# Incompressible Navier-Stokes-Fourier limit from the Landau equation

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**Abstract**

In this work, we provide a result on the derivation of the incompressible Navier-Stokes-Fourier system from the Landau equation for hard, Maxwellian and moderately soft potentials. To this end, we first investigate the Cauchy theory associated to the rescaled Landau equation for small initial data. Our approach is based on proving estimates of some adapted Sobolev norms of the solution that are uniform in the Knudsen number. These uniform estimates also allow us to obtain a result of weak convergence towards the fluid limit system.

# Contents



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# 1 Introduction

## 1.1 The model.

We start by introducing the Landau equation. This equation is a kinetic model in plasma physics that describes the evolution of the density function $f_\varepsilon = f_\varepsilon(t,x,v)$ representing at time $t \in \mathbb{R}^+$ the density of particles at position $x \in \mathbb{T}^3$ the 3-dimensional unit periodic box and velocity $v \in \mathbb{R}^3$. This equation is given by

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon) \\ f_{\varepsilon|t=0} = f_{\varepsilon,0}, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is the Knudsen number which is the inverse of the average number of collisions for each particle per unit time and $Q$ is the so-called Landau collision operator which acts on the variable $v$ and which contains diffusion in velocity. More precisely, the Landau operator is defined by

$$Q(G,F) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v-v_*)[G_* \partial_j F - F \partial_j G_*] \, \mathrm{d}v_*, \tag{2}$$

and we use the convention of summation of repeated indices, and the usual derivatives are in the velocity variable $v$ i.e. $\partial_i = \partial_{v_i}$. Hereafter we use the shorthand notations $G_* = G(v_*)$, $F = F(v)$, $\partial_j G_* = \partial_{v_{*j}} G(v_*)$, $\partial_j F = \partial_{v_j} F(v)$, etc. The matrix $A(v) = (a_{ij}(v))_{1 \le i,j \le 3}$ is symmetric, positive, definite, depends on the interaction between particles and is given by

$$a_{ij}(v) = |v|^{\gamma+2} \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad \gamma \in [-3,1].$$

We recall the standard classification: we call hard potentials if $\gamma \in (0,1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2,0)$, very soft potentials if $\gamma \in (-3,-2)$ and Coulombian potential if $\gamma = -3$. Hereafter we shall consider the cases of hard potentials, Maxwellian molecules and moderately soft potentials, i.e. $\gamma \in [-2,1]$. We consider the fluctuation around the centered normalized Maxwellian distribution

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

by setting $f_\varepsilon(t,x,v) = \mu + \varepsilon \mu^{1/2} g_\varepsilon(t,x,v)$, and

$$\begin{aligned} \Gamma(f,g) &= \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} g) \\ &= \partial_i \Big[ (a_{ij} * \mu^{1/2} f) \partial_j g \Big] - (a_{ij} * \frac{v_i}{2} \mu^{1/2} f) \partial_j g \\ &\quad - \partial_i \Big[ (a_{ij} * \mu^{1/2} \partial_j f) g \Big] + (a_{ij} * \frac{v_i}{2} \mu^{1/2} \partial_j f) g, \end{aligned}$$

the homogeneous linearized Landau operator $\mathcal{L}$ takes the form

$$\begin{aligned} \mathcal{L} &= -\Gamma(\sqrt{\mu}, f) - \Gamma(f, \sqrt{\mu}) \\ &:= -\mathcal{L}_1 - \mathcal{L}_2. \end{aligned}$$

The operator $\mathcal{L}$ acts only in variable $v$, is selfadjoint, and consists of a diffusion part and a compact part. Using for example [15], [30], we show that the diffusion part $\mathcal{L}_1$ writes as follows

$$\mathcal{L}_1 f = \nabla_v \cdot [\mathbf{A}(v) \nabla_v f] - \left( \mathbf{A}(v) \frac{v}{2} \cdot \frac{v}{2} \right) f + \nabla_v \cdot \left[ \mathbf{A}(v) \frac{v}{2} \right] f,$$

where $\mathbf{A}(v) = (\overline{a}_{ij}(v))_{1\le i,j\le 3}$ is a symmetric matrix defined through

$$\overline{a}_{ij} = a_{ij} *_v \mu,$$

and the compact part $\mathcal{L}_2$ is given by

$$\mathcal{L}_2 f = -\mu^{-1/2}\partial_i \Big\{\mu \Big[a_{ij} *_v \Big\{\mu^{1/2}\Big[\partial_j f + \frac{v_j}{2} f\Big]\Big\}\Big]\Big\}.$$

Now the original problem (1) is reduced to the Cauchy problem for the fluctuation $g_\varepsilon$

$$\begin{cases} \partial_t g_\varepsilon + \frac{1}{\varepsilon} v\cdot\nabla_x g_\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L} g_\varepsilon = \frac{1}{\varepsilon}\Gamma(g_\varepsilon, g_\varepsilon) \\ g_{\varepsilon|t=0} = g_{\varepsilon,0}, \end{cases} \tag{3}$$

where $g_{\varepsilon,0}$ is given by $f_{\varepsilon,0} = \mu + \varepsilon\mu^{1/2} g_{\varepsilon,0}$. From now on, we will always assume that

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f_{\varepsilon,0}(x,v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}x\mathrm{d}v = \int_{\mathbb{T}^3\times\mathbb{R}^3}\mu(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}x\mathrm{d}v, \tag{4}$$

which implies that

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} g_\varepsilon(t,x,v)\mu^{1/2}(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}x\mathrm{d}v = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad \text{for all } t\ge 0,$$

since our equation preserves the total mass, momentum and energy.

## 1.2 Notations and functional spaces.

Throughout the paper we shall adopt the following notations. For $v\in\mathbb{R}^3$ we denote $\langle v\rangle = (1+|v|^2)^{1/2}$, where we recall that $|v|$ is the canonical Euclidian norm of $v$ in $\mathbb{R}^3$. The gradient in velocity (resp. space) will be denoted by $\partial_v$ (resp. $\partial_x$). For simplicity of notations, $a\sim b$ means that there exist constants $c_1$, $c_2>0$ depending only on fixed number such that $c_1 b\le a\le c_2 b$; we abbreviate "$\le C$ " to "$\lesssim$", where $C$ is a positive constant depending only on fixed number that may change from line to line.
In what follows, we shall write for $p\in[1,+\infty]$

$$L^p_x = L^p(\mathbb{T}^3),\quad L^p_v = L^p(\mathbb{R}^3),\quad L^p_xL^p_v = L^p(\mathbb{T}^3\times\mathbb{R}^3).$$

For $p=2$, we use the notations $(\cdot,\cdot)_{L^2_x}$, $(\cdot,\cdot)_{L^2_v}$ and $(\cdot,\cdot)_{L^2_xL^2_v}$ to represent the inner product on the Hilbert spaces $L^2_x$, $L^2_v$ and $L^2_{x,v}$ respectively.
For $m=m(v)$ a positive Borel weight function and $1\le p,q\le\infty$, we define the space $L^q_xL^p_v(m)$ as the Lebesgue space associated to the norm, for $f=f(x,v)$

$$\|f\|_{L^q_xL^p_v(m)} = \|\|f\|_{L^p_v(m)}\|_{L^q_x} = \|\|mf\|_{L^p_v}\|_{L^q_x}.$$

We define for $s\in\mathbb{N}$ the spaces $H^s_x$ to be the usual Sobolev space on $\mathbb{T}^3$ and $H^s_xL^2_v$ by the norm

$$\|f\|_{H^s_xL^2_v} = \Big(\sum_{|k|\le s}\|\partial^k_x f\|^2_{L^2_{x,v}}\Big)^{\frac12}.$$

It is well known that the null space $\mathcal{N}$ of $\mathcal{L}$ is spanned by the set of collision invariants:

$$\mathcal{N}(\mathcal{L}) = \mathrm{Span}\left\{\sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, |v|^2\sqrt{\mu}\right\}. \tag{5}$$

We also let $\mathcal{N}^\perp$ denote the orthogonal space of $\mathcal{N}$ with respect to the standard inner product $(\cdot,\cdot)_{L^2_v}$. Following [15], we introduce the $H^1_{v,*}$-norm defined by

$$\|f\|^2_{H^1_{v,*}} := \|\langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}+1}(I-P_v)\nabla_v f\|^2_{L^2_v}, \tag{6}$$

this norm naturally arises in the study of the Landau equation. $P_v$ is the projection on $v$, i.e. $P_v w = \left(w\cdot \frac{v}{|v|}\right)\frac{v}{|v|}$. We define the space $H^s_x H^1_{v,*}$ for $s\in\mathbb{N}$ associated to the norm

$$\|f\|^2_{\mathcal{X}^s(\mathbb{T}^3_x\times\mathbb{R}^3_v)} = \sum_{|\alpha|\le s}\int_{\mathbb{T}^3} \|\partial^\alpha_x f\|^2_{H^1_{v,*}}\,\mathrm{d}x. \tag{7}$$

We also define the space $H^s_x(H^1_{v,*})'$ (is the dual of $H^s_x H^1_{v,*}$ w.r.t. $H^s_x L^2_v$) in the following way

$$\begin{aligned}\|f\|_{H^s_x(H^1_{v,*})'} &:= \sup_{\|\phi\|_{H^s_x H^1_{v,*}}\le 1} (f,\phi)_{H^s_x L^2_v}\\ &:= \sup_{\|\phi\|_{H^s_x H^1_{v,*}}\le 1} \sum_{|\beta|\le s}\left(\partial^\beta_x f, \partial^\beta_x \phi\right)_{L^2_x L^2_v}.\end{aligned} \tag{8}$$

Let us recall the so called macro-micro decomposition of solutions

$$g = \Pi_0 g + (I-\Pi_0)g := g_1 + g_2, \tag{9}$$

où $\Pi_0$ est appelée la projection macroscopique de $\mathcal{N}$, $g_1 = \Pi_0 g$ is called the macroscopic projection of $g$ and $g_2 = (I-\Pi_0)g$ is called the kinetic or microscopic part of $g$. Furthermore, we often use the following notation:

$$\Pi_0 g(t,x,v) = \{a(t,x) + v\cdot b(t,x) + |v|^2 c(t,x)\}\sqrt{\mu}, \quad \mathcal{A}(g) = (a,b,c), \tag{10}$$

where

$$a := \int_{\mathbb{R}^3} g\Big(\frac{5}{2} - \frac{|v|^2}{2}\Big)\mu^{1/2}\,\mathrm{d}v, \quad b := \int_{\mathbb{R}^3} g v \mu^{1/2}\,\mathrm{d}v, \quad c := \int_{\mathbb{R}^3} g\Big(\frac{|v|^2}{6} - \frac{1}{2}\Big)\mu^{1/2}\,\mathrm{d}v.$$

We introduce the following instant energy functional and dissipation rate functional respectively

$$\begin{aligned}\mathcal{E}^2(g) &= \|g\|^2_{H^3_x L^2_v} = \|g_1\|^2_{H^3_x L^2_v} + \|g_2\|^2_{H^3_x L^2_v}\\ &\sim \|\mathcal{A}(g)\|^2_{H^3_x} + \|g_2\|^2_{H^3_x L^2_v},\\ \mathcal{D}(g) &= \|g_2\|_{\mathcal{X}^3(\mathbb{T}^3_x\times\mathbb{R}^3_v)},\\ \mathcal{C}(g) &= \|\nabla_x \Pi_0 g\|_{H^2_x L^2_v} \sim \|\nabla_x \mathcal{A}(g)\|_{H^2_x}.\end{aligned} \tag{11}$$

These quantities will be in the heart of the coming study and the main results we present below.

## 1.3 Main results.

The first result is about the existence and uniqueness of the solution of the Landau equation. Notice that in the subsequent analysis, the Knudsen number $\varepsilon$ is always supposed to be less than 1 and in our Cauchy theory, we have a smallness condition on our initial data, which is independent of $\varepsilon$.

**Theorem 1.1.** *There exists $M_0 > 0$ such that for $\varepsilon \in (0,1)$ and $\|g_{\varepsilon,0}\|_{H^3_x L^2_v} \leq M_0$, the Cauchy problem (3) admits a unique global solution*

$$g_\varepsilon \in L^\infty([0,\infty); H^3_x L^2_v)$$

*with the global energy estimate*

$$\sup_{t\geq 0} \mathcal{E}^2(t) + C_0 \int_0^\infty \frac{1}{\varepsilon^2} \mathcal{D}^2(t)\mathrm{d}t + C_0 \int_0^\infty \mathcal{C}^2(t)\mathrm{d}t \leq C_0' \mathcal{E}^2(0), \tag{12}$$

*where $C_0, C_0' > 0$ are independent of $\varepsilon$.*

In the second part of this work, we study the limit to the incompressible Navier-Stokes-Fourier system associated with the Boussinesq equation which writes

$$\begin{cases} \partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta_x \theta, \\ \nabla_x \cdot u = 0, \\ \rho + \theta = 0. \end{cases} \tag{13}$$

In this system $\theta$ (the temperature), $\rho$ (the density) and $p$ (the pressure) are scalar unknowns and $u$ (the velocity) is a 3-component unknown vector field. The pressure can actually be eliminated from the equations by applying to the momentum equation the Leray projector $\mathcal{P}$ onto the space of divergence free vector field (precisely which is defined in (96)). This projector is bounded over $H^N_x$ for all $N$, and in $L^p_x$ for all $1 < p < \infty$. The viscosity coefficients are fully determined by the linearized Landau operator $\mathcal{L}$ (see Section 4).
The derivation theorem from (3) to (13) is the following.

