteq a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2) \tag{9.24}$$

$$\mathbb{H}_L^{1,p} \subseteq \dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2 \tag{9.25}$$

in the range $1_* < p \leq 2$.

The main observation is the following inclusion for DB-adapted spaces. The result appears already in [22, Sec. 4.4] but for convenience we include a proof.

Lemma 9.13. *If $p \in (1_*, 2]$, then $\mathbb{H}^p_{DB} \subseteq \mathrm{H}^p \cap \mathrm{L}^2$ and the inclusion is continuous for the p-quasinorms.*

Proof. The claim holds for $p = 2$, see (8.4), and the interpolation theorem for inclusions of adapted Hardy spaces [3, Thm. 4.32] yields that the set of exponents for which the claim holds is an interval. Hence, we only have to treat the case $p \leq 1$.

We use the molecular decomposition for $\mathbb{H}^p_{DB}$ (Theorem 8.17) for some admissible M and $\varepsilon = 1$. It suffices to check that there is a constant c such that $\|m\|_{\mathrm{H}^p} \leq c$ for every $(\mathbb{H}^p_{DB}, 1, M)$-molecule. Writing $m = D(B(DB)^{-1}m)$, we see that m is a generic multiple of an $(\mathbb{H}^p_D, 1, 1)$-molecule. The required bound follows from Corollary 8.29. □

Now, we can use Figure 8 as follows to complete Part 4. Moving from the third to the fourth row, we obtain for $f \in \mathbb{H}^p_L \cap \mathsf{R}(a^{-1} \operatorname{div}_x)$ that

$$\|af\|_{\mathrm{H}^p} \lesssim \left\| \begin{bmatrix} af \\ 0 \end{bmatrix} \right\|_{\mathbb{H}^p_{DB}} \lesssim \|f\|_{\mathbb{H}^p_L}.$$

The bound extends to $f \in \mathbb{H}^p_L$ by density, which gives (9.24). Likewise, moving from the third to the second row, we get

$$\|f\|_{\dot{\mathrm{H}}^{1,p}} = \|\nabla_x f\|_{\mathrm{H}^p} \lesssim \left\| \begin{bmatrix} 0 \\ \nabla_x f \end{bmatrix} \right\|_{\mathbb{H}^p_{DB}} \lesssim \|f\|_{\mathbb{H}^{1,p}_L},$$

first for $f \in \mathbb{H}^{1,p}_L \cap \mathsf{D}(L^{1/2})$ and then for all $f \in \mathbb{H}^{1,p}_L$, which gives (9.25).

Going one step further to the first row gives an additional Riesz transform bound , which is of independent interest. It extends [58, Prop. 5.6] beyond semigroup generators.

Proposition 9.14. *If $p \in (1_*, 2]$, then*

$$\|\nabla_x L^{-1/2} f\|_{\mathrm{H}^p} \lesssim \|f\|_{\mathbb{H}^p_L} \quad (f \in \mathbb{H}^p_L \cap \mathsf{R}(L^{1/2})).$$

Part 5: Injection of classical spaces into L-adapted spaces for $p \leq 1$. We complement the previous section by proving the reverse continuous inclusions

$$a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2) \subseteq \mathbb{H}^p_L \qquad (p_-(L) < p \leq 1) \tag{9.26}$$

and

$$\dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2 \subseteq \mathbb{H}^{1,p}_L \qquad ((p_-(L)_* \vee 1_*) < p \leq 1), \tag{9.27}$$

if these intervals of exponents are non-empty.

The strategy is the same for both inclusions and relies on the atomic decompositions. We use the auxiliary function $\psi(z) := z^\alpha(1+z)^{-2\alpha}$, where $\alpha \in \mathbb{N}$ will be chosen large later on, and introduce the square functions

$$S^{(0)}_{\psi,L} f(x) := \left(\iint_{|x-y|<t} |\psi(t^2L)f(y)|^2 \frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}} \right)^{1/2}, \tag{9.28}$$

$$S^{(1)}_{\psi,L} f(x) := \left(\iint_{|x-y|<t} |t^{-1}\psi(t^2L)f(y)|^2 \frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}} \right)^{1/2}. \tag{9.29}$$

Then $\|S^{(0)}_{\psi,L}(\cdot)\|_p$ and $\|S^{(1)}_{\psi,L}(\cdot)\|_p$ are equivalent norms on $\mathbb{H}^p_L$ and $\mathbb{H}^{1,p}_L$ provided that we take at least $\alpha > {}^{n}/_{(2p)} - {}^{n}/_{4}$.

We shall establish the following bounds.

Lemma 9.15. *Let $p \in (p_-(L)_* \vee 1_*, 1]$ and α sufficiently large depending on $n, p, p_-(L)$. For all L^2-atoms m for $\dot{\mathrm{H}}^{1,p}$ it follows that*

$$\|S^{(1)}_{\psi,L}(m)\|_p \lesssim 1.$$

Lemma 9.16. *Let $p \in (p_-(L), 1]$ and α sufficiently large depending on $n, p, p_-(L)$. For all L^2-atoms m for H^p it follows that*

$$\|S^{(0)}_{\psi,L}(a^{-1}m)\|_p \lesssim 1.$$

Let us take these estimates for granted and complete the objective of this part first. Given $f \in \mathrm{L}^2$ such that $af \in \mathrm{H}^p$, we write the latter as an L^2-convergent atomic decomposition $af = \sum_i \lambda_i m_i$ with $\|(\lambda_i)\|_{\ell^p} \lesssim \|af\|_{\mathrm{H}^p}$. We use Fatou's lemma as in the proof of Proposition 9.11 to obtain

$$S^{(0)}_{\psi,L}(f)(x) \leq \sum_i |\lambda_i| S^{(0)}_{\psi,L}(a^{-1}m_i)(x) \quad (x \in \mathbb{R}^n)$$

and we conclude by Lemma 9.16 and as $p \leq 1$,

$$\|S^{(0)}_{\psi,L}(f)\|_p^p \leq \sum_i |\lambda_i|^p \|S^{(0)}_{\psi,L}(a^{-1}m_i)\|_p^p \lesssim \sum_i |\lambda_i|^p \lesssim \|af\|_{\mathrm{H}^p}^p.$$

The left-hand side is equivalent to $\|f\|_{\mathbb{H}^p_L}$ and (9.26) follows.

As for (9.27), it suffices to prove $\|u\|_{\mathbb{H}^{1,p}_L} \lesssim \|\nabla_x u\|_p$ for all $u \in \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$. Indeed, since $\mathcal{Z}$ is dense in $\dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2$, this is yet another application of the Fatou argument above. Now, we can take a $\dot{\mathrm{W}}^{1,2}$-convergent atomic decomposition $u = \sum_i \lambda_i m_i$ as in Proposition 8.31. By the solution of the Kato problem we have L^2-convergence of

$$\psi(t^2T)u = \sum_i \lambda_i L^{-1/2}\psi(t^2L)L^{1/2}m_i$$

and the same argument as before applies.

Proof of Lemma 9.15. Let m be an L^2-atom for $\dot{\mathrm{H}}^{1,p}$ associated with a cube Q of sidelength ℓ as in Definition 8.30.

We begin with a local bound. By the solution of the Kato problem we have $m \in \mathsf{D}(L^{1/2})$. It follows that

$$S_{\psi,L}^{(1)} m(x) = \left(\iint_{|x-y|<t} |\varphi(t^2 L) L^{1/2} m(y)|^2 \, \frac{\mathrm{d}t \mathrm{d}y}{t^{1+n}} \right)^{1/2} =: S_{\varphi,L}(L^{1/2} m)(x) \quad (x \in \mathbb{R}^n),$$

where $\varphi(z) \coloneqq z^{\alpha-1/2}(1+z)^{-2\alpha}$. Hölder's inequality and the L^2-bound for the square function with φ (McIntosh's theorem) yield

$$\begin{aligned} \|S_{\psi,L}^{(1)}(m)\|_{\mathrm{L}^p(4Q)} &\leq |4Q|^{\frac{1}{p}-\frac{1}{2}} \|S_{\psi,L}^{(1)}(m)\|_{\mathrm{L}^2(4Q)} \\ &\lesssim |Q|^{\frac{1}{p}-\frac{1}{2}} \|L^{1/2} m\|_2 \\ &\simeq |Q|^{\frac{1}{p}-\frac{1}{2}} \|\nabla_x m\| \\ &= 1. \end{aligned} \tag{9.30}$$

In preparation of the global bound, we pick some $q \in (p_-(L), p^*) \cap (1,2]$. This is possible by the assumption on p. We also take α large enough in dependence of q and $p_-(L)$ in order to have $\mathrm{L}^q - \mathrm{L}^2$ off-diagonal estimates of arbitrarily large order for $(\psi(t^2 L))_{t>0}$ at our disposal. This is possible due to Lemma 7.4.(ii) since we can expand

$$\psi(z) = z^\alpha (1+z)^{-2\alpha} = ((1+z)^{-1} - (1+z)^{-2})^\alpha \tag{9.31}$$

Consequently, we have for all $x \in \mathbb{R}^n$ the estimate

$$\begin{aligned} \|\psi(t^2 L) m\|_{\mathrm{L}^2(B(x,t))} &\lesssim t^{\frac{n}{2}-\frac{n}{q}} \left(1 + \frac{\mathrm{d}(B(x,t),Q)}{t}\right)^{-\gamma} \|m\|_q \\ &\simeq t^{\frac{n}{2}-\frac{n}{q}} \left(1 + \frac{\mathrm{d}(x,Q)}{t}\right)^{-\gamma} \|m\|_q, \end{aligned} \tag{9.32}$$

where $\gamma > 0$ is at our disposal and the second step uses $\mathrm{d}(B(x,t),Q) \geq {}^{\mathrm{d}(x,Q)}/_2$ for $t \leq {}^{\mathrm{d}(x,Q)}/_2$ and $1 \geq {}^{2d(x,Q)}/_t$ for $t \geq {}^{\mathrm{d}(x,Q)}/_2$. Squaring and integrating this bound with respect to ${}^{\mathrm{d}t}/_{t^{n+3}}$ gives

$$\begin{aligned} S_{\psi,L}^{(1)} m(x) &\lesssim \left(\int_0^\infty t^{-\frac{2n}{q}-2} \left(1 + \frac{\mathrm{d}(x,Q)}{t}\right)^{-2\gamma} \frac{\mathrm{d}t}{t} \right)^{1/2} \|m\|_q \\ &\simeq \mathrm{d}(x,Q)^{-\frac{n}{q}-1} \|m\|_q, \end{aligned}$$

where the last step follows by a change of variable $s = t\mathrm{d}(x,Q)$ and we have taken $2\gamma > {}^{2n}/_q - 2$ in order to have a finite integral in s. Thus,

$$\begin{aligned} \|S_{\psi,L}^{(1)} m\|_{\mathrm{L}^p({}^c(4Q))} &\lesssim \left(\int_{{}^c(4Q)} \mathrm{d}(x,Q)^{-\frac{np}{q}-p} \, \mathrm{d}x \right)^{\frac{1}{p}} \|m\|_q \\ &\lesssim \ell^{\frac{n}{p}-\frac{n}{q}-1} \|m\|_q, \end{aligned}$$

where we have used ${}^{np}/_q + p > n$ to calculate the integral in x. Since m is supported in Q, we obtain from Hölder's and Poincaré's inequality that

$$\|S^{(1)}_{\psi,L}m\|_{\mathrm{L}^p({}^c(4Q))} \lesssim \ell^{\frac{n}{p}-\frac{n}{2}-1}\|m\|_{\mathrm{L}^2(Q)} \lesssim \ell^{\frac{n}{p}-\frac{n}{2}}\|\nabla_x m\|_{\mathrm{L}^2(Q)} = 1,$$

which is the required global bound. □

Proof of Lemma 9.16. Let m be an L^2-atom for H^p associated with a cube Q of sidelength ℓ, see Definition 2.5.

As before, the local bound $\|S^{(0)}_{\psi,L}(a^{-1}m)\|_{\mathrm{L}^p(4Q)} \lesssim 1$ follows from Hölder's inequality and the L^2-bound for the square function.

To prepare the global bound, we pick exponents $p_-(L) < s < r < q < p$. The resolvents of L are $a^{-1}\,\mathrm{H}^s$-bounded and also $\mathrm{L}^\varrho - \mathrm{L}^2$-bounded for some $\varrho < 2$ thanks to Lemmata 6.3 and 6.4. Keeping in mind the expansion (9.31), we take α large and conclude from Lemma 4.4 that $(\psi(t^2L))_{t>0}$ is $a^{-1}\,\mathrm{H}^r - \mathrm{L}^2$-bounded. Together with the usual L^2 off-diagonal estimates we obtain for all $x \in \mathbb{R}^n$ that

$$\begin{aligned}
&\|\psi(t^2L)(a^{-1}m)\|_{\mathrm{L}^2(B(x,t))} \\
&\qquad = \|\psi(t^2L)(a^{-1}m)\|^{1-\theta}_{\mathrm{L}^2(B(x,t))}\|\psi(t^2L)(a^{-1}m)\|^{\theta}_{\mathrm{L}^2(B(x,t))} \\
&\qquad \lesssim \Big(t^{\frac{n}{2}-\frac{n}{r}}\|m\|_{\mathrm{H}^r}\Big)^{1-\theta}\bigg(\Big(1+\frac{\mathrm{d}(B(x,t),Q)}{t}\Big)^{-\gamma}\|m\|_2\bigg)^{\theta},
\end{aligned}$$

where $\theta \in (0,1)$ and $\gamma > 0$ are still at our disposal. Since $|Q|^{1/p-1/r}m$ is an L^2-atom for H^r, we have $\|m\|_{\mathrm{H}^r} \lesssim |Q|^{1/r-1/p}$. Picking θ such that ${}^{(1-\theta)}/_r + {}^{\theta}/_2 = {}^1/_q$, we obtain

$$\|\psi(t^2L)(a^{-1}m)\|_{\mathrm{L}^2(B(x,t))} \lesssim t^{\frac{n}{2}-\frac{n}{q}}\Big(1+\frac{\mathrm{d}(B(x,t),Q)}{t}\Big)^{-\gamma\theta}|Q|^{\frac{1}{q}-\frac{1}{p}}.$$

This estimate is of the exact same type as (9.32) and we can repeat the previous proof from thereon. Indeed, we integrate the square with respect to ${}^{\mathrm{d}t}/_{t^{1+n}}$ to obtain

$$S^{(0)}_{\psi,L}(a^{-1}m)(x) \lesssim \mathrm{d}(x,Q)^{-\frac{n}{q}}\ell^{\frac{n}{q}-\frac{n}{p}},$$

and then the required global bound

$$\|S^{(0)}_{\psi,L}(a^{-1}m)\|_{\mathrm{L}^p({}^c(4Q))} \lesssim \ell^{\frac{n}{p}-\frac{n}{q}}\ell^{\frac{n}{q}-\frac{n}{p}} = 1,$$

follows since ${}^{np}/_q > n$. □

Part 6: $\boldsymbol{h_-(L) \leq p_-(L)}$. Let $p \in (p_-(L),2]$. We have to prove that $a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2) = \mathbb{H}^p_L$ with equivalent Hardy norms.

The inclusion '$\subseteq$' was obtained in Part 5 for $p \in (p_-(L),1]$ and in Proposition 9.11 for $p \in (p_-(L) \vee 1, 2]$. The converse was obtained in Part 4 in the range $p \in (1_*, 2]$.

Part 7: $h^1_-(L) \leq (p_-(L)_* \vee 1_*)$. Let $p \in (p_-(L)_* \vee 1_*, 2]$. We have to prove $\dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2 = \mathbb{H}^{1,p}_L$ with equivalent Hardy norms.

We have obtained '$\subseteq$' in Part 5 for $p \in (p_-(L)_* \vee 1_*, 1]$ and in Proposition 9.12 for $p \in (p_-(L)_* \vee 1, 2]$. The converse follows again from Part 4.

Part 8: $h_+(L) \geq p_+(L)$. Let $p \in (2, p_+(L))$. In Part 2 we have obtained $\mathrm{L}^p \cap \mathrm{L}^2 \subseteq \mathbb{H}^p_L$ with continuous inclusion for the p-norms. It remains to establish the opposite inclusion and this will follow by duality.

To this end, we recall from Section 3.5 that L^* is an operator in the same class as $\widetilde{L}$ and similar to an operator $L^\sharp$ in the same class as L under conjugation with a^*. By duality and similarity we have $p' \in (p_-(L^\sharp) \vee 1, 2)$. Replacing systematically L with $L^\sharp$, the result of Part 6 entails $\mathbb{H}^{p'}_{L^\sharp} = \mathrm{L}^{p'} \cap \mathrm{L}^2$ with equivalent p'-norms and from Figure 8 we can read off

$$\mathbb{H}^{p'}_{L^*} = a^* \mathbb{H}^{p'}_{L^\sharp} = \mathrm{L}^{p'} \cap \mathrm{L}^2 .$$

Given $f \in \mathbb{H}^p_L$, we use Proposition 8.9 for second-order operators to give

$$|\langle f, g \rangle| \lesssim \|f\|_{\mathbb{H}^p_L} \|g\|_{\mathbb{H}^{p'}_{L^*}} \simeq \|f\|_{\mathbb{H}^p_L} \|g\|_{p'} \quad (g \in \mathrm{L}^{p'} \cap \mathrm{L}^2).$$

We conclude $f \in \mathrm{L}^p \cap \mathrm{L}^2$ along with $\|f\|_p \lesssim \|f\|_{\mathbb{H}^p_L}$ as required.

Part 9: $h^1_+(L) \geq q_+(L)$. We have to show that $\dot{\mathrm{W}}^{1,p} \cap \mathrm{L}^2 = \mathbb{H}^{1,p}_L$ with equivalent p-norms for $p \in (2, q_+(L))$. In fact, we shall establish continuous inclusions for the p-Hardy norms

$$\dot{\mathrm{W}}^{1,p} \cap \mathrm{L}^2 \supseteq \mathbb{H}^{1,p}_L \qquad (2 < p < q_+(L)) \tag{9.33}$$

and

$$\dot{\mathrm{W}}^{1,p} \cap \mathrm{L}^2 \subseteq \mathbb{H}^{1,p}_L \qquad (2 < p < p_+(L)), \tag{9.34}$$

which is a more general result since by Theorem 6.2 we have $p_+(L) \geq q_+(L)^*$.

In the following let $p \in (2, p_+(L))$. Part 8 implies $p < h_+(L)$. Hence, we can identify $\mathbb{H}^p_L = \mathrm{L}^p \cap \mathrm{L}^2$ and the ubiquitous Figure 8 tells us that

$$\|f\|_{\mathbb{H}^{1,p}_L} \simeq \|L^{1/2} f\|_{\mathbb{H}^p_L} \simeq \|L^{1/2} f\|_p \quad (f \in \mathbb{H}^{1,p}_L \cap \mathsf{D}(L^{1/2})). \tag{9.35}$$

Proof of (9.33). If even $p < q_+(L)$, then the Riesz transform is L^p-bounded according to Theorem 7.3 and we obtain from (9.35) that

$$\|f\|_{\mathbb{H}^{1,p}_L} \gtrsim \|\nabla_x f\|_p \quad (f \in \mathbb{H}^{1,p}_L \cap \mathsf{D}(L^{1/2})).$$

A general $f \in \mathbb{H}^{1,p}_L$ can be approximated by a sequence $(f_j) \subseteq \mathbb{H}^{1,p}_L \cap \mathsf{D}(L^{1/2})$ simultaneously in $\mathbb{H}^{1,p}_L$ and L^2, see Section 8.1. Then $(\nabla_x f_j)$

is a Cauchy sequence in L^p whose limit coincides with $\nabla_x f$ thanks to L^2-convergence of (f_j). Hence, the previous estimate extends to f.

Proof of (9.34). It suffices to establish the bound

$$\|u\|_{\mathbb{H}^{1,p}_L} \lesssim \|\nabla_x u\|_p \quad (u \in \dot{\mathrm{W}}^{1,p} \cap \mathrm{W}^{1,2}). \tag{9.36}$$

Indeed, a general $u \in \dot{\mathrm{W}}^{1,p} \cap \mathrm{L}^2$ can be approximated in $\dot{\mathrm{W}}^{1,p} \cap \mathrm{L}^2$ by a sequence $(u_j) \subseteq \mathcal{Z}$ and L^2-convergence suffices to infer $\|u\|_{\mathbb{H}^{1,p}_L} \leq \liminf_{j\to\infty} \|u_j\|_{\mathbb{H}^{1,p}_L}$, see the proof of Proposition 9.11.

We rely on a duality argument using the same notation as in Part 8. Again, we have $p' \in (p_-(L^\sharp) \vee 1, 2)$ and we obtain from Theorem 7.3 that the Riesz transform for $L^\sharp$ is $\mathrm{L}^{p'}$-bounded. For any $g \in \mathsf{R}(L^*) \cap \mathsf{D}(L^*) \cap \mathrm{L}^{p'}$ it follows that

$$\begin{aligned} \langle L^{1/2}u, g\rangle &= \langle u, (L^*)^{1/2} g\rangle \\ &= \langle u, L^*(L^*)^{-1/2} g\rangle \\ &= \langle \nabla_x u, d^* \nabla_x (a^*)^{-1} (L^*)^{-1/2} g\rangle \\ &= \langle d\nabla_x u, \nabla_x (L^\sharp)^{-1/2} (a^*)^{-1} g\rangle, \end{aligned}$$

where the third step is just the definition of L^* and the final step uses that the similarity of operators $L^* = a^* L^\sharp (a^*)^{-1}$ carries over to the functional calculi by construction. Hölder's inequality yields

$$|\langle L^{1/2}u, g\rangle| \lesssim \|\nabla_x u\|_p \|\nabla_x (L^\sharp)^{-1/2} (a^*)^{-1} g\|_{p'} \lesssim \|\nabla_x u\|_p \|g\|_{p'}.$$

Since g was taken from a dense subspace of $\mathrm{L}^{p'}$ (as is granted by Lemma 7.2 applied to $L^\sharp$ and similarity), the bound $\|L^{1/2}u\|_p \lesssim \|\nabla_x u\|_p$ follows. Now, (9.36) is a consequence of (9.35).

Part 10: $\boldsymbol{h^1_+(L) \leq q_+(L)}$. Suppose that the interval $\mathcal{H}^1(L)$ contains some exponent $p \geq q_+(L)$. In particular, $q_+(L)$ is finite.

Since we have $q_+(L) < p_+(L)$ by Theorem 6.2, we can assume $p < p_+(L)$ and by the result of Part 8 this implies $p \in \mathcal{H}(L)$. Therefore, we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{H}^p_L \cap \mathsf{R}(L^{1/2}) & \xrightarrow{L^{-1/2}} & \mathbb{H}^{1,p}_L \cap \mathsf{D}(L^{1/2}) & \hookrightarrow & \dot{\mathrm{W}}^{1,p} \cap \mathsf{D}(L^{1/2}) \\ \uparrow & & & & \downarrow \nabla_x \\ \mathrm{L}^p \cap \mathsf{R}(L^{1/2}) & & \xrightarrow{\nabla_x L^{-1/2}} & & \mathrm{L}^p \cap \mathrm{L}^2, \end{array}$$

where the mapping of $L^{-1/2}$ follows from Figure 8 and the unlabeled arrows indicate continuous inclusions for the p-norms. Lemma 7.2 guarantees that $\mathrm{L}^p \cap \mathsf{R}(L^{1/2})$ is dense in $\mathrm{L}^p \cap \mathrm{L}^2$ and we conclude that the Riesz transform is L^p-bounded. But then we must have $p \leq q_+(L)$ according to Theorem 7.3 and therefore $p = q_+(L)$.

This argument has two consequences. First, $q_+(L) \in \mathcal{H}^1(L)$ is possible only if the Riesz transform is $\mathrm{L}^{q_+(L)}$-bounded. We shall see in the next section that this is never the case. Second, $\mathcal{H}^1(L)$ cannot contain exponents $p > q_+(L)$ and hence that we have $h^1_+(L) \le q_+(L)$. At this stage the proof of Theorem 9.6 is complete. □

9.3. **Consequences for square functions.** By definition of $\mathbb{H}^p_L$, the identification theorem (Theorem 9.6) can be reformulated in terms of L^p-bounds for conical square functions of type

$$S_{\psi,L}f(x) \coloneqq S(\mathbb{Q}_{\psi,L}f)(x) = \bigg(\iint_{|x-y|<t} |\psi(t^2L)f(y)|^2 \, \frac{\mathrm{d}t\mathrm{d}y}{t^{1+n}} \bigg)^{1/2}.$$

Here, we collect and improve these bounds with an emphasis on the decay for the auxiliary function $\psi \in \Psi^+_+$ at $|z| = 0$ and $|z| = \infty$ within a sector. This will be important for the applications to boundary value problems.

When $p \ge 2$, we will use the simple fact that the conical square functions S can be controlled by the vertical square function defined for $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ as

$$V(F)(x) \coloneqq \bigg(\int_0^\infty |F(t,x)|^2 \, \frac{\mathrm{d}t}{t} \bigg)^{1/2},$$

see for instance [13, Prop. 2.1] for the following lemma.

Lemma 9.17. *Let $p \in [2, \infty)$. There is a constant c depending on p and n such that for all $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$,*

$$\|S(F)\|_p \le c\|V(F)\|_p.$$

Upper bounds for vertical square functions are provided by an abstract theorem due Cowling–Doust–McIntosh–Yagi [36, Thm. 6.6]. We state the quantitative version found in the textbook [64], but inspection of the original argument would yield the same dependence of the constants. We continue to write

$$(\mathbb{Q}_{\psi,T}f)(t,x) = (\psi(t^2T)f)(x)$$

as in Section 8, even though T need not act on L^2, and we note that up to a norming factor of 2 the vertical square function $V(\mathbb{Q}_{\psi,T}f)$ does not change if instead we use first-order scaling $(\mathbb{Q}_{\psi,T}f)(t,x) = (\psi(tT)f)(x)$.

Theorem 9.18 ([64, Thm. 10.4.23])**.** *Let $p \in (1,\infty)$ and let T be a sectorial operator in $\mathrm{L}^p(\mathbb{R}^n; W)$, where W is a finite-dimensional Hilbert space. Suppose that T has a bounded H^∞-calculus of angle $\omega \in (0,\pi)$ on $\overline{\mathsf{R}(T)}$. Let $\mu \in (0, {}^{(\pi-\omega)}/{}_2)$ and choose decay parameters $\sigma, \tau > 0$. Then for all $\psi \in \Psi^\tau_\sigma(\mathrm{S}^+_{\pi-2\mu})$ and all $f \in \overline{\mathsf{R}(T)}$,*

$$\|V(\mathbb{Q}_{\psi,T}f)\|_p \lesssim \|\psi\|_{\sigma,\tau,\mu}\|f\|_p,$$

where the implicit constant depends on T through $M_{T,\nu}$ and $M^\infty_{T,\nu}$ for some $\nu \in (\omega, \pi - 2\mu)$.

Remark 9.19. The numbers $M_{T,\nu}$ and $M^\infty_{T,\nu}$ correspond to resolvent and functional calculus bounds, see (3.6) and (3.13). The theorem remains true for all $f \in \mathrm{L}^p(\mathbb{R}^n; W)$ since we have $\psi(t^2 T)f = 0$ if $f \in \mathsf{N}(T)$ and $t > 0$.

With this at hand, we obtain abstract square function bounds. We largely follow the idea for second-order elliptic operators $-\operatorname{div}_x d\nabla_x$ in [13], see also [19,22], but with a more direct interpolation argument in tent spaces.

Proposition 9.20. *Let T be a sectorial operator that satisfies the standard assumptions (8.5). Let $p \in [2, \infty)$ and suppose that*

$$\|f\|_{\mathbb{H}^p_T} \simeq \|f\|_p \quad (f \in \mathbb{H}^p_T).$$

Let $\theta \in (0, 1]$, fix an angle $\mu \in (0, \theta(\pi-\omega)/2)$ and let $\psi \in \Psi^\tau_\sigma(\mathrm{S}^+_{\pi-2\mu})$ with $\sigma, \tau > 0$. Consider the square function bound

$$\|\mathbb{Q}_{\psi,T} f\|_{\mathrm{T}^q} \lesssim \|f\|_q \quad (f \in \mathrm{L}^q \cap \mathrm{L}^2),$$

with an implicit constant that depends on T only through (8.5) and the comparison constant for the p-norms in the assumption. Then this bound is valid provided that

$$q \geq 2 \quad \text{and} \quad \frac{1}{q} > \frac{1}{p} - \frac{[p, 2]_\theta}{p} \frac{2\sigma}{n}.$$

Proof. We organize the proof in five steps.

Step 1: H^∞-calculus for the L^p-realization of T. It follows from Proposition 8.10 and the assumption on p that

$$\|\eta(T)f\|_p \lesssim \|\eta\|_\infty \|f\|_p \tag{9.37}$$

for all $f \in \mathrm{L}^p \cap \overline{\mathsf{R}(T)}$ and all admissible $\eta \in \mathrm{H}^\infty$.

Let $\nu \in [0, \frac{\pi-\omega}{2})$ and $\zeta \in \mathrm{S}^+_\nu$. For the special choice $\eta(z) \coloneqq (1+\zeta^2 z)^{-1}$ the operator $\eta(T)$ acts as the identity on $\mathsf{N}(T)$. Hence, the bound above extends to all $f \in \mathrm{L}^p \cap \mathrm{L}^2$, that is to say, $((1+\zeta^2 T)^{-1})_{\zeta \in \mathrm{S}^+_\nu}$ is L^p-bounded. Hence, T has an L^p-realization described in Proposition B.1 and this is a sectorial operator in L^p of the same angle ω as T.

For $\eta \in \Psi^+_+$ the bound (9.37) also remains true for general $f \in \mathrm{L}^p \cap \mathrm{L}^2$ since $\eta(T)$ vanishes on $\mathsf{N}(T)$. We have $\eta(T)f = \eta(T_p)f$ since these operators are given by the same Cauchy integral. Since $\mathrm{L}^p \cap \mathrm{L}^2$ is dense in L^p it follows that T_p has a bounded H^∞-calculus of angle ω on the closure of its range.

The idea of proof is now to interpolate between two square function bounds that we have seen before: Theorem 9.18 for T_p and Lemma 9.7 for T.

Step 2: Definition of an interpolating family. For $\alpha \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ we define

$$\psi_\alpha : \mathrm{S}^+_{\pi-2\mu} \to \mathbb{C}, \quad \psi_\alpha(z) := \left(\frac{z}{1+z}\right)^{\alpha-\sigma} \psi(z). \tag{9.38}$$

As ${}^{z}/_{(1+z)} = (1+z^{-1})^{-1} \in \mathrm{S}^+_{\pi-2\mu}$ and $\operatorname{Re}\alpha > 0$, we obtain

$$\sup_{z \in \mathrm{S}_{\pi-2\mu}} \left| \left(\frac{z}{1+z}\right)^{\alpha-\sigma} \right| \lesssim \mathrm{e}^{(\pi-2\mu)|\operatorname{Im}\alpha|}(|z|^{\operatorname{Re}\alpha-\sigma} \wedge 1), \tag{9.39}$$

where the implicit constant is independent of α. Consequently, we have $\psi_\alpha \in \Psi^\tau_{\operatorname{Re}\alpha}(\mathrm{S}^+_{\pi-2\mu})$ and

$$\|\psi_\alpha\|_{\operatorname{Re}\alpha,\tau,\mu} \lesssim \mathrm{e}^{(\pi-2\mu)|\operatorname{Im}\alpha|}\|\psi\|_{\sigma,\tau,\mu}. \tag{9.40}$$

Combining Lemma 9.17 and Theorem 9.18 leads to the following bound for $q := p$ and all $f \in \mathrm{L}^q \cap \mathrm{L}^2$:

$$\begin{aligned}\|\mathbb{Q}_{\psi_\alpha,T} f\|_{\mathrm{T}^q} &= \|S(\mathbb{Q}_{\psi_\alpha,T_p} f)\|_q \\ &\lesssim \|V(\mathbb{Q}_{\psi_\alpha,T_p} f)\|_q \\ &\lesssim \mathrm{e}^{(\pi-2\mu)|\operatorname{Im}\alpha|}\|\psi\|_{\sigma,\tau,\mu}\|f\|_q.\end{aligned}$$

The implicit constant is independent of ψ and α. By McIntosh's theorem the same holds for $q = 2$ and hence for all $q \in [2,p]$ by interpolation. If, however, $\operatorname{Re}\alpha > \frac{n}{2[p,2]_\theta}$, then Lemma 9.7 provides the same bound for *all* $q \in [2,\infty)$, so that in total we obtain

$$\|\mathbb{Q}_{\psi_\alpha,T} f\|_{\mathrm{T}^q} \lesssim \mathrm{e}^{(\pi-2\mu)|\operatorname{Im}\alpha|}\|\psi\|_{\sigma,\tau,\mu}\|f\|_{\mathrm{L}^q} \quad (f \in \mathrm{L}^q \cap \mathrm{L}^2) \tag{9.41}$$

if $(\operatorname{Re}\alpha, {}^1/_q)$ belongs to the interior of the gray shaded region in Figure 9.

Step 3: Abstract Stein interpolation. For technical reasons it will be more convenient to work with the 'truncated' operators

$$\mathbb{Q}^{(k)}_{\psi_\alpha,T} f = \mathrm{e}^{\alpha^2}\mathbf{1}_{K_k}(\mathbb{Q}_{\psi_\alpha,T} f) \quad (k \in \mathbb{N}),$$

where $K_k := (k^{-1},k) \times B(0,k) \subseteq \mathbb{R}^{1+n}_+$. For fixed z the map $\alpha \to \psi_\alpha(z)$ is holomorphic in the half plane $\mathbb{C}^+$. Writing out the Cauchy integral for $\psi_\alpha(t^2 T)$ and applying the dominated convergence theorem (justified by (9.40)), we obtain that

$$\mathbb{C}^+ \to \mathrm{L}^2(K_k), \quad \alpha \mapsto \mathbb{Q}^{(k)}_{\psi_\alpha,T} f$$

is holomorphic, whenever $f \in \mathrm{L}^2$. Moreover, thanks to the factor e^{α^2} this mapping is *qualitatively* bounded on any strip $\{\alpha \in \mathbb{C} : c_0 \leq \operatorname{Re}\alpha \leq c_1\} \subseteq \mathbb{C}^+$ with a bound depending on all parameters at stake. By the choice of K_k, the square function $S(\mathbb{Q}^{(k)}_{\psi_\alpha,L} f)(x)$ vanishes for $x \in {}^c B(0,2k)$. Hence we get for any $p \in (1,\infty)$ that

$$\|\mathbb{Q}^{(k)}_{\psi_\alpha,L} f\|_{\mathrm{T}^p} \leq |B(0,2k)|^{\frac{1}{p}} k^{\frac{1+n}{2}} \|\mathbb{Q}^{(k)}_{\psi_\alpha,T} f\|_{\mathrm{L}^2(K_k)},$$

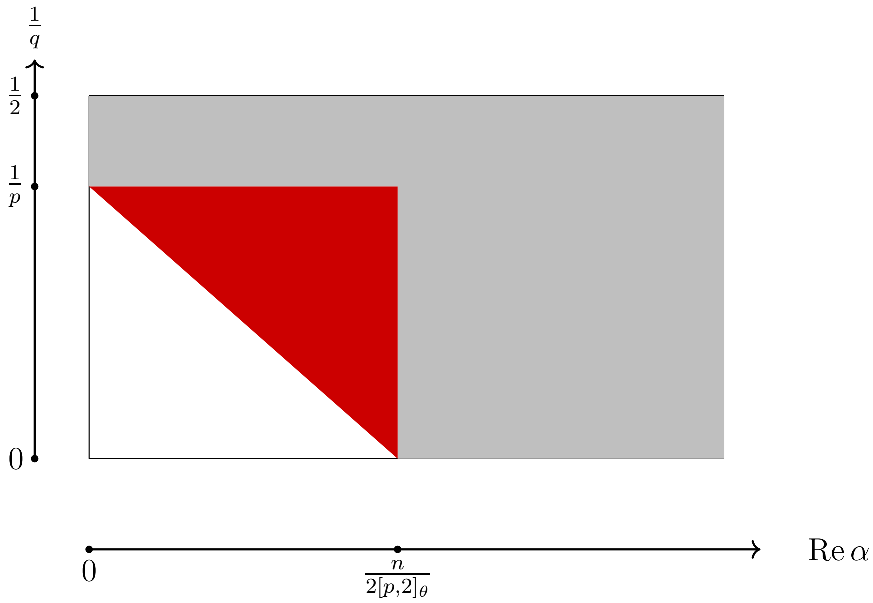

FIGURE 9. Visualization of the interpolation in Proposition 9.20. For $(\operatorname{Re}\alpha, {}^1\!/\!_q)$ in the interior of the grey shaded region, $\mathbb{Q}_{\psi_\alpha,T}$ is bounded $\mathrm{L}^q \to \mathrm{T}^q$ with a bound $C\mathrm{e}^{(\pi-2\mu)\operatorname{Im}\alpha}$, where C is independent of α. Stein interpolation in Step 3 provides boundedness $\mathrm{L}^q \to \mathrm{T}^q$ in the interior of the red triangular region, the lower boundary of which is given by $\frac{1}{q} = \frac{1}{p} - \frac{[p,2]_\theta}{p}\frac{2\operatorname{Re}\alpha}{n}$.

which shows that the qualitative mapping properties remain valid if we replace the target space $\mathrm{L}^2(K_k)$ by T^p.

If in addition $(\operatorname{Re}\alpha, {}^1\!/\!_q)$ belongs to the interior of the gray shaded region in Figure 9 and $f \in \mathrm{L}^q \cap \mathrm{L}^2$, then we obtain the *quantitative* bound

$$\|\mathbb{Q}^{(k)}_{\psi_\alpha,T} f\|_{\mathrm{T}^q} \leq |\mathrm{e}^{\alpha^2}| \|\mathbb{Q}_{\psi_\alpha,T} f\|_{\mathrm{T}^q} \lesssim \mathrm{e}^{(\operatorname{Re}\alpha)^2} \|\psi\|_{\sigma,\tau,\mu} \|f\|_{\mathrm{L}^q},$$

where in the second step we have used (9.41) and the implicit constant is independent of ψ, α, k.

Now, let $(\operatorname{Re}\alpha_j, {}^1\!/\!_{q_j})$, $j = 0, 1$, belong to the interior of the gray shaded region in Figure 9. We intend to use to Proposition 4.11 for

$$T(z) \coloneqq \mathbb{Q}^{(k)}_{\psi_{\gamma(z)},L}, \quad \gamma(z) \coloneqq (\operatorname{Re}\alpha_0)(1-z) + (\operatorname{Re}\alpha_1)z,$$

and the interpolation couples $X_j \coloneqq \mathrm{L}^{q_j}$ and $Y_j \coloneqq \mathrm{T}^{q_j}$. The dense subspace is $Z \coloneqq \mathrm{L}^2 \cap \mathrm{L}^{q_0} \cap \mathrm{L}^{q_1}$. The *qualitative* bounds above yield (i) and the continuity part of (ii) in Proposition 4.11. The *quantitative* bounds determine the constants M_j in (ii). Hence, we get for any

$(\operatorname{Re}\alpha, 1/q)$ on the segment connecting the $(\operatorname{Re}\alpha_j, 1/q_j)$ a bound

$$\|\mathbb{Q}^{(k)}_{\psi_\alpha,T} f\|_{\mathrm{T}^q} \lesssim \|\psi\|_{\sigma,\tau,\mu}\|f\|_{\mathrm{L}^q} \qquad (f \in \mathrm{L}^q \cap \mathrm{L}^2),$$

where the implicit constant is independent of ψ and k. Finally, we can pass to the limit as $k \to \infty$ via Fatou's lemma to obtain the same type of bound with $\mathbb{Q}_{\psi_\alpha,T} f$ on the left-hand side. We have now completed Figure 9 by adding the triangular region.

Step 4: Conclusion. We specialize to $\alpha = \sigma$, so that $\psi_\alpha = \psi$. The corresponding boundedness properties for $\mathbb{Q}_{\psi,T}$ are dictated by Figure 9. If $\sigma \leq \frac{n}{2[p,2]_\theta}$, then $\frac{1}{q} > \frac{1}{p} - \frac{[p,2]_\theta}{p}\frac{2\sigma}{n}$ is needed. If $\sigma > \frac{n}{2[p,2]_\theta}$, then every $q \in [2, \infty)$ is admissible and this coincides with the range obtained in the first case. $\square$

We single out the conclusion for the operator L and the most common auxiliary functions ψ. Note that we can allow any $\psi \in \Psi_+^+$ when $p \geq 2$, which is a significant improvement compared to what is predicted by the abstract theory in Section 8.2.

Theorem 9.21. *Let $p_-(L) < p < p_+(L)$ and let $\sigma, \tau > 0$. Let ψ be of class Ψ_σ^τ on any sector. Then*

$$\|S_{\psi,L} f\|_p \simeq \|af\|_{\mathrm{H}^p} \quad (f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)),$$

provided that

- $\tau > |n/4 - n/(2p)|$ *and* $\sigma > 0$ *if* $p \leq 2$,
- $\tau > 0$ *and* $\sigma > 0$ *if* $p \geq 2$.

Moreover, the upper square function bound '$\lesssim$' remains to hold for $p_+(L) \leq p < np_+(L)/(n - 2\sigma p_+(L))$, where the upper exponent bound is interpreted as ∞ if $2\sigma p_+(L) > n$.

Proof of Theorem 9.21. If $p \leq 2$, then the assumption means that ψ is an admissible auxiliary function for $\mathbb{H}^p_L$, see Section 8.2. Hence,

$$\|S_{\psi,L} f\|_p = \|\mathbb{Q}_{\psi,L} f\|_{\mathrm{T}^p} \simeq \|f\|_{\mathbb{H}^p_L} \simeq \|af\|_{\mathrm{H}^p},$$

where the final step is due to Theorem 9.6.

If $2 < p < p_+(L)$, then our assumptions on ψ are less restrictive than the ones predicted by the abstract theory.

We begin with the upper bounds. By Theorem 9.6 we have $\mathbb{H}^p_L = \mathrm{L}^p \cap \mathrm{L}^2$ with equivalent p-norms. Hence, we can apply Proposition 9.20 for any $p \in (2, p_+(L))$ and by assumption on ψ we may do so for any $\theta \in (0, 1)$. This leads to

$$\|S_{\psi,L} f\|_q \lesssim \|f\|_q \quad (f \in \mathrm{L}^q \cap \mathrm{L}^2)$$

for any $q \geq 2$ that satisfies $1/q > 1/p_+(L) - 2\sigma/n$, which is the range stated in the theorem.

For the lower bound we let $f \in \mathrm{L}^p \cap \mathrm{L}^2$ and take $\varphi \in \Psi_\infty^\infty$ as in Remark 3.3 so that we have the reproducing formula

$$f = \int_0^\infty \varphi(t^2 L)\psi(t^2 L) f \, \frac{\mathrm{d}t}{t}.$$

Now, we refine the duality argument of Part 8 in the proof of Theorem 9.6. We write again $L^* = a^* L^\sharp (a^*)^{-1}$, with $L^\sharp$ an operator in the same class as L and $p' \in (p_-(L^\sharp) \vee 1, 2)$. For all $g \in \mathrm{L}^{p'} \cap \mathrm{L}^2$ we get

$$\begin{aligned} \langle f, g \rangle &= \int_0^\infty \langle \psi(t^2 L) f, a^* \varphi^*(t^2 L^\sharp)(a^*)^{-1} g \rangle \, \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \int_{\mathbb{R}^n} \mathbb{Q}_{\psi, L} f \cdot \overline{a^* \mathbb{Q}_{\varphi^*, L^\sharp} (a^*)^{-1} g} \, \frac{\mathrm{d}x \mathrm{d}t}{t}, \end{aligned}$$

where $\langle \cdot \, , \cdot \rangle$ denotes the L^2 inner product. Thus,

$$\begin{aligned} |\langle f, g \rangle| &\leq \|\mathbb{Q}_{\psi, L} f\|_{\mathrm{T}^p} \|a^* \mathbb{Q}_{\varphi^*, L^\sharp}(a^*)^{-1} g\|_{\mathrm{T}^{p'}} \\ &\lesssim \|S_{\psi, L} f\|_p \|S_{\varphi^*, L^\sharp}(a^*)^{-1} g\|_{p'} \\ &\lesssim \|S_{\psi, L} f\|_p \|g\|_{p'}, \end{aligned}$$

where the first step is by the $\mathrm{T}^p - \mathrm{T}^{p'}$ duality and the third step uses the upper square function bound with $\varphi^* \in \Psi_\infty^\infty$ for $L^\sharp$ on $\mathrm{L}^{p'}$. Since $g \in \mathrm{L}^{p'} \cap \mathrm{L}^2$ was arbitrary, the lower bound $\|S_{\psi, L} f\|_p \gtrsim \|f\|_p$ follows. ☐

10. A digression: H^∞-calculus and analyticity

In this short section we present two consequences of the identification theorem for operator-adapted Hardy spaces that are of independent interest. One concerns analyticity, the other one concerns the H^∞-calculus for L.

Recall that the standard assumptions (8.5) that we use to build the L-adapted spaces depend only on the configuration on L^2: sectoriality, H^∞-calculus and off-diagonal estimates for the resolvents $(1 + t^2 L)^{-1}$ with real t. By the sectorial version of Proposition 8.10 discussed in Section 8.2, all L-adapted spaces inherit the H^∞-calculus with the same angle as on L^2.

It follows from Theorem 9.6 that we obtain H^∞-calculi for L on classical H^p and $\dot{\mathrm{H}}^{1,q}$-spaces with the best possible angle. In the range $p \in (1, \infty)$, such results on L^p could in principle be obtained from Blunck and Kunstmann's theorem [30]. This is the road taken in [6, Sec. 5] when $a = 1$. We are not aware of an analog of the Blunck–Kunstmann result on Hardy–Sobolev spaces. In fact, we are not even aware of any general results for $p \leq 1$ or $q \leq 1$ or even of functional calculus away from the Banach space range.

We summarize this discussion in the following result.

Theorem 10.1. *Let $p_-(L) < p < p_+(L)$ and $(p_-(L)_* \vee 1_*) < q < q_+(L)$. For every $\nu \in (\omega, \pi)$ the functional calculus bounds*

$$\|a\eta(L)a^{-1}f\|_{\mathrm{H}^p} \lesssim \|\eta\|_\infty \|f\|_{\mathrm{H}^p}$$
$$\|\eta(L)g\|_{\dot{\mathrm{H}}^{1,q}} \lesssim \|\eta\|_\infty \|g\|_{\dot{\mathrm{H}}^{1,q}}$$

hold for all $\eta \in \mathrm{H}^\infty(\mathrm{S}_\nu^+)$ and all $f \in \mathrm{H}^p \cap \mathrm{L}^2$, $g \in \dot{\mathrm{H}}^{1,q} \cap \mathrm{L}^2$.

The open p-interval in Theorem 10.1 is the largest possible one since $\eta(\zeta) = (1+t^2\zeta)^{-1}$ with real t is admissible. An example that illustrates the less familiar second inequality is $\|\nabla_x(1+t^2L)^{-1}g\|_{\mathrm{H}^q} \lesssim \|\nabla_x g\|_{\mathrm{H}^q}$, which is of a different nature than the bounds defining $\mathcal{N}(L)$ and is valid for q in a bigger set.

This also leads us to analyticity, that is, resolvent bounds for parameters in a sector in the complex plane. According to Section 3.2, L is sectorial in L^2 with angle ω_L not exceeding $2\omega_{DB} < \pi$. We obtain that for X being any one of the spaces in the statement above and every $\mu \in (\omega_L, \pi)$ extensions by density with operator norm bounds

$$\sup_{z \in \mathbb{C}\setminus\overline{\mathrm{S}_\mu^+}} \|z(z-L)^{-1}\|_{\mathrm{X}\to\mathrm{X}} < \infty.$$

This means that H^p-boundedness of resolvents $(1+t^2L)^{-1}$ with real t alone self-improves to the same properties for the resolvents $(1+z^2L)^{-1}$ for $z \in \mathrm{S}_\mu^+$ and $\mu \in (0, {}^{(\pi-\omega_L)}/2)$.

A similar discussion applies to L^p off-diagonal estimates for $T(z) := (1+z^2L)^{-1}$, $z \in \mathrm{S}_\mu^+$, when $(p_-(L) \vee 1) < p < p_+(L)$. For a small and p-dependent angle they can be obtained from the Stein interpolation theorem for analytic families of operators, see Lemma 4.13. Having the L^p-boundedness and the L^2 off-diagonal estimates for the (p-independent) optimal angle implies by complex interpolation applied to each single operator $T(z)$ the L^p off-diagonal estimates for $T(z)$, see Lemma 4.14. If $p_-(L) < 1$ (resp. $p_-(L^\sharp) < 1$), we shall see in Section 14 that we may also include L^1 (resp. L^∞) off-diagonal estimates here.

In the same manner, we could obtain self-improvements for other families. Of particular interest is the analytic Poisson semigroup generated by $-L^{1/2}$, which has angle ${}^\pi/2 - {}^{\omega_L}/2$, and when $\omega_L < {}^\pi/2$ — that is, for instance when $a = 1$ — the analytic heat semigroup e^{-zL} with angle ${}^\pi/2 - \omega_L$.

11. Riesz transform estimates: Part II

We come back to the Riesz transform interval

$$\mathcal{I}(L) := \{p \in (1_*, \infty) : R_L \text{ is } a^{-1}\,\mathrm{H}^p - \mathrm{H}^p\text{-bounded}\},$$

defined in (7.1), the endpoints of which we have denoted by $r_\pm(L)$. In Section 7 we have characterized the endpoints of the part of $\mathcal{I}(L)$ in $(1, \infty)$. The identification theorem for adapted Hardy spaces allows us to complete the discussion through the following theorem.

Theorem 11.1. *It follows that*

$$\mathcal{I}(L) = (p_-(L), q_+(L)).$$

Moreover, the following hold true:

(i) *The map* $aL^{1/2} : \dot{\mathrm{H}}^{1,p} \cap \dot{\mathrm{W}}^{1,2} \to \mathrm{H}^p \cap \mathrm{L}^2$ *is well-defined and bounded for the* p*-quasinorms if* $p_-(L)_* \vee 1_* < p < p_+(L)$.

(ii) *An exponent* $p \in (1_*, \infty)$ *belongs to* $\mathcal{I}(L)$ *if and only if the map in (i) extends by density to an isomorphism* $\dot{\mathrm{H}}^{1,p} \to \mathrm{H}^p$ *whose inverse agrees with* $L^{-1/2}a^{-1}$ *on* $\mathrm{H}^p \cap \mathrm{L}^2$*. In particular, if* $p \in \mathcal{I}(L)$*, then*

$$\|R_L f\|_{\mathrm{H}^p} \simeq \|af\|_{\mathrm{H}^p} \quad (f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)).$$

The reader may wonder if the separate discussion in Section 7 could have been avoided. The answer is that it can not, since Theorem 7.3 was used in proving Theorem 9.6.

Proof. The Hardy space theory yields for $p_-(L)_* \vee 1_* < p < p_+(L)$ continuous inclusions for the p-quasinorms,

$$\begin{aligned} \dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2 &\subseteq \mathbb{H}^{1,p}_L, \\ \mathbb{H}^p_L &\subseteq a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2). \end{aligned} \tag{11.1}$$

More precisely, by Theorem 9.6 the first inclusion is an equality up to equivalent norms if $p < q_+(L)$ and the second one is an equality if $p > p_-(L)$. The first inclusion for $q_+(L) \leq p < p_+(L)$ is due to (9.34) and the second inclusion for $p_-(L)_* \vee 1_* < p \leq p_-(L)$ is due to (9.24).

Step 1: Proof of (i). As Figure 8 tells us that $L^{1/2} : \mathbb{H}^{1,p}_L \cap \mathrm{W}^{1,2} \to \mathbb{H}^p_L$ is bounded for the p-quasinorms, we conclude from the inclusions above that $aL^{1/2} : \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2} \to \mathrm{H}^p \cap \mathrm{L}^2$ is well-defined and bounded for the respective p-quasinorms. The extension to $\dot{\mathrm{H}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$ follows by density.

Step 2: Bounds for R_L*.* Let $p_-(L) < p < q_+(L)$. Then the inclusions in (11.1) become equalities and Figure 8 tells us that

$$\|\nabla_x L^{-1/2} f\|_{\mathrm{H}^p} \simeq \|af\|_{\mathrm{H}^p} \quad (f \in \mathbb{H}^p_L \cap \mathsf{R}(L^{1/2})).$$

Since $\mathbb{H}^p_L \cap \mathsf{R}(L^{1/2})$ is dense in $\mathbb{H}^p_L$ for the norm $\|\cdot\|_{\mathbb{H}^p_L} + \|\cdot\|_2$, we obtain by approximation and the various quasinorm equivalences that

$$\|R_L f\|_{\mathrm{H}^p} \simeq \|af\|_{\mathrm{H}^p} \quad (f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)).$$

In particular, R_L is $a^{-1}\,\mathrm{H}^p - \mathrm{H}^p$-bounded.

Step 3: Identification of the endpoints of $\mathcal{I}(L)$*.* In view of Theorem 7.3 it remains to show $p_-(L) = r_-(L)$ in the case that one of these exponents is smaller than 1. In Step 2 we have already shown $r_-(L) \leq p_-(L)$ without any such restrictions. The only task remaining is to prove that $r_-(L) < \varrho < 1$ implies $(\varrho, 1] \subseteq \mathcal{J}(L)$.

We may assume $\varrho < 2_*$ since otherwise the claim already follows from Proposition 6.7. Since $r_-(L) \vee 1 < \varrho^* < 2$, Theorem 7.3 yields $p_-(L) \vee 1 < \varrho^* < 2$ and hence $(tL^{1/2}(1+t^2L)^{-1})_{t>0}$ is L^{ϱ^*}-bounded, see Lemma 4.16.(i). Now, let $f \in \mathrm{H}^\varrho \cap \mathrm{L}^2$. Then $f \in \mathsf{R}(aL^{1/2})$ thanks to Lemma 7.7, so that we can estimate

$$\begin{aligned}\|(1+t^2L)^{-1}a^{-1}f\|_{\varrho^*} &\lesssim t^{-1}\|L^{-1/2}a^{-1}f\|_{\varrho^*} \\ &\lesssim t^{-1}\|\nabla_x L^{-1/2}a^{-1}f\|_{\mathrm{H}^\varrho} \\ &\lesssim t^{-1}\|f\|_{\mathrm{H}^\varrho},\end{aligned}$$

where we used the assumption $r_-(L) < \varrho$ in the last line. This means that the resolvents are $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^{\varrho^*}$-bounded. According to Lemma 4.15 they satisfy L^{ϱ^*} off-diagonal estimates of arbitrarily large order and for compactly supported $f \in \mathrm{L}^2$ with mean value zero we recall from Corollary 5.4 that $\int_{\mathbb{R}^n} a(1+t^2L)^{-1}(a^{-1}f)\,\mathrm{d}x = 0$. With these properties at hand, the required H^p-boundedness of the resolvents for $p \in (\varrho, 1]$ follows from Lemma 4.9.

Step 4: Proof of (ii). If $aL^{1/2}$ extends to an isomorphism with the given property, then

$$\|R_L f\|_{\mathrm{H}^p} = \|L^{-1/2}a^{-1}(af)\|_{\dot{\mathrm{H}}^{1,p}} \simeq \|af\|_{\mathrm{H}^p} \quad (f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2))$$

as required.

Conversely, suppose that $p \in \mathcal{I}(L)$. This means that $R_L = \nabla_x L^{-1/2}$ is $a^{-1}\,\mathrm{H}^p - \mathrm{H}^p$-bounded and hence $L^{-1/2}a^{-1} : \mathrm{H}^p \cap \mathrm{L}^2 \to \dot{\mathrm{H}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$ is well-defined and bounded for the p-quasinorms. According to Step 3 the exponent p must be contained in $[p_-(L), q_+(L)] \cap (1_*, \infty)$, which, in view of Theorem 6.2, is a subset of the interval considered in (i). Therefore $aL^{1/2} : \dot{\mathrm{H}}^{1,p} \cap \dot{\mathrm{W}}^{1,2} \to \mathrm{H}^p \cap \mathrm{L}^2$ is also bounded for the p-quasinorms and hence it extends to an isomorphism with the required properties.

Step 5: Conclusion. We already know the endpoints of $\mathcal{I}(L)$ and it remains to show that this interval is open. The map in (i) is defined and continuous for p in an open interval I that contains $\mathcal{I}(L)$ and the isomorphism property in (ii) characterizes $\mathcal{I}(L)$ as a subset of I. Since the scales of spaces $(\dot{\mathrm{H}}^{1,p})_{p\in(1_*,\infty)}$ and $(\mathrm{H}^p)_{p\in(1_*,\infty)}$ interpolate by the complex method, the openness of $\mathcal{I}(L)$ is a consequence of Šneĭberg's stability theorem [68, Thm. 8.1]. See also [68, Thm. 8.1] for the fact that compatibility of the inverses is preserved. □

In Part 10 of the proof of Theorem 9.6 we have seen that $q_+(L) \in \mathcal{H}^1(L)$ is possible only if the Riesz transform is $\mathrm{L}^{q_+(L)}$-bounded. Hence, we can note:

Corollary 11.2. *The interval $\mathcal{H}^1(L)$ is open at the upper endpoint, that is, $q_+(L) \notin \mathcal{H}^1(L)$.*

The statement (ii) in Theorem 11.1 can be strengthened to a Riesz transform characterization of abstract and concrete Hardy spaces. For operators of type $-\operatorname{div}_x d\nabla_x$ such results first appeared in [58, Sec. 5]. Interestingly, this observation allows us to strengthen the identification theorem for $\mathbb{H}^p_L$ itself in that $\mathcal{H}(L)$ is open and hence identification fails at the endpoints.

Theorem 11.3. *Let $p \in (p_-(L)_* \vee 1_*, q_+(L))$. Then*

$$\mathbb{H}^p_L = \{f \in \mathrm{L}^2 : R_L f \in \mathrm{H}^p\}$$

with equivalent quasinorms $\|\cdot\|_{\mathbb{H}^p_L} \simeq \|R_L \cdot\|_{\mathrm{H}^p}$. In particular, it follows that

$$\mathcal{H}(L) = (p_-(L), p_+(L)).$$

Proof. Let $p \in (p_-(L)_* \vee 1_*, q_+(L))$. We first prove the quasinorm equivalence for $f \in \mathbb{H}^p_L$. To this end, we argue as in Step 2 of the proof of Theorem 11.1, except that in the given range of exponents only the first inclusion in (11.1) is an equality but we cannot identify $\mathbb{H}^p_L$ unless $p > p_-(L)$. This yields

$$\|R_L f\|_{\mathrm{H}^p} \simeq \|f\|_{\mathbb{H}^p_L} \quad (f \in \mathbb{H}^p_L)$$

and we can replace $\mathbb{H}^p_L$ with $a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$ if in addition $p > p_-(L)$.

Conversely, let $f \in \mathrm{L}^2$ satisfy $R_L f \in \mathrm{H}^p$. Arguing as in Step 1 of the proof of Theorem 11.1, we find that $L^{1/2} : \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2} \to \mathbb{H}^p_L$ is bounded for the p-quasinorms. The only difference is again that we cannot identify $\mathbb{H}^p_L$. By assumption we have $L^{-1/2} f \in \dot{\mathrm{H}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$. Let $(u_k) \subseteq \mathcal{Z}$ be a sequence with $u_k \to L^{-1/2} f$ in $\dot{\mathrm{H}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$ as $k \to \infty$ and set $f_k := L^{1/2} u_k$. Then (f_k) is a Cauchy sequence in (the possibly non-complete space) $\mathbb{H}^p_L$ that converges to f in L^2. Let $\mathbb{H}^p_L$ be defined by the auxiliary function ψ. By L^2 convergence

$$\int_{B(x,t)} |\psi(t^2 L) f(y)|^2 \,\mathrm{d}y = \lim_{k \to \infty} \int_{B(x,t)} |\psi(t^2 L) f_k(y)|^2 \,\mathrm{d}y$$

holds for all $(t, x) \in \mathbb{R}^{1+n}$ and Fatou's lemma yields

$$\begin{aligned} \|f\|^p_{\mathbb{H}^p_L} &\leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} \left(\iint_{|x-y|<t} |\psi(t^2 L) f_k(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}} \right)^{p/2} \mathrm{d}x \\ &= \liminf_{k \to \infty} \|f_k\|^p_{\mathbb{H}^p_L}. \end{aligned}$$

The final expression is finite by the Cauchy property in $\mathbb{H}^p_L$, which means that $f \in \mathbb{H}^p_L$.

Concerning the final statement, we recall from Theorem 9.6 that $p_\pm(L)$ are the endpoints of $\mathcal{H}(L)$. For the sake of a contradiction, suppose $p := p_-(L) \in \mathcal{H}(L)$. The first part yields $\|R_L f\|_{\mathrm{H}^p} \simeq \|af\|_{\mathrm{H}^p}$ for all $f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$, which contradicts Theorem 11.1. Likewise, suppose $p_+(L) \in \mathcal{H}(L)$. Since $\mathcal{H}(L) \subseteq (1_*, \infty)$, we must have $p_+(L) < \infty$

and therefore $p_-(L^\sharp) = p_+(L)' > 1$ by duality and similarity. Proposition 8.9 implies with $p = p_-(L^\sharp)$ that $\mathbb{H}^p_{L^\sharp} = \mathrm{L}^p \cap \mathrm{L}^2$ with equivalent p-norms, that is $p \in \mathcal{H}(L^\sharp)$, which is impossible as we have already seen. □

12. Critical numbers for Poisson and heat semigroups

For the applications to boundary value problems we are mainly interested in estimates for the *Poisson semigroup* $(\mathrm{e}^{-tL^{1/2}})_{t>0}$. It would have been equally natural to try building the theory in Section 6 from the intervals

$$\mathcal{J}^{\mathrm{Pois}}(L) \coloneqq \big\{p \in (1_*, \infty) : (\mathrm{e}^{-tL^{1/2}})_{t>0} \text{ is } a^{-1}\,\mathrm{H}^p\text{-bounded}\big\},$$
$$\mathcal{N}^{\mathrm{Pois}}(L) \coloneqq \big\{p \in (1_*, \infty) : (t\nabla_x \mathrm{e}^{-tL^{1/2}})_{t>0} \text{ is } a^{-1}\,\mathrm{H}^p - \mathrm{H}^p\text{-bounded}\big\}.$$

Note that both intervals contain $p = 2$. Indeed, $2 \in \mathcal{J}^{\mathrm{Pois}}(L)$ follows from the functional calculus on L^2 and to prove $2 \in \mathcal{N}^{\mathrm{Pois}}(L)$ we additionally use ellipticity to obtain

$$\|t\nabla_x \mathrm{e}^{-tL^{1/2}} f\|_2^2 \lesssim \operatorname{Re}\langle a t^2 L \mathrm{e}^{-tL^{1/2}} f, \mathrm{e}^{-tL^{1/2}} f\rangle \lesssim \|f\|_2^2$$

for all $t > 0$ and all $f \in \mathrm{L}^2$. This gives rise to a definition of critical 'Poisson' numbers.

Definition 12.1. The lower and upper endpoints of $\mathcal{J}^{\mathrm{Pois}}(L)$ are denoted by $p_-^{\mathrm{Pois}}(L)$ and $p_+^{\mathrm{Pois}}(L)$, respectively. Likewise, $q_\pm^{\mathrm{Pois}}(L)$ denote the endpoints of $\mathcal{N}^{\mathrm{Pois}}(L)$.

The reason why we use $\mathcal{J}(L)$ and $\mathcal{N}(L)$ is that Poisson semigroups offer very limited off-diagonal decay (think of the Poisson kernel for the Laplacian), whereas the resolvents offer exponential decay. One main result in this section is that while the decay properties are strikingly different, the associated critical numbers are the same.

Theorem 12.2. *$p_\pm^{\mathrm{Pois}}(L) = p_\pm(L)$ and $q_\pm^{\mathrm{Pois}}(L) = q_\pm(L)$.*

Aiming in a similar direction, we note that the unperturbed operator $L_0 = -\operatorname{div}_x d\nabla_x$ is sectorial of angle $\omega_{L_0} \in (0, \pi/2)$ and hence it generates a holomorphic semigroup $(\mathrm{e}^{-t^2 L_0})_{t>0}$ on L^2, called *heat semigroup*. The associated intervals

$$\mathcal{J}^{\mathrm{heat}}(L_0) \coloneqq \big\{p \in (1_*, \infty) : (\mathrm{e}^{-t^2 L_0})_{t>0} \text{ is } \mathrm{H}^p\text{-bounded}\big\},$$
$$\mathcal{N}^{\mathrm{heat}}(L_0) \coloneqq \big\{p \in (1_*, \infty) : (t\nabla_x \mathrm{e}^{-t^2 L_0})_{t>0} \text{ is } \mathrm{H}^p\text{-bounded}\big\}$$

contain $p = 2$ by the same argument as for the Poisson semigroup and their endpoint are the critical 'heat' numbers.

Definition 12.3. The lower and upper endpoints of $\mathcal{J}^{\mathrm{heat}}(L_0)$ are denoted by $p_-^{\mathrm{heat}}(L_0)$ and $p_+^{\mathrm{heat}}(L_0)$, respectively. Likewise $q_\pm^{\mathrm{heat}}(L_0)$ denote the endpoints of $\mathcal{N}^{\mathrm{heat}}(L_0)$.

We refer to [3] for a systematic treatise of critical heat numbers in the range $p \in (1,\infty)$ and their relation to Riesz transforms, H^∞-calculus and square function estimates.

The second main result in this section shows that critical numbers and critical heat numbers are the same in the full interval of exponents. Since also the critical numbers for L_0 and L are the same (Theorem 6.9), this provides a means of characterizing all intervals of exponents in the monograph through properties of a heat semigroup, even though L itself need not be a generator.