**Theorem 1.2.** *Let $M_0$ be as in Theorem 1.1. For any $\varepsilon \in (0,1)$, assume that the initial data $g_{\varepsilon,0}$ in (3) satisfy*

1) $g_{\varepsilon,0} \in H^3_x L^2_v$,

2) $\|g_{\varepsilon,0}\|_{H^3_x L^2_v} \leq M_0$,

3) *there exist scalar functions $\rho_0$, $\theta_0 \in H^3_x$ and vector-valued function $u_0 \in H^3_x$ such that*

$$g_{\varepsilon,0} \longrightarrow g_0, \quad \text{strongly in } H^3_x L^2_v \tag{14}$$

*as $\varepsilon \longrightarrow 0$, where $g_0(x,v)$ is of the form*

$$g_0(x,v) = \rho_0(x)\sqrt{\mu}(v) + u_0(x) \cdot v\sqrt{\mu}(v) + \theta_0(x)\Big(\frac{|v|^2}{2} - \frac{3}{2}\Big)\sqrt{\mu}(v). \tag{15}$$

*Let now $g_\varepsilon$ be the family of solutions to the Landau equation (3) constructed in Theorem 1.1. Then, as $\varepsilon \longrightarrow 0$,*

$$g_\varepsilon \longrightarrow \rho\sqrt{\mu} + u\cdot v\sqrt{\mu} + \theta\Big(\frac{|v|^2}{2} - \frac{3}{2}\Big)\sqrt{\mu} \tag{16}$$

*weakly-$\star$ in $L^\infty([0,\infty); H^3_x L^2_v)$. Here*

$$(\rho, u, \theta) \in C(\mathbb{R}^+; H^2_x) \cap L^\infty(\mathbb{R}^+; H^3_x) \tag{17}$$

*is a solution of the incompressible Navier-Stokes-Fourier equation (13) with initial data:*

$$u|_{t=0} = \mathcal{P}u_0(x), \;\; \theta|_{t=0} = \frac{3}{5}\theta_0(x) - \frac{2}{5}\rho_0(x), \tag{18}$$

*where $\mathcal{P}$ is the Leray projection. Moreover, the following convergence of the moments holds:*

$$\mathcal{P}\left(g_\varepsilon, v\sqrt{\mu}\right)_{L^2_v} \longrightarrow u, \tag{19}$$

$$\left(g_\varepsilon, \Big(\frac{|v|^2}{5} - 1\Big)\sqrt{\mu}\right)_{L^2_v} \longrightarrow \theta, \tag{20}$$

*strongly in $C(\mathbb{R}^+; H^2_x)$, weakly-$\star$ in $L^\infty(\mathbb{R}^+; H^3_x)$ as $\varepsilon \longrightarrow 0$.*

The history of derivation of Navier-Stokes equation from kinetic equations is very rich. This has been an active research field since the 70's with some major breakthrough. In particular Bardos, Golse and Levermore [4] initiated this program in the 80's, and the first convergence result without compactness assumption was given by Golse and Saint-Raymond in [13], following a series of hard works by Bardos, Golse, Levermore, Lions, Masmoudi and Saint-Raymond [3, 12, 23, 24] the list given here is not exhaustive. More recently this program was tackled in various geometries (with bondary in [18, 19, 26]). All these results are obtained in a framework of weak solutions: the renormalized solutions for the Boltzmann equation (from DiPerna-Lions [10] or Mischler [27] for bounded domains) and the Leray solutions for the Navier-Stokes equations. Let us also mention [2] and [29] in which the theory of derivation of macroscopic fluid equations from kinetic equations has been treated exhaustively (although the Landau equation is not studied there). In what follows, we present previous results obtained in a framework of strong solutions and also comment the results obtained in the present paper.

In this framework of strong solutions, we first refer to [1, 6, 7] for Boltzmann equation with cutoff (with inelastic collisions for [1]) and to [20] for Boltzmann equation without cutoff in which a result of weak convergence to the fluid model is obtained. Briant in [6] justified the convergence from the Boltzmann equation with cutoff in the case of hard or Maxwellian potential to the incompressible Navier-Stokes equations on the torus. He used hypocoercivity to obtain a proof of existence and exponential decay (uniformly in the Knudsen number) for solutions around a global equilibrium in Sobolev spaces $H^N_{x,v}$ for $N \geq 3$. These results allow him to obtain a derivation of the incompressible Navier-Stokes equations as the Knudsen number tends to 0. In addition, he obtained strong convergence by adding additional conditions on the initial

data and using a decomposition of the semigroup associated to the linearized equation coming from [11]. Roughly speaking, this analysis is based on the study of the spectrum of the Fourier transform in the space variable of the linearized operator. Let us also mention that all the hypocoercivity theory assumptions hold for several different kinetic models which include the Landau equation with hard, Maxwellian and moderately soft potentials. Briant, Merino-Aceituno and Mouhot in [7] obtained similar results in larger Sobolev spaces with polynomial weights using an "enlargement method" coming from [14] that takes into account the dependencies on the Knudsen number (which allowed them to weaken the assumptions on the data down to Sobolev spaces with polynomial weights). Notice that in the present paper, since our linearization is based on the following decomposition: $f_\varepsilon = \mu + \varepsilon\mu^{1/2}g_\varepsilon$, our original data $f_{\varepsilon,0}$ are supposed to enjoy an exponential decay.
We also mention that there is another type of results about the connection to fluid equations that are also obtained in the context of classical solutions. The general idea is to obtain first the solutions for the limiting fluid equations, then constructing a sequence of solutions (around the Maxwellian) of the scaled Kinetic equations (Boltzmann or Landau) for small Knudsen number $\varepsilon$. These solutions are of the form $f_\varepsilon = \mu+\varepsilon\mu^{1/2}(g^1+\varepsilon g^2+\cdots+\varepsilon^n g^n_\varepsilon)$, where $g^1, g^2, \cdots$ can be determined by the Hilbert expansion, and $g^n_\varepsilon$ is the error term. There is a large literature in this context, let us point out the work [17] in which Guo verified the diffusive expansion (around the Maxwellian) for the Boltzmann equation with angular cutoff (hard and soft potential) and for Landau equation in the Coulombian case on the torus using a nonlinear energy method. This expansion is beyond the Navier–Stokes limit in the following sense: the coefficient $g^1$ is determined by the incompressible Navier–Stokes–Fourier equations and $g^2 \cdots g^{n-1}$ are determined by the linearized incompressible Navier–Stokes–Fourier equations with source term from the known kinetic part. In addition, he studied the decay in time uniformly in $\varepsilon$ for the error term by nonlinear energy method, such a result is obtained in $H^N_{x,v}$ for $N \geq 8$.
Our aim in this article is to study the weak convergence process for the Landau equation in the cases of hard potentials, Maxwellian molecules and moderately soft potentials and verify its limit to the incompressible Navier-Stokes-Fourier equation with the Boussinesq relation on the torus. This equation has strong diffusion properties, which implies difficulties linked to the functional spaces. The convergence proof is based on a nonlinear energy method which is similar to [15, 17]. The idea is to establish microscopic and macroscopic estimates (thanks to the study of the classical 13-moments) uniformly in $\varepsilon$ for our equation. These estimates allow us to obtain a global energy estimate (uniformly in $\varepsilon$) in $H^3_x L^2_v$. Notice that in [15,17], this type of results was obtained in $H^N_{x,v}$ with $N \geq 8$, we have thus enlarge the space in which such a result is available. In terms of functional spaces, our result is comparable to the one in [20] in which the authors used this method to derive the incompressible Navier-Stokes-Fourier equation but from Boltzmann without cutoff on the whole space. Hence the difference with our case concerns the choice of the instant energy and the dissipation rate functional to capture the structure of the rescaled equation, see (6), (7) and (11) for our case. We shall sometimes follow the line of the proof of [20]. But we pay very much attention to the construction of local solutions (uniformly with respect to the Knudsen number), the linear problem being handled thanks to Lion's theorem (see [21]). A special focus is also given on the weak-$\star$ limit at the very end of this article using in particular the Aubin-Lions-Simon theorem and some compensated compactness argument from [22]

(see [1, 13] for similar approaches).
We finally emphasize that a great challenge in this field would be to get strong convergence, that we hope to do in the future using the hypoelliptic properties of the Landau operator.

**Organization of the article.** In Section 2, we study the construction of the local solution. In Section 3, we show the global existence and we establish the uniform energy estimate. Section 4 is devoted for the incompressible Navier-Stokes limit. In Appendix A, we give a proof for the existence of a local solution for the linear Landau equation.

# 2 Construction of Local Solutions

In this section, we show the existence of a local solution to equation (3). First, we start with nonlinear estimates where we use the idea of the proof of Lemma 3.5 in [8]. Then, we are interested in the construction of local solutions for Landau equation where we use the technique presented in section 2 in [20].

## 2.1 Nonlinear estimates.

We prove in this section some estimates on the nonlinear operator $\Gamma$.

**Lemma 2.1.** *We have:*

$$|(a_{ij}*\mu^{1/2}f)(v)| + \left|\left(a_{ij}*\frac{v_i}{2}\mu^{1/2}f\right)(v)\right| + |(a_{ij}*\mu^{1/2}\partial_j f)(v)| + \left|\left(a_{ij}*\frac{v_i}{2}\mu^{1/2}\partial_j f\right)(v)\right| \\ + |(a_{ij}*\mu^{1/2}f)(v)v_iv_j| + |(a_{ij}*\mu^{1/2}f)(v)v_j| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v}.$$

*Proof.* For the first term, we have

$$|(a_{ij}*\mu^{1/2}f)(v)| = \left|\int_{v_*} a_{ij}(v-v_*)\mu^{1/2}(v_*)f_*\right| \lesssim \int_{v_*}\langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+2}\mu_*^{1/2}|f_*| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v}.$$

In a similar way we get

$$|(a_{ij}*\frac{v_i}{2}\mu^{1/2}f)(v)| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v}.$$

For the third term, we have

$$a_{ij}*\mu^{1/2}\partial_j f = \partial_j a_{ij}*\mu^{1/2}f - a_{ij}*\partial_j\mu^{1/2}f,$$

then

$$\begin{aligned}|(a_{ij}*\mu^{1/2}\partial_j f)(v)| &\lesssim |\partial_j a_{ij}*\mu^{1/2}f(v)| + |a_{ij}*\partial_j\mu^{1/2}f(v)| \\ &\lesssim \int_{v_*}\langle v\rangle^{\gamma+1}\langle v_*\rangle^{\gamma+1}\mu_*^{1/2}|f_*| + \int_{v_*}\langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+3}\mu_*^{1/2}|f_*| \\ &\lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v}.\end{aligned}$$

In a similar way we get

$$\left|\left(a_{ij} * \frac{v_i}{2}\mu^{1/2}\partial_j f\right)(v)\right| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v}.$$

Recall that 0 is an eigenvalue of the matrix $A(v) = (a_{ij}(v))_{1\le i,j\le 3}$ with corresponding eigenvector $v$ so that $a_{ij}(v-v_*)v_j = a_{ij}(v-v_*)v_{*j}$ and $a_{ij}(v-v_*)v_iv_j = a_{ij}(v-v_*)v_{*i}v_{*j}$. Using this we can easily obtain,

$$\begin{aligned}|(a_{ij} * \mu^{1/2}f)(v)v_iv_j| &= \left|\int_{v_*} a_{ij}(v-v_*)v_iv_j\mu_*^{1/2}f_*\right| = \left|\int_{v_*} a_{ij}(v-v_*)v_{*i}v_{*j}\mu_*^{1/2}f_*\right| \\ &\lesssim \int_{v_*} \langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+4}\mu_*^{1/2}|f_*| \\ &\lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v}.\end{aligned}$$

In a similar way we get

$$|(a_{ij} * \mu^{1/2}f)(v)v_j| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v}.$$

□

**Lemma 2.2.** *The following estimate holds:*

$$(\Gamma(f,g),h)_{L^2_v} \lesssim \|f\|_{L^2_v}\|g\|_{H^1_{v,*}}\|h\|_{H^1_{v,*}}. \tag{21}$$

*Proof.* We write

$$\begin{aligned}(\Gamma(f,g),h)_{L^2_v} &= -\int (a_{ij} * \mu^{1/2}f)\partial_j g\partial_i h - \int \left(a_{ij} * \frac{v_i}{2}\mu^{1/2}f\right)\partial_j g\, h \\ &\quad + \int (a_{ij} * \mu^{1/2}\partial_j f)g\partial_i h + \int \left(a_{ij} * \frac{v_i}{2}\mu^{1/2}\partial_j f\right)gh \\ &= -\int (A(v) * \mu^{1/2}f)\nabla_v g\cdot\nabla_v h - \int X(v)\cdot\nabla_v g\, h \\ &\quad + \int Y(v)\cdot\nabla_v h\, g + \int \left(a_{ij} * \frac{v_i}{2}\mu^{1/2}\partial_j f\right)gh \\ &=: T_1 + T_2 + T_3 + T_4.\end{aligned}$$

Here $X(v)$ (resp. $Y(v)$) are vectors with coefficients $X_j(v)$ (resp. $Y_i(v)$) which are written in the following form:

$$X_j(v) = \sum_{i=1}^3 a_{ij} * \frac{v_i}{2}\mu^{1/2}f \quad\text{and}\quad Y_i(v) = \sum_{j=1}^3 a_{ij} * \mu^{1/2}\partial_j f.$$

Step 1. For the first term, since the estimate for $|v|\le 1$ is evident, we only consider the case $|v|\ge 1$. We decompose $\nabla_v g = P_v\nabla_v g + (I-P_v)\nabla_v g$, and similarly for $\nabla_v h$ where we recall that $P_v\nabla_v g = v|v|^{-2}(v\cdot\nabla_v g)$. We hence write

$$\begin{aligned}T_1 &= \int (A(v) * \mu^{1/2}f)P_v\nabla_v g\cdot P_v\nabla_v h + \int (A(v) * \mu^{1/2}f)P_v\nabla_v g\cdot(I-P_v)\nabla_v h \\ &\quad + \int (A(v) * \mu^{1/2}f)(I-P_v)\nabla_v g\cdot P_v\nabla_v h + \int (A(v) * \mu^{1/2}f)(I-P_v)\nabla_v g\cdot(I-P_v)\nabla_v h \\ &:= T_{11} + T_{12} + T_{13} + T_{14}.\end{aligned}$$

Therefore we have

$$T_{11} = \int (A(v) * \mu^{1/2} f) \frac{(v \cdot \nabla_v g)}{|v|^2} v \cdot \frac{(v \cdot \nabla_v h)}{|v|^2} v,$$

thanks to Lemma 2.1, we obtain

$$\begin{aligned} |T_{11}| &\lesssim \|f\|_{L^2_v} \int \langle v \rangle^{\gamma+2} |v|^{-2} \, |\nabla_v g| \, |\nabla_v h| \\ &\lesssim \|f\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L^2_v}. \end{aligned}$$

Moreover

$$T_{12} = \int (A(v) * \mu^{1/2} f) \frac{(v \cdot \nabla_v g)}{|v|^2} v \cdot (I - P_v) \nabla_v h,$$

then

$$\begin{aligned} |T_{12}| &\lesssim \|f\|_{L^2_v} \int \langle v \rangle^{\gamma+2} |v|^{-1} \, |\nabla_v g| \, |(I - P_v) \nabla_v h| \\ &\lesssim \|f\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v h\|_{L^2_v}. \end{aligned}$$

Similarly

$$|T_{13}| \lesssim \|f\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v h\|_{L^2_v}.$$

For the term $T_{14}$ we obtain

$$T_{14} = \int (A(v) * \mu^{1/2} f) (I - P_v) \nabla_v g \cdot (I - P_v) \nabla_v h,$$

then

$$\begin{aligned} |T_{14}| &\lesssim \|f\|_{L^2_v} \int \langle v \rangle^{\gamma+2} |(I - P_v) \nabla_v g| \, |(I - P_v) \nabla_v h| \\ &\lesssim \|f\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v g\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v h\|_{L^2_v}. \end{aligned}$$

Step 2. Let us investigate the second term $T_2$, and again we only consider $|v| > 1$. The same argument as for $T_1$ gives us

$$\begin{aligned} T_2 &= - \int X(v) \cdot \{P_v \nabla_v g + (I - P_v) \nabla_v g\} h \\ &:= T_{21} + T_{22}. \end{aligned}$$

We have

$$T_{21} = - \int X(v) \cdot v \frac{(v \cdot \nabla_v g)}{|v|^2} \, h,$$

then

$$\begin{aligned} |T_{21}| &\lesssim \|f\|_{L^2_v} \int \langle v \rangle^{\gamma+2} |v|^{-1} \, |\nabla_v g| \, |h| \\ &\lesssim \|f\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L^2_v} \|\langle v \rangle^{\frac{\gamma}{2}+1} h\|_{L^2_v}. \end{aligned}$$

For the other term we get

$$T_{22} = -\int X(v)\cdot (I-P_v)\nabla_v g\, h$$

then

$$\begin{aligned}|T_{22}| &\lesssim \|f\|_{L^2_v}\int \langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v g|\,|h| \\ &\lesssim \|f\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}(I-P_v)\nabla_v g\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}h\|_{L^2_v}.\end{aligned}$$

Step 3. For the term $T_3$,

$$\begin{aligned}T_3 &= -\int Y(v)\cdot\{P_v\nabla_v h + (I-P_v)\nabla_v h\}g \\ &:= T_{31}+T_{32}.\end{aligned}$$

Using the same proof in step 2, we have

$$T_{31}\lesssim \|f\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}g\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v h\|_{L^2_v},$$

and

$$|T_{32}|\lesssim \|f\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}g\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}(I-P_v)\nabla_v h\|_{L^2_v}.$$

Step 4. We finally investigate the term $T_4$ and we have

$$T_4 = -\int\Big(a_{ij}*\frac{v_i}{2}\mu^{1/2}\partial_j f\Big)h\, g,$$

then

$$|T_4|\lesssim \|f\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}g\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}h\|_{L^2_v}.$$

□

The next lemma gives an estimate on the nonlinear collision operator $\Gamma$ in terms of instant energy functional and dissipation rate.

**Lemma 2.3.** *Consider $f$ such that $\int_{\mathbb{T}^3}\Pi_0 f\,\mathrm{d}x = 0$, we have*

$$(\Gamma(f,f),h)_{H^3_x L^2_v}\lesssim \mathcal{E}(f)\{\mathcal{C}(f)+\mathcal{D}(f)\}\mathcal{D}(h), \tag{22}$$

*therefore*

$$\|\Gamma(f,f)\|_{H^3_x(H^1_{v,*})'}\lesssim \|f\|_{H^3_x L^2_v}\|f\|_{H^3_x H^1_{v,*}}, \tag{23}$$

*where we recall that the space $H^3_x(H^1_{v,*})'$ is defined in (8) and $\mathcal{E},\mathcal{D},\mathcal{C}$ are defined in (11).*

*Proof.* We have that

$$(\Gamma(f,f),h)_{H^3_xL^2_v} = (\Gamma(f,f),h_2)_{L^2_xL^2_v} + \sum_{1\le|\beta|\le 3} \left(\partial^\beta_x\Gamma(f,g),\partial^\beta_x h_2\right)_{L^2_xL^2_v}$$

because

$$(\Gamma(f,f),\boldsymbol{\phi})_{L^2_v} = 0 \quad \text{for } \boldsymbol{\phi} = \sqrt{\mu}, v_i\sqrt{\mu}, |v|^2\sqrt{\mu}$$

and

$$\partial^\beta_x\Gamma(f,f) = \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2}\Gamma(\partial^{\beta_1}_x f, \partial^{\beta_2}_x f).$$

The proof of the lemma is a consequence of Lemma 2.2 together with the following inequalities, that we shall use in the sequel when integrating in $x\in\mathbb{T}^3$

$$\|u\|_{L^\infty(\mathbb{T}^3)} \lesssim \|u\|_{H^2(\mathbb{T}^3)}, \quad \|u\|_{L^6(\mathbb{T}^3)} \lesssim \|u\|_{H^1(\mathbb{T}^3)}, \quad \|u\|_{L^3(\mathbb{T}^3)} \lesssim \|u\|^{1/2}_{H^1(\mathbb{T}^3)}\|u\|^{1/2}_{L^2(\mathbb{T}^3)}.$$

Step 1. Using Lemma 2.2 we easily get,

$$\begin{aligned}(\Gamma(f,f),h_2)_{L^2_xL^2_v} &\lesssim \int_{\mathbb{T}^3}\left(\|f\|_{L^2_v}\|f_2\|_{H^1_{v,*}}\|h_2\|_{H^1_{v,*}} + \|f\|_{L^2_v}\|f_1\|_{L^2_v}\|h_2\|_{H^1_{v,*}}\right)\\ &\lesssim \|f\|_{H^2_xL^2_v}\|f_2\|_{L^2_xH^1_{v,*}}\|h_2\|_{L^2_xH^1_{v,*}} + \|f\|_{H^2_xL^2_v}\|f_1\|_{L^2_xL^2_v}\|h_2\|_{L^2_xH^1_{v,*}},\end{aligned}$$

furthermore, $\Pi_0 f$ has zero mean on the torus, thus we can apply Poincaré inequality on the torus and we obtain

$$(\Gamma(f,f),h_2)_{L^2_xL^2_v} \lesssim \mathcal{E}(f)\{\mathcal{C}(f)+\mathcal{D}(f)\}\mathcal{D}(h).$$

Step 2. Case $|\beta|=1$. Arguing as in the previous step, from Lemma 2.2, it follows

$$\begin{aligned}&\left(\Gamma(f,\partial^\beta_x f),\partial^\beta_x h_2\right)_{L^2_xL^2_v}\\ &\lesssim \int_{\mathbb{T}^3}\left(\|f\|_{L^2_v}\|\nabla_x f_2\|_{H^1_{v,*}}\|\nabla_x h_2\|_{H^1_{v,*}} + \|f\|_{L^2_v}\|\nabla_x f_1\|_{L^2_v}\|\nabla_x h_2\|_{H^1_{v,*}}\right)\\ &\lesssim \|f\|_{H^2_xL^2_v}\|\nabla_x f_2\|_{L^2_xH^1_{v,*}}\|\nabla_x h_2\|_{L^2_xH^1_{v,*}} + \|f\|_{H^2_xL^2_v}\|\nabla_x f_1\|_{L^2_xL^2_v}\|\nabla_x h_2\|_{L^2_xH^1_{v,*}}\\ &\lesssim \mathcal{E}(f)\{\mathcal{C}(f)+\mathcal{D}(f)\}\mathcal{D}(h).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\left(\Gamma(\partial^\beta_x f,f),\partial^\beta_x h_2\right)_{L^2_xL^2_v}\\ &\lesssim \int_{\mathbb{T}^3}\left(\|\nabla_x f\|_{L^2_v}\|f_2\|_{H^1_{v,*}}\|\nabla_x h_2\|_{H^1_{v,*}} + \|\nabla_x f\|_{L^2_v}\|f_1\|_{L^2_v}\|\nabla_x h_2\|_{H^1_{v,*}}\right)\\ &\lesssim \|\nabla_x f\|_{H^2_xL^2_v}\|f_2\|_{L^2_xH^1_{v,*}}\|\nabla_x h_2\|_{L^2_xH^1_{v,*}} + \|\nabla_x f\|_{H^2_xL^2_v}\|f_1\|_{L^2_xL^2_v}\|\nabla_x h_2\|_{L^2_xH^1_{v,*}},\end{aligned}$$

using Poincaré inequality on the torus, we get

$$\left(\Gamma(\partial^\beta_x f,f),\partial^\beta_x h_2\right)_{L^2_xL^2_v} \lesssim \mathcal{E}(f)\{\mathcal{C}(f)+\mathcal{D}(f)\}\mathcal{D}(h).$$

Step 3. Case $|\beta|=2$. When $\beta_2=\beta$ we have

$$\begin{aligned}&\left(\Gamma(f,\partial^\beta_x f),\partial^\beta_x h_2\right)_{L^2_xL^2_v}\\ &\lesssim \int_{\mathbb{T}^3}\left(\|f\|_{L^2_v}\|\nabla^2_x f_2\|_{H^1_{v,*}}\|\nabla^2_x h_2\|_{H^1_{v,*}} + \|f\|_{L^2_v}\|\nabla^2_x f_1\|_{L^2_v}\|\nabla^2_x h_2\|_{H^1_{v,*}}\right)\\ &\lesssim \|f\|_{H^2_xL^2_v}\|\nabla^2_x f_2\|_{L^2_xH^1_{v,*}}\|\nabla^2_x h_2\|_{L^2_xH^1_{v,*}} + \|f\|_{H^2_xL^2_v}\|\nabla^2_x f_1\|_{L^2_xL^2_v}\|\nabla^2_x h_2\|_{L^2_xH^1_{v,*}}\\ &\lesssim \mathcal{E}(f)\{\mathcal{C}(f)+\mathcal{D}(f)\}\mathcal{D}(h).\end{aligned}$$

If $|\beta_1| = |\beta_2| = 1$ then we obtain

$$
\begin{aligned}
&\Big(\Gamma(\partial_x^{\beta_1} f, \partial_x^{\beta_2} f), \partial_x^{\beta} h_2\Big)_{L^2_x L^2_v} \\
&\lesssim \int_{\mathbb{T}^3} \Big( \|\nabla_x f\|_{L^2_v} \, \|\nabla_x f_2\|_{H^1_{v,*}} \, \|\nabla_x^2 h_2\|_{H^1_{v,*}} + \|\nabla_x f\|_{L^2_v} \, \|\nabla_x f_1\|_{L^2_v} \, \|\nabla_x^2 h_2\|_{H^1_{v,*}} \Big) \\
&\lesssim \|\nabla_x f\|_{H^2_x L^2_v} \, \|\nabla_x f_2\|_{L^2_x H^1_{v,*}} \, \|\nabla_x^2 h_2\|_{L^2_x H^1_{v,*}} + \|\nabla_x f\|_{H^2_x L^2_v} \, \|\nabla_x f_1\|_{L^2_x L^2_v} \, \|\nabla_x^2 h_2\|_{L^2_x H^1_{v,*}} \\
&\lesssim \mathcal{E}(f)\{\mathcal{C}(f) + \mathcal{D}(f)\}\mathcal{D}(h).
\end{aligned}
$$