Theorem 12.4. *$p_\pm^{\mathrm{heat}}(L_0) = p_\pm(L_0)$ and $q_\pm^{\mathrm{heat}}(L_0) = q_\pm(L_0)$.*

This second result tells us that the theory in [6] relying on the critical heat numbers is in coherence with the one here when restricted to the range $p \in (1,\infty)$.

The proofs of both theorems follow the same pattern. If we assume resolvent bounds, then semigroup bounds follow immediately from the functional calculus bound in Theorem 10.1, whereas in the opposite direction we can represent resolvents via Laplace transforms of the semigroup. For the Poisson semigroup these formulæ become more technical since we want to estimate the resolvents of the square of the semigroup generator $L^{1/2}$. It is for this reason that we shall showcase the strategy for the heat semigroup first although the Poisson semigroup is of greater importance to us.

For both proofs we need part (i) of the following proposition. The extensions in (ii) and (iii) will be needed much later in Section 20.

Proposition 12.5. *Let $p_-(L) < p \leq q < p_+(L)$ and consider the families $(a\mathrm{e}^{-tL^{1/2}}a^{-1})_{t>0}$ and $(\mathrm{e}^{-t^2L_0})_{t>0}$.*

(i) *If $p \leq q < p_+(L)$, then they are $\mathrm{H}^p - \mathrm{H}^q$-bounded.*
(ii) *If $p_-(L^\sharp) < 1$ and $0 < \alpha < n(1/p_-(L^\sharp) - 1)$, then the first one is $\mathrm{H}^p - a\dot{\Lambda}^\alpha$-bounded and the second one is $\mathrm{H}^p - \dot{\Lambda}^\alpha$-bounded*
(iii) *If $p_-(L^\sharp) < 1$, then they are $\mathrm{H}^p - \mathrm{L}^\infty$-bounded.*

Proof. We prove the three statements in order.

Proof of (i). We recall from Theorem 6.9 that $p_\pm(L) = p_\pm(L_0)$. Hence, H^p-boundedness follows directly from Theorem 10.1.

As we have $p_-(L) < 2_*$ (Proposition 6.7), we can use a Sobolev embedding followed by Theorem 11.1.(ii) and Theorem 10.1 with exponent $p = 2_*$ in order to obtain for all $t > 0$ and all $f \in \mathrm{H}^{2_*} \cap \mathrm{L}^2$ that

$$\begin{aligned} \|a\mathrm{e}^{-tL^{1/2}}a^{-1}f\|_2 &\lesssim \|\nabla_x \mathrm{e}^{-tL^{1/2}}a^{-1}f\|_{\mathrm{H}^{2_*}} \\ &\simeq \|aL^{1/2}\mathrm{e}^{-tL^{1/2}}a^{-1}f\|_{\mathrm{H}^{2_*}} \\ &\lesssim t^{-1}\|f\|_{\mathrm{H}^{2_*}}. \end{aligned} \tag{12.1}$$

Hence, $(a\mathrm{e}^{-tL^{1/2}}a^{-1})_{t>0}$ is $\mathrm{H}^{2_*} - \mathrm{L}^2$-bounded. By the first part and Lemma 4.4 we obtain for each $p \in (p_-(L), 2)$ an integer β such that $(a\mathrm{e}^{-t\beta L^{1/2}}a^{-1})_{t>0}$ is $\mathrm{H}^p - \mathrm{L}^2$-bounded. Interpolation with the first part yields $\mathrm{H}^p - \mathrm{H}^q$-boundedness for all exponents $p_-(L) < p \leq q \leq 2$.

Applying this result to $L^\sharp$ and using $L^* = a^* L^\sharp (a^*)^{-1}$ yields in particular $\mathrm{L}^{q'} - \mathrm{L}^{p'}$-boundedness for $(\mathrm{e}^{-t(L^*)^{1/2}})_{t>0}$ if $(p_-(L^\sharp) \vee 1) < q' \leq p' \leq 2$. Hence, $\mathrm{L}^p - \mathrm{L}^q$ boundedness of $(a\mathrm{e}^{-tL^{1/2}}a^{-1})_{t>0}$ follows for $2 \leq p \leq q < p_+(L)$ by duality and ellipticity of a^*. In the remaining case that p and q are on opposite side of 2 we can use the semigroup property and combine $\mathrm{H}^p - \mathrm{L}^2$ and $\mathrm{L}^2 - \mathrm{L}^q$-boundedness.

The proof for the heat semigroup is *mutadis mutandis* the same since the second-order scaling guarantees that the third step in (12.1) remains valid.

Proof of (ii). Let $\alpha = n(1/\varrho - 1)$ with $p_-(L^\sharp) < \varrho < 1$. Part (i) yields that $(a^*\mathrm{e}^{-t(L^\sharp)^{1/2}}(a^*)^{-1})_{t>0}$ is $\mathrm{H}^\varrho - \mathrm{L}^{p'}$-bounded if $p_-(L) \vee 1 < p < p_+(L)$. By similarity and duality $(\mathrm{e}^{-tL^{1/2}})_{t>0}$ is $\mathrm{L}^p - \dot{\Lambda}^\alpha$-bounded. This is the claim under the additional assumption $p > 1$. The full result follows from (i) by the semigroup property.

The same argument applies to the heat semigroup.

Proof of (iii). By the semigroup property and (i) it suffices to treat the case $p > 1$. The claim has nothing to do with semigroups and simply follows from (i), (ii) and the following interpolation inequality. □

Lemma 12.6. *Let $1 \leq p < \infty$ and $0 < \alpha < 1$. If $g \in \mathrm{L}^p \cap \dot{\Lambda}^\alpha$, then $g \in \mathrm{L}^\infty$ and*

$$\|g\|_\infty \leq 2|B(0,1)|^\theta \|g\|_p^\theta \|g\|_{\dot{\Lambda}^\alpha}^{1-\theta}, \quad \theta = \frac{\alpha}{\alpha + n/p}.$$

Proof. For $x, y \in \mathbb{R}^n$ we have

$$|g(x)| \leq |g(y)| + |x-y|^\alpha \|g\|_{\dot{\Lambda}^\alpha}.$$

We take the average in y over some ball $B(x,r)$ and use Hölder's inequality to give

$$|g(x)| \leq (|B(0,1)|r^n)^{-1/p}\|g\|_p + r^\alpha \|g\|_{\dot{\Lambda}^\alpha}.$$

We conclude by picking r such that the terms on the right are equal. □

12.1. **Identification of the critical heat numbers.**

We turn to the proof of the second principal results of this section.

Proof of Theorem 12.4. We break the argument into three steps.

Step 1: From the resolvent to the semigroup. Proposition 12.5.(i) implies $(p_-(L_0), p_+(L_0)) \subseteq \mathcal{J}^{\mathrm{heat}}(L_0)$.

Next, we let $p \in (q_-(L_0), q_+(L_0))$. Then $p \in (p_-(L_0), p_+(L_0))$ by Theorem 6.2. Combining Theorem 11.1.(ii) and Theorem 10.1, we get

$$\|t\nabla_x \mathrm{e}^{-t^2 L_0} f\|_{\mathrm{H}^p} \simeq \|t L_0^{1/2} \mathrm{e}^{-t^2 L_0} f\|_{\mathrm{H}^p} \lesssim \|f\|_{\mathrm{H}^p},$$

which proves $p \in \mathcal{N}^{\mathrm{heat}}(L_0)$. We conclude that $(q_-(L_0), q_+(L_0)) \subseteq \mathcal{N}^{\mathrm{heat}}(L_0)$.

Step 2: From $\mathcal{J}^{\mathrm{heat}}(L_0)$ to $\mathcal{J}(L_0)$. For every $t > 0$ the operator

$$T \coloneqq 1 + t^2 L_0$$

is invertible and sectorial of angle $\omega_{L_0} < \pi/2$. By the Calderón reproducing formula we have for $f \in \mathrm{L}^2$ as an improper Riemann integral,

$$f = \int_0^\infty T \mathrm{e}^{-sT} f \,\mathrm{d}s.$$

Applying T^{-1} on both sides gives the classical formula

$$(1 + t^2 L_0)^{-1} f = \int_0^\infty \mathrm{e}^{-s} \mathrm{e}^{-st^2 L_0} f \,\mathrm{d}s \tag{12.2}$$

and the integral converges absolutely in L^2 since the heat semigroup is uniformly bounded.

Let now $r \in \mathcal{J}^{\mathrm{heat}}(L_0)$ and take any p between r and 2. We shall show that $((1 + t^2 L_0)^{-1})_{t>0}$ is H^p-bounded, that is, $p \in \mathcal{J}(L_0)$. Then, together with Step 1, $p_\pm^{\mathrm{heat}}(L_0) = p_\pm(L_0)$ follows.

Step 2a: The Lebesgue case $p > 1$. Since the heat semigroup is L^p-bounded, the integral in (12.2) converges absolutely in L^p for all $t > 0$ and all $f \in \mathrm{L}^p \cap \mathrm{L}^2$ and we obtain

$$\|(1 + t^2 L_0)^{-1} f\|_p \lesssim \|f\|_p$$

as required.

Step 2b: The Hardy case $p \leq 1$. We appeal to Lemma 4.9 in order to show that the resolvents are H^p-bounded.

For $f \in \mathrm{L}^2$ with compact support and mean value zero we have $\int_{\mathbb{R}^n} (1+t^2 L_0)^{-1} f \,\mathrm{d}x = 0$ by Corollary 5.4. For the other two assumptions in Lemma 4.9 we use exponents $\varrho \in (r, p)$ and $q \in (1, 2)$ with $n/\varrho - n/q < 1$. In particular, ϱ, q are interior points of $\mathcal{J}^{\mathrm{heat}}(L_0)$.

From Step 2a we obtain $q \in (p_-(L_0), 2)$. Hence, $((1 + t^2 L_0)^{-1})_{t>0}$ satisfies L^q off-diagonal estimates of arbitrarily large order by interpolation with the L^2 off-diagonal decay.

It remains to show that $((1 + t^2 L_0)^{-1})_{t>0}$ is $\mathrm{H}^\varrho - \mathrm{L}^q$-bounded. The following boundedness properties hold for the heat semigroup: first H^r and H^ϱ (by assumption), second $\mathrm{L}^q - \mathrm{L}^2$ (by Proposition 12.5), third $\mathrm{H}^\varrho - \mathrm{L}^2$ (by Lemma 4.4 and the semigroup law), fourth $\mathrm{H}^\varrho - \mathrm{L}^q$ (by interpolation). This allows us to take L^q-norms in (12.4) and obtain for all $t > 0$ and all $f \in \mathrm{H}^\varrho \cap \mathrm{L}^2$,

$$\begin{aligned} \|(1+t^2L_0)^{-1}f\|_q &\lesssim \int_0^\infty \mathrm{e}^{-s}(s^{\frac{1}{2}}t)^{\frac{n}{q}-\frac{n}{\varrho}}\|f\|_{\mathrm{H}^\varrho}\,\mathrm{d}s \\ &\lesssim t^{\frac{n}{q}-\frac{n}{\varrho}}\|f\|_{\mathrm{H}^\varrho}, \end{aligned} \tag{12.3}$$

where the integral in s is finite by the choice of our exponents. This completes Step 2b.

Step 3: From $\mathcal{N}^{\mathrm{heat}}(L_0)$ to $\mathcal{N}(L_0)$. Let $r \in \mathcal{N}^{\mathrm{heat}}(L_0)$ and take any p between r and 2. We shall show that $(t\nabla_x(1+t^2L_0)^{-1})_{t>0}$ is H^p-bounded, that is, $p \in \mathcal{N}(L_0)$. Then, together with Step 1, $q_\pm^{\mathrm{heat}}(L_0) = q_\pm(L_0)$ follows.

Step 3a: The Lebesgue case $p > 1$. We apply $t\nabla_x$ on both sides of (12.2) and take L^p-norms in order to get

$$\|t\nabla_x(1+t^2L_0)^{-1}f\|_p \lesssim \int_0^\infty s^{-\frac{1}{2}}\mathrm{e}^{-s}\|f\|_p\,\mathrm{d}s \lesssim \|f\|_p$$

as required for all $t > 0$ and all $f \in \mathrm{L}^p \cap \mathrm{L}^2$.

Step 3b: The Hardy case $p \leq 1$. By Theorem 6.2 the intervals $\mathcal{J}(L_0)$ and $\mathcal{N}(L_0)$ have the same lower endpoint. Hence, it suffices to prove $p \in \mathcal{J}(L_0)$, that is, H^p-boundedness of $((1+t^2L_0)^{-1})_{t>0}$. Once again we appeal to Lemma 4.9. We fix any $\varrho \in (r,p)$ and let $q := r^* \in (1,2)$. In particular, $n/\varrho - n/q < 1$ and ϱ, q are interior points of $\mathcal{N}^{\mathrm{heat}}(L_0)$.

For $f \in \mathrm{L}^2$ with compact support and mean value zero we have $\int_{\mathbb{R}^n}(1+t^2L_0)^{-1}f\,\mathrm{d}x = 0$ by Corollary 5.4. From Step 3a we obtain that q is an interior point of $\mathcal{N}(L_0)$, hence of $\mathcal{J}(L_0)$. By interpolation with the L^2 off-diagonal decay we find that $((1+t^2L_0)^{-1})_{t>0}$ satisfies L^q off-diagonal estimates of arbitrarily large order.

Finally, we obtain by a Sobolev embedding for all $t > 0$ and all $f \in \mathrm{H}^r \cap \mathrm{L}^2$,

$$\|\mathrm{e}^{-t^2L_0}f\|_{\mathrm{L}^q} \lesssim \|\nabla_x\mathrm{e}^{-t^2L_0}f\|_{\mathrm{H}^r} \lesssim t^{-1}\|f\|_{\mathrm{H}^r},$$

which is $\mathrm{H}^r - \mathrm{L}^q$-boundedness of the heat semigroup. Since q is an interior point of $\mathcal{J}(L_0)$, we also have L^q-boundedness of the heat semigroup from Step 1 and hence we obtain $\mathrm{H}^\rho - \mathrm{L}^q$-boundedness by interpolation. This being said, we can take again L^q-norms in (12.2) and conclude the missing $\mathrm{H}^\rho - \mathrm{L}^q$-boundedness of $((1+t^2L_0)^{-1})_{t>0}$ as in (12.3). □

12.2. **Identification of the critical Poisson numbers.** We present the proof for the Poisson semigroup *vis-à-vis* and focus on where the argument gets technically more involved.

Proof of Theorem 12.2. We break the argument again in three steps.

Step 1: From the resolvent to the semigroup. We get $(p_-(L), p_+(L)) \subseteq \mathcal{J}^{\mathrm{Pois}}(L)$ and $(q_-(L), q_+(L)) \subseteq \mathcal{N}^{\mathrm{Pois}}(L)$ by repeating the argument for the heat semigroup *mutadis mutandis*.

Step 2: From $\mathcal{J}^{\mathrm{Pois}}(L)$ to $\mathcal{J}(L)$. We shall always get from Poisson semigroup bounds to resolvents of $L^{1/2}$ on the imaginary axis and then to resolvents of L via the decomposition

$$(1 + t^2 L)^{-1} = (1 - \mathrm{i}tL^{1/2})^{-1}(1 + \mathrm{i}tL^{1/2})^{-1} \quad (t > 0).$$

As a substitute for (12.2) we need Laplace transform formulæ on the imaginary axis that we are going to derive next.

Let $\varepsilon \in (0, {}^{(\pi-\omega_L)}/_4)$ and $t > 0$. Since $L^{1/2}$ is sectorial of angle ${}^{\omega_L}/_2$, the operator

$$T \coloneqq \mathrm{e}^{\mathrm{i}(\varepsilon - \frac{\pi}{2})} + t\mathrm{e}^{\mathrm{i}\varepsilon} L^{1/2} = -\mathrm{i}\mathrm{e}^{\mathrm{i}\varepsilon}(1 + \mathrm{i}tL^{1/2})$$

is invertible and sectorial of angle ${}^{\pi}/_2 - \varepsilon$. By the Calderón reproducing formula we have for $f \in \mathrm{L}^2$ as an improper Riemann integral,

$$f = \int_0^\infty T\mathrm{e}^{-sT} f \,\mathrm{d}s.$$

Applying T^{-1} on both sides gives the formula

$$(1 + \mathrm{i}tL^{1/2})^{-1} f = -\mathrm{i}\mathrm{e}^{\mathrm{i}\varepsilon} \int_0^\infty \mathrm{e}^{\mathrm{i}s\mathrm{e}^{\mathrm{i}\varepsilon}} \mathrm{e}^{-st\mathrm{e}^{\mathrm{i}\varepsilon} L^{1/2}} f \,\mathrm{d}s. \tag{12.4}$$

The latter integral converges absolutely in L^2 since by the functional calculus on L^2 the Poisson semigroup is uniformly bounded on $\mathrm{e}^{\mathrm{i}\varepsilon}\mathbb{R}^+$ and $\mathrm{Re}(\mathrm{i}\mathrm{e}^{\mathrm{i}\varepsilon}) = -\sin(\varepsilon) < 0$. A similar formula holds for $(1 - \mathrm{i}tL^{1/2})^{-1} f$ upon replacing i by $-$i at each occurrence.

Let now $r \in \mathcal{J}^{\mathrm{Pois}}(L)$ and take any p between r and 2. We shall show that $((1+t^2L)^{-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^p$-bounded, that is, $p \in \mathcal{J}(L)$. Then, together with Step 1, $p_\pm^{\mathrm{Pois}}(L) = p_\pm(L)$ follows.

Step 2a: The Lebesgue case $p > 1$. Interpolation (Lemma 4.13) of the L^r-bound for $(0, \infty)$ and the L^2-bound on some sector provides us with a smaller $\varepsilon > 0$ such that $\mathrm{e}^{-zL^{1/2}}$ is L^p-bounded for $z \in \overline{\mathrm{S}_\varepsilon^+}$. Hence, the integral on the right-hand side in (12.4) converges absolutely in L^p for all $t > 0$ and all $f \in \mathrm{L}^p \cap \mathrm{L}^2$ and we obtain

$$\|(1 + \mathrm{i}tL^{1/2})^{-1} f\|_p \lesssim \|f\|_p.$$

The same argument applies to $(1 - \mathrm{i}tL^{1/2})^{-1}$ and L^p-boundedness of $(1 + t^2L)^{-1}$ follows by composition.

Step 2b: The Hardy case $p \leq 1$. As in the case of the heat semigroup we appeal to Lemma 4.9 and use exponents $\varrho \in (r, p)$ and $q \in (1, 2)$ with ${}^{n}/_{\varrho} - {}^{n}/_{q} < 1$. In particular, ϱ, q are interior points of $\mathcal{J}^{\mathrm{Pois}}(L)$.

The vanishing moments condition and the L^q off-diagonal estimates of arbitrarily large order for $((1 + t^2L)^{-1})_{t>0}$ follow exactly as for the heat semigroup and it remains to show $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^q$-boundedness. As before, we arrive at $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^q$-boundedness for $(\mathrm{e}^{-tL^{1/2}})_{t>0}$ but we need to extend the property to some small sector in order to use (12.4).

Again by Step 2a, we know that there exists a smaller $\varepsilon > 0$ such that the Poisson semigroup $\mathrm{e}^{-zL^{1/2}}$ is L^q-bounded for $z \in \overline{\mathrm{S}^+_{2\varepsilon}}$. Now, let $z \in \overline{\mathrm{S}^+_\varepsilon}$ and decompose

$$z = t + z' \quad \text{with} \quad t > 0, z' \in \overline{\mathrm{S}^+_{2\varepsilon}}, |z| \simeq |z'| \simeq t.$$

By composition, $\mathrm{e}^{-zL^{1/2}} = \mathrm{e}^{-z'L^{1/2}}\mathrm{e}^{-tL^{1/2}}$ is $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^q$-bounded for $z \in \overline{\mathrm{S}^+_\varepsilon}$. Taking L^q-norms in (12.4), obtain for all $t > 0$ and all $f \in \mathrm{H}^\varrho \cap \mathrm{L}^2$,

$$\begin{aligned}\|(1+\mathrm{i}tL^{1/2})^{-1}a^{-1}f\|_q &\lesssim \int_0^\infty \mathrm{e}^{-s\sin(\varepsilon)}(st)^{n/q-n/\varrho}\|f\|_{\mathrm{H}^\varrho}\,\mathrm{d}s \\ &\lesssim t^{n/q-n/\varrho}\|f\|_{\mathrm{H}^\varrho},\end{aligned}$$

where the integral in s is finite by the choice of our exponents. Hence, $((1+\mathrm{i}tL^{1/2})^{-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^q$-bounded. In Step 2a we have seen that $((1-\mathrm{i}tL^{1/2})^{-1})_{t>0}$ is L^q-bounded. Thus, $((1+t^2L)^{-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^q$-bounded. This completes Step 2b.

Step 3: From $\mathcal{N}^{\mathrm{Pois}(L)}$ to $\mathcal{N}(L)$. We cannot work with the representation (12.4): once the gradient is inside the integral, we would have to deal with a function that behaves like s^{-1} in L^2-norm near $s = 0$.

For $\varepsilon \in (0, (\pi-\omega_L)/4)$, $t > 0$, and $T := -\mathrm{i}\mathrm{e}^{\mathrm{i}\varepsilon}(1+\mathrm{i}tL^{1/2})$ as before, we use instead the reproducing formula

$$f = \int_0^\infty sT^2\mathrm{e}^{-sT}f\,\mathrm{d}s$$

for $f \in \mathrm{L}^2$. Applying T^{-2} on both sides, we find the absolutely convergent representation

$$(1+\mathrm{i}tL^{1/2})^{-2}f = -\mathrm{e}^{2\mathrm{i}\varepsilon}\int_0^\infty \mathrm{e}^{\mathrm{i}s\mathrm{e}^{\mathrm{i}\varepsilon}}\,s\,\mathrm{e}^{-st\mathrm{e}^{\mathrm{i}\varepsilon}L^{1/2}}f\,\mathrm{d}s \tag{12.5}$$

with an additional factor of s. Again, an analogous representation is available for $(1-\mathrm{i}tL^{1/2})^{-2}f$.

Let now $r \in \mathcal{N}^{\mathrm{Pois}}(L)$ and take any p between r and 2. We shall show that $(t\nabla_x(1+t^2L)^{-1})_{t>0}$ is $a^{-1}\,\mathrm{H}^p - \mathrm{H}^p$-bounded, that is, $p \in \mathcal{N}(L)$. Then, together with Step 1, $q_\pm^{\mathrm{Pois}}(L) = q_\pm(L)$ follows.

In constrast to the proof for the heat semigroup we also need to distinguish the case $p > 2$ from the rest.

Step 3a: The case $1 < p \leq 2$. We can further assume $p < 2_*$ (and hence $n \geq 3$), since otherwise we can directly conclude by Proposition 6.7.

We claim that for every $q \in [p, 2]$ there exists a smaller $\varepsilon > 0$ such that the following boundedness properties hold for all $z \in \overline{\mathrm{S}^+_\varepsilon}$:

$$\mathrm{L}^q - \mathrm{L}^q \;\text{ for } z\nabla_x\mathrm{e}^{-zL^{1/2}} \tag{12.6}$$

$$\mathrm{L}^q - \mathrm{L}^{q^*} \ \text{ for } \mathrm{e}^{-zL^{1/2}}. \tag{12.7}$$

For (12.6) we use interpolation between the L^r-result on $(0,\infty)$ and the L^2-result on some sector. As for (12.7), we use the assumption and a Sobolev embedding to give

$$\|\mathrm{e}^{-zL^{1/2}} f\|_{r^*} \leq \|\nabla_x \mathrm{e}^{-zL^{1/2}} f\|_r \lesssim |z|^{-1}\|f\|_r$$

for all $z \in (0,\infty)$ and all $f \in \mathrm{L}^r \cap \mathrm{L}^2$. This means $\mathrm{L}^r - \mathrm{L}^{r^*}$-boundedness. The same argument works for z in a sector if we replace the exponent r by 2 and we can conclude by interpolation as before.

We use (12.6) for $q = p^*$. This choice is admissible since we assume $p < 2_*$. Applying $t\nabla_x$ to (12.5) and taking L^{p^*}-norms, we obtain for all $t > 0$ and all $f \in \mathrm{L}^{p^*} \cap \mathrm{L}^2$,

$$\|t\nabla_x(1 + \mathrm{i}tL^{1/2})^{-2} f\|_{p^*} \lesssim \int_0^\infty \mathrm{e}^{-s\sin(\varepsilon)}\|f\|_{p^*} \,\mathrm{d}s.$$

Hence, $(t\nabla_x(1 + \mathrm{i}tL^{1/2})^{-2})_{t>0}$ is L^{p^*}-bounded. In the same manner, (12.7) for $q = p$ implies that $((1 - \mathrm{i}tL^{1/2})^{-2})_{t>0}$ is $\mathrm{L}^p - \mathrm{L}^{p^*}$-bounded. By composition, $(t\nabla_x(1 + t^2L)^{-2})_{t>0}$ is $\mathrm{L}^p - \mathrm{L}^{p^*}$-bounded.

Since this works for all $p \in (r, 2_*)$, we get L^p-boundedness of $(t\nabla_x(1+t^2L)^{-2})_{t>0}$. Indeed, it suffices to interpolate with the L^2 off-diagonal estimates and then use Lemma 4.7. But then we can apply Lemma 6.5 in order to get L^p-boundedness of $(t\nabla_x(1 + t^2L)^{-1})_{t>0}$ as required.

Step 3b: The case $1_* < p \leq 1$. As in Step 3b for the heat semigroup we see that it suffices to prove H^p-boundedness of $((1 + t^2L)^{-1})_{t>0}$. As usual, we rely on Lemma 4.9. We fix any $\varrho \in (r, p)$ and let $q := r^* \in (1, 2)$. In particular, $n/\varrho - n/q < 1$ and ϱ, q are interior points of $\mathcal{N}^{\mathrm{Pois}}(L_0)$.

Repeating the argument from Step 3b for the heat semigroup *mutadis mutandis*, we get the vanishing moments condition and the L^q off-diagonal estimates of arbitrarily large order for $((1 + t^2L)^{-1})_{t>0}$ and we get $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^q$-boundedness of the Poisson semigroup $(\mathrm{e}^{-tL^{1/2}})_{t>0}$. From Step 3a we know that q is an interior point of $\mathcal{N}(L)$, hence of $\mathcal{J}(L)$. Theorem 10.1 yields L^q-boundedness of $(\mathrm{e}^{-zL^{1/2}})_{z\in\overline{\mathrm{S}_\varepsilon^+}}$ for any admissible $\varepsilon > 0$. Consequently, we are back in the Situation of Step 2b of the ongoing proof and obtain the missing $a^{-1}\,\mathrm{H}^\varrho - \mathrm{L}^q$-boundedness of $((1 + t^2L)^{-1})_{t>0}$.

Step 3c: The case $2 < p < \infty$. We claim that there exists a smaller $\varepsilon > 0$ and an exponent $q \in (1, p]$ with $n/q - n/p < 1$ such that the following boundedness properties hold for $z \in \overline{\mathrm{S}_\varepsilon^+}$:

$$\mathrm{L}^p - \mathrm{L}^p \ \text{ for } z\nabla_x \mathrm{e}^{-zL^{1/2}} \tag{12.8}$$

$$\mathrm{L}^q - \mathrm{L}^p \ \text{ for } \mathrm{e}^{-zL^{1/2}}. \tag{12.9}$$

The first part follows by interpolation between the L^r and the L^2-result. For the second part we first note that $2^* \leq p^+(L) \leq p_+^{\mathrm{Pois}}(L)$ and $q_-^{\mathrm{Pois}}(L) \leq q_-(L) < 2_*$ by Step 1 and Proposition 6.7. In dimensions $n \leq 2$ we have $2^* = \infty$, hence $2 < p < r < p_+^{\mathrm{Pois}}(L)$. We take $q := p$ and obtain the claim by interpolation between the L^r-result on $(0,\infty)$ and the L^2-result on a sector. In dimension $n \geq 3$, we have $r_* \in (2_*, r) \subseteq \mathcal{N}^{\mathrm{Pois}}(L)$ and we obtain $\mathrm{L}^{r_*} - \mathrm{L}^r$-boundedness on $(0,\infty)$ by the Sobolev embedding as in Step 3a. Now, (12.9) follows by interpolation with the L^2-boundedness on a sector for the choice $q := [2, r_*]_\theta$ given that $p = [2, r]_\theta$. Note that $n/q - n/p = \theta < 1$.

Equipped with (12.8) and (12.9), we can take L^p-norms in (12.5) after having applied the gradient as well as in the analogous formula for $(1 - \mathrm{i}tL^{1/2})^{-1}$. We obtain L^p-boundedness of $(t\nabla_x(1+\mathrm{i}tL^{1/2})^{-2})_{t>0}$ and $\mathrm{L}^q - \mathrm{L}^p$-boundedness of $((1 - \mathrm{i}tL^{1/2})^{-2})_{t>0}$. In the second case the restriction on q guarantees again that the integral in s converges. Hence, $(t\nabla_x(1 + t^2L)^{-2})_{t>0}$ is $\mathrm{L}^q - \mathrm{L}^p$-bounded.

At this point we can repeat the argument in the last paragraph of Step 3a to conclude L^p-boundedness. □

12.3. **More on off-diagonal decay for the Poisson semigroup.** We include an exemplary result to illustrate the poor off-diagonal decay of the Poisson semigroup. In general, and in stark contrast to the resolvents, there is not enough decay to bridge between $\mathrm{L}^q - \mathrm{L}^2$-estimates and $\mathrm{L}^q - \mathrm{L}^q$-estimates via Lemma 4.7.

Proposition 12.7. *If* $(p_-(L) \vee 1) < q \leq 2$, *then* $(tL^{1/2}\mathrm{e}^{-tL^{1/2}})_{t>0}$ *satisfies* $\mathrm{L}^q - \mathrm{L}^2$ *off-diagonal estimates of order* $n/q - n/2 + 1$.

Proof. We pick $p \in (p_-(L) \vee 1, q)$ and let $\theta \in (0,1)$ be such that $q = [p, 2]_\theta$. For a parameter $\alpha > 1$, to be chosen later on, we consider the family

$$t^\alpha L^{\alpha/2}\mathrm{e}^{-tL^{1/2}} = t^\alpha L^{\alpha/2}\mathrm{e}^{-\frac{t}{2}L^{1/2}}\mathrm{e}^{-\frac{t}{2}L^{1/2}} \quad (t > 0).$$

From the left-hand side and Lemma 4.16.(i) we obtain L^2 off-diagonal estimates of order α, whereas from the right-hand side and Proposition 12.5 we obtain $\mathrm{L}^p - \mathrm{L}^2$-boundedness. This implies $\mathrm{L}^q - \mathrm{L}^2$ off-diagonal estimates of order $\theta\alpha$, see Lemma 4.14.

Now, let $E, F \subseteq \mathbb{R}^n$ be measurable, $f \in \mathrm{L}^q \cap \mathrm{L}^2$ and $t > 0$. We use the Calderón reproducing formula

$$f = c_\alpha \int_0^\infty s^{\alpha-1}\mathrm{e}^{-sL^{1/2}} \frac{\mathrm{d}s}{s}$$

in order to give

$$\mathbf{1}_F\big(tL^{1/2}\mathrm{e}^{-tL^{1/2}}\big)\mathbf{1}_E f = c_\alpha \int_0^\infty \frac{ts^{\alpha-1}}{(s+t)^\alpha}\mathbf{1}_F\big((s+t)^\alpha L^{\alpha/2}\mathrm{e}^{-(s+t)L^{1/2}}\big)\mathbf{1}_E f \,\frac{\mathrm{d}s}{s}.$$

Thus, setting $\gamma := {}^{n}/_{q} - {}^{n}/_{2} \geq 0$, we get

$$\begin{aligned}
&\|\mathbf{1}_F(tL^{1/2}\mathrm{e}^{-tL^{1/2}})\mathbf{1}_E f\|_2 \\
&\quad\lesssim \|f\|_q \int_0^\infty \frac{ts^{\alpha-1}}{(s+t)^{\alpha+\gamma}}\Big(1+\frac{\mathrm{d}(E,F)}{s+t}\Big)^{-\theta\alpha}\,\frac{\mathrm{d}s}{s} \\
&\quad= \|f\|_q t^{-\gamma}\int_0^\infty \frac{\sigma^{\alpha-1}}{(1+\sigma)^{\alpha+\gamma}}\Big(1+\frac{\mathrm{d}(E,F)/t}{1+\sigma}\Big)^{-\theta\alpha}\,\frac{\mathrm{d}\sigma}{\sigma}.
\end{aligned}$$

We let $X := {}^{\mathrm{d}(E,F)}/_{t}$. It remains to show that we can choose $\alpha > 1$ in such a way that with an implicit constant independent of X,

$$\int_0^\infty \frac{\sigma^{\alpha-1}}{(1+\sigma)^{\alpha+\gamma}}\Big(1+\frac{X}{1+\sigma}\Big)^{-\theta\alpha}\,\frac{\mathrm{d}\sigma}{\sigma} \lesssim (1+X)^{-\gamma-1}.$$

In the case $X \leq 1$, we simply bound the left-hand side by

$$\int_0^\infty \frac{\sigma^{\alpha-1}}{(1+\sigma)^{\alpha+\gamma}}\,\frac{\mathrm{d}\sigma}{\sigma} \lesssim 1 \lesssim (1+X)^{-\gamma-1}.$$

In the case $X > 1$, we split the integral into three pieces and obtain a bound (up to a multiplicative constant depending on α, γ, θ) by

$$\begin{aligned}
\int_0^1 \sigma^{\alpha-1}X^{-\theta\alpha}\,\frac{\mathrm{d}\sigma}{\sigma} &+ \int_1^X \frac{\sigma^{\alpha-1}}{\sigma^{\alpha+\gamma}}\Big(\frac{X}{\sigma}\Big)^{-\theta\alpha}\,\frac{\mathrm{d}\sigma}{\sigma} + \int_X^\infty \frac{\sigma^{\alpha-1}}{\sigma^{\alpha+\gamma}}\,\frac{\mathrm{d}\sigma}{\sigma} \\
&\lesssim X^{-\theta\alpha} + X^{-\theta\alpha}(X^{\theta\alpha-\gamma-1}+1) + X^{-\gamma-1} \\
&\lesssim (1+X)^{-\gamma-1},
\end{aligned}$$

provided that we pick $\alpha > {}^{(\gamma+1)}/_{\theta} \vee 1$. □

13. L^p boundedness of the Hodge projector

Let $p \in (1,\infty)$. The well-known *Leray–Helmholtz decomposition* states that every vector field $f \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{C}^{mn})$ can be decomposed into a divergence-free part and a gradient field. In order to set the stage for studying operator-adapted counterparts, it will be convenient to reproduce the simple proof.

Definition 13.1. For $p \in (1,\infty)$ let

$$\begin{aligned}
\mathsf{N}_p(\operatorname{div}_x) &:= \{g \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{C}^{nm}) : \operatorname{div}_x g = 0\}, \\
\mathsf{R}_p(\nabla_x) &:= \{\nabla_x h : h \in \mathrm{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^m)\}.
\end{aligned}$$

Lemma 13.2 (Leray–Helmholtz decomposition). *Let $p \in (1,\infty)$. There is a topological decomposition*

$$\mathrm{L}^p(\mathbb{R}^n;\mathbb{C}^{nm}) = \mathsf{N}_p(\operatorname{div}_x) \oplus \overline{\mathsf{R}_p(\nabla_x)}$$

and the projection onto $\overline{\mathsf{R}_p(\nabla_x)}$ is given by the L^p-bounded Fourier multiplication operator $-\nabla_x(-\Delta_x^{-1})\operatorname{div}_x$.

Proof. The Fourier symbol $\xi \mapsto |\xi|^{-2}\xi \otimes \xi$ of $-\nabla_x(-\Delta_x^{-1}) \operatorname{div}_x$ is homogeneous of degree 0 and hence fits into the scope of the Mihlin multiplier theorem. Hence, this operator is defined on $\mathcal{Z}'$ and restricts to bounded map on L^p that we call $\mathbb{P}_p$. As $\mathbb{P}_p$ is a projection on L^p, it induces the topological decomposition $\mathrm{L}^p = \mathsf{R}(1-\mathbb{P}_p) \oplus \mathsf{R}(\mathbb{P}_p)$. By construction, we have $\mathsf{R}(1-\mathbb{P}_p) \subseteq \mathsf{N}_p(\operatorname{div}_x)$ and $\mathsf{R}(\mathbb{P}_p) \subseteq \overline{\mathsf{R}_p(\nabla_x)}$. Equality in both inclusions follows provided that $\mathsf{N}_p(\operatorname{div}_x) \cap \overline{\mathsf{R}_p(\nabla_x)} = \{0\}$. But if f belongs to this intersection, then $f = \nabla_x h$, where $h \in \dot{\mathrm{W}}^{1,p}$ satisfies $\Delta_x h = 0$ in $\mathcal{Z}'$. Therefore $h = 0$ in $\mathcal{Z}'$, so h must be a polynomial and hence a constant, which in turn means that $f = 0$. □

In view of the explicit formula for the projection in Lemma 13.2, the Leray–Helmholtz decomposition is also called *Hodge decomposition* associated with $-\Delta_x$. Following [6, Sec. 4.5], we look for similar decompositions adapted to divergence form operators $-\operatorname{div}_x d\nabla_x$. These operators are defined in the sense of distributions modulo constants as bounded operators

$$-\operatorname{div}_x d\nabla_x : \dot{\mathrm{W}}^{1,p}(\mathbb{R}^n; \mathbb{C}^m) \to \dot{\mathrm{W}}^{-1,p}(\mathbb{R}^n; \mathbb{C}^m) \tag{13.1}$$

for every $p \in (1, \infty)$. Their action is consistent for different values of p and for $p = 2$ we find the operator Λ defined in (3.5). The adjoint to (13.1) is given by

$$-\operatorname{div}_x d^*\nabla_x : \dot{\mathrm{W}}^{1,p'}(\mathbb{R}^n; \mathbb{C}^m) \to \dot{\mathrm{W}}^{-1,p'}(\mathbb{R}^n; \mathbb{C}^m).$$

When $p = 2$, it corresponds to the operator $L_0^\sharp = L_0^*$ in the same way that $-\operatorname{div}_x d\nabla_x$ corresponds to L_0.

The interval that we are mainly interested in this section concerns the bounded extension to L^p of the L^2-bounded *Hodge projector* $\nabla_x \Lambda^{-1} \operatorname{div}_x$.

Definition 13.3. Introduce the interval

$$\mathcal{P}(L_0) \coloneqq \big\{p \in (1, \infty) : \nabla_x \Lambda^{-1} \operatorname{div}_x \text{ is } \mathrm{L}^p\text{-bounded}\big\}.$$

A priori, there are two possibilities to incorporate the matrix d into the Leray–Helmholtz decomposition:

$$\mathrm{L}^p(\mathbb{R}^n; \mathbb{C}^{nm}) = \mathsf{N}_p(\operatorname{div}_x) \oplus \overline{d\,\mathsf{R}_p(\nabla_x)}, \tag{13.2}$$

and

$$\mathrm{L}^p(\mathbb{R}^n; \mathbb{C}^{nm}) = \mathsf{N}_p(\operatorname{div}_x d) \oplus \overline{\mathsf{R}_p(\nabla_x)}, \tag{13.3}$$

where closures are taken in L^p and $\mathsf{N}_p(\operatorname{div}_x d) \coloneqq \{f \in \mathrm{L}^p(\mathbb{R}^n; \mathbb{C}^{nm}) : \operatorname{div}_x(df) = 0\}$. We shall see that these *topological* decompositions always hold when $p = 2$ and that this directly relates to (13.1) being an isomorphism for $p = 2$. We say that such a topological decomposition *compatibly holds* if in addition for every $f \in \mathrm{L}^p \cap \mathrm{L}^2$ the decomposition in L^p is the same as in L^2.

Compatibility with the theory for $p = 2$ is a key issue here and we take the occasion to clarify some points that had been left unclear in the literature. The central question is whether the set of $p \in (1,\infty)$ for which $-\operatorname{div}_x d\nabla_x : \dot{\mathrm{W}}^{1,p} \to \dot{\mathrm{W}}^{-1,p}$ is an isomorphism is an open interval. While openness turns out to be true in general, connectedness requires more specific arguments.

As a cautionary tale, let us remark that in general the compatibility of the inverse does not come for free and hence the property of being an isomorphism does not interpolate. To give a simple example, consider the dilation $f \mapsto (t \mapsto f(\frac{t}{2}))$ on the real line. Its restriction T_p to $\mathrm{L}^p(\mathbb{R})$ is invertible and $\|T_p\|_{p\to p} = 2^{1/p} = \|T_p^{-1}\|_{p\to p}^{-1}$. Hence, the spectrum $\sigma(T_p)$ is contained in the circle of radius $2^{1/p}$. Now, pick $\lambda \in \sigma(T_3)$. Then $\lambda - T$ is invertible on $\mathrm{L}^2(\mathbb{R})$ and $\mathrm{L}^4(\mathbb{R})$ but not on $\mathrm{L}^3(\mathbb{R})$ and therefore the inverses cannot be compatible.

Concerning the isomorphism property for $-\operatorname{div}_x d\nabla_x$, the formulation in [6, Cor. 4.24] is ambiguous. As far as Hodge decompositions are concerned, a general statement in [48, Prop. 2.17] asserts (when restricted to our setup) that the set of exponents for which they are valid is an interval, but their proof offers no specific argument. In view of our discussion below, connectedness should still be considered unproved at this stage and compatible invertibility and compatible Hodge decompositions only hold in the connected component that contains $p = 2$. The fact that this connected component enters the discussion has previously been noticed in [21, Section 3].

13.1. **Compatible adapted Hodge decompositions.** The following discussion extends and streamlines the presentation in [6, Sec. 4.5].

Lemma 13.4. *Let $p \in (1,\infty)$. Then $p \in \mathcal{P}(L_0)$ if and only if Λ extends by density from $\dot{\mathrm{W}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$ to an isomorphism $\dot{\mathrm{W}}^{1,p} \to \dot{\mathrm{W}}^{-1,p}$ whose inverse agrees with Λ^{-1} on $\dot{\mathrm{W}}^{-1,p} \cap \dot{\mathrm{W}}^{-1,2}$. In particular, $\mathcal{P}(L_0)$ is an open set.*

Proof. We need some preliminary observations on the Leray–Helmholtz decompositions of L^p and L^2 in Lemma 13.2. As they are being achieved through projections that coincide on the dense subset $\mathrm{L}^p \cap \mathrm{L}^2$, we also have a direct decomposition

$$\mathrm{L}^p \cap \mathrm{L}^2 = \big(\mathsf{N}_p(\operatorname{div}_x) \cap \mathsf{N}_2(\operatorname{div}_x)\big) \oplus \big(\overline{\mathsf{R}_p(\nabla_x)} \cap \overline{\mathsf{R}_2(\nabla_x)}\big) \tag{13.4}$$

that is topological with respect to L^p and L^2-norms. Moreover, the subspaces on the right are dense in $\mathsf{N}_p(\operatorname{div}_x)$ and $\overline{\mathsf{R}_p(\nabla_x)}$ for the L^p-norm, respectively. Now,

$$\nabla_x : \dot{\mathrm{W}}^{1,p} \cap \dot{\mathrm{W}}^{1,2} \to \overline{\mathsf{R}_p(\nabla_x)} \cap \overline{\mathsf{R}_2(\nabla_x)} \tag{13.5}$$

is bijective and bounded from above and below for the respective p-norms. The same is true for

$$\operatorname{div}_x : \overline{\mathsf{R}_p(\nabla_x)} \cap \overline{\mathsf{R}_2(\nabla_x)} \to \dot{\mathrm{W}}^{-1,p} \cap \dot{\mathrm{W}}^{-1,2}. \tag{13.6}$$

Indeed, the upper bound follows right away, injectivity is due to (13.4) and surjectivity and the lower bound follow since $\nabla_x \Delta_x^{-1}$ is an explicit right inverse.

We turn to the actual proof. Since $\Lambda : \dot{\mathrm{W}}^{1,p} \cap \dot{\mathrm{W}}^{1,2} \to \dot{\mathrm{W}}^{-1,p} \cap \dot{\mathrm{W}}^{-1,2}$ is well-defined and bounded for the p-norms, it follows that it extends to an isomorphism as claimed precisely if $\Lambda^{-1} : \dot{\mathrm{W}}^{-1,p} \cap \dot{\mathrm{W}}^{-1,2} \to \dot{\mathrm{W}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$ is well-defined and bounded for the p-norms. Composition with the maps in (13.5) and (13.6) yields equivalence to well-definedness and boundedness in p-norm for

$$\nabla_x \Lambda^{-1} \operatorname{div}_x : \overline{\mathsf{R}_p(\nabla_x)} \cap \overline{\mathsf{R}_2(\nabla_x)} \to \overline{\mathsf{R}_p(\nabla_x)} \cap \overline{\mathsf{R}_2(\nabla_x)}.$$

Due to (13.4) this is the same as saying $p \in \mathcal{P}(L_0)$.

Finally, the set of exponents $p \in (1, \infty)$ with the isomorphism property for Λ with compatible inverse is open in $(1, \infty)$ thanks to Šneĭberg's stability theorem, using that the scales $(\dot{\mathrm{W}}^{1,p})_{p \in (1,\infty)}$ and $(\dot{\mathrm{W}}^{-1,p})_{p \in (1,\infty)}$ interpolate by the complex method. See for instance [10, 86] and also [68, Thm. 8.1] for the compatibility. □

Lemma 13.5. *If $p \in \mathcal{P}(L_0)$, then the Hodge decompositions (13.2) and (13.3) compatibly hold. The projections onto $\overline{d\,\mathsf{R}_p(\nabla_x)}$ and $\overline{\mathsf{R}_p(\nabla_x)}$ are the extensions (by density) of $-d\nabla_x \Lambda^{-1} \operatorname{div}_x$ and $-\nabla_x \Lambda^{-1} \operatorname{div}_x d$, respectively.*

Proof. On $\mathrm{L}^2(\mathbb{R}^n; \mathbb{C}^{nm})$ we consider the bounded projection operators

$$\mathbb{P}_2 := -d\nabla_x \Lambda^{-1} \operatorname{div}_x,$$
$$\widetilde{\mathbb{P}}_2 := -\nabla_x \Lambda^{-1} \operatorname{div}_x d.$$

They are L^p-bounded since we assume $p \in \mathcal{P}(L_0)$. We call $\mathbb{P}_p$ and $\widetilde{\mathbb{P}}_p$ their extensions by density from $\mathrm{L}^p \cap \mathrm{L}^2$ to bounded projections on L^p, which induce the topological decompositions

$$\mathrm{L}^p = \mathsf{R}(1 - \mathbb{P}_p) \oplus \mathsf{R}(\mathbb{P}_p), \quad \mathrm{L}^p = \mathsf{R}(1 - \widetilde{\mathbb{P}}_p) \oplus \mathsf{R}(\widetilde{\mathbb{P}}_p). \tag{13.7}$$

By construction, we have

$$\mathsf{R}(\mathbb{P}_p) \subseteq \overline{d\,\mathsf{R}_p(\nabla_x)}, \quad \mathsf{R}(\widetilde{\mathbb{P}}_p) \subseteq \overline{\mathsf{R}_p(\nabla_x)}$$

and from $\operatorname{div}_x \mathbb{P}_p f = \operatorname{div}_x f$ and $\operatorname{div}_x(d\widetilde{\mathbb{P}}_p f) = \operatorname{div}_x(df)$ for $f \in \mathrm{L}^p \cap \mathrm{L}^2$ we also conclude

$$\mathsf{R}(1 - \mathbb{P}_p) \subseteq \mathsf{N}_p(\operatorname{div}_x), \quad \mathsf{R}(1 - \widetilde{\mathbb{P}}_p) \subseteq \mathsf{N}_p(\operatorname{div}_x d).$$

It remains to establish equality in all four inclusions and owing to (13.7) we only have to show that $\mathsf{N}_p(\operatorname{div}_x) \cap \overline{d\,\mathsf{R}_p(\nabla_x)}$ and $\mathsf{N}_p(\operatorname{div}_x d) \cap \overline{\mathsf{R}_p(\nabla_x)}$ are trivial.

Let $f \in \mathsf{N}_p(\operatorname{div}_x) \cap \overline{d\,\mathsf{R}_p(\nabla_x)}$. By density, we find $h_j \in \mathrm{W}^{1,p} \cap \mathrm{W}^{1,2}$ such that $d\nabla_x h_j \to f$ in L^p as $j \to \infty$. Then $\operatorname{div}_x(d\nabla_x h_j) \to 0$ in $\dot{\mathrm{W}}^{-1,p}$, whereupon Lemma 13.4 yields $h_j \to 0$ in $\dot{\mathrm{W}}^{1,p}$. Consequently, we have $f = 0$.

Likewise, if $f \in \mathsf{N}_p(\operatorname{div}_x d) \cap \overline{\mathsf{R}_p(\nabla_x)}$, then we pick $h_j \in \mathrm{W}^{1,p} \cap \mathrm{W}^{1,2}$ with $\nabla_x h_j \to f$ in L^p as $j \to \infty$ and conclude $f = 0$ as before. □

We shall see momentarily that $p \in \mathcal{P}(L_0)$ also entails the following property.

Definition 13.6. Let $p \in (1, \infty)$. Then d is said to satisfy *p-lower bounds* if

$$\|df\|_p \gtrsim \|f\|_p \quad (f \in \overline{\mathsf{R}_p(\nabla_x)}).$$

While this is trivially fulfilled for a strictly elliptic matrix (and probably for that reason has not even been mentioned in [6, 48]), in the realm of elliptic systems it imposes a structural condition on d.

Lemma 13.7. *If $p \in \mathcal{P}(L_0)$, then d satisfies p-lower bounds.*

Proof. By density it suffices to verify the p-lower bound for $f = \nabla_x h$ with $h \in \mathrm{W}^{1,p} \cap \mathrm{W}^{1,2}$. Then $df \in \mathrm{L}^p \cap \mathrm{L}^2$ and

$$f = \nabla_x(-\operatorname{div}_x d\nabla_x)^{-1} \operatorname{div}_x d\nabla_x h = (\nabla_x \Lambda^{-1} \operatorname{div}_x) df.$$

The assumption $p \in \mathcal{P}(L_0)$ implies $\|f\|_p \lesssim \|df\|_p$. □

Altogether, we arrive at the following characterization.

Proposition 13.8. *Let $p \in (1, \infty)$. The followings are equivalent:*

(i) $p \in \mathcal{P}(L_0)$.
(ii) $-\operatorname{div}_x d\nabla_x : \dot{\mathrm{W}}^{1,p} \to \dot{\mathrm{W}}^{-1,p}$ *is an isomorphism whose inverse agrees with Λ^{-1} on $\dot{\mathrm{W}}^{-1,p} \cap \dot{\mathrm{W}}^{-1,2}$.*
(iii) *d satisfies p-lower bounds and (13.2) compatibly holds.*
(iv) *d^* satisfies p'-lower bounds and (13.3) compatibly holds.*

Proof. We show the following implications.

(i) $\Longleftrightarrow$ *(ii)*. This is Lemma 13.4.

(i) $\Longrightarrow$ *(iii), (iv)*. The compatible Hodge decompositions are due to Lemma 13.5 and the p-lower bound for d is due to Lemma 13.7. Moreover, we have $p' \in \mathcal{P}(L_0^*)$ by duality and Lemma 13.7 yields the p'-lower bound for d^*.

(iii) $\Longrightarrow$ *(i)*. We have $2 \in \mathcal{P}(L_0)$ and according to Lemma 13.5 the decomposition holds for $p = 2$ in virtue of the projection $-d\nabla_x \Lambda^{-1} \operatorname{div}_x$. The compatibility of the Hodge decomposition implies that this operator is L^p-bounded. Using the p-lower bounds, we obtain for all $f \in \mathrm{L}^p \cap \mathrm{L}^2$,

$$\|\nabla_x \Lambda^{-1} \operatorname{div}_x f\|_p \lesssim \|d\nabla_x \Lambda^{-1} \operatorname{div}_x f\|_p \lesssim \|f\|_p.$$

(iv) $\Longrightarrow$ *(i)*. As in the previous step, we get that $-\nabla_x \Lambda^{-1} \operatorname{div}_x d$ is L^p-bounded. By duality, $-d^* \nabla_x (\Lambda^*)^{-1} \operatorname{div}_x$ is $\mathrm{L}^{p'}$-bounded and the p'-lower bound implies $p' \in \mathcal{P}(L_0^*)$. Again by duality, $p \in \mathcal{P}(L_0)$ follows. □

13.2. **Adapted Hodge decompositions.** We drop the compatibility assumption and ask under which conditions the d-adapted Hodge decompositions hold.

Proposition 13.9. *Let $p \in (1,\infty)$. The followings are equivalent:*

(i) $-\operatorname{div}_x d\nabla_x : \dot{\mathrm{W}}^{1,p} \to \dot{\mathrm{W}}^{-1,p}$ *is an isomorphism.*
(ii) d *satisfies p-lower bounds and* (13.2) *holds.*
(iii) d^* *satisfies p'-lower bounds and* (13.3) *holds.*

Remark 13.10. As (i) is equivalent to the adjoint statement that $-\operatorname{div}_x d^*\nabla_x : \dot{\mathrm{W}}^{1,p'} \to \dot{\mathrm{W}}^{-1,p'}$ is an isomorphism, we could add to the list three more items.

Proof. We establish the following implications.

$(i) \Longrightarrow (ii), (iii)$. Set Λ_p the operator in (i). The Hodge decomposition follows by a *verbatim* repetition of the proof of Lemma 13.5. In fact, it is even easier using the operator $\Lambda_p^{-1} : \dot{\mathrm{W}}^{-1,p} \to \dot{\mathrm{W}}^{1,p}$ provided by assuming (i). We can directly define the bounded projections

$$\mathbb{P}_p \coloneqq -d\nabla_x \Lambda_p^{-1} \operatorname{div}_x,$$
$$\widetilde{\mathbb{P}}_p \coloneqq -\nabla_x \Lambda_p^{-1} \operatorname{div}_x d.$$

on L^p and use (i) in place of Lemma 13.4 in the proof. Likewise, for the p-lower bound for d we can repeat the proof of Lemma 13.7 with Λ_p^{-1} in place of Λ^{-1} and working with $f = \nabla_x h \in \mathrm{W}^{1,p}$. By duality, (i) also implies that $\Lambda_p^* : \dot{\mathrm{W}}^{1,p'} \to \dot{\mathrm{W}}^{-1,p'}$ is an isomorphism and hence the p'-lower bound for d^* follows as well.

$(ii) \Longrightarrow (i)$. The p-lower bound implies $\overline{d\,\mathsf{R}_p(\nabla_x)} = d\overline{\mathsf{R}_p(\nabla_x)}$. Hence, $\mathsf{N}_{p'}(\operatorname{div}_x d^*)$ annihilates $\overline{d\,\mathsf{R}_p(\nabla_x)}$ in the $\mathrm{L}^p - \mathrm{L}^{p'}$-duality. In the same duality, $\overline{\mathsf{R}_{p'}(\nabla_x)}$ annihilates $\mathsf{N}_p(\operatorname{div}_x)$. The Hodge decomposition (13.2) implies $\mathsf{N}_{p'}(\operatorname{div}_x d^*) \cap \overline{\mathsf{R}_{p'}(\nabla_x)} = \{0\}$. As we have $\overline{\mathsf{R}_{p'}(\nabla_x)} = \{\nabla_x h : h \in \dot{\mathrm{W}}^{1,p'}\}$, injectivity of

$$\Lambda_p^* = -\operatorname{div}_x d^*\nabla_x : \dot{\mathrm{W}}^{1,p'} \to \dot{\mathrm{W}}^{-1,p'} \tag{13.8}$$

follows. From $\overline{d\,\mathsf{R}_p(\nabla_x)} = d\overline{\mathsf{R}_p(\nabla_x)}$ and (13.2) we also obtain directly the injectivity of

$$\Lambda_p = -\operatorname{div}_x d\nabla_x : \dot{\mathrm{W}}^{1,p} \to \dot{\mathrm{W}}^{-1,p}. \tag{13.9}$$

Hence, both maps have dense range and they become isomorphisms once we have shown that the first map has closed range. To this end, let $h' \in \dot{\mathrm{W}}^{1,p'}$ and $F \in \mathrm{L}^p$. We decompose $F = G + d\nabla_x f$ according to (13.2) and obtain

$$\begin{aligned} |\langle \nabla_x h', F\rangle| &= |\langle \nabla_x h', d\nabla_x f\rangle| \\ &= |\langle d^*\nabla_x h', \nabla_x f\rangle| \\ &\lesssim \|\operatorname{div}_x d^*\nabla_x h'\|_{\dot{\mathrm{W}}^{-1,p'}} \|\nabla_x f\|_p \end{aligned}$$

$$\lesssim \| \operatorname{div}_x d^* \nabla_x h' \|_{\dot{\mathrm{W}}^{-1,p'}} \| d \nabla_x f \|_p$$
$$\lesssim \| \operatorname{div}_x d^* \nabla_x h' \|_{\dot{\mathrm{W}}^{-1,p'}} \| F \|_p,$$

where the third line is just the identification of $\dot{\mathrm{W}}^{-1,p'}$ with the dual space of $\dot{\mathrm{W}}^{1,p}$, the fourth is by the d-lower bounds on L^p in (ii) and the fifth uses that the splitting (13.2) is topological in (ii). Taking the supremum over all F yields $\|h'\|_{\dot{\mathrm{W}}^{1,p'}} \lesssim \| \operatorname{div}_x d \nabla_x h' \|_{\dot{\mathrm{W}}^{-1,p'}}$, which implies closed range in (13.8).

(iii) $\implies$ *(i)*. The argument is almost identical to the previous step. This time we get $\overline{d^* \mathsf{R}_{p'}(\nabla_x)} = d^* \overline{\mathsf{R}_{p'}(\nabla_x)}$, which annihilates $\mathsf{N}_p(\operatorname{div}_x d)$. By (13.3) we find $\mathsf{N}_{p'}(\operatorname{div}_x) \cap d^* \overline{\mathsf{R}_{p'}(\nabla_x)} = \{0\}$ and therefore the map in (13.8) is injective. Injectivity in (13.9) follows directly from (13.3). In order to see that we have closed range in (13.8), we let h' and F as before and decompose $F = G + \nabla_x f$ according to (13.3). Then

$$\begin{aligned} |\langle d^* \nabla_x h', F \rangle| &= |\langle d^* \nabla_x h', \nabla_x f \rangle| \\ &\lesssim \| \operatorname{div}_x d^* \nabla_x h' \|_{\dot{\mathrm{W}}^{-1,p'}} \| \nabla_x f \|_p \\ &\lesssim \| \operatorname{div}_x d^* \nabla_x h' \|_{\dot{\mathrm{W}}^{-1,p'}} \| F \|_p, \end{aligned}$$

which yields $\|d^* \nabla_x h'\|_{p'} \lesssim \| \operatorname{div}_x d^* \nabla_x h' \|_{\dot{\mathrm{W}}^{-1,p'}}$. Using the p'-lower bounds for d^* leads to

$$\|h\|_{\dot{\mathrm{W}}^{1,p'}} = \|\nabla_x h'\|_{p'} \lesssim \|d^* \nabla_x h'\|_{p'} \lesssim \| \operatorname{div}_x d^* \nabla_x h' \|_{\dot{\mathrm{W}}^{-1,p'}}. \qquad \square$$

A comparison between Proposition 13.9 and Proposition 13.8 shows that compatibility in one of the Hodge decompositions directly relates to compatibility of the inverse of $-\operatorname{div}_x d \nabla_x$ on $\dot{\mathrm{W}}^{-1,p}$ with the inverse found by the Lax–Milgram lemma on $\dot{\mathrm{W}}^{-1,2}$. To the best of our knowledge, the question whether incompatibility of the inverses is possible for the operators $-\operatorname{div}_x d \nabla_x$ is still open. A more illuminating comparison between the two results is as follows.

Lemma 13.11. *Let $\widetilde{\mathcal{P}}(L_0) \subseteq (1, \infty)$ be the set of exponents p such that $\Lambda_p = -\operatorname{div}_x d \nabla_x : \dot{\mathrm{W}}^{1,p} \to \dot{\mathrm{W}}^{-1,p}$ is an isomorphism. Then $\widetilde{\mathcal{P}}(L_0)$ is open and $\mathcal{P}(L_0)$ is its connected component that contains 2.*

Proof. All relies on the fact that $(\dot{\mathrm{W}}^{1,p})_{p \in (1,\infty)}$ and $(\dot{\mathrm{W}}^{-1,p})_{p \in (1,\infty)}$ interpolate by the complex method and have a universal approximation technique. Šneĭberg's stability theorem yields that $\widetilde{\mathcal{P}}(L_0)$ is open. If $p_0, p_1 \in \widetilde{\mathcal{P}}(L_0)$ are such that the inverses agree with the one on $\dot{\mathrm{W}}^{-1,2}$, then by interpolation of the mapping property for the inverses the same is true for all $p \in (p_0, p_1)$. Hence, the subset of exponents with this property is the connected component that contains 2. In Lemma 13.4 we have identified it to $\mathcal{P}(L_0)$. $\square$

13.3. **Characterizations of $\mathcal{P}(L_0)$.** For equations ($m = 1$) it has been asserted in [6, Cor. 4.24] that $\mathcal{P}(L_0)$ coincides with the interval $((q_+^{\text{heat}}(L_0^*))', q_+^{\text{heat}}(L_0))$, albeit being implicit on questions of compatibility. Given Theorem 12.4, this is the same interval as $((q_+(L_0^*))', q_+(L_0))$. We take the opportunity to give the full argument and make compatibilities explicit.

Theorem 13.12. *$\mathcal{P}(L_0) = ((q_+(L_0^*))', q_+(L_0))$.*

For the proof, we need a particular Sobolev-type inequality and a factorization of Λ^{-1} via Riesz transforms.

Lemma 13.13. *If $q \in \mathcal{P}(L_0) \cap (1^*, n)$, then*

$$\|\Lambda^{-1} g\|_{q^*} \lesssim \|g\|_{q_*} \quad (g \in \mathrm{L}^{q_*} \cap \dot{\mathrm{W}}^{-1,2})$$

Proof. We use that $\mathcal{Z}$ is dense in $\mathrm{L}^{q_*} \cap \dot{\mathrm{W}}^{-1,2}$, see Section 2.5. Since $\Lambda^{-1} : \dot{\mathrm{W}}^{-1,2} \to \dot{\mathrm{W}}^{1,2}$ is bounded, we may assume $g \in \mathcal{Z}$. Hence, $f := \nabla_x (\Delta_x)^{-1} g$ is defined in $\mathcal{Z}$ and we have $g = \operatorname{div}_x f$. From the assumption and Sobolev embeddings, we get

$$\begin{aligned} \|\Lambda^{-1} g\|_{q^*} &\lesssim \|\nabla_x \Lambda^{-1} \operatorname{div}_x f\|_q \\ &\lesssim \|f\|_q \\ &\lesssim \|\nabla_x^2 (\Delta_x)^{-1} g\|_{q_*} \\ &\lesssim \|g\|_{q_*}, \end{aligned}$$

where the final step is due to the Mihlin multiplier theorem. □

Lemma 13.14. *Let $R_{L_0} = \nabla_x L_0^{-1/2}$ and $R_{L_0^*} = \nabla_x (L_0^*)^{-1/2}$ be the bounded Riesz transforms on L^2 associated with L_0 and L_0^*, respectively. Then*

$$-R_{L_0} (R_{L_0^*})^* = \nabla_x \Lambda^{-1} \operatorname{div}_x \tag{13.10}$$

as bounded operators on L^2.

Proof. The factorization formally follows but some (tedious) density arguments are necessary to make this precise.

Let $f \in \mathrm{L}^2$. The decomposition (13.2) with $p = 2$ allows us to write $f = f_0 + df_1$, where $f_0 \in \mathsf{N}(\operatorname{div}_x)$ and $f_1 \in \overline{\mathsf{R}(\nabla_x)}$. As usual, our notation indicates kernels and ranges of the operators in L^2 with maximal domain. Since $\mathsf{R}(R_{L_0^*}) \subseteq \overline{\mathsf{R}(\nabla_x)} = \mathsf{N}(\operatorname{div}_x)^\perp$ by construction, the left-hand side of (13.10) sends f_0 to 0. Obviously the same is true for the right-hand side. As for the action on df_1, we may assume $f_1 = \nabla_x u$ for $u \in \mathsf{D}(L)$. Indeed, the general case follows by density since $\mathsf{D}(L_0)$ is dense in $\mathsf{D}(L_0^{1/2}) = \mathrm{W}^{1,2}$. We obtain $\operatorname{div}_x(df_1) = -\Lambda u$, so that

$$\nabla_x \Lambda^{-1} \operatorname{div}_x (df_1) = -\nabla_x u.$$

Moreover, for $g \in \mathsf{R}(L_0^*)$ we get

$$\begin{aligned}\langle (R_{L_0^*})^*(df_1), g\rangle &= \langle df_1, \nabla_x (L_0^*)^{-1/2} g\rangle \\ &= \langle L_0 u, (L_0^*)^{-1/2} g\rangle \\ &= \langle L_0^{1/2} u, g\rangle.\end{aligned}$$

Since this holds for all g in a dense subspace of L^2, we first obtain $(R_{L_0^*})^*(df_1) = L_0^{1/2} u$ and then $-R_{L_0}(R_{L_0^*})^*(df_1) = -\nabla_x u$. Altogether, we have justified (13.10). □

Proof of Theorem 13.12. Recall that in the case of L_0 the duality relations (6.1) yield $(1 \vee p_-(L_0^*)) = p_+(L_0)'$ and $(1 \vee p_-(L_0))' = p_+(L_0^*)$. The proof of the theorem is organized in 4 Steps.