Finally, when $\beta_1 = \beta$ we get

$$
\begin{aligned}
&\Big(\Gamma(\partial_x^{\beta} f, f), \partial_x^{\beta} h_2\Big)_{L^2_x L^2_v} \\
&\lesssim \int_{\mathbb{T}^3} \Big( \|\nabla_x^2 f\|_{L^2_v} \, \|f_2\|_{H^1_{v,*}} \, \|\nabla_x^2 h_2\|_{H^1_{v,*}} + \|\nabla_x^2 f\|_{L^2_v} \, \|f_1\|_{L^2_v} \, \|\nabla_x^2 h_2\|_{H^1_{v,*}} \Big) \\
&\lesssim \|\nabla_x^2 f\|_{L^6_x L^2_v} \, \|f_2\|_{L^3_x H^1_{v,*}} \, \|\nabla_x^2 h_2\|_{L^2_x H^1_{v,*}} + \|\nabla_x^2 f\|_{L^6_x L^2_v} \, \|f_1\|_{L^3_x L^2_v} \, \|\nabla_x^2 h_2\|_{L^2_x H^1_{v,*}} \\
&\lesssim \|\nabla_x^2 f\|_{H^1_x L^2_v} \, \|f_2\|^{1/2}_{L^2_x H^1_{v,*}} \, \|f_2\|^{1/2}_{H^1_x H^1_{v,*}} \, \|\nabla_x^2 h_2\|_{L^2_x H^1_{v,*}} \\
&\qquad + \|\nabla_x^2 f\|_{H^1_x L^2_v} \, \|f_1\|^{1/2}_{L^2_x L^2_v} \, \|f_1\|^{1/2}_{H^1_x H^1_{v,*}} \, \|\nabla_x^2 h_2\|_{L^2_x H^1_{v,*}} \\
&\lesssim \mathcal{E}(f)\{\mathcal{C}(f) + \mathcal{D}(f)\}\mathcal{D}(h).
\end{aligned}
$$

Step 4. Case $|\beta| = 3$. When $\beta_2 = \beta$ we have obtained

$$
\begin{aligned}
&\Big(\Gamma(f, \partial_x^{\beta} f), \partial_x^{\beta} h_2\Big)_{L^2_x L^2_v} \\
&\lesssim \|f\|_{H^2_x L^2_v} \, \|\nabla_x^3 f_2\|_{L^2_x H^1_{v,*}} \, \|\nabla_x^3 h_2\|_{L^2_x H^1_{v,*}} + \|f\|_{H^2_x L^2_v} \, \|\nabla_x^3 f_1\|_{L^2_x L^2_v} \, \|\nabla_x^3 h_2\|_{L^2_x H^1_{v,*}} \\
&\lesssim \mathcal{E}(f)\{\mathcal{C}(f) + \mathcal{D}(f)\}\mathcal{D}(h).
\end{aligned}
$$

If $|\beta_1| = 1$ and $|\beta_2| = 2$ then we obtain

$$
\begin{aligned}
&\Big(\Gamma(\partial_x^{\beta_1} f, \partial_x^{\beta_2} f), \partial_x^{\beta} h_2\Big)_{L^2_x L^2_v} \\
&\lesssim \int_{\mathbb{T}^3} \Big( \|\nabla_x f\|_{L^2_v} \, \|\nabla_x^2 f_2\|_{H^1_{v,*}} \, \|\nabla_x^3 h_2\|_{H^1_{v,*}} + \|\nabla_x f\|_{L^2_v} \, \|\nabla_x^2 f_1\|_{L^2_v} \, \|\nabla_x^3 h_2\|_{H^1_{v,*}} \Big) \\
&\lesssim \|\nabla_x f\|_{H^2_x L^2_v} \, \|\nabla_x^2 f_2\|_{L^2_x H^1_{v,*}} \, \|\nabla_x^3 h_2\|_{L^2_x H^1_{v,*}} + \|\nabla_x f\|_{H^2_x L^2_v} \, \|\nabla_x^2 f_1\|_{L^2_x L^2_v} \, \|\nabla_x^3 h_2\|_{L^2_x H^1_{v,*}} \\
&\lesssim \mathcal{E}(f)\{\mathcal{C}(f) + \mathcal{D}(f)\}\mathcal{D}(h).
\end{aligned}
$$

When $|\beta_1| = 2$ and $|\beta_2| = 1$ then we get

$$
\begin{aligned}
&\Big(\Gamma(\partial_x^{\beta_1} f, \partial_x^{\beta_2} f), \partial_x^{\beta} h_2\Big)_{L^2_x L^2_v} \\
&\lesssim \int_{\mathbb{T}^3} \Big( \|\nabla_x^2 f\|_{L^2_v} \, \|\nabla_x f_2\|_{H^1_{v,*}} \, \|\nabla_x^3 h_2\|_{H^1_{v,*}} + \|\nabla_x^2 f\|_{L^2_v} \, \|\nabla_x f_1\|_{L^2_v} \, \|\nabla_x^3 h_2\|_{H^1_{v,*}} \Big) \\
&\lesssim \|\nabla_x^2 f\|_{H^1_x L^2_v} \, \|\nabla_x f_2\|^{1/2}_{L^2_x L^2_v} \, \|\nabla_x f_2\|^{1/2}_{H^1_x H^1_{v,*}} \, \|\nabla_x^3 h_2\|_{L^2_x H^1_{v,*}} \\
&\qquad + \|\nabla_x f\|_{H^1_x L^2_v} \, \|\nabla_x f_1\|^{1/2}_{L^2_x L^2_v} \, \|\nabla_x f_1\|^{1/2}_{H^1_x L^2_v} \, \|\nabla_x^3 h_2\|_{L^2_x H^1_{v,*}} \\
&\lesssim \mathcal{E}(f)\{\mathcal{C}(f) + \mathcal{D}(f)\}\mathcal{D}(h).
\end{aligned}
$$

Finally, when $\beta_1 = \beta$, it follows

$$
\begin{aligned}
&\Big(\Gamma(\partial_x^{\beta} f, f), \partial_x^{\beta} h_2\Big)_{L^2_x L^2_v} \\
&\lesssim \int_{\mathbb{T}^3} \Big( \|\nabla_x^3 f\|_{L^2_v} \, \|f_2\|_{H^1_{v,*}} \, \|\nabla_x^3 h_2\|_{H^1_{v,*}} + \|\nabla_x^3 f\|_{L^2_v} \, \|f_1\|_{L^2_v} \, \|\nabla_x^3 h_2\|_{H^1_{v,*}} \Big) \\
&\lesssim \|\nabla_x^3 f\|_{L^2_x L^2_v} \, \|f_2\|_{H^2_x H^1_{v,*}} \, \|\nabla_x^3 h_2\|_{L^2_x H^1_{v,*}} + \|\nabla_x^3 f\|_{L^2_x L^2_v} \, \|f_1\|_{H^2_x L^2_v} \, \|\nabla_x^3 h_2\|_{L^2_x H^1_{v,*}} \\
&\lesssim \mathcal{E}(f)\{\mathcal{C}(f) + \mathcal{D}(f)\}\mathcal{D}(h).
\end{aligned}
$$

Regarding estimate (23): Using (22), we have

$$(\Gamma(f,f),h)_{H^3_xL^2_v} \lesssim \|f\|_{H^3_xL^2_v}\|f\|_{H^3_xH^1_{v,*}}\|h\|_{H^3_xH^1_{v,*}},$$

using now (8), we obtain for $\|h\|_{H^3_xH^1_{v,*}} \leq 1$,

$$\|\Gamma(f,f)\|_{H^3_x(H^1_{v,*})'} \lesssim \|f\|_{H^3_xL^2_v}\|f\|_{H^3_xH^1_{v,*}}.$$

□

## 2.2 Linear problem

We consider now the linear Cauchy problem

$$\begin{cases} \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g + \frac{1}{\varepsilon^2}\mathcal{L} g = \frac{1}{\varepsilon}\Gamma(f,f) \\ g_{|t=0} = g_0 \in H^3_xL^2_v \end{cases} \tag{24}$$

where $f$ is a given function such that

$$\sup_{0\leq t\leq T} \mathcal{E}^2(f(t)) + \int_0^T \mathcal{D}^2(f(t))\mathrm{d}t < \infty \tag{25}$$

for some $T > 0$ and verifies

$$\int_{\mathbb{T}^3} \Pi_0 f \,\mathrm{d}x = 0. \tag{26}$$

We study the existence of a solution to the equation (24) in the space $L^\infty([0,T]; H^3_xL^2_v) \cap L^2([0,T]; H^3_xH^1_{v,*})$.

**Remark 2.4.** We note that by using (23), (25) and (26), we obtain that $\Gamma(f,f) \in L^2([0,T]; H^3_x(H^1_{v,*})')$.

**Definition 2.5.** We call weak solution of (24) any function $g \in L^2([0,T]; H^3_xL^2_v)$ satisfying

$$\int_0^T (g, \mathcal{Q}\phi)_{H^3_xL^2_v}\,\mathrm{d}t - (g_0, \phi(0))_{H^3_xL^2_v} = \int_0^T \left\langle \frac{1}{\varepsilon}\Gamma(f,f), \phi\right\rangle_{H^3_x(H^1_{v,*})', H^3_xH^1_{v,*}} \mathrm{d}t \tag{27}$$

for all $\phi \in C^\infty_c([0,T[\times\mathbb{T}^3_x\times\mathbb{R}^3_v)$, where $\mathcal{Q}$ is defined in (142) in Appendix A.

**Proposition 2.6.** *Let $g_0 \in H^3_xL^2_v$, and $f$ satisfy (25) and (26). Then the Cauchy problem (24) admits a weak solution*

$$g \in L^\infty([0,T]; H^3_xL^2_v) \cap L^2([0,T]; H^3_xH^1_{v,*})$$

*which satisfies*

$$\sup_{0\leq t\leq T} \mathcal{E}^2(g(t)) + \frac{C_\gamma}{\varepsilon^2}\int_0^T \mathcal{D}^2(g(t))\mathrm{d}t \leq \frac{4C^2}{C_\gamma}\sup_{0\leq t\leq T}\mathcal{E}^2(f)\left\{\sup_{0\leq t\leq T}\mathcal{E}^2(f) + \int_0^T \mathcal{D}^2(f)\mathrm{d}t\right\} + 2\mathcal{E}^2(g_0) \tag{28}$$

*for some $T > 0$, where $C, C_\gamma$ are independent of $0 < \varepsilon \leq 1$.*

*Proof.* We will show the existence of a solution to the Cauchy problem (24) by using Proposition A.1 in Appendix A. By applying Proposition A.1 with $\mathrm{U}_\varepsilon = \frac{1}{\varepsilon}\Gamma(f,f)$ (we have from Remark 2.4 $\frac{1}{\varepsilon}\Gamma(f,f) \in L^2([0,T]; H^3_x(H^1_{v,*})')$ for $\varepsilon$ a fixed parameter, then there exists a solution $g \in L^2([0,T]; H^3_x H^1_{v,*})$ such that for any $\phi \in C^\infty_c([0,T[\times\mathbb{T}^3_x\times\mathbb{R}^3_v)$

$$\int_0^T (g, \mathcal{Q}\phi)_{H^3_x L^2_v}\,\mathrm{d}t = \int_0^T \left\langle \frac{1}{\varepsilon}\Gamma(f,f), \phi \right\rangle_{H^3_x(H^1_{v,*})', H^3_x H^1_{v,*}} \mathrm{d}t + (g_0, \phi(0))_{H^3_x L^2_v}, \tag{29}$$

where $\mathcal{Q}$ is defined in (142). Then, $g \in L^2([0,T]; H^3_x H^1_{v,*})$ is a weak solution of the Cauchy problem (24). Now, we will show that the weak solution $g$ satisfies (28). We only sketch the proof which could be done using mollifiers of $g$ and supposing therefore $\partial_t g, v\cdot\nabla_x g$ and $\mathcal{L}g$ are in $L^2([0,T]; H^3_x L^2_v)$ and that $g$ and its derivatives in $x$ have traces in times. For elements of proof see [9, Appendix A], [16] and [20]. We define the operator $\mathcal{G}$ by

$$\mathcal{G}g := \partial_t g + \frac{1}{\varepsilon} v\cdot\nabla_x g + \frac{1}{\varepsilon^2}\mathcal{L}g = \frac{1}{\varepsilon}\Gamma(f,f), \tag{30}$$

we also suppose $\mathcal{G}g \in L^2([0,T]; H^3_x L^2_v)$. We have $g \in L^2([0,T]; H^3_x L^2_v)$, then

$$(\mathcal{G}g, g)_{L^2([0,T];H^3_x L^2_v)} = \frac{1}{\varepsilon}(\Gamma(f,f), g)_{L^2([0,T];H^3_x L^2_v)}.$$

Using (30) and the fact that $v\cdot\nabla_x$ is skew-adjoint, we have

$$\begin{aligned}(\mathcal{G}g, g)_{H^3_x L^2_v} &= (\partial_t g, g)_{H^3_x L^2_v} + \frac{1}{\varepsilon}(v\cdot\nabla_x g, g)_{H^3_x L^2_v} + \frac{1}{\varepsilon^2}(\mathcal{L}g, g)_{H^3_x L^2_v} \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{H^3_x L^2_v} + \frac{1}{\varepsilon^2}(\mathcal{L}g, g)_{H^3_x L^2_v}.\end{aligned} \tag{31}$$

Returning to Theorem 1.2 in [28], we have the following estimate:

$$(\mathcal{L}f, f)_{L^2_v} \geq C_\gamma \|f\|^2_{H^1_{v,*}}, \ \ \forall f \in N(\mathcal{L})^\perp, \tag{32}$$

where $C_\gamma > 0$. Then,

$$\frac{1}{\varepsilon^2}(\mathcal{L}g, g)_{H^3_x L^2_v} \geq \frac{C_\gamma}{\varepsilon^2}\mathcal{D}^2(g). \tag{33}$$

From the above, we get for $t \in ]0,T[$,

$$\int_0^t (\mathcal{G}g, g)_{H^3_x L^2_v}\,\mathrm{d}s \geq \frac{1}{2}\|g(t)\|^2_{H^3_x L^2_v} - \frac{1}{2}\|g(0)\|^2_{H^3_x L^2_v} + \frac{C_\gamma}{\varepsilon^2}\int_0^t \mathcal{D}^2(g)\mathrm{d}s.$$

Using Lemma 2.3 and the fact that $\mathcal{C}(f) \leq \mathcal{E}(f)$, we have

$$\begin{aligned}&\frac{1}{2}\|g(t)\|^2_{H^3_x L^2_v} + \frac{C_\gamma}{\varepsilon^2}\int_0^t \mathcal{D}^2(g)\mathrm{d}s \\ &\leq \frac{C}{\varepsilon}\int_0^t \mathcal{E}(f)\{\mathcal{E}(f) + \mathcal{D}(f)\}\mathcal{D}(g)\mathrm{d}s + \frac{1}{2}\mathcal{E}^2(g_0) \\ &\leq \frac{C}{\varepsilon}\int_0^t \{\mathcal{E}^2(f) + \mathcal{E}(f)\mathcal{D}(f)\}\mathcal{D}(g)\mathrm{d}s + \frac{1}{2}\mathcal{E}^2(g_0) \\ &\leq \frac{C^2}{C_\gamma}\int_0^t \mathcal{E}^2(f)\{\mathcal{E}^2(f) + \mathcal{D}^2(f)\}\mathrm{d}s + \frac{C_\gamma}{2\varepsilon^2}\int_0^t \mathcal{D}^2(g)\mathrm{d}s + \frac{1}{2}\mathcal{E}^2(g_0).\end{aligned}$$

Thus, we get

$$\mathcal{E}^2(g(t)) + \frac{C_\gamma}{\varepsilon^2}\int_0^t \mathcal{D}^2(g(t))\mathrm{d}s \le \frac{2C^2}{C_\gamma}\int_0^T \mathcal{E}^2(f)\{\mathcal{E}^2(f)+\mathcal{D}^2(f)\}\mathrm{d}s + \mathcal{E}^2(g_0).$$

This implies that for $T \le 1$

$$\sup_{0\le t\le T}\mathcal{E}^2(g(t)) + \frac{C_\gamma}{\varepsilon^2}\int_0^T \mathcal{D}^2(g(t))\mathrm{d}t \le \frac{4C^2}{C_\gamma}\sup_{0\le t\le T}\mathcal{E}^2(f)\left\{\sup_{0\le t\le T}\mathcal{E}^2(f) + \int_0^T \mathcal{D}^2(f)\mathrm{d}t\right\} + 2\mathcal{E}^2(g_0).$$

Finally, (28) and (29) show that $g \in L^\infty([0,T]; H^3_x L^2_v) \cap L^2([0,T]; H^3_x H^1_{v,*})$ is a weak solution of the Cauchy problem (24). □

## 2.3 Local existence for nonlinear problem.

We consider now the following iteration

$$\begin{cases} \partial_t g^{n+1} + \frac{1}{\varepsilon} v\cdot\nabla_x g^{n+1} + \frac{1}{\varepsilon^2}\mathcal{L} g^{n+1} = \frac{1}{\varepsilon}\Gamma(g^n, g^n) \\ g^{n+1}|_{t=0} = g_0, \end{cases} \tag{34}$$

with $g^0 = 0$.

**Proposition 2.7.** *There exists $0 < \delta_0 \le 1$, $0 < T \le 1$, such that for any $0 < \varepsilon \le 1$, $g_0 \in H^3_x L^2_v$ with*

$$\|g_0\|_{H^3_x L^2_v} \le \delta_0$$

*then the iteration problem (34) admits a sequence of solutions $\{g^n\}_{n\ge 1}$ satisfying*

$$\sup_{0\le t\le T}\mathcal{E}^2(g^n) + \frac{C_\gamma}{\varepsilon^2}\int_0^T \mathcal{D}^2(g^n)\mathrm{d}t \le 4\delta_0^2. \tag{35}$$

*Proof.* For the linear Cauchy problem (34), given $g^n$ satisfying (35), the existence of $g^{n+1}$ is obtained by the Proposition 2.6. So that it is enough to prove (35) by induction. Using estimate (28) with $g^n = f$ and $g^{n+1} = g$, we obtain

$$\sup_{0\le t\le T}\mathcal{E}^2(g^{n+1}) + \frac{C_\gamma}{\varepsilon^2}\int_0^T \mathcal{D}^2(g^{n+1})\mathrm{d}t \le \frac{4C^2}{C_\gamma}\sup_{0\le t\le T}\mathcal{E}^2(g^n)\{\sup_{0\le t\le T}\mathcal{E}^2(g^n) + \int_0^T \mathcal{D}^2(g^n)\mathrm{d}t\} + 2\mathcal{E}^2(g_0).$$

Then, using (35), we obtain

$$\sup_{0\le t\le T}\mathcal{E}^2(g^{n+1}) + \frac{C_\gamma}{\varepsilon^2}\int_0^T \mathcal{D}^2(g^{n+1})\mathrm{d}t \le \delta_0^2\Big(2 + 64\,\delta_0^2\frac{C^2}{C_\gamma}\Big).$$

We complete the proof of the Proposition by choosing $\delta_0$ such that

$$2 + 64\,\delta_0^2\frac{C^2}{C_\gamma} \le 4.$$

□

**Theorem 2.8.** *For $T > 0$, such that for any $0 < \varepsilon < 1$, $g_{\varepsilon,0} \in H^3_x L^2_v$ with*

$$\|g_{\varepsilon,0}\|_{H^3_x L^2_v} \le \delta_0,$$

*then the Cauchy problem (3) admits a unique solution $g \in L^\infty([0,T]; H^3_x L^2_v)$ satisfying*

$$\sup_{t\in[0,T]} \mathcal{E}^2(g_\varepsilon) + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(g_\varepsilon) \mathrm{d}t \le 4\delta_0^2. \tag{36}$$

*Proof.* We will show that $\{g^n\}$ defined in Proposition 2.7 is a Cauchy sequence in $L^\infty([0,T]; L^2(\mathbb{T}^3_x \times \mathbb{R}^3_v))$. We set $w^n = g^{n+1} - g^n$ and deduce from (34),

$$\begin{cases} \partial_t w^n + \frac{1}{\varepsilon} v \cdot \nabla_x w^n + \frac{1}{\varepsilon^2} \mathcal{L} w^n = \frac{1}{\varepsilon}[\Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1})] \\ w^n{}_{|t=0} = 0. \end{cases} \tag{37}$$

Using the fact that, for any $h \in L^2$

$$\begin{aligned} &\Big(\Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}), \Pi_0 h\Big)_{L^2_v} = 0, \\ &\Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}) = \Gamma(g^n, g^n - g^{n-1}) + \Gamma(g^n, g^{n-1}) - \Gamma(g^{n-1}, g^{n-1}) \\ &\qquad\qquad\qquad\qquad\qquad\quad = \Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}), \end{aligned}$$

we obtain

$$\Big(\Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}), w^n\Big)_{L^2_{x,v}} = \Big(\Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}), w^n_2\Big)_{L^2_{x,v}},$$

where $w^n_2 = (I - \Pi_0) w^n$. Using Lemma 2.2, we have

$$\begin{aligned} \frac{1}{\varepsilon}\left|\Big(\Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}), w^n_2\Big)_{L^2_{x,v}}\right| &\le \frac{C}{\varepsilon} \mathcal{E}(g^n)\Big(\|w^{n-1}_1\|_{L^2_{x,v}} + \|w^{n-1}_2\|_{\chi^0}\Big)\|w^n_2\|_{\chi^0} \\ &\quad + \frac{C}{\varepsilon}\|w^{n-1}\|_{L^2_{x,v}}\Big(\|g^{n-1}_1\|_{H^3_x L^2_v} + \mathcal{D}(g^{n-1})\Big)\|w^n_2\|_{\chi^0} \\ &\le C_\delta \mathcal{E}^2(g^n)\Big(\|w^{n-1}_1\|^2_{L^2_{x,v}} + \|w^{n-1}_2\|^2_{\chi^0}\Big) \\ &\quad + C_\delta \|w^{n-1}\|^2_{L^2_{x,v}}\Big(\|g^{n-1}_1\|^2_{H^3_x L^2_v} + \mathcal{D}^2(g^{n-1})\Big) \\ &\quad + \frac{\delta}{\varepsilon^2}\|w^n_2\|^2_{\chi^0}. \end{aligned}$$

Thus, fix a small $\delta > 0$, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^n\|^2_{L^2_{x,v}} + \frac{C_\gamma}{2\varepsilon^2}\|w^n_2\|^2_{\chi^0} &\le C_\delta \mathcal{E}^2(g^n)\Big(\|w^{n-1}_1\|^2_{L^2_{x,v}} + \|w^{n-1}_2\|^2_{\chi^0}\Big) \\ &\quad + C_\delta \|w^{n-1}\|^2_{L^2_{x,v}}\Big(\|g^{n-1}_1\|^2_{H^3_x L^2_v} + \mathcal{D}^2(g^{n-1})\Big). \end{aligned}$$

Using the fact that

$$\|w^{n-1}_1\|^2_{L^2_{x,v}} = \|\Pi_0 w^{n-1}\|^2_{L^2_{x,v}} \le C\|w^{n-1}\|^2_{L^2_{x,v}}, \quad \|g^{n-1}_1\|^2_{H^3_x L^2_v} \le C\mathcal{E}^2(g^{n-1}),$$

we obtain $t \in ]0, T[$,

$$\begin{aligned} &\|w^n\|^2_{L^2_{x,v}} + \frac{1}{\varepsilon^2}\int_0^t \|w^n_2\|^2_{\chi^0}\,\mathrm{d}s \\ &\le C \sup_{t\in[0,T]} \mathcal{E}^2(g^n)\Big(T\|w^{n-1}\|^2_{L^\infty([0,T];L^2_{x,v})} + \int_0^T \|w^{n-1}_2\|^2_{\chi^0}\,\mathrm{d}s\Big) \\ &+ C\|w^{n-1}\|^2_{L^\infty([0,T];L^2_{x,v})}\Big(T \sup_{t\in[0,T]} \mathcal{E}^2(g^{n-1}) + \int_0^T \mathcal{D}^2(g^{n-1})\,\mathrm{d}s\Big). \end{aligned}$$

Using now (35) with $\delta_0 > 0$ small enough, we get that for any $0 < \varepsilon \le 1$

$$\|w^n\|^2_{L^\infty([0,T];L^2_{x,v})} + \frac{1}{\varepsilon^2}\int_0^T \|w^n_2\|^2_{\mathcal{X}^0}\,\mathrm{d}t \le \frac{1}{2}\Big(\|w^{n-1}\|^2_{L^\infty([0,T];L^2_{x,v})} + \frac{1}{\varepsilon^2}\int_0^T \|w^{n-1}_2\|^2_{\mathcal{X}^0}\,\mathrm{d}t\Big). \tag{38}$$

Thus we have proved that $g^n$ is a Cauchy sequence in $L^\infty([0,T]; L^2_{x,v})$. Combining with the estimate (35), the limit $g$ is in $L^\infty([0,T]; H^3_x L^2_v)$. If we note

$$\mathcal{F}(g^n) = \sup_{0\le t\le T} \mathcal{E}^2(g^n) + \frac{C_\gamma}{\varepsilon^2}\int_0^T \mathcal{D}^2(g^n)\mathrm{d}t,$$

then from weak lower semicontinuity, we have

$$\mathcal{F}(g) \le \liminf_{n\to\infty} \mathcal{F}(g^n) \le 4\delta_0^2.$$

Then we obtain the estimate (36). Now, we will show the uniqueness of the local solution. Let $g^{(1)}_\varepsilon$ and $g^{(2)}_\varepsilon$ be two solutions to (3) with same initial data $g^{(1)}_{\varepsilon,0} = g^{(2)}_{\varepsilon,0}$ that satisfy (36). The difference $\mathbf{h}_\varepsilon = g^{(1)}_\varepsilon - g^{(2)}_\varepsilon$ satisfies

$$\begin{cases} \partial_t \mathbf{h}_\varepsilon + \frac{1}{\varepsilon} v\cdot\nabla_x \mathbf{h}_\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L}\mathbf{h}_\varepsilon = \frac{1}{\varepsilon}\Gamma(g^{(2)}_\varepsilon, \mathbf{h}_\varepsilon) + \frac{1}{\varepsilon}\Gamma(\mathbf{h}_\varepsilon, g^{(1)}_\varepsilon) \\ \mathbf{h}_{\varepsilon|t=0} = 0, \end{cases} \tag{39}$$

using the same argument in Theorem 3.10 in [8], we can show that $\mathbf{h}_\varepsilon = 0$, hence the uniqueness of the weak solution. □

# 3 Uniform estimate and global solutions

In this section, we establish microscopic and macroscopic energy estimates for the Landau equation. These estimates with Theorem 2.8 allow to prove the existence of global solutions in the space $L^\infty([0,\infty); H^3_x L^2_v)$.