Step 1: Sufficient condition for $\mathcal{P}(L_0)$. Let $(q_+(L_0^*))' < p < q_+(L_0)$. We demonstrate that $p \in \mathcal{P}(L_0)$.

Theorem 6.2 yields $q_+(L_0) \le p_+(L_0)$ and $q_+(L_0^*) \le p_+(L_0^*)$. Hence, we obtain from (6.1) that $p_-(L_0) < p < q_+(L_0)$ and $p_-(L_0^*) < p' < q_+(L_0^*)$. Theorem 7.3 yields that R_{L_0} is L^p-bounded and that $R_{L_0^*}$ is $\mathrm{L}^{p'}$-bounded. By composition and duality $R_{L_0}(R_{L_0^*})^*$ is L^p-bounded and the previous lemma yields the claim.

Step 2: Necessary condition for $\mathcal{P}(L_0) \cap (2,\infty)$. We let $p \in \mathcal{P}(L_0) \cap (2,\infty)$ and prove that $p \le q_+(L_0)$.

To begin with, we claim that

$$[2_*, p) \subseteq \mathcal{J}(L_0). \tag{13.11}$$

Thanks to Proposition 6.7 there is nothing to do if $p \le 2^*$. Hence, we may assume $p > 2^*$ (and therefore $n \ge 3$ implicitly). We set $p_0 := p$, define iteratively $p_k := (p_{k-1})_{**} := ((p_{k-1})_*)_*$ and stop at the first exponent $k^- \ge 0$ with $p_{k^-} \in [2_*, 2^*)$. Again by Proposition 6.7 we have $[2_*, p_{k^-}) \subseteq \mathcal{J}(L)$. Now, suppose $[2_*, p_k) \subseteq \mathcal{J}(L_0)$ and pick any $\varrho \in (p_k \vee 2^*, p_{k-1})$. Let $f \in \mathrm{L}^{\varrho_{**}} \cap \mathrm{L}^2$. The function

$$g := \Lambda(1+t^2L_0)^{-1} f = L_0(1+t^2L_0)^{-1} f = t^{-2}(1-(1+t^2L_0)^{-1}) f$$

belongs to $\dot{\mathrm{W}}^{-1,2}$ since it is contained in the range of Λ and it belongs to $\mathrm{L}^{\varrho_{**}}$ since we have $\varrho_{**} \in (2_*, p_k) \subseteq \mathcal{J}(L_0)$ by assumption. We also have $\varrho_* \in (2, p \wedge n) \subseteq \mathcal{P}(L_0) \cap (1^*, n)$, so we can apply Lemma 13.13 with $q = \varrho_*$ in order to obtain

$$\|(1+t^2L_0)^{-1} f\|_\varrho = \|\Lambda^{-1} g\|_\varrho \lesssim \|g\|_{\varrho_{**}} \lesssim t^{-2}\|f\|_{\varrho_{**}}.$$

This means that the resolvents of L_0 are $\mathrm{L}^{\varrho_{**}} - \mathrm{L}^\varrho$-bounded. Since $\varrho \in (p_k \vee 2^*, p_{k-1})$ was arbitrary, interpolation with the L^2 off-diagonal estimates leads to $\mathrm{L}^{\varrho_{**}} - \mathrm{L}^\varrho$ off-diagonal estimates of arbitrarily large order and L^ϱ-boundedness follows, see Lemma 4.14 and Lemma 4.7. Hence, we have $(p_k \vee 2^*, p_{k-1}) \subseteq \mathcal{J}(L_0)$ and since the latter is an interval, we also have $[2_*, p_{k-1}) \subseteq \mathcal{J}(L_0)$. Now, (13.11) follows by backward induction.

So far, we know that $2 < p \leq p_+(L_0)$ but as $\mathcal{P}(L_0)$ is open (see Lemma 13.4) we have in fact $2 < p < p_+(L_0)$. By (6.1) we get $(p_-(L_0^*)\vee 1) < p' < 2$, so that Theorem 11.3 applied to L_0^* yields the two-sided estimate

$$\|R_{L_0^*} g\|_{p'} \simeq \|g\|_{p'} \quad (g \in \mathrm{L}^2),$$

where one (and hence both) sides can be infinite. On the other hand, $p \in \mathcal{P}(L_0)$ implies $p' \in \mathcal{P}(L_0^*)$ by duality, that is to say,

$$R_{L_0^*}(R_{L_0})^* = -\nabla_x (\Lambda^*)^{-1} \operatorname{div}_x$$

is $\mathrm{L}^{p'}$-bounded. Here, we used Lemma 13.14 with the roles of L_0 and L_0^* reversed. Altogether, we find for all $f \in \mathrm{L}^{p'} \cap \mathrm{L}^2$ that

$$\|(R_{L_0})^* f\|_{p'} \simeq \|R_{L_0^*}(R_{L_0})^* f\|_{p'} \lesssim \|f\|_{p'}.$$

This means that $(R_{L_0})^*$ is $\mathrm{L}^{p'}$-bounded. By duality, R_{L_0} is L^p-bounded and according to Theorem 7.3 this can only happen if $p \leq q_+(L_0)$.

Step 3: Necessary condition for $\mathcal{P}(L_0) \cap (1,2)$. Let $p \in \mathcal{P}(L_0) \cap (1,2)$. By duality we get $p' \in \mathcal{P}(L_0^*)$ and Step 2 applied to L_0^* gives $p' \leq q_+(L_0^*)$. Hence, we have $(q_+(L_0^*))' \leq p$.

Step 4: Conclusion. Steps 1-3 show that $(q_+(L_0^*))'$ and $q_+(L_0)$ are the endpoints of $\mathcal{P}(L_0)$. The latter being an open set by Lemma 13.4, we can conclude. □

14. Critical numbers and kernel bounds

In this section, we work out a precise relation between kernel bounds and critical numbers $p_-(L)$ *strictly* below 1. Except for Section 14.5 this is an intermezzo not needed for the application to boundary value problems. However, it nicely illustrates the usefulness of our choice for the interval $\mathcal{J}(L)$ compared to [6] and connects with the theory of Gaussian estimates in the first chapter of [24]. In particular, we obtain resolvent kernels from those of high powers of the resolvent without using heat semigroups (which exist only if $\omega_L < \pi/2$).

It will be convenient to introduce the following notation.

Definition 14.1. Given $1_* < p < 1$ and $0 < \eta < 1$, write $\eta_p := n/p - n$ and conversely $p_\eta := \frac{\eta}{n+\eta}$.

14.1. Consequences of $p_-(L) < 1$.

The following result is the core of this section.

Theorem 14.2. *The following assertions are equivalent:*

(i) *There exists $p \in (1_*, 1)$ such that $(a(1+t^2L)^{-1}a^{-1})_{t>0}$ is H^p-bounded.*

(ii) *There exist $\eta \in (0,1)$ and $\beta(n,\eta) \geq 1$ such that for all integers $\beta \geq \beta(n,\eta)$ the family $((1+t^2 L^\sharp)^{-\beta})_{t>0}$ satisfies $\mathrm{L}^2 - \mathrm{L}^\infty$ off-diagonal estimates of exponential order and is $\mathrm{L}^2 - \dot{\Lambda}^\eta$-bounded.*

Moreover, $p_-(L) = p_{\eta(L^\sharp)}$, where $\eta(L^\sharp)$ is the supremum of those η for which the second property holds.

For the proof we need an auxiliary result.

Lemma 14.3. *Let $(T(t))_{t>0}$ be a family of operators that satisfies L^2 off-diagonal estimates of arbitrarily large (resp. exponential) order and that is $\mathrm{L}^2 - \dot{\Lambda}^\eta$-bounded for some $\eta \in (0,1)$. Then $(T(t))_{t>0}$ satisfies $\mathrm{L}^2 - \mathrm{L}^\infty$ off-diagonal estimates of arbitrarily large (resp. exponential) order.*

Proof. Lemma 12.6 yields $\mathrm{L}^2 - \mathrm{L}^\infty$-boundedness. Hence, it suffices to check the off-diagonal estimates when $E, F \subseteq \mathbb{R}^n$ are measurable sets with $\mathrm{d} := \mathrm{d}(E,F) \geq t$.

We let $f \in \mathrm{L}^2$ with support in E and $\|f\|_2 = 1$, set $G := \{x \in \mathbb{R}^n : \mathrm{d}(x,F) \leq d/2\}$, and pick a Lipschitz function φ with $\mathbf{1}_F \leq \varphi \leq \mathbf{1}_G$ and $\|\nabla \varphi\|_\infty \leq 4/\mathrm{d}$. Lemma 12.6 yields

$$\|\mathbf{1}_F T(t) f\|_\infty \lesssim \|\varphi T(t) f\|_2^\theta \|\varphi T(t) f\|_{\dot{\Lambda}^\eta}^{1-\theta},$$

where $\theta \in (0,1)$ is such that $(1-\theta)(n/2 + \eta) = n/2$. On the right, the first term is bounded by $\|\mathbf{1}_G T(t) f\|_2^\theta$ and as $\mathrm{d}(G,E) \geq \mathrm{d}/2$, this gives the required off-diagonal decay. The second term is controlled by

$$\begin{aligned}\big(\|\varphi\|_\infty \|T(t)f\|_{\dot{\Lambda}^\eta} + \|\varphi\|_{\dot{\Lambda}^\eta} \|T(t)f\|_\infty\big)^{1-\theta} &\lesssim \big(t^{-\eta-\frac{n}{2}} + \mathrm{d}^{-\eta} t^{-\frac{n}{2}}\big)^{1-\theta} \\ &\leq t^{-n/2},\end{aligned}$$

using the $\mathrm{L}^2 - \dot{\Lambda}^\eta$-bound, the $\mathrm{L}^2 - \mathrm{L}^\infty$-bound and $\mathrm{d} \geq t$. □

Proof of Thm 14.2. Let us recall from Corollary 3.12 that the resolvents of L (and hence of $L^\sharp$) satisfy L^2 off-diagonal estimates of exponential order.

(i) $\Longrightarrow$ (ii). The family $(a(1+t^2L)^{-1}a^{-1})_{t>0}$ is also $\mathrm{H}^\varrho - \mathrm{L}^2$-bounded for some $\varrho < 2$ depending on the dimension n, see Lemmata 6.3 and 6.4. Thus, Lemma 4.4 implies that for any $p < q \leq 1$ there exists $\beta(n,q)$ such that for all $\beta \geq \beta(n,q)$ the family $(a(1+t^2L)^{-\beta}a^{-1})_{t>0}$ is $\mathrm{H}^q - \mathrm{L}^2$-bounded. By duality (Lemma 4.3) and the fact that $L^\sharp = (a^*)^{-1} L^* a^*$, it follows that $((1+t^2L^\sharp)^{-\beta})_{t>0}$ is $\mathrm{L}^2 - \dot{\Lambda}^\eta$-bounded with $\eta := \eta_q$. It remains to apply Lemma 14.3.

(ii) $\Longrightarrow$ (i). Let $p := p_\eta$. By duality, $(a(1+t^2L)^{-\beta}a^{-1})_{t>0}$ is $\mathrm{H}^p - \mathrm{L}^2$-bounded. This family satisfies L^2 off-diagonal estimates of exponential order and $\int_{\mathbb{R}^n} a(1+t^2L)^{-\beta}(a^{-1}f)\,\mathrm{d}x = 0$ holds for all $f \in \mathrm{L}^2$ with compact support and mean value 0, see Corollary 5.4. Lemma 4.9 yields H^q-boundedness for any $p < q \leq 1$, hence for any $p < q \leq 2$ by interpolation with the L^2-boundedness.

Let $p < q \leq 1$. By interpolation with the original $\mathrm{H}^p - \mathrm{L}^2$-boundedness we obtain $\mathrm{H}^q - \mathrm{L}^r$-boundedness for some $r > 1$ with $0 \leq {}^n/_q - {}^n/_r < 1$. Now we apply the formula

$$(14.1) \quad (1+t^2L)^{-1}(a^{-1}f) = (\beta - 1)\int_0^\infty (1+u+t^2uL)^{-\beta}(a^{-1}f)\, \mathrm{d}u,$$

which follows for $f \in \mathrm{H}^q \cap \mathrm{L}^2$ by applying $(1+t^2L)^{-1}$ to both sides of (6.2) and conclude

$$\begin{aligned}\|(1+t^2L)^{-1}(a^{-1}f)\|_r &\lesssim \int_0^\infty (1+u)^{-\beta}\left(\frac{1+u}{t^2u}\right)^{\frac{n}{2q}-\frac{n}{2r}} \|f\|_{\mathrm{H}^q}\, \mathrm{d}u \\ &\lesssim t^{\frac{n}{r}-\frac{n}{q}}\|f\|_{\mathrm{H}^q},\end{aligned}$$

where $0 \leq {}^n/_q - {}^n/_r < 1$ has guaranteed that the integral in u is finite. This proves that $(a(1+t^2L)^{-1}a^{-1})_{t>0}$ is $\mathrm{H}^q - \mathrm{L}^r$-bounded.

In the same manner we can start with L^r-boundedness for the higher-order resolvents when $1 < r \leq 2$ and obtain first L^r-boundedness of $(a(1+t^2L)^{-1}a^{-1})_{t>0}$ and then L^r off-diagonal estimates of exponential order by interpolation with the L^2-result.

Now, we apply again Lemma 4.9 to conclude H^q-boundedness of $(a(1+t^2L)^{-1}a^{-1})_{t>0}$ whenever $p < q \leq 1$.

Re-examination of the proof shows that the stated relation holds for $p_-(L)$. □

The following corollary is interesting because L^1 and L^∞ are not part of our $\mathcal{J}(L)$-theory.

Corollary 14.4. *If $p_-(L) < 1$, then $((1+t^2L)^{-1})_{t>0}$ satisfies L^1 off-diagonal estimates of exponential order and $((1+t^2L^\sharp)^{-1})_{t>0}$ satisfies L^∞ off-diagonal estimates of exponential order. In particular, these families are L^1-bounded and L^∞-bounded, respectively.*

Proof. It directly follows from Theorem 14.2 and Remark 4.8 that for $\beta \geq 2$ large enough $((1+t^2L^\sharp)^{-\beta})_{t>0}$ satisfies L^∞ off-diagonal estimates of exponential order. Hence, $((1+t^2L)^{-\beta})_{t>0}$ satisfies L^1 off-diagonal estimates of exponential order.

By the formula (14.1) applied to $f \in \mathrm{L}^1 \cap \mathrm{L}^2$ and using that a is bounded and invertible in L^∞, we see that $(1+t^2L)^{-1}$ has the desired property. Indeed, if f has support in E and F is another measurable set, then with the change of variable $v = (\frac{1+u}{u})^{1/2}$ in the integral we obtain

$$\begin{aligned}&\|(1+t^2L)^{-1}f\|_{\mathrm{L}^1(F)} \\ &\quad \leq (\beta-1)\int_0^\infty (1+u)^{-\beta}\left\|\left(1+t^2\frac{u}{1+u}L\right)^{-\beta}f\right\|_{\mathrm{L}^1(F)} \mathrm{d}u \\ &\quad = 2(\beta-1)\int_1^\infty (v^2-1)^{\beta-2}v^{1-2\beta}\left\|\left(1+\frac{t^2}{v^2}L\right)^{-\beta}f\right\|_{\mathrm{L}^1(F)} \mathrm{d}v\end{aligned}$$

$$\lesssim \int_1^\infty v^{-3} \mathrm{e}^{-c\frac{\mathrm{d}(E,F)v}{t}} \|f\|_1 \,\mathrm{d}v$$
$$\lesssim \mathrm{e}^{-c\frac{\mathrm{d}(E,F)}{t}} \|f\|_1,$$

where we used $\beta \geq 2$ in the third step.

Finally, the claim for $((1 + t^2 L^\sharp)^{-1})_{t>0}$ follows by duality and similarity. □

14.2. **Equivalence with kernel estimates.** Going one step further, we shall now incorporate pointwise kernel estimates into the machinery. The convention on the variables for integral kernels is that we always look for representations in the form

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \,\mathrm{d}y.$$

We rely on the following lemma.

Lemma 14.5. *Let $(T(t))_{t>0}$ be a family of bounded operators on L^2 and denote by $K_t(x,y)$ their distribution kernels. For every $\eta \in (0,1)$ the following assertions are equivalent:*

(i) *$(T(t))_{t>0}$ satisfies $\mathrm{L}^2 - \mathrm{L}^\infty$ off-diagonal estimates of exponential order and is $\mathrm{L}^2 - \dot{\Lambda}^\eta$-bounded.*
(ii) *For each $t > 0$, $K_t(x,y)$ agrees with a measurable function and there are constants $C, c > 0$ that do not depend on t such that for all $x, h \in \mathbb{R}^n$ and all measurable sets E,*

$$\int_E |K_t(x,y)|^2 \,\mathrm{d}y \leq C t^{-n} \mathrm{e}^{-c\frac{d(x,E)}{t}} \tag{14.2}$$

$$\int_{\mathbb{R}^n} |K_t(x+h,y) - K_t(x,y)|^2 \,\mathrm{d}y \leq C|h|^{2\eta} t^{-n-2\eta}. \tag{14.3}$$

Proof. The implication (ii) $\Longrightarrow$ (i) is a direct consequence of the Cauchy–Schwarz inequality.

Next, assume that (i) holds. Fix $t > 0$. As pointed out in [4, Thm. 1.3], any linear operator $T(t)$ that is bounded from L^2 to L^∞ has an integral representation

$$T(t)f(x) = \int_{\mathbb{R}^n} K_t(x,y) f(y) \,\mathrm{d}y \quad (f \in \mathrm{L}^2,\ \text{a.e. } x \in \mathbb{R}^n)$$

with a measurable kernel that belongs to $\mathrm{L}^\infty(\mathrm{L}^2)$ with norm equal to the operator norm. Hence, $K_t(x,y)$ can indeed be identified to a measurable function that satisfies (14.2). For $h \in \mathbb{R}^n$ let τ_h be the translation operator $f \mapsto f(\cdot + h)$. Since $T(t)f$ is also Hölder continuous of exponent η, the family $((t/|h|)^\eta (1 - \tau_h) T(t))_{t>0}$ is $\mathrm{L}^2 - \mathrm{L}^\infty$-bounded, uniformly in h, and we may apply the above result again to obtain (14.3). □

We introduce two auxiliary functions that naturally appear in kernel estimates for the resolvents.

Definition 14.6. Define functions $\omega_n, \widetilde{\omega}_n : (0,\infty) \to (0,\infty)$ by

$$\omega_n(s) := \begin{cases} 1 & \text{if } n = 1 \\ |\ln s| + 1 & \text{if } n = 2 \\ s^{2-n} & \text{if } n \geq 3 \end{cases} \quad \text{and} \quad \widetilde{\omega}_n(s) := \begin{cases} 1 & \text{if } n = 1,2 \\ s^{2-n} & \text{if } n \geq 3 \end{cases}.$$

Combining Theorem 14.2 with Lemma 14.5 allows us to characterize the property $p_-(L) < 1$ through L^2 kernel bounds of a large power of the resolvent. What is missing to get to pointwise kernel bounds is dual information on $L^\sharp$.

Theorem 14.7. *The following assertions are equivalent:*

(i) *There exists $p \in (1_*, 1)$ such that $(a(1+t^2L)^{-1}a^{-1})_{t>0}$ and $(a^*(1+t^2L^\sharp)^{-1}(a^*)^{-1})_{t>0}$ are H^p-bounded.*

(ii) *There exists $\eta \in (0,1)$ such that for all $t > 0$ the operator $(1+t^2L)^{-1}a^{-1}$ is given by a measurable kernel $G_t(x,y)$ that satisfies, for some constants $C, c > 0$, the following bounds:*

$$|G_t(x,y)| \leq Ct^{-n}\omega_n\left(\frac{|x-y|}{t}\right)\mathrm{e}^{-c\frac{|x-y|}{t}}, \tag{14.4}$$

$$|G_t(x,y+h) - G_t(x,y)| \leq Ct^{-n}\left(\frac{|h|}{|x-y|}\right)^\eta \widetilde{\omega}_n\left(\frac{|x-y|}{t}\right)\mathrm{e}^{-c\frac{|x-y|}{t}}, \tag{14.5}$$

$$|G_t(x+h,y) - G_t(x,y)| \leq Ct^{-n}\left(\frac{|h|}{|x-y|}\right)^\eta \widetilde{\omega}_n\left(\frac{|x-y|}{t}\right)\mathrm{e}^{-c\frac{|x-y|}{t}}, \tag{14.6}$$

provided that $2|h| \leq |x-y|$.

Moreover, if either condition holds, then

$$p_-(L) = p_{\eta(L^\sharp)} \quad \& \quad p_-(L^\sharp) = p_{\eta(L)},$$

where $\eta(L^\sharp)$ and $\eta(L)$ are the suprema of those η for which (14.5) *and* (14.6) *hold, respectively.*

Proof. We argue in three steps.

Step 1: (i) $\Longrightarrow$ (ii). We apply Theorem 14.2 and Lemma 14.5 to both L and $L^\sharp$. Hence, there is an even integer β such that $(1+t^2L)^{-\beta/2}$ and $(1+t^2L^\sharp)^{-\beta/2}a^{-1}$ are given by measurable kernels $K_t^{(L)}(x,y)$ and $K_t^{(L^\sharp)}(x,y)$, respectively, and both kernels satisfy (14.2) and (14.3). By duality and composition, we see that $(1+t^2L)^{-\beta}a^{-1}$ is an integral operator given by the kernel

$$G_t^\beta(x,y) := \int_{\mathbb{R}^n} K_t^{(L)}(x,z)K_t^{(L^\sharp)}(y,z)\,\mathrm{d}z.$$

We claim that there are constants $C, c \in (0,\infty)$, $\eta \in (0,1)$ such that for all x, y, h,

$$|G_t^\beta(x,y)| \le Ct^{-n}\mathrm{e}^{-c\frac{|x-y|}{t}}, \tag{14.7}$$

$$|G_t^\beta(x,y+h) - G_t^\beta(x,y)| \le Ct^{-n}\Big(\frac{|h|}{t}\Big)^\eta, \tag{14.8}$$

$$|G_t^\beta(x+h,y) - G_t^\beta(x,y)| \le Ct^{-n}\Big(\frac{|h|}{t}\Big)^\eta. \tag{14.9}$$

Indeed, (14.8) and (14.9) follow directly from (14.2), (14.3) and the Cauchy–Schwarz inequality. The same argument yields the first estimate if we split integration in z into the parts where $|x-z| \ge {}^{|x-y|}/_2$ and $|y-z| \ge {}^{|x-y|}/_2$ beforehand. Note that η in (14.8) and (14.9) can be any exponent such that $p_\eta > p_-(L)$ and $p_\eta > p_-(L^\sharp)$, respectively.

Taking logarithmic convex combinations of (14.7) with (14.8) and (14.9), we obtain in the same ranges of η but with different constants $C, c > 0$ the following Hölder estimates with exponential decay when $2|h| \le |x-y|$:

$$|G_t^\beta(x,y+h) - G_t^\beta(x,y)| \le Ct^{-n}\Big(\frac{|h|}{t}\Big)^\eta \mathrm{e}^{-c\frac{|x-y|}{t}}, \tag{14.10}$$

$$|G_t^\beta(x+h,y) - G_t^\beta(x,y)| \le Ct^{-n}\Big(\frac{|h|}{t}\Big)^\eta \mathrm{e}^{-c\frac{|x-y|}{t}}. \tag{14.11}$$

From there, it suffices to use again the formula (14.1) for all $f \in \mathrm{L}^1 \cap \mathrm{L}^2$ and (14.7) to see that $(1+t^2L)^{-1}a^{-1}$ is given by a kernel with bound

$$\begin{aligned}
|G_t(x,y)| &\le Ct^{-n}\int_0^\infty (1+u)^{-\beta}(1+\tfrac{1}{u})^{\frac{n}{2}}\mathrm{e}^{-c\frac{|x-y|}{t}(1+\frac{1}{u})^{1/2}}\,\mathrm{d}u \\
&= 2Ct^{-n}\int_1^\infty v^{n-2\beta+1}(v^2-1)^{\beta-2}\mathrm{e}^{-c\frac{|x-y|}{t}v}\,\mathrm{d}v \\
&\le 2Ct^{-n}\int_1^\infty v^{n-3}\mathrm{e}^{-c\frac{|x-y|}{t}v}\,\mathrm{d}v,
\end{aligned}$$

where we used the change of variable $v = (1+\frac{1}{u})^{1/2}$ and $\beta \ge 2$. The latter integral is controlled by $\omega_n({}^{|x-y|}/_t)\mathrm{e}^{-c|x-y|/2t}$ and (14.4) follows.

Next, we use the same strategy starting from (14.11). In that case, we assume $2|h| \le |x-y|$ and we obtain

$$|G_t(x+h,y) - G_t(x,y)| \le 2C|h|^\eta t^{-n-\eta}\int_1^\infty v^{n+\eta-3}\mathrm{e}^{-c\frac{|x-y|}{t}v}\,\mathrm{d}v$$

and conclude readily for (14.6). The argument to obtain (14.5) from (14.10) is the same.

Step 2: $(ii) \Longrightarrow (i)$. For the converse, let η be given in the estimates and let $1 > p > p_\eta$. It is enough to show that $(a(1+t^2L)^{-1}a^{-1})_{t>0}$ is H^p-bounded. The argument for the adjoint is the same.

To this end, it suffices to establish for some $C, \varepsilon > 0$ the molecular bounds

$$\|a(1+t^2L)^{-1}a^{-1}m\|_{\mathrm{L}^2(C_j(B))} \leq C(2^j r(B))^{\frac{n}{2}-\frac{n}{p}} 2^{-\varepsilon j} \quad (j \geq 1), \tag{14.12}$$

whenever $t > 0$ and m is an L^2-atom for H^p associated with a ball B. Indeed, since $a(1+t^2L)^{-1}a^{-1}m$ has integral zero by Corollary 5.4, we can first use Lemma 4.10 to get a uniform H^p-bound and then conclude by the (L^2-convergent) atomic decomposition for functions in $\mathrm{H}^p \cap \mathrm{L}^2$.

For $j = 1$ we have as required

$$\|a(1+t^2L)^{-1}a^{-1}m\|_{\mathrm{L}^2(C_1(B))} \leq C\|m\|_2 \leq Cr(B)^{\frac{n}{2}-\frac{n}{p}}.$$

For $j \geq 2$ we use the mean value property of m to write

$$a(1+t^2L)^{-1}a^{-1}m(x) = \int_B a(x)(G_t(x,y) - G_t(x,y_B))m(y)\,\mathrm{d}y$$

with y_B the center of B and obtain for $x \in C_j(B)$ that

$$\begin{aligned}|a(1+t^2L_0)^{-1}a^{-1}m(x)| &\leq Ct^{-n}2^{-j\eta}\widetilde{\omega}_n\Big(\frac{2^{j-1}r(B)}{t}\Big)\mathrm{e}^{-c\frac{2^{j-1}r(B)}{t}}\|m\|_1 \\ &\leq CC'(2^{j-1}r(B))^{-n}2^{-j\eta}r(B)^{n-\frac{n}{p}},\end{aligned}$$

where $C' \coloneqq \sup_{s>0} s^n\widetilde{\omega}_n(s)\mathrm{e}^{-cs}$. Integrating the square of this inequality on $C_j(B)$ and sorting powers of 2^j and $r(B)$ gives us (14.12) with $\varepsilon \coloneqq {}^n\!/_p - n - \eta$. Now $\varepsilon > 0$ is equivalent to $p > p_\eta$, which we have assumed.

Step 3: The formulæ for the critical numbers. In Step 1 we have obtained (14.5) if $p_\eta > p_-(L)$, whereas in Step 2 we have obtained H^p-boundedness if $p > p_{\eta(L^\sharp)}$. Thus, we have $p_-(L) = p_{\eta(L^\sharp)}$. We have also seen the same conclusions with the roles of L and $L^\sharp$ interchanged. □

Remark 14.8. In dimension $n = 1$ it is shown in [23] that the first-order derivatives of $G_t(x,y)$ in x and y exist and have an exponentially decaying pointwise bound in $|x-y|$. In particular, $\eta(L) = 1$ is attained.

Remark 14.9. Under one of the conditions of Theorem 14.7 one can also obtain pointwise and Hölder bounds for the kernel $G(x,y)$ of $L^{-1}a^{-1}$ when $n \geq 2$. Since $L^{-1}a^{-1} = L_0^{-1}$, this kernel G is just the Green kernel of L_0 and does not depend on a. To see the estimates, it suffices to replace the formula (14.1) by the Calderón reproducing formula $L_0^{-1}f = (\beta - 1)\int_0^\infty (1+uL_0)^{-\beta} f\,\mathrm{d}u$ that is valid for $f \in \mathsf{R}(L_0)$ and to plug in the estimates (14.7), (14.10) and (14.11). This does not work for $n = 1$.

14.3. **Dirichlet property, stability and examples.** Having made the link between critical numbers strictly below one and kernel estimates for the resolvent, opens the door to further characterizations of either property in terms of regularity theory for the corresponding elliptic system in $\mathbb{R}^n$.

We shall use the notion of weak solutions and Caccioppoli's inequality. A reader who is not familiar with these tools will find all necessary background material (written for systems in $\mathbb{R}^{1+n}$) in Section 16 below. The Dirichlet property for $L_0 = -\operatorname{div}_x d\nabla_x$ is the following quantitative regularity property.

Definition 14.10. The operator L_0 satisfies the *Dirichlet property* if there are $\mu \in (0,1)$ and $C_0 \in (0,\infty)$ such that for all $R > 0$ and all $x_0 \in \mathbb{R}^n$ it follows that any weak solution $v \in \mathrm{W}^{1,2}(B(x_0,R))$ to $\operatorname{div}_x d\nabla_x u = 0$ in $B(x_0,R)$ satisfies

$$(14.13) \qquad \int_{B(x_0,\rho)} |\nabla v|^2 \,\mathrm{d}x \le C_0 \left(\frac{\rho}{R}\right)^{n-2+2\mu} \int_{B(x_0,R)} |\nabla v|^2 \,\mathrm{d}x,$$

when $0 < \rho \le R$. The supremum of those μ for which this property holds is denoted by $\mu(L_0)$.

Remark 14.11. The Dirichlet property has been discussed in detail in [24, Sec. 1] for elliptic equations ($m = 1$). Amongst others, it was shown that it is stable under small L^∞-perturbations of the coefficients d and that it holds when $n = 1$ with $\mu(L_0) = 1$, when $n = 2$ with $\mu(L_0) > 0$ and when $n \ge 3$ for real-valued d with $\mu(L_0) > 0$ or with $\mu(L_0) = 1$ when d has small enough BMO-norm. The latter example includes in particular the case of constant coefficients. More exotic examples are given by coefficients d that depend only on one coordinate. In this case $\mu(L_0) = 1$, see [24, App. B]. For systems ($m \ge 2$) all examples but the case of real-valued coefficients can be adapted.

Let us prove that hat critical numbers below 1 are also characterized through the Dirichlet property for the adjoint.

Theorem 14.12. *$p_-(L_0) < 1$ if and only if L_0^* satisfies the Dirichlet property. Moreover, $p_-(L_0) = p_{\mu(L_0^*)}$.*

By Theorem 14.12 the critical numbers for L and L_0 are the same. Hence, we immediately obtain

Corollary 14.13. *The condition $p_-(L) < 1$ is satisfied exactly when L_0^* has the Dirichlet property.*

Proof of Theorem 14.12. By Theorem 14.2 we can replace the assertion $p_-(L_0) < 1$ by the existence of $0 < \eta < 1$ and $\beta(n,\eta) \ge 1$ such that for all integers $\beta \ge \beta(n,\eta)$ the family $((1+t^2L_0^*)^{-\beta})_{t>0}$ is $\mathrm{L}^2 - \dot{\Lambda}^\eta$-bounded and satisfies $\mathrm{L}^2 - \mathrm{L}^\infty$ off-diagonal estimates of exponential order.

We shall prove that under this assumption the Dirichlet property for L_0^* holds for any $\mu \in (0,\eta)$ and that conversely the Dirichlet property for L_0^* with exponent μ implies the above for any $\eta \in (0,\mu)$. Once this is done, also $p_-(L_0) = p_{\mu(L_0^*)}$ follows from Theorem 14.2.

Step 1: From $p_-(L) < 1$ to property (H). Let $0 < \mu < \eta$. We prove that L_0^* has the property (H) with exponent μ: There is a constant

C depending on L_0^* such that for any ball B of radius $R > 0$ and any $u \in \mathrm{W}^{1,2}(B)$ with $\operatorname{div}_x d^* \nabla_x u = 0$ on B in the weak sense it follows that

$$\sup_{\frac{1}{4}B} |u| + R^\mu \sup_{(x,y) \in \frac{1}{4}B,\, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\mu} \leq C\Big(\fint_B |u|^2 \,\mathrm{d}x \Big)^{1/2}.$$

The proof is a modification of an argument in [24, Sec. 1.4.2].

Let $u \in \mathrm{W}^{1,2}(B)$ be a weak solution to $\operatorname{div}_x d^* \nabla_x u = 0$ in B. Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ be supported in $\frac{8}{9}B$ with $\chi = 1$ on $\frac{7}{8}B$ and $\|\nabla \chi\|_\infty \leq cR^{-1}$ for a dimensional constant c. Let $v \coloneqq u\chi$. Since $v = u$ on $\frac{7}{8}B$, it suffices to show that for any $\varphi \in \mathrm{C}_0^\infty(\frac{1}{4}B)$ and any $h \in \frac{1}{2}B$ we have

$$\Big| \int_{\mathbb{R}^n} v(x)\overline{\varphi(x)} \,\mathrm{d}x \Big| \leq CR^{-\frac{n}{2}} \|\varphi\|_1 \|u\|_{\mathrm{L}^2(B)} \tag{14.14}$$

and

$$\Big| \int_{\mathbb{R}^n} (v(x+h) - v(x))\overline{\varphi(x)} \,\mathrm{d}x \Big| \leq C|h|^\mu R^{-\mu-\frac{n}{2}} \|\varphi\|_1 \|u\|_{\mathrm{L}^2(B)}. \tag{14.15}$$

We abbreviate inner products in $\mathrm{L}^2(\mathbb{R}^n)$ by $\langle v, \varphi \rangle \coloneqq \int_{\mathbb{R}^n} v\overline{\varphi} \,\mathrm{d}x$ and set $T(t) \coloneqq (1 + t^2 L_0)^{-1}$. Since $T(t)^\beta \varphi \in \mathrm{W}^{1,2}(\mathbb{R}^n)$ and $v \in \mathrm{W}^{1,2}(\mathbb{R}^n)$, we can write

$$\begin{aligned} \langle v, \varphi \rangle &= \langle u\chi, T(R)^\beta \varphi \rangle - \int_0^R \Big\langle u\chi, \frac{\mathrm{d}}{\mathrm{d}t} T(t)^\beta \varphi \Big\rangle \,\mathrm{d}t \\ &= \langle u\chi, T(R)^\beta \varphi \rangle - 2\beta \int_0^R \langle \nabla_x(u\chi), d\nabla_x T(t)^{\beta+1} \varphi \rangle \, t\mathrm{d}t. \end{aligned} \tag{14.16}$$

By duality the family $(T(t)^\beta)_{t>0}$ satisfies $\mathrm{L}^1 - \mathrm{L}^2$ off-diagonal estimates of exponential order. In particular, it is $\mathrm{L}^1 - \mathrm{L}^2$-bounded and we obtain

$$|\langle u\chi, T(R)^\beta \varphi \rangle| \leq \|u\chi\|_2 \|T(R)^\beta \varphi\|_2 \lesssim \|u\|_{\mathrm{L}^2(B)} R^{-\frac{n}{2}} \|\varphi\|_1. \tag{14.17}$$

Next, we rewrite the inner product inside the integral in (14.16) as

$$\begin{aligned} &\langle d^* \nabla_x u, \nabla_x(\chi T(t)^{\beta+1} \varphi) \rangle \\ &\quad + \langle d^*(\nabla_x \chi \otimes u), \nabla_x T(t)^{\beta+1} \varphi \rangle \\ &\quad - \langle d^* \nabla_x u, \nabla_x \chi \otimes T(t)^{\beta+1} \varphi \rangle \\ &\quad =: \mathrm{I} + \mathrm{II} - \mathrm{III}, \end{aligned}$$

where $\nabla_x \chi \otimes u$ is short for the vector in $(\mathbb{C}^m)^n$ that comes from the product rule when calculating $\nabla_x(\chi u)$.

The term I vanishes thanks to the equation for u.

For the term II we note that $(t\nabla_x T(t)^{\beta+1})_{t>0}$ satisfies $\mathrm{L}^1 - \mathrm{L}^2$ off-diagonal estimates of exponential order by composing the $\mathrm{L}^1 - \mathrm{L}^2$-estimates for $(T(t)^\beta)_{t>0}$ and the L^2-estimates for $(t\nabla_x T(t))_{t>0}$ from

Corollary 3.12. As the supports of φ and $\nabla_x \chi$ have distance at least $\frac{5}{8}R$, we obtain for some $\alpha > 0$ that

$$|t\,\mathrm{II}\,| \lesssim R^{-1} t^{-\frac{n}{2}} \mathrm{e}^{-\frac{\alpha R}{t}} \|u\|_{\mathrm{L}^2(B)} \|\varphi\|_1$$

Similarly, we get

$$|t\,\mathrm{III}\,| \lesssim R^{-1} \|\nabla u\|_{\mathrm{L}^2(\frac{8}{9}B)} t^{1-\frac{n}{2}} \mathrm{e}^{-\frac{\alpha R}{t}} \|\varphi\|_1$$

and hence by the Caccioppoli inequality

$$|t\,\mathrm{III}\,| \lesssim R^{-2} t^{1-\frac{n}{2}} \mathrm{e}^{-\frac{\alpha R}{t}} \|u\|_{\mathrm{L}^2(B)} \|\varphi\|_1.$$

Going back to (14.16), we obtain by integration that

$$\left| \int_0^R \langle d\nabla_x(u\chi), \nabla_x T(t)^{\beta+1}\varphi\rangle \; t\mathrm{d}t \right| \lesssim R^{-\frac{n}{2}} \|u\|_{\mathrm{L}^2(B)} \|\varphi\|_1$$

as desired. Together with (14.17) this proves (14.14).

The integral in (14.15) can be interpreted as $\langle v, \varphi_h\rangle$, where $\varphi_h := (1-\tau_{-h})\varphi$ and τ_h is the translation operator $f \mapsto f(\,\cdot\, + h)$ as before. We replace $T(t)^\beta \varphi$ by $T(t)^\beta \varphi_h$ and run the same argument since we still have the necessary bounds, namely:

- By duality $(({}^{t}/_{|h|})^\mu T(t)^\beta (1-\tau_{-h}))_{t>0}$ is $\mathrm{L}^1 - \mathrm{L}^2$-bounded, uniformly in h.
- When $|h| \leq {}^{R}/_{2}$ and $S(t)$ is one of $T(t)^{\beta+1}$ or $t\nabla_x T(t)^{\beta+1}$, then $({}^{t}/_{|h|})^\mu S(t)(1-\tau_{-h})$ is bounded from $\mathrm{L}^1(\frac{1}{4}B)$ into $\mathrm{L}^2(\frac{8}{9}B \setminus \frac{7}{8}B)$ with norm controlled by $t^{-n/2}\mathrm{e}^{-cR/t}$.

This completes the proof of property (H).

Step 2: From property (H) to the Dirichlet property. Condition (H) for L_0^* implies the Dirichlet property for L_0^* with the same μ. This argument is done in [24, p.45].

Step 3: From the Dirichlet property to resolvent kernel bounds. Assuming the Dirichlet property for L_0^* with exponent μ, it suffices to follow line by line the argument in [24, Sec. 1.4.3] up until the intermediate result of formula (38) which, in particular, states that for large enough integer k_0 the family $((1+t^2L_0^*)^{-k_0})_{t>0}$ is $\mathrm{L}^2 - \dot{\Lambda}^\eta$-bounded for any $\eta < \mu$. Then we conclude using Lemma 14.3. □

14.4. **Remarks on multiplicative perturbations.** It is instructive to put our results in perspective with Theorem 6.9, which states that the numbers $p_-(L)$ are a-independent, that is $p_-(L) = p_-(L_0)$ if we write L as a multiplicative perturbation $L = a^{-1}L_0$. There is no other conditions on a than the standing ellipticity condition from Section 3.1. This implies that the set of estimates on the kernel for $(1+t^2L_0)^{-1}$ in Theorem 14.7 is equivalent to the similar ones for the kernel of $(1+t^2L)^{-1}a^{-1}$.

Prior to that there were works on multiplicative perturbations involving semigroups. Duong and Ouhabaz [41, 80] proved that semigroup kernel estimates for e^{-tL_0} imply semigroup kernel estimates for $\mathrm{e}^{-ta^{-1}L_0}a^{-1}$ if d is a $n \times n$ matrix with real valued coefficients (so $m = 1$) under the additional assumption that d is symmetric or, more generally, that the sectoriality angle of $a^{-1}L_0$ does not exceed $\pi/2$. This condition is of course necessary to define a holomorphic semigroup and allows one to use contour integrals.

Before that, work of McIntosh-Nahmod dealt with the specific case of $L = -a^{-1}\Delta_x$, see [78]. It was shown in [20] that the only restriction to transfer a set of estimates called condition (G) on the semigroup kernel of e^{-tL_0} to the corresponding ones for $\mathrm{e}^{-tL}a^{-1}$ is the sectoriality of L.

The conclusion is that if estimates on the resolvent kernels or their high powers suffice for an application, then the existence of the semigroup generated by $-L$ can be removed. Besides, the arguments are somewhat less involved than those passing through semigroups.

14.5. **Kernel estimates for $\boldsymbol{L = -a^{-1}\Delta_x}$.** We close this section with kernel estimates in the special case of $L = -a^{-1}\Delta_x$ that are used later in this monograph. Some of them are due to [78]. Interestingly, we use a much simpler method than the original proof and we obtain further estimates, notably those on mixed second-order derivatives. Corollary 6.10 yields $p_-(L) = 1_*$ and so we could try to apply the previous theory. However, we wish to give a complete argument with the minimal tools.

Proposition 14.14. *For all integers $\beta > n/2 + 2$ the following properties hold for the kernel $H_t^\beta(x, y)$ of the higher-order resolvents $(1 - t^2a^{-1}\Delta_x)^{-\beta}a^{-1}$.*

(i) *There are $C, c > 0$, depending on ellipticity, dimensions and β, such that one has for for all $t > 0$ and $x, y \in \mathbb{R}^n$,*

$$|H_t^\beta(x,y)| + |t\nabla_x H_t^\beta(x,y)| + |t\nabla_y H_t^\beta(x,y)| + |t^2\nabla_x\nabla_y H_t^\beta(x,y)| \le Ct^{-n}\mathrm{e}^{-\frac{c|x-y|}{t}}.$$

(ii) *For all $\eta \in (0,1)$, the kernels*

$$t\nabla_x H_t^\beta(x,y), \quad t\nabla_y H_t^\beta(x,y), \quad t^2\nabla_x\nabla_y H_t^\beta(x,y)$$

are Hölder continuous in both variables with exponent η and norms in this space of the order of $t^{-\eta-n}$. In particular, $H_t^\beta \in \mathrm{C}^{1,\eta}(\mathbb{R}^n \times \mathbb{R}^n)$, the space of C^1-functions having Hölder continuous first order derivatives of exponent η.

Proof. We set $L := -a^{-1}\Delta_x$ and $L_0 := -\Delta_x$ is acting componentwise on $\mathbb{C}^m$-valued functions. It suffices to prove the properties of H_t^β for $t = 1$ with implicit constants that depend on dimensions and ellipticity.

Indeed, a change of variables yields that $H_t^\beta(x,y) = t^{-n}\widetilde{H}_1^\beta(x/t, y/t)$, where $\widetilde{H}_1^\beta$ corresponds to the coefficients $a_t(x) \coloneqq a(tx)$, which has the same ellipticity constant as a. We split the proof into four steps.

Step 1: Pointwise estimates for H^β. Let $s > 0$. When $m = 1$, $(1 - s\Delta_x)^{-1}$ is given by convolution with a classical Bessel potential, that is, a positive function with integral 1 that is in L^r whenever $1/r > 1 - 2/n$, see for instance [88, Sec.V.3]. When $m \geq 2$, $(1 + sL_0)^{-1}$ is given by componentwise convolution with the same potential.

By positivity, we get for $f \in \mathrm{L}^2$ and $s > 0$ the pointwise bound

$$|(1 + sL_0)^{-1}f| \leq (1 - s\Delta_x)^{-1}|f|, \tag{14.18}$$

where $|\cdot|$ is the $\mathbb{C}^m$-norm and the resolvent on the right-hand side is scalar-valued. In particular, $(1 + sL_0)^{-1}$ is a contraction on L^2.

We can write

$$a = \tau(1 - b) \tag{14.19}$$

for some $\tau > 0$ and $b \in \mathrm{L}^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^m))$ with $\|b\|_\infty < 1$. We shall give the well-known argument in the final step of the proof.

Using the above decomposition of a, we find

$$(a + L_0)^{-1} = \tfrac{1}{\tau}(1 - (1 + \tfrac{1}{\tau}L_0)^{-1}b)^{-1}(1 + \tfrac{1}{\tau}L_0)^{-1}$$

as operators on L^2, and the first term on the right can be computed by a Neumann series. Expanding this series explicitly and applying (14.18) inductively with $s = 1/\tau$, we have

$$|(a + L_0)^{-1}f| \leq \tfrac{1}{\tau}\sum_{k=0}^{\infty}((1 - \tfrac{1}{\tau}\Delta_x)^{-1}\|b\|_\infty)^k(1 - \tfrac{1}{\tau}\Delta_x)^{-1}|f|,$$

so that summing backward, we obtain the pointwise bound

$$|(a + L_0)^{-1}f| \leq (\alpha - \Delta_x)^{-1}|f|, \quad \text{where} \quad \alpha = \tau(1 - \|b\|_\infty).$$

Applying this estimate to af in place of f, we get

$$|(1 + L)^{-1}f| \leq \|a\|_\infty(\alpha - \Delta_x)^{-1}|f| \tag{14.20}$$

and obtain extensions by density with (possibly infinite) operator-norm bounds

$$\|(1 + L)^{-1}\|_{\mathrm{L}^p \to \mathrm{L}^q} \leq \|a\|_\infty\|(\alpha - \Delta_x)^{-1}\|_{\mathrm{L}^p \to \mathrm{L}^q}, \tag{14.21}$$

whenever $1 \leq p \leq q \leq \infty$. By Young's inequality for convolutions, the latter controlled if $1/p - 1/q < 2/n$. This gives $\mathrm{L}^1 - \mathrm{L}^\infty$-boundedness of $(1 + L)^{-\beta}a^{-1}$ provided that $\beta > n/2$.

By the Dunford–Pettis theorem [4, Thm. 1.3] we obtain that $(1 + L)^{-\beta}a^{-1}$ is given by a bounded kernel $H^\beta(x,y)$ and the bound depends only on dimensions and ellipticity. Iterating (14.20), we see that $|H^\beta(x,y)|$ is dominated by the kernel of $(\alpha - \Delta_x)^{-\beta}$ up to a factor $\|a\|_\infty^\beta\|a^{-1}\|_\infty$. The latter operator is given by convolution with a higher-order Bessel potential. Since $\beta > n/2$, we get exponential decay and no singularity at $x = y$ as stated, see [88, Sec.V.3].

Step 2: Proof of (ii) and the other bounds in (i) with $c = 0$. Write

$$\Delta_x(1+L)^{-1}a^{-1} = -1 + a(1+L)^{-1}a^{-1},$$

the Laplacian acting componentwise, so that

$$\begin{aligned} &\|\Delta_x(1+L)^{-1}a^{-1}\|_{\mathrm{L}^p\to\mathrm{L}^p} \\ &\qquad\leq 1 + \|a\|_\infty \|(\alpha-\Delta_x)^{-1}\|_{\mathrm{L}^p\to\mathrm{L}^p}. \end{aligned} \tag{14.22}$$

The operator norm in the line above is controlled for all $p \in [1,\infty]$. If $1 < p < \infty$, then by the Mihlin multiplier theorem (14.21) and (14.22) imply that $(1+L)^{-1}a^{-1}$ and $\nabla_x(1+L)^{-1}a^{-1}$ are bounded from L^p to $\mathrm{W}^{1,p}$. In particular, for $p > n$, we have the inhomogeneous Sobolev embedding $\mathrm{W}^{1,p} \subseteq \dot{\Lambda}^\eta \cap \mathrm{L}^\infty$, $\eta = 1 - n/p$. The same applies with a^* in place of a and by duality $(1+L)^{-1}a^{-1}\operatorname{div}_x$ is bounded from L^1 to $\mathrm{L}^{p'}$. By composition, we obtain that for $\beta > n/2 + 2$ the operators

$$\nabla_x(1+L)^{-\beta}a^{-1}, \quad -(1+L)^{-\beta}a^{-1}\operatorname{div}_x, \quad -\nabla_x(1+L)^{-\beta}a^{-1}\operatorname{div}_x$$

are bounded from L^1 into $\dot{\Lambda}^\eta \cap \mathrm{L}^\infty$. In particular they are bounded from L^1 into L^∞ and, invoking again the Dunford–Pettis theorem, they correspond to the kernels $\nabla_x H^\beta(x,y)$, $\nabla_y H^\beta(x,y)$, $\nabla_x\nabla_y H^\beta(x,y)$, which therefore are bounded measurable functions.

We can then use the mapping properties from L^1 into $\dot{\Lambda}^\eta$ and once more the Dunford–Pettis theorem, in order to obtain first Hölder continuity of the kernels in x (with any exponent $\eta \in (0,1)$), uniformly in y, and then by duality the same with the roles of x and y reversed. This proves (ii) and finishes the proof of (i) with $c = 0$.

Step 3: Exponential decay for the other kernels. We begin with $\partial_{x_i} H^\beta$, where $1 \leq i \leq n$. Let $e_i \in \mathbb{R}^n$ be the i-th standard unit vector and let $h > 0$. By the fundamental theorem of calculus we have

$$\begin{aligned} &\frac{1}{h}(H^\beta(x+he_i,y) - H^\beta(x,y)) \\ &= \fint_0^h \partial_{x_i} H^\beta(x+se_i,y)\,\mathrm{d}s \\ &= \partial_{x_i} H^\beta(x,y) + \fint_0^h \partial_{x_i} H^\beta(x+se_i,y) - \partial_{x_i} H^\beta(x,y)\,\mathrm{d}s, \end{aligned}$$

where $x,y \in \mathbb{R}^n$. If $2|h| \leq |x-y|$, then $|x+he_i-y| \geq |x-y|/2$ and we get from (i) for H^β and (ii) for $\nabla_x H^\beta$ that

$$|\partial_{x_i} H^\beta(x,y)| \leq \frac{C}{h}\Big(\mathrm{e}^{-\frac{c}{2}|x-y|} + \mathrm{e}^{-c|x-y|}\Big) + h^\eta \|\nabla_x H^\beta(\cdot,y)\|_{\dot{\Lambda}^\eta}.$$

Since in Step 2 we have already obtained a uniform bound for $\partial_{x_i} H^\beta$, it suffices to prove the decay for $|x-y|$ large, say $|x-y|\mathrm{e}^{\frac{c}{4}|x-y|} \geq 2$.

This restriction is manufactured such that we can take $h := \mathrm{e}^{-\frac{c}{4}|x-y|}$, resulting in the desirable estimate

$$|\partial_{x_i} H^\beta(x,y)| \leq C' \mathrm{e}^{-\frac{\eta c}{4}|x-y|}$$

for some new constant C' that depends on a only through ellipticity. This completes the proof for $\nabla_x H^\beta$.

The argument above has only used the exponential decay for H^β, the uniform boundedness of $\nabla_x H^\beta$ and the $\dot{\Lambda}^\eta$-estimate for $\nabla_x H^\beta$ in the x-variable uniformly in the y-variable, in order to give exponential decay for $\nabla_x H^\beta$. Thus, it can be repeated *verbatim* for the decay of $\nabla_y H^\beta$. Then, replacing H^β by $\nabla_y H^\beta$ gives decay of $\nabla_x \nabla_y H^\beta$.

Step 4: Proof of (14.19). We let $\tau := \lambda^{-1}\|a\|_\infty^2$ and $b := 1 - \tau^{-1} a$. If $\xi \in \mathbb{C}^m$ is normalized to $|\xi| = 1$, then

$$\begin{aligned} |b(x)\xi|^2 &= 1 + \tau^{-2}|a(x)\xi|^2 - 2\tau^{-1}\operatorname{Re}\langle a(x)\xi, \xi\rangle \\ &\leq 1 + \tau^{-2}\|a\|_\infty^2 - 2\tau^{-1}\lambda \\ &= 1 - \lambda^2\|a\|_\infty^{-2} < 1. \end{aligned}$$

□

15. Comparison with the Auscher–Stahlhut interval

The identification of adapted-Hardy spaces as a key tool to treating boundary value problems has appeared first in [22]. Although we argue independently of this reference concerning this particular issue, we need to make the bridge and the results of this section are explicitly used in Section 22 on Neumann problems.

In [22, Thm. 5.1] an interval of values of p is constructed, where one has the identification $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ even for more general operators DB. (The matrix B need not be block-diagonal.) Its upper endpoint is denoted by $p_+(DB)$ and the lower endpoint is at most the lower Sobolev conjugate of another exponent $p_-(DB)$. To avoid confusion, we denote these exponents by $p_\pm^{\mathrm{AS}}(DB)$ here. They have a precise meaning that we recall next. The following material is all taken from [22, Sec. 3.2].

Let

$$\mathsf{D}_p(D) := \{f \in \mathrm{L}^p : Df \in \mathrm{L}^p\},$$

where $\mathrm{L}^p = \mathrm{L}^p(\mathbb{R}^n; \mathbb{C}^m \times \mathbb{C}^{mn})$ and the action of D is in the sense of distributions. Then BD is defined as an unbounded operator in L^p with domain $\mathsf{D}_p(BD) = \mathsf{D}_p(D)$, null space $\mathsf{N}_p(BD)$ and range $\mathsf{R}_p(BD)$. Similar to Definition 13.6, one introduces the set of exponents with p-lower bounds

$$\mathcal{I}(BD) := \{p \in (1,\infty) : \|Bf\|_p \gtrsim \|f\|_p \text{ for all } f \in \mathsf{R}_p(D)\}$$

and the analogous set with B^* replacing B. They are open but possibly non-connected and $\mathcal{I}_2$ denotes the connected component of $\mathcal{I}(BD) \cap \mathcal{I}(B^*D)'$ that contains $p = 2$. Here, $I' = \{p' : p \in I\}$ is the dual set of a given $I \subseteq (1, \infty)$. In passing, we point out that the use of $\mathcal{I}(B^*D)$ instead of its dual set in [22, Rem. 3.5] is a typo that does not appear in the original reference [21, Sec. 5].

Then $(p_-^{\mathrm{AS}}(DB), p_+^{\mathrm{AS}}(DB))$ is the interval of exponents $q \in \mathcal{I}_2$ such that for all p between 2 and q there is a topological decomposition

$$\mathrm{L}^p = \mathsf{N}_p(BD) \oplus \overline{\mathsf{R}_p(BD)}, \tag{15.1}$$

see [22, Thm. 3.6]. It is proved in [22, Thm. 5.1] that for $(p_-^{\mathrm{AS}}(DB))_* < p < p_+^{\mathrm{AS}}(DB)$ one has $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$. It was not proved that this interval is optimal for the identification in the class of DB-operators there and for some examples it was shown that this is not the case, especially for the lower endpoint. Hence, [22] does not provide the whole identification interval, yet [22, Prop. 6.4 & 6.5] there describe it as an open interval.

Using the same framework as [22], it became clear in the classification theorems of [19] as well as in the uniqueness statements of [11] that the full interval of identification is the object of interest. Both references introduce the set of exponents $p \in (1_*, p_+^{\mathrm{AS}}(DB))$ for which $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ holds with equivalent p-quasinorms. It is called I_L in [19] and $\mathcal{H}_{\mathcal{L}}$ in [11]. Hence, either of these intervals is of the form

$$(a^{\mathrm{AS}}(DB), p_+^{\mathrm{AS}}(DB))$$

for some number $a^{\mathrm{AS}}(DB) \geq 1_*$ which could be in particular less than $(p_-^{\mathrm{AS}}(DB))_*$.

In the block situation of this monograph, we proceeded differently and introduced the set of identification $\mathcal{H}(DB)$ in (9.1) directly as the largest set of exponents $p \in (1_*, \infty)$ for which $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ holds with equivalent p-quasinorms. Then we proved that it is an open interval and characterized its endpoints as $h_-(DB) = p_-(L)$ and $h_+(DB) = q_+(L)$, see Theorems 9.6 and 11.3. Hence, in order to be able to apply the results in [11, 19] within the interval of identification $\mathcal{H}(DB)$, we need to connect both approaches.

The discussion above already shows that $a^{\mathrm{AS}}(DB) = p_-(L)$ and $q_+(L) = h_+(DB) \geq p_+^{\mathrm{AS}}(DB)$. Identifying the upper endpoints requires a specific argument.

Proposition 15.1. *In the block case setting of this monograph the number $p_+(DB) = p_+^{\mathrm{AS}}(DB)$ of [22] coincides with $h_+(DB) = q_+(L)$.*

Proof. As said, it remains to prove $q_+(L) \leq p_+^{\mathrm{AS}}(DB)$.

Let $2 \leq p < q_+(L)$. First, we recall that $q_+(L) = q_+(L_0)$ from Theorem 6.9, so that by Theorem 13.12 we have $p \in \mathcal{P}(L_0)$. Hence, Proposition 13.8 implies p-lower bounds for d and p'-lower bounds for d^*, as well as the topological Hodge decomposition (13.2).

To reinterpret this, we recall that

$$B = \begin{bmatrix} a^{-1} & 0 \\ 0 & d \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix}.$$

Using the notation of Section 13, we have

$$\mathsf{N}_p(D) = \{0\} \times \mathsf{N}_p(\operatorname{div}_x), \quad \overline{\mathsf{R}_p(D)} = \mathrm{L}^p \times \overline{\mathsf{R}_p(\nabla_x)}.$$

Since a^{-1} is strictly elliptic, we see that the conditions $p \in \mathcal{I}(BD)$ and $p \in \mathcal{I}(B^*D)'$ are equivalent to p-lower bounds for d and p'-lower bounds for d^*, respectively. Moreover, using the p-lower bounds to determine the null space, we find

$$\mathsf{N}_p(BD) = \{0\} \times \mathsf{N}_p(\operatorname{div}_x), \quad \overline{\mathsf{R}_p(BD)} = \mathrm{L}^p \times \overline{d\,\mathsf{R}_p(\nabla_x)}.$$

In turn, this shows that (15.1) is equivalent to the Hodge decomposition (13.2).

Altogether, we have shown that $p \subseteq \mathcal{I}(BD) \cap \mathcal{I}(B^*D)'$ as well as the Hodge decomposition (15.1). As we have done this for all p in the interval $[2, q_+(L))$, this proves that $q_+(L) \leq p_+^{\mathrm{AS}}(DB)$. □

Summarizing, we have obtained

Corollary 15.2. *In the block case setting of this monograph the open intervals $\mathcal{H}_{\mathcal{L}}$ from [11] and I_L from [19] both equal $(p_-(L), q_+(L))$.*

16. Basic properties of weak solutions

At this point in the monograph we begin to slightly change our perspective from Hardy spaces adapted to $L = -a^{-1} \operatorname{div}_x d\nabla_x$ to weak solutions to the elliptic system

$$\mathcal{L}u = -\operatorname{div}(A\nabla u) = -\partial_t(a\partial_t u) - \operatorname{div}_x d\nabla_x u = 0 \tag{16.1}$$

in $\mathbb{R}^{1+n}$, where as before we write

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

for the coefficient matrix in block form. In this section, we gather well-known properties of weak solutions that will frequently be used in the further course.

As usual, a *weak solution* to the equation

$$\mathcal{L}u = g \in \mathrm{L}^2_{\mathrm{loc}}(O)$$

in an open set $O \subseteq \mathbb{R}^{1+n}$ is a function $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(O; \mathbb{C}^m)$ that satisfies for all $\phi \in \mathrm{C}_0^\infty(O; \mathbb{C}^m)$,

$$\iint_O A\nabla u \cdot \overline{\nabla \phi} \,\mathrm{d}t\mathrm{d}x = \iint_O g \cdot \overline{\phi} \,\mathrm{d}t\mathrm{d}x$$

16.1. **Energy solutions.** The most common construction of weak solutions is by the Lax–Milgram lemma, using the *energy class*

$$\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+) := \{v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+) : \nabla v \in \mathrm{L}^2(\mathbb{R}^{1+n}_+)\}/\mathbb{C}^m$$

This is a Hilbert space for the inner product $\langle \nabla \cdot\,, \nabla \cdot \rangle$ and it contains the restrictions of $\mathrm{C}_0^\infty(\mathbb{R}^{1+n})$-functions to $\mathbb{R}^{1+n}_+$ as a dense subspace, see for instance [14, Lem. 3.1].

We recall the well-known trace and extension results. For convenience and a later use we include elementary proofs in our homogeneous Sobolev setting.

Lemma 16.1. *Every equivalence class $v \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ has a representative that is continuous on $[0,\infty)$ with values in $\mathrm{L}^2_{\mathrm{loc}}$. In this sense $v \in \mathrm{C}_0([0,\infty); \dot{\mathrm{H}}^{1/2,2})$ and*

$$\sup_{t \geq 0} \|v(t,\cdot)\|_{\dot{\mathrm{H}}^{1/2,2}} \lesssim \|\nabla v\|_2.$$

Conversely, every $f \in \dot{\mathrm{H}}^{1/2,2}$ can be extended to a function $v \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ with $v(0,\cdot) = f$ and $\|\nabla v\|_2 \simeq \|f\|_{\dot{\mathrm{H}}^{1/2,2}}$.

Proof. That v has a representative that is continuous on $[0,\infty)$ valued in $\mathrm{L}^2_{\mathrm{loc}}$ is just the one-dimensional Sobolev embedding in the t-variable. This property is not affected by adding constants to v and amounts to re-defining v a.e. on $\mathbb{R}^{1+n}_+$.

By density it suffices to prove the embedding into $\mathrm{C}_0([0,\infty); \dot{\mathrm{H}}^{1/2,2})$ in the case that v is the restriction of a function in $\mathrm{C}_0^\infty(\mathbb{R}^{1+n})$. For all $t \geq 0$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta_x)^{1/4} v(t,\cdot)\|_2^2 &= 2\,\mathrm{Re}\langle (-\Delta_x)^{1/4} v(t,\cdot), (-\Delta_x)^{1/4}\partial_t v(t,\cdot)\rangle \\
&\leq 2\|(-\Delta_x)^{1/2} v(t,\cdot)\|_2 \|\partial_t v(t,\cdot)\|_2 \\
&\lesssim \|\nabla_x v(t,\cdot)\|_2^2 + \|\partial_t v(t,\cdot)\|_2^2,
\end{aligned}$$

where the final step is by the solution of the Kato problem. Integration in t gives

$$\|(-\Delta_x)^{1/4} v(t,\cdot)\|_2^2 \lesssim \|\nabla v\|_2^2 \quad (t \geq 0)$$

and the left-hand side is comparable to $\|v(t,\cdot)\|^2_{\dot{\mathrm{H}}^{1/2,2}}$ by Corollary 3.8.

Again by density it suffices to prove the extension part for $f \in \dot{\mathrm{H}}^{1/2,2} \cap \mathrm{L}^2$. We set $v(t,\cdot) := \mathrm{e}^{-t(-\Delta_x)^{1/2}} f$. Clearly v is continuous on $[0,\infty)$

valued in L^2 with $v(0,\cdot)=f$. Moreover, we have

$$\begin{aligned}
\|\nabla v\|_2^2 &= \int_0^\infty \|\partial_t v(t,\cdot)\|_2^2 + \|\nabla_x v(t,\cdot)\|_2^2 \,\mathrm{d}t \\
&= \int_0^\infty \|(-\Delta_x)^{1/2}\mathrm{e}^{-t(-\Delta_x)^{1/2}} f\|_2^2 + \|\nabla_x \mathrm{e}^{-t(-\Delta_x)^{1/2}} f\|_2^2 \,\mathrm{d}t \\
&\simeq \int_0^\infty \|(-\Delta_x)^{1/2}\mathrm{e}^{-t(-\Delta_x)^{1/2}} f\|_2^2 \,\mathrm{d}t \qquad (16.2)\\
&= \int_0^\infty \|(-t^2\Delta_x)^{1/4}\mathrm{e}^{-(-t^2\Delta_x)^{1/2}} (-\Delta_x)^{1/4} f\|_2^2 \,\frac{\mathrm{d}t}{t} \\
&\simeq \|(-\Delta_x)^{1/4} f\|_2^2 \\
&\simeq \|f\|_{\dot{\mathrm{H}}^{1/2,2}}^2,
\end{aligned}$$

where the fourth step is by McIntosh's theorem. □

We also obtain the usual characterization of the subspace with trace zero at the boundary.

Lemma 16.2. *The subspace*

$$\dot{\mathrm{W}}_0^{1,2}(\mathbb{R}^{1+n}_+) \coloneqq \{u \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+) : u(0,\cdot) = 0 \text{ in } \dot{\mathrm{H}}^{1/2,2}\}$$

coincides with the closure of $\mathrm{C}_0^\infty(\mathbb{R}^{1+n}_+)$ *in* $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$.

Proof. Since the restriction $R : \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+) \to \dot{\mathrm{H}}^{1/2,2}$ to $t=0$ is bounded, $\dot{\mathrm{W}}_0^{1,2}(\mathbb{R}^{1+n}_+)$ is a closed subspace and it contains $\mathrm{C}_0^\infty(\mathbb{R}^{1+n}_+)$.