## 3.1 Microscopic energy estimate.

**Proposition 3.1.** *Let $g \in L^\infty([0,T]; H^3_x L^2_v)$ be a solution of the equation (3) constructed in Theorem 2.8, then there exists a constant $C$ independent of $\varepsilon$ such that the following estimate holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^2 + \frac{C_\gamma}{\varepsilon^2}\mathcal{D}^2 \le C\left\{\frac{1}{\varepsilon}\mathcal{E}\mathcal{D}^2 + (\mathcal{E}\mathcal{C})^2\right\}, \tag{40}$$

*where for simplicity, we note $\mathcal{E} = \mathcal{E}(g)$, $\mathcal{D} = \mathcal{D}(g)$ and $\mathcal{C} = \mathcal{C}(g)$.*

*Proof.* We apply $\partial^\alpha_x$ to (3) and take the $L^2(\mathbb{T}^3\times\mathbb{R}^3)$ inner product with $\partial^\alpha_x g$. Since the innerproduct including $v\cdot\nabla_x g$ vanishes by integration by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^2 + \frac{1}{\varepsilon^2}\sum_{|\alpha|\le 3}\left(\mathcal{L}\partial^\alpha_x g, \partial^\alpha_x g\right)_{L^2_{x,v}} = \frac{1}{\varepsilon}\sum_{|\alpha|\le 3}\left(\partial^\alpha_x\Gamma(g,g), \partial^\alpha_x g\right)_{L^2_{x,v}}.$$

Returning to Theorem 1.2 in [28], we have

$$\sum_{|\alpha|\le 3}(\mathcal{L}\partial_x^\alpha g,\partial_x^\alpha g)_{L^2_{x,v}}\ge C_\gamma\|g_2\|^2_{\chi^3(\mathbb{T}^3_x\times\mathbb{R}^3_v)}=C_\gamma\mathcal{D}^2.$$

Since $\int_{\mathbb{T}^3}\Pi_0 g\,\mathrm{d}x=0$, Lemma 2.3 implies that,

$$\left|\frac{1}{\varepsilon}\sum_{|\alpha|\le 3}(\partial_x^\alpha\Gamma(g,g),\partial_x^\alpha g)_{L^2_{x,v}}\right|\le\frac{C_1}{\varepsilon}\mathcal{E}(\mathcal{C}\mathcal{D}+\mathcal{D}^2)\le\frac{C_1}{\varepsilon}\mathcal{E}\mathcal{D}^2+\frac{C_1}{2\eta}(\mathcal{E}\mathcal{C})^2+\frac{\eta}{2}\frac{C_1}{\varepsilon^2}\mathcal{D}^2.$$

Taking $\eta=\dfrac{C_\gamma}{C_1}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^2+\frac{C_\gamma}{2}\frac{1}{\varepsilon^2}\mathcal{D}^2\le\frac{C_1}{\varepsilon}\mathcal{E}\mathcal{D}^2+\frac{C_1^2}{2C_\gamma}(\mathcal{E}\mathcal{C})^2,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^2+\frac{C_\gamma}{\varepsilon^2}\mathcal{D}^2\le C\left\{\frac{1}{\varepsilon}\mathcal{E}\mathcal{D}^2+(\mathcal{E}\mathcal{C})^2\right\}.$$

□

## 3.2 Macroscopic energy estimates

We study now the energy estimate for the macroscopic part $\Pi_0 g$ where $g$ is a solution of the equation (3). First we decompose the equation (3) into microscopic and macroscopic parts, i.e. rewrite it into the following equation

$$\partial_t\{a+bv+c|v|^2\}\mu^{1/2}+\frac{1}{\varepsilon}v\cdot\nabla_x\{a+bv+c|v|^2\}\mu^{1/2}=-\partial_t g_2-\frac{1}{\varepsilon}v\cdot\nabla_x g_2-\frac{1}{\varepsilon^2}\mathcal{L}g_2+\frac{1}{\varepsilon}\Gamma(g,g). \tag{41}$$

**Lemma 3.2.** *Let $\partial^\alpha=\partial_x^\alpha$, $\alpha\in\mathbb{N}^3$, $|\alpha|\le 2$. If $g$ is a solution of the Landau equation (3), and $\mathcal{A}=(a,b,c)$ defined in (10), then*

$$\varepsilon\|\partial_t\partial^\alpha\mathcal{A}\|_{L^2_x}\lesssim\mathcal{C}+\mathcal{D}, \tag{42}$$

*where $\mathcal{E}=\mathcal{E}(g)$ and $\mathcal{D}=\mathcal{D}(g)$.*

*Proof.* Let $g$ be a solution to solution of the scaled Landau equation (3). The second set of equations we consider are the local conservation laws satisfied by $(a,b,c)$. To derive these we multiply (41) by the collision invariants that are the elements of $\mathcal{N}(\mathcal{L})$ in (5) and integrate only in the velocity variables to obtain

$$\partial_t(a+3c)+\frac{1}{\varepsilon}\nabla_x\cdot b=0, \tag{43}$$

$$\partial_t b+\frac{1}{\varepsilon}(\nabla_x a+5\nabla_x c)=-\frac{1}{\varepsilon}\left(v\cdot\nabla_x g_2,v\sqrt{\mu}\right)_{L^2_v}, \tag{44}$$

$$\partial_t(3a+15c)+\frac{5}{\varepsilon}\nabla_x\cdot b=-\frac{1}{\varepsilon}\left(v\cdot\nabla_x g_2,|v|^2\sqrt{\mu}\right)_{L^2_v}. \tag{45}$$

Taking linear combinations of the first and third local conservation laws results in

$$\begin{aligned}\partial_t a &= \frac{1}{2\varepsilon}\left(v\cdot\nabla_x g_2, |v|^2\sqrt{\mu}\right)_{L^2_v},\\ \partial_t b + \frac{1}{\varepsilon}(\nabla_x a + 5\nabla_x c) &= -\frac{1}{\varepsilon}\left(v\cdot\nabla_x g_2, v\sqrt{\mu}\right)_{L^2_v},\\ \partial_t c + \frac{1}{3\varepsilon}\nabla_x\cdot b &= -\frac{1}{6\varepsilon}\left(v\cdot\nabla_x g_2, |v|^2\sqrt{\mu}\right)_{L^2_v}.\end{aligned}$$

These are the local conservation laws that we will study below. From the first equation, we have

$$\|\partial_t\partial^\alpha a\|_{L^2_x} \lesssim \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha g_2\|_{L^2_x L^2_v}, \tag{46}$$

the second equation gives that

$$\begin{aligned}\|\partial_t\partial^\alpha b\|_{L^2_x} &\lesssim \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha a\|_{L^2_x} + \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha c\|_{L^2_x} + \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha g_2\|_{L^2_x L^2_v}\\ &\lesssim \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha \mathcal{A}\|_{L^2_x} + \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha g_2\|_{L^2_x L^2_v}\end{aligned}$$

and the last equation implies that

$$\begin{aligned}\|\partial_t\partial^\alpha c\|_{L^2_x} &\lesssim \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha b\|_{L^2_x} + \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha g_2\|_{L^2_x L^2_v}\\ &\lesssim \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha \mathcal{A}\|_{L^2_x} + \frac{1}{\varepsilon}\|\nabla_x\partial^\alpha g_2\|_{L^2_x L^2_v}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\varepsilon\|\partial_t\partial^\alpha \mathcal{A}\|_{L^2_x} &\lesssim \|\nabla_x\partial^\alpha \mathcal{A}\|_{L^2_x} + \|\nabla_x\partial^\alpha g_2\|_{L^2_x L^2_v}\\ &\lesssim \mathcal{C} + \mathcal{D}.\end{aligned}$$

□

We start with the macroscopic energy estimate where we use the so-called 13-moments. The set of 13-moments is 13-dimensional subspace of $L^2(\mathbb{R}^3_v)$ and given by

$$\{e_j\}_{j=1}^{13} = \left\{\sqrt{\mu}, v_i\sqrt{\mu}, v_i v_j\sqrt{\mu}, v_i|v|^2\sqrt{\mu}\right\}.$$

It is well-known [16] that the macroscopic component $g_1 = \Pi_0 g \sim \mathcal{A} = (a,b,c)$, satisfies the following set of equations

$$\begin{cases} v_i|v|^2\mu^{1/2} : & \frac{1}{\varepsilon}\nabla_x c = -\partial_t r_c + \frac{1}{\varepsilon}m_c + \frac{1}{\varepsilon^2}\ell_c + \frac{1}{\varepsilon}h_c,\\ v_i^2\mu^{1/2} : & \partial_t c + \frac{1}{\varepsilon}\partial_i b_i = -\partial_t r_i + \frac{1}{\varepsilon}m_i + \frac{1}{\varepsilon^2}\ell_i + \frac{1}{\varepsilon}h_i,\\ v_i v_j\mu^{1/2} : & \frac{1}{\varepsilon}\partial_i b_j + \frac{1}{\varepsilon}\partial_j b_i = -\partial_t r_{ij} + \frac{1}{\varepsilon}m_{ij} + \frac{1}{\varepsilon^2}\ell_{ij} + \frac{1}{\varepsilon}h_{ij}, \quad i\neq j,\\ v_i\mu^{1/2} : & \partial_t b_i + \frac{1}{\varepsilon}\partial_i a = -\partial_t r_{bi} + \frac{1}{\varepsilon}m_{bi} + \frac{1}{\varepsilon^2}\ell_{bi} + \frac{1}{\varepsilon}h_{bi},\\ \mu^{1/2} : & \partial_t a = -\partial_t r_a + \frac{1}{\varepsilon}m_a + \frac{1}{\varepsilon^2}\ell_a + \frac{1}{\varepsilon}h_a \end{cases} \tag{47}$$

where

$$\begin{aligned} &r = (g_2, e)_{L^2_v}, \quad m = (-v\cdot\nabla_x g_2, e)_{L^2_v}, \quad h = (\Gamma(g,g), e)_{L^2_v},\\ &\ell = -(\mathcal{L}g_2, e)_{L^2_v} \end{aligned} \tag{48}$$

stand for $r_c, \cdots, h_a$, while

$$e \in \mathrm{Span}\{v_i|v|^2\mu^{1/2}, v_i^2\mu^{1/2}, v_i v_j\mu^{1/2}, v_i\mu^{1/2}, \mu^{1/2}\}, \text{ for } i,j = 1,2,3.$$

**Lemma 3.3.** *Let $r, m, \ell, h$ be the ones defined by (48), $\partial^\alpha = \partial_x^\alpha$, $\partial_i = \partial_{x_i}$ $\alpha \in \mathbb{N}^3$, $|\alpha| \leq 2$. Then, one has*

$$\|\partial_i \partial^\alpha r\|_{L^2_x} \lesssim \min\{\|g_2\|_{H^3_x L^2_v}, \mathcal{D}\}, \tag{49}$$

$$\|\partial^\alpha m\|_{L^2_x} \lesssim \min\{\|g_2\|_{H^3_x L^2_v}, \mathcal{D}\}, \tag{50}$$

$$\|\partial^\alpha \ell\|_{L^2_x} \lesssim \min\{\|g_2\|_{H^2_x L^2_v}, \|g_2\|_{\chi^2(\mathbb{T}^3_x \times \mathbb{R}^3_v)}\}, \tag{51}$$

$$\|\partial^\alpha h\|_{L^2_x} \lesssim \|g\|_{H^2_x L^2_v}(\|g_2\|_{\chi^2(\mathbb{T}^3_x \times \mathbb{R}^3_v)} + \|\nabla_x \Pi_0 g\|_{H^1_x L^2_v}). \tag{52}$$

*Proof.* We have

$$\|\partial_i \partial^\alpha r\|_{L^2_x} = \|\,(\partial_i \partial^\alpha g_2, e)_{L^2_v}\,\|_{L^2_x} \lesssim \|\partial_i \partial^\alpha g_2\|_{L^2_x L^2_v}\,\|e\|_{L^2_v} \lesssim \|g_2\|_{H^3_x L^2_v},$$

$$\|\partial^\alpha \ell\|_{L^2_x} = \|\,(\partial^\alpha g_2, \mathcal{L}e)_{L^2_v}\,\|_{L^2_x} \lesssim \|\partial^\alpha g_2\|_{L^2_x L^2_v}\,\|\mathcal{L}e\|_{L^2_v} \lesssim \|g_2\|_{H^2_x L^2_v},$$

$$\|\partial^\alpha m\|_{L^2_x} = \|\,(\nabla_x \partial^\alpha g_2, ve)_{L^2_v}\,\|_{L^2_x} \lesssim \|\nabla_x \partial^\alpha g_2\|_{L^2_x L^2_v}\,\|ve\|_{L^2_v} \lesssim \|g_2\|_{H^3_x L^2_v}.$$

The fact that for $f \in H^1_{v,*}$ and $\gamma \in [-2, 1]$,

$$\|f\|_{L^2_v(\langle v\rangle^{1+\gamma/2})} \leq \|f\|_{H^1_{v,*}},$$

implies (49), (50) and (51). For the estimate (52), we have

$$\partial^\alpha h = (\partial^\alpha \Gamma(g, g), e)_{L^2_v},$$

then using the same method of proof of Lemma 2.3, we get (52). □

**Lemma 3.4.** *Let $|\alpha| \leq 2$, and let $g$ be a solution of the scaled Landau equation (3). Then there exists a positive constant $\tilde{C}$ independent of $\varepsilon$, such that the following estimate holds:*

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{|\alpha|\leq 2}(\partial^\alpha r, \nabla_x \partial^\alpha(a,b,c))_{L^2_x} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2_x}\right\} + \mathcal{C}^2 \leq \tilde{C}\left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{E}(\mathcal{C}^2 + \mathcal{D}^2)\right\}. \tag{53}$$

*Proof.* Recall that

$$\mathcal{C}^2 = \sum_{|\alpha|\leq 2}\|\nabla_x \partial^\alpha \mathcal{A}\|^2_{L^2_x} = \sum_{|\alpha|\leq 2}\|\nabla_x \partial^\alpha a\|^2_{L^2_x} + \|\nabla_x \partial^\alpha b\|^2_{L^2_x} + \|\nabla_x \partial^\alpha c\|^2_{L^2_x}.$$