Conversely, let $u \in \dot{\mathrm{W}}_0^{1,2}(\mathbb{R}^{1+n}_+)$. Let $E : \dot{\mathrm{H}}^{1/2,2} \to \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ be the extension operator from the proof of Lemma 16.1. We pick a sequence $(u_k) \subseteq \mathrm{C}_0^\infty(\mathbb{R}^{1+n})$ with $u_k \to u$ in $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ as $k \to \infty$ and set $v_k \coloneqq (1-ER)u_k$. Then $Rv_k = 0$ and $v_k \to u$ in $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$. Therefore it suffices to approximate each v_k by $\mathrm{C}_0^\infty(\mathbb{R}^{1+n}_+)$-functions. In fact, it suffices to find approximants with compact support in $\mathbb{R}^{1+n}_+$ since then we can conclude via convolution with smooth kernels.

To this end, we note that $Ru_k \in \mathrm{L}^2$ together with the explicit construction of E implies $v_k \in \mathrm{C}_0([0,\infty);\mathrm{L}^2)$ with $v_k(0,\cdot)=0$. Extending v_k to $\mathbb{R}^{1+n}$ by 0 and using the L^2-continuity of the translation in the t-direction, we obtain approximants w_k with the same properties that have their support in $\mathbb{R}^{1+n}_+$. Now, we take $\eta \in \mathrm{C}_0^\infty(\mathbb{R}^{1+n})$ with $\eta(0,0)=1$ and set $\eta_\varepsilon(t,x) \coloneqq \eta(\varepsilon t, \varepsilon x)$. We can bound

$$\|\nabla(\eta_\varepsilon w_k) - \nabla w_k\|_2 \lesssim \|(1-\eta_\varepsilon)\nabla w_k\|_2 + \varepsilon^{\frac{1}{2}} \|w_k\|_{\mathrm{L}^\infty((0,\infty);\mathrm{L}^2)}.$$

In the limit as $\varepsilon \to 0$ the first term on the right vanishes by dominated convergence, whereas the second one vanishes thanks to the additional information $w_k \in \mathrm{L}^\infty((0,\infty);\mathrm{L}^2)$. □

We can now use the Lax–Milgram lemma to prove the following well-posedness result. Neither the block structure of A nor its t-independence are needed in the argument. We call u the *energy solution* to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ with Dirichlet data f.

Proposition 16.3. *For all $f \in \dot{\mathrm{H}}^{1/2,2}$ there exists a unique solution u (modulo constants) to the problem*

$$\begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \nabla u \in \mathrm{L}^2(\mathbb{R}^{1+n}_+), & \\ u(0,\cdot) = f & (\text{in } \dot{\mathrm{H}}^{1/2,2}). \end{cases}$$

Moreover, $\|\nabla u\|_2 \lesssim \|f\|_{\dot{\mathrm{H}}^{1/2,2}}$ *and* $\lim_{t\to\infty} u(t,\cdot) = 0$ *in* $\dot{\mathrm{H}}^{1/2,2}$.

Proof. If u is any solution, then we obtain by density and Lemma 16.2 that

$$\iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi} \,\mathrm{d}t\mathrm{d}x = 0 \quad (\phi \in \mathrm{W}^{1,2}_0(\mathbb{R}^{1+n}_+)).$$

Since A is elliptic, $u \in \mathrm{W}^{1,2}_0(\mathbb{R}^{1+n}_+)$ implies $\nabla u = 0$. Hence, solutions are unique modulo constants. In order to construct a solution, let $v \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ be an extension of f as in Lemma 16.1. By the Lax–Milgram lemma, there exists $w \in \mathrm{W}^{1,2}_0(\mathbb{R}^{1+n}_+)$ solving

$$\iint_{\mathbb{R}^{1+n}_+} A\nabla w \cdot \overline{\nabla \phi} \,\mathrm{d}t\mathrm{d}x = -\iint_{\mathbb{R}^{1+n}_+} A\nabla v \cdot \overline{\nabla \phi} \,\mathrm{d}t\mathrm{d}x \quad (\phi \in \mathrm{W}^{1,2}_0(\mathbb{R}^{1+n}_+)).$$

Hence, $u := v + w$ is a solution to the given problem and Lemma 16.1 yields the limit at $t = \infty$ as well as the bound

$$\|\nabla u\|_2 \leq \|\nabla v\|_2 + \|\nabla w\|_2 \lesssim \|\nabla v\|_2 \simeq \|f\|_{\dot{\mathrm{H}}^{1/2,2}}. \qquad \square$$

16.2. **Semigroup solutions.** In the specific situations of coefficients in block form, we can also use the Poisson semigroup for L to construct weak solutions. Here, the natural boundary space is L^2 rather than $\dot{\mathrm{H}}^{1/2,2}$.

Proposition 16.4. *Let $f \in \mathrm{L}^2$. Then $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$ is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ of class $\mathrm{C}_0([0,\infty);\mathrm{L}^2) \cap \mathrm{C}^\infty((0,\infty);\mathrm{L}^2)$ with $u(0,\cdot) = f$.*

Proof. The regularity in t follows directly from the functional calculus. In particular, $u(t,\cdot)$ is in the domain of L for every $t > 0$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2} u(t,\cdot) = Lu(t,\cdot)$. Since a is bounded and independent of t, the function au has the same properties and we have $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(au(t,\cdot)) = aLu(t,\cdot)$. Let now $\phi \in \mathrm{C}^\infty_0(\mathbb{R}^{1+n}_+)$. For any $t > 0$, the Lax–Milgram interpretation of aL in (3.5) yields

$$\int_{\mathbb{R}^n} \frac{\mathrm{d}^2}{\mathrm{d}t^2}(au(t,\cdot)) \cdot \overline{\phi(t,\cdot)} \,\mathrm{d}x = \int_{\mathbb{R}^n} d\nabla_x u(t,\cdot) \cdot \overline{\nabla_x \phi(t,\cdot)} \,\mathrm{d}x$$

and the claim follows by integrating both sides in t and then integrating by parts in t on the left-hand side. $\square$

We have the following *compatibility* between semigroup and energy solutions. This could be deduced from more general results in [14] but in the block situation there is a particularly simple proof.

Proposition 16.5. *If* $f \in \dot{\mathrm{H}}^{1/2,2} \cap \mathrm{L}^2$*, then* $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$ *is the energy solution to* $\mathcal{L}u = 0$ *in* $\mathbb{R}^{1+n}_+$ *with Dirichlet data* f*.*

Proof. We already know that u is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ with $u(0,\cdot) = f$ in the sense of $\mathrm{C}_0([0,\infty);\mathrm{L}^2)$. Furthermore, $\nabla u \in \mathrm{L}^2(\mathbb{R}^{1+n}_+)$ follows by a literal repetition of the argument in (16.2), replacing $-\Delta_x$ by L at each occurrence. In fact, this is why we have justified (16.2) by abstract arguments instead of using the Fourier transform. □

16.3. **Interior estimates.** We continue with the standard interior estimates. All this is well-known but precise references for systems with our ellipticity assumption are hard to find. One is [27, Cor. 22], where even systems of higher order are treated, but for the reader's convenience we include the simple arguments in the second-order case. Again the block structure of A and its t-independence are not needed for this part.

We call $W \subseteq \mathbb{R}^{1+n}_+$ a *cylinder of radius* r if $W = I \times B$, where $I \subseteq (0,\infty)$ is an interval of length r and $B \subseteq \mathbb{R}^n$ ball of radius r (or a cube of sidelength r). As usual, we write αW for the concentrically scaled version of W.

Lemma 16.6 (Caccioppoli)**.** *Let* $O \subseteq \mathbb{R}^{1+n}$ *be open,* $g \in \mathrm{L}^2_{\mathrm{loc}}(O)$ *and* u *a weak solution to* $\mathcal{L}u = g$ *in* O*. Let* $W \subseteq \mathbb{R}^{1+n}$ *a cylinder of radius* r *and* $\alpha > 1$ *be such that* $\alpha W \Subset O$*. Then there is a constant* C *depending on dimensions, ellipticity and* α*, such that*

$$\iint_W |\nabla u|^2 \,\mathrm{d}s\mathrm{d}y \leq C \iint_{\alpha W} r^{-2}|u|^2 + r^2|g|^2 \,\mathrm{d}s\mathrm{d}y.$$

Proof. Fix $\eta \in \mathrm{C}_0^\infty(\mathbb{R}^{1+n})$ with $\mathbf{1}_W \leq \eta \leq \mathbf{1}_{\alpha W}$ and $|\nabla \eta| \leq c_n r^{-1}$ for a purely dimensional constant c_n. We write $\langle \cdot\,,\cdot \rangle$ for the inner product on $\mathrm{L}^2(\mathbb{R}^{1+n})$. By ellipticity and multiple applications of the product rule, we have

$$\begin{aligned}\lambda \|\nabla(\eta u)\|_2^2 &\leq |\langle A\nabla(\eta u), \nabla(\eta u)\rangle| \\ &\leq |\langle A\nabla u, \nabla(\eta^2 u)\rangle| + |\langle \eta A \nabla u, u \otimes \nabla \eta\rangle| \\ &\quad + |\langle A(u \otimes \nabla\eta), \nabla(\eta u)\rangle| \\ &=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,\end{aligned}$$

where our notation is $\nabla(\eta u) := \eta \nabla u + u \otimes \nabla \eta$ in the sense prescribed by the product rule. By the equation for u, the Cauchy–Schwarz inequality and the elementary bound $xy \leq \frac{\varepsilon}{2}x^2 + \frac{1}{2\varepsilon}y^2$ for positive numbers x, y, ε, we have

$$|\,\mathrm{I}_1| = |\langle g, \eta^2 u\rangle| \leq \frac{1}{2r^2}\|\eta u\|_2^2 + \frac{r^2}{2}\|\eta g\|_2^2.$$

Similarly, we get

$$\begin{aligned}\mathrm{I}_2 + \mathrm{I}_3 &\leq \frac{\lambda}{4}\|\eta\nabla u\|_2^2 + \frac{\lambda}{4}\|\nabla(\eta u)\|_2^2 + C\|u \otimes \nabla\eta\|_2^2 \\ &\leq \frac{3\lambda}{4}\|\nabla(\eta u)\|_2^2 + \Big(C + \frac{\lambda}{2}\Big)\|u \otimes \nabla\eta\|_2^2,\end{aligned}$$

where C depends on dimensions and ellipticity. Rearranging terms leads to

$$\frac{\lambda}{4}\|\nabla(\eta u)\|_2^2 \leq \Big(C + \frac{\lambda}{2}\Big)\|u \otimes \nabla\eta\|_2^2 + \frac{1}{2r^2}\|\eta u\|_2^2 + \frac{r^2}{2}\|\eta g\|_2^2$$

and by choice of η we are done. □

Lemma 16.7 (Reverse Hölder)**.** *Let u be a weak solution to $\mathcal{L}u = 0$ in an open set $O \subseteq \mathbb{R}^{1+n}$ and let $\alpha > 1$. There is a constant C depending on dimensions, ellipticity and α, such that for all cylinders W with $\alpha W \Subset O$ it follows that*

$$\Big(⨌_W |\nabla u|^2 \,\mathrm{d}s\mathrm{d}y\Big)^{1/2} \leq C ⨌_{\alpha W} |\nabla u| \,\mathrm{d}s\mathrm{d}y.$$

Moreover, with $q := \frac{2(n+1)}{n-1}$ in dimension $n \geq 2$ and $q \in (2, \frac{2(n+1)}{n-1})$ arbitrary in dimension $n = 1$, it follows that

$$\Big(⨌_W |u|^q \,\mathrm{d}s\mathrm{d}y\Big)^{1/q} \leq C ⨌_{\alpha W} |u| \,\mathrm{d}s\mathrm{d}y,$$

where C also depends on q.

Proof. We begin with the first inequality. Let $c := ⨌_{\alpha W} u$ and $p := \frac{2(n+1)}{n+3}$, the lower Sobolev conjugate of 2 in dimension $n+1$. We apply the Caccioppoli inequality to $u - c$ and bound the right-hand side by the Sobolev–Poincaré inequality in order to give:

$$\Big(⨌_W |\nabla u|^2 \,\mathrm{d}s\mathrm{d}y\Big)^{1/2} \leq C\Big(⨌_{\alpha W} |\nabla u|^p \,\mathrm{d}s\mathrm{d}y\Big)^{1/p}.$$

As we have $p < 2$, this is a reverse Hölder inequality for ∇u. It remains to lower the exponent to $p = 1$ but this is always possible by a general feature of such inequalities, see [65, Thm. 2]. Strictly speaking, this reference is for $W = I \times B$ with B a cube and the case of a ball then follows by a straightforward covering argument.

For the second inequality we let $c := ⨌_W u$ and note that $\frac{2(n+1)}{n-1}$ is the upper Sobolev conjugate of 2 in dimension $n+1$. It follows that

$$\Big(⨌_W |u|^q \,\mathrm{d}s\mathrm{d}y\Big)^{1/q} \leq \Big(⨌_W |u-c|^q \,\mathrm{d}s\mathrm{d}y\Big)^{1/q} + ⨌_W |u| \,\mathrm{d}s\mathrm{d}y$$

$$\leq C\bigg(\iint_{\alpha W} |u|^2 \,\mathrm{d}s\mathrm{d}y\bigg)^{1/2},$$

where the second step follows again by combining the Sobolev–Poincaré inequality with the Caccioppoli inequality. The exponent on the right-hand side can be lowered as before. □

We close with a simple but important approximation result for weak solutions.

Lemma 16.8. *Let (u_k) be a sequence of weak solutions to $\mathcal{L}u_k = 0$ in an open set $O \subseteq \mathbb{R}^{1+n}$ that converges to u in $\mathrm{L}^1_{\mathrm{loc}}(O)$. Then u is a weak solution to (16.1) in O and (u_k) tends to u in $\mathrm{W}^{1,2}_{\mathrm{loc}}(O)$.*

Proof. The Cauchy property in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ follows by applying the reverse Hölder and the Caccioppoli inequality to $u_k - u_j$ on arbitrary admissible cylinders. Hence, we can pass to the limit in k in the weak formulation of the equation for u_k. □

Corollary 16.9. *If u is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$, then so is $\partial_t u$. In particular, u is of class $\mathrm{C}^\infty((0,\infty); \mathrm{L}^2_{\mathrm{loc}})$.*

Proof. For $\varepsilon > 0$ and $h \in (-\varepsilon, \varepsilon)$ define $v_h(t,x) := \frac{1}{h}(u(t+h,x) - u(t,x))$ in $\mathbb{R}^{n+1}_{+,\varepsilon} := \{(s,y) \in \mathbb{R}^{1+n} : s > \varepsilon\}$. All v_h are weak solutions in $\mathbb{R}^{n+1}_{+,\varepsilon}$ since the coefficients of $\mathcal{L}$ are independent of t and we have $v_h \to \partial_t u$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_{+,\varepsilon})$ as $h \to 0$. By the preceding lemma, $\partial_t u$ is a weak solution in $\mathbb{R}^{1+n}_+$, so that in particular $\partial_t^2 u \in \mathrm{L}^2_{\mathrm{loc}}((0,\infty); \mathrm{L}^2_{\mathrm{loc}})$. By iteration the same applies to $\partial_t^k u$ for any $k \in \mathbb{N}$ and the claimed regularity follows by (one-dimensional) Sobolev embeddings. □

17. Existence in H^p Dirichlet and Regularity problems

In this section we establish the existence part in our main results on the Dirichlet and Regularity problems with H^p-data, Theorems 1.1 and 1.2. When the data f additionally belongs to L^2, the (eventually unique) solution is given by the Poisson semigroup. Hence, we proceed in two steps: First, we establish the required semigroup estimates for data $f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$ and $f \in \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$, respectively. Second, we obtain existence of a solution by a density argument for the full class of data.

17.1. **Estimates towards the Dirichlet problem.** We begin with the square function bound.

Proposition 17.1. *Let $p_-(L) < p < p_+(L)^*$. If $f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$, then $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$ satisfies*

$$\|S(t\nabla u)\|_p \simeq \|af\|_{\mathrm{H}^p}.$$

Proof. We organize the argument in three steps. For $p \leq 2$ we will be able to use Hardy space theory 'off-the-shelf' but for $p > 2$ different arguments on the level of the second-order equation for u are needed since p might lie outside of $\mathcal{H}(L)$.

Step 1: The case $p_-(L) < p \leq 2$. We have

$$t\partial_t u = -tL^{1/2}\mathrm{e}^{-tL^{1/2}} f =: \psi(t^2 L)f,$$

and, recalling (3.2) - (3.4),

$$t\nabla_x u = t\nabla_x a^{-1}\mathrm{e}^{-t\widetilde{L}^{1/2}}(af) = (-tDB\mathrm{e}^{-t[DB]}g)_\parallel =: (\varphi(tDB)g)_\parallel$$

where $g = [af, 0]^\top$. We recall from Proposition 8.2 and the corresponding result for sectorial operators in Section 8.2 that $\psi \in \Psi_{1/2}^\infty$ and $\varphi \in \Psi_1^\infty$ are admissible auxiliary functions for $\mathbb{H}_L^p$ and $\mathbb{H}_{DB}^p$, respectively. By Theorem 9.6 we have $p \in \mathcal{H}(L) \cap \mathcal{H}(DB)$ and hence we get as required

$$\begin{aligned}
\|af\|_{\mathrm{H}^p} &\simeq \|f\|_{\mathbb{H}_L^p} \\
&\simeq \|S(t\partial_t u)\|_p \\
&\leq \|S(t\nabla u)\|_p \\
&\lesssim \|f\|_{\mathbb{H}_L^p} + \|g\|_{\mathbb{H}_{DB}^p} \\
&\simeq \|af\|_{\mathrm{H}^p} + \|g\|_{\mathrm{H}^p} \\
&\lesssim \|af\|_{\mathrm{H}^p}.
\end{aligned}$$

Step 2: Upper bound for $2 < p < p_+(L)^*$. Consider the auxiliary function $\phi(z) := \mathrm{e}^{-\sqrt{z}} - (1+z)^{-2}$. Then $\phi \in \Psi_{1/2}^2$ on any sector. Differentiating the resolvent twice, we find that $v := \phi(t^2 L)f$ solves the following equation in $\mathbb{R}_+^{1+n}$ in the weak sense:

$$\begin{aligned}
&(a\partial_t^2 + \operatorname{div}_x d\nabla_x)v \\
&\quad = 4aL(1+t^2L)^{-3}f - 24at^2L^2(1+t^2L)^{-4}f - aL(1+t^2L)^{-2}f \\
&\quad =: t^{-2}a\psi(t^2L)f,
\end{aligned}$$

where $\psi \in \Psi_1^1$ on any sector. For $x \in \mathbb{R}^n$ and $t > 0$ consider Whitney boxes $W(t,x) := (t, 2t) \times B(x, 2t)$ and $\widetilde{W}(t,x) := (t/2, 4t) \times B(x, 4t)$. The Caccioppoli inequality yields

$$\iint_{W(t,x)} |s\nabla v|^2 \,\mathrm{d}s\mathrm{d}y \lesssim \iint_{\widetilde{W}(t,x)} |v|^2 + |\psi(s^2L)f|^2 \,\mathrm{d}s\mathrm{d}y. \tag{17.1}$$

Summing up these estimates for $t = 2^{-k}$, $k \in \mathbb{Z}$, leads to

$$\iint_{|x-y|<s} |s\nabla v|^2 \frac{\mathrm{d}s\mathrm{d}y}{s^{1+n}} \lesssim \iint_{|x-y|<8s} |v|^2 + |\psi(s^2L)f|^2 \frac{\mathrm{d}s\mathrm{d}y}{s^{1+n}},$$

where we have used that at most 3 of the enlarged boxes $\widetilde{W}(2^{-k}, x)$ overlap in order to get the term on the right. By definition of v we conclude

$$\|S(t\nabla u)\|_p \lesssim \|S_{\phi,L}f\|_p + \|S_{\psi,L}f\|_p + \|S(t\nabla(1+t^2L)^{-2}f)\|_p, \tag{17.2}$$

where as usual $S_{\phi,L}f$ denotes the square function of $\phi(t^2L)f(x)$.

Since $\phi \in \Psi^2_{1/2}$ and $\psi \in \Psi^1_1$, Theorem 9.21 applies in our range of exponents and yields

$$\|S_{\phi,L}f\|_p + \|S_{\psi,L}f\|_p \lesssim \|f\|_p.$$

The analogous bound for the third square function in (17.2) is a consequence of Remark 9.8. Indeed, the family

$$\begin{aligned} t\nabla(1+t^2L)^{-2} &= \begin{bmatrix} -2t^2L(1+t^2L)^{-3} \\ t\nabla_x(1+t^2L)^{-2} \end{bmatrix} \\ &= \begin{bmatrix} 2((1+t^2L)^{-3}-(1+t^2L)^{-2}) \\ t\nabla_x(1+t^2L)^{-2} \end{bmatrix} \end{aligned}$$

satisfies L^2 off diagonal estimates of arbitrary large order by composition and we have for all $t > 0$

$$\|t\nabla(1+t^2L)^{-2}f\|_2^2 \simeq \|t^2L(1+t^2L)^{-3}f\|_2^2 + \|tL^{1/2}(1+t^2L)^{-2}f\|_2^2$$

by the solution of the Kato problem, so that the theorems of Fubini and McIntosh yield the L^2-bound

$$\|S(t\nabla(1+t^2L)^{-2}f)\|_2^2 \simeq \int_0^\infty \|t\nabla(1+t^2L)^{-2}f\|_2^2 \,\frac{\mathrm{d}t}{t} \simeq \|f\|_2^2.$$

Step 3: Lower bound for $2 < p < p_+(L)^*$. Introduce the adapted Laplacian $H := -(a^*)^{-1}\Delta_x$ and for $f \in \mathrm{L}^p \cap \mathrm{L}^2$ and $g \in \mathrm{L}^{p'} \cap \mathrm{L}^2$ set

$$\Phi : (0,\infty) \to \mathbb{C}, \quad \Phi(t) := \langle a\mathrm{e}^{-tL^{1/2}}f, \mathrm{e}^{-tH^{1/2}}g\rangle,$$

where $\langle\cdot\,,\cdot\rangle$ is the L^2 inner product. By the functional calculus on L^2, this is a smooth function and we have

$$\begin{aligned} \Phi'(t) &= -\langle aL^{1/2}\mathrm{e}^{-tL^{1/2}}f, \mathrm{e}^{-tH^{1/2}}g\rangle - \langle a\mathrm{e}^{-tL^{1/2}}f, H^{1/2}\mathrm{e}^{-tH^{1/2}}g\rangle \\ \Phi''(t) &= \langle aL\mathrm{e}^{-tL^{1/2}}f, \mathrm{e}^{-tH^{1/2}}g\rangle + 2\langle aL^{1/2}\mathrm{e}^{-tL^{1/2}}f, H^{1/2}\mathrm{e}^{-tH^{1/2}}g\rangle \\ &\quad + \langle a\mathrm{e}^{-tL^{1/2}}f, H\mathrm{e}^{-tH^{1/2}}g\rangle \\ &= \langle d\nabla_x\mathrm{e}^{-tL^{1/2}}f, \nabla_x\mathrm{e}^{-tH^{1/2}}g\rangle + 2\langle aL^{1/2}\mathrm{e}^{-tL^{1/2}}f, H^{1/2}\mathrm{e}^{-tH^{1/2}}g\rangle \\ &\quad + \langle\nabla_x\mathrm{e}^{-tL^{1/2}}f, \nabla_x\mathrm{e}^{-tH^{1/2}}g\rangle, \end{aligned}$$

as well as

$$\lim_{t\to\infty}\Phi(t) = \lim_{t\to\infty} t\Phi'(t) = \lim_{t\to 0} t\Phi'(t) = 0, \quad \lim_{t\to 0}\Phi(t) = \langle af, g\rangle.$$

Putting all together and integrating by parts twice in t, we obtain

$$\begin{aligned} \langle af, g\rangle &= \int_0^\infty t^2\Phi''(t)\,\frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \langle dt\nabla_x\mathrm{e}^{-tL^{1/2}}f, t\nabla_x\mathrm{e}^{-tH^{1/2}}g\rangle\,\frac{\mathrm{d}t}{t} \end{aligned}$$

$$+ 2\int_0^\infty \langle atL^{1/2}\mathrm{e}^{-tL^{1/2}}f, tH^{1/2}\mathrm{e}^{-tH^{1/2}}g\rangle \,\frac{\mathrm{d}t}{t}$$
$$+ \int_0^\infty \langle t\nabla_x\mathrm{e}^{-tL^{1/2}}f, t\nabla_x\mathrm{e}^{-tH^{1/2}}g\rangle \,\frac{\mathrm{d}t}{t}.$$

We regard the right-hand side as $\mathrm{T}^p - \mathrm{T}^{p'}$ duality pairings in order to give

$$\begin{aligned}|\langle af, g\rangle| &\lesssim \|S(t\nabla_x\mathrm{e}^{-tL^{1/2}}f)\|_p\|S(t\nabla_x\mathrm{e}^{-tH^{1/2}}g)\|_{p'}\\ &\quad + \|S(tL^{1/2}\mathrm{e}^{-tL^{1/2}}f)\|_p\|S(tH^{1/2}\mathrm{e}^{-tH^{1/2}}g)\|_{p'}\\ &\leq 2\|S(t\nabla\mathrm{e}^{-tL^{1/2}}f)\|_p\|S(t\nabla\mathrm{e}^{-tH^{1/2}}g)\|_{p'}.\end{aligned}$$

We know that $p_-(H) = 1_*$ from Corollary 6.10. Hence, Step 1 for H on $\mathrm{L}^{p'}$ yields $\|S(t\nabla\mathrm{e}^{-tH^{1/2}}g)\|_{p'} \lesssim \|g\|_{p'}$ and since $g \in \mathrm{L}^{p'} \cap \mathrm{L}^2$ was arbitrary, we conclude

$$\|af\|_p \lesssim \|S(t\nabla\mathrm{e}^{-tL^{1/2}}f)\|_p. \qquad \square$$

We turn to bounds for the non-tangential maximal function and begin by recalling the respective L^2-bound for our perturbed Dirac operators.

Theorem 17.2 ([22, Thm. 9.9])**.** *Let T be one of DB or BD. Then*

$$\|\widetilde{N}_*(\mathrm{e}^{-t[T]}f)\|_2 \simeq \|f\|_2 \quad (f \in \overline{\mathsf{R}(T)})$$

and for every $f \in \mathrm{L}^2$ the Whitney averages converge in the L^2-sense

$$\lim_{t\to 0} \iint_{W(t,x)} |\mathrm{e}^{-t[T]}f - f(x)|^2 \,\mathrm{d}s\mathrm{d}y = 0 \quad (a.e.\ x \in \mathbb{R}^n).$$

We remark that the result above for $T = BD$ is originally due to Rosén [83, Thm. 5.1].

Proposition 17.3. *Let $p_-(L) < p < p_+(L)^*$. If $f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$, then $u(t,x) \coloneqq \mathrm{e}^{-tL^{1/2}}f(x)$ satisfies*

$$\|\widetilde{N}_*(u)\|_p \simeq \|af\|_{\mathrm{H}^p}$$

and

$$\lim_{t\to 0} \iint_{W(t,x)} |u(s,y) - f(x)|^2 \,\mathrm{d}s\mathrm{d}y = 0 \quad (a.e.\ x \in \mathbb{R}^n).$$

Proof. We recall from (3.3) that L is incorporated in the matrix operator $(BD)^2$. Hence, we have

$$\mathrm{e}^{-t[BD]}\begin{bmatrix} f \\ 0\end{bmatrix} = \begin{bmatrix} u \\ 0\end{bmatrix}$$

and the claim for $p = 2$ as well as the convergence of averages follows from Theorem 17.2.

Step 1: Upper bound. If $p \in (p_-(L), 2)$, then according to Proposition 8.27 and Theorem 9.6 we have

$$\|\widetilde{N}_*(u)\|_p \lesssim \|f\|_{\mathbb{H}^p_L} \simeq \|af\|_{\mathrm{H}^p}.$$

If $p \in (2, p_+(L)^*)$, we first introduce $\psi(z) := \mathrm{e}^{-\sqrt{z}} - (1+z)^{-1}$ and split

$$u = v + w := \psi(t^2L)f + (1+t^2L)^{-1}f.$$

We have $\psi \in \Psi^1_{1/2}$ on any sector. Combining Lemma 8.26 and Theorem 9.21, we find that

$$\|\widetilde{N}_*(v)\|_p \lesssim \|S_{\psi,L}f\|_p \lesssim \|f\|_p.$$

As for w, we use that the resolvents satisfy off-diagonal estimates of arbitrarily large order. Consequently, Lemma 8.23 and the $\mathrm{L}^{p/2}$-bound for the Hardy–Littlewood maximal operator yield

$$\|\widetilde{N}_*(w)\|_p \le \|(\mathcal{M}(|f|^2))^{1/2}\|_p \lesssim \|f\|_p.$$

Step 2: Lower bound for $p > 1$. The convergence of Whitney averages implies $\widetilde{N}_*(u) \ge f$ a.e. on $\mathbb{R}^n$ and $\|\widetilde{N}_*(u)\|_p \ge \|f\|_p$ follows.

Step 3: Lower bound for $p_-(L) < p \le 1$. We calculate the H^p-norm of af using the Fefferman–Stein characterization of H^p. This argument works for all $p \in ({}^n/_{(n+1)}, 1]$, not only $p \in (p_-(L), 1]$.

Fix $\phi \in \mathrm{C}^\infty_0(\mathbb{R}^n; \mathbb{R})$ with support in $B(0,1)$ and $\int_{\mathbb{R}^n} \phi = 1$ and let $\phi_t(y) := t^{-n}\phi({}^y/_t)$. Then a function $h \in \mathrm{L}^2$ belongs to H^p if and only if the maximal function

$$(\mathcal{M}_\phi\, h)(x) := \sup_{t>0} |h * \phi_t|(x) \quad (x \in \mathbb{R}^n)$$

is in L^p and in this case $\|h\|_{\mathrm{H}^p} \simeq \|\mathcal{M}_\phi\, h\|_p$, see e.g. [51, Thm. 6.4.4]

Temporarily fix $t > 0$ and $x \in \mathbb{R}^n$. Let $\chi : [0,\infty) \to [0,1]$ be smooth with $\mathbf{1}_{[0,1/2]} \le \chi \le \mathbf{1}_{[0,2)}$, set $\chi_t(s) := \chi({}^s/_t)$ and introduce $\Phi(s,y) := \phi_t(x-y)\chi_t(s)$. The functional calculus on L^2 and the compact support of Φ justify writing

$$\begin{aligned}(af * \phi_t)(x) &= \int_{\mathbb{R}^n} (af)(y)\phi_t(x-y)\,\mathrm{d}y \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} -\bigg(\int_\varepsilon^\infty \partial_s(\Phi a u)\,\mathrm{d}s\bigg)\mathrm{d}y.\end{aligned}$$

For $\varepsilon < {}^t/_2$ we expand, integrate by parts and use $a\partial_s^2 u = Lu$, to give

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_\varepsilon^\infty \partial_s(\Phi a u)\,\mathrm{d}s\mathrm{d}y \\ &\quad= \int_{\mathbb{R}^n}\int_\varepsilon^\infty (\partial_s\Phi)au + \Phi a\partial_s u\,\mathrm{d}s\mathrm{d}y \\ &\quad= \int_{\mathbb{R}^n}\int_\varepsilon^\infty (\partial_s\Phi)au - (\partial_s\Phi)sa\partial_s u - \Phi sa\partial_s^2 u\,\mathrm{d}s\mathrm{d}y\end{aligned}$$

$$+\int_{\mathbb{R}^n} \Phi(\varepsilon, y)\varepsilon a(y)\partial_s u(\varepsilon, y)\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^n}\int_\varepsilon^\infty (\partial_s\Phi)au - (\partial_s\Phi)as\partial_s u + \nabla_y\Phi\cdot sd\nabla_y u\,\mathrm{d}s\mathrm{d}y$$
$$+\int_{\mathbb{R}^n} \phi_t(x-y)\varepsilon a(y)\partial_s u(\varepsilon, y)\,\mathrm{d}y.$$

By the functional calculus for L we have as a limit in L^2,

$$\lim_{\varepsilon\to 0} \varepsilon a\partial_s u(\varepsilon,\cdot) = -\lim_{\varepsilon\to 0}\varepsilon a L^{1/2}\mathrm{e}^{-\varepsilon L^{1/2}} f = 0.$$

By Young's convolution inequality we get $\phi_t * (\varepsilon a\partial_s u(\varepsilon,\cdot)) \to 0$ uniformly on $\mathbb{R}^n$ as $\varepsilon\to 0$. Altogether,

$$\begin{aligned}
&|(af*\phi_t)(x)| \\
&\le \iint_{\mathbb{R}^{1+n}_+} |(\partial_s\Phi)au|\,\mathrm{d}s\mathrm{d}y + \iint_{\mathbb{R}^{1+n}_+} |(\partial_s\Phi)as\partial_s u|\,\mathrm{d}s\mathrm{d}y \\
&\quad + \iint_{\mathbb{R}^{1+n}_+} |\nabla_y\Phi\cdot sd\nabla_y u|\,\mathrm{d}s\mathrm{d}y \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{17.3}$$

Since $\partial_s\Phi$ is bounded by t^{-1-n} and supported in $W(t,x)$, we get

$$|\,\mathrm{I}\,| + |\,\mathrm{II}\,| \lesssim \widetilde{N}_*(u)(x) + \widetilde{N}_*(t\partial_t u)(x).$$

As for III, we get

$$|\,\mathrm{III}\,| \lesssim t^{-1-n}\iint_{\mathbb{R}^{1+n}_+} |\mathbf{1}_{(0,2t)\times B(x,t)} s\nabla_x u|\,\mathrm{d}s\mathrm{d}y,$$

so that Lemma A.3 applied to $F := |\mathbf{1}_{(0,2t)\times B(x,t)} s\nabla_x u|$ with $r=1$ and $p = {}^{n}/_{(n+1)}$ yields

$$|\,\mathrm{III}\,| \lesssim t^{-1-n}\|\widetilde{N}_*(F)\|_{\frac{n}{n+1}}.$$

If a Whitney ball $W(r,z)$ intersects the support of F at some $(s,y)\in\mathbb{R}^{1+n}_+$, then

$$|x-z| \le |x-y| + |y-z| \le t + r \le t + 2s \le 5t,$$

which means that $\widetilde{N}_*(F)$ has support in $B(x,5t)$. Thus, we have

$$\begin{aligned}
|\,\mathrm{III}\,| &\lesssim \Big(t^{-n}\int_{B(x,5t)} |\widetilde{N}_*(F)|^{\frac{n}{n+1}}\Big)^{\frac{n+1}{n}} \\
&\lesssim \mathcal{M}(|\widetilde{N}_*(t\nabla_x u)|^{\frac{n}{n+1}})^{\frac{n+1}{n}}(x).
\end{aligned}$$

Going back to (17.3) and taking the supremum in t, leads us to

$$\mathcal{M}_\phi(af) \lesssim \widetilde{N}_*(u) + \widetilde{N}_*(t\partial_t u) + \mathcal{M}(|\widetilde{N}_*(t\nabla_x u)|^{\frac{n}{n+1}})^{\frac{n+1}{n}}.$$

By assumption we have $p > {}^{n}/_{(n+1)}$. Hence, $\mathcal{M}$ is bounded on $\mathrm{L}^{p(n+1)/n}$ and it follows that

$$\|af\|_{\mathrm{H}^p} \simeq \| \mathcal{M}_\phi(af)\|_p \lesssim \|\widetilde{N}_*(u)\|_p + \|\widetilde{N}_*(t\nabla u)\|_p \lesssim \|\widetilde{N}_*(u)\|_p,$$

where the final step is due to Caccioppoli's inequality. □

Finally, we establish uniform bounds and strong continuity at $t = 0$.

Proposition 17.4. *Let $p_-(L) < p < p_+(L)$. If $f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$ and $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$, then au is of class*

$$\mathrm{C}_0([0,\infty); \mathrm{H}^p) \cap \mathrm{C}^\infty((0,\infty); \mathrm{H}^p)$$

and satisfies

$$\sup_{t>0} \|au(t,\cdot)\|_{\mathrm{H}^p} \simeq \|af\|_{\mathrm{H}^p}$$

and for all $k \in \mathbb{N}$,

$$\sup_{t>0} \|t^{\frac{k}{2}} \partial_t^k (au(t,\cdot))\|_{\mathrm{H}^p} \lesssim (\tfrac{k}{2})^{\frac{k}{2}} \mathrm{e}^{-k} \|af\|_{\mathrm{H}^p}.$$

Proof. According to Theorem 9.6 we have $a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2) = \mathbb{H}^p_L$ with equivalent p-quasinorms $\|f\|_{\mathbb{H}^p_L} \simeq \|af\|_{\mathrm{H}^p}$.

The upper bounds for u and $\partial_t^k u$ now follow immediately from the bounded H^∞-calculus on $\mathbb{H}^p_L$, see Section 8.2. Likewise, Proposition 8.13 provides the limits $au(t,\cdot) \to af$ as $t \to 0$ and $au(t,\cdot) \to 0$ as $t \to \infty$ in H^p and the limit at $t = 0$ implies lower bound for u. □

For exponents $p \geq p_+(L)$, the space $\mathbb{H}^p_L$ does not equal $a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$ and the previous argument breaks down, see Theorem 11.3. However, using off-diagonal estimates, we can still obtain the continuity at the boundary $t = 0$ with values in $\mathrm{L}^2_{\mathrm{loc}}$ if $p_+(L) \leq p < p_+(L)^*$.

Lemma 17.5. *If $p_+(L) \leq p < p_+(L)^*$, then for all $f \in \mathrm{L}^p \cap \mathrm{L}^2$, all balls $B \subseteq \mathbb{R}^n$ and all $t > 0$,*

$$\|\mathrm{e}^{-tL^{1/2}} f - f\|_{\mathrm{L}^2(B)} \lesssim r(B)^{\frac{n}{2} - \frac{n}{p} - 1}(r(B) + t)\|f\|_p.$$

Proof. We can pick q such that $2 \leq q < p_+(L)$ and ${}^1/_q - {}^1/_p < {}^1/_n$. We split $f = \sum_{j \geq 1} f_j$, where $f_j := \mathbf{1}_{C_j(B)} f$, and obtain from Hölder's inequality that

$$\begin{aligned}
\|\mathrm{e}^{-tL^{1/2}} f - f\|_{\mathrm{L}^2(B)} &\leq \|\mathrm{e}^{-tL^{1/2}} f_1 - f_1\|_{\mathrm{L}^2(B)} \\
&\quad + r(B)^{\frac{n}{2} - \frac{n}{q}} \sum_{j \geq 2} \|\mathrm{e}^{-tL^{1/2}} f_j\|_{\mathrm{L}^q(B)} \\
&\leq r(B)^{\frac{n}{2} - \frac{n}{p}} \|f\|_p \\
&\quad + r(B)^{\frac{n}{2} - \frac{n}{q}} \sum_{j \geq 2} \|\mathrm{e}^{-tL^{1/2}} f_j\|_{\mathrm{L}^q(B)}.
\end{aligned}$$

Since the Poisson semigroup satisfies L^q off-diagonal estimates of order 1, see Corollary 4.17, we can bound the sum in j by

$$\sum_{j\geq 2} t2^{-j} r(B)^{-1} \|f_j\|_{\mathrm{L}^q(B)} \lesssim t r(B)^{\frac{n}{q}-\frac{n}{p}-1} \sum_{j\geq 2} 2^{j(\frac{n}{q}-\frac{n}{p}-1)} \|f\|_p,$$

where the right-hand side is finite by choice of q. The claim follows. □

17.2. **Estimates towards the Regularity problem.** We begin again with the square function bounds.

Proposition 17.6. *Let $(p_-(L)_* \vee 1_*) < p < q_+(L)$. If $f \in \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$, then $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$ satisfies*

$$\|S(t\nabla\partial_t u)\|_p \simeq \|\nabla_x f\|_{\mathrm{H}^p}.$$

Proof. Let us first interpret the exponents. The identification Theorem 9.6 tells us that we have $\mathbb{H}^{1,p}_L = \dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2$ with equivalent p-quasinorms and then $\|g\|_{\mathbb{H}^p_{\widetilde{M}}} \simeq \|g\|_{\mathrm{H}^p}$ for all $g \in \mathbb{H}^p_{\widetilde{M}}$ follows from Figure 8. The square function we have to control contains

$$t\nabla_x\partial_t u = -t\nabla_x L^{1/2}\mathrm{e}^{-tL^{1/2}} f = -t\widetilde{M}^{1/2}\mathrm{e}^{-t\widetilde{M}^{1/2}}\nabla_x f =: \psi(t^2\widetilde{M})\nabla_x f,$$

where $\psi \in \Psi^\infty_{1/2}$ on any sector and we used an intertwining relation for the functional calculus on L^2, as well as

$$t\partial_t^2 u = -\psi(t^2L)L^{1/2}f = t^{-1}(t^2L\mathrm{e}^{-tL^{1/2}})f =: t^{-1}\phi(t^2L)f,$$

where $\phi \in \Psi^\infty_1$ on any sector.

If $p \leq 2$, then ϕ and ψ are admissible auxiliary functions for defining $\mathbb{H}^{1,p}_L$ and $\mathbb{H}^p_{\widetilde{M}}$, respectively. Thus, we get

$$\begin{aligned}\|S(t\nabla\partial_t u)\|_p &\simeq \|S(t^{-1}\phi(t^2L)f)\|_p + \|S(\psi(t^2\widetilde{M})\nabla_x f)\|_p\\ &\simeq \|f\|_{\mathbb{H}^{1,p}_L} + \|\nabla_x f\|_{\mathbb{H}^p_{\widetilde{M}}}\\ &\simeq \|\nabla_x f\|_{\mathrm{H}^p}\end{aligned}$$

right away. If $p \geq 2$, then Proposition 9.20 applies to $\widetilde{M}$ with auxiliary function ψ and $q = p$. The same holds for L since from Theorem 9.6 and the general bound $q_+(L) < p_+(L)$ in Theorem 6.2 we obtain $\mathbb{H}^p_L = \mathrm{L}^p \cap \mathrm{L}^2$ with equivalent p-norms. Consequently, we have

$$\begin{aligned}\|S(t\nabla\partial_t u)\|_p &\simeq \|S(\psi(t^2L)L^{1/2}f)\|_p + \|S(\psi(t^2\widetilde{M})\nabla_x f)\|_p\\ &\lesssim \|L^{1/2}f\|_p + \|\nabla_x f\|_p\\ &\simeq \|\nabla_x f\|_p,\end{aligned}$$

where the final equivalence is due to Theorem 11.1. As for the lower bound, we note that $t\partial_t^2 u = -t\partial_t v$ with $v := \mathrm{e}^{-tL^{1/2}}(L^{1/2}f)$ and $L^{1/2}f \in \mathrm{L}^p \cap \mathrm{L}^2$. Hence, we can apply Proposition 17.1 in order to get

$$\|S(t\nabla\partial_t u)\|_p \geq \|S(t\partial_t v)\|_p \simeq \|L^{1/2}f\|_p \simeq \|\nabla_x f\|_p$$

as required. □

We continue with the non-tangential maximal function bounds.

Proposition 17.7. *Let* $(p_-(L)_* \vee 1_*) < p < q_+(L)$. *If* $f \in \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$, *then* $u(t,x) \coloneqq \mathrm{e}^{-tL^{1/2}} f(x)$ *satisfies*

$$\|\widetilde{N}_*(\nabla u)\|_p \simeq \|\nabla_x f\|_{\mathrm{H}^p}$$

and

$$\lim_{t\to 0} \iint_{W(t,x)} \left| \begin{bmatrix} a\partial_t u \\ \nabla_x u \end{bmatrix} - \begin{bmatrix} -aL^{1/2} f(x) \\ \nabla_x f(x) \end{bmatrix} \right|^2 \mathrm{d}s\mathrm{d}y = 0 \quad (a.e.\ x \in \mathbb{R}^n).$$

Proof. We use the intertwining property to write $\nabla_x u = \mathrm{e}^{-t\widetilde{M}^{1/2}} \nabla_x f$. Moreover, we have $\partial_t u = \mathrm{e}^{-tL^{1/2}}(-L^{1/2} f)$, so that by similarity $a\partial_t u = \mathrm{e}^{-t\widetilde{L}^{1/2}}(-aL^{1/2} f)$. We recall from (3.4) that $\widetilde{M}$ and $\widetilde{L}$ are incorporated in the matrix operator $(DB)^2$. Hence, we have

$$\mathrm{e}^{-t[DB]} \begin{bmatrix} -aL^{1/2} f \\ \nabla_x f \end{bmatrix} = \begin{bmatrix} a\partial_t u \\ \nabla_x u \end{bmatrix}.$$

The claim for $p = 2$ as well as the convergence of averages now follows from Theorem 17.2 and the comparison $\|aL^{1/2} f\|_2 \simeq \|\nabla_x f\|_2$.

Step 1: Upper bound for $p \neq 2$. As in the proof of Proposition 17.6 we have $\mathbb{H}^{1,p}_L = \dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2$ with equivalent p-quasinorms.

If $p \in (p_-(L)_* \vee 1_*, 2]$, then Proposition 8.27 applied to $\widetilde{M}$ and L directly yields

$$\begin{aligned} \|\widetilde{N}_*(\nabla u)\|_p &\leq \|\widetilde{N}_*(\nabla_x u)\|_p + \|\widetilde{N}_*(\partial_t u)\|_p \\ &\lesssim \|\nabla_x f\|_{\mathbb{H}^p_{\widetilde{M}}} + \|L^{1/2} f\|_{\mathbb{H}^p_L} \end{aligned}$$

and the ubiquitous Figure 8 allows us to compare with

$$\begin{aligned} &\simeq \|f\|_{\mathbb{H}^{1,p}_L} \\ &\simeq \|\nabla_x f\|_{\mathrm{H}^p} \end{aligned}$$

as required. If $p \in (2, q_+(L))$, we first introduce $\psi(z) \coloneqq \mathrm{e}^{-\sqrt{z}} - (1+z)^{-1}$ and split

$$\nabla u = v + w \coloneqq \begin{bmatrix} -\psi(t^2 L) L^{1/2} f \\ \psi(t^2 \widetilde{M}) \nabla_x f \end{bmatrix} + \begin{bmatrix} -(1+t^2 L)^{-1} L^{1/2} f \\ (1+t^2\widetilde{M})^{-1} \nabla_x f \end{bmatrix}.$$

We have $\psi \in \Psi^1_{1/2}$ on any sector. As in the preceding proof, Proposition 9.20 with $q = p$ and auxiliary function ψ applies to both $\widetilde{M}$ and L in our range of exponents. Along with Lemma 8.26, we find that

$$\begin{aligned} \|\widetilde{N}_*(v)\|_p &\leq \|\widetilde{N}_*(\mathbb{Q}_{\psi,\widetilde{M}} \nabla_x f)\|_p + \|\widetilde{N}_*(\mathbb{Q}_{\psi,L} L^{1/2} f)\|_p \\ &\lesssim \|S_{\psi,\widetilde{M}}(\nabla_x f)\|_p + \|S_{\psi,L}(L^{1/2} f)\|_p \\ &\lesssim \|\nabla_x f\|_p + \|L^{1/2} f\|_p \\ &\simeq \|\nabla_x f\|_p, \end{aligned}$$

where the final equivalence is due to Theorem 11.1. As for w, we use that the resolvents of L and $\widetilde{M}$ satisfy off-diagonal estimates of arbitrarily large order. Consequently, Lemma 8.23 and the $\mathrm{L}^{p/2}$-bound for the Hardy–Littlewood maximal operator yield

$$\begin{aligned}\|\widetilde{N}_*(w)\|_p &\leq \|(\mathcal{M}(|\nabla_x f|^2)^{1/2}\|_p + \|(\mathcal{M}(|L^{1/2} f|^2)^{1/2}\|_p \\ &\lesssim \|\nabla_x f\|_p + \|L^{1/2} f\|_p\end{aligned}$$

and we conclude as before. Combining these estimates gives the required upper bound for $\widetilde{N}_*(\nabla u)$.

Step 2: Lower bound for $p > 1$. Since $f \in \mathrm{L}^2$, we obtain from the convergence of Whitney averages that $\widetilde{N}_*(\nabla u) \geq |\nabla_x f|$ a.e. on $\mathbb{R}^n$ and $\|\widetilde{N}_*(\nabla u)\|_p \geq \|\nabla_x f\|_p$ follows.

Step 3: Lower bound for $p \leq 1$. As in Step 3 of the proof of Proposition 17.3 we calculate the H^p-norm of $\nabla_x f$ through the Fefferman–Stein characterization of H^p. The argument works again for all $p \in (\frac{n}{n+1}, 1]$.

Fix $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^n; \mathbb{R})$ with support in $B(0,1)$ and $\int_{\mathbb{R}^n} \phi = 1$ and let $\phi_t(y) := t^{-n} \phi(y/t)$. We need to control the L^p-norm of

$$\mathcal{M}_\phi(\nabla_x f)(x) := \sup_{t>0} |\nabla_x f * \phi_t|(x) \quad (x \in \mathbb{R}^n).$$

Temporarily fix $t > 0$ and $x \in \mathbb{R}^n$. Let $\chi : [0,\infty) \to [0,1]$ be smooth with $\mathbf{1}_{[0,1/2]} \leq \chi \leq \mathbf{1}_{[0,2)}$, set $\chi_t(s) := \chi(s/t)$ and introduce $\Phi(s,y) := \phi_t(x-y)\chi_t(s)$. As $\nabla_x u(s,y) = \mathrm{e}^{-t\widetilde{M}^{1/2}} \nabla_x f(y)$, the functional calculus on L^2 and the compact support of Φ justify writing

$$\begin{aligned}(\nabla_x f * \phi_t)(x) &= \int_{\mathbb{R}^n} \nabla_x f(y) \phi_t(x-y) \,\mathrm{d}y \\ &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} -\bigg(\int_\varepsilon^\infty \partial_s(\Phi \nabla_x u) \,\mathrm{d}s \bigg) \mathrm{d}y \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} -\partial_s \Phi \nabla_x u - \Phi \partial_s \nabla_x u \,\mathrm{d}y \mathrm{d}s,\end{aligned}$$

so that

$$\begin{aligned}|(\nabla_x f * \phi_t)(x)| &\leq \iint_{\mathbb{R}^{1+n}_+} |\partial_s \Phi \nabla_x u| \,\mathrm{d}s\mathrm{d}y + \iint_{\mathbb{R}^{1+n}_+} |\nabla_x \Phi \otimes \partial_s u| \,\mathrm{d}s\mathrm{d}y \\ &=: \mathrm{I} + \mathrm{II},\end{aligned}$$

where $\nabla_x \Phi \otimes \partial_s u$ is the vector in $(\mathbb{C}^m)^n$ coming from integration by parts in x. Now, we can literally repeat the arguments in Step 3 of the proof of Proposition 17.3 and arrive at

$$\mathrm{I} \lesssim \widetilde{N}_*(\nabla_x u)(x)$$

and

$$\mathrm{II} \lesssim \mathcal{M}(|\widetilde{N}_*(\partial_t u)|^{\frac{n}{n+1}})^{\frac{n+1}{n}}(x)$$

for all $x \in \mathbb{R}^n$. Consequently, we have a pointwise bound

$$\mathcal{M}_\phi(\nabla_x f) \lesssim \widetilde{N}_*(\nabla_x u) + \mathcal{M}(|\widetilde{N}_*(\partial_t u)|^{\frac{n}{n+1}})^{\frac{n+1}{n}}$$

and since $\mathcal{M}$ is bounded on $\mathrm{L}^{p(n+1)/n}$ we get $\|\nabla_x f\|_{\mathrm{H}^p} \lesssim \|\widetilde{N}_*(\nabla u)\|_p$ as required. □

Uniform boundedness and strong continuity follow again by abstract semigroup theory.

Proposition 17.8. *Let $(p_-(L)_* \vee 1_*) < p < q_+(L)$. If $f \in \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$, then $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$ satisfies*

(i) $\nabla_x u \in \mathrm{C}_0([0,\infty); \mathrm{H}^p) \cap \mathrm{C}^\infty((0,\infty); \mathrm{H}^p)$ *with*

$$\sup_{t>0} \|\nabla_x u(t,\cdot)\|_{\mathrm{H}^p} \simeq \|\nabla_x f\|_{\mathrm{H}^p}$$

and, for every $k \in \mathbb{N}$,

$$\sup_{t>0} \|t^{\frac{k}{2}} \partial_t^k \nabla_x u(t,\cdot)\|_{\mathrm{H}^p} \lesssim (\tfrac{k}{2})^{\frac{k}{2}} \mathrm{e}^{-k} \|\nabla_x f\|_{\mathrm{H}^p}.$$

(ii) *If $p < n$, then $u \in \mathrm{C}_0([0,\infty); \mathrm{L}^{p^*}) \cap \mathrm{C}^\infty((0,\infty); \mathrm{L}^{p^*})$ with*

$$\|f\|_{p^*} \le \sup_{t>0} \|u(t,\cdot)\|_{p^*} \lesssim \|\nabla_x f\|_{\mathrm{H}^p} + \|f\|_{p^*}.$$

Proof. From the proofs of Propositions 17.7 and 17.6 we know $\nabla_x u = \mathrm{e}^{-t\widetilde{M}^{1/2}} \nabla_x f$ and that in the given range of exponents $\|g\|_{\mathbb{H}^p_{\widetilde{M}}} \simeq \|g\|_{\mathrm{H}^p}$ holds for all $g \in \mathbb{H}^p_{\widetilde{M}}$. Hence, (i) follows *verbatim* as for the Dirichlet problem in Proposition 17.4 by appealing to the abstract theory for $\widetilde{M}$ instead of L.

For $p < n$ we have the Sobolev embedding $\dot{\mathrm{H}}^{1,p} \subseteq \mathrm{L}^{p^*}/\mathbb{C}^m$ but since $\mathrm{L}^{p^*} + \mathrm{L}^2$ does not contain any constants but 0 we also have $\dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2 \subseteq \mathrm{L}^{p^*}$. This yields the regularity statement in (ii) and the upper bound, whereas the lower bound follows again from the continuity at $t = 0$. □

17.3. **Conclusion of the existence part.** We now guide the reader through collecting and extending by density the respective estimates in order to obtain the existence part in our main results.

Existence of a solution with the properties in Theorem 1.1. First, let $f \in \mathrm{L}^p \cap \mathrm{L}^2$ if $p > 1$ and $f \in a^{-1}(\mathrm{H}^1 \cap \mathrm{L}^2)$ if $p = 1$. Then $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$ is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$, see Proposition 16.4, and parts of (i) - (iv) are contained in the previous sections:

Part	Obtained in
(i)	Propositions 17.3 & 17.1
(ii)	Proposition 17.3

(iii)	Proposition 17.4 (including quantitative bounds on the t-derivatives)
(iv)	Lemma 17.5 & qualitative continuity with values in L^2

The non-tangential convergence with L^2-averages in (ii) is stronger than what is asked for in $(D)_p^{\mathcal{L}}$. Hence, u solves $(D)_p^{\mathcal{L}}$ with data f.

Now, consider general data $f \in \mathrm{L}^p$ if $p > 1$ and $f \in a^{-1}\,\mathrm{H}^1$ if $p = 1$. Take any sequence of data $(f_k) \subseteq \mathrm{L}^2$ that approximates f in the data space as $k \to \infty$. Here, $a^{-1}\,\mathrm{H}^1$ is considered as a subspace of L^1 with natural norm $\|a \cdot\|_{\mathrm{H}^1}$. Denote the corresponding solutions by u_k.

By (i), we have that (u_k) is a Cauchy sequence in $\mathrm{T}^{0,p}_\infty$ and that $(t\nabla u_k)$ is a Cauchy sequence in T^p. Both topologies are stronger than $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$. Hence, (u_k) has a limit u in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ that satisfies (i) and it follows from Lemma 16.8 that u is a weak solution to $\mathcal{L}u = 0$. Note that this construction is independent of the choice of the (f_k). In the same way we obtain (iii) and (iv) for u since we can identify limits for the respective topologies in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$.

Property (ii) for u can be obtained by a well-known argument for maximal functions. More precisely, we obtain from (ii) for the u_k that for a.e. $x \in \mathbb{R}^n$,

$$\limsup_{t \to 0} \left(\fint\!\!\fint_{W(t,x)} |u - f(x)|^2 \,\mathrm{d}s\mathrm{d}y \right)^{\frac{1}{2}} \leq \widetilde{N}_*(u - u_k)(x) + |f(x) - f_k(x)|. \tag{17.4}$$

If the left-hand side exceeds a fixed threshold $\varepsilon > 0$, then at least one of the terms on the right exceeds $\varepsilon/2$. By (i) applied to $u - u_k$ and Markov's inequality, this can only happen on a set of measure

$$C\varepsilon^{-p}(\|a(f - f_k)\|_{\mathrm{H}^p} + \|f - f_k\|_{\mathrm{L}^p}),$$

which tends to 0 as $k \to \infty$ since $p \geq 1$. Hence, the left-hand side of (17.4) vanishes for a.e. $x \in \mathbb{R}^n$.

Finally, suppose that f is also an admissible datum for energy solutions. In the case $p > 1$ this means that we assume $f \in \mathrm{L}^p \cap \dot{\mathrm{H}}^{1/2,2}$ and by the universal approximation technique in Hardy–Sobolev spaces we can take the f_k above in such a way that $f_k \to f$ also in $\dot{\mathrm{H}}^{1/2,2}$. We know from Proposition 16.5 that u_k is the energy solution with Dirichlet data f_k and it follows from Proposition 16.3 that u is the energy solution with Dirichlet data f.

In the case $p = 1$ we assume $f \in (a^{-1}\,\mathrm{H}^1) \cap \dot{\mathrm{H}}^{1/2,2}$. We claim that this is a subspace of L^2. Taking the claim for granted, no approximation is necessary to construct the solution $u(t,x) = \mathrm{e}^{-tL^{1/2}} f(x)$ and by Proposition 16.5 this is the energy solution with data f. The easiest way to see the claim is to note that $f \in \mathrm{L}^1 \cap \dot{\mathrm{H}}^{1/2,2}$ and hence its Fourier transform satisfies

$$\int_{\mathbb{R}^n} |\mathcal{F}f(\xi)|^2 \, \mathrm{d}\xi \le \int_{B(0,1)} \|f\|_1^2 \, \mathrm{d}\xi + \int_{^cB(0,1)} |\xi||\mathcal{F}f(\xi)|^2 \, \mathrm{d}\xi$$
$$\le C\|f\|_1^2 + \|f\|_{\dot{\mathrm{H}}^{1/2,2}}^2.$$

Existence of a solution with the properties in Theorem 1.2. First, recall from Theorem 6.2 that $p_-(L) = q_-(L)$. Let $f \in \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$. As before, $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$ is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ from Proposition 16.4 and (i) as well as (iii) - (v) are contained in the previous sections. Part (ii) will mostly follow from a general trace theorem that we comment on below:

Part	Obtained in
(i)	Propositions 17.7 & 17.6 & Theorem 11.1.(i)
(iii)	Proposition 17.7
(iv)	Proposition 17.8
(v)	Proposition 17.3 & Proposition 17.4 & Theorem 11.1 since $\partial_t u = -\mathrm{e}^{-tL^{1/2}}(L^{1/2}f)$ with $L^{1/2}f \in a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$

As for the extension to a general data $f \in \dot{\mathrm{H}}^{1,p}$, we first treat the case $p < n$. We can assume $f \in \mathrm{L}^{p^*}$ since the general case follows by modifying data and solution by the same additive constant.

Take any sequence $(f_k) \subseteq \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$ with $f_k \to f$ as $k \to \infty$ in $\dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^{p^*}$. It follows from (iv) that (u_k) is a Cauchy sequence in $\mathrm{C}([0,\infty); \mathrm{L}^{p^*})$, hence in $\mathrm{L}^1_{\mathrm{loc}}$. Lemma 16.8 asserts that (u_k) converges in $\mathrm{W}^{1,2}_{\mathrm{loc}}$ to a weak solution to $\mathcal{L}u = 0$. The properties (i), (iv), (v) for u follow by identifying limits as before and for (iii) we rely on the same type of density argument as in (17.4). In particular, (iv) implies $\lim_{t\to 0} u(t,\cdot) = f$ in $\mathcal{D}'$ as claimed in (ii). This being said, the non-tangential limit in (ii) follows from the Kenig–Pipher trace theorem (Proposition A.5).

In the case $p \ge n$ we can only take a sequence $(f_k) \subseteq \dot{\mathrm{H}}^{1,p} \cap \mathrm{W}^{1,2}$ with $f_k \to f$ in $\dot{\mathrm{H}}^{1,p}$ as $k \to \infty$. We use (i) to infer that for the corresponding solutions (∇u_k) converges in $\mathrm{T}^{0,p}_\infty$, hence in $\mathrm{L}^2_{\mathrm{loc}}$. Define the averages $c_k := (u_k)_W$ with $W \subseteq \mathbb{R}^{1+n}_+$ a fixed cube. By Poincaré's inequality $(u_k - c_k)$ is bounded in $\mathrm{W}^{1,2}_{\mathrm{loc}}$. By compactness, we can define, up to passing to a subsequence,

$$u := \lim_{k\to\infty} u_k - c_k \quad (\text{in } \mathrm{L}^2_{\mathrm{loc}}).$$

Lemma 16.8 asserts again that u is a weak solution to $\mathcal{L}u = 0$ and modulo constants the construction of u is independent of the particular choice of the (f_k). With this definition all properties but (ii) follow as before. For the latter we fix the representative for f. Since $n > p_-(L)$, see Proposition 6.7, we obtain from (v) that $\partial_t u \in \mathrm{C}_0([0,\infty); \mathrm{L}^p)$. Hence, $u(t,\cdot)$ has a limit in $\mathcal{D}'$ as $t \to 0$. By (iv) we can fix the free constant for u such that this limit is f and the non-tangential convergence follows again from Proposition A.5.

Finally, if $f \in \dot{\mathrm{H}}^{1,p} \cap \dot{\mathrm{H}}^{1/2,2}$, then the same argument as for the Dirichlet problem yields that modulo constants u is the energy solution with Dirichlet datum f.

18. Existence in the Dirichlet problems with $\dot{\Lambda}^\alpha$-data

Here, we establish the existence part of Theorem 1.3, our main result on the Dirichlet problems $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ and $(\widetilde{D})^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ with boundary data in $\dot{\Lambda}^\alpha$. Let us stress that in accordance with the formulation of these problems the data space is *not* considered modulo constants.

Since $\dot{\Lambda}^\alpha \cap \mathrm{L}^2$ is not dense in $\dot{\Lambda}^\alpha$ for the strong topology, we cannot proceed in two well-separated steps as in the previous section. Instead, given $f \in \dot{\Lambda}^\alpha$, we directly define

$$u(t,\cdot) := \sum_{j=1}^\infty \mathrm{e}^{-tL^{1/2}}(\mathbf{1}_{C_j(Q)} f) \quad (t>0), \tag{18.1}$$

where $Q \subseteq \mathbb{R}^n$ is any cube, and check that this is a solution with all required properties for both Dirichlet problems. More concisely, we can write

$$u(t,\cdot) = \lim_{j\to\infty} \mathrm{e}^{-tL^{1/2}}(\mathbf{1}_{2^{j+1}Q} f) \quad (t>0),$$

but the representation as a series will be advantageous for most considerations. In fact, the assumptions of Theorem 1.3 are already required to prove convergence in $\mathrm{L}^2_{\mathrm{loc}}$ via off-diagonal estimates. More precisely, we work with the following exponents for most of the section:

(18.2)
- $p_+(L) > n$ and $0 \le \alpha < 1 - n/p_+(L)$.
- When α is fixed, p denotes a fixed exponent with $2 \le p < p_+(L)$ and $\alpha < 1 - n/p$.

We break the argument into six parts.

Part 1: Well-definedness of the solution. We begin with an elementary oscillation estimate.

Lemma 18.1. *Let $\alpha \in [0,1)$ and $p \in [1,\infty)$. For all $f \in \dot{\Lambda}^\alpha$, all cubes $Q \subseteq \mathbb{R}^n$ and all $j \ge 1$, it follows that*

$$\bigg(\fint_{2^jQ} |f - (f)_Q|^p \,\mathrm{d}y \bigg)^{\frac{1}{p}} \lesssim \gamma_j \ell(Q)^\alpha \|f\|_{\dot{\Lambda}^\alpha},$$

where $\gamma_j := \ln(j) + 1$ if $\alpha = 0$ and $\gamma_j := 2^{\alpha j}$ if $\alpha > 0$.

Proof. If $\alpha = 0$, then $\dot{\Lambda}^\alpha = \mathrm{BMO}$ and hence for all cubes $Q \subseteq \mathbb{R}^n$,

$$\Big(\fint_{2Q} |f - (f)_Q|^p \,\mathrm{d}y \Big)^{\frac{1}{p}} \lesssim \|f\|_{\dot{\Lambda}^0}.$$

A telescopic sum of the estimates for $Q, 2Q, \ldots, 2^{j-1}Q$ yields the claim. If $\alpha > 0$, then $|f(x) - f(y)| \lesssim (2^j \ell(Q))^\alpha$ for $x \in Q$ and $y \in 2^j Q$ and the claim follows immediately. □

The oscillation estimate allows us to prove convergence of the right-hand side in (18.1) and obtain further useful representations of u.