(a) **Estimate of $\nabla_x \partial^\alpha a$.** From the macroscopic equations (47),

$$\begin{aligned}
\|\nabla_x \partial^\alpha a\|^2_{L^2_x} &= (\nabla_x \partial^\alpha a, \nabla_x \partial^\alpha a)_{L^2_x} \\
&= \left(\partial^\alpha\Big(-\varepsilon \partial_t b - \varepsilon \partial_t r + m + \frac{1}{\varepsilon}\ell + h\Big), \nabla_x \partial^\alpha a\right)_{L^2_x} \\
&\lesssim \varepsilon R_1 + |\,(\partial^\alpha m, \nabla_x \partial^\alpha a)_{L^2_x}\,| + \frac{1}{\varepsilon}|\,(\partial^\alpha \ell, \nabla_x \partial^\alpha a)_{L^2_x}\,| + |\,(\partial^\alpha h, \nabla_x \partial^\alpha a)_{L^2_x}\,|.
\end{aligned}$$

Here,

$$\begin{aligned}\varepsilon R_1 &= -\varepsilon \left(\partial^\alpha \partial_t b + \partial^\alpha \partial_t r, \nabla_x \partial^\alpha a\right)_{L^2_x} \\ &= -\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \left(\partial^\alpha (b+r), \nabla_x \partial^\alpha a\right)_{L^2_x} - \varepsilon \left(\nabla_x \partial^\alpha (b+r), \partial_t \partial^\alpha a\right)_{L^2_x}.\end{aligned}$$

Note that, from (46) and (49) $\varepsilon \left(\nabla_x \partial^\alpha r, \partial_t \partial^\alpha a\right)_{L^2_x} \lesssim \mathcal{D}^2$, and

$$\varepsilon \left(\nabla_x \partial^\alpha b, \partial_t \partial^\alpha a\right)_{L^2_x} \lesssim \eta \|\nabla_x \partial^\alpha b\|^2_{L^2_x} + \frac{1}{2\eta}\mathcal{D}^2.$$

Furthermore, Lemma 3.3 implies that

$$\begin{aligned}&|\left(\partial^\alpha m, \nabla_x \partial^\alpha a\right)_{L^2_x}| \lesssim \mathcal{D} \|\nabla_x \partial^\alpha a\|_{L^2_x} \lesssim \mathcal{D} \|\nabla_x \partial^\alpha \mathcal{A}\|_{H^2_x} \lesssim \mathcal{D}\mathcal{C}, \\ &\frac{1}{\varepsilon}|\left(\partial^\alpha \ell, \nabla_x \partial^\alpha a\right)_{L^2_x}| \lesssim \frac{1}{\varepsilon}\|g_2\|_{\chi^2(\mathbb{T}^3_x\times\mathbb{R}^3_v)} \|\nabla_x \partial^\alpha \mathcal{A}\|_{H^2_x} \lesssim \frac{1}{\varepsilon}\mathcal{D}\mathcal{C}, \\ &|\left(\partial^\alpha h, \nabla_x \partial^\alpha a\right)_{L^2_x}| \lesssim \mathcal{E}(\mathcal{C}+\mathcal{D})\mathcal{C}.\end{aligned}$$

Hence, for some small $0<\eta<1$

$$\begin{aligned}\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \left(\partial^\alpha (b+r), \nabla_x \partial^\alpha a\right)_{L^2_x} + \|\nabla_x \partial^\alpha a\|^2_{L^2_x} &\lesssim \eta \|\nabla_x \partial^\alpha b\|^2_{L^2_x} + \frac{1}{2\eta}\mathcal{D}^2 + \mathcal{D}\mathcal{C} \\ &\quad + \frac{1}{\varepsilon}\mathcal{D}\mathcal{C} + \mathcal{E}(\mathcal{C}+\mathcal{D})\mathcal{C} \\ &\lesssim \eta\mathcal{C}^2 + \frac{1}{\eta}\left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{E}(\mathcal{C}^2+\mathcal{D}^2)\right\}.\end{aligned} \tag{54}$$

(b) **Estimate of $\nabla_x \partial^\alpha b$.** Recall $b=(b_1,b_2,b_3)$. From the macroscopic equations (47),

$$\begin{aligned}\Delta_x \partial^\alpha b + \partial_i^2 \partial^\alpha b_i &= \partial^\alpha \Big[\sum_{j\neq i} \partial_j(\partial_j b_i + \partial_i b_j) + \partial_i(2\partial_i b_i - \sum_{j\neq i}\partial_j b_j)\Big] \\ &= \partial^\alpha \Big[\sum_{j\neq i} \partial_j \Big(-\varepsilon \partial_t r_{ij} + m_{ij} + \frac{1}{\varepsilon}\ell_{ij} + h_{ij}\Big) \\ &\qquad + \partial_i \Big(-2\varepsilon \partial_t r_i + 2m_i + \frac{2}{\varepsilon}\ell_i + 2h_i + \sum_{j\neq i} \varepsilon \partial_t r_j - m_j - \frac{1}{\varepsilon}\ell_j - h_j\Big)\Big] \\ &= \partial^\alpha \Big[\partial_i \Big(-\varepsilon \partial_t r + m + \frac{1}{\varepsilon}\ell + h\Big)\Big]\end{aligned}$$

where $r, \ell, h$ stand for linear combinations of $r_i, \ell_i, h_i$ and $r_{ij}, \ell_{ij}, h_{ij}$ for $i,j=1,2,3$ respectively. Then

$$\begin{aligned}\|\nabla_x \partial^\alpha b_i\|^2_{L^2_x} + \|\partial_i \partial^\alpha b_i\|^2_{L^2_x} &= -\left(\Delta_x \partial^\alpha b + \partial_i^2 \partial^\alpha b_i, \partial^\alpha b_i\right)_{L^2_x} \\ &= \varepsilon R_2 + R_3 + R_4 + R_5,\end{aligned}$$

where

$$\begin{aligned}\varepsilon R_2 &= \varepsilon \left(\partial_i \partial^\alpha \partial_t r, \partial^\alpha b_i\right)_{L^2_x} \\ &= -\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \left(\partial^\alpha r, \partial_i \partial^\alpha b_i\right)_{L^2_x} + \varepsilon \left(\partial^\alpha r, \partial_t \partial_i \partial^\alpha b_i\right)_{L^2_x},\end{aligned}$$

moreover, we have from (42) and (49) that

$$\varepsilon|\,(\partial_i\partial^\alpha r,\partial_t\partial^\alpha b_i)_{L^2_x}\,|\lesssim\eta\mathcal{C}^2+\frac{1}{\eta}\mathcal{D}^2.$$

Furthermore, Lemma 3.3 implies that

$$\begin{aligned}
|\,(\partial^\alpha m,\partial_i\partial^\alpha b_i)_{L^2_x}\,| &\lesssim \frac{1}{\eta}\|\partial^\alpha m\|^2_{L^2_x}+\eta\|\partial_i\partial^\alpha b_i\|^2_{L^2_x}\\
&\lesssim \frac{1}{\eta}\|\partial^\alpha m\|^2_{L^2_x}+\eta\|\nabla_x\partial^\alpha b_i\|^2_{L^2_x}\\
&\lesssim \frac{1}{\eta}\mathcal{D}^2+\eta\|\nabla_x\partial^\alpha b_i\|^2_{L^2_x},\\
\frac{1}{\varepsilon}|\,(\partial^\alpha \ell,\partial_i\partial^\alpha b_i)_{L^2_x}\,| &\lesssim \frac{1}{\varepsilon}\|\partial^\alpha\ell\|_{L^2_x}\|\nabla_x\partial^\alpha\mathcal{A}\|_{H^2_x}\lesssim\frac{1}{\varepsilon}\mathcal{D}\mathcal{C},\\
|\,(\partial^\alpha h,\partial_i\partial^\alpha b_i)_{L^2_x}\,| &\lesssim \mathcal{E}(\mathcal{C}+\mathcal{D})\mathcal{C}\\
&\lesssim \mathcal{E}(\mathcal{C}^2+\mathcal{D}^2).
\end{aligned}$$

Hence, for some small $0<\eta<1$

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\,(\partial^\alpha r,\partial_i\partial^\alpha b_i)_{L^2_x}+\|\nabla_x\partial^\alpha b_i\|^2_{L^2_x}\lesssim\eta\mathcal{C}^2+\frac{1}{\eta}\left\{\frac{1}{\varepsilon^2}\mathcal{D}^2+\mathcal{E}(\mathcal{C}^2+\mathcal{D}^2)\right\},$$

summing up for $1\le i\le 3$, we obtain that

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\,(\partial^\alpha r,\nabla_x\partial^\alpha b)_{L^2_x}+\|\nabla_x\partial^\alpha b\|^2_{L^2_x}\lesssim\eta\mathcal{C}^2+\frac{1}{\eta}\left\{\frac{1}{\varepsilon^2}\mathcal{D}^2+\mathcal{E}(\mathcal{C}^2+\mathcal{D}^2)\right\}. \tag{55}$$

(c) **Estimate of $\nabla_x\partial^\alpha c$.** From the macroscopic equations (47),

$$\begin{aligned}
\|\nabla_x\partial^\alpha c\|^2_{L^2_x} &= (\nabla_x\partial^\alpha c,\nabla_x\partial^\alpha c)_{L^2_x}\\
&=\Big(\partial^\alpha(-\varepsilon\partial_t r+m+\frac{1}{\varepsilon}\ell+h),\nabla_x\partial^\alpha c\Big)_{L^2_x}\\
&\lesssim \varepsilon R_6+\eta\mathcal{C}^2+\frac{1}{\eta}\mathcal{D}^2+\frac{1}{\eta}\frac{1}{\varepsilon^2}\mathcal{D}^2+\mathcal{E}(\mathcal{C}^2+\mathcal{D}^2),
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon R_6 &= -\varepsilon\,(\partial^\alpha\partial_t r,\nabla_x\partial^\alpha c)_{L^2_x}\\
&= -\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\,(\partial^\alpha r,\nabla_x\partial^\alpha c)_{L^2_x}-\varepsilon\,(\nabla_x\partial^\alpha r,\partial_t\partial^\alpha c)_{L^2_x}\\
&\lesssim -\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\,(\partial^\alpha r,\nabla_x\partial^\alpha c)_{L^2_x}+\eta\mathcal{C}^2+\frac{2}{\eta}\frac{1}{\varepsilon^2}\mathcal{D}^2.
\end{aligned}$$

Thus

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\,(\partial^\alpha r,\nabla_x\partial^\alpha c)_{L^2_x}+\|\nabla_x\partial^\alpha c\|^2_{L^2_x}\lesssim\eta\mathcal{C}^2+\frac{1}{\eta}\left\{\frac{1}{\varepsilon^2}\mathcal{D}^2+\mathcal{E}(\mathcal{C}^2+\mathcal{D}^2)\right\}. \tag{56}$$

By combining the above estimates (54), (55), (56) and taking $\eta>0$ sufficiently small, then we get the estimate (53) uniformly for $0<\varepsilon<1$ , thus complete the proof of Lemma 3.4 . $\square$

Let

$$E := \left[\mathcal{E}^2 + \eta_1 \varepsilon \left\{ \sum_{|\alpha|\le 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a,b,c))_{L^2_x} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2_x} \right\}\right]^{1/2}, \tag{57}$$

with $\eta_1 > 0$ to be chosen later.

**Theorem 3.5 (Global Energy Estimate).** *If $g$ is a solution of the scaled Landau equation (3), then there exist constants $c_0, c_3 > 0$ independents of $\varepsilon$ such that if $E \le 1$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} E^2 + c_3 \Big( \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \Big) \le c_0 E \left\{ \frac{1}{\varepsilon} \mathcal{D}^2 + \mathcal{C}^2 \right\} \tag{58}$$

*holds as far as $g$ exists.*

*Proof.* Based on the microscopic estimate (40) and the macroscopic estimate (53), we can derive the uniform energy estimate. Indeed, estimates (40) and (53), imply that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left( \mathcal{E}^2 + \eta_1 \varepsilon \left\{ \sum_{|\alpha|\le 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a,b,c))_{L^2_x} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2_x} \right\} \right) + \frac{C_\gamma}{\varepsilon^2} \mathcal{D}^2 + \eta_1 \mathcal{C}^2 \\
&\le C \Big( \frac{1}{\varepsilon} \mathcal{E} \mathcal{D}^2 + (\mathcal{E}\mathcal{C})^2 \Big) + \frac{\eta_1 \tilde{C}}{\varepsilon^2} \mathcal{D}^2 + \eta_1 \tilde{C} \mathcal{E} (\mathcal{C}^2 + \mathcal{D}^2).
\end{aligned}$$

We first choose $\eta_1$ small enough so that $C_\gamma - \eta_1 \tilde{C} > 0$, it gives that for $0 < \varepsilon < 1$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \mathcal{E}^2 + \eta_1 \varepsilon \sum_{|\alpha|\le 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a,b,c))_{L^2_x} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2_x} \right\} + (C_\gamma - \eta_1 \tilde{C}) \frac{1}{\varepsilon^2} \mathcal{D}^2 + \eta_1 \mathcal{C}^2 \\
&\le (C + \eta_1 \tilde{C}) \frac{1}{\varepsilon} \mathcal{E} \mathcal{D}^2 + (C\mathcal{E} + \eta_1 \tilde{C}) \mathcal{E} \mathcal{C}^2.
\end{aligned} \tag{59}$$

Using (49), we have

$$\begin{aligned}
\Big| \sum_{|\alpha|\le 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a,b,c))_{L^2_x} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2_x} \Big| &\lesssim \|\nabla_x \partial^\alpha r\|^2_{L^2_x} + \|\mathcal{A}\|^2_{H^3_x} \\
&\lesssim \|g_2\|^2_{H^3_x L^2_v} + \|\mathcal{A}\|^2_{H^3_x} \lesssim \mathcal{E}^2.
\end{aligned}$$

Thus, we can choose $\eta_1 > 0$ small such that, for any $0 < \varepsilon < 1$

$$c_1 \, \mathcal{E} \le E \le c_2 \, \mathcal{E}, \tag{60}$$

for some positive constants $c_1$ and $c_2$. Finally, using (59) and (60) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E^2 + (C_\gamma - \eta_1 \tilde{C}) \frac{1}{\varepsilon^2} \mathcal{D}^2 + \eta_1 \mathcal{C}^2 \le \Big( \frac{C + \eta_1 \tilde{C}}{c_1} \Big) \frac{1}{\varepsilon} E \mathcal{D}^2 + \Big( \frac{C}{c_1^2} + \frac{\eta_1 \tilde{C}}{c_1} \Big) E \mathcal{C}^2,$$

then there exist constants $c_0, c_3 > 0$ independent of $\varepsilon$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} E^2 + c_3 \Big( \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \Big) \le c_0 E \left\{ \frac{1}{\varepsilon} \mathcal{D}^2 + \mathcal{C}^2 \right\}.$$

□

We recall that $\delta_0$, $c_2$, $c_0$, $c_3$ are defined respectively in Proposition 2.7 and Theorem 3.5.