Lemma 18.2. *Assume* (18.2). *Then the following hold true.*

(i) *The sum defining u converges absolutely in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n)$, locally uniformly in t. In particular, u is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$.*

(ii) *If a family $(\eta_j) \subseteq \mathrm{L}^\infty(\mathbb{R}^n; \mathbb{C})$ satisfies* (5.2), *then $u(t, \cdot) = \sum_{j=1}^\infty \mathrm{e}^{-tL^{1/2}}(\eta_j f)$ with absolute convergence in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n)$, locally uniformly in t. In particular, u is independent of Q.*

(iii) *If $f = c$ is constant, then $u = c$ almost everywhere.*

Proof. By Corollary 4.17 the Poisson semigroup satisfies L^p off-diagonal estimates of order 1. Let $K \subseteq \mathbb{R}^n$ be any compact set and set $\ell := \ell(Q)$. For j large enough we have $\mathrm{d}(K, C_j(Q)) \geq 2^{j-1}\ell$ and hence

$$\begin{aligned}
&\|\mathrm{e}^{-tL^{1/2}}(\mathbf{1}_{C_j(Q)} f)\|_{\mathrm{L}^p(K)} \\
&\quad\lesssim t(2^j\ell)^{-1} \|f\|_{\mathrm{L}^p(C_j(Q))} \\
&\quad\lesssim t(2^j\ell)^{-1} \Big(\|f - (f)_Q\|_{\mathrm{L}^p(2^{j+1}Q)} + (2^j\ell)^{\frac{n}{p}} |(f)_Q| \Big) \\
&\quad\lesssim t(2^j\ell)^{\frac{n}{p}-1} \Big(\ell^\alpha \gamma_j \|f\|_{\dot{\Lambda}^\alpha} + |(f)_Q| \Big),
\end{aligned} \tag{18.3}$$

where we have used Lemma 18.1 in the final step. The right-hand side is summable in j since $\alpha < 1 - n/p$, which proves convergence of the series in (18.1) in $\mathrm{L}^p_{\mathrm{loc}}$, locally uniformly in t. Since all partial sums are weak solutions to the equation for $\mathcal{L}$ in $\mathbb{R}^{1+n}_+$, the same is true for u, see Proposition 16.5 and Lemma 16.8. This completes the proof of (i).

Now, (ii) follows by repeating the proof of Proposition 5.1 word by word up to incorporating the off-diagonal estimate above. Finally, (iii) is due to the conservation property for Poisson semigroups (Proposition 5.6). □

Part 2: Proof of (ii). We start by proving continuity and convergence towards the boundary data in $\mathrm{L}^2_{\mathrm{loc}}$.

Lemma 18.3. *The solution u is of class $\mathrm{C}([0,T]; \mathrm{L}^2_{\mathrm{loc}})$ with $u(0, \cdot) = f$ for every $T > 0$.*

Proof. Continuity on $(0,T]$ is a general property of weak solutions, see Corollary 16.9. We fix an arbitrary cube Q of sidelength ℓ and prove the limit at $t=0$ in $\mathrm{L}^2(Q)$.

Set $f_j := (f-(f)_Q)\mathbf{1}_{C_j(Q)}$. By Lemma 18.2 we have, whenever $y \in Q$ and $s>0$,

$$\begin{aligned} u(s,y)-f(y) &= \sum_{j=1}^\infty \mathrm{e}^{-sL^{1/2}} f_j(y) + (f)_Q - f(y) \\ &= \sum_{j=2}^\infty \mathrm{e}^{-sL^{1/2}} f_j(y) + (\mathrm{e}^{-sL^{1/2}} f_1(y) - f_1(y)). \end{aligned} \tag{18.4}$$

For the error terms with $j \geq 2$ we use again that the Poisson semigroup satisfies L^p off-diagonal estimates of order 1, see Corollary 4.17. Here, p is as in (18.2). Together with Lemma 18.1, we obtain

$$\begin{aligned} \Big\| \sum_{j=2}^\infty \mathrm{e}^{-sL^{1/2}} f_j \Big\|_{\mathrm{L}^p(Q)} &\leq \sum_{j=2}^\infty \|\mathrm{e}^{-s^{1/2}} f_j\|_{\mathrm{L}^p(Q)} \\ &\lesssim \sum_{j=2}^\infty \frac{s}{2^j \ell} \|f_j\|_{\mathrm{L}^p(Q)} \\ &\lesssim \frac{s}{\ell^{1-\frac{n}{p}-\alpha}} \sum_{j=2}^\infty 2^{j(\frac{n}{p}-1)} \gamma_j \|f\|_{\dot{\Lambda}^\alpha}, \end{aligned} \tag{18.5}$$

where the sum in j is finite by the choice of p. In particular, we have by Hölder's inequality that

$$\Big\| \sum_{j=2}^\infty \mathrm{e}^{-sL^{1/2}} f_j \Big\|_{\mathrm{L}^2(Q)} \lesssim \frac{s}{\ell^{1-\frac{n}{2}-\alpha}} \|f\|_{\dot{\Lambda}^\alpha},$$

which in combination with (18.4) leads us to

$$\|u(s,\cdot)-f\|_{\mathrm{L}^2(Q)} \lesssim \frac{s}{\ell^{1-\frac{n}{2}-\alpha}} \|f\|_{\dot{\Lambda}^\alpha} + \|\mathrm{e}^{-sL^{1/2}} f_1 - f_1\|_2.$$

The right-hand side tends to 0 in the limit as $s \to 0$ since we have $f \in \dot{\Lambda}^\alpha$ and $f_1 \in \mathrm{L}^2$. □

We turn to non-tangential convergence towards the boundary data and control of the corresponding sharp functional on Whitney averages. In the case $\alpha > 0$ this would come for free from Proposition A.8 once we have established the upper bound for the Carleson functional as stated in (i) but the following direct argument also works for $\alpha = 0$.

Lemma 18.4. *The solution u satisfies*

$$\lim_{t \to 0} \iint_{W(t,x)} |u(s,y)-f(x)|^2 \,\mathrm{d}s\mathrm{d}y = 0 \quad (a.e.\ x \in \mathbb{R}^n)$$

and

$$\|\widetilde{N}_{\sharp,\alpha}(u-f)\|_\infty \lesssim \|f\|_{\dot\Lambda^\alpha}.$$

Proof. We only need a slight refinement of the previous argument. To this end let $x \in \mathbb{R}^n$, $2\ell \geq 2t \geq s$ and let Q be the axis-parallel cube of sidelength ℓ centered at x.

For any $(s,y) \in W(t,x) = (t/2, 2t) \times B(x,t)$ we can use (18.4) and (18.5) with this choice of Q and the same definition of f_j, $j \geq 1$, in order to obtain

$$\begin{aligned}
&\|u(s,\cdot) - f\|_{\mathrm{L}^2(B(x,t))} \\
&\quad\lesssim t^{\frac{n}{2}-\frac{n}{p}} \Big\| \sum_{j=2}^\infty \mathrm{e}^{-sL^{1/2}} f_j \Big\|_{\mathrm{L}^p(Q)} + \|\mathrm{e}^{-sL^{1/2}} f_1 - f_1\|_{\mathrm{L}^2(B(x,t))} \\
&\quad\lesssim \frac{st^{\frac{n}{2}-\frac{n}{p}}}{\ell^{1-\frac{n}{p}-\alpha}} \|f\|_{\dot\Lambda^\alpha} + \|\mathrm{e}^{-sL^{1/2}} f_1 - f_1\|_{\mathrm{L}^2(B(x,t))}.
\end{aligned}$$

Thus, we get our key estimate

$$\begin{aligned}
&\bigg(\iint_{W(t,x)} |u(s,\cdot) - f|^2 \,\mathrm{d}s\mathrm{d}y \bigg)^{1/2} \\
&\quad\lesssim \frac{t^{1-\frac{n}{p}}}{\ell^{1-\frac{n}{p}-\alpha}} \|f\|_{\dot\Lambda^\alpha} + \bigg(\iint_{W(t,x)} |\mathrm{e}^{-sL^{1/2}} f_1 - f_1|^2 \,\mathrm{d}s\mathrm{d}y \bigg)^{1/2}.
\end{aligned} \tag{18.6}$$

For the first claim it suffices (by the Lebesgue differentiation theorem) to prove that the left-hand side in (18.6) vanishes in the limit as $t \to 0$ for a.e. $x \in \mathbb{R}^n$. But passing to the limit on the right-hand side, the first term vanishes since we have $p > n$ by (18.2) and the second term vanishes for a.e. $x \in \mathbb{R}^n$ thanks to the Lebesgue differentiation theorem and Proposition 17.3 applied to $f_1 \in \mathrm{L}^2$.

In order to bound the sharp functional, we use (18.6) with $t = \ell$. This yields for all $t > 0$ and all $x \in \mathbb{R}^n$ the required uniform bound

$$\begin{aligned}
&\frac{1}{t^\alpha} \bigg(\iint_{W(t,x)} |u(s,\cdot) - f|^2 \,\mathrm{d}s\mathrm{d}y \bigg)^{1/2} \\
&\quad\lesssim \|f\|_{\dot\Lambda^\alpha} + \frac{1}{t^\alpha} \bigg(\iint_{W(t,x)} |\mathrm{e}^{-sL^{1/2}} f_1 - f_1|^2 \,\mathrm{d}s\mathrm{d}y \bigg)^{1/2} \\
&\quad\lesssim \|f\|_{\dot\Lambda^\alpha} + \frac{1}{t^{\alpha+\frac{n}{2}}} \sup_{s>0} \|\mathrm{e}^{-sL^{1/2}} f_1 - f_1\|_2 \\
&\quad\lesssim \|f\|_{\dot\Lambda^\alpha} + \frac{1}{t^{\alpha+\frac{n}{2}}} \|f_1\|_2 \\
&\quad\lesssim \|f\|_{\dot\Lambda^\alpha},
\end{aligned}$$

where the final step is due to Lemma 18.1, keeping in mind that by definition $f_1 = (f - (f)_Q)\mathbf{1}_{4Q}$ and that t is the sidelength of Q. □

Part 3: The upper bound for the Carleson functional. In this part we prove the upper bound $\|C_\alpha(t\nabla u)\|_\infty \lesssim \|f\|_{\dot{\Lambda}^\alpha}$. It will be convenient to use cubes instead of balls for the Carleson functional and to show that for all cubes $Q \subseteq \mathbb{R}^n$ of sidelength ℓ we have

$$\left(\int_0^\ell \fint_Q |s\nabla u|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s}\right)^{1/2} \lesssim \ell^\alpha \|f\|_{\dot{\Lambda}^\alpha}. \tag{18.7}$$

From now on Q is fixed. Since both sides stay the same under adding constants to u and f, we can assume $(f)_Q = 0$. For $j \geq 1$ we introduce

$$f_j := \mathbf{1}_{C_j(Q)} f, \quad u_j(t,\cdot) := \mathrm{e}^{-tL^{1/2}} f_j.$$

Step 1: The local bound. By Lemma 18.1 we have $\|f_1\|_2^2 \lesssim |Q|^{-1}\ell^{2\alpha}\|f\|_{\dot{\Lambda}^\alpha}$. Hence, the local term u_1 can be handled via the L^2-bound for the square function in Proposition 17.1:

$$\begin{aligned}\int_0^\ell \fint_Q |s\nabla u_1|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s} &\leq |Q|^{-1} \iint_{\mathbb{R}^{1+n}_+} |s\nabla \mathrm{e}^{-tL^{1/2}} f_1|^2 \, \frac{\mathrm{d}s\mathrm{d}y}{s} \\ &\lesssim \ell^{2\alpha}\|f\|_{\dot{\Lambda}^\alpha}^2.\end{aligned}$$

Step 2: Decomposition of the non-local terms. Set $W(t,x) := (t,2t) \times Q(x,t)$ and $\widetilde{W}(t,x) := (t/2, 4t) \times Q(x,2t)$. Let $\phi(z) := \mathrm{e}^{-\sqrt{z}} - (1+z)^{-2}$ and recall from (17.1) the Caccioppoli estimate

$$\begin{aligned}&\iint_{W(t,x)} |s\nabla\phi(s^2L)f_j|^2 \, \frac{\mathrm{d}s\mathrm{d}y}{s} \\ &\qquad \lesssim \iint_{\widetilde{W}(t,x)} |\phi(s^2L)f_j|^2 + |\psi(s^2L)f_j|^2 \, \frac{\mathrm{d}s\mathrm{d}y}{s},\end{aligned} \tag{18.8}$$

where $\psi \in \Psi_1^1$ on any sector. Let the regions $(W(t_k,x_k))_k$ cover $(0,\ell) \times Q$ modulo a set of measure zero such that the $(\widetilde{W}(t_k,x_k))_k$ are contained in $(0,2\ell) \times 2Q$ and at most 2^{n+1} of them overlap at each point. Summing up in k yields

$$\begin{aligned}&\int_0^\ell \int_Q |s\nabla\phi(s^2L)f_j|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s} \\ &\qquad \lesssim \int_0^{2\ell} \int_{2Q} |\phi(s^2L)f_j|^2 + |\psi(s^2L)f_j|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s},\end{aligned}$$

so that in total

$$\begin{aligned}&\left(\int_0^\ell \fint_Q |s\nabla u_j|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s}\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_0^{2\ell} \fint_{2Q} |\phi(s^2L)f_j|^2 + |\psi(s^2L)f_j|^2 + |s\nabla(1+s^2L)^{-2}f_j|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s}\right)^{\frac{1}{2}}.\end{aligned}$$

From Lemma 18.2 and Caccioppoli's inequality we obtain that $u = \sum_{j=1}^{\infty} u_j$ converges in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$. We can use Fatou's lemma to conclude

$$\left(\int_0^\ell \fint_Q |s\nabla u|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s}\right)^{\frac{1}{2}} \lesssim \ell^\alpha \|f\|_{\dot{\Lambda}^\alpha} + \sum_{j=2}^{\infty} \mathrm{I}_j + \mathrm{II}_j + \mathrm{III}_j, \tag{18.9}$$

where

$$\mathrm{I}_j := \left(\int_0^{2\ell} \fint_{2Q} |\phi(s^2L)f_j|^2 \frac{\mathrm{d}y\mathrm{d}s}{s}\right)^{\frac{1}{2}},$$
$$\mathrm{II}_j := \left(\int_0^{2\ell} \fint_{2Q} |\psi(s^2L)f_j|^2 \frac{\mathrm{d}y\mathrm{d}s}{s}\right)^{\frac{1}{2}},$$
$$\mathrm{III}_j := \left(\int_0^{2\ell} \fint_{2Q} |s\nabla(1+s^2L)^{-2}f_j|^2 \frac{\mathrm{d}y\mathrm{d}s}{s}\right)^{\frac{1}{2}}.$$

Step 3: Bounds for the off-diagonal pieces. We begin with the bound for I_j. The family $(\phi(t^2L))_{t>0}$ satisfies L^p off-diagonal estimates of order 1. This is due to Lemma 4.16 since $\phi \in \Psi^2_{1/2}$ on any sector. Hence,

$$\begin{aligned}\left(\fint_{2Q} |\phi(s^2L)f_j|^2 \, \mathrm{d}y\right)^{\frac{1}{2}} &\le \left(\fint_{2Q} |\phi(s^2L)f_j|^p \, \mathrm{d}y\right)^{\frac{1}{p}} \\ &\lesssim \left(\frac{2^j\ell}{s}\right)^{-1} 2^{j\frac{n}{p}} \left(\fint_{2^jQ} |f|^p \, \mathrm{d}y\right)^{\frac{1}{p}} \\ &\lesssim s\ell^{\alpha-1}\gamma_j 2^{j(\frac{n}{p}-1)} \|f\|_{\dot{\Lambda}^\alpha},\end{aligned}$$

where the final step is again due to Lemma 18.1. We take L^2-norms with respect to $\frac{\mathrm{d}s}{s}$ on both sides to give

$$\le \ell^\alpha \|f\|_{\dot{\Lambda}^\alpha} \gamma_j 2^{j(\frac{n}{p}-1)}. \tag{18.10}$$

Summing these estimates in j leads to a desirable bound in (18.9).

In estimating I_j we have only used $\phi \in \Psi^\tau_{1/2}$ on any sector for some $\tau > 0$. Hence, we can use the same strategy for II_j and the first component of

$$s\nabla(1+s^2L)^{-2}f_j = \begin{bmatrix} -4s^2L(1+s^2L)^{-3}f_j \\ s\nabla_x(1+s^2L)^{-2}f_j \end{bmatrix}$$

in III_j. As for the second component, we have L^2 off-diagonal estimates of arbitrarily large order $\gamma > 0$ for $(t\nabla_x(1+t^2L)^{-2})_{t>0}$ by composition. Therefore, we can run the same argument as before but with $p = 2$ in

Lemma 18.1 and obtain

$$\Big(\int_0^{2\ell} \fint_{2Q} |s \nabla_x (1+s^2L)^{-2} f_j|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s} \Big)^{\frac{1}{2}} \leq \ell^\alpha \|f\|_{\dot{\Lambda}^\alpha} \gamma_j 2^{j(\frac{n}{2}-\gamma)}.$$

We take $\gamma := n/2 - n/p + 1$ and conclude a desirable bound for III_j in (18.9). This completes the proof of (18.7).

Part 4: Compatibility. In this section we work with $\dot{\Lambda}^\alpha$ as a homogeneous smoothness space modulo constants. In view of Lemma 18.2 this determines u modulo constants.

Our goal is to establish compatibility of u with the energy class, that is, we assume $f \in \dot{\Lambda}^\alpha \cap \dot{\mathrm{H}}^{1/2,2}$ and have to show that modulo constants u is the energy solution with Dirichlet data f. This is a delicate matter since no density argument can help us here. We shall rely on the following two lemmata.

Lemma 18.5. *Let $g_1 \in \mathrm{L}^2$ and $g_2 \in \mathrm{T}^{-1,\infty;\alpha}$ for some $\alpha \in [0,1)$ be such that $g_1 - g_2$ is constant on $\mathbb{R}^{1+n}_+$. Then $g_1 = g_2$ almost everywhere.*

Proof. Let $g_1 - g_2 = c$ almost everywhere. We obtain for all $r > 0$ that

$$\begin{aligned} |c|^2 &\simeq r^{-1-n} \int_r^{2r} \int_{B(0,2r)} |g_1 - g_2|^2 \, \mathrm{d}x\mathrm{d}t \\ &\lesssim r^{-1-n} \|g_1\|_2^2 + r^{2\alpha-2} \|g_2\|_{\mathrm{T}^{-1,\infty;\alpha}}^2. \end{aligned}$$

As $\alpha < 1$, sending $r \to \infty$ yields $c = 0$. □

Lemma 18.6. *Let $\alpha \in [0,1)$. Each $f \in \dot{\Lambda}^\alpha \cap \dot{\mathrm{H}}^{1/2,2}$ can be decomposed in $\dot{\Lambda}^\alpha \cap \dot{\mathrm{H}}^{1/2,2}$ as $f = f_{\mathrm{loc}} + f_{\mathrm{glob}}$, where $f_{\mathrm{loc}} \in \dot{\mathrm{W}}^{1,2}$ and $f_{\mathrm{glob}} \in \mathrm{L}^2$.*

Proof. We pick $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^n; \mathbb{R})$ such that $\mathbf{1}_{B(0,1)} \leq \varphi \leq \mathbf{1}_{B(0,2)}$ and set

$$f_{\mathrm{loc}} := \mathcal{F}^{-1}(\varphi \mathcal{F} f), \quad f_{\mathrm{glob}} := \mathcal{F}^{-1}((1-\varphi)\mathcal{F} f).$$

Then obviously $f = f_{\mathrm{loc}} + f_{\mathrm{glob}}$ and since φ and $1-\varphi$ are smooth Fourier multipliers in the scope of the Mihlin multiplier theorem, both f_{loc} and f_{glob} remain in $\dot{\Lambda}^\alpha \cap \dot{\mathrm{H}}^{1/2,2}$. Moreover, $m_{\mathrm{loc}}(\xi) := |\xi|^{1/2} \varphi(\xi)$ and $m_{\mathrm{glob}}(\xi) := |\xi|^{-1/2}(1-\varphi(\xi))$ are Mihlin multipliers and since we have $g := \mathcal{F}^{-1}(|\xi|^{1/2} \mathcal{F} f) \in \mathrm{L}^2$ by assumption, we obtain that

$$\mathcal{F}^{-1}(|\xi| \mathcal{F} f_{\mathrm{loc}}) = \mathcal{F}^{-1}(m_{\mathrm{loc}} \mathcal{F} g) \in \mathrm{L}^2, \quad f_{\mathrm{glob}} = \mathcal{F}^{-1}(m_{\mathrm{glob}} \mathcal{F} g) \in \mathrm{L}^2$$

as required. □

As we are dealing with a linear problem, the benefit from Lemma 18.6 is that it suffices to prove compatibility under the additional assumption that either $f \in \mathrm{L}^2$ or $f \in \dot{\mathrm{W}}^{1,2}$.

If additionally $f \in \mathrm{L}^2$, then $\sum_{j=1}^\infty \mathbf{1}_{C_j(Q)} f$ converges to f in L^2 and from (18.1) we get back

$$u(t, \cdot) = \mathrm{e}^{-tL^{1/2}} f \quad (t > 0).$$

According to Proposition 16.5 this is the energy solution with Dirichlet data f.

Now, suppose that additionally $f \in \dot{\mathrm{W}}^{1,2}$ and let $\widetilde{u}$ be the energy solution with Dirichlet data f. We claim that it suffices to show that for all $g \in \mathrm{C}_0^\infty$ with $\int_{\mathbb{R}^n} g \,\mathrm{d}x = 0$ and all $t > 0$ we have

$$\langle u(t,\cdot), g\rangle = \langle \widetilde{u}(t,\cdot), g\rangle, \tag{18.11}$$

where the angular brackets denote the (extended) inner product on L^2. Indeed, the claim implies that $u - \widetilde{u}$ is independent of the x-variable but looking at the equation $\mathcal{L}(u - \widetilde{u}) = 0$ in $\mathbb{R}^{1+n}_+$, we also obtain $a\partial_t^2(u - \widetilde{u}) = 0$, so $\partial_t u - \partial_t \widetilde{u}$ is constant. By definition we have $\partial_t \widetilde{u} \in \mathrm{L}^2$ and by the Carleson bound in Part 3 we have $\partial_t u \in \mathrm{T}^{-1,\infty;\alpha}$. Lemma 18.5 yields $\partial_t u - \partial_t \widetilde{u} = 0$ and the desired compatibility $u = \widetilde{u}$ (modulo constants) follows.

In order to prove (18.11), we pick a cube Q that contains the support of g and use Lemma 18.2 to write

$$u(t,x) = \sum_{j=1}^\infty \mathrm{e}^{-tL^{1/2}}(\eta_j f)(x) \quad ((t,x) \in \mathbb{R}^{1+n}_+), \tag{18.12}$$

with $(\eta_j)_j$ a smooth partition of unity on $\mathbb{R}^n$ subordinate to the sets $D_1 := 4Q$ and $D_j := 2^{j+1}Q \setminus 2^{j-1}Q$, $j \geq 2$, such that $\|\eta_j\|_\infty + 2^j \ell(Q)\|\nabla_x \eta_j\|_\infty \leq C$ for a dimensional constant C.

Since g has integral 0, we can write $g = \operatorname{div}_x G$ with $G \in \mathrm{C}_0^\infty(Q)$. Indeed, in dimension $n = 1$ it suffices to take a suitable primitive of g and in dimension $n \geq 2$ this is Bogovskiĭ's lemma [49, Lemma III.3.1]. By duality and the intertwining relations, we obtain

$$\begin{aligned}
\langle \mathrm{e}^{-tL^{1/2}}(\eta_j f), g\rangle &= \langle \eta_j f, \mathrm{e}^{-t(L^*)^{1/2}} \operatorname{div}_x G\rangle \\
&= \langle \eta_j f, \operatorname{div}_x \mathrm{e}^{-t(M^\sharp)^{1/2}} G\rangle \\
&= -\langle \eta_j \nabla_x f, \mathrm{e}^{-t(M^\sharp)^{1/2}} G\rangle \\
&\quad - \langle \nabla_x \eta_j \otimes f, \mathrm{e}^{-t(M^\sharp)^{1/2}} G\rangle \\
&=: -\mathrm{I}_j - \mathrm{II}_j,
\end{aligned} \tag{18.13}$$

where $M^\sharp := -d^* \nabla_x (a^*)^{-1} \operatorname{div}_x$ intertwines with L^* in the same ways as M intertwines with $\widetilde{L}$. Our notation is $\nabla_x(\eta_j f) = \eta_j \nabla_x f + \nabla_x \eta_j \otimes f$ as predicted by the product rule. The assumption $\nabla_x f \in \mathrm{L}^2$ and the fact that $\mathrm{e}^{-t(M^\sharp)^{1/2}} G \in \mathrm{L}^2$ allow us to sum up

$$\sum_{j=1}^\infty \mathrm{I}_j = \langle \nabla_x f, \mathrm{e}^{-t(M^\sharp)^{1/2}} G\rangle. \tag{18.14}$$

As for the error terms II_j, we shall need the qualitative information

$$e^{-t(M^\sharp)^{1/2}} G \in \mathrm{L}^q \quad (\text{for some } q < 2). \tag{18.15}$$

In each of the following steps we take q as close to 2 as necessary for the respective result to apply. First, we write $G = G_1 + G_2$ with $G_1 \in \mathsf{N}(\operatorname{div}_x)$ and $G_2 \in \overline{\mathsf{R}(d^*\nabla_x)}$ as in the Hodge decomposition (13.2) with d^* replacing d. By Proposition 13.8 and Lemma 13.4, this decomposition can be taken topological in L^q. The identification Theorem 9.6 tells us that we can have $\mathbb{H}^{1,q}_{L^\sharp} = \dot{\mathrm{W}}^{1,q} \cap \mathrm{L}^2$ with equivalent q-norms and then $\mathbb{H}^q_{M^\sharp} = \mathrm{L}^q \cap \overline{\mathsf{R}(d^*\nabla_x)}$ follows by moving from the second to the fourth row in Figure 8. Proposition 8.10 yields $\mathrm{e}^{-t(M^\sharp)^{1/2}} G_2 \in \mathrm{L}^q$ and from $G_1 \in \mathsf{N}(M^\sharp)$ we obtain by the functional calculus in L^2 that $\mathrm{e}^{-t(M^\sharp)^{1/2}} G_1 = G_1$, which also belongs to L^q. Hence, (18.15) follows.

Now, we go back to (18.13). We pick exponents $r, s \in (1,\infty)$ such that $1/q + 1/r + 1/s = 1$ and obtain for all $J \geq 1$ that

$$\Big| \sum_{j=1}^J \mathrm{II}_j \Big| \lesssim \Big\| \sum_{j=1}^J \nabla_x \eta_j \Big\|_r \|f\|_{\mathrm{L}^s(2^{J+1}Q)} \|\mathrm{e}^{-t(M^\sharp)^{1/2}} G\|_q$$
$$\lesssim 2^{J(\frac{n}{r}-1)} \|f\|_{\mathrm{L}^s(2^{J+1}Q)},$$

where we have used that $\sum_{j=1}^J \nabla_x \eta_j$ has support in $2^{J+1}Q$ and is controlled in L^∞-norm by 2^{-J}. The implicit constant depends on all variables but J. The choice of s depends on Sobolev embeddings. In dimension $n \geq 3$ we can assume $f \in \mathrm{L}^{2^*}$ up to modifying f (and hence u) by a constant. Then we pick $s := 2^*$ and obtain

$$\Big| \sum_{j=1}^J \mathrm{II}_j \Big| \lesssim 2^{J(\frac{n}{r}-1)} = 2^{J(\frac{n}{2}-\frac{n}{q})},$$

which tends to 0 as $J \to \infty$ since $q < 2$. In dimension $n \leq 2$ we can assume $f \in \dot{\Lambda}^{1-n/2}$ and also change f to $f - (f)_Q$ in (18.12), which changes u by a constant. With this modification, we obtain together with Lemma 18.1 that

$$\Big| \sum_{j=1}^J \mathrm{II}_j \Big| \lesssim 2^{J(\frac{n}{r}-1)} 2^{J\frac{n}{s}} (\gamma_J + 1) = \begin{cases} 2^{J(\frac{1}{2}-\frac{1}{q})} & \text{if } n = 1 \\ 2^{J(1-\frac{2}{q})}(1 + \ln J) & \text{if } n = 2 \end{cases},$$

which also tends to 0 as $J \to \infty$. Together with (18.12) - (18.14), we arrive at

$$\langle u(t,\cdot), g \rangle = -\langle \nabla_x f, \mathrm{e}^{-t(M^\sharp)^{1/2}} G \rangle.$$

Since $f \in \dot{\mathrm{W}}^{1,2} \cap \dot{\mathrm{H}}^{1/2,2}$, the universal approximation technique lets us pick a sequence $(f)_k \subseteq \dot{\mathrm{W}}^{1,2} \cap \dot{\mathrm{H}}^{1/2,2} \cap \mathrm{L}^2$ with $f_k \to f$ in both $\dot{\mathrm{W}}^{1,2}$ and $\dot{\mathrm{H}}^{1/2,2}$. We let u_k be the energy solution with Dirichlet data f_k. Then (u_k) tends to the energy solution $\widetilde{u}$ with data f in $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$. By Lemma 16.1, this implies $u_k(t,\cdot) \to u(t,\cdot)$ in the sense of distributions modulo constants. On the other hand, we know from Proposition 16.5

that $u_k(t,\cdot) = \mathrm{e}^{-tL^{1/2}} f_k$ and we can undo the duality and intertwining in order to give

$$\begin{aligned}\langle u(t,\cdot), g\rangle &= -\lim_{k\to\infty} \langle \nabla_x f_k, \mathrm{e}^{-t(M^\sharp)^{1/2}} G\rangle \\ &= \lim_{k\to\infty} \langle \mathrm{e}^{-tL^{1/2}} f_k, g\rangle \\ &= \lim_{k\to\infty} \langle u_k(t,\cdot), g\rangle \\ &= \langle \widetilde{u}(t,\cdot), g\rangle.\end{aligned}$$

This establishes the remaining claim (18.11) and the proof is complete.

Part 5: The lower bound for the Carleson functional. Our goal is to show that for all $g \in \mathrm{C}_0^\infty$ with $\int_{\mathbb{R}^n} g \,\mathrm{d}x = 0$ the solution u in (18.1) satisfies

$$|\langle f, g\rangle| \lesssim \|C_\alpha(t\nabla u)\|_\infty \|g\|_{\mathrm{H}^\varrho}, \tag{18.16}$$

where $\varrho \in (1_*, 1]$ is such that $n(1/\varrho - 1) = \alpha$ and $\langle \cdot\,, \cdot\rangle$ is the extended L^2-duality pairing. Indeed, then density and duality yield the lower bound

$$\|f\|_{\dot{\Lambda}^\alpha} \lesssim \|C_\alpha(t\nabla u)\|_\infty.$$

We suggest that the reader recalls the argument of Step 3 of Proposition 17.1 beforehand. The proof here follows the same line of thought but since $u(t,\cdot)$ and f may not be globally in L^2, we cannot as directly rely on the functional calculus in L^2 as before. This is the major technical challenge.

From now on we fix g and pick a cube Q that contains its support. Since both sides in (18.16) do not change when adding constants to f or u, we can assume $(f)_Q = 0$ and write u as

$$u(t,\cdot) = \sum_{j=1}^\infty \mathrm{e}^{-tL^{1/2}} f_j, \quad f_j \coloneqq \mathbf{1}_{C_j(Q)} f.$$

Next, we introduce again $H \coloneqq -(a^*)^{-1}\Delta_x$ and set

$$v(t,\cdot) \coloneqq (1 + t^2 H)^{-\beta}((a^*)^{-1} g)$$

for an integer $\beta > n/2 + 2$. Then the kernel estimates in Proposition 14.14 become available and this is why we use the resolvents of H and not the Poisson semigroup as in the proof of Proposition 17.1. The auxiliary function we are working with is

$$\Phi : (0,\infty) \to \mathbb{C}, \quad \Phi(t) \coloneqq \sum_{j=1}^\infty \langle \mathrm{e}^{-tL^{1/2}} f_j, a^* v(t,\cdot)\rangle. \tag{18.17}$$

This turns out to be the appropriate way of defining $\langle u(t,\cdot), a^* v(t,\cdot)\rangle$ as we shall see momentarily. We divide the proof of (18.16) into eight steps.

Step 1: Qualitative growth bounds for v. We claim that there are $c > 0$ and $C > 0$ depending also on β, g and Q such that

(18.18)
$$|v(t,x)| + |t\nabla v(t,x)| \leq C(1 \wedge t^{-n-1})\mathrm{e}^{-c\frac{\mathrm{d}(x,Q)}{t}} \quad ((t,x) \in \mathbb{R}^{1+n}_+).$$

To this end we recall that $(1+t^2H)^{-\beta}(a^*)^{-1}$ is given by an integral kernel denoted by $H_t^\beta(x,y)$ (up to replacing a to a^*) with bounds

$$\begin{aligned} |H_t^\beta(x,y)| + |t\nabla_x H_t^\beta(x,y)| + |t\nabla_y H_t^\beta(x,y)| \\ +|t^2\nabla_x\nabla_y H_t^\beta(x,y)| \leq Ct^{-n}\mathrm{e}^{-c\frac{|x-y|}{t}}, \end{aligned}$$

see Proposition 14.14. Hence, by the support of g and an L^1-bound on the kernel,

$$\begin{aligned} |v(t,x)| &\leq \mathrm{e}^{-\frac{c}{2}\frac{\mathrm{d}(x,Q)}{t}} \int_{\mathbb{R}^n} \mathrm{e}^{\frac{c}{2}\frac{|x-y|}{t}} |H_t^\beta(x,y)||g(y)| \,\mathrm{d}y \\ &\lesssim \mathrm{e}^{-\frac{c}{2}\frac{\mathrm{d}(x,Q)}{t}} \|g\|_\infty. \end{aligned}$$

Since $g \in \mathrm{C}_0^\infty(Q)$ has mean value 0, we can also write $g = \operatorname{div}_x F$ with $F \in \mathrm{C}_0^\infty(Q)$, using a suitable primitive in dimension $n = 1$ and Bogovskiĭ's lemma in higher dimensions. Thus,

$$v(t,x) = -\int_{\mathbb{R}^n} \nabla_y H_t^\beta(x,y) \cdot F(y) \,\mathrm{d}y$$

and the L^∞-bound for the kernel yields

$$|v(t,x)| \leq Ct^{-n-1}\mathrm{e}^{-c\frac{\mathrm{d}(x,Q)}{t}}\|F\|_1.$$

This completes the estimate in (18.18) for v. The bounds for $t\nabla_x v$ follow *mutadis mutandis*, using the kernel bounds for $t\nabla_x H_t^\beta$ and $t^2\nabla_x\nabla_y H_t^\beta$. Eventually,

$$\begin{aligned} t\partial_t v(t,\cdot) &= -2\beta t^2 H(1+t^2H)^{-\beta-1}((a^*)^{-1}g) \\ &= -2\beta((1+t^2H)^{-\beta} - (1+t^2H)^{-\beta-1})((a^*)^{-1}g) \end{aligned}$$

is a linear combination of two functions of the same type as v.

Step 2: Φ *is well-defined.* More precisely, we shall show the qualitative bound

$$\sum_{j=2}^\infty \||\mathrm{e}^{-tL^{1/2}} f_j||a^*v(t,\cdot)|\|_1 \leq C(t \wedge t^{-\frac{n}{p}}) < \infty, \tag{18.19}$$

where C is independent of $t > 0$ but may depend on all other parameters.

By Hölder's inequality, we have

$$\begin{aligned} \||\mathrm{e}^{-tL^{1/2}} f_j||a^*v(t,\cdot)|\|_1 \leq \|\mathbf{1}_{2^{j-1/2}Q}\mathrm{e}^{-tL^{1/2}} f_j\|_p \|a^*v(t,\cdot)\|_{p'} \\ + \|\mathrm{e}^{-tL^{1/2}} f_j\|_p \|\mathbf{1}_{^c(2^{j-1/2}Q)} a^*v(t,\cdot)\|_{p'}. \end{aligned} \tag{18.20}$$

Since $p \in [2, p_+(L))$, the Poisson semigroup satisfies L^p off-diagonal estimates of order 1, see Corollary 4.17. From the support of f_j and Lemma 18.1 we obtain for $j \geq 2$ with implicit constants independent of j and t,

$$\|\mathbf{1}_{2^{j-1/2}Q} \mathrm{e}^{-tL^{1/2}} f_j\|_p \lesssim t2^{-j}\|f_j\|_p \lesssim t\gamma_j 2^{j(\frac{n}{p}-1)}. \tag{18.21}$$

and

$$\|\mathrm{e}^{-tL^{1/2}} f_j\|_p \lesssim \|f_j\|_p \lesssim \gamma_j 2^{j\frac{n}{p}}.$$

Likewise, integrating the p'-th powers of both sides of (18.18) gives

$$\|a^* v(t,\cdot)\|_{p'} \lesssim (1 \wedge t^{-n-1})(1 + t^{\frac{n}{p'}}) \lesssim 1 \wedge t^{-1-\frac{n}{p}}.$$

and, with a smaller constant c then in (18.18),

$$\|\mathbf{1}_{{}^c(2^{j-1/2}Q)} a^* v(t,\cdot)\|_{p'} \lesssim (1 \wedge t^{-n-1}) t^{\frac{n}{p'}} \mathrm{e}^{-c\frac{2^j}{t}} \lesssim (t^{n+1-\frac{n}{p}} \wedge t^{-\frac{n}{p}})2^{-j},$$

where in the final step we have used the crude bound $\mathrm{e}^{-s} \lesssim s^{-1}$ for $s > 0$ in order to restore the right homogeneity in t. Using these bounds on the right-hand side of (18.20), leads us to

$$\||\mathrm{e}^{-tL^{1/2}} f_j||a^* v(t,\cdot)|\|_1 \lesssim (t \wedge t^{-\frac{n}{p}})\gamma_j 2^{j\frac{n}{p}-1}. \tag{18.22}$$

Since $\alpha < 1 - {}^n\!/_p$, we can sum in j and conclude (18.19).

As a matter of fact, the same estimate holds if in the definition of Φ we replace $v(t,\cdot)$ by $t\nabla v(t,\cdot)$, which satisfies the same pointwise bounds. We can also replace $\mathrm{e}^{-tL^{1/2}}$ by $(tL^{1/2})^k \mathrm{e}^{-tL^{1/2}}$ for an integer $k \geq 1$ since the latter satisfies again L^p off-diagonal estimates of order 1, see Lemma 4.16. All such sums are called *of Φ-type*. We also remark that it was only the bound (18.20) that required $j \geq 2$. All other estimates in this step also work for $j = 1$.

Step 3: Integration by parts in t. Since we have left out the term for $j = 1$ in Step 2, the full estimate for $\Phi(t)$ is

$$|\Phi(t) - \langle \mathrm{e}^{-tL^{1/2}} f_1, a^* v(t,\cdot)\rangle| \lesssim t \wedge t^{-\frac{n}{p}}$$

By the functional calculus on L^2 we have

$$\lim_{t\to 0} \langle \mathrm{e}^{-tL^{1/2}} f_1, a^* v(t,\cdot)\rangle = \langle f_1, g\rangle = \langle f, g\rangle,$$

where in the final step we used the support of f_1, and likewise

$$\lim_{t\to\infty} \langle \mathrm{e}^{-tL^{1/2}} f_1, a^* v(t,\cdot)\rangle = 0.$$

We conclude $\lim_{t\to 0} \Phi(t) = \langle f, g\rangle$ and $\lim_{t\to\infty} \Phi(t) = 0$. Next,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathrm{e}^{-tL^{1/2}} f_j, a^* v(t,\cdot)\rangle = -\langle L^{1/2} \mathrm{e}^{-tL^{1/2}} f_j, a^* v(t,\cdot)\rangle \\ + \langle \mathrm{e}^{-tL^{1/2}} f_j, a^* \partial_t v(t,\cdot)\rangle\end{aligned}$$

gives rise to two sums of Φ-type (times a factor of t^{-1}), which converge locally uniformly in t by Step 2. Hence, we can differentiate Φ term by term. The upshot is that we can integrate Φ by parts to obtain

$$\langle f, g\rangle = \int_0^\infty \Phi'(t)\,\mathrm{d}t := \int_0^\infty \Phi^{(1)}(t)\,\mathrm{d}t - \int_0^\infty \Psi^{(1)}(t)\,\mathrm{d}t, \tag{18.23}$$

where $\Phi^{(1)}(t), \Psi^{(1)}(t) : (0,\infty) \to \mathbb{C}$ are given by

$$\Phi^{(1)}(t) := \sum_{j=1}^\infty \langle L^{1/2}\mathrm{e}^{-tL^{1/2}} f_j, a^* v(t,\cdot)\rangle,$$
$$\Psi^{(1)}(t) := \sum_{j=1}^\infty \langle \mathrm{e}^{-tL^{1/2}} f_j, a^* \partial_t v(t,\cdot)\rangle$$

and $t\Phi^{(1)}$ and $t\Psi^{(1)}$ are of Φ-type. The idea here is that $\Phi^{(1)}$ is the bad term that we have to keep, whereas the part involving $\Psi^{(1)}$ can be treated directly.

Step 4: Integral estimate for $\Psi^{(1)}$. We introduce

$$\widetilde{v}(t,\cdot) := 2\beta(1+t^2H)^{-\beta-1}((a^*)^{-1}g),$$

which is of the same type as v but with a higher resolvent power. The objective in this step is to establish the bound

$$\int_0^\infty |\Psi^{(1)}(t)|\,\mathrm{d}t \leq \iint_{\mathbb{R}^{1+n}_+} |t\nabla_x u| \cdot |t\nabla_x \widetilde{v}| \,\frac{\mathrm{d}t\mathrm{d}x}{t}. \tag{18.24}$$

Let $\eta \in \mathrm{C}_0^\infty(\mathbb{R}^n;\mathbb{R})$ be such that $\mathbf{1}_Q \leq \eta \leq \mathbf{1}_{2Q}$ and for $R > 0$ set $\eta_R(x) := \eta(x/R)$. We note that

$$\begin{aligned} a^*\partial_t v(t,\cdot) &= -2\beta a^* tH(1+t^2H)^{-\beta-1}((a^*)^{-1}g) \\ &= -2\beta t\Delta_x(1+t^2H)^{-\beta-1}((a^*)^{-1}g) \\ &=: -t\Delta_x\widetilde{v}(t,\cdot) \end{aligned}$$

and, having split

$$\Psi^{(1)}(t) = \sum_{j=1}^\infty \langle \eta_R \mathrm{e}^{-tL^{1/2}} f_j, -t\Delta_x\widetilde{v}(t,\cdot)\rangle + \langle (1-\eta_R)\mathrm{e}^{-tL^{1/2}} f_j, a^*\partial_t v(t,\cdot)\rangle,$$

we can integrate by parts the term with η_R to give

$$\begin{aligned} \Psi^{(1)}(t) &= \frac{1}{t}\langle \eta_R t\nabla_x u(t,\cdot), t\nabla_x\widetilde{v}(t,\cdot)\rangle \\ &\quad + \frac{1}{t}\sum_{j=1}^\infty \langle (t\nabla_x\eta_R)\otimes \mathrm{e}^{-tL^{1/2}} f_j, t\nabla_x\widetilde{v}(t,\cdot)\rangle \\ &\quad + \frac{1}{t}\sum_{j=1}^\infty \langle (1-\eta_R)\mathrm{e}^{-tL^{1/2}} f_j, ta^*\partial_t v(t,\cdot)\rangle. \end{aligned} \tag{18.25}$$

Our notation is

$$\nabla_x(\eta_R \mathrm{e}^{-tL^{1/2}} f_j) = \eta_R \nabla_x \mathrm{e}^{-tL^{1/2}} f_j + (\nabla_x \eta_R) \otimes \mathrm{e}^{-tL^{1/2}} f_j$$

as predicted by the product rule and for the sum with $\eta_R \nabla_x \mathrm{e}^{-tL^{1/2}} f_j$ we have used that the series that defines $u(t, \cdot)$ converges in $\mathrm{W}^{1,2}_{\mathrm{loc}}$ as a consequence of $\mathrm{L}^2_{\mathrm{loc}}$-convergence and the Caccioppoli inequality.

So far, (18.25) holds for any $t > 0$ and any $R > 0$. We let now $k \geq 2$, set $R := (1 \vee t)k$ and integrate in t to obtain

(18.26)
$$\begin{aligned}
&\int_0^\infty |\Psi^{(1)}(t)| \,\mathrm{d}t \\
&\quad \leq \iint_{\mathbb{R}^{1+n}_+} |t\nabla_x u| \cdot |t\nabla_x \widetilde{v}| \,\frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\quad + c_n \int_0^\infty \int_{^c(2Q)} |t\nabla_x \eta_{(1\vee t)k}| \sum_{j=1}^\infty |\mathrm{e}^{-tL^{1/2}} f_j||t\nabla_x \widetilde{v}| \,\frac{\mathrm{d}x\mathrm{d}t}{t} \\
&\quad + c_n \|a^*\|_\infty \int_0^\infty \int_{^c(2Q)} |1 - \eta_{(1\vee t)k}| \sum_{j=1}^\infty |\mathrm{e}^{-tL^{1/2}} f_j||t\partial_t v| \,\frac{\mathrm{d}x\mathrm{d}t}{t},
\end{aligned}$$

where c_n only depends on n. We also used that the terms with η vanish on $2Q$ and interchanged the sum with the integral in x using the monotone convergence theorem.

The sums in j are of Φ-type and when using the bounds from Step 2 for such sums only on $^c(2Q)$, we can allow $j = 1$ and pick up the same behavior in t. Indeed, on the right-hand side of (18.20) we would only get the second term when $j = 1$. It follows that

$$\sum_{j=1}^\infty \| |\mathrm{e}^{-tL^{1/2}} f_j||t\nabla_x \widetilde{v}| \|_{\mathrm{L}^1(^c(2Q))} \lesssim t \wedge t^{-\frac{n}{p}}$$

and likewise with $\partial_t v$ replacing $t\nabla_x \widetilde{v}$. Hence, in (18.26) the sums in j are of class $\mathrm{L}^1((0,\infty) \times {}^c(2Q); \frac{\mathrm{d}x\mathrm{d}t}{t})$. Since $|t\nabla_x \eta_{(1\vee t)k}|$ and $|1 - \eta_{(1\vee t)k}|$ are bounded by dimensional constants and tend to 0 pointwise as $k \to \infty$, we can use the dominated convergence theorem in (18.26) to conclude (18.24).

Step 5: Completing the treatment of $\Psi^{(1)}$ by duality. We can interpret the right-hand side in (18.24) as a $\mathrm{T}^{0,\infty;\alpha} - \mathrm{T}^\varrho$ duality pairing, where $\varrho \in (1_*, 1]$ is such that $\alpha = n(1/\varrho - 1)$, see Section 2.2. Consequently, we have

$$\int_0^\infty |\Psi^{(1)}(t)| \,\mathrm{d}t \lesssim \|C_\alpha(t\nabla_x u)\|_\infty \|S(t\nabla_x \widetilde{v})\|_\varrho.$$

In order to bound the square function, let

$$B_H \coloneqq \begin{bmatrix} (a^*)^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

be the matrix that corresponds to H in the same way as B corresponds to L. Recalling (3.2) and the intertwining relation (3.15), we write

$$\begin{aligned} \begin{bmatrix} 0 \\ t\nabla_x \widetilde{v} \end{bmatrix} &= -2\beta t D(1+(tB_H D)^2)^{-\beta-1} B_H \begin{bmatrix} g \\ 0 \end{bmatrix} \\ &=: \psi(tDB_H) \begin{bmatrix} g \\ 0 \end{bmatrix}, \end{aligned} \tag{18.27}$$

where $\psi(z) = -2\beta z(1+z^2)^{-\beta-1}$ is of class $\Psi_1^{2\beta+1}$ on any sector. As $\beta > {}^n\!/2+2$, this is an admissible auxiliary function for $\mathbb{H}^\varrho_{DB_H}$. From $p_-(H) = 1_*$ (Corollary 6.10) and the identification theorem (Theorem 9.6) we obtain

$$\|S(t\nabla_x \widetilde{v})\|_\varrho \simeq \left\| \begin{bmatrix} g \\ 0 \end{bmatrix} \right\|_{\mathbb{H}^\varrho_{DB_H}} \simeq \|g\|_{\mathrm{H}^\varrho}.$$

Thus, we have found

$$\int_0^\infty |\Psi^{(1)}(t)| \, \mathrm{d}t \lesssim \|C_\alpha(t\nabla_x u)\|_\infty \|g\|_{\mathrm{H}^\varrho}, \tag{18.28}$$

which is a desirable bound for the second term in (18.23).

Step 6: Setting up an iteration on $\Psi^{(1)}$. At this point we are left with proving

$$\left| \int_0^\infty \Phi^{(1)}(t) \, \mathrm{d}t \right| \lesssim \|C_\alpha(t\nabla u)\|_\infty \|g\|_{\mathrm{H}^\varrho},$$

where

$$\Phi^{(1)}(t) = \sum_{j=1}^\infty \langle L^{1/2} \mathrm{e}^{-tL^{1/2}} f_j, a^* v(t,\cdot)\rangle.$$

Since $t\Phi^{(1)}(t)$ is of Φ-type, we can repeat Step 3 with this function replacing $\Phi(t)$. The only difference is that now $\lim_{t\to 0} t\Phi^{(1)}(t) = 0$ and we can integrate by parts without boundary terms to give

$$\int_0^\infty \Phi^{(1)}(t) \, \mathrm{d}t = \int_0^\infty t\Phi^{(2)}(t) \, \mathrm{d}t - \int_0^\infty t\Psi^{(2)}(t) \, \mathrm{d}t,$$

where

$$\Phi^{(2)}(t) \coloneqq \sum_{j=1}^\infty \langle L\mathrm{e}^{-tL^{1/2}} f_j, a^* v(t,\cdot)\rangle,$$

$$\Psi^{(2)}(t) \coloneqq \sum_{j=1}^\infty \langle L^{1/2}\mathrm{e}^{-tL^{1/2}} f_j, a^* \partial_t v(t,\cdot)\rangle.$$

Now, $t^2\Phi^{(2)}$ and $t^2\Psi^{(2)}$ are of Φ-type and $t\Psi^{(2)}$ is of the same structure as $\Psi^{(1)}$ except for an extra t-derivative on the Poisson semigroup. Hence, we can repeat Step 4 and Step 5 *mutadis mutandis* for $\Psi^{(2)}$ and arrive at

$$\int_0^\infty |t\Psi^{(2)}(t)| \, \mathrm{d}t \lesssim \|C_\alpha(t^2\nabla_x\partial_t u)\|_\infty \|g\|_{\mathrm{H}^\varrho}$$

as replacement for (18.28). But since $\partial_t u$ is a weak solution to the same equation, we can use Caccioppoli's inequality on Carleson boxes $(0, \ell(Q)) \times Q$ as in Part 3 to bound

$$\|C_\alpha(t^2\nabla_x\partial_t u)\|_\infty \lesssim \|C_\alpha(t\partial_t u)\|_\infty$$

and conclude with a desirable bound.

The upshot is that we can iterate this scheme until for some large N, depending on the dimension, we can control

$$\left|\int_0^\infty t^{N-1}\Phi^{(N)}(t)\, \mathrm{d}t\right| \lesssim \iint_{\mathbb{R}^{1+n}_+} |t^{N-1}\nabla_x\partial_t^{N-2}u| \cdot |t\nabla_x v| \, \frac{\mathrm{d}t\mathrm{d}x}{t}, \tag{18.29}$$

where

$$\Phi^{(N)}(t) := \sum_{j=1}^\infty \langle L^{N/2}\mathrm{e}^{-tL^{1/2}} f_j, a^* v(t,\cdot)\rangle.$$

Indeed, a desirable bound for the right-hand side of (18.29) follows by $\mathrm{T}^{0,\infty;\alpha} - \mathrm{T}^\varrho$-duality and Caccioppoli's inequality as before.

Step 7: Reduction to a final estimate of Φ-type. We shall establish (18.29) for the first integer that satisfies $N > n/2 + 2$. As

$$\langle L^{N/2}\mathrm{e}^{-tL^{1/2}} f_j, a^* v(t,\cdot)\rangle = \langle -\operatorname{div}_x d\nabla_x L^{N/2-1}\mathrm{e}^{-tL^{1/2}} f_j, v(t,\cdot)\rangle,$$

we can integrate by parts as in Step 4 but in the opposite direction. Using the same notation, the replacement for (18.25) is

$$\begin{aligned} &t^{N-1}\Phi^{(N)}(t) \\ &\quad = \frac{(-1)^{N-2}}{t}\langle dt^{N-1}\nabla_x\partial_t^{N-2}u(t,\cdot), \eta_R t\nabla_x v(t,\cdot)\rangle \\ &\qquad + \frac{1}{t}\sum_{j=1}^\infty \langle dt\nabla_x(tL^{1/2})^{N-2}\mathrm{e}^{-tL^{1/2}} f_j, (t\nabla_x\eta_R)\otimes v(t,\cdot)\rangle \\ &\qquad + \frac{1}{t}\sum_{j=1}^\infty \langle (tL^{1/2})^N\mathrm{e}^{-tL^{1/2}} f_j, (1-\eta_R)a^* v(t,\cdot)\rangle \end{aligned}$$

and the replacement for (18.26) is

$$\begin{aligned} &\int_0^\infty |t^{N-1}\Phi^{(N)}(t)| \, \mathrm{d}t \\ &\quad \le \|d\|_\infty \iint_{\mathbb{R}^{1+n}_+} |t^{N-1}\nabla_x\partial_t^{N-2}u| \cdot |t\nabla_x v| \, \frac{\mathrm{d}t\mathrm{d}x}{t} \end{aligned}$$

$$+ c_n \int_0^\infty \int_{^c(2Q)} |t\nabla_x \eta_{(1\vee t)k}| \sum_{j=1}^\infty |td\nabla_x (tL^{1/2})^{N-2} \mathrm{e}^{-tL^{1/2}} f_j||v| \, \frac{\mathrm{d}x\mathrm{d}t}{t}$$
$$+ c_n \|a^*\|_\infty \int_0^\infty \int_{^c(2Q)} |1 - \eta_{(1\vee t)k}| \sum_{j=1}^\infty |(tL^{1/2})^N \mathrm{e}^{-tL^{1/2}} f_j||v| \, \frac{\mathrm{d}x\mathrm{d}t}{t},$$

where c_n only depends on n. Thus, we have to prove that the second and third term on the right vanish in the limit as $k \to \infty$. The third term contains a sum of Φ-type, so that we can use dominated convergence as in Step 4. The middle term is not of Φ-type since we do not have L^p off-diagonal estimates for the gradient of the Poisson semigroup. We claim that nonetheless there are $\sigma, \tau > 0$ such that

$$\sum_{j=1}^\infty \| |t\nabla_x (tL^{1/2})^{N-2} \mathrm{e}^{-tL^{1/2}} f_j||v| \|_{\mathrm{L}^1(^c(2Q))} \lesssim t^\sigma \wedge t^{-\tau}. \tag{18.30}$$

Taking this estimate for granted, dominated convergence also applies to the middle term and (18.29) follows.

Step 8: Conclusion. In order to prove the final missing bound (18.30), we argue as in Step 2 but with $p = 2$. To simplify notation, let

$$T(t) \coloneqq t\nabla_x (tL^{1/2})^{N-2} \mathrm{e}^{-tL^{1/2}} \quad (t > 0).$$

This family satisfies L^2 off-diagonal estimates of order $N-1$ by composition and Lemma 4.16 since we can write

$$T(t) = \Big(t\nabla_x (1 + t^2 L)^{-1}\Big)\Big((tL^{1/2})^{N-2}(1 + t^2 L)\mathrm{e}^{-tL^{1/2}}\Big).$$

By Hölder's inequality, we have

$$\begin{aligned} &\| |T(t) f_j||v(t,\cdot)| \|_{\mathrm{L}^1(^c(2Q))} \\ &\qquad \le \|\mathbf{1}_{2^{j-1/2}Q} T(t) f_j\|_{\mathrm{L}^2(^c(2Q))} \|v(t,\cdot)\|_{\mathrm{L}^2(^c(2Q))} \\ &\qquad\quad + \|T(t) f_j\|_2 \|\mathbf{1}_{^c(2^{j-1/2}Q)} v(t,\cdot)\|_2, \end{aligned} \tag{18.31}$$

where the first term on the right vanishes for $j = 1$. From the support of f_j and Lemma 18.1 we obtain for $j \ge 2$ that

$$\|\mathbf{1}_{2^{j-1/2}Q} T(t) f_j\|_{\mathrm{L}^2(^c(2Q))} \lesssim t^\gamma 2^{-j\gamma} \|f_j\|_2 \lesssim t^\gamma \gamma_j 2^{j(\frac{n}{2} - \gamma)}$$

and for $j \ge 1$ that

$$\|T(t) f_j\|_2 \lesssim \|f_j\|_2 \lesssim \gamma_j 2^{j\frac{n}{2}}$$

with $\gamma \in (0, N-1]$ at our disposal and implicit constants independent of j and t. The bounds for v are obtained by squaring both sides of (18.18) and integrating. They take the form

$$\|v(t,\cdot)\|_{\mathrm{L}^2(^c(2Q))} \lesssim (1 \wedge t^{-n-1}) t^{\frac{n}{2}} = t^{\frac{n}{2}} \wedge t^{-1-\frac{n}{2}}$$

and

$$\|\mathbf{1}_{^c(2^{j-1/2}Q)} v(t,\cdot)\|_2 \lesssim (1 \wedge t^{-n-1}) t^{\frac{n}{2}} \mathrm{e}^{-\frac{c}{2}\frac{2^j}{t}} \lesssim (t^{\frac{n}{2}+\gamma} \wedge t^{\gamma - \frac{n}{2} - 1}) 2^{-j\gamma},$$

where in the final step we have used the crude bound $\mathrm{e}^{-s} \lesssim s^{-\gamma}$ for $s > 0$ to restore the homogeneity in t. Using these bounds on the right-hand side of (18.31), we find

$$\| |T(t)f_j| |v(t,\cdot)| \|_{\mathrm{L}^1({}^c(2Q))} \lesssim (t^{\frac{n}{2}+\gamma} \wedge t^{\gamma-\frac{n}{2}-1})\gamma_j 2^{j(\frac{n}{2}-\gamma)}.$$

We need $\gamma > n/2+\alpha$ to be able to sum in j and $\gamma < n/2+1$ to pick up decay at $t = \infty$. Such γ exists since $\alpha < 1$ and the choice is admissible because we have assumed $N > n/2 + 2$. It is only at this point where we need the size of N. Now, (18.30) follows from (18.31) and the proof is complete.

Part 6: Proof of (iii). Instead of (18.2) we work with the following exponents in this part:

(18.32)
- $p_-(L^\sharp) < 1$ and $0 \leq \alpha < n(1/p_-(L^\sharp) - 1)$.
- When α is fixed, $p_-(L^\sharp) < p \leq 1$ is such that $\alpha = n(1/p - 1)$.

This is a stronger assumption than in the previous parts since $p_-(L^\sharp) < 1$ implies $p_+(L) = \infty$ by duality and similarity.

In particular, $(\mathrm{e}^{-t(L^\sharp)^{1/2}})_{t>0}$ is $(a^*)^{-1}\,\mathrm{H}^p$-bounded by Theorem 12.2 and we can define $(\mathrm{e}^{-tL^{1/2}})_{t>0}$ as a bounded semigroup on $\dot{\Lambda}^\alpha$ via duality and similarity:

$$\langle \mathrm{e}^{-tL^{1/2}} f, g\rangle \coloneqq \langle f, a^* \mathrm{e}^{-t(L^\sharp)^{1/2}} (a^*)^{-1} g\rangle \quad (f \in \dot{\Lambda}^\alpha, g \in \mathrm{H}^p \cap \mathrm{L}^2).$$

Next, we identify the solution u from (18.1) with such a semigroup extension.

Lemma 18.7. *Assume* (18.32)*. If* $g \in \mathrm{C}_0^\infty$ *with* $\int_{\mathbb{R}^n} g \,\mathrm{d}x = 0$*, then*

$$\langle u(t,\cdot), g\rangle = \langle f, a^* \mathrm{e}^{-t(L^\sharp)^{1/2}} (a^*)^{-1} g\rangle \quad (t > 0),$$

where the left-hand side is the extended $\mathrm{L}^2 - \mathrm{L}^2$*-duality and the right-hand side is the* $\dot{\Lambda}^\alpha - \mathrm{H}^p$*-duality.*

Proof. We fix t and g and let Q be a cube that contains the support of g. As $a^* \mathrm{e}^{-t(L^\sharp)^{1/2}} (a^*)^{-1} g \in \mathrm{H}^p \cap \mathrm{L}^2 \subseteq \mathrm{H}^1$ we have in particular that $\int_{\mathbb{R}^n} a^* \mathrm{e}^{-t(L^\sharp)^{1/2}} (a^*)^{-1} g \,\mathrm{d}x = 0$. Therefore we can assume $(f)_Q = 0$. In the following, C denotes a constant that may depend on all parameters but on $j \geq 1$ used for the annuli $C_j(Q)$.

Since $p \in (1_*, 1)$, we can fix $q \in (1,2)$ such that $\varepsilon \coloneqq n/q - n/p + 1 > 0$. Then $(\mathrm{e}^{-t(L^\sharp)^{1/2}})_{t>0}$ is L^q-bounded and satisfies L^q off-diagonal estimates of order 1, see Corollary 4.17. We conclude that

$$\|a^* \mathrm{e}^{-t(L^\sharp)^{1/2}} (a^*)^{-1} g\|_{\mathrm{L}^q(C_j(Q))} \leq C 2^{-j} = C 2^{j(\frac{n}{q}-\frac{n}{p})} 2^{-\varepsilon j}.$$

Hence, we can use Lemma 4.10 in order to write

$$a^* \mathrm{e}^{-t(L^\sharp)^{1/2}} (a^*)^{-1} g = \sum_{j=1}^\infty C 2^{-\varepsilon j} a_j \quad (\text{in } \mathrm{H}^p \cap \mathrm{L}^1_{\mathrm{loc}}),$$

where the a_j are L^q-atoms for H^p with support in $C_{j+1}(Q) \cup C_j(Q)$. Using Lemma 18.1 with exponent q' and the atomic bounds, we obtain

$$|\langle f, C2^{-\varepsilon j} a_j \rangle| \leq C\gamma_j 2^{j\frac{n}{q'}} 2^{-\varepsilon j} \|a_j\|_q \leq C\gamma_j 2^{-\alpha j} 2^{-\varepsilon j}. \tag{18.33}$$

Now, we use the definition of u, duality for the semigroups on L^2 and absolute convergence of the series following from (18.33) in order to write, setting $a_0 \coloneqq 0$,

$$\begin{aligned}
\langle u(t,\cdot), g\rangle &= \sum_{j=1}^\infty \langle \mathrm{e}^{-tL^{1/2}} \mathbf{1}_{C_j(Q)} f, g\rangle \\
&= \sum_{j=1}^\infty \langle \mathbf{1}_{C_j(Q)} f, a^* \mathrm{e}^{-t(L^\sharp)^{1/2}} (a^*)^{-1} g\rangle \\
&= \sum_{j=1}^\infty C\langle f, \mathbf{1}_{C_j(Q)} (2^{-\varepsilon j} a_j + 2^{-\varepsilon(j-1)} a_{j-1})\rangle \\
&= \sum_{j=1}^\infty C\langle f, (\mathbf{1}_{C_j(Q)} + \mathbf{1}_{C_{j+1}(Q)}) 2^{-\varepsilon j} a_j\rangle \\
&= \sum_{j=1}^\infty \langle f, C2^{-\varepsilon j} a_j\rangle = \langle f, a^* \mathrm{e}^{-t(L^\sharp)^{1/2}} (a^*)^{-1} g\rangle. \qquad \square
\end{aligned}$$

Since C_0^∞-functions with integral zero are dense in H^p, we obtain from the lemma and Proposition 17.4 applied to $L^\sharp$ that u is of class

$$\mathrm{C}_0([0,\infty); \dot{\Lambda}^\alpha_{\text{weak}^*}) \cap \mathrm{C}^\infty((0,\infty); \dot{\Lambda}^\alpha_{\text{weak}^*}),$$

where the subscript indicates that $\dot{\Lambda}^\alpha$ carries the weak* topology as the dual of H^p, with bound

$$\sup_{t>0} \|u(t,\cdot)\|_{\dot{\Lambda}^\alpha} \lesssim \|f\|_{\dot{\Lambda}^\alpha}.$$

In the opposite direction, Part 2 implies for all $g \in \mathrm{C}_0^\infty$ with integral zero that

$$|\langle f, g\rangle| = \lim_{t\to 0} |\langle u(t,\cdot), g\rangle| \leq \sup_{t>0} \|u(t,\cdot)\|_{\dot{\Lambda}^\alpha} \|g\|_{\mathrm{H}^p}$$

and $\|f\|_{\dot{\Lambda}^\alpha} \leq \sup_{t>0} \|u(t,\cdot)\|_{\dot{\Lambda}^\alpha}$ follows. Hence, we have

$$\sup_{t>0} \|u(t,\cdot)\|_{\dot{\Lambda}^\alpha} \simeq \|f\|_{\dot{\Lambda}^\alpha}. \tag{18.34}$$

For the global $\dot{\Lambda}^\alpha(\overline{\mathbb{R}^{1+n}_+})$ upper bound we need a variant of the Poincaré inequality that we prove at the end of the section.

Lemma 18.8. *Let $v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ with $t^{1/2}\nabla v \in \mathrm{L}^2_{\mathrm{loc}}(\overline{\mathbb{R}^{1+n}_+})$. There is a dimensional constant c such that for all cubes $Q \subseteq \mathbb{R}^n$,*

$$⨌_{T(Q)} |v - (v)_{T(Q)}|^2 \,\mathrm{d}s\mathrm{d}y \leq c \int_0^{\ell(Q)} ⨍_Q s|\nabla v|^2 \,\mathrm{d}y\mathrm{d}s,$$

where $T(Q) := (0, \ell(Q)) \times Q$. In particular, $v \in \mathrm{L}^2_{\mathrm{loc}}(\overline{\mathbb{R}^{1+n}_+})$. The same inequality holds with balls instead of cubes.

Together with the upper Carleson bound of Part 3 we now obtain, for all cubes $Q \subseteq \mathbb{R}^n$,

$$\begin{aligned}\Big(\fint\!\!\fint_{T(Q)} |u - (u)_{T(Q)}|^2 \,\mathrm{d}s\mathrm{d}y\Big)^{\frac{1}{2}} &\lesssim \Big(\int_0^{\ell(Q)} \fint_Q s|\nabla u|^2 \,\mathrm{d}y\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\leq \ell(Q)^\alpha \|C_\alpha(t\nabla u)\|_\infty \\ &\lesssim \ell(Q)^\alpha \|f\|_{\dot{\Lambda}^\alpha}.\end{aligned}$$

This is an oscillation estimate at the boundary of $\mathbb{R}^{1+n}_+$. In order to replace $T(Q)$ by an arbitrary cube $T(Q) + (t_0, 0)$ with $t_0 > 0$, we use that according to Lemma 18.7 we have the semigroup property $u(t + t_0, \cdot) = \mathrm{e}^{-tL^{1/2}} f_{t_0} =: u_{t_0}(t, \cdot)$, where $f_{t_0} := u(t_0, \cdot) \in \dot{\Lambda}^\alpha$. The previous estimate with u_{t_0} in place of u becomes

$$\begin{aligned}\Big(\fint\!\!\fint_{T(Q)+(t_0,0)} |u - (u)_{T(Q)+(t_0,0)}|^2 \,\mathrm{d}s\mathrm{d}y\Big)^{\frac{1}{2}} &\lesssim \ell(Q)^\alpha \|f_{t_0}\|_{\dot{\Lambda}^\alpha} \\ &\lesssim \ell(Q)^\alpha \|f\|_{\dot{\Lambda}^\alpha},\end{aligned}$$

where the final step is due to (18.34). By definition of the BMO-norm if $\alpha = 0$ and by the Morrey–Campanato characterization of Hölder continuity if $\alpha \in (0, 1)$, see [79], we conclude

$$\|u\|_{\dot{\Lambda}^\alpha(\overline{\mathbb{R}^{1+n}_+})} \lesssim \|f\|_{\dot{\Lambda}^\alpha}.$$

The proof of (iii) is complete, modulo the

Proof of Lemma 18.8. We can assume that Q is the unit cube centered at the origin, as a scaling argument gives the general result. Let $T_\varepsilon(Q) := (\varepsilon, 1) \times (1 - \varepsilon)Q$ for $\varepsilon \in (0, 1)$. We apply first the Hardy–Poincaré inequality of Boas–Straube [31]:

$$\iint_{T_\varepsilon(Q)} |v - (v)_{T_\varepsilon(Q)}|^2 \,\mathrm{d}s\mathrm{d}y \leq c_\varepsilon \iint_{T_\varepsilon(Q)} \mathrm{d}((s, y), \partial T_\varepsilon(Q))|\nabla v|^2 \,\mathrm{d}s\mathrm{d}y.$$

A priori, the constant c_ε depends on $T_\varepsilon(Q)$ but scaling and translation to $(1, 2) \times Q$ reveals that we can take $c_\varepsilon = (1 - \varepsilon)c$, where c is dimensional. We conclude

$$\iint_{T_\varepsilon(Q)} |v - (v)_{T_\varepsilon(Q)}|^2 \,\mathrm{d}s\mathrm{d}y \leq c \iint_{T(Q)} s|\nabla v|^2 \,\mathrm{d}s\mathrm{d}y, \tag{18.35}$$

where the right-hand side is assumed to be finite.

Now, consider a decreasing sequence of values $\varepsilon \in (0, {}^1\!/_2)$ with $\varepsilon \to 0$. Since $T_{1/2}(Q) \subseteq T_\varepsilon(Q)$ and $v \in \mathrm{L}^2(T_{1/2}(Q))$, it follows from (18.35) that the numerical sequence $((v)_{T_\varepsilon(Q)})_\varepsilon$ is bounded. Let C be one of its accumulation points. Via Fatou's lemma we can pass to the limit in (18.35) along a subsequence of ε to give

$$\iint_{T(Q)} |v - C|^2 \, \mathrm{d}s\mathrm{d}y \leq c \iint_{T(Q)} s|\nabla v|^2 \, \mathrm{d}s\mathrm{d}y.$$

This implies that v is (square) integrable on $T(Q)$ and therefore we have $C = (v)_{T(Q)}$ by dominated convergence.

The argument for balls instead of cubes is the same. □

19. Existence in Dirichlet problems with fractional regularity data

In this section we prove the compatible existence on Dirichlet problems with data in homogeneous Hardy–Sobolev and Besov spaces of fractional smoothness that have been announced in Section 1.6. We also compare them to what can be obtained by the general first-order approach [3] when specialized to elliptic systems in block form. We recall the color code for our various exponent regions and segments:

- Gray corresponds to what can be obtained from the theory of DB-adapted spaces in [3] and our identification of the interval from [11, 19] in Corollary 15.2.
- Blue shows extra information obtained from the theory of L-adapted spaces.
- Red indicates results outside of the theory of operator-adapted spaces.

When we speak of 'colored' points or regions, we always mean points or regions that are displayed in one of these three colors.

19.1. **Fractional identification regions.** As in Section 8 we treat adapted Hardy–Sobolev and Besov spaces simultaneously by letting X denote one of B or H. As before, 'identification' means 'equality of sets with equivalent p-quasinorms'.

Proposition 19.1. *Identification* $\mathbb{X}^{s,p}_L = \dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2$ *holds for all exponents corresponding to the interior of the colored trapezoidal region in Figure 10.*

Proof. Theorem 9.6 yields $\mathbb{H}^{1,p}_L = \dot{\mathrm{H}}^{1,p} \cap \mathrm{L}^2$ and $\mathbb{H}^p_L = a^{-1}(\mathrm{H}^p \cap \mathrm{L}^2) = \mathrm{L}^p \cap \mathrm{L}^2$ if $(1/p, s)$ belongs to the open segments that join $(1/q_+(L), 1)$ to $(1/(p_-(L)_* \vee 1_*), 1)$ and $(1/p_+(L), 0)$ to $(1/(p_-(L) \vee 1), 0)$, respectively. Both cases can be summarized as saying $\mathbb{H}^{s,p}_L = \mathbb{H}^{s,p}_{-\Delta_x}$, see Figure 6. By real and complex interpolation ([3, Thm.4.32] or equivalently the argument in the proof of Lemma 9.4) we conclude $\mathbb{X}^{s,p}_L = \mathbb{X}^{s,p}_{-\Delta_x}$ in the interior of the convex hull of the two segments and the claim follows by using Figure 6 again. □

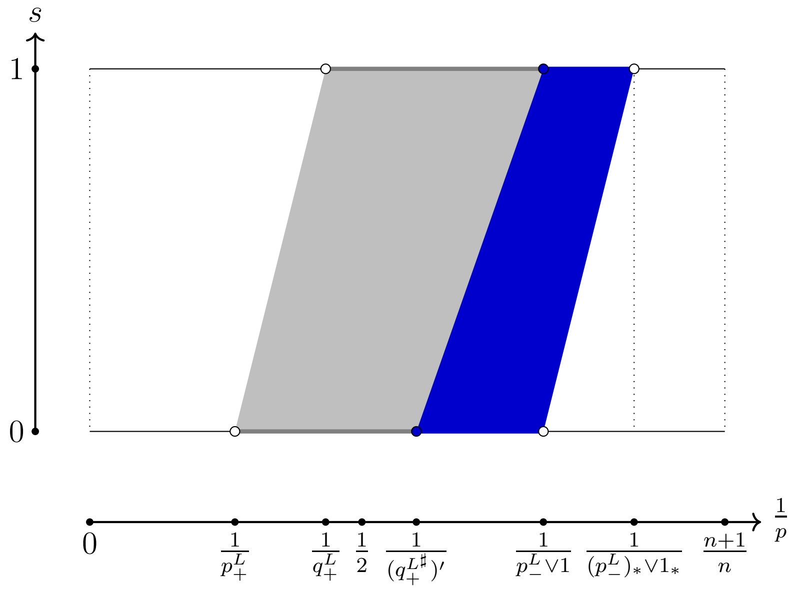

FIGURE 10. Identification $\mathbb{X}_L^{s,p} = \dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2$ up to equivalent p-quasinorms holds for all exponents corresponding to the interior of the colored trapezoidal region. The picture is up to scale when $p_-(L) \geq 1$. When $p_-(L) < 1$, the top blue point is situated at $({}^1\!/_{p_-(L)}, 1)$.

Remark 19.2. If $p_-(L) < 1$, then we could also combine Theorem 9.6 with Corollary 6.10 and write $\mathbb{H}_L^{s,p} = \mathbb{H}_{-a^{-1}\Delta_x}^{s,p}$ on the top segment, which in this case joins $({}^1\!/_{q_+(L)}, 1)$ to $({}^1\!/_{1_*}, 1)$, and the full bottom segment joining $({}^1\!/_{p_+(L)}, 0)$ to $({}^1\!/_{p_-(L)}, 0)$. Extending Figure 10 to the left by the triangle with vertices $({}^1\!/_{1_*}, 1), (1, 0), ({}^1\!/_{p_-(L)}, 0)$, the same interpolation argument yields identification $\mathbb{X}_L^{s,p} = \mathbb{X}_{-a^{-1}\Delta_x}^{s,p}$ in the interior of that extended region. The reason why we do not use this extension is that we do not know whether $\mathbb{X}_{-a^{-1}\Delta_x}^{s,p} = \mathbb{X}_{-\Delta_x}^{s,p}$ and not even whether a completion of $\mathbb{X}_{-a^{-1}\Delta_x}^{s,p}$ can be realized as a space of distributions, except if $a = 1$ of course. In the first-order DB-theory this phenomenon does not appear as B is applied first. As a cautionary tale we remark that even when $a = 1$ not all of our arguments for solvability of Dirichlet problems would go through in the extended gray region, notably the non tangential trace used in Proposition 19.7.

Identification in the interior of the gray region in Figure 10 has previously been obtained (implicitly) in [3, Sec. 7.2.4] and we shall next explain why.

Let us first recall that in Theorem 9.6 we have identified $\mathbb{H}_{DB}^{0,p} = \mathbb{H}_D^{0,p}$ for $p \in (p_-(L), q_+(L))$. For $p \in (1, \infty)$, the $\heartsuit$-*duality* from [3, Cor. 5.14] states that $\mathbb{H}_{DB^*}^{0,p'} = \mathbb{H}_D^{0,p'}$ implies $\mathbb{H}_{DB}^{-1,p} = \mathbb{H}_D^{-1,p}$. Thus, the latter follows for $p \in (q_+(L^\sharp)', p_+(L^\sharp))$ by duality and similarity. As before, interpolation leads to the identification region that is shown in Figure 11. Lemma 9.1 'maps' the gray region in Figure 11 onto the gray region in Figure 10 since $\mathbb{X}^{s,p}_{DB} = \mathbb{X}^{s,p}_{D}$ implies $\mathbb{X}^{s+1,p}_{L} = \dot{\mathrm{X}}^{s+1,p} \cap \mathrm{L}^2$.

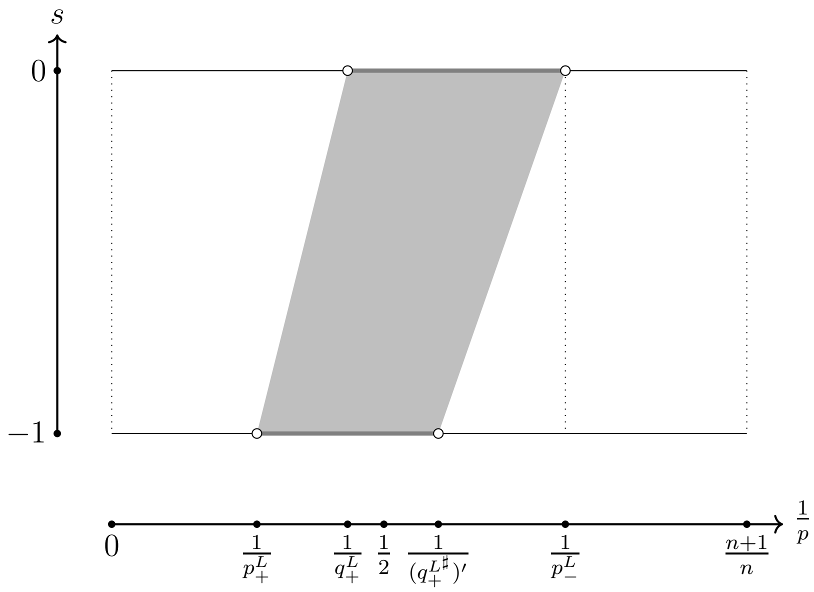

FIGURE 11. In the interior of the gray region $\mathbb{X}^{s,p}_{DB} = \mathbb{X}^{s,p}_{D}$ holds (up to equivalent p-quasinorms). By ♡-duality [3, Cor. 5.14] this is equivalent to $\mathbb{X}^{-s-1,p'}_{DB^*} = \mathbb{X}^{-s-1,p'}_{D}$.

In the particular case $p_-(L^\sharp) < 1$ we have $p_+(L) = \infty$ by duality and similarity and hence the left lower vertex of the identification regions is situated at the origin. However, results can be improved further as follows. We reproduce the argument from [3, Sec. 7.2.1] for the sake of clarity.

Proposition 19.3. *If $p_-(L^\sharp) < 1$, then identification $\mathbb{X}^{s,p}_{DB} = \mathbb{X}^{s,p}_{D}$ and $\mathbb{X}^{s,p}_{L} = \dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2$ hold in the interior of the extended gray regions of Figure 12 and Figure 13, respectively.*

Proof. It suffices to argue for Figure 12 since the extension of Figure 13 follows from Lemma 9.1 as before.

Consider the analog of Figure 11 but for B^*. Since we assume $p_-(L^\sharp) < 1$, the right-hand segment of the gray trapezoid described by

$$\frac{1}{p} = \frac{-s}{q_+(L)'} + \frac{s+1}{p_-(L^\sharp)}$$

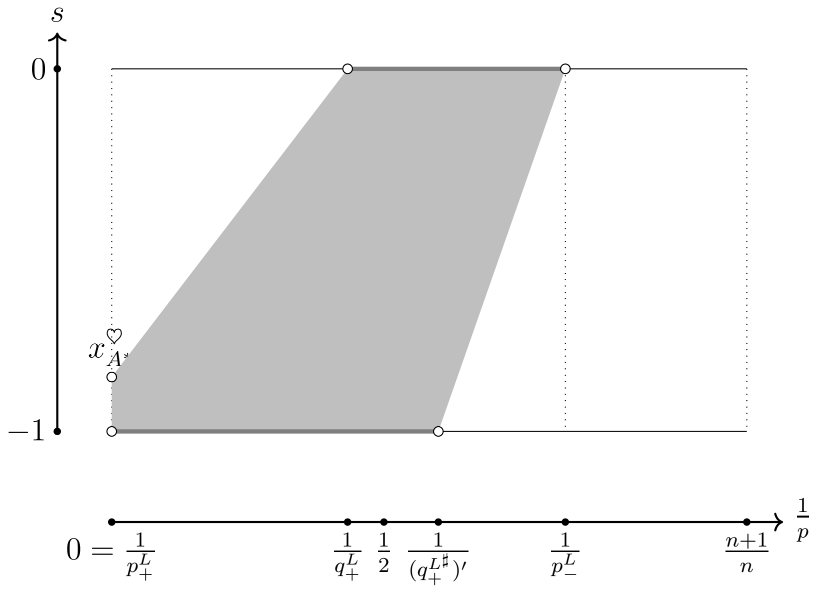

FIGURE 12. Extension of Figure 11 to the left in the case $p_-(L^\sharp) < 1$. The extension only concerns exponents with $p \geq q_+(L) > 2$. The length of the vertical segment on the left is at most $n/p_-(L^\sharp) - n$.

intersects the vertical line $1/p = 1$ at a point that is called x_{A^*} in [3]. Let $x^{\heartsuit}_{A^*}$ be the symmetric point with respect to $(1/2, -1/2)$. By $\heartsuit$-duality this is a boundary point of the identification region for $\mathbb{X}^{s,p}_{DB} = \mathbb{X}^{s,p}_{D}$. Interpolation with the exponents that have already been obtained in Figure 11 yields the extension that is displayed in Figure 12.

The length of the vertical segment that we have been able to add on the line $1/p = 0$ is given by σ, where

$$\sigma\left(\frac{1}{p_-(L^\sharp)} - \frac{1}{q_+(L)'}\right) = \frac{1}{p_-(L^\sharp)} - 1.$$

Since Theorem 6.2 for $L^\sharp$ yields $p_-(L^\sharp) \leq (q_+(L)')_*$, the left-hand side is bounded from below by σ/n and we obtain

$$\sigma \leq \frac{n}{p_-(L^\sharp)} - n$$

as we have claimed. □

Let us illustrate these diagrams in special cases. When $m = 1$, $n \geq 3$ and d is real-valued, we know that $p_-(L) = q_-(L) < 1$ and $p_+(L) = \infty$ (Remark 14.11). Thus we are in the case of Figure 13 for the blue and gray identification regions. This is also the generic situation in dimension $n = 2$ for any L (Proposition 6.7).

In dimension $n = 1$, Proposition 6.7 yields $p_-(L) = q_-(L) = 1_*(= 1/2)$ and $p_+(L) = q_+(L) = \infty$. The same holds for $L^\sharp$ in place of L

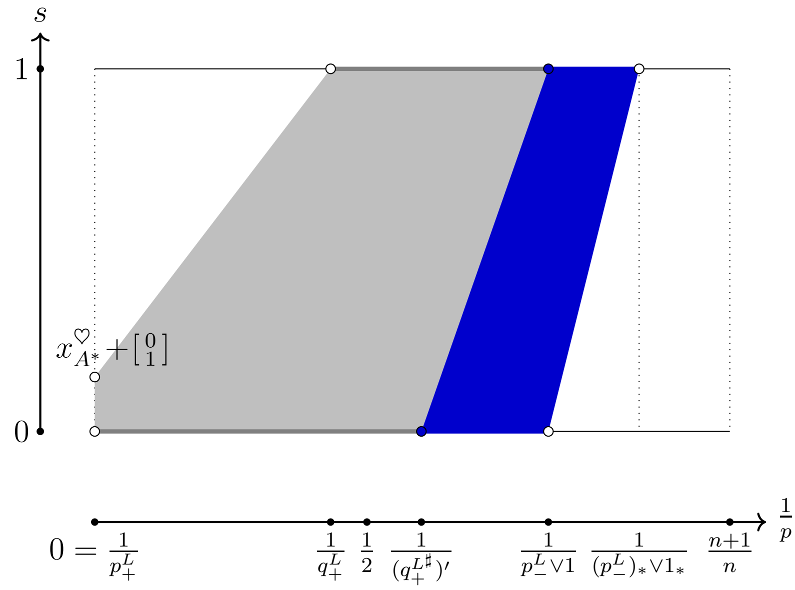

FIGURE 13. Extension of Figure 10 to the left in the case $p_-(L^\sharp) < 1$. The extension only concerns exponents with $p \geq q_+(L) > 2$. The length of the vertical segment on the left is at most $n/p_-(L^\sharp) - n$.

and therefore $x^{\heartsuit}_{A^*} = [0,0]^\top$. Consequently, we already have the largest possible gray region shown in Figure 14 and there is no additional blue region. In any dimension, the same situation occurs for operators of type $-a^{-1}\Delta_x$ (Corollary 6.10) or more generally when d depends only on one coordinate (Remark 14.11).

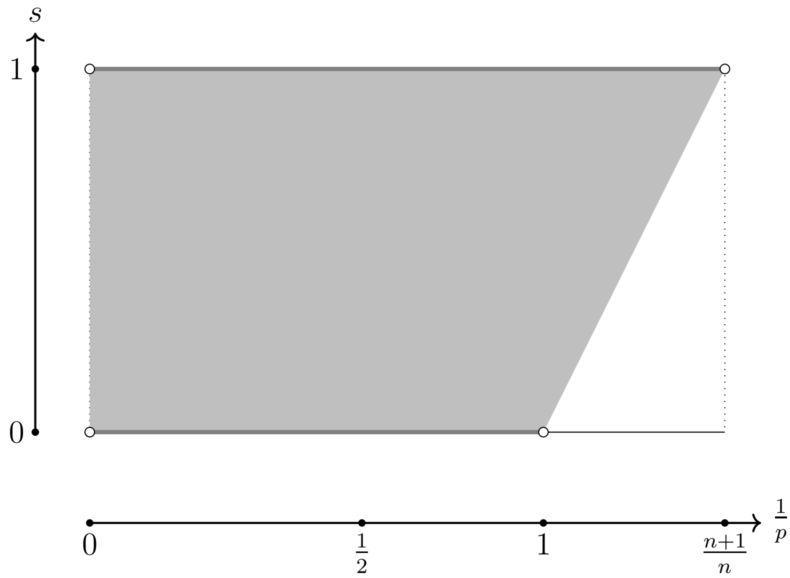

FIGURE 14. Figure 13 in dimension $n = 1$ and in any dimension for the special case $L = -a^{-1}\Delta_x$ or more generally when d depends only on one coordinate.

19.2. **Solvability for fractional regularity data.** We turn to solvability of the Dirichlet problems $(D)^{\mathcal{L}}_{\dot{\mathrm{H}}^{s,p}}$ and $(D)^{\mathcal{L}}_{\dot{\mathrm{B}}^{s,p}}$ when $0 < s < 1$ and $0 < p \leq \infty$ satisfy $1/p < 1 + s/n$. The restrictions on s and p guarantee that all distributions in $\dot{\mathrm{H}}^{s,p}$ and $\dot{\mathrm{B}}^{s,p}$ are locally integrable functions. Indeed, for $p = \infty$ we have $\mathrm{H}^{s,\infty} \subseteq \mathrm{B}^{s,\infty} = \dot{\Lambda}^s$, whereas for $p < \infty$ both are interpolation spaces between $\dot{\mathrm{H}}^{0,p_0} = \mathrm{L}^{p_0}$ and $\dot{\mathrm{H}}^{1,p_1} \subseteq \mathrm{L}^{(p_1)^*}$ for some exponents $p_0 > 1, p_1 > 1_*$.

In the formulation of the Dirichlet problems for fractional regularity data we consider the data spaces as classes of measurable functions and do not factor out constants. We use the pair $(\mathrm{Y}, \dot{\mathrm{X}})$ to denote either $(\mathrm{Z}, \dot{\mathrm{B}})$ or $(\mathrm{T}, \dot{\mathrm{H}})$. By definition of tent and Z-spaces, all problems that appear in Section 1.6 can simultaneously be phrased as asking for given $f \in \dot{\mathrm{X}}^{s,p}$ to find a solution to

$$(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \nabla u \in \mathrm{Y}^{s-1,p}, \\ \lim_{t \to 0} \fint\!\!\fint_{W(t,x)} |u(s,y) - f(x)| \,\mathrm{d}s\mathrm{d}y = 0 & (\text{a.e. } x \in \mathbb{R}^n). \end{cases}$$

Let us mention that another way of formulating the boundary condition is $\lim_{t\to 0} u(t,\cdot) = f$ in $\mathcal{D}'(\mathbb{R}^n)/\mathbb{C}^m$, see [3, 28]. In all cases, we recover this condition in the construction of our solutions. We do not impose a condition at $t = \infty$, contrarily to [3].

Remark 19.4. For $(1/p, s) = (1/2, 1/2)$ we obtain $\dot{\mathrm{X}}^{1/2,2} = \dot{\mathrm{H}}^{1/2,2}$ by Fubini's theorem and $\mathrm{Y}^{-1/2,2} = \mathrm{L}^2$ by the averaging trick, so that $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{1/2,2}}$

is a Dirichlet problem for the energy class. The energy solution given by Proposition 16.3 is (modulo a constant) a solution to this problem. Indeed, consider $f \in \dot{\mathrm{H}}^{1/2,2}$ and let u be the energy solution. It converges to f as $t \to 0$ in $\dot{\mathrm{X}}^{1/2,2}$. By Proposition A.8, there exists a non-tangential trace u_0 and the Cesàro means of $u(t,\cdot)$ converge in $\mathcal{D}'$ to u_0 as $t \to 0$. It follows that $f = u_0 + c$ for some $c \in \mathbb{C}^m$. From now on, we call $u + c$ the *energy solution* with Dirichlet datum f.

Solvability of $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ means that for any given data there exists a solution. *Compatible solvability* means that the energy solution is a solution if the data is also in $\dot{\mathrm{H}}^{1/2,2}$. This notion of (compatible) solvability differs from parts of the literature in that we do not require an *a priori* estimates for solutions by the data, compare with [28, Section 2.4]. Such estimate usually holds since a specific method was used to construct solutions. We find it natural to separate these two aspects of solvability theory by using the concept of solution operators. This notion is manufactured in a way that is amenable to interpolation, independently of any uniqueness result.

Definition 19.5. Let $s \in (0,1)$ and $p \in (1_*, \infty]$ satisfy $1/p < 1 + s/n$. Consider $\dot{\mathrm{X}}^{s,p}$ as a (quasi-)Banach space modulo constants. A *solution operator* for $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ is a linear map $\mathrm{sol} : \dot{\mathrm{X}}^{s,p} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$ such that for all $f \in \dot{\mathrm{X}}^{s,p}$ the function $u := \mathrm{sol}\, f$ satisfies

$$\begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \|\nabla u\|_{\mathrm{Y}^{s-1,p}} \lesssim \|f\|_{\dot{\mathrm{X}}^{s,p}}, & \\ \lim_{t\to 0} u(t,\cdot) = f & (\text{in } \mathcal{D}'(\mathbb{R}^n)/\mathbb{C}^m), \end{cases} \tag{19.1}$$

where the implicit constant in the second line is independent of f. The solution operator is *compatible* if it agrees on $\dot{\mathrm{X}}^{s,p} \cap \dot{\mathrm{H}}^{1/2,2}$ with the solution operator for the energy class (Proposition 16.3).

Recall that a weak solution of $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$, is in $\mathrm{W}^{1,2}_{\mathrm{loc}}$ by definition and of class $\mathrm{C}^\infty((0,\infty); \mathrm{L}^2_{\mathrm{loc}})$ by Corolllary 16.9. Hence, all conditions in our definition make sense. The second line implies that $\mathrm{sol} : \dot{\mathrm{X}}^{s,p} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$ is continuous. In passing, we note that in the existence parts of both Theorem 1.1 (Section 17.3) and Theorem 1.3 (Section 18) we have already encountered such operators for different classes of data without using the terminology. Proposition 16.3 provides a solution operator for $(D)^{\mathcal{L}}_{\dot{\mathrm{H}}^{1/2,2}}$.

Lemma 19.6. *Let $s \in (0,1)$ and $p \in (1_*, \infty]$ satisfy $1/p < 1 + s/n$. If there is a (compatible) solution operator for $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$, then $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ is (compatibly) solvable.*

Proof. For solvability we do not consider $\dot{\mathrm{X}}^{s,p}$ modulo constants. Given $f \in \dot{\mathrm{X}}^{s,p}$, the assumption yields a solution u to (19.1). Now, u has a non-tangential trace u_0 and the Cesàro means of $u(t,\cdot)$ converge to u_0

in $\mathcal{D}'$ as $t \to 0$, see Proposition A.8. Thus, $f = u_0 + c$ for some $c \in \mathbb{C}^m$ and $u + c$ is a solution to $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ with data f.

If the solution operator is compatible and f also belongs to $\dot{\mathrm{H}}^{1/2,2}$, then $u + c$ is the energy solution, see Remark 19.4. □

We shall now construct solution operators in a series of results, enlarging the range of boundary spaces step by step.

We begin with exponents in the blue and gray identification regions from the previous section. Note that the H^p regularity problem $(R)^{\mathcal{L}}_p$ does not fit into the scheme of problems $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ because of the missing square function control for ∇u. Hence, no interpolation argument between the existence parts of Theorem 1.1 and 1.2 can help us here. Instead, we rely on the first-order theory and adapted Hardy spaces as in Section 17.

Proposition 19.7. *Suppose that $(1/p, s)$ is contained in the interior of the colored region described in Figure 10 and Figure 13 in the particular case $p_-(L^\sharp) < 1$. Then $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ is solvable. There is a compatible solution operator that assigns to each $f \in \dot{\mathrm{X}}^{s,p}$ a solution of class $\mathrm{C}_0([0,\infty); \dot{\mathrm{X}}^{s,p}) \cap \mathrm{C}^\infty((0,\infty); \dot{\mathrm{X}}^{s,p})$ with $u(0,\cdot) = f$ and comparability*

$$\sup_{t>0} \|u(t,\cdot)\|_{\dot{\mathrm{X}}^{s,p}} \simeq \|f\|_{\dot{\mathrm{X}}^{s,p}} \simeq \|\nabla u\|_{\mathrm{Y}^{s-1,p}}.$$

Proof. In view of Lemma 19.6 it suffices to construct the solution operator. We first consider $f \in \dot{\mathrm{X}}^{s,p} \cap \mathrm{W}^{1,2}$. In this case, we set of course $u(t,x) := \mathrm{e}^{-tL^{1/2}} f(x)$.

Step 1: Regularity and the first comparability. Since we have $\mathbb{X}^{s,p}_L = \dot{\mathrm{X}}^{s,p} \cap \mathrm{L}^2$ with equivalent p-quasinorms, the regularity for u and the first comparability immediately follow from the bounded H^∞-calculus and the semigroup properties on $\mathbb{X}^{s,p}_L$, see Section 8.2. This argument also yields quantitative bounds for $\|t^{k/2}\partial_t u(t,\cdot)\|_{\dot{\mathrm{X}}^{s,p}}$ that will be needed to carry the C^∞-property over to general data $f \in \dot{\mathrm{X}}^{s,p}$ in Step 4.

Step 2: The second comparability when $p \leq 2$. By means of the intertwining property we find

$$\begin{aligned}
\|\nabla u\|_{\mathrm{Y}^{s-1,p}} &\simeq \|L^{1/2}\mathrm{e}^{-tL^{1/2}} f\|_{\mathrm{Y}^{s-1,p}} + \|\nabla_x \mathrm{e}^{-tL^{1/2}} f\|_{\mathrm{Y}^{s-1,p}} \\
&\simeq \|tL^{1/2}\mathrm{e}^{-tL^{1/2}} f\|_{\mathrm{Y}^{s,p}} + \|\mathrm{e}^{-t\widetilde{M}^{1/2}} \nabla_x f\|_{\mathrm{Y}^{s-1,p}} \\
&=: \|\phi(t^2 L) f\|_{\mathrm{Y}^{s,p}} + \|\psi(t^2 \widetilde{M}) \nabla_x f\|_{\mathrm{Y}^{s-1,p}}
\end{aligned}$$

and the auxiliary functions are of class $\phi \in \Psi^\infty_{1/2}$ and $\psi \in \Psi^\infty_0$. They are admissible for defining $\mathbb{X}^{s,p}_L$ and $\mathbb{X}^{s-1,p}_{\widetilde{M}}$, respectively, since we have $p \leq 2$ and $s < 1$. Hence, we can continue with

$$\begin{aligned}
&\simeq \|f\|_{\mathbb{X}^{s,p}_L} + \|\nabla_x f\|_{\mathbb{X}^{s-1,p}_{\widetilde{M}}} \\
&\simeq \|f\|_{\mathbb{X}^{s,p}_L}
\end{aligned}$$

$$\simeq \|f\|_{\dot{\mathrm{X}}^{s,p}},$$

where we used Figure 7 in the second step.

Step 3: The second comparability when $p > 2$. In this case we are in the gray identification region. We know from Figure 11 (or Figure 12) that we can identify $\mathbb{X}_{DB}^{s-1,p} = \mathbb{X}_D^{s-1,p}$ and therefore the Cauchy characterization of adapted spaces in [3, Thm. 5.26] and [3, Rem. 5.28] yields

$$\|\mathrm{e}^{-t[DB]}\mathbf{1}_{\mathbb{C}^+}(DB)g\|_{\mathrm{Y}^{s-1,p}} \simeq \|g\|_{\mathbb{X}_D^{s-1,p}} \quad (g \in \overline{\mathsf{R}(DB)}). \tag{19.2}$$

We pick

$$g := \begin{bmatrix} 0 \\ \nabla_x f \end{bmatrix} = DB \begin{bmatrix} -af \\ 0 \end{bmatrix}.$$

As for the right-hand side in (19.2), Figure 6 yields $\|g\|_{\mathbb{X}_D^{s-1,p}} \simeq \|f\|_{\dot{\mathrm{X}}^{s,p}}$. Next, we use the identity $2(\mathbf{1}_{\mathbb{C}^+}(z)) = 1 + \sqrt{z^2}/z$ to write

$$2(\mathbf{1}_{\mathbb{C}^+}(DB)g) = \begin{bmatrix} 0 \\ \nabla_x f \end{bmatrix} + [DB] \begin{bmatrix} -af \\ 0 \end{bmatrix} = \begin{bmatrix} -\widetilde{L}^{1/2} af \\ \nabla_x f \end{bmatrix}.$$

The intertwining relation and the similarity of L and $\widetilde{L}$ lead to

$$2\mathrm{e}^{-t[DB]}\mathbf{1}_{\mathbb{C}^+}(DB)g = \begin{bmatrix} -\mathrm{e}^{-t\widetilde{L}^{1/2}}\widetilde{L}^{1/2}af \\ \mathrm{e}^{-t\widetilde{M}^{1/2}}\nabla_x \end{bmatrix} = \begin{bmatrix} -aL^{1/2}\mathrm{e}^{-t\,\mathrm{L}^{1/2}} f \\ \nabla_x \mathrm{e}^{-t\widetilde{L}^{1/2}} f \end{bmatrix} = \begin{bmatrix} a\partial_t u \\ \nabla_x u \end{bmatrix}.$$

Thus, the left-hand side in (19.2) is comparable to $\|\nabla u\|_{\mathrm{Y}^{s-1,p}}$.

Step 4: Extension to a solution operator. By the same density argument as for the regularity problem in Section 17.3 when $p \geq n$, we can construct for general $f \in \dot{\mathrm{X}}^{s,p}$ a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ that has all the properties stated in the proposition. The construction depends linearly on the data and since $u(t, \cdot) \to f$ in $\dot{\mathrm{X}}^{s,p} \subseteq \mathcal{D}'/\mathbb{C}^m$, we see that u solves (19.1). This means that we have constructed a compatible solution operator. □

If $p_+(L) < \infty$, then the (existence part of) Theorem 1.1 contains existence of the Dirichlet problem $(D)_p^{\mathcal{L}}$ in a range of exponents that exceeds the identification region for $\mathbb{H}_L^{0,p}$ by up to one Sobolev conjugate. This leads to the following improvement of the previous result in that case.

Proposition 19.8. *Suppose that $p_+(L) < \infty$. If $(1/p, s)$ is contained in the interior of the colored region in Figure 15, then there is a compatible solution operator for $(D)_{\dot{\mathrm{X}}^{s,p}}^{\mathcal{L}}$. In particular, the problem is compatibly solvable.*

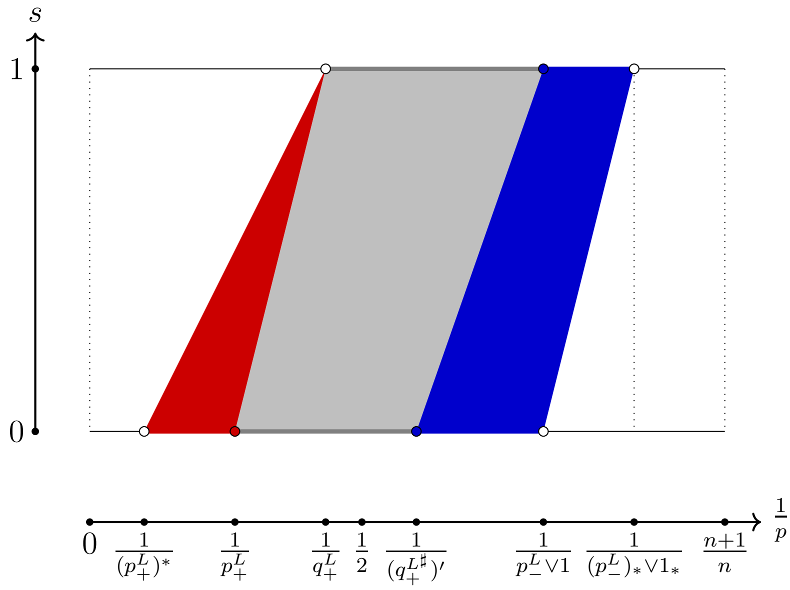

Figure 15. Extended region for compatible solvability of $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ when $p_+(L) < \infty$. Recall that $p_+(L)^* = \infty$ if $p_+(L) \geq n$.

Proof. The blue and gray regions have been treated in Proposition 19.7. We need to add the red triangle to the picture. It suffices to show for any $P_0 \coloneqq (1/p_0, 0)$ with $p_+(L) \leq p_0 < p_+(L)^*$ (bottom red segment) and any $P_1 \coloneqq (1/p_1, s_1)$ in the interior of the gray region that a compatible solution operator exists for all points on the open segment $\overline{P_0 P_1}$. Compatible solvability then follows by Lemma 19.6.

We argue by interpolation and consider the data classes as Banach spaces embedded into $\mathcal{D}'/\mathbb{C}^m$. In Section 17.3 we have established existence of a solution with the properties (i) and (iv) of Theorem 1.1. This furnishes a continuous linear solution operator $\mathrm{sol}_0 : \dot{\mathrm{H}}^{0,p_0} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$ such that $u = \mathrm{sol}_0 f$ solves

$$\begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \|\nabla u\|_{\mathrm{T}^{-1,p_0}} \lesssim \|f\|_{\dot{\mathrm{H}}^{0,p_0}}, & \\ \lim_{t\to 0} u(t,\cdot) = f & (\text{in } \mathcal{D}'(\mathbb{R}^n)/\mathbb{C}^m), \end{cases}$$

whereas Proposition 19.7 furnishes a continuous linear solution operator $\mathrm{sol}_1 : \dot{\mathrm{H}}^{s_1,p_1} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$ such that $u = \mathrm{sol}_1 f$ solves

$$\begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \|\nabla u\|_{\mathrm{T}^{s_1-1,p_1}} \lesssim \|f\|_{\dot{\mathrm{H}}^{s_1,p_1}}, & \\ \lim_{t\to 0} u(t,\cdot) = f & (\text{in } \mathcal{D}'(\mathbb{R}^n)/\mathbb{C}^m). \end{cases}$$

Since both operators produce compatible solutions, the universal approximation technique implies that they coincide on $\dot{\mathrm{H}}^{0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}$.

Hence, we have a well-defined continuous linear operator

$$\mathrm{sol} : \dot{\mathrm{H}}^{0,p_0} + \dot{\mathrm{H}}^{s_1,p_1} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$$

such that $u = \mathrm{sol}\, f$ solves $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ and satisfies $u(t,\cdot) \to f$ as $t \to 0$ in $\mathcal{D}'/\mathbb{C}^m$.

Pick any point $(1/p, s)$ on the open segment $\overline{P_0P_1}$. Since the real and complex interpolation spaces of an interpolation couple continuously embed into the sum space, we obtain that $\mathrm{sol} : \dot{\mathrm{X}}^{s,p} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$ is continuous. The map sol and the continuous solution map for energy solutions from Proposition 16.3 agree on $\dot{\mathrm{X}}^{s,p} \cap \dot{\mathrm{H}}^{1/2,2} \cap \dot{\mathrm{H}}^{s_1,p_1}$ and hence on $\dot{\mathrm{X}}^{s,p} \cap \dot{\mathrm{H}}^{1/2,2}$. Since the maps $\nabla\, \mathrm{sol} : \dot{\mathrm{H}}^{0,p_0} \to \mathrm{T}^{-1,p_0}$ and $\nabla\, \mathrm{sol} : \dot{\mathrm{H}}^{s_1,p_1} \to \mathrm{T}^{s_1-1,p_0}$ are bounded, we obtain by real and complex interpolation that $\nabla\, \mathrm{sol} : \dot{\mathrm{X}}^{s,p} \to \mathrm{Y}^{s-1,p}$ is bounded. This means that we have constructed a solution operator for $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$. □

In the case $p_+(L) > n$ we can go one step further and study endpoint problems $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{\alpha,\infty}}$ for $0 < \alpha < 1 - n/p_+(L)$. We have $\dot{\mathrm{B}}^{\alpha,\infty} = \dot{\Lambda}^\alpha$ with equivalent norms, so that $(D)^{\mathcal{L}}_{\dot{\mathrm{B}}^{\alpha,\infty}}$ is a third way of posing a Dirichlet problem with Hölder continuous data. The other endpoint problem uses the data space $\dot{\mathrm{H}}^{\alpha,\infty} = \dot{\mathrm{BMO}}^\alpha$, which is continuously embedded into $\dot{\Lambda}^\alpha$ and carries the equivalent norm (2.9). The upshot is that, given $f \in \dot{\mathrm{X}}^{\alpha,\infty}$, the existence part of Theorem 1.3 already shows that u defined in (18.1) is a compatible solution that converges to f at the boundary in the non-tangential sense. The following addendum guarantees that this solution also solves the new endpoint problem and that (18.1) defines a compatible solution operator to $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{\alpha,\infty}}$.

Proposition 19.9. *Suppose that $p_+(L) > n$ and that $0 < \alpha < 1 - n/p_+(L)$. Then the Dirichlet problem $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{\alpha,\infty}}$ is compatibly solvable. More precisely, given $f \in \dot{\mathrm{X}}^{\alpha,\infty}$, the same solution u that was defined in (18.1) and solves $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ and $(\widetilde{D})^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$, also solves $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{\alpha,\infty}}$ and satisfies*

$$\|\nabla u\|_{\mathrm{Y}^{\alpha-1,\infty}} \simeq \|f\|_{\dot{\mathrm{X}}^{\alpha,\infty}}.$$

Remark 19.10. Combining the existence part of Theorem 1.3 with Proposition 19.9 yields comparability

$$\|\nabla u\|_{\mathrm{T}^{-1,\infty;\alpha}} = \|C_\alpha(t\nabla u)\|_\infty \simeq \|W(t^{1-\alpha}\nabla u)\|_\infty = \|\nabla u\|_{\mathrm{Z}^{\alpha-1,\infty}},$$

whenever u is a solution to $(D)_{\dot{\Lambda}^\alpha}$. A simple comparison of the two functionals shows that estimate '$\gtrsim$' holds for any $\mathrm{L}^2_{\mathrm{loc}}$-function F in place of ∇u. The converse is a special property of weak solutions to $\mathcal{L}u = 0$.

Remark 19.11. If $p_-(L^\sharp) < 1$ and $\alpha < n(1/p_-(L^\sharp) - 1)$, then u is given by a weak*-continuous semigroup on $\dot{\Lambda}^\alpha$ as the dual of H^p, $\alpha = n(1/p - 1)$, see Lemma 18.7. In essence, this followed from the identification $\mathbb{H}^p_{L^\sharp} = (a^*)^{-1}(\mathrm{H}^p \cap \mathrm{L}^2)$. By interpolation one can obtain a subregion of the red region where the (unique) solution to $(D)^{\mathcal{L}}_{\mathrm{B}^{s,p}}$ is given by a C_0-semigroup.

An analogous result for $\dot{\mathrm{BMO}}^\alpha$ would require boundedness of the Poisson semigroup for $L^\sharp$ on $(a^*)^{-1}(\dot{\mathrm{H}}^{-\alpha,1} \cap \mathrm{L}^2)$, which we do not know when $\alpha > 0$. One can use the first-order approach to obtain the semigroup property of the solution to $(D)^{\mathcal{L}}_{\dot{\mathrm{H}}^{\alpha,\infty}}$ for $0 < \alpha < \theta$, where θ appears in Figure 3 or equivalently as the upper endpoint of the vertical boundary segment of the gray region in Figure 13. The semigroup property for $\theta \leq \alpha < n(1/p_-(L^\sharp) - 1)$ is unclear. These observations will not be needed in the further course, so we do not detail them.

Proof of Proposition 19.9. We fix (a representative for) $f \in \dot{\mathrm{X}}^{\alpha,\infty}$ and let u be the solution to both $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ and $(\widetilde{D})^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ defined in (18.1). Since we are working within the same or even a smaller class of boundary data, we have at our disposal all properties for u from Section 18 and only at distinguished places we have to intervene in order to obtain the additional features that we claimed above. More precisely, we have to modify Part 3 for the upper bound of $\|\nabla u\|_{\mathrm{Y}^{\alpha-1,\infty}}$ and Part 5 for the converse.

Modification of Part 3: The bound '$\lesssim$'. In the case $\mathrm{X} = \mathrm{B}$ it suffices to combine the observation from Remark 19.10 and the existence part of Theorem 1.3 in order to obtain

$$\|\nabla u\|_{\mathrm{Z}^{\alpha-1,\infty}} \lesssim \|C_\alpha(t\nabla u)\|_\infty \lesssim \|f\|_{\dot{\Lambda}^\alpha}.$$

We turn to the case $\mathrm{X} = \mathrm{H}$. We have to prove that for all cubes $Q \subseteq \mathbb{R}^n$ of sidelength ℓ we have

$$\left(\int_0^\ell \int_Q |s^{1-\alpha}\nabla u|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s}\right)^{1/2} \lesssim |Q| \|f\|_{\dot{\mathrm{BMO}}^\alpha}. \tag{19.3}$$

From now on Q is fixed. Since both sides stay the same under adding constants to u and f, we can assume $(f)_Q = 0$.

In contrast to Section 18 we use a smooth resolution for f in order to represent u. We let $(\eta_j)_j$ be a smooth partition of unity on $\mathbb{R}^n$ subordinate to the sets $D_1 := 4Q$ and $D_j := 2^{j+1}Q \setminus 2^{j-1}Q$, $j \geq 2$, such that $\|\eta_j\|_\infty + 2^j \ell(Q)\|\nabla_x \eta_j\|_\infty \leq C$ for a dimensional constant C. For $j \geq 1$ we introduce

$$f_j := \eta_j f, \quad u_j(t, \cdot) := \mathrm{e}^{-tL^{1/2}} f_j.$$

The main difficulty is to handle the local term for $j = 1$. For the moment, let us take for granted the estimate

$$\|f_1\|^2_{\dot{\mathrm{H}}^{\alpha,2}} \lesssim |Q| \|f\|^2_{\dot{\mathrm{BMO}}^\alpha}. \tag{19.4}$$

This is where the smoothness of η_1 is needed and we include the argument at the end. Thus, it suffices to prove the local bound

$$\int_0^\ell \int_Q |s^{1-\alpha}\nabla u_1|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s} \lesssim \|f_1\|_{\dot{\mathrm{H}}^{\alpha,2}}^2. \tag{19.5}$$

In doing so, we can work under the qualitative assumption $f_1 \in \mathrm{W}^{1,2}$ which can be removed afterwards via density of $\mathrm{W}^{1,2}$ in $\dot{\mathrm{H}}^{\alpha,2} \cap \mathrm{L}^2$ and Fatou's lemma. We use the intertwining property to write

$$s^{1-\alpha}\nabla u_1(s,y) = \begin{bmatrix} -s^{1-\alpha}L^{1/2}\mathrm{e}^{-sL^{1/2}}f_1 \\ s^{1-\alpha}\mathrm{e}^{-s\widetilde{M}^{1/2}}\nabla_x f_1 \end{bmatrix} =: \begin{bmatrix} s^{-\alpha}\phi(s^2L)f_1 \\ s^{1-\alpha}\psi(s^2\widetilde{M})\nabla_x f_1 \end{bmatrix},$$

where $\phi \in \Psi_{1/2}^\infty$ and $\psi \in \Psi_0^\infty$. These auxiliary functions are admissible for $\mathbb{H}_L^{\alpha,2}$ and $\mathbb{H}_{\widetilde{M}}^{\alpha-1,2}$, respectively. Hence, we get as required

$$\begin{aligned}
\int_0^\ell \int_Q |s^{1-\alpha}\nabla u_1|^2 \, \frac{\mathrm{d}y\mathrm{d}s}{s} &\leq \iint_{\mathbb{R}^{1+n}_+} \left| \begin{bmatrix} s^{-\alpha}\phi(s^2L)f_1 \\ s^{1-\alpha}\psi(s^2\widetilde{M})\nabla_x f_1 \end{bmatrix} \right|^2 \frac{\mathrm{d}y\mathrm{d}s}{s} \\
&\simeq \|f_1\|_{\mathbb{H}_L^{\alpha,2}}^2 + \|\nabla_x f_1\|_{\mathbb{H}_{\widetilde{M}}^{\alpha-1,2}}^2 \\
&\simeq \|f_1\|_{\mathbb{H}_L^{\alpha,2}}^2 \\
&\simeq \|f_1\|_{\dot{\mathrm{H}}^{\alpha,2}}^2,
\end{aligned}$$

where the third step is due to Figure 7 and the final step uses that $(1/2, \alpha)$ belongs to the identification region of Figure 10.

For the non-local terms with $j \geq 2$ we can now follow Steps 2 and 3 *verbatim*, the only modification being that we multiply the Caccioppoli estimate (18.8) by $t^{-2\alpha}$ before summing. This leads to (18.9) with the local bound $\ell^\alpha \|f\|_{\dot{\Lambda}^\alpha}$ replaced by $\|f\|_{\mathrm{B\dot{M}O}^\alpha}$ and additional powers $s^{-2\alpha}$ in each of the off-diagonal pieces, so that the power ℓ^α in (18.10) disappears. Thus, we control the sum of the off-diagonal pieces by $\|f\|_{\dot{\Lambda}^\alpha} \lesssim \|f\|_{\mathrm{B\dot{M}O}^\alpha}$.

The proof is complete modulo the argument for (19.4) that we give now. By translation we can assume that Q is centered at the origin. A classical argument using the Fourier transform of $f_1 \in \mathrm{L}^2$ yields

$$\|f_1\|_{\dot{\mathrm{H}}^{\alpha,2}}^2 \simeq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_1(y) - f_1(z)|^2}{|y-z|^{n+2\alpha}} \, \mathrm{d}z\mathrm{d}y =: \mathrm{I},$$

see for example [26, Prop. 1.3.7]. According to (2.9) it suffices to prove $\mathrm{I} \lesssim A$, where

$$A := \int_{4Q} \int_{4Q} \frac{|f(y) - f(z)|^2}{|y-z|^{n+2\alpha}} \, \mathrm{d}z\mathrm{d}y.$$

By symmetry, we have

$$\mathrm{I} = 2 \iint_{|y| \geq |z|} \frac{|f_1(y) - f_1(z)|^2}{|y-z|^{n+2\alpha}} \, \mathrm{d}z\mathrm{d}y.$$

We write the numerator as $\eta_1(y)(f(z)-f(y))+f(z)(\eta_1(z)-\eta_1(y))$. The first term vanishes unless $y \in 4Q$ and in that case $z \in 4Q$ follows from $|y| \geq |z|$. Hence, the integral of this part is controlled by A. Likewise, $z \in {}^c(4Q)$ implies $y \in {}^c(4Q)$ and the second term vanishes. Altogether, we obtain

$$\mathrm{I} \leq A + \int_{\mathbb{R}^n} \int_{4Q} \frac{|f(z)|^2 |\eta_1(z) - \eta_1(y)|^2}{|y-z|^{n+2\alpha}} \,\mathrm{d}z\mathrm{d}y,$$

where we bound $|\eta_1(z) - \eta_1(y)|$ via the mean value theorem if $|x-y| \leq \ell(Q)$ and in L^∞-norm if not, in order to get

$$\begin{aligned}
&\leq A + \frac{C^2}{\ell(Q)^2} \int_{4Q} \int_{|y-z| \leq \ell(Q)} \frac{|f(z)|^2}{|y-z|^{n+2\alpha-2}} \,\mathrm{d}y\mathrm{d}z \\
&\quad + 4C^2 \int_{4Q} \int_{|y-z| \geq \ell(Q)} \frac{|f(z)|^2}{|y-z|^{n+2\alpha}} \,\mathrm{d}y\mathrm{d}z \\
&\lesssim A + \frac{1}{\ell(Q)^{2\alpha}} \int_{4Q} |f(z)|^2 \,\mathrm{d}z \\
&= A + \frac{1}{\ell(Q)^{2\alpha}} \int_{4Q} \left| \fint_Q f(z) - f(y) \,\mathrm{d}y \right|^2 \mathrm{d}z \\
&\lesssim A.
\end{aligned}$$

We used $(f)_Q = 0$ in the second to last step and Jensen's inequality and $|y-z| \lesssim \ell(Q)$ in the final step.

Modification of Part 5: The bound '$\gtrsim$'. We fix $g \in \mathrm{C}_0^\infty$ with $\int_{\mathbb{R}^n} g \mathrm{d}x = 0$ and consider the extended L^2-duality pairing $\langle f, g \rangle$. We use the same notation as in Part 5 of Section 18. The only difference in the argument appears in Step 5, where we have to handle

$$\iint_{\mathbb{R}^{1+n}_+} |t\nabla_x u| \cdot |t \nabla_x \widetilde{v}| \,\frac{\mathrm{d}t\mathrm{d}x}{t} \tag{19.6}$$

by a duality. The argument is repeated twice in Step 6 for t-derivatives of u. The control of these integrals determines the bound for $|\langle f, g \rangle|$. We recall from (18.27) the notation

$$\begin{bmatrix} 0 \\ t\nabla_x \widetilde{v} \end{bmatrix} = \psi(tDB_H) \begin{bmatrix} g \\ 0 \end{bmatrix},$$

where $\psi \in \Psi_1^{2\beta+1}$ with $\beta > {}^n\!/\!_2 + 2$ and DB_H correspond to $H = -(a^*)^{-1}\Delta_x$ in the same way as DB corresponds to L. In Section 18 we have interpreted (19.6) as a $\mathrm{T}^{0,\infty;\alpha} - \mathrm{T}^{\varrho}$ duality pairing, where $\varrho \in (1_*, 1]$ is such that $\alpha = n(1/\varrho - 1)$, in order to bring $C_\alpha(t\nabla u)$ into play.

Now, we use the $\mathrm{Y}^{\alpha,\infty} - \mathrm{Y}^{-\alpha,1}$ pairing, see Sections 2.2 and 2.3, in order to give

$$\iint_{\mathbb{R}^{1+n}_+} |t\nabla_x u| \cdot |t \nabla_x \widetilde{v}| \,\frac{\mathrm{d}t\mathrm{d}x}{t} \lesssim \|\nabla_x u\|_{\mathrm{Y}^{\alpha-1,\infty}} \|t\nabla_x \widetilde{v}\|_{\mathrm{Y}^{-\alpha,1}}.$$

Since $\beta > n/2 + 2$, the function ψ is admissible for defining $\mathbb{X}^{-\alpha,1}_{DB_H}$. We have $p_-(H) = 1_*$ and $q_+(H) = \infty$ (Corollary 6.10) and consequently the identification region for DB_H in Figure 11 contains the full open segment that joins $(1,-1)$ to $(1,0)$, see also Figure 14. In particular, $\mathbb{X}^{-\alpha,1}_{DB_H} = \mathbb{X}^{-\alpha,1}_{D}$ and together with Figure 6 we obtain

$$\|t\nabla_x \widetilde{v}\|_{\mathrm{Y}^{-\alpha,1}} \simeq \left\| \begin{bmatrix} g \\ 0 \end{bmatrix} \right\|_{\mathbb{X}^{-\alpha,1}_{DB_H}} \simeq \|g\|_{\dot{\mathrm{X}}^{-\alpha,1}}.$$

Thus, we control (19.6) by $\|\nabla u\|_{\mathrm{Y}^{\alpha-1,\infty}} \|g\|_{\dot{\mathrm{X}}^{-\alpha,1}}$ and we conclude for all $g \in \mathrm{C}^\infty_0$ with $\int_{\mathbb{R}^n} g \,\mathrm{d}x = 0$ that

$$|\langle f, g\rangle| \lesssim \|\nabla u\|_{\mathrm{Y}^{\alpha-1,\infty}} \|g\|_{\dot{\mathrm{X}}^{-\alpha,1}}.$$

These g form a dense subclass of $\dot{\mathrm{X}}^{-\alpha,1}$. There are probably many ways to see this – one is to use the smooth atomic decomposition for $\dot{\mathrm{X}}^{-\alpha,1}$ in [46, Thm. 5.11 & 5.18]. By duality, we obtain the lower bound

$$\|f\|_{\dot{\mathrm{X}}^{\alpha,\infty}} \lesssim \|\nabla_x u\|_{\mathrm{Y}^{\alpha-1,\infty}}. \qquad \square$$

Let us come back to Figure 15 but for $p_+(L) > n$, so that the left lower vertex of the red triangle is situated at the origin. Proposition 19.9 allows us to add a segment on the line $1/p = 0$ and we can try to interpolate again to enlarge the region for compatible solvability as illustrated in Figure 16. This is the content of the final result in this section.

Proposition 19.12. *Suppose that $p_+(L) > n$. If $(1/p, s)$ is contained in the interior of the colored region in Figure 16, then there is a compatible solution operator for $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$. In particular, the problem is compatibly solvable.*

Proof. As before, it suffices to construct the compatible solution operator. In view of Proposition 19.8 it remains to consider points in the interior of the triangle ORX and on the open segment $\overline{OR}$. Our starting point is that by Proposition 19.9 there is a compatible solution operator for the problems corresponding to the open segment $\overline{OX}$ and that the constructed solution has all the properties listed in Theorem 1.3.

Fix any $P = (0, \alpha) \in \overline{OX}$. At $E := (1/2, 1/2)$ the corresponding problem is the Dirichlet problem for the energy class and we have the solution operator $\mathrm{sol}_E : \dot{\mathrm{X}}^{1/2,2} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$ from Proposition 16.3, which is compatible with the solution operator $\mathrm{sol}_P : \dot{\mathrm{X}}^{\alpha,\infty} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$ at P. Hence, we obtain a well-defined linear operator

$$\mathrm{sol} : \dot{\mathrm{X}}^{\alpha,\infty} + \dot{\mathrm{X}}^{1/2,2} \to \mathcal{D}'(\mathbb{R}^{1+n}_+)/\mathbb{C}^m$$

such that $u = \mathrm{sol}\, f$ solves $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ and satisfies $u(t,\cdot) \to f$ as $t \to 0$ in $\mathcal{D}'(\mathbb{R}^n)/\mathbb{C}^m$. This time the compatibility with sol_E

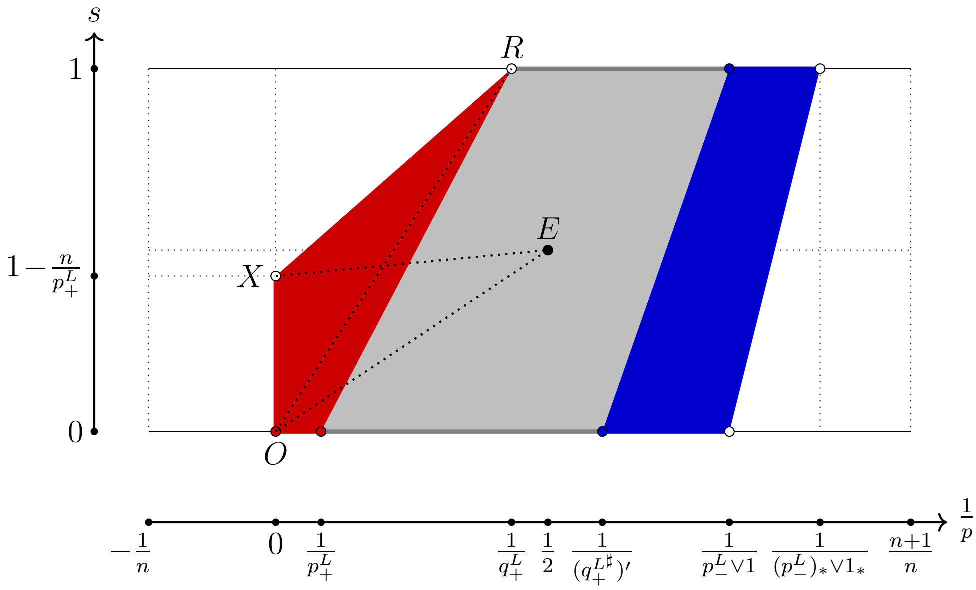

FIGURE 16. Extended region for compatible solvability if $p_+(L) > n$ via a two-step interpolation argument. The picture is up to scale when $p_+(L) < \infty$. If $p_+(L) = \infty$, then red region becomes the triangle ORX with $X = (0, 1)$. If furthermore $p_-(L^\sharp) < 1$, then parts of the red region also belong to the extended gray identification region of Figure 13. This special situation has already been showcased in Figure 3 in the introduction, where on the bottom line also the exponents 'beyond infinity' corresponding to $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ appear.

already holds by construction and no density argument is needed. Real and complex interpolation of the mapping properties at the endpoints yields that $\nabla \operatorname{sol} : \dot{\mathrm{X}}^{s,p} \to \mathrm{Y}^{s-1,p}$ is bounded provided $(1/p, s)$ belongs to the open segment $\overline{PE}$. This yields the required solution operator for $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ and we can add the interior of the triangle OEX in Figure 16 to the region of compatible solvability.

Now that we have successfully moved away from the line $1/p = 0$ of infinite exponents, we can repeat the argument in the proof of Proposition 19.8 once more for any P_0 in the interior of OEX and any P_1 in the interior of the gray region. In particular, we reach any point in the interior of ORX and on the open segment $\overline{OR}$. □

20. SINGLE LAYER OPERATORS FOR $\mathcal{L}$ AND ESTIMATES FOR $\mathcal{L}^{-1}$

This section is needed to prepare the next section on uniqueness. We consider the divergence form operator

$$\mathcal{L}u = -\operatorname{div} A\nabla u = -\partial_t(a\partial_t u) - \operatorname{div}_x d\nabla_x u$$

on $\mathbb{R}^{1+n}$. It is of the same class as aL in (3.5) but in one dimension higher. Hence, $\mathcal{L}$ is defined on $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n})$ via the Lax-Milgram lemma and invertible onto $\dot{\mathrm{W}}^{-1,2}(\mathbb{R}^{1+n})$. It turns out that the inverse $\mathcal{L}^{-1}$ on particular test functions can explicitly be constructed using abstract single layer operators $\mathcal{S}_t^{\mathcal{L}}$. All this relies on the fundamental observation of Rosén [83] that what is called *single layer potential* in the classical context of elliptic operators with real coefficients can abstractly be defined using the H^∞-calculus for the perturbed Dirac operator DB in (3.2). Here, we cite the equivalent formulation from [11], which is somewhat closer to our terminology.

We define the *conormal gradient* $\nabla_A := [a\partial_t, \nabla_x]^\top$. For all $f \in \mathrm{L}^2$ and $t > 0$ there is a unique distribution (up to a constant) that we denote by $\mathcal{S}_t^{\mathcal{L}} f$ such that

$$\nabla_A \mathcal{S}_t^{\mathcal{L}} f := \begin{cases} +\mathrm{e}^{-tDB} \mathbf{1}_{\mathbb{C}^+}(DB) \begin{bmatrix} f \\ 0 \end{bmatrix} & \text{if } t > 0, \\ -\mathrm{e}^{-tDB} \mathbf{1}_{\mathbb{C}^-}(DB) \begin{bmatrix} f \\ 0 \end{bmatrix} & \text{if } t < 0. \end{cases} \tag{20.1}$$

Note that $[f, 0]^\top \in \mathcal{H} = \overline{\mathsf{R}(D)} = \overline{\mathsf{R}(DB)}$ so that the right-hand side is defined in the same space via the bounded H^∞-calculus. Then, we have the following result.

Proposition 20.1 ([11, Prop. 4.5]). *Assume $\widetilde{G} = \operatorname{div}_x G^\sharp$ with $G^\sharp \in \mathrm{C}_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^{mn})$. Let $H := \mathcal{L}^{-1}(\partial_t \widetilde{G})$. Then H is given for all $t \in \mathbb{R}$ as an L^2-valued Bochner integral*

$$H(t, \cdot) = \int_{\mathbb{R}} \partial_t \mathcal{S}_{t-s}^{\mathcal{L}} \widetilde{G}(s, \cdot) \, \mathrm{d}s.$$

The reader may be surprised that the representation by convolution with the single layer is not a singular integral. This is due to a hidden integration by parts because we represent $H := \mathcal{L}^{-1}(\partial_t \widetilde{G})$ and not $\mathcal{L}^{-1}(\widetilde{G})$, see [11, Rem. 4.6]. We also note that $\partial_t \widetilde{G} \in \dot{\mathrm{W}}^{-1,2}(\mathbb{R}^{1+n}_+)$ because it is a test function with integral zero and hence defines a tempered distribution modulo constants.

For our purpose it will be more convenient to write the single layer operators in terms of the second-order operator L. This is the content of the following proposition.