**Lemma 3.6.** *If we choose the initial data $g_{\varepsilon,0}$ such that*

$$\mathcal{E}(0) = \|g_{\varepsilon,0}\|_{H^3_x L^2_v} \leq M,$$

*where $M$ is defined as*

$$M = \min\left\{\delta_0, \frac{1}{4c_2}, \frac{c_3}{4c_0c_2}\right\}, \tag{61}$$

*then,*

$$E^2(T) + \frac{c_3}{2}\int_0^T \left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{C}^2\right\} \mathrm{d}t \leq E^2(0), \tag{62}$$

*for some $T > 0$.*

*Proof.* Note that $\mathcal{E}(0) \leq M \leq \delta_0$, then from Theorem 2.8 there exists a solution $g_\varepsilon \in L^\infty([0,T]; H^3_x L^2_v)$ for some $T > 0$, and from the local estimate (36), we have $\mathcal{E}(t) \leq 2M$ for $0 < t < T$. Note that on $[0,T]$, $E(t) \leq c_2\mathcal{E}(t) \leq 2c_2M < 1$. Then the global energy estimate (58) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^2 + (c_3 - 2c_0c_2M)\left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{C}^2\right\} \leq 0.$$

From the choice of $M$, $c_3 - 2c_0c_2M \geq \frac{c_3}{2}$. Thus

$$E^2(T) + \frac{c_3}{2}\int_0^T \left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{C}^2\right\} \mathrm{d}t \leq E^2(0).$$

□

**Proof of Theorem 1.1.** Now, we are ready to prove Theorem 1.1 by the usual continuation argument. We set $M_0 := \frac{c_1}{c_2}M$, where $M$ defined in (61). Let

$$\mathcal{E}(0) = \|g_{\varepsilon,0}\|_{H^3_x L^2_v} \leq M_0,$$

which implies that $\mathcal{E}(0) \leq M \leq \delta_0$. Then from Theorem 2.8 there exists a solution $g$ on $[0,T]$ for some $T > 0$. Furthermore, using (63), we have $E(T) \leq c_2M_0$ then $\mathcal{E}(T) \leq M \leq \delta_0$. Using again Theorem 2.8 with initial data $\mathcal{E}(T)$, we obtain the existence of the solution on $[T, 2T]$ and so on. Finally, the local solution constructed in Theorem 2.8 can be extended globally. Now, we will show the estimate (12). Using (63) and the fact that we can iterate the process in $[T,2T], [2T,3T]\ldots$, we get for all $t \in \mathbb{R}^+$

$$E^2(t) + \frac{c_3}{2}\int_0^t \left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{C}^2\right\} \mathrm{d}s \leq E^2(0). \tag{63}$$

We get therefore

$$\sup_{t\geq 0} E^2(t) + \frac{c_3}{2}\int_0^\infty \left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{C}^2\right\} \mathrm{d}s \leq 2E^2(0). \tag{64}$$

Next, using (60) we get

$$\sup_{t\geq 0} \mathcal{E}^2(t) + \frac{c_3}{2c_1^2}\int_0^\infty \left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{C}^2\right\} \mathrm{d}s \leq \frac{2c_2^2}{c_1^2}\mathcal{E}^2(0). \tag{65}$$

Finally, we can choose $C_0 = \dfrac{c_3}{2c_1^2}$ and $C_0' = \dfrac{2c_2^2}{c_1^2}$, hence the estimate (12) is true.

## 4 Limit to fluid incompressible Navier-Stokes-Fourier

In this section, we study the convergence of the perturbed Landau equation (3) to the fluid incompressible Navier-Stokes-Fourier system (13) as $\varepsilon \longrightarrow 0$. We will present a well explained and detailed proof. The approach is reminiscent of the one in [20] (see also [1] for a related problem) but here, we aim to provide a fully rigorous proof. Note that most of the arguments have already been used in [13] but our framework of strong solution allows us to develop a simpler proof than the one in [13].
In the rest of the paper, our convergences hold up to extracting subsequences. Note that for a vector function $\mathbf{w} = \mathbf{w}(x) \in \mathbb{R}^3$, $\nabla_x \mathbf{w}$ is a matrix defined by $(\partial_{x_j}\mathbf{w}_i)_{1\le i,j\le 3}$. We also write $(\nabla_x \cdot M)^i = \sum_j \partial_{x_j} M_{ij}(x)$, if $M$ is a matrix defined by $M = (M_{ij}(x))_{1\le i,j\le 3}$.

### 4.1 Local conservation laws

We first introduce the following fluid variables

$$\rho_\varepsilon = (g_\varepsilon, \sqrt{\mu})_{L^2_v}, \;\; u_\varepsilon = (g_\varepsilon, v\sqrt{\mu})_{L^2_v}, \;\; \theta_\varepsilon = \left(g_\varepsilon, \left(\frac{|v|^2}{3} - 1\right)\sqrt{\mu}\right)_{L^2_v}. \tag{66}$$

Then we can derive the following local conservation laws from the solutions $g_\varepsilon$ constructed in Theorem 1.1.

**Lemma 4.1.** *Assume that $g_\varepsilon$ is the solutions to the perturbed Landau equation (3) constructed in Theorem 1.1. Then the following local conservation laws hold*

$$\begin{cases} \partial_t \rho_\varepsilon + \frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon = 0, \\ \partial_t u_\varepsilon + \frac{1}{\varepsilon}\nabla_x(\rho_\varepsilon + \theta_\varepsilon) + \frac{1}{\varepsilon}\nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\right)_{L^2_v} = 0, \\ \partial_t \theta_\varepsilon + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot \left(\hat{B}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\right)_{L^2_v} = 0. \end{cases} \tag{67}$$

*Proof.* Step 1. Conservation law of $\rho_\varepsilon$. We multiply the first $g_\varepsilon$-equation of (3) by $\sqrt{\mu} \in \mathcal{N}(\mathcal{L})$ and integrate over $v \in \mathbb{R}^3$. Then we obtain

$$\partial_t \rho_\varepsilon + \underbrace{\frac{1}{\varepsilon}(v \cdot \nabla_x g_\varepsilon, \sqrt{\mu})_{L^2_v}}_{I_1} + \underbrace{\frac{1}{\varepsilon^2}(\mathcal{L}g_\varepsilon, \sqrt{\mu})_{L^2_v}}_{I_2=0} = \underbrace{\frac{1}{\varepsilon^2}(\Gamma(g_\varepsilon, g_\varepsilon), \sqrt{\mu})_{L^2_v}}_{I_3=0}. \tag{68}$$

For the term $I_1$, we have

$$I_1 = \frac{1}{\varepsilon}\nabla_x \cdot (g_\varepsilon, v\sqrt{\mu})_{L^2_v} = \frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon. \tag{69}$$

Collecting the above relations, we deduce that

$$\partial_t \rho_\varepsilon + \frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon = 0, \tag{70}$$

hence the first equation of (67) holds.
Step 2. Conservation law of $u_\varepsilon$. We multiply the first $g_\varepsilon$-equation of (3) by $v\sqrt{\mu} \in \mathcal{N}(\mathcal{L})$ and integrate over $v \in \mathbb{R}^3$, we have

$$\partial_t u_\varepsilon + \underbrace{\frac{1}{\varepsilon}(v \cdot \nabla_x g_\varepsilon, v\sqrt{\mu})_{L^2_v}}_{\mathrm{II}_1} + \underbrace{\frac{1}{\varepsilon^2}(\mathcal{L}g_\varepsilon, v\sqrt{\mu})_{L^2_v}}_{\mathrm{II}_2=0} = \underbrace{\frac{1}{\varepsilon^2}(\Gamma(g_\varepsilon, g_\varepsilon), v\sqrt{\mu})_{L^2_v}}_{\mathrm{II}_3=0}. \tag{71}$$

For the term $\mathrm{II}_1$, we introduce the matrix $A(v) := v \otimes v - \frac{|v|^2}{3}\mathbb{I}_3$. We have that $A\sqrt{\mu} \in \mathcal{N}^{\perp}(\mathcal{L})$ and there exists $\hat{A}(v)$ such that $\mathcal{L}(\sqrt{\mu}\hat{A}) = \sqrt{\mu}A$ with $\hat{A}\sqrt{\mu} \in \mathcal{N}^{\perp}(\mathcal{L})$ and $\mathbb{I}_3$ is the $3 \times 3$ unitary matrix (see for exemple [13]). Then we can write that

$$
\begin{aligned}
\mathrm{II}_1 &= \frac{1}{\varepsilon}\nabla_x \cdot (g_\varepsilon, v \otimes v\sqrt{\mu})_{L^2_v} \\
&= \frac{1}{\varepsilon}\nabla_x \cdot \left(g_\varepsilon, \left(v \otimes v - \frac{|v|^2}{3}\mathbb{I}_3\right)\sqrt{\mu}\right)_{L^2_v} + \frac{1}{\varepsilon}\nabla_x \cdot \left(g_\varepsilon, \frac{|v|^2}{3}\mathbb{I}_3\sqrt{\mu}\right)_{L^2_v} \\
&= \frac{1}{\varepsilon}\nabla_x \cdot (g_\varepsilon, A(v)\sqrt{\mu})_{L^2_v} + \frac{1}{\varepsilon}\nabla_x\Big( (g_\varepsilon, \sqrt{\mu})_{L^2_v} + \left(g_\varepsilon, (\frac{|v|^2}{3} - 1)\sqrt{\mu}\right)_{L^2_v}\Big) \\
&= \frac{1}{\varepsilon}\nabla_x \cdot (g_\varepsilon, A(v)\sqrt{\mu})_{L^2_v} + \frac{1}{\varepsilon}\nabla_x\theta_\varepsilon + \frac{1}{\varepsilon}\nabla_x\rho_\varepsilon.
\end{aligned}
\tag{72}
$$

Finally, using that $\mathcal{L}(\sqrt{\mu}\hat{A}) = \sqrt{\mu}A$ and that $\mathcal{L}$ is self-adjoint in $L^2_v$, we have

$$
\mathrm{II}_1 = \frac{1}{\varepsilon}\nabla_x \cdot \Big(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\Big)_{L^2_v} + \frac{1}{\varepsilon}\nabla_x(\theta_\varepsilon + \rho_\varepsilon).
$$

Collecting the previous calculations, we obtain

$$
\partial_t u_\varepsilon + \frac{1}{\varepsilon}\nabla_x(\rho_\varepsilon + \theta_\varepsilon) + \frac{1}{\varepsilon}\nabla_x \cdot \Big(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\Big)_{L^2_v} = 0, \tag{73}
$$

then the second equations of (67) holds.

Step 3. Conservation law of $\theta_\varepsilon$. We multiply the first $g_\varepsilon$-equation of (3) by $\Big(\frac{|v|^2}{3} - 1\Big)\sqrt{\mu} \in \mathcal{N}(\mathcal{L})$ and integrate over $v \in \mathbb{R}^3$, we have

$$
\begin{aligned}
\partial_t\theta_\varepsilon + \underbrace{\frac{1}{\varepsilon}\left(v \cdot \nabla_x g_\varepsilon, \Big(\frac{|v|^2}{3} - 1\Big)\sqrt{\mu}\right)_{L^2_v}}_{\mathrm{III}_1} &+ \underbrace{\frac{1}{\varepsilon^2}\left(\mathcal{L}g_\varepsilon, \Big(\frac{|v|^2}{3} - 1\Big)\sqrt{\mu}\right)_{L^2_v}}_{\mathrm{III}_2=0} \\
&= \underbrace{\frac{1}{\varepsilon^2}\left(\Gamma(g_\varepsilon, g_\varepsilon), \Big(\frac{|v|^2}{3} - 1\Big)\sqrt{\mu}\right)_{L^2_v}}_{\mathrm{III}_3=0}.
\end{aligned}
\tag{74}
$$

For the term $\mathrm{III}_2$, we introduce $B(v) := v\Big(\frac{|v|^2}{2} - \frac{5}{2}\Big)$. We have that $B\sqrt{\mu} \in \mathcal{N}^{\perp}(\mathcal{L})$ and there exists $\hat{B}(v)$ such that $\mathcal{L}(\sqrt{\mu}\hat{B}) = \sqrt{\mu}B$ with $\hat{B}\sqrt{\mu} \