Proposition 20.2. *Let $t \in \mathbb{R}, t \neq 0$ and $f \in \mathrm{L}^2$. Then*

$$\partial_t \mathcal{S}_t^{\mathcal{L}} f = \frac{1}{2} \operatorname{sgn}(t) \mathrm{e}^{-|t| L^{1/2}} (a^{-1} f)$$

Proof. We have $[z] = \sqrt{z^2} = \pm z$ in the complex half-planes $z \in \mathbb{C}_\pm$. Hence, we can write the $\perp$-component of (20.1) as

$$a\partial_t \mathcal{S}_t^{\mathcal{L}} f := \operatorname{sgn}(t) \begin{cases} \left(\mathrm{e}^{-|t|[DB]} \mathbf{1}_{\mathbb{C}^+}(DB) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp & \text{if } t > 0, \\ \left(\mathrm{e}^{-|t|[DB]} \mathbf{1}_{\mathbb{C}^-}(DB) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp & \text{if } t < 0. \end{cases} \tag{20.2}$$

If $[g_\perp, g_\parallel]^\top$ is in the range of $[DB]$, then the functional calculus on $\overline{\mathsf{R}(D)}$ translates the identity of functions $\mathbf{1}_{\mathbb{C}^\pm}(z) = 1/2(1 \pm z/\sqrt{z^2})$ into

$$\begin{aligned} &\mathbf{1}_{\mathbb{C}^\pm}(DB) \begin{bmatrix} g_\perp \\ g_\parallel \end{bmatrix} \\ &= \frac{1}{2} \left(\begin{bmatrix} g_\perp \\ g_\parallel \end{bmatrix} \pm \begin{bmatrix} 0 & \operatorname{div}_x d \\ -\nabla_x a^{-1} & 0 \end{bmatrix} \begin{bmatrix} (\widetilde{L})^{-1/2} g_\perp \\ (\widetilde{M})^{-1/2} g_\parallel \end{bmatrix} \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} g_\perp \\ g_\parallel \end{bmatrix} \pm \begin{bmatrix} \operatorname{div}_x d(\widetilde{M})^{-1/2} g_\parallel \\ -\nabla_x L^{-1/2} a^{-1} g_\perp \end{bmatrix} \right), \end{aligned} \tag{20.3}$$

compare with the matrix representations in (3.2) and (3.4). We set $g_\parallel = 0$ and apply the $[DB]$ semigroup to give

$$\left(\mathrm{e}^{-|t|[DB]} \mathbf{1}_{\mathbb{C}^\pm}(DB) \begin{bmatrix} g_\perp \\ 0 \end{bmatrix} \right)_\perp = \frac{1}{2} \mathrm{e}^{-|t|\widetilde{L}^{1/2}} g_\perp = \frac{1}{2} a \mathrm{e}^{-|t|L^{1/2}} a^{-1} g_\perp.$$

This identity extends to general $f \in \mathrm{L}^2$ in place of $g_\perp$ since $\widetilde{L}$ has dense range in L^2 and the claim follows from (20.2). □

Combining the previous two results gives us the following representation.

Corollary 20.3. *Assume* $G = \partial_t \operatorname{div}_x G^\sharp$ *with* $G^\sharp \in \mathrm{C}_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^{mn})$ *and set* $\widetilde{G} = \operatorname{div}_x G^\sharp$*. Let* $H := \mathcal{L}^{-1}(G)$*. Then for all* $t \in \mathbb{R}$*,* $H(t, \cdot)$ *is given as an* L^2*-valued Bochner integral by*

$$H(t, \cdot) = \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(t - s) \mathrm{e}^{-|t-s|L^{1/2}} (a^{-1} \widetilde{G}(s, \cdot)) \, \mathrm{d}s. \tag{20.4}$$

As the formula for H only uses the Poisson semigroup, we can use the range where the semigroup enjoys L^p-estimates. This leads to additional estimates as compared to [11] in the non-block case.

Lemma 20.4. *Let* G, H *be as in the corollary above and suppose in addition that* $\operatorname{supp} G^\sharp \subseteq [1/\beta, \beta] \times \mathbb{R}^n$ *for some* $\beta > 1$*. Let* $h := H(0, \cdot)$*. Then if* $r \in (p_-(L) \vee 1, p_+(L))$*, there is some* $\gamma > 0$ *such that for all*

$t > 0$,

$$\begin{aligned} \|H(t,\cdot) - \mathrm{e}^{-tL^{1/2}} h\|_r &\lesssim t \wedge t^{-\gamma} \\ \|\partial_t (H(t,\cdot) - \mathrm{e}^{-tL^{1/2}} h)\|_r &\lesssim 1 \wedge t^{-1-\gamma}. \end{aligned} \tag{20.5}$$

If, in addition, $p_-(L^\sharp) < 1$, then this also holds for $r = \infty$.

Proof. We remark that $a^{-1}\widetilde{G}(s,\cdot)$ belongs to any L^q-space, uniformly in $s \in [1/\beta, \beta]$. We will choose q at our convenience.

We treat the case $r < \infty$ first. For the exponent r we have at hand the estimates for the Poisson semigroup from Proposition 12.5 and the H^∞-calculus on $\mathrm{L}^r \cap \mathrm{L}^2$, see Theorem 10.1.

Step 1. We begin with the estimate for $H - \mathrm{e}^{-tL^{1/2}} h$ using (20.4). For $0 < t \leq 1/4\beta$, we have

$$\begin{aligned} H(t,\cdot) &- \mathrm{e}^{-tL^{1/2}} h \\ &= -\frac{1}{2} \int_{1/\beta}^{\beta} \Big(\mathrm{e}^{-(s-t)L^{1/2}} - \mathrm{e}^{-(s+t)L^{1/2}}\Big)(a^{-1}\widetilde{G}(s,\cdot))\,\mathrm{d}s. \end{aligned}$$

Writing

$$\Big(\mathrm{e}^{-(s-t)L^{1/2}} - \mathrm{e}^{-(s+t)L^{1/2}}\Big) = \mathrm{e}^{-(s-2t)L^{1/2}}\Big(\mathrm{e}^{-tL^{1/2}} - \mathrm{e}^{-3tL^{1/2}}\Big), \tag{20.6}$$

the operator on the far right is L^r-bounded with bound Ct by the H^∞-calculus and the operator to its left is L^r-bounded, uniformly.

For $1/4\beta < t < 4\beta$, we see that

$$\begin{aligned} &H(t,\cdot) - \mathrm{e}^{-tL^{1/2}} h \\ &= \frac{1}{2} \int_{1/\beta}^{\beta} \Big(\operatorname{sgn}(t-s)\mathrm{e}^{-|t-s|L^{1/2}} + \mathrm{e}^{-(s+t)L^{1/2}}\Big)(a^{-1}\widetilde{G}(s,\cdot))\,\mathrm{d}s, \end{aligned} \tag{20.7}$$

and we get a uniform L^r-bound.

Finally for $t \geq 4\beta$, we have

$$\begin{aligned} &H(t,\cdot) - \mathrm{e}^{-tL^{1/2}} h \\ &= \frac{1}{2} \int_{1/\beta}^{\beta} \Big(\mathrm{e}^{-(t-s)L^{1/2}} + \mathrm{e}^{-(s+t)L^{1/2}}\Big)(a^{-1}\widetilde{G}(s,\cdot))\,\mathrm{d}s, \\ &= \frac{1}{2} \int_{1/\beta}^{\beta} \mathrm{e}^{-(t-2\beta)L^{1/2}}\Big(\mathrm{e}^{-(2\beta-s)L^{1/2}} + \mathrm{e}^{-(2\beta+s)L^{1/2}}\Big)(a^{-1}\widetilde{G}(s,\cdot))\,\mathrm{d}s. \end{aligned} \tag{20.8}$$

We pick any $q \in (p_-(L) \vee 1, r)$. In the last line, the operator in brackets is L^q-bounded, uniformly, and the operator to its left is $\mathrm{L}^q - \mathrm{L}^r$-bounded with norm controlled by $t^{-n/q+n/r}$. We use then that $a^{-1}\widetilde{G}(s,\cdot)$ belongs to L^q, uniformly.

Step 2. We turn to estimates for $\partial_t(H - \mathrm{e}^{-tL^{1/2}}h)$ on differentiating (20.4). For $t > 4\beta$ we have

$$\begin{aligned}&\partial_t(H(t,\cdot) - \mathrm{e}^{-tL^{1/2}}h)\\&= -\frac{1}{2}\int_{1/\beta}^{\beta} L^{1/2}\Big(\mathrm{e}^{-(t-s)L^{1/2}} + \mathrm{e}^{-(s+t)L^{1/2}}\Big)(a^{-1}\widetilde{G}(s,\cdot))\,\mathrm{d}s.\end{aligned}$$

We expand the kernel as

$$\begin{aligned}&L^{1/2}\Big(\mathrm{e}^{-(t-s)L^{1/2}} + \mathrm{e}^{-(s+t)L^{1/2}}\Big)\\&= \Big(\mathrm{e}^{-(\frac{t}{2}-\beta)L^{1/2}}\Big)\Big(L^{1/2}\mathrm{e}^{-(\frac{t}{2}-\beta)L^{1/2}}\Big)\Big(\mathrm{e}^{-(2\beta-s)L^{1/2}} + \mathrm{e}^{-(2\beta+s)L^{1/2}}\Big)\end{aligned}$$

and pick again any $q \in (p_-(L) \vee 1, r)$. On the right-hand side the third operator is uniformly L^q-bounded, the second one is L^q-bounded with bound controlled by t^{-1} and the first one is $\mathrm{L}^q - \mathrm{L}^r$-bounded with bound controlled by $t^{-n/q+n/r}$. We use then that $a^{-1}\widetilde{G}(s,\cdot)$ belongs to L^q, uniformly.

For $0 < t \leq 4\beta$ we need a uniform L^r-bound. We are integrating over the singularity at $t = s$ in the second line of (20.4) but using the convolution structure in the first line of (20.4), we can compute with $G = \partial_s\widetilde{G}$,

$$\begin{aligned}&\partial_t(H(t,\cdot) - \mathrm{e}^{-tL^{1/2}}h)\\&= \frac{1}{2}\int_{1/\beta}^{\beta} \operatorname{sgn}(t-s)\mathrm{e}^{-|t-s|L^{1/2}}(a^{-1}G(s,\cdot))\,\mathrm{d}s\\&\quad - \frac{1}{2}\int_{1/\beta}^{\beta} L^{1/2}\mathrm{e}^{-(s+t)L^{1/2}}(a^{-1}\widetilde{G}(s,\cdot))\,\mathrm{d}s.\end{aligned} \tag{20.9}$$

The operators inside the integrals are L^r-bounded, uniformly for s, t in the prescribed range.

Finally, we establish the L^∞-bounds under the additional assumption $p_-(L^\sharp) < 1$. This implies $p_+(L) = \infty$ by duality and similarity.

Step 3. We modify Step 1 as follows.

If $t \leq 1/4\beta$, then we pick any $r \in (p_-(L) \vee 1, \infty)$ and on the left-hand side of (20.6) we use the $\mathrm{L}^r - \mathrm{L}^\infty$-bound for $\mathrm{e}^{-(s-2t)L^{1/2}}$, see Proposition 12.5.(iii), which is uniform in s and t since $s - 2t \in [1/2\beta, \beta]$.

If $1/4\beta < t < 4\beta$, then the operator inside the integral in (20.7) is $\mathrm{L}^q - \mathrm{L}^\infty$-bounded with norm controlled by $|s-t|^{-n/q}$. Since $p_+(L) = \infty$, we can pick $q > n$ and this bound becomes integrable on $[1/\beta, \beta]$. Then we use that $a^{-1}\widetilde{G}(s,\cdot)$ belongs to L^q, uniformly.

Likewise, if $t \geq 4\beta$, then $\mathrm{e}^{-(t-2\beta)L^{1/2}}$ in (20.8) is $\mathrm{L}^q - \mathrm{L}^\infty$-bounded with norm controlled by $t^{-n/q}$.

Step 4. We modify Step 2 as follows.

If $t > 4\beta$, then thanks to Proposition 12.5.(iii) the same argument as before applies with $r = \infty$.

If $0 < t \leq 4\beta$, then the operator inside the first integral in (20.9) is $\mathrm{L}^q - \mathrm{L}^\infty$-bounded with norm controlled by $|s-t|^{-n/q}$ and choosing $q > n$ gives an integrable singularity. In the second integral we write

$$L^{1/2}\mathrm{e}^{-(s+t)L^{1/2}} = \mathrm{e}^{-(\frac{s}{2}+\frac{t}{2})L^{1/2}}\Big(L^{1/2}\mathrm{e}^{-(\frac{s}{2}+\frac{t}{2})L^{1/2}}\Big).$$

The operator on the far right is L^q-bounded and the one to its left is $\mathrm{L}^q - \mathrm{L}^\infty$-bounded, both with uniform bounds since $s/2 + t/2 \in [1/2\beta, 5\beta/2]$. □

21. Uniqueness in regularity and Dirichlet problems

This section complements Section 17, 18 and 19. We shall prove the uniqueness parts in Theorems 1.1, 1.2, 1.3 and 1.4.

In [11], we developed a strategy to prove uniqueness for elliptic systems without regularity assumptions and with coefficients not necessarily in block form. We streamline the strategy in the case of the block system $\mathcal{L}u = 0$ to obtain uniqueness of solutions in much greater generality.

21.1. **Review of the strategy of proof of uniqueness.** Throughout, we denote by $\langle \cdot, \cdot \rangle$ the sesquilinear duality pairing between distributions and test functions in $\mathbb{R}^{1+n}_+$. Since we are dealing with a linear equation, it suffices to assume that u solves one of

$$(R)^{\mathcal{L}}_p, (D)^{\mathcal{L}}_p, (D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}, (\widetilde{D})^{\mathcal{L}}_{\dot{\Lambda}^\alpha}, (D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$$

with boundary data 0 and show that this forces u to vanish almost everywhere.

It begins with the following lemma in order to restrict the class of necessary testing conditions for u. The possible combinations of an interior control with a boundary limit cover all cases that can occur in our BVPs.

Lemma 21.1. *Let u be a weak solution to $\mathcal{L}u = -\operatorname{div} A\nabla u = 0$ on $\mathbb{R}^{1+n}_+$. Let $\alpha \in [0,1)$ and $p \in (0,\infty)$. Assume one of the interior controls*

- $\widetilde{N}_*(u) \in \mathrm{L}^p$
- $\widetilde{N}_*(\nabla u) \in \mathrm{L}^p$
- $W(t^{1-\alpha}\nabla u) \in \mathrm{L}^p$
- $S(t^{1-\alpha}\nabla u) \in \mathrm{L}^p$
- $\widetilde{N}_{\sharp,\alpha}(u) \in \mathrm{L}^\infty$
- $C_\alpha(t\nabla u) \in \mathrm{L}^\infty$
- $C_0(t^{1-\alpha}\nabla u) \in \mathrm{L}^\infty$

and one of the boundary limits

$$\lim_{t\to 0} \iint_{W(t,x)} |u(s,y)| \,\mathrm{d}s\mathrm{d}y = 0 \quad (a.e.\ x \in \mathbb{R}^n), \tag{21.1}$$

$$\lim_{t\to 0} \fint_{t/2}^{2t} |u(s,\cdot)|\, \mathrm{d}s = 0 \quad (\textit{in } \mathrm{L}^2_{\mathrm{loc}}). \tag{21.2}$$

If $\langle u, G\rangle = 0$ *for all test functions of the form* $G = \partial_t \operatorname{div}_x G^\sharp$ *with* $G^\sharp \in \mathrm{C}_0^\infty(\mathbb{R}^{1+n}_+; \mathbb{C}^{mn})$*, then* $u = 0$ *almost everywhere.*

Proof. We have $\langle \nabla_x \partial_t u, G^\sharp\rangle = 0$, where $G^\sharp$ is an arbitrary test function in $\mathbb{R}^{1+n}_+$. Hence, $\partial_t u \in \mathrm{L}^2_{\mathrm{loc}}$ is independent of x and we obtain

$$u(t,x) = g(t) + f(x)$$

with $f \in \mathrm{L}^2_{\mathrm{loc}}$ and $g : (0,\infty) \to \mathbb{C}^m$ smooth (Corollary 16.9). If (21.1) holds, then we write

$$\fint\!\!\fint_{W(t,x)} u(s,y)\, \mathrm{d}s\mathrm{d}y = \fint_{t/2}^{2t} g(s)\, \mathrm{d}s + \fint_{B(x,t)} f(y)\, \mathrm{d}y,$$

where in the limit as $t \to 0$ the left-hand side tends to 0 for a.e. $x \in \mathbb{R}^n$ by assumption and the second term on the right-hand side tends to $f(x)$ by Lebesgue's differentiation theorem. Hence, $\fint_{t/2}^{2t} g(s)\mathrm{d}s$ has a limit as $t \to 0$ that we call $\beta \in \mathbb{C}^m$ and we have $f(x) = -\beta$ almost everywhere. The same conclusion holds under the assumption (21.2) since then

$$\fint_{t/2}^{2t} u(s,\cdot)\, \mathrm{d}s = \fint_{t/2}^{2t} g(s)\, \mathrm{d}s + f(\cdot)$$

tends to 0 in $\mathrm{L}^2_{\mathrm{loc}}$ as $t \to 0$.

So far we know that $u(t,x) = g(t) - \beta$. The equation for u yields $a\partial_t^2 g = \mathcal{L}u = 0$. Consequently, g is a linear function. By definition of β we get $g(t) = \gamma t + \beta$ for some $\gamma \in \mathbb{C}^m$ and therefore $u(t,x) = \gamma t$. If $\gamma \neq 0$, then we get for all $x \in \mathbb{R}^n$ and all $t > 0$ that

- $\widetilde{N}_*(u)(x) = \infty$
- $\widetilde{N}_*(\nabla u)(x) = |\gamma|$
- $W(t^{1-\alpha}\nabla u)(t,x) = t^{1-\alpha}|\gamma|$
- $S(t^{1-\alpha}\nabla u)(x) = \infty$
- $\widetilde{N}_{\sharp,\alpha}(u)(x) = \infty$
- $C_\alpha(t\nabla u)(x) = \infty$
- $C_0(t^{1-\alpha}\nabla u)(x) = \infty$

and none of the interior controls is satisfied. Thus, $\gamma = 0$. □

Now, let u be a solution to $\mathcal{L}u = -\operatorname{div} A\nabla u = 0$ on $\mathbb{R}^{1+n}_+$. We take G as above. To compute $\langle u, G\rangle$, we then pick a second function θ, compactly supported in $\mathbb{R}^{1+n}_+$, real-valued, Lipschitz continuous and equal to 1 on the support of G. Finally, we let $H := (\mathcal{L}^*)^{-1}G$, which is a weak solution to the adjoint equation

$$\mathcal{L}^* H = -\operatorname{div} A^* \nabla H = G \quad (\text{on } \mathbb{R}^{1+n}).$$

As $u\theta$ is a test function for this equation, we have

$$\langle u, G\rangle = \langle u\theta, G\rangle = \langle A\nabla(u\theta), \nabla H\rangle.$$

Next,

$$\begin{aligned}\langle A\nabla(u\theta),\nabla H\rangle &= \langle A(u\otimes\nabla\theta),\nabla H\rangle + \langle A(\theta\nabla u),\nabla H\rangle\\ &= \langle A(u\otimes\nabla\theta),\nabla H\rangle - \langle A\nabla u, H\otimes\nabla\theta\rangle + \langle A\nabla u,\nabla(\theta H)\rangle,\end{aligned}$$

and the last term vanishes because θH is a test function for $\mathcal{L}u = 0$. Our notation $\nabla(u\theta) = u\otimes\nabla\theta + \theta\nabla u$ is as predicted by the product rule.

We let $h := H(0,\cdot) \in \mathrm{L}^2$ (see Corollary 20.3) and take

$$H_1(t,x) := \mathrm{e}^{-t(L^\sharp)^{1/2}} h(x), \tag{21.3}$$

where we recall that $L^\sharp$ corresponds to $\mathcal{L}^*$ in the same way as L corresponds to $\mathcal{L}$. In particular, H_1 a solution to the adjoint problem $\mathcal{L}^* H_1 = 0$ on $\mathbb{R}^{1+n}_+$ with boundary condition h, see Proposition 16.4. We can apply the same decomposition to $\langle A\nabla(u\theta),\nabla H_1\rangle$ and remark that this term vanishes since $u\theta$ is a test function for $\mathcal{L}^* H_1 = 0$. Hence, we obtain

$$\langle u, G\rangle = \langle A(u\otimes\nabla\theta),\nabla(H-H_1)\rangle - \langle A\nabla u,(H-H_1)\otimes\nabla\theta\rangle. \tag{21.4}$$

We remark that u and $H - H_1$ vanish at the boundary. In fact, the reason to introduce H_1 is to help convergence near the boundary.

Lemma 21.1 implies the following reformulation of uniqueness in the four BVPs.

Proposition 21.2. *Suppose that u solves one of the problems $(R)^{\mathcal{L}}_p$, $(D)^{\mathcal{L}}_p$, $(D)^{\mathcal{L}}_{\dot\Lambda^\alpha}$, $(\widetilde{D})^{\mathcal{L}}_{\dot\Lambda^\alpha}$, $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ with boundary data $f = 0$. If the right hand side of (21.4) converges to 0 as $\theta \to 1$ everywhere on $\mathbb{R}^{1+n}_+$, then $u = 0$ almost everywhere.*

We prepare the limit procedure by picking θ in a more explicit way. For the rest of the section the following parameters will be used:

(21.5)

- $G^\sharp \in \mathrm{C}^\infty_0(\mathbb{R}^{1+n}_+;\mathbb{C}^{mn})$ with support in $[1/\beta,\beta]\times B(0,\beta) \subseteq \mathbb{R}^{1+n}_+$ and $G := \partial_t \operatorname{div}_x G^\sharp$,
- $\chi \in \mathrm{C}^\infty_0(\mathbb{R}^n;\mathbb{R})$ with $\mathbf{1}_{B(0,1)} \leq \chi \leq \mathbf{1}_{B(0,2)}$,
- η the continuous piecewise linear function, which is equal to 0 on $[0,\frac{2}{3}]$, equal to 1 on $[\frac{3}{2},\infty)$, and linear in between,
- $M > 8\beta$ and $0 < \varepsilon < 1/4\beta$ and $8\beta < R < \infty$ to finally set
$$\theta(t,x) := \chi\Big(\frac{x}{M}\Big)\eta\Big(\frac{t}{\varepsilon}\Big)\Big(1-\eta\Big(\frac{t}{R}\Big)\Big).$$

We also use the block structure of A to write

$$A(u\otimes\nabla\theta) = \begin{bmatrix} au\partial_t\theta \\ d(u\otimes\nabla_x\theta)\end{bmatrix}, \quad A\nabla u = \begin{bmatrix} a\partial_t u \\ d\nabla_x u\end{bmatrix}.$$

Due to the explicit form of θ, we obtain for the first term on the right hand side of (21.4) that

$$|\langle A(u \otimes \nabla\theta), \nabla(H - H_1)\rangle| \lesssim I_{M,\varepsilon,R} + J_{\varepsilon,M} + J_{R,M}, \tag{21.6}$$

with

$$I_{M,\varepsilon,R} := \frac{1}{M} \int_{M \le |y| \le 2M} \int_{2\varepsilon/3}^{3R/2} |u||\nabla_x(H - H_1)| \,\mathrm{d}s\mathrm{d}y \tag{21.7}$$

and

$$J_{\tau,M} := \int_{|y| \le 2M} \fint_{2\tau/3}^{3\tau/2} |u||\partial_t(H - H_1)| \,\mathrm{d}s\mathrm{d}y. \tag{21.8}$$

For the second term, we have

$$|\langle A\nabla u, (H - H_1) \otimes \nabla\theta\rangle| \lesssim \widetilde{I}_{M,\varepsilon,R} + \widetilde{J}_{\varepsilon,M} + \widetilde{J}_{R,M}, \tag{21.9}$$

with

$$\widetilde{I}_{M,\varepsilon,R} := \frac{1}{M} \int_{M \le |y| \le 2M} \int_{2\varepsilon/3}^{3R/2} |\nabla_x u||H - H_1| \,\mathrm{d}s\mathrm{d}y \tag{21.10}$$

and

$$\widetilde{J}_{\tau,M} := \int_{|y| \le 2M} \fint_{2\tau/3}^{3\tau/2} |\partial_t u||H - H_1| \,\mathrm{d}s\mathrm{d}y. \tag{21.11}$$

Implicit constants depend only on dimensions and ellipticity. The way how the parameters M, R tend to ∞ and ε tend to 0 will be specified to make the terms on the right of (21.6) and (21.9) tend to 0.

21.2. **Uniqueness for $(R)_p^{\mathcal{L}}$ – conclusion of the proof of Theorem 1.2.** We shall obtain uniqueness of solutions to $(R)_p^{\mathcal{L}}$ in the range

$$p_-(L)_* \vee 1_* < p < p_+(L).$$

By Theorem 6.2 we have $q_+(L) \le p_+(L)$, so that this is even a larger range than for existence of a solution in Theorem 1.2. We assume the interior control $\widetilde{N}_*(\nabla u) \in \mathrm{L}^p$ and the convergence at the boundary (21.1) for almost every x. Then we distinguish two cases:

- $(p_-(L)_* \vee 1_*) < p \le (p_-(L) \vee 1)$,
- $(p_-(L) \vee 1) < p < p_+(L)$.

Case 1: $p_-(L)_* \vee 1_* < p \le (p_-(L) \vee 1)$. To implement the strategy in Section 21.1, we begin with the following lemma.

Lemma 21.3. *If $0 < p < r \le 2$, then for any weak solution u to $\mathcal{L}u = 0$ on $\mathbb{R}^{1+n}_+$,*

$$\Big(\iint_{\mathbb{R}^{1+n}_+} |\nabla u|^r t^{n(\frac{r}{p}-1)} \frac{\mathrm{d}t\mathrm{d}x}{t} \Big)^{\frac{1}{r}} \lesssim \|\widetilde{N}_*(\nabla u)\|_p. \tag{21.12}$$

Moreover, if (21.1) *holds, then*

$$\left(\iint_{\mathbb{R}^{1+n}_+} |u|^r t^{n(\frac{r}{p}-1)-r} \, \frac{\mathrm{d}t\mathrm{d}x}{t} \right)^{\frac{1}{r}} \lesssim \|\widetilde{N}_*(\nabla u)\|_p. \tag{21.13}$$

Proof. The first inequality is due to Lemma A.3 applied to $F \coloneqq |\nabla u|$. For the second inequality, Proposition A.5.(iii) yields $\|\widetilde{N}_{*,1}({}^u\!/\!{}_t)\|_p \lesssim \|\widetilde{N}_*(\nabla u)\|_p$, where $\widetilde{N}_{*,1}$ is a non-tangential maximal function that uses L^1-averages instead of L^2-averages. But as u is a weak solution to $\mathcal{L}u = 0$, it satisfies reverse Hölder inequalities. Hence, $\|\widetilde{N}_*({}^u\!/\!{}_t)\|_p \lesssim \|\widetilde{N}_*(\nabla u)\|_p$, where we also used a change of parameters in non-tangential maximal functions (Lemma A.1). Applying Lemma A.3 to $F \coloneqq {}^u\!/\!{}_t$ concludes the proof. □

We fix an exponent r such that $(p_-(L) \vee 1) < r \leq 2$. Then the assumption $p < r$ in Lemma 21.3 holds automatically and we have $2 \leq r' < p_+(L^\sharp)$ by duality and similarity. Next, we recall that $H_1(t, \cdot) = \mathrm{e}^{-t(L^\sharp)^{1/2}} h$, where h is the trace of H at $t = 0$. We have at our disposal the estimates of Lemma 20.4 with $L^\sharp$ replacing L. In particular, we obtain for some $\gamma > 0$ and all $t > 0$,

$$\begin{aligned} \|H(t,\cdot) - H_1(t,\cdot)\|_{r'} &\lesssim t \wedge t^{-\gamma}, \\ \|\partial_t(H(t,\cdot) - H_1(t,\cdot))\|_{r'} &\lesssim 1 \wedge t^{-1-\gamma}. \end{aligned} \tag{21.14}$$

We come to taking limits in (21.6) and (21.9). We shall send $M \to \infty$ for ε, R fixed and then send $\varepsilon \to 0$ and $R \to \infty$. We start with the terms on the right hand side of (21.6).

The term $I_{M,\varepsilon,R}$. We can bound $MI_{M,\varepsilon,R}$ by a finite number (depending on ε, R) of integrals

$$K_{\tau,M} \coloneqq \int_{|y|\geq M} \int_{2\tau/3}^{3\tau/2} |u||\nabla_x(H - H_1)| \,\mathrm{d}s\mathrm{d}y \quad (\varepsilon \leq \tau \leq R)$$

and it suffices to bound each of them uniformly for M large, say $M \geq 10R$.

Because we do not have global bounds on $\nabla_x(H - H_1)$, we argue as follows. We let $w(\tau, x) \coloneqq ({}^{2\tau}\!/\!{}_3, {}^{3\tau}\!/\!{}_2) \times B(x, {}^{\tau}\!/\!{}_2)$ denote slightly smaller Whitney boxes and use an averaging trick to give

$$\begin{aligned} K_{\tau,M} &\lesssim \int_{|x|\geq M/2} \left(\fint\!\!\fint_{w(\tau,x)} |u||\tau\nabla_x(H - H_1)| \right) \mathrm{d}x \\ &\lesssim \int_{|x|\geq M/2} \left(\fint\!\!\fint_{w(\tau,x)} |u|^2 \right)^{\frac{1}{2}} \left(\fint\!\!\fint_{w(\tau,x)} |\tau\nabla_x(H - H_1)|^2 \right)^{\frac{1}{2}} \mathrm{d}x \\ &\lesssim \int_{|x|\geq M/2} \left(\fint\!\!\fint_{W(\tau,x)} |u|^r \right)^{\frac{1}{r}} \left(\fint\!\!\fint_{W(\tau,x)} |H - H_1|^{r'} \right)^{\frac{1}{r'}} \mathrm{d}x, \end{aligned}$$

where for the last line we used reverse Hölder estimates for u and the Caccioppoli estimate followed by Hölder's inequality for $H-H_1$, which is a weak solution to $\mathcal{L}^*(H-H_1)=0$ in a neighborhood of each $W(\tau,x)$. Indeed, $\mathcal{L}^*(H-H_1)=G$ on $\mathbb{R}^{1+n}_+$ but as $W(\tau,x)\subseteq\{(t,y):|y|\geq 4R\}$, all Whitney boxes are outside the support of G, see (21.5). By Hölder's inequality in x, we have

$$\begin{aligned}K_{\tau,M} &\lesssim \left(\int_{\mathbb{R}^n} \fint\!\!\fint_{W(\tau,x)} |u|^r \mathrm{d}y\mathrm{d}t\mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} \fint\!\!\fint_{W(\tau,x)} |H-H_1|^{r'} \mathrm{d}y\mathrm{d}t\mathrm{d}x\right)^{\frac{1}{r'}}\\ &\lesssim \left(\int_{\tau/2}^{2\tau}\int_{\mathbb{R}^n} |u|^r \,\frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{\frac{1}{r}} \left(\fint_{\tau/2}^{2\tau}\int_{\mathbb{R}^n} |H-H_1|^{r'}\,\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{r'}}\\ &\lesssim \tau^{1-n(\frac{1}{p}-\frac{1}{r})}\|\widetilde{N}_*(\nabla u)\|_p\,(\tau\wedge\tau^{-\gamma})\end{aligned}$$

using (21.13) and (21.14). Summing up in τ, we conclude $MI_{M,\varepsilon,R}\lesssim \|\widetilde{N}_*(\nabla u)\|_p$ with an implicit constant that depends on ε, R but not on M. Thus, $I_{M,\varepsilon,R}\to 0$ in the limit as $M\to\infty$.

The term $J_{\varepsilon,M}$. When $M\to\infty$, we have to take the dx-integral on all of $\mathbb{R}^n$ and we can use Hölder's inequality directly to obtain a bound for the limit by

$$\begin{aligned}&\left(\fint_{2\varepsilon/3}^{3\varepsilon/2}\int_{\mathbb{R}^n} |u|^r\,\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{r}} \left(\fint_{2\varepsilon/3}^{3\varepsilon/2}\int_{\mathbb{R}^n} |\partial_t(H-H_1)|^{r'}\,\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{r'}}\\ &\lesssim \varepsilon^{1-n(\frac{1}{p}-\frac{1}{r})}\left(\int_{2\varepsilon/3}^{3\varepsilon/2}\int_{\mathbb{R}^n} |u|^r t^{n(\frac{r}{p}-1)-r}\,\frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{\frac{1}{r}}\end{aligned} \tag{21.15}$$

using the estimate (21.14) when $\varepsilon<1$. At this point we have to discuss the choice of r.

In dimension $n\geq 2$ we set $r:=p^*$. In order to see that this choice is admissible, we first note that Proposition 6.7 yields $(p_-(L)\vee 1)\leq 2_*$ and therefore $r\leq 2$ follows from the upper bound on p. Likewise, the lower bound on p implies $r>(p_-(L)\vee 1)$. For this choice of r the exponent of ε in (21.15) vanishes and we conclude from (21.13) that the remaining integral converges to 0 as $\varepsilon\to 0$.

In dimension $n=1$ we have $1^*=\infty$ and hence we must argue differently. Proposition 6.7 yield $p_-(L)={}^1/_2=1_*$. Hence, our assumption on p is ${}^1/_2<p\leq 1$ and this allows us to pick $r>1$ sufficiently close to 1 such that ${}^1/_r\geq{}^1/_p-1$. Consequently, the exponent for ε in (21.15) is non-negative and we conclude as before.

The term $J_{R,M}$. Similarly, we have a bound for the limit as $M \to \infty$ by

$$
\begin{aligned}
&\Big(\fint_{2R/3}^{3R/2} \int_{\mathbb{R}^n} |u|^r \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{r}} \Big(\fint_{2R/3}^{3R/2} \int_{\mathbb{R}^n} |\partial_t (H - H_1)|^{r'} \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{r'}} \\
&\lesssim R^{-n(\frac{1}{p}-\frac{1}{r})-\gamma} \Big(\int_{2R/3}^{3R/2} \int_{\mathbb{R}^n} |u|^r t^{n(\frac{r}{p}-1)-r} \,\frac{\mathrm{d}y\mathrm{d}t}{t}\Big)^{\frac{1}{r}},
\end{aligned} \tag{21.16}
$$

using (21.14) when $R > 1$. Since we have $r > p$ in any case, we get a negative power of R in front of the integral and in view of (21.13) this term tends to 0 as $R \to \infty$.

We next consider the terms on the right hand side of (21.9).

The term $\widetilde{I}_{M,\varepsilon,R}$. Hölder's inequality yields that $M\widetilde{I}_{M,\varepsilon,R}$ is bounded by

$$
\Big(\int_{2\varepsilon/3}^{3R/2}\int_{\mathbb{R}^n} |t\nabla_x u|^r \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{r}} \Big(\int_{2\varepsilon/3}^{3R/2}\int_{\mathbb{R}^n} \Big|\frac{H - H_1}{t}\Big|^{r'} \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{r'}}.
$$

Using (21.12) and (21.14), we thus obtain $M\widetilde{I}_{M,\varepsilon,R} \lesssim \|\widetilde{N}_*(\nabla u)\|_p$ with an implicit constant that depends on ε, R but not on M. Hence, we have $\widetilde{I}_{M,\varepsilon,R} \to 0$ in the limit as $M \to \infty$.

The term $\widetilde{J}_{\varepsilon,M}$. We have again the following bound for the limit as $M \to \infty$ by taking the $\mathrm{d}x$-integral on $\mathbb{R}^n$ and using Hölder's inequality directly:

$$
\begin{aligned}
&\Big(\fint_{2\varepsilon/3}^{3\varepsilon/2} \int_{\mathbb{R}^n} |t\partial_t u|^r \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{r}} \Big(\fint_{2\varepsilon/3}^{3\varepsilon/2} \int_{\mathbb{R}^n} \Big|\frac{H - H_1}{t}\Big|^{r'} \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{r'}} \\
&\lesssim \varepsilon^{1-n(\frac{1}{p}-\frac{1}{r})} \Big(\int_{2\varepsilon/3}^{3\varepsilon/2} \int_{\mathbb{R}^n} |\partial_t u|^r t^{n(\frac{r}{p}-1)} \,\frac{\mathrm{d}y\mathrm{d}t}{t}\Big)^{\frac{1}{r}}
\end{aligned} \tag{21.17}
$$

where the second step is due to (21.14). The exponent for ε is the same as in (21.15) and thus becomes non-negative for the same choice of r as before. It follows from (21.12) that the remaining integral tends to 0 as $\varepsilon \to 0$.

The term $\widetilde{J}_{R,M}$. Similarly, for the limit of $\widetilde{J}_{R,M}$ as $M \to \infty$, we have the bound

$$
\begin{aligned}
&\Big(\fint_{2R/3}^{3R/2} \int_{\mathbb{R}^n} |t\partial_t u|^r \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{r}} \Big(\fint_{2R/3}^{3R/2} \int_{\mathbb{R}^n} \Big|\frac{H - H_1}{t}\Big|^{r'} \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{r'}} \\
&\lesssim R^{-n(\frac{1}{p}-\frac{1}{r})-\gamma} \Big(\int_{2R/3}^{3R/2} \int_{\mathbb{R}^n} |t\partial_t u|^r t^{n(\frac{r}{p}-1)} \,\frac{\mathrm{d}y\mathrm{d}t}{t}\Big)^{\frac{1}{r}},
\end{aligned} \tag{21.18}
$$

using (21.14) when $R > 1$. The exponent for R is negative and in the limit as $R \to \infty$, the right-hand side tends to 0, taking into account (21.12). The argument is complete.

Case 2: $(p_-(L) \vee 1) < p < p_+(L)$. For this case we organize the limit procedure differently. We set $R = M$ and first send $\varepsilon \to 0$ and then $M \to \infty$ in (21.6) and (21.9).

The interior control $\widetilde{N}_*(\nabla u) \in \mathrm{L}^p$ and the boundary limit (21.1) enter the calculations in a particularly concise form via the trace estimate

$$\fint\!\!\fint_{W(t,x)} |u| \,\mathrm{d}s\mathrm{d}y \lesssim t\widetilde{N}_*(\nabla u)(x) \quad ((t,x) \in \mathbb{R}^{1+n}_+)$$

from Proposition A.5. The non-tangential maximal function $\widetilde{N}_*(\nabla u)$ has no further meaning to our argument and we can proceed without any additional effort under the following general assumption: Besides (21.1) we assume that there exists $\Theta \in \mathrm{L}^p$ and $\alpha \in [0,1]$ such that

$$U_t(x) \coloneqq \fint\!\!\fint_{W(t,x)} |u| \,\mathrm{d}s\mathrm{d}y \tag{21.19}$$

is controlled via

$$U_t(x) \leq t^\alpha \Theta(x) \quad ((t,x) \in \mathbb{R}^{1+n}_+). \tag{21.20}$$

This generalization will have fruitful implications for some of the other boundary value problems.

We begin with the terms in (21.6).

The term $J_{\varepsilon,M}$. We let $w(\tau, x) \coloneqq (2\tau/3, 3\tau/2) \times B(x, \tau/2)$ denote slightly smaller Whitney boxes and use an averaging trick to give

$$\begin{aligned} J_{\varepsilon,M} &\lesssim \int_{|x|\leq 3M} \bigg(\fint\!\!\fint_{w(\varepsilon,x)} |u||\partial_t(H-H_1)| \bigg) \,\mathrm{d}x \\ &\lesssim \int_{|x|\leq 3M} \bigg(\fint\!\!\fint_{w(\varepsilon,x)} |u|^2 \bigg)^{\frac{1}{2}} \bigg(\fint\!\!\fint_{w(\varepsilon,x)} |\partial_t(H-H_1)|^2 \bigg)^{\frac{1}{2}} \,\mathrm{d}x \\ &\lesssim \int_{|x|\leq 3M} \bigg(\fint\!\!\fint_{W(\varepsilon,x)} |u| \bigg) \bigg(\fint\!\!\fint_{W(\varepsilon,x)} \bigg| \frac{H-H_1}{\varepsilon} \bigg| \bigg) \,\mathrm{d}x \\ &= \int_{|x|\leq 3M} U_\varepsilon(x) \bigg(\fint\!\!\fint_{W(\varepsilon,x)} \bigg| \frac{H-H_1}{\varepsilon} \bigg| \,\mathrm{d}t\mathrm{d}y \bigg) \,\mathrm{d}x. \end{aligned} \tag{21.21}$$

The third line is the combination of Caccioppoli's estimate and the reverse Hölder inequality for $H - H_1$, which solves $\mathcal{L}^*(H - H_1) = G$ on $\mathbb{R}^{1+n}_+$ but $W(\varepsilon, x) \subseteq \{(t, y) : 0 < t < 1/2\beta\}$ is outside the support of G, see (21.5).

Next, we bring into play the maximal function $\mathcal{M}^\varepsilon$ restricted to balls with radii not exceeding ε. The averaging trick followed by Hölder's inequality yields

$$
\begin{aligned}
J_{\varepsilon,M} &\lesssim \int_{|y|\le 4M} \mathcal{M}^\varepsilon(U_\varepsilon)(y) \fint_{\varepsilon/2}^{2\varepsilon} \left|\frac{H-H_1}{t}\right| \mathrm{d}t\mathrm{d}y \\
&\lesssim \bigg(\int_{|y|\le 4M} \mathcal{M}^\varepsilon(U_\varepsilon)^p \,\mathrm{d}y\bigg)^{\frac{1}{p}} \bigg(\int_{\mathbb{R}^n} \bigg(\fint_{\varepsilon/2}^{2\varepsilon} \left|\frac{H-H_1}{t}\right| \mathrm{d}t\bigg)^{p'} \mathrm{d}y\bigg)^{\frac{1}{p'}} \\
&\lesssim \bigg(\int_{|y|\le 4M} \mathcal{M}^\varepsilon(U_\varepsilon)^p \,\mathrm{d}y\bigg)^{\frac{1}{p}} \bigg(\int_{\mathbb{R}^n} \fint_{\varepsilon/2}^{2\varepsilon} \left|\frac{H-H_1}{t}\right|^{p'} \mathrm{d}t\mathrm{d}y\bigg)^{\frac{1}{p'}} \\
&\lesssim \bigg(\int_{|y|\le 5M} U_\varepsilon^p \,\mathrm{d}y\bigg)^{\frac{1}{p}} \bigg(\fint_{\varepsilon/2}^{2\varepsilon} \int_{\mathbb{R}^n} \left|\frac{H-H_1}{t}\right|^{p'} \mathrm{d}y\mathrm{d}t\bigg)^{\frac{1}{p'}},
\end{aligned}
$$

where we have used $\mathcal{M}^\varepsilon(U_\varepsilon) \le \mathcal{M}(\mathbf{1}_{B(0,5M)}U_\varepsilon)$ on $B(0,4M)$ and the maximal theorem in the last line.

The assumption on p implies $(p_-(L^\sharp) \vee 1) < p' < p_+(L^\sharp)$ by duality and similarity. Thus, we may use Lemma 20.4 for $H - H_1$ with $r = p'$ and obtain for some $\gamma > 0$ and all $t > 0$ the bound

$$
\|H(t,\cdot) - H_1(t,\cdot)\|_{p'} \lesssim t \wedge t^{-\gamma}. \tag{21.22}
$$

Thus, the second integral on the right in the estimate above is uniformly bounded in $\varepsilon \le 1$ and we are left with

$$
J_{\varepsilon,M} \lesssim \bigg(\int_{|y|\le 5M} U_\varepsilon^p \,\mathrm{d}y\bigg)^{\frac{1}{p}} \qquad (\varepsilon \le 1).
$$

According to (21.20) we have $U_\varepsilon \le \Theta \in \mathrm{L}^p$ for all $\varepsilon \le 1$, so that we can use the dominated convergence theorem when passing to the limit as $\varepsilon \to 0$. By assumption (21.1) we have $U_\varepsilon(x) \to 0$ for a.e. $x \in \mathbb{R}^n$ and $J_{\varepsilon,M} \to 0$ follows. This completes the treatment of this term.

The terms $I_{M,\varepsilon,M}$ and $J_{M,M}$. Having sent $\varepsilon \to 0$, we have to estimate

$$
\lim_{\varepsilon\to 0} I_{M,\varepsilon,M} + J_{M,M} =: I_M + J_M,
$$

where

$$
\begin{aligned}
I_M &:= \frac{1}{M} \int_{M\le|y|\le 2M} \int_0^{3M/2} |u||\nabla_x(H-H_1)| \,\mathrm{d}s\mathrm{d}y \\
J_M &:= \int_{|y|\le 2M} \fint_{2M/3}^{3M/2} |u||\partial_t(H-H_1)| \,\mathrm{d}s\mathrm{d}y.
\end{aligned} \tag{21.23}
$$

We begin with I_M. In the following we use small Whitney regions $w(\tau,x) = (2\tau/3, 3\tau/2) \times B(x, 2\tau/9)$. Let $\tau_j := (9/4)^j$ for $j \in \mathbb{Z}$ and let j_M be the unique integer with $\tau_{j_M-1} \le M < \tau_{j_M}$. Then $\tau_{j_M} \le 9M/4$ and

$$
MI_M \le \sum_{j=-\infty}^{j_M} K_{\tau_j,M} \tag{21.24}
$$

with

$$K_{\tau,M} := \int_{M\leq|y|\leq 2M}\int_{2\tau/3}^{3\tau/2} |u||\nabla_x(H-H_1)|\,\mathrm{d}s\mathrm{d}y. \tag{21.25}$$

Applying Caccioppoli and reverse Hölder inequalities as usual, we obtain for $\tau \leq 9M/4$ that

$$\begin{aligned} K_{\tau,M} &\lesssim \int_{\frac{M}{2}\leq|x|\leq\frac{5M}{2}} \bigg(\fint\!\!\fint_{w(\tau,x)} |u||\tau\nabla_x(H-H_1)|\bigg)\,\mathrm{d}x \\ &\lesssim \int_{\frac{M}{2}\leq|x|\leq\frac{5M}{2}} \bigg(\fint\!\!\fint_{w(\tau,x)} |u|^2\bigg)^{\frac{1}{2}}\bigg(\fint\!\!\fint_{w(\tau,x)} |\tau\nabla_x(H-H_1)|^2\bigg)^{\frac{1}{2}}\,\mathrm{d}x \\ &\lesssim \int_{\frac{M}{2}\leq|x|\leq\frac{5M}{2}} \bigg(\fint\!\!\fint_{W(\tau,x)} |u|\bigg)\bigg(\fint\!\!\fint_{W(\tau,x)} |H-H_1|\bigg)\,\mathrm{d}x \\ &= \int_{\frac{M}{2}\leq|x|\leq\frac{5M}{2}} U_\tau(x)\bigg(\fint\!\!\fint_{W(\tau,x)} |H-H_1|\bigg)\,\mathrm{d}x. \end{aligned}$$

To justify the interior estimates for $H-H_1$ on $w(\tau,x)$, we remark that $W(\tau,x)$ lies outside the support of G. Indeed, if $\tau \leq M/4$, then $|y| \geq M/4 = R/4 > 2\beta$ for all $(s,y) \in W(\tau,x)$, and if $\tau \geq M/4$, then $s \geq M/8 > \beta$, see (21.5).

Now, we bring again the maximal functions into play and use the averaging trick and (21.20) to give

$$K_{\tau,M} \lesssim \tau^\alpha \int_{|y|\leq 5M} \mathcal{M}(\Theta)(y)\bigg(\fint_{\tau/2}^{2\tau} |H-H_1|\,\mathrm{d}t\bigg)\mathrm{d}y. \tag{21.26}$$

We continue by

$$\begin{aligned} K_{\tau,M} &\lesssim \tau^\alpha\bigg(\int_{\mathbb{R}^n} \mathcal{M}(\Theta)(y)^p\,\mathrm{d}y\bigg)^{\frac{1}{p}}\bigg(\int_{\mathbb{R}^n}\bigg|\fint_{\tau/2}^{2\tau}|H-H_1|\,\mathrm{d}t\bigg|^{p'}\mathrm{d}y\bigg)^{\frac{1}{p'}} \\ &\lesssim \tau^\alpha\bigg(\int_{\mathbb{R}^n}\Theta(y)^p\,\mathrm{d}y\bigg)^{\frac{1}{p}}\bigg(\fint_{\tau/2}^{2\tau}\int_{\mathbb{R}^n}|H-H_1|^{p'}\,\mathrm{d}y\mathrm{d}t\bigg)^{\frac{1}{p'}} \\ &\lesssim \tau^\alpha(\tau\wedge\tau^{-\gamma})\|\Theta\|_p, \end{aligned}$$

where we have used Hölder's inequality in the first line, the maximal theorem and Jensen's inequality in the second one and (21.22) in the third one. At this point we can go back to (21.24) and sum up the estimates for $\tau = \tau_j$ in order to obtain

$$MI_M \lesssim (1+M^{\alpha-\gamma})\|\Theta\|_p.$$

By assumption we have $\alpha \leq 1$ and $\gamma > 0$. Hence, M appears with exponent smaller than 1 on the right-hand side and we conclude $I_M \to 0$ in the limit as $M \to \infty$.

For J_M in (21.23) we can argue just as for $K_{\tau,M}$ with $\tau = M$ since we have not used the lower bound on $|y|$ to justify the interior estimates in (21.26) when $\tau \geq M/4$. This leads to

$$\begin{aligned} MJ_M &\lesssim \int_{|x|\leq \frac{5M}{2}} U_M(x)\bigg(\fint\!\!\fint_{W(M,x)} |H-H_1|\bigg)\,\mathrm{d}x \\ &\lesssim M^\alpha \int_{|y|\leq 5M} \mathcal{M}(\Theta)(y)\bigg(\fint_{\tau/2}^{2\tau} |H-H_1|\,\mathrm{d}t\bigg)\mathrm{d}y \end{aligned} \tag{21.27}$$

and repeating the argument from (21.26) onward yields the same bound

$$MJ_M \lesssim (1+M^{\alpha-\gamma})\|\Theta\|_p.$$

As before, we conclude $J_M \to 0$ as $M \to \infty$.

At this point we have handled the terms in (21.6). The argument for the terms in (21.9) is *verbatim* the same. Indeed, all of our estimates concerning (21.6) have used reverse Hölder estimates on u and $H-H_1$ and the Caccioppoli inequality to replace $\nabla(H-H_1)$ by $\frac{H-H_1}{t}$ in the L^2-averages. Now, we simply use Caccioppoli inequalities to replace $t\nabla u$ by u and obtain the same bounds. The proof of Theorem 1.2 is complete.

21.3. **Uniqueness for $(D)^{\mathcal{L}}_p$ – conclusion of the proof of Theorem 1.1.** We shall implement again the formalism of Section 21.1. The interval of allowable exponents is $p_-(L) < p < p_+(L)^*$ if $p_-(L) \geq 1$ and $1 \leq p < p_+(L)^*$ if $p_-(L) < 1$. Hence, we assume $\widetilde{N}_*(u) \in \mathrm{L}^p$ and that (21.1) holds. We distinguish three cases:

- $(p_-(L) \vee 1) < p < p_+(L)$,
- $p = 1$ if $p_-(L) < 1$,
- $p_+(L) \leq p < p_+(L)^*$.

Case 1: $(p_-(L) \vee 1) < p < p_+(L)$. This is the range of exponents for the generic argument under the assumptions (21.1) and (21.20). In our concrete setting the latter holds with $\alpha = 0$ and $\Theta = \widetilde{N}_*(u)$ and there is nothing more to do.

Case 2: $p_-(L) < 1 = p$. We basically follow the generic argument in Case 2 for the regularity problem with $\alpha = 0$ and $\Theta = \widetilde{N}_*(u) \in \mathrm{L}^1$. In addition, we incorporate the following estimate for $H - H_1$ that comes from Lemma 20.4 in the case $r = \infty$: for some $\gamma > 0$ and all $t > 0$,

$$\|H(t,\cdot) - H_1(t,\cdot)\|_\infty \lesssim t \wedge t^{-\gamma}. \tag{21.28}$$

This uniform bound will allow us to avoid the maximal operator.

The term $J_{\varepsilon,M}$. By (21.21) we have

$$J_{\varepsilon,M} \lesssim \int_{|x|\leq 3M} U_\varepsilon(x)\bigg(\fint\!\!\fint_{W(\varepsilon,x)} \bigg|\frac{H-H_1}{\varepsilon}\bigg|\,\mathrm{d}t\mathrm{d}y\bigg)\,\mathrm{d}x$$

and thanks to (21.28) we get for $\varepsilon \le 1$

$$J_{\varepsilon,M} \lesssim \int_{|x|\le 3M} U_\varepsilon(x)\,\mathrm{d}x.$$

The assumption (21.1) together with the pointwise bound $U_\varepsilon \le \widetilde{N}_*(u)$ and the dominated convergence theorem yield again $J_{\varepsilon,M} \to 0$ as $\varepsilon \to 0$.

The terms I_M and J_M. We have to estimate I_M and J_M in (21.23). Once again, we intervene before introducing the maximal operator (21.26) and simply use (21.28). In this way we obtain

$$K_{\tau,M} \lesssim (\tau \wedge \tau^{-\gamma})\|\widetilde{N}_*(u)\|_1.$$

Since the right-hand side is summable for $\tau = \tau_j$, $j \in \mathbb{Z}$, we conclude $MI_M \lesssim \|\widetilde{N}_*(u)\|_1$. Thus, we have $I_M \to 0$ in the limit as $M \to \infty$. For MJ_M we obtain the same type of bound by arguing as for $K_{\tau,M}$ with $\tau = M$.

At this point we have handled the terms in (21.6) and the argument at the end of Case 2 for the regularity problem explains why our proof automatically covers the terms in (21.9).

Case 3: $p_+(L) \le p < p_+(L)^*$. We fine-tune the strategy in Case 2 for the regularity problem. Once again, working under the general assumptions (21.1) and (21.20) does not pose any additional difficulty. However, the range of admissible exponents now changes with the parameter α in (21.20) and we need to assume

$$0 < \frac{1}{p_+(L)} - \frac{1}{p} < \frac{1-\alpha}{n}. \tag{21.29}$$

For the Dirichlet problem $(D)^{\mathcal{L}}_p$ we have $\Theta = \widetilde{N}_*(u)$ and $\alpha = 0$, so that this is the range that we are aiming at.

The term $J_{\varepsilon,M}$. By (21.21) we have

$$J_{\varepsilon,M} = \int_{|x|\le 3M} U_\varepsilon(x)\Big(\fint\!\!\fint_{W(\varepsilon,x)} \Big|\frac{H-H_1}{\varepsilon}\Big|\,\mathrm{d}t\mathrm{d}y\Big)\,\mathrm{d}x.$$

We introduce the maximal function $\mathcal{M}^\varepsilon$ restricted to balls with radii not exceeding ε and use the Cauchy–Schwarz inequality to give

$$\begin{aligned}
J_{\varepsilon,M} &\lesssim \Big(\int_{|y|\le 4M} \mathcal{M}^\varepsilon(U_\varepsilon)^2\,\mathrm{d}y\Big)^{\frac12}\Big(\int_{\mathbb{R}^n}\Big(\fint_{\varepsilon/2}^{2\varepsilon}\Big|\frac{H-H_1}{t}\Big|\,\mathrm{d}t\Big)^2\mathrm{d}y\Big)^{\frac12}\\
&\lesssim \Big(\int_{|y|\le 4M} \mathcal{M}^\varepsilon(U_\varepsilon)^2\,\mathrm{d}y\Big)^{\frac12}\Big(\int_{\mathbb{R}^n}\fint_{\varepsilon/2}^{2\varepsilon}\Big|\frac{H-H_1}{t}\Big|^2\mathrm{d}t\mathrm{d}y\Big)^{\frac12}\\
&\lesssim \Big(\int_{|y|\le 5M} U_\varepsilon^2\,\mathrm{d}y\Big)^{\frac12}\Big(\fint_{\varepsilon/2}^{2\varepsilon}\int_{\mathbb{R}^n}\Big|\frac{H-H_1}{t}\Big|^2\mathrm{d}y\mathrm{d}t\Big)^{\frac12},
\end{aligned}$$

where we have used $\mathcal{M}^\varepsilon(U_\varepsilon) \le \mathcal{M}(\mathbf{1}_{B(0,5M)}U_\varepsilon)$ on $B(0,4M)$ and the maximal theorem in the last line. The second integral on the right is uniformly bounded in $\varepsilon \le 1$ by Lemma 20.4 applied with $r = 2$ and we are left with

$$J_{\varepsilon,M} \lesssim \bigg(\int_{|y|\le 5M} U_\varepsilon^2 \, \mathrm{d}y \bigg)^{\frac{1}{2}} \qquad (\varepsilon \le 1), \tag{21.30}$$

so far under the mere assumption that u is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$. Hölder's inequality yields

$$J_{\varepsilon,M} \lesssim M^{\frac{n}{2}-\frac{n}{p}} \bigg(\int_{\mathbb{R}^n} U_\varepsilon^p \, \mathrm{d}y \bigg)^{\frac{1}{p}},$$

which goes to 0 as $\varepsilon \to 0$, using (21.1), the pointwise bound $U_\varepsilon \le \Theta$ and the dominated convergence theorem.

The terms I_M and J_M. We are left with treating the terms I_M and J_M in (21.23). To this end, we recall the generic decomposition from (21.24) and (21.26):

$$MI_M \le \sum_{j=-\infty}^{j_M} K_{\tau_j,M}$$

where $\tau_j = (9/4)^j$, j_M is the unique integer with $\tau_{j_M-1} \le M < \tau_{j_M}$ and for $\tau \le 9M/4$,

$$K_{\tau,M} \lesssim \tau^\alpha \int_{|y|\le 5M} \mathcal{M}(\Theta)(y) \bigg(\fint_{\tau/2}^{2\tau} |H - H_1| \, \mathrm{d}t \bigg) \mathrm{d}y.$$

With a choice of $(p_-(L) \vee 1) < r < p_+(L)$ that will specified later on, we obtain

$$\begin{aligned}
K_{\tau,M} &\lesssim \tau^\alpha \bigg(\int_{\mathbb{R}^n} \mathcal{M}(\Theta)(y)^p \, \mathrm{d}y \bigg)^{\frac{1}{p}} \bigg(\int_{|y|\le 5M} \bigg| \fint_{\tau/2}^{2\tau} |H - H_1| \, \mathrm{d}t \bigg|^{p'} \mathrm{d}y \bigg)^{\frac{1}{p'}} \\
&\lesssim \tau^\alpha M^{\frac{n}{p'}-\frac{n}{r'}} \bigg(\int_{\mathbb{R}^n} \Theta(y)^p \, \mathrm{d}y \bigg)^{\frac{1}{p}} \bigg(\int_{|y|\le 5M} \bigg| \fint_{\tau/2}^{2\tau} |H - H_1| \, \mathrm{d}t \bigg|^{r'} \mathrm{d}y \bigg)^{\frac{1}{r'}} \\
&\le \tau^\alpha M^{\frac{n}{p'}-\frac{n}{r'}} \|\Theta\|_p \bigg(\fint_{\tau/2}^{2\tau} \int_{\mathbb{R}^n} |H - H_1|^{r'} \, \mathrm{d}y\mathrm{d}t \bigg)^{\frac{1}{r'}} \\
&\lesssim \tau^\alpha (\tau \wedge \tau^{-\gamma}) M^{\frac{n}{p'}-\frac{n}{r'}} \|\Theta\|_p \\
&= \tau^\alpha (\tau \wedge \tau^{-\gamma}) M^{\frac{n}{r}-\frac{n}{p}} \|\Theta\|_p.
\end{aligned}$$

We have used Hölder's inequality in the first line, the maximal theorem and again Hölder's inequality in the second one, Jensen's inequality in the third one and Lemma 20.4 with exponent r' in the fourth one. The exponent γ is positive and depends on r. Summing up the estimates for $\tau = \tau_j$ yields

$$MI_M \lesssim M^{\frac{n}{r}-\frac{n}{p}}(1+M^{\alpha-\gamma})\|\Theta\|_p.$$

The assumption (21.29) guarantees that we can pick r such that ${}^1\!/_r - {}^1\!/_p < {}^{(1-\alpha)}\!/_n$. In this case M appears with exponent smaller than 1 on the right-hand side and we conclude $I_M \to 0$ in the limit as $M \to \infty$.

For J_M we have the bound

$$MJ_M \lesssim M^\alpha \int_{|y|\leq 5M} \mathcal{M}(\Theta)(y)\Big(\fint_{\tau/2}^{2\tau} |H - H_1| \,\mathrm{d}t\Big)\mathrm{d}y$$

see (21.27). The steps above with $\tau = M$ yield the same bound

$$MJ_M \lesssim M^{\frac{n}{r}-\frac{n}{p}}(1+M^{\alpha-\gamma})\|\Theta\|_p$$

from which we conclude $J_M \to 0$ as $M \to \infty$.

We have handled the terms in (21.6) and once again the discussion at the the end of Case 2 for the regularity problem explains why our proof automatically covers the terms in (21.9). This completes the proof of Theorem 1.1.

21.4. **Uniqueness for $(\widetilde{D})^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$.** We turn to the situation when $p_+(L) > n$ and prove that solutions to $(\widetilde{D})^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ are unique in the range of exponents $0 \leq \alpha < 1 - {}^n\!/_{p_+(L)}$. Hence, we assume (21.1) and $\widetilde{N}_{\sharp,\alpha}(u) \in \mathrm{L}^\infty$. The control of the sharp functional means that we have

$$U_t(x) \leq t^\alpha \widetilde{N}_{\sharp,\alpha}(u)(x) \quad ((t,x) \in \mathbb{R}^{1+n}_+), \tag{21.31}$$

which is an assumption of the same type as (21.20) but for $p = \infty$. Fortunately, this only requires a slight modification of the generic argument in the previous section.

The term $J_{\varepsilon,M}$. According to (21.30) we have

$$J_{\varepsilon,M} \lesssim \Big(\int_{|x|\leq 5M} U_\varepsilon(x)^2 \,\mathrm{d}y\Big)^{\frac{1}{2}}$$

and (21.31) still allows us to use the dominated convergence theorem when passing to the limit as $\varepsilon \to 0$. Hence, $J_{\varepsilon,M} \to 0$ follows.

The terms I_M and J_M. For the terms in (21.23) we start out with the usual decomposition from (21.24) and the estimate before (21.26):

$$MI_M \leq \sum_{j=-\infty}^{j_M} K_{\tau_j,M}$$

where $\tau_j = (9/4)^j$, j_M is the unique integer with $\tau_{j_M - 1} \le M < \tau_{j_M}$ and for $\tau \le 9M/4$,

$$K_{\tau,M} \lesssim \int_{\frac{M}{2} \le |x| \le \frac{5M}{2}} U_\tau(x) \bigg(\fint\!\!\fint_{W(\tau,x)} |H - H_1| \bigg) \, \mathrm{d}x. \tag{21.32}$$

We use (21.31), Hölder's inequality and Lemma 20.4 with an exponent $(p_-(L^\sharp) \vee 1) < r' < p_+(L^\sharp)$ to be specified yet in order to give

$$\begin{aligned} K_{\tau,M} &\lesssim \tau^\alpha \|\widetilde{N}_{\sharp,\alpha}(u)\|_\infty \int_{|y| \le 5M} \bigg(\fint_{\tau/2}^{2\tau} |H - H_1| \, \mathrm{d}t \bigg) \mathrm{d}y \\ &\lesssim \tau^\alpha \|\widetilde{N}_{\sharp,\alpha}(u)\|_\infty M^{\frac{n}{r}} \fint_{\tau/2}^{2\tau} \bigg(\int_{\mathbb{R}^n} |H - H_1|^{r'} \, \mathrm{d}y \bigg)^{\frac{1}{r'}} \mathrm{d}t \\ &\lesssim \tau^\alpha (\tau \wedge \tau^{-\gamma}) M^{\frac{n}{r}} \|\widetilde{N}_{\sharp,\alpha}(u)\|_\infty, \end{aligned}$$

where $\gamma > 0$ depends on r. Summing up the estimates for $\tau = \tau_j$ leads to

$$M I_M \lesssim M^{\frac{n}{r}} (1 + M^\alpha) \|\widetilde{N}_{\sharp,\alpha}(u)\|_\infty.$$

By assumption on p we can pick $(p_-(L) \vee 1) < r < p_+(L)$ such that $\alpha < 1 - n/r$. Then the exponent for M on the right-hand side becomes smaller than 1 and $I_M \to 0$ in the limit as $M \to \infty$ follows.

For J_M we recall from (21.27) the bound

$$M J_M \lesssim \int_{|x| \le \frac{5M}{2}} U_M(x) \bigg(\fint\!\!\fint_{W(M,x)} |H - H_1| \bigg) \, \mathrm{d}x$$

and the previous argument for $\tau = M$ yields $J_M \to 0$ in the limit as $M \to \infty$.

At this point we have handled the terms in (21.6) and as in the earlier steps the limits for the terms in (21.9) come for free.

21.5. **Uniqueness for $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ – conclusion of the proof of Theorem 1.3.** We turn to uniqueness of solutions to the Dirichlet problem $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$ with interior Carleson control. We work under the same assumptions $p_+(L) > n$ and $0 \le \alpha < 1 - n/p_+(L)$ as in the previous section.

The case $\alpha > 0$ is particularly simple. We merely need the following general lemma to compare several functionals that all measure smoothness of order $\alpha - 1$.

Lemma 21.4. *Let $\alpha \in (0,1)$. There is dimensional constant ω_n such that for all $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$,*

$$\|W(t^{1-\alpha} \nabla u)\|_\infty \le \omega_n 2^\alpha \|C_\alpha(t \nabla u)\|_\infty \le \omega_n 2^\alpha \|C_0(t^{1-\alpha} \nabla u)\|_\infty.$$

Moreover, if (21.1) holds, then

$$\|\widetilde{N}_{\sharp,\alpha}(u)\|_\infty \lesssim \|W(t^{1-\alpha} \nabla u)\|_\infty.$$

Proof. For the first claim we simply note that for any $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$,

$$\Big(\iint_{W(t,x)} |s^{1-\alpha} F(s,y)|^2 \,\mathrm{d}s\mathrm{d}y\Big)^{\frac{1}{2}} \leq \frac{\omega_n 2^\alpha}{t^\alpha} \Big(\int_0^{2t} \fint_{B(x,2t)} |sF(s,y)|^2 \,\frac{\mathrm{d}y\mathrm{d}s}{s}\Big)^{\frac{1}{2}} \leq \omega_n 2^\alpha \Big(\int_0^{2t} \fint_{B(x,2t)} |s^{1-\alpha}F(s,y)|^2 \,\frac{\mathrm{d}y\mathrm{d}s}{s}\Big)^{\frac{1}{2}}$$

and taking suprema in t and x on both sides yields the claim. Under assumption (21.1) the second claim follows from the trace theorem in Proposition A.8.(iii). □

To prove uniqueness of solutions to $(D)^{\mathcal{L}}_{\dot{\Lambda}^\alpha}$, we assume $C_\alpha(t\nabla u) \in \mathrm{L}^\infty$ and that (21.1) holds. Lemma 21.4 yields $\widetilde{N}_{\sharp,\alpha}(u) \in \mathrm{L}^\infty$ and under this weaker assumption we have already shown $u = 0$ in the previous section.

It remains to treat the BMO Dirichlet problem $(D)^{\mathcal{L}}_{\dot{\Lambda}^0}$. We assume therefore $C_0(t\nabla u) \in \mathrm{L}^\infty$ and for the first time (21.2). We implement the strategy of Section 21.1 with $R = M$ and first send $\varepsilon \to 0$ and then $M \to \infty$ in (21.6) and (21.9).

The terms $J_{\varepsilon,M}$ and $\widetilde{J}_{\varepsilon,M}$. The Cauchy–Schwarz inequality yields

(21.33)
$$J_{\varepsilon,M} \lesssim \Big(\int_{|x|\leq 2M} \fint_{2\varepsilon/3}^{3\varepsilon/2} |u|^2 \,\mathrm{d}t\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \fint_{2\varepsilon/3}^{3\varepsilon/2} |\partial_t(H-H_1)|^2 \,\mathrm{d}t\mathrm{d}x\Big)^{\frac{1}{2}}.$$

By covering $B(0,2M)$ up to a set of measure zero by pairwise disjoint cubes Q_k of sidelength ε with $2Q_k \subseteq B(0,2M+1)$ and using reverse Hölder inequalities for u, we obtain

$$\begin{aligned}\int_{B(0,2M)} \fint_{2\varepsilon/3}^{3\varepsilon/2} |u|^2 \,\mathrm{d}t\mathrm{d}x &\lesssim \sum_k |Q_k| \fint_{Q_k} \fint_{2\varepsilon/3}^{3\varepsilon/2} |u|^2 \,\mathrm{d}t\mathrm{d}x \\ &\lesssim \sum_k |Q_k| \Big(\fint_{2Q_k} \fint_{\varepsilon/2}^{2\varepsilon} |u| \,\mathrm{d}t\mathrm{d}x\Big)^2 \\ &\lesssim \sum_k |Q_k| \fint_{2Q_k} \Big(\fint_{\varepsilon/2}^{2\varepsilon} |u| \,\mathrm{d}t\Big)^2 \mathrm{d}x \\ &\leq \int_{B(0,2M+1)} \Big(\fint_{\varepsilon/2}^{2\varepsilon} |u| \,\mathrm{d}t\Big)^2 \mathrm{d}x.\end{aligned}$$

By assumption (21.2), this integral tends to 0 as $\varepsilon \to 0$. As for the term with $H - H_1$ in (21.33), we use Lemma 20.4 with $r = 2$ to deduce a uniform bound in $\varepsilon \in (0,1)$.

The estimate for $\widetilde{J}_{\varepsilon,M}$ is very similar. Indeed, $t\partial_t u$ is handled via the same argument and incorporating the Caccioppoli inequality, whereas for ${}^{(H-H_1)}/t$ we use Lemma 20.4 again.

The terms I_M, $\widetilde{I}_M$ and J_M, $\widetilde{J}_M$. We estimate the terms in (21.23). Only one change to the corresponding argument for $(\widetilde{D})^{\mathcal{L}}_{\Lambda^0}$ in Section 21.4 will be necessary. In particular, the estimates for the tilde terms that correspond to (21.9) come again for free.

The argument for I_M with $\alpha = 0$ in the previous section uses the interior control only once, namely to bound $U_\tau(x)$ in (21.32) uniformly by $\|\widetilde{N}_{\sharp,\alpha}(u)\|_\infty$. This bound is not available under our current assumption but the following lemma provides a substitute that still suits our purpose.

Lemma 21.5. *If $v \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ is such that $C_0(t\nabla v) \in \mathrm{L}^\infty(\mathbb{R}^n)$, then*

$$\fint\!\!\fint_{W(t,x)} |v| \,\mathrm{d}s\mathrm{d}y \lesssim 1 + |\ln(t)| + \ln(1+|x|) \quad ((t,x) \in \mathbb{R}^{1+n}_+).$$

We defer the proof and use Lemma 21.5 to bound $U_\tau(x)$ in (21.32). This yields an additional factor $(1 + |\ln(\tau)| + \ln(M))$ compared to the estimates in the previous section and hence we obtain

$$K_{\tau,M} \lesssim (1 + |\ln(\tau)| + \ln(M))M^{\frac{n}{r}}(\tau \wedge \tau^{-\gamma})$$

with $r > n$ and $\gamma > 0$. Summing up the estimates for $\tau = \tau_j$ yields

$$MI_M \leq \sum_{j=-\infty}^{j_M} K_{\tau_j,M} \lesssim \ln(M)M^{\frac{n}{r}},$$

which still implies that I_M tends to 0 as $M \to \infty$.

Likewise, using Lemma 21.5 to control $U_M(x)$ in (21.27) leads to $MJ_M \lesssim \ln(M)M^{\frac{n}{r}}$ and we conclude as before. The proof of Theorem 1.3 is complete modulo the

Proof of Lemma 21.5. Set $w := |v|$, which satisfies the same assumptions. Suppose that $W_j = W(t_j, x_j)$ and $W_k = W(t_k, x_k)$ are two Whitney regions with non-empty intersection and suppose that $t_j \leq t_k$. Then W_j and W_k are comparable in measure and the cylinder $W := ({}^{t_j}/2, 8t_j) \times B(x_k, 8t_j)$ contains both W_j and W_k. Hence, we can use Poincaré's inequality in order to give

$$\begin{aligned} |(w)_{W_j} - (w)_{W_k}| &\lesssim \fint\!\!\fint_W |w - (w)_{W_k}| \,\mathrm{d}t\mathrm{d}x \\ &\lesssim \fint\!\!\fint_W |t\nabla w| \,\mathrm{d}t\mathrm{d}x \\ &\lesssim \|C_0(t\nabla w)\|_\infty \end{aligned}$$

with a implicit constant that depends only on n. If $W_1, \dots, W_k$ is a chain of Whitney regions with the property that each region intersects its successor, then a telescopic sum yields

$$|(w)_{W_1} - (w)_{W_k}| \lesssim k \|C_0(t\nabla w)\|_\infty.$$

We write $W_1 \to W_k$ in that case.

Now, we fix $(t, x) \in \mathbb{R}^n$. Since w is locally integrable, it suffices to construct a chain $W(t, x) \to W(1, 0)$ of length controlled by $1 + |\ln(t)| + \ln(1 + |x|)$. One possible construction is as follows. Successively halving or doubling t, we obtain a chain $W(t, x) \to W(1, x)$ of length comparable to $1 + |\ln(t)|$. If $|x| < 1$, then $W(1, x)$ and $W(1, 0)$ intersect and we are done. If $|x| \geq 1$, then in the same manner we obtain chains $W(1, x) \to W(2|x|, x)$ and $W(1, 0) \to W(2|x|, 0)$ of length comparable to $\ln(1 + |x|)$. Moreover, $W(2|x|, x)$ and $W(2|x|, 0)$ intersect. $\square$

21.6. **Uniqueness for $(D)^{\mathcal{L}}_{\dot{\mathrm{X}}^{s,p}}$ – conclusion of the proof of Theorem 1.4.** The last uniqueness result concerns the problems $(D)_{\dot{\mathrm{X}}^{s,p}}$ with fractional regularity data. As usual, X denotes B or H and Y is the corresponding solution space of type Z or T. Figure 17 and Figure 18 show the regions of exponents that we are aiming at in an $(1/p, s)$-plane. In the previous sections we have already obtained uniqueness on the bottom and top segments.

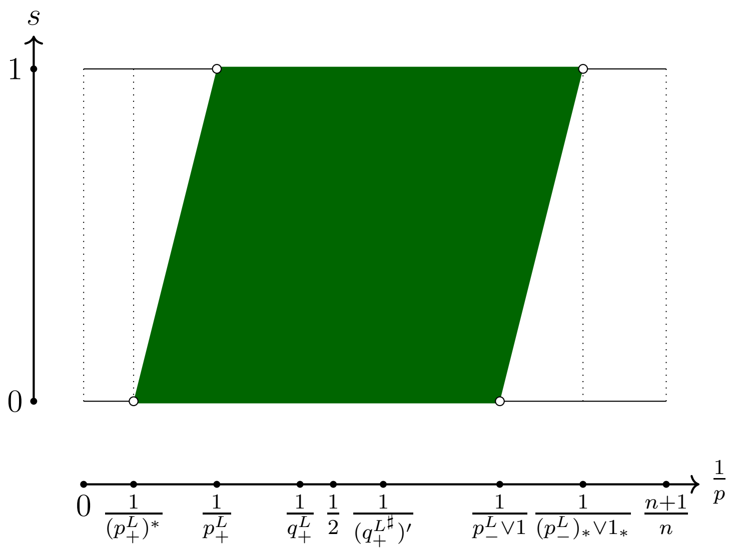

FIGURE 17. Exponents for uniqueness in Dirichlet and regularity problems in the case $p_+(L) \leq n$. Uniqueness holds on the open bottom and top segments (Sections 21.3 and 21.2) and the interior of the trapezoidal region (Section 21.6).

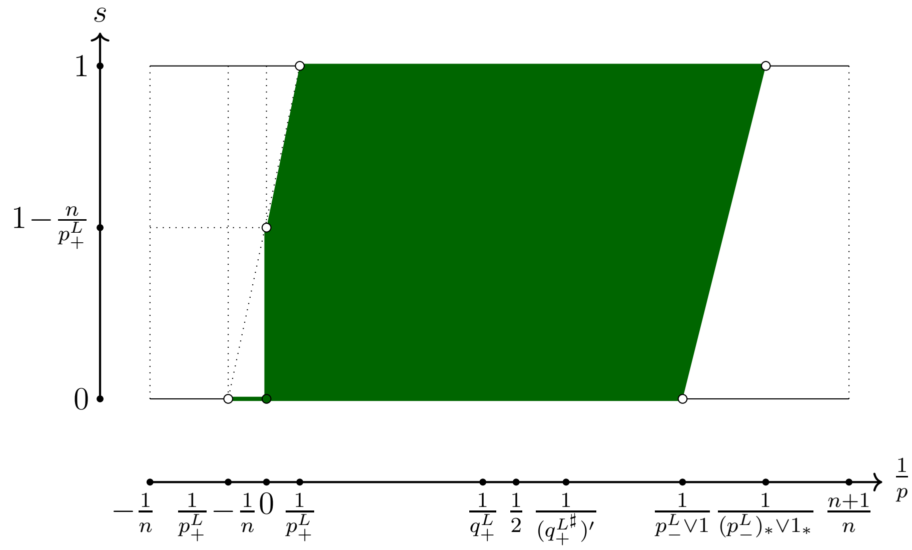

FIGURE 18. Exponents for uniqueness in Dirichlet and regularity problems in the case $p_+(L) > n$. Uniqueness holds on the open bottom and top segments (Sections 21.3, 21.2 and 21.5), the open vertical segment at $1/p = 0$ and the interior of the trapezoidal region (Section 21.6). Exponents with $1/p < 0$ correspond to the spaces $\dot{\Lambda}^\alpha$ with $\alpha = 1 - n/p$ as usual.

We distinguish four cases.

- The rectangle

$$(p_-(L) \vee 1) < p < p_+(L) \quad \& \quad s \in (0,1),$$

- the left-hand triangle ($p_+(L) \leq n$) or trapezoid ($p_+(L) > n$)

$$p_+(L) \leq p < p_+(L)^* \quad \& \quad \frac{1}{p_+(L)} - \frac{1}{p} < \frac{1-s}{n},$$

- $p_+(L) > n$ and the vertical segment

$$p = \infty \quad \& \quad 0 < s < 1 - \frac{n}{p_+(L)},$$

- the right-hand triangle

$$(p_-(L)_* \vee 1_*) < p \leq (p_-(L) \vee 1) \quad \& \quad \frac{1}{p} - \frac{1}{p_-(L) \vee 1} < \frac{s}{n}.$$

In any case we assume (21.1) and $\|\nabla u\|_{\mathrm{Y}^{s-1,p}} < \infty$, which by definition of tent and Z-spaces corresponds to one of the interior controls in Lemma 21.1.

Case 1: The rectangle. According to the trace theorem from Proposition A.8, there exists a function $\Theta \in \mathrm{L}^p$ such that

$$U_t(x) \leq t^s \Theta(x) \quad ((t,x) \in \mathbb{R}^{1+n}_+).$$

Hence, (21.20) holds with $\alpha = s$ and the general result from Case 2 for the regularity problem applies directly.

Case 2: The left-hand triangle or trapezoid. Since we still work with finite exponents, assumption (21.20) holds as in Case 1 with exponent $\alpha = s$. Thus, we can apply the general result from Case 3 for the Dirichlet problem provided that the exponents satisfy the respective assumption (21.29). But this is exactly the restriction that defines this region.

Case 3: $p_+(L) > n$ and the vertical segment. Let $0 < \alpha < 1 - {}^n/_{p_+(L)}$. We assume one of $C_0(t^{1-\alpha}\nabla u) \in \mathrm{L}^\infty$ or $W(t^{1-\alpha}\nabla u) \in \mathrm{L}^\infty$ and in any case that (21.1) holds at the boundary. Lemma 21.4 yields $\widetilde{N}_{\sharp,\alpha}(u) \in \mathrm{L}^\infty$ and under this weaker assumption we have already shown $u = 0$ in Section 21.4.

Case 4: The right-hand triangle. The argument in Case 1 for the regularity problem implicitly contains a more general result that applies here. In view of the technicalities concerning the choice of exponents in that argument we have decided to stick with the version at regularity $s = 1$ earlier on and here we provide the required generalization.

We begin with the substitute for Lemma 21.3.

Lemma 21.6. *If $s \in (0,1)$ and $0 < p < r \leq 2$, then for any weak solution u to $\mathcal{L}u = 0$ on $\mathbb{R}^{1+n}_+$,*

$$\left(\iint_{\mathbb{R}^{1+n}_+} |\nabla u|^r t^{n(\frac{r}{p}-1)+(1-s)r} \, \frac{\mathrm{d}t\mathrm{d}x}{t} \right)^{\frac{1}{r}} \lesssim \|\nabla u\|_{\mathrm{Y}^{s-1,p}}. \tag{21.34}$$

Moreover, if (21.1) holds, then

$$\left(\iint_{\mathbb{R}^{1+n}_+} |u|^r t^{n(\frac{r}{p}-1)-sr} \, \frac{\mathrm{d}t\mathrm{d}x}{t} \right)^{\frac{1}{r}} \lesssim \|\nabla u\|_{\mathrm{Y}^{s-1,p}}. \tag{21.35}$$

Proof. Since $p < r$, we can use the mixed embedding for tent and Z-spaces from [3, Thm. 2.34] to the effect that $\mathrm{Y}^{s,p} \subseteq \mathrm{Z}^{\alpha,r}$ if $\alpha - s = n({}^1/_r - {}^1/_p)$. This means that

$$\left(\iint_{\mathbb{R}^{1+n}_+} W(t^{-\alpha}F)(\tau,y)^r \, \frac{\mathrm{d}\tau\mathrm{d}y}{\tau} \right)^{\frac{1}{r}} \lesssim \|F\|_{\mathrm{Y}^{s,p}}.$$

As $r \leq 2$, Hölder's inequality implies

$$⨌_{W(\tau,y)} |t^{-\alpha}F(t,x)|^r \, \mathrm{d}t\mathrm{d}x \leq W(t^{-\alpha}F)(\tau,y)^r \tag{21.36}$$

and applying the averaging trick backwards yields

$$\left(\iint_{\mathbb{R}^{1+n}_+} |t^{-\alpha} F(t,x)|^r \, \frac{\mathrm{d}t\mathrm{d}x}{t} \right)^{\frac{1}{r}} \lesssim \|F\|_{\mathrm{Y}^{s,p}}.$$

If $F := |t\nabla u|$, then $\|F\|_{\mathrm{Y}^{s,p}} = \|\nabla u\|_{\mathrm{Y}^{s-1,p}}$ and sorting out the exponent for t on the left-hand side yields (21.34).

Again since $r \leq 2$ we can use part (i) of the trace theorem in Proposition A.8 for $\nabla u \in \mathrm{Z}^{\alpha-1,r}$ with the same exponent r. Owing to (21.1), we obtain

$$⨌_{W(\tau,y)} |t^{-\alpha} u(t,x)|^r \, \mathrm{d}t\mathrm{d}x \lesssim \Theta(y)^r$$

for some function Θ with $\|\Theta\|_r \lesssim \|\nabla u\|_{\mathrm{Z}^{\alpha-1,r}}$. Integrating in y and applying the averaging trick backwards yields

$$\left(\iint_{\mathbb{R}^{1+n}_+} |t^{-\alpha} u(t,x)|^r \, \frac{\mathrm{d}t\mathrm{d}x}{t} \right)^{\frac{1}{r}} \lesssim \|\nabla u\|_{\mathrm{Z}^{\alpha-1,r}} \lesssim \|\nabla u\|_{\mathrm{Y}^{s-1,p}}$$

and as before $-\alpha r$ reveals itself as the same exponent than in the claim. $\square$

Lemma 21.6 allows us to control ∇u and u in certain Lebesgue norms exactly as it was the case with Lemma 21.3, except that we have different powers of t to compensate: $t^{n(\frac{r}{p}-1)+(1-s)r}$ and $t^{n(\frac{r}{p}-1)-sr}$ replace $t^{n(\frac{r}{p}-1)}$ and $t^{n(\frac{r}{p}-1)-r}$, respectively, that is to say, we have an additional power $t^{(1-s)r}$. Armed with this observation, we pick again $(1 \vee p_-(L)) < r \leq 2$ and follow the proof in Case 1 of Section 21.2 *verbatim*. We only have to check that the additional power of t still allows us to pass to the limits.

As for $I_{M,\varepsilon,R}$ and $\widetilde{I}_{M,\varepsilon,R}$, the different power of t only changes the implicit constant that depends on ε, R. Hence, these terms vanish when sending $M \to \infty$ as before.

The estimates for $J_{\varepsilon,M}$ and $\widetilde{J}_{\varepsilon,M}$ are more delicate since now we obtain $\varepsilon^{s-n(\frac{1}{p}-\frac{1}{r})}$ as factor in (21.15) and (21.17) if we want to control the respective integral on the right via Lemma 21.6. We need to pick an admissible r such that the exponent is non-negative.

In dimension $n \geq 2$ we pick $r := \frac{np}{n-ps}$ since then the exponent of ε vanishes. In particular, using also the restriction on p, we have

$$\frac{1}{p} - \frac{1}{r} = \frac{s}{n} > \frac{1}{p} - \frac{1}{p_-(L) \vee 1},$$

which in turn implies that $r > (p_-(L) \vee 1)$. On the other hand, $s \leq 1$ implies $r \leq p^*$ and at the same time we have $p \leq (p_-(L) \vee 1) \leq 2_*$ by Proposition 6.7. Thus, $r \leq 2$ and we conclude that r is admissible.

In dimension $n = 1$, Proposition 6.7 yields $p_-(L) = {}^1\!/_2 = 1_*$. Hence, our assumption on p is ${}^1\!/_{(s+1)} < p \leq 1$ and this allows us to pick

$r > 1$ sufficiently close to 1 such that ${}^1\!/\!_r \geq {}^1\!/\!_p - s$. Consequently, the exponent for ε in (21.15) is non-negative and we conclude as before.

Likewise, we obtain for $J_{R,M}$ and $\widetilde{J}_{R,M}$ the new factor $R^{(s-1)-\gamma-n(\frac{1}{p}-\frac{1}{r})}$ in (21.16) and (21.18). The exponent is negative since we have $s < 1$, $\gamma > 0$ and $r > p$. This completes the proof.

22. The Neumann problem

We begin by recalling the construction of energy solutions to the Neumann problem. We use again the energy space $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ from Section 16.1.

If $u \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$, then there exists a unique element $\partial_{\nu_A} u(0,\cdot) \in \dot{\mathrm{H}}^{-1/2,2}$ such that

$$\iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi} \, \mathrm{d}t\mathrm{d}x = -\langle \partial_{\nu_A} u(0,\cdot), \phi(0,\cdot)\rangle \quad (\phi \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)),$$

where on the right-hand side we use the duality pairing between $\dot{\mathrm{H}}^{-1/2,2}$ and $\dot{\mathrm{H}}^{1/2,2}$. Indeed, by assumption on u and Lemma 16.2, the left-hand side is a bounded anti-linear functional on $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ that vanishes whenever $\phi(0,\cdot) = 0$ and therefore it defines a bounded anti-linear functional on the trace space $\dot{\mathrm{H}}^{1/2,2}$. We call $\partial_{\nu_A} u(0,\cdot)$ the (inward pointing) *conormal derivative* of u at the boundary.

Proposition 22.1. *For all $f \in \dot{\mathrm{H}}^{-1/2,2}$ there exists a unique solution u (modulo constants) to the problem*

$$\begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \nabla u \in \mathrm{L}^2(\mathbb{R}^{1+n}_+), \\ \partial_{\nu_A} u(0,\cdot) = f & (\text{in } \dot{\mathrm{H}}^{-1/2,2}). \end{cases}$$

Moreover, $\|\nabla u\|_2 \lesssim \|f\|_{\dot{\mathrm{H}}^{-1/2,2}}$ and $\lim_{t\to\infty} u(t,\cdot) = 0$ in $\dot{\mathrm{H}}^{1/2,2}$.

Proof. This is just the Lax–Milgram lemma applied in $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$. The limit at $t = \infty$ follows from Lemma 16.1. □

In the situation above we call u the *energy solution* to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ with Neumann data f. Much alike to Section 16.1 the energy solution coincides with the Poisson semigroup extension for suitable data. Throughout, we use the (extension to an) isomorphism $L^{1/2}$: $\dot{\mathrm{W}}^{1,2} \to \mathrm{L}^2$ with inverse $L^{-1/2}$. By duality and similarity we also obtain an (extension to an) isomorphism $aL^{1/2} : \mathrm{L}^2 \to \dot{\mathrm{W}}^{-1,2}$.

Proposition 22.2. *If $f \in \mathrm{L}^2 \cap \dot{\mathrm{W}}^{-1,2}$, then the energy solution with Neumann data f is given by $u(t,x) = -\mathrm{e}^{-tL^{1/2}}(aL^{1/2})^{-1} f(x)$.*

Proof. Set $g \coloneqq -(aL^{1/2})^{-1} f$. Then $g \in \dot{\mathrm{W}}^{1,2} \cap \mathrm{L}^2$ and, by interpolation, $g \in \dot{\mathrm{H}}^{1/2,2}$. It follows from Proposition 16.5 that $u(t,x) \coloneqq \mathrm{e}^{-tL^{1/2}} g(x)$ is an energy solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$. In order to determine its

Neumann datum, we let $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^{1+n})$. By the functional calculus on L^2 we have $au \in \mathrm{C}^1([0,\infty);\mathrm{L}^2)$ with $a\partial_t u(0,\cdot) = f$. Hence, we can integrate by parts in t and use the definition of L to give

$$\iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi} \,\mathrm{d}t\mathrm{d}x = - \int_{\mathbb{R}^n} f \cdot \overline{\phi(0,\cdot)} \,\mathrm{d}x.$$

The L^2-pairing on the right-hand side can also be viewed as the $\dot{\mathrm{H}}^{-1/2,2}$–$\dot{\mathrm{H}}^{1/2,2}$-duality. Then the identity can be extended to all $\phi \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$ and we conclude $\partial_{\nu_A} u(0,\cdot) = f$. □

The semigroup construction provides solutions to the Neumann problem $(N)_p^{\mathcal{L}}$ in an appropriate range of exponents.

Proposition 22.3. *Let $q_-(L) < p < q_+(L)$. If $f \in \mathrm{H}^p \cap \mathrm{L}^2 \cap \dot{\mathrm{W}}^{-1,2}$, then the energy solution u with Neumann data f satisfies*

$$\|\widetilde{N}_*(\nabla u)\|_p \simeq \|f\|_p.$$

Proof. We have $q_-(L) = p_-(L)$ and $q_+(L) < p_+(L)$, see Theorem 6.2. Letting $g := -(aL^{1/2})^{-1} f \in \mathrm{W}^{1,2}$ as before, we obtain from Proposition 17.7 and Theorem 11.1 that

$$\|\widetilde{N}_*(\nabla u)\|_p \simeq \|\nabla_x g\|_{\mathrm{H}^p} \simeq \|aL^{1/2} g\|_{\mathrm{H}^p} = \|f\|_{\mathrm{H}^p}. \qquad \square$$

Proof of Theorem 1.5. Let $q_-(L) < p < q_+(L)$. According to Corollary 15.2 this range is the same as what is called I_L in [19]. We have seen in the introduction (Section 1.7) that it suffices to prove the bound $\|\widetilde{N}_*(\nabla u)\|_p \lesssim \|f\|_{\mathrm{H}^p}$, whenever $f \in \mathrm{H}^p \cap \dot{\mathrm{H}}^{-1/2,2}$ and u is the energy solution with Neumann data f.

By the universal approximation technique we an pick for any given f a sequence $(f_k) \subseteq \mathrm{H}^p \cap \mathrm{L}^2 \cap \dot{\mathrm{W}}^{-1,2}$ with $f_k \to f$ as $k \to \infty$ in both H^p and $\dot{\mathrm{H}}^{-1/2,2}$. It follows from Proposition 22.1 that the corresponding energy solutions u_k tend to u in $\dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+n}_+)$, whereas Proposition 22.3 implies that $(\nabla u_k)_k$ is a Cauchy sequence in $\mathrm{T}^{0,p}_\infty$. The limits can be identified in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ and the conclusion follows. □

Let us conclude with an additional uniqueness result for the Neumann problem. We remark that in our formulation of the Neumann problem the convergence of the conormal derivative to its trace is in the sense of distributions. By [19, Cor. 1.2], the Whitney averages convergence

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} a\partial_t u \,\mathrm{d}s\mathrm{d}y = g(x) \quad (\text{a.e. } x \in \mathbb{R}^n)$$

of the conormal derivative of the unique solution to its trace comes as a bonus if $p \geq 1$ with $q_-(L) < p < q_+(L)$. In the case of block systems, one can reverse these interpretations of the boundary behavior and still obtain uniqueness, hence compatible well-posedness.

Theorem 22.4. *If $p \geq 1$ with $q_-(L) < p < q_+(L)$, then the following Neumann problem with non-tangential boundary trace is compatibly well-posed (modulo constants). Given $g \in \mathrm{H}^p(\mathbb{R}^n; \mathbb{C}^m)$, solve*

$$(\widetilde{N})_p^{\mathcal{L}} \qquad \begin{cases} \mathcal{L}u = 0 & (\text{in } \mathbb{R}^{1+n}_+), \\ \widetilde{N}_*(\nabla u) \in \mathrm{L}^p(\mathbb{R}^n), \\ \lim_{t\to 0} \fint\!\!\fint_{W(t,x)} a\partial_t u \, \mathrm{d}s\mathrm{d}y = g(x) & (a.e.\ x \in \mathbb{R}^n). \end{cases}$$

Proof. In view of the preceding discussion we only need to establish uniqueness.

According to [19, Thm. 1.1] and our identification of I_L, the condition $\widetilde{N}_*(\nabla u) \in \mathrm{L}^p(\mathbb{R}^n)$ implies (is equivalent to, in fact) the representation of the conormal gradient of u via the $[DB]$-semigroup:

$$\nabla_A u(t, \cdot) = \mathrm{e}^{-t[DB]} \nabla_A u\,|_{t=0} \qquad (t > 0),$$

where $F_0 \coloneqq \nabla_A u\,|_{t=0} \in \mathrm{H}^p_D$ is characterized by $\mathbf{1}_{\mathbb{C}^+}(DB)F_0 = F_0$ and the functional calculus is extended from $\mathbb{H}^p_D$ to its completion H^p_D for the H^p-norm as $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$. It follows from (20.3) that $F_0 = [g, -\nabla_x L^{-1/2}(a^{-1}g)]^\top$ for some $g \in \mathrm{H}^p$. Assume now that the Whitney averages of $a\partial_t u$ converge to 0 almost everywhere at the boundary. By [19, Cor. 1.2], we know that this limit agrees with g almost everywhere. Thus, $g = 0$. We conclude that F_0 vanishes identically and it follows that u is constant in $\mathbb{R}^{1+n}_+$. □

Appendix A. Non-tangential maximal functions and traces

In this section we collect some technical results involving non-tangential maximal functions with a focus on non-tangential trace theorems.

Throughout, we consider the Whitney parameters $c_0 > 1$ and $c_1 > 0$ fixed, write $W(t,x) \coloneqq (c_0^{-1}t, c_0 t) \times B(x, c_1 t)$ for $(t,x) \in \mathbb{R}^{1+n}_+$ and for $q > 0$ we use the q-adapted non-tangential maximal functions

$$\widetilde{N}_{*,q}(F)(x) \coloneqq \sup_{t>0} \left(\fint\!\!\fint_{W(t,x)} |F(s,y)|^q \, \mathrm{d}s\mathrm{d}y \right)^{1/q} \qquad (x \in \mathbb{R}^n)$$

defined for measurable functions on $\mathbb{R}^{1+n}_+$. In the case $q = 2$ we simply write $\widetilde{N}_*$ as before. Implicit constants always depend only on the Whitney parameters, dimensions and the exponents at stake. We shall not mention this at each occurrence.

It is common knowledge that different choices of Whitney parameters yield maximal functions with comparable L^p-norms. For the reader's convenience we include a proof.

Lemma A.1 (Change of Whitney parameters)**.** *Let $0 < p, q < \infty$. Let c_0, c_1 and d_0, d_1 be two pairs of Whitney parameters and let $\widetilde{N}^{(c)}_{*,q}$ and*

$\widetilde{N}^{(d)}_{*,q}$ *be the corresponding maximal functions. Then,*

$$\|\widetilde{N}^{(d)}_{*,q}(F)\|_p \simeq \|\widetilde{N}^{(c)}_{*,q}(F)\|_p$$

for all measurable functions $F : \mathbb{R}^{1+n}_+ \to \mathbb{R}$.

Proof. By symmetry it suffices to prove the estimate '$\lesssim$'. We write $W_{c_0,c_1}(t,x) := (c_0^{-1}t, c_0 t) \times B(x, c_1 t)$. By compactness, we find points $(t_i, x_i) \in W_{d_0,d_1}(1,0)$, $i = 1, \ldots, N$, such that the sets $W_{c_0,c_1/2}(t_i, x_i)$ cover $W_{d_0,d_1}(1,0)$. Using the affine transformation $(s,y) \mapsto (ts, x+ty)$, we obtain

$$W_{d_0,d_1}(t,x) \subseteq \bigcup_{i=1}^N W_{c_0,c_1/2}(t_i t, x + t x_i) \tag{A.1}$$

for any $(t,x) \in \mathbb{R}^{1+n}_+$. Since $|x - (x + tx_i)| \leq t d_1 \leq d_0 d_1 t_i t$, we get

$$\Big(\fint\!\!\!\fint_{W_{d_0,d_1}(t,x)} |F|^q\Big)^{1/q} \leq C \sum_{i=1}^N \sup_{|x-y|<d_0 d_1 t_i t} \Big(\fint\!\!\!\fint_{W_{c_0,c_1/2}(t_i t, y)} |F|^q\Big)^{1/q}$$

for an admissible constant. For measurable $H : \mathbb{R}^{1+n}_+ \to \mathbb{R}$ let $H_{*,\eta}(x) := \sup_{|x-y|<\eta t} |H(t,y)|$ be the pointwise non-tangential maximal function with aperture η. With $H(t,y) := (\fint\!\!\fint_{W_{c_0,c_1/2}(t,y)} |F|^q)^{1/q}$ the previous bound yields

$$\widetilde{N}^{(d)}_{*,q}(F)(x) \leq C H^*_{d_0 d_1}(x) \quad (x \in \mathbb{R}^n).$$

On the other hand, $|y - x| < {}^{tc_1}/_2$ implies $B(y, {}^{tc_1}/_2) \subseteq B(x, c_1 t)$, so that

$$H_{*,c_1/2}(x) \leq C \widetilde{N}^{(c)}_{*,q}(F)(x) \quad (x \in \mathbb{R}^n).$$

For the classical pointwise non-tangential maximal functions we can change the aperture [89, II.2.5.1]: There is $C = C(n, c_0, c_1, d_0, d_1)$ such that

$$|\{x : \mathbb{R}^n : H_{*,d_0 d_1}(x) > \alpha\}| \leq C |\{x : \mathbb{R}^n : H_{*,c_1/2}(x) > \alpha\}| \quad (\alpha > 0).$$

The claim follows from the previous three bounds and the layer cake formula. □

Remark A.2. The covering argument in (A.1) implies as well that different choices of Whitney parameters for the Whitney average functionals yield equivalent Z-space norms.

We continue with a useful non-tangential embedding.

Lemma A.3 ([19, Lem. 2.2] & [61, Lem. A.2])**.** *If* $0 < p < r \leq 2$, *then there is a constant* C *such that for all measurable functions* $F : \mathbb{R}^{1+n}_+ \to \mathbb{R}$,

$$\Big(\iint_{\mathbb{R}^{1+n}_+} |F(t,x)|^r \, t^{n(\frac{r}{p}-1)} \, \frac{\mathrm{d}t\mathrm{d}x}{t}\Big)^{1/r} \leq C \|\widetilde{N}_*(F)\|_p.$$

We turn our attention to non-tangential trace theorems.

Definition A.4. A locally integrable function u on $\mathbb{R}^{1+n}_+$ is said to have a *non-tangential trace* (in the sense of Whitney averages) if there exists a function u_0 on $\mathbb{R}^n$ such that for almost every $x \in \mathbb{R}^n$,

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} u(s,y)\,\mathrm{d}s\mathrm{d}y = u_0(x).$$

As a pointwise limit of measurable functions, such a trace is necessarily measurable. The following is a variation of Kenig–Pipher's trace theorem [72, Thm. 3.2] that covers exponents $p \leq 1$ and applies to averaged non-tangential maximal functions. This has appeared (without proof) in many earlier works and we take the opportunity to close the gap.

Proposition A.5. *Let $p \in ({}^{n}/_{(n+1)}, \infty)$ and $q \in [1,\infty)$. Let $u \in \mathrm{W}^{1,q}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ satisfy $\|\widetilde{N}_{*,q}(\nabla u)\|_p < \infty$. Then there exists a non-tangential trace u_0 with the following properties.*

(i) *Let $r \in (0,\infty)$ and assume $r \leq \frac{(n+1)q}{n+1-q}$ if $q < n+1$. For almost every $x \in \mathbb{R}^n$ and all $t > 0$,*

$$\bigg(\fint\!\!\fint_{W(t,x)} |u(s,y) - u_0(x)|^r\,\mathrm{d}s\mathrm{d}y\bigg)^{\frac{1}{r}} \leq Ct\widetilde{N}_{*,q}(\nabla u)(x).$$

In particular, the left-hand side tends to 0 as $t \to 0$ and u_0 does not depend (in the almost everywhere sense) on the choice of the Whitney parameters.

(ii) *u_0 of class $\dot{\mathrm{H}}^{1,p}(\mathbb{R}^n)$ with $\|\nabla_x u_0\|_{\mathrm{H}^p} \leq C\|\widetilde{N}_{*,q}(\nabla u)\|_p$.*

(iii) *Let r be as in (i) and suppose in addition that $r < \frac{np}{n-p}$ if $p < n$. Then,*

$$\bigg\|\widetilde{N}_{*,r}\Big(\frac{u-u_0}{t}\Big)\bigg\|_p \leq C\|\widetilde{N}_{*,q}(\nabla u)\|_p.$$

(iv) *Suppose that either $p \geq 1$ or that $p < 1$ and that there exists $\varepsilon > 0$ such that $\sup_{0<t<\varepsilon} \|u(t,\cdot)\|_{\frac{np}{n-p}} < \infty$. Then,*

$$\lim_{t\to 0} \fint_{(c_0)^{-1}t}^{c_0 t} u(s,\cdot)\,\mathrm{d}s = u_0 \quad (\textit{in } \mathcal{D}'(\mathbb{R}^n)).$$

Remark A.6. In applications we usually have $q = 2$ and $r \in [0,2]$, which is admissible in (i). Also $r \in (0,1]$ is always admissible in (iii). Identification of the non-tangential trace with a distributional limit seems to be far from obvious in the case $p < 1$. We got the idea to impose the additional condition on u from [61, Lem. 5.2]. In our applications to the regularity problem $(R)^{\mathcal{L}}_p$ it follows from Sobolev embeddings and strong continuity of the Poisson semigroup.

For the proof we need a simple lemma on real functions.

Lemma A.7. *Let $h : (0,\infty) \to \mathbb{R}$ be a function for which there are constants $\theta > 1$, $\alpha > 0$ and $C \geq 0$ such that $|h(t) - h(\tau)| \leq Ct^\alpha$, whenever $\tau \in [\theta^{-1}t, t]$. Then $h(0) = \lim_{s\to 0} h(s)$ exists and satisfies*

$$|h(t) - h(0)| \leq \frac{Ct^\alpha}{1-\theta^{-\alpha}} \quad (t > 0).$$

Proof. Given $0 < \tau \leq t$, let k be the smallest integer with $\tau \leq \theta^{-k}t$. By a telescopic sum we find

$$\begin{aligned}|h(t) - h(\tau)| &\leq |h(\theta^{-k}t) - h(\tau)| + \sum_{j=1}^{k} |h(\theta^{-j+1}t) - h(\theta^{-j}t)| \\ &\leq \sum_{j=1}^{k+1} C\theta^{\alpha(-j+1)}t^\alpha \leq \frac{Ct^\alpha}{1-\theta^{-\alpha}}.\end{aligned}$$

This proves the Cauchy property for h at 0. Hence, $h(0)$ is defined and the estimate follows by sending $s \to 0$. □

Proof of Proposition A.5. Throughout the proof we write $\widetilde{N}_{*,q}^{\text{large}}$ for a non-tangential maximal function with Whitney parameters $c_0^{\text{large}} > c_0$ and $c_1^{\text{large}} \geq c_1$ that will be further specified if needed. We denote the associated Whitney regions by $W^{\text{large}}(t,x)$.

Proof of (i). Let $\theta > 1$ be such that $c_0^{\text{large}} = \theta c_0$. If $\tau \in [\theta^{-1}t, t]$, then both $W(\tau,x)$ and $W(t,x)$ are contained in $W^{\text{large}}(t,x)$ and we can estimate

$$\begin{aligned}&|(u)_{W(\tau,x)} - (u)_{W(t,x)}| \\ &\quad\leq \fint\!\!\fint_{W(s,x)} |u - (u)_{W(t,x)}| \,\mathrm{d}s\mathrm{d}y \\ &\quad\lesssim \bigg(\fint\!\!\fint_{W^{\text{large}}(t,x)} |u - (u)_{W(t,x)}|^q \,\mathrm{d}s\mathrm{d}y\bigg)^{\frac{1}{q}} \\ &\quad\lesssim t\bigg(\fint\!\!\fint_{W^{\text{large}}(t,x)} |\nabla u|^q \,\mathrm{d}s\mathrm{d}y\bigg)^{\frac{1}{q}} \\ &\quad\leq t\widetilde{N}_{*,q}^{\text{large}}(\nabla u)(x),\end{aligned} \tag{A.2}$$

where the third step is due to the (Sobolev–)Poincaré inequality on cylinders. From the assumption on u and Lemma A.1 we obtain that $\widetilde{N}_{*,q}^{\text{large}}(\nabla u)(x)$ is finite for a.e. $x \in \mathbb{R}^n$. In this case Lemma A.7 yields the existence of a non-tangential trace $u_0(x)$ with control

$$|(u)_{W(t,x)} - u_0(x)| \leq Ct\widetilde{N}_{*,q}^{\text{large}}(\nabla u)(x). \tag{A.3}$$

This argument works for any choice of Whitney parameters. In order to see that u_0 is always the same, it suffices (by transitivity) to verify that the trace u_0^{large} corresponding to the regions $W^{\text{large}}(t,x)$ agrees with u_0. By the argument in (A.2) we have

$$|(u)_{W(t,x)} - (u)_{W^{\text{large}}(t,x)}| \lesssim t\widetilde{N}_{*,q}^{\text{large}}(\nabla u)(x)$$

and hence the limits as $t \to 0$ are the same almost everywhere.

As for the estimate in (i) we pick some smaller Whitney parameters with associated regions $w(t,x)$ such that $W(t,x) = w^{\text{large}}(t,x)$. In this scenario (A.3) becomes

$$|(u)_{w(t,x)} - u_0(x)| \leq Ct\widetilde{N}_{*,q}(\nabla u)(x)$$

and the restriction on r allows us to use the Sobolev–Poincaré inequality in order to give

$$\begin{aligned}\Big(\fint\!\!\!\fint_{W(t,x)} &|u - u_0(x)|^r \,\mathrm{d}s\mathrm{d}y\Big)^{\frac{1}{r}} \\ &\lesssim \Big(\fint\!\!\!\fint_{W(t,x)} |u - (u)_{w(t,x)}|^r \,\mathrm{d}s\mathrm{d}y\Big)^{\frac{1}{r}} + |(u)_{w(t,x)} - u_0(x)| \\ &\lesssim t\widetilde{N}_{*,q}(\nabla u)(x).\end{aligned}$$

Proof of (ii). We use the following result: If there is $g \in \mathrm{L}^p(\mathbb{R}^n)$ such that for almost every $x, y \in \mathbb{R}^n$,

$$|u_0(x) - u_0(y)| \leq |x-y|(g(x)+g(y)), \tag{A.4}$$

then $u_0 \in \dot{\mathrm{H}}^{1,p}$ with $\|\nabla_x u_0\|_{\mathrm{H}^p} \lesssim \|g\|_p$. For $p > 1$ this is Hajlasz's Sobolev space characterization [54, Thm. 1] and the result for exponents $n/(n+1) < p \leq 1$ has been obtained in [73, Thm. 1 & Prop. 5].

Now, let $x, y \in \mathbb{R}^n$ and set $t := |x-y|$. We take $c_1^{\text{large}} \geq 1 + c_1$. Since $B(y, c_1 t) \subseteq B(x, (1+c_1)t)$, we have $W(t,y) \subseteq W^{\text{large}}(t,x)$ and Poincaré's inequality yields again

$$|(u)_{W(t,y)} - (u)_{W(t,x)}| \leq Ct\widetilde{N}_{*,q}^{\text{large}}(\nabla u)(x).$$

Together with (A.3), we see that we can take $g := 3C\widetilde{N}_{*,q}^{\text{large}}(\nabla u)$. Note that $\|g\|_p \simeq \|\widetilde{N}_{*,q}(\nabla u)\|_p$ by Lemma A.1.

Proof of (iii). It suffices to find a function h with $\|h\|_p \leq C\|\widetilde{N}_{*,q}(\nabla u)\|_p$ such that for a.e. $x \in \mathbb{R}^n$ and all $t > 0$,

$$\Big(\fint\!\!\!\fint_{W(t,x)} |u - u_0|^r \,\mathrm{d}y\mathrm{d}s\Big)^{\frac{1}{r}} \lesssim th(x). \tag{A.5}$$

Indeed, since we are integrating s on $(c_0^{-1}t, c_0 t)$ on the left-hand side, the bound required in (iii) follows immediately. The argument slightly differs depending on whether or not we have $p > 1$. Let us first assume that this is the case.

The additional restriction on r makes sure that we can find some $\varrho \in (1, p \wedge n)$ such that $n\varrho/(n-\varrho) \geq r$. Hence, by Hölder's inequality followed by the Sobolev–Poincaré inequality, we have

$$\begin{aligned}&\bigg(\fint_{B(x,c_1t)} |u_0-(u_0)_{B(x,c_1t)}|^r \,\mathrm{d}y\bigg)^{\frac{1}{r}} \\ &\qquad\leq \bigg(\fint_{B(x,c_1t)} |u_0-(u_0)_{B(x,c_1t)}|^{\frac{n\varrho}{n-\varrho}} \,\mathrm{d}y\bigg)^{\frac{1}{\varrho}-\frac{1}{n}} \\ &\qquad\leq Ct\bigg(\fint_{B(x,c_1t)} |\nabla_x u_0|^\varrho \,\mathrm{d}y\bigg)^{\frac{1}{\varrho}}.\end{aligned} \tag{A.6}$$

Since $n\varrho/(n-\varrho) \geq 1$, we can also argue as in (A.2), using the Sobolev–Poincaré inequality in the second step, to get whenever $\tau \in [t/2, t]$,

$$|(u_0)_{B(x,c_1t)} - (u_0)_{B(x,c_1\tau)}| \lesssim t\,\mathcal{M}(|\nabla_x u_0|^\varrho)(x)^{\frac{1}{\varrho}}.$$

Lemma A.7 yields

$$|(u_0)_{B(x,c_1t)} - u_0(x)| \lesssim t\,\mathcal{M}(|\nabla_x u_0|^\varrho)(x)^{\frac{1}{\varrho}} \tag{A.7}$$

for a.e. $x \in \mathbb{R}^n$. Using the decomposition

$$u(s,y)-u_0(y) = u(s,y)-u_0(x)+u_0(x)-(u_0)_{B(x,c_1t)}+(u_0)_{B(x,c_1t)}-u_0(y)$$

and combining (i), (A.6) and (A.7), we arrive at

$$\begin{aligned}&\bigg(\fint\!\!\fint_{W(t,x)} |u(s,y)-u_0(y)|^r \,\mathrm{d}s\mathrm{d}y\bigg)^{\frac{1}{r}} \\ &\qquad\lesssim t\Big(\widetilde{N}_{*,q}(\nabla u)(x) + \mathcal{M}(|\nabla_x u_0|^\varrho)(x)^{\frac{1}{\varrho}}\Big)\end{aligned}$$

for a.e. $x \in \mathbb{R}^n$ and all $t > 0$. The right-hand side is admissible for (A.5) by assumption on u, the $\mathrm{L}^{p/\varrho}$-boundedness of the maximal function and the result of (ii).

We turn to the case $p \leq 1$. Since $p > n/(n+1)$, we can pick $\varrho \in (n/(n+1), p)$ with $n\varrho/(n-\varrho) \geq (r \vee 1)$. Since the function g in (A.4) is locally ϱ-integrable, we have Hajlasz's Sobolev–Poincaré inequality

$$\bigg(\fint_{B(x,c_1t)} |u_0-(u_0)_{B(x,c_1t)}|^{\frac{n\varrho}{n-\varrho}} \,\mathrm{d}y\bigg)^{\frac{1}{\varrho}-\frac{1}{n}} \lesssim t\bigg(\fint_{B(x,2c_1t)} g^\varrho \,\mathrm{d}y\bigg)^{\frac{1}{\varrho}},$$

see [55, Thm. 8.7]. Hence, except for replacing $\nabla_x u_0$ by g, the argument stays the same.

Proof of (iv). Let $B \subseteq \mathbb{R}^n$ be a ball and let $\phi \in \mathrm{C}_0^\infty(B)$. We use the averaging trick to write

$$\begin{aligned}\int_{\mathbb{R}^n} \Big(\fint_{(c_0)^{-1}t}^{c_0 t} u(s,y)\,\mathrm{d}s - u_0(y) \Big) \phi(y)\,\mathrm{d}y \\ = \int_{\mathbb{R}^n} \Big(\iint\!\!\!\!\!\!-_{W(t,x)} (u-u_0)\phi\,\mathrm{d}s\mathrm{d}y \Big)\mathrm{d}x \\ =: \int_{\mathbb{R}^n} F_t^\phi(x)\,\mathrm{d}x.\end{aligned} \tag{A.8}$$

We have to show that the right-hand side tends to 0 as $t \to 0$. From now on, we require $t < {r(B)}/{c_1}$, so that all functions F_t^ϕ have support in $2B$.

If $p \geq 1$, then (A.5) for the admissible choice $r = 1$ gives us $|F_t^\phi(x)| \leq \|\phi\|_\infty t h(x)$ and h is locally integrable, so we are done.

In the case $p < 1$ we need a different argument and this is where the additional assumption $C := \sup_{0<t<\varepsilon} \|u(t,\cdot)\|_{np/(n-p)} < \infty$ comes into play. We abbreviate $p^* := {np}/{(n-p)} > 1$. We can restrict ourselves to $t < {(\varepsilon \wedge r(B))}/{c_1}$ and $x \in 2B$. In this case, $B(x, c_1 t) \subseteq 3B$ and by Hölder's inequality we can crudely bound

$$\begin{aligned}\iint\!\!\!\!\!\!-_{W(t,x)} |u-u_0|\,\mathrm{d}s\mathrm{d}y &\lesssim t^{1-\frac{n}{p}} \fint_{c_0^{-1}t}^{c_0 t} \|u(s,\cdot) - u_0\|_{\mathrm{L}^{p^*}(3B)}\,\mathrm{d}s \\ &\leq 2Ct^{1-\frac{n}{p}},\end{aligned}$$

as $u_0 \in \mathrm{L}^{p^*}_{\mathrm{loc}}$ from the Hardy–Sobolev embedding or by the following direct argument. We have, for t small enough, using Hölder's inequality and averaging,

$$\int_{3B} |(u)_{W(t,x)}|^{p^*}\,\mathrm{d}x \leq \fint_{c_0^{-1}t}^{c_0 t} \int_{4B} |u(s,y)|^{p^*}\,\mathrm{d}y\mathrm{d}s \leq C^{p^*}.$$

Thus by Fatou's lemma, we obtain $\int_{3B} |u_0|^{p^*}\,\mathrm{d}x \leq C^{p^*}$. Now, we use the p-th power of (A.5) (with $r = 1$) and the $(1-p)$-th power of the crude bound in order to get for a.e. $x \in 2B$ that

$$|F_t^\phi(x)| \leq \|\phi\|_\infty \iint\!\!\!\!\!\!-_{W(t,x)} |u-u_0|\,\mathrm{d}s\mathrm{d}y \leq \|\phi\|_\infty (2C)^{1-p} t^{1+n-\frac{n}{p}} h(x)^p.$$

On the right the power of t is positive since $p > {n}/{(n+1)}$ and we have $h^p \in \mathrm{L}^1$. Thus, we get the desired convergence in (A.8) when passing to the limit as $t \to 0$. □

Next, we present variants of the non-tangential trace theorem for tent and Z-spaces. In our application we shall only encounter functionals based on L^2-averages such as S and W that used to define tent and Z-spaces, respectively. For simplicity we stick to that case. The following result have a appeared in [28, Thm. 6.3] ($p = \infty$) and [3, Sec. 6.6] ($p < \infty$). For the sake of self-containedness we include a proof that follows the same pattern as before. The lower bound on p, notably to identify the non-tangential trace with a distributional limit, is now related to fractional Sobolev embeddings and the argument turns out to be conceptually simpler than in Proposition A.5.

As usual, we treat both scales of spaces simultaneously and let Y denote one of T or Z.

Proposition A.8. *Let $\alpha \in (0,1)$ and ${}^{n}/_{(n+\alpha)} < p < \infty$. Let $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ satisfy $\|\nabla u\|_{\mathrm{Y}^{\alpha-1,p}} < \infty$. Then there exists a non-tangential trace u_0 with the following properties.*

(i) *Let $r \in (0,\infty)$ and assume $r \leq \frac{2(n+1)}{n-1}$ if $n > 1$. For all $x \in \mathbb{R}^n$ and all $t > 0$,*

$$\left(\fint\!\!\!\fint_{W(t,x)} |u(s,y) - u_0(x)|^r \,\mathrm{d}s\mathrm{d}y \right)^{\frac{1}{r}} \leq C t^\alpha \Theta(x)$$

with $\|\Theta\|_p \leq C\|\nabla u\|_{\mathrm{Y}^{\alpha-1,p}}$. In particular, the left-hand side tends to 0 almost everywhere as $t \to 0$ and u_0 does not depend on the choice of the Whitney parameters.

(ii) *There is convergence*

$$\lim_{t\to 0} \fint_{(c_0)^{-1}t}^{c_0 t} u(s,\cdot)\,\mathrm{d}s = u_0 \quad (\textit{in } \mathcal{D}'(\mathbb{R}^n)).$$

(iii) *The results above continue to hold for $p = \infty$ and $\nabla u \in \mathrm{Z}^{\alpha-1,\infty}$. In that case $\Theta(x) = \|\nabla u\|_{\mathrm{Z}^{\alpha-1,\infty}}$ and u_0 is of class $\dot{\Lambda}^\alpha$ with $\|u_0\|_{\dot{\Lambda}^\alpha} \leq C\|\nabla u\|_{\mathrm{Z}^{\alpha-1,\infty}}$.*

The following lemma contains the construction of the function Θ in part (i) for finite p.

Lemma A.9. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$ and $F \in \mathrm{Y}^{\alpha-1,p}$. There exists a measurable function $\Theta : \mathbb{R}^n \to [0,\infty)$ with $\|\Theta\|_p \leq C\|F\|_{\mathrm{Y}^{\alpha-1,p}}$ such that*

$$\left(\fint\!\!\!\fint_{W(t,x)} |s^{1-\alpha}F|^2 \,\mathrm{d}s\mathrm{d}y \right)^{\frac{1}{2}} \leq C\Theta(x) \quad ((t,x) \in \mathbb{R}^{1+n}_+).$$

Proof. We begin with the case Y = Z and set

$$\Theta(x) := \left(\int_0^\infty \left(\fint\!\!\!\fint_{W^{\mathrm{large}}(t,x)} |s^{1-\alpha}F|^2 \,\mathrm{d}s\mathrm{d}y \right)^{\frac{p}{2}} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}},$$

where $W^{\mathrm{large}}(t,x)$ are Whitney regions with Whitney parameter $c_0^{\mathrm{large}} := 2c_0$. Since

$$W(t,x) \subseteq W^{\mathrm{large}}(\tau,x) \quad (\tau \in [{}^{t}/_{2}, t]),$$

we can infer that

$$\left(\fint\!\!\!\fint_{W(t,x)} |s^{1-\alpha}F|^2 \,\mathrm{d}s\mathrm{d}y \right)^{\frac{p}{2}}$$

$$\lesssim \int_{t/2}^{t} \bigg(\fint\!\!\fint_{W^{\mathrm{large}}(\tau,x)} |s^{1-\alpha} F|^2 \,\mathrm{d}s\mathrm{d}y \bigg)^{\frac{p}{2}} \frac{\mathrm{d}\tau}{\tau}$$

and the right-hand side is bounded by $\Theta(x)^p$. Moreover, a change of Whitney parameters for Z-space norms yields $\|\Theta\|_p \simeq \|F\|_{\mathrm{Z}^{\alpha-1,p}}$.

In the case $\mathrm{Y} = \mathrm{T}$ we can simply set

$$\Theta(x) := \bigg(\iint_{|x-y|<2c_1 s} |s^{1-\alpha} F|^2 \frac{\mathrm{d}s\mathrm{d}y}{s^{n+1}} \bigg)^{\frac{1}{2}}$$

since $W(t,x)$ is contained in the cone appearing in the integral. By a change of aperture in tent space norms we conclude that $\|\Theta\|_p \simeq \|\nabla u\|_{\mathrm{T}^{\alpha-1,p}}$. □

Proof of Proposition A.8. We use the same notation as in the proof of Proposition A.5 and follow the same line of thoughts.

Proof of (i). Let $c_0^{\mathrm{large}} := 2c_0$. If $\tau \in [t/2, t]$, then both $W(\tau,x)$ and $W(t,x)$ are contained in $W^{\mathrm{large}}(t,x)$ and using the Poincaré inequality with $q = 2$ as in (A.2), we obtain

$$\begin{aligned} |(u)_{W(\tau,x)} - (u)_{W(t,x)}| &\lesssim t \bigg(\fint\!\!\fint_{W^{\mathrm{large}}(t,x)} |\nabla u|^2 \,\mathrm{d}s\mathrm{d}y \bigg)^{\frac{1}{2}} \\ &\lesssim t^{\alpha} \bigg(\fint\!\!\fint_{W^{\mathrm{large}}(t,x)} |s^{1-\alpha} \nabla u|^2 \,\mathrm{d}s\mathrm{d}y \bigg)^{\frac{1}{2}}. \end{aligned}$$

Lemma A.9 applied to the 'large' Whitney regions yields a function Θ with $\|\Theta\|_p \lesssim \|\nabla u\|_{\mathrm{Y}^{\alpha-1,p}}$ such that

$$|(u)_{W(\tau,x)} - (u)_{W(t,x)}| \lesssim t^{\alpha} \Theta(x).$$

Now, we can apply Lemma A.7 to obtain a non-tangential trace $u_0(x)$ with control

$$|(u)_{W(t,x)} - u_0(x)| \lesssim t^{\alpha} \Theta(x), \tag{A.9}$$

whenever $\Theta(x) < \infty$, that is, almost everywhere. That u_0 is independent of the choice of Whitney parameters follows as in the proof of Proposition A.5 and the restriction on r allows us to use the Sobolev–Poincaré inequality again in order to conclude

$$\begin{aligned} &\bigg(\fint\!\!\fint_{W(t,x)} |u - u_0(x)|^r \,\mathrm{d}s\mathrm{d}y \bigg)^{\frac{1}{r}} \\ &\qquad \lesssim \bigg(\fint\!\!\fint_{W(t,x)} |u - (u)_{W(t,x)}|^r \,\mathrm{d}s\mathrm{d}y \bigg)^{\frac{1}{r}} + |(u)_{W(t,x)} - u_0(x)| \\ &\qquad \lesssim t^{\alpha} \bigg(\fint\!\!\fint_{W(t,x)} |s^{1-\alpha} \nabla u|^2 \,\mathrm{d}s\mathrm{d}y \bigg)^{\frac{1}{2}} + |(u)_{W(t,x)} - u_0(x)| \\ &\qquad \lesssim t^{\alpha} \Theta(x). \end{aligned}$$

Proof of (ii). We begin with the case $p > 1$. With the notation of the proof of Proposition A.5.(iv) we have to show that $\int_{\mathbb{R}^n} |F_t^\phi(x)| \, dx$ converges to 0 as $t \to 0$. Recall that the expression F_t^ϕ defined in (A.8) is supported in $2B$ if the support of ϕ is contained in B and $t \leq 1$.

We record two elementary observations.

- If y belongs to a ball $B(x, c_1 t)$, then
$$|(u)_{W(t,y)} - (u)_{W(t,x)}| \lesssim t^\alpha \Theta(x).$$
Indeed, we take $c_1^{\text{large}} \geq 1 + c_1$. Since $B(y, c_1 t) \subseteq B(x, (1+c_1)t)$, we have $W(t,y) \subseteq W^{\text{large}}(t,x)$ and Poincaré's inequality yields this inequality as before.
- For almost every $y \in B(x, c_1 t)$ the first observation together with (A.9) yields
$$\begin{aligned} |u_0(y) - u_0(x)| &\leq |u_0(y) - (u)_{W(t,y)}| + |(u)_{W(t,y)} - (u)_{W(t,x)}| \\ &\quad + |(u)_{W(t,x)} - u_0(x)| \\ &\lesssim t^\alpha(\Theta(x) + \Theta(y)). \end{aligned}$$

The second observation implies
$$|u(s,y) - u_0(y)| \lesssim |u(s,y) - u_0(x)| + t^\alpha(\Theta(x) + \Theta(y))$$
and taking into account (i) with $r = 1$, we are left with
$$|F_t^\phi(x)| \lesssim \|\phi\|_\infty t^\alpha \mathcal{M}(\Theta)(x).$$
The maximal theorem ensures that $\mathcal{M}(\Theta) \in \mathrm{L}^p$ and since F_t^ϕ is supported in $2B$ we conclude $\int_{\mathbb{R}^n} |F_t^\phi(x)| \, dx \to 0$ in the limit as $t \to 0$.

In the case $p \leq 1$ we use the embedding $\mathrm{Y}^{\alpha-1,p} \subseteq \mathrm{Y}^{\beta-1,q}$ for $0 < p < q < \infty$ and $\alpha - \beta = n(1/p - 1/q)$, see [3, Thm. 2.34]. We have $\alpha > n(1/p - 1)$ by assumption, which allows us to pick $q > 1$ and $0 < \beta < \alpha$. Hence, we are back in the case of integrability above 1.

Proof of (iii). If $p = \infty$ and $\nabla u \in \mathrm{Z}^{\alpha-1,\infty}$, then the constant function $\Theta(x) := C\|\nabla u\|_{\mathrm{Z}^{\alpha-1,\infty}}$ has the properties stated in Lemma A.9 by definition of the $\mathrm{Z}^{\alpha-1,\infty}$-norm. Hence, we can repeat the first two steps and the second observation in the proof of (ii) yields $\|u_0\|_{\dot{\Lambda}^\alpha} \leq C\|\nabla u\|_{\mathrm{Z}^{\alpha-1,\infty}}$. □

Appendix B. The L^p-realization of a sectorial operator in L^2

The following result is folklore but we could not find a precise statement in the literature.

Proposition B.1. *Let T be a sectorial operator in L^2 and let $p \in (1,\infty)$. Suppose that there exists $\mu \in (\omega_T, \pi)$ such that*

$$\|z(z-T)^{-1}f\|_p \lesssim \|f\|_p \quad (f \in \mathrm{L}^p \cap \mathrm{L}^2,\ z \in \mathbb{C} \setminus \overline{\mathrm{S}_\mu^+}). \tag{B.1}$$

The case $\mu = \pi$ with the convention that $\mathbb{C} \setminus \overline{\mathrm{S}_\pi^+} := (-\infty, 0)$ is also permitted. Then there is a (unique) sectorial operator T_p in L^p of angle smaller than μ that satisfies

$$(z - T_p)^{-1} f = (z - T)^{-1} f \quad (f \in \mathrm{L}^p \cap \mathrm{L}^2,\ z \in \mathbb{C} \setminus \overline{\mathrm{S}_\mu^+}). \tag{B.2}$$

Moreover, $T_p f = T f$ for $f \in \mathsf{D}(T_p) \cap \mathsf{D}(T)$ and if T is injective then so is T_p. The corresponding statement for bisectorial operators also holds.

Remark B.2. The assumption with $\mu = \pi$ simply means that T satisfies $\|(1 + t^2 T)^{-1} f\|_p \lesssim \|f\|_p$ for all $f \in \mathrm{L}^p \cap \mathrm{L}^2$ and all $t > 0$.

The operator T_p is usually called L^p*-realization* of T. We have tried to avoid passing to an L^p-realization whenever possible, but knowing that we always can turns out helpful when dealing with abstract results that do not need a distinguished space such as L^2 to start with. One such example is Theorem 9.18.

Condition (B.1) is obviously necessary for the existence of a L^p-realization with consistent resolvents as in (B.2) and the latter uniquely determines T. We also obtain consistency of T_p and T, whereas consistency of general invertible operators does not imply consistency of their inverses, compare with the discussion in Section 13.2.

Proof. By (B.1) we can define $R(z)$ as the extension by density of $(z - T)^{-1}$ to L^p. Then $(zR(z))_{z \in \mathbb{C} \setminus \mathrm{S}_\mu^+}$ is a uniformly bounded family in L^p with the property

$$R(z) - R(z') = (z' - z) R(z) R(z') \quad (z, z' \in \mathbb{C} \setminus \mathrm{S}_\mu^+). \tag{B.3}$$

We claim that for $f \in \mathrm{L}^p$ we have

$$\lim_{z \in (-\infty, 0), z \to -\infty} zR(z) f = f \quad (\text{weakly in } \mathrm{L}^p) \tag{B.4}$$

and if in addition T is injective, also that

$$\lim_{z \to 0} zR(z) f = 0 \quad (\text{weakly in } \mathrm{L}^p). \tag{B.5}$$

Indeed, since T is sectorial in L^2, the limits exist strongly in L^2 if $f \in \mathrm{L}^p \cap \mathrm{L}^2$, see [53, Prop. 2.1.1(a)]. The extension then follows by uniform boundedness and density.

By (B.3), $R(-1) f = 0$ implies $R(z) f = 0$ for all z. Then $f = 0$ follows from (B.4), so $R(-1)$ is injective. We show that $T_p := -R(-1)^{-1} - 1$ has the required properties.

For $f \in \mathsf{D}(T_p)$ we have

$$R(z)(z - T_p) f = R(z)((z + 1) R(-1) + 1) R(-1)^{-1} f = f,$$

where the final step uses (B.3). Likewise, for $g \in \mathrm{L}^p$ we have

$$R(z) g = R(-1)(g - (z + 1) R(z) g) \in \mathsf{D}(T_p)$$

and

$$(z - T_p) R(z) g = (z + 1 + R(-1)^{-1}) R(z) g = g.$$

This proves $(z-T_p)^{-1} = R(z)$, so (B.2) holds. By a Neumann series, the uniform boundedness of the family $(zR(z))$ implies that T_p is a sectorial operator of angle smaller than μ. Now, suppose that $f \in \mathsf{D}(T_p) \cap \mathsf{D}(T)$. Then

$$z(z-T_p)^{-1}T_p f = z(z-T)^{-1}Tf \tag{B.6}$$

since both terms can be expanded in terms of $R(z)$. When $z \in (-\infty, 0)$ tends to $-\infty$, the left-hand side tends to $T_p f$ weakly in L^p and the right-hand side tends to Tf strongly in L^2, see (B.4). This proves $T_p f = Tf$. Finally, if $f \in \mathsf{N}(T_p)$, then $f = zR(z)f$ for all z and if T is injective, then $f = 0$ follows from (B.5).

The argument for a bisectorial operator is exactly the same, using $z \in \mathrm{i}(0, \infty)$ instead of $z \in (-\infty, 0)$ for the limits. In this case we can allow $\mu = \pi/2$ with the convention that $\mathbb{C} \setminus \overline{\mathrm{S}_{\pi/2}} := \mathrm{i}\mathbb{R}$. □

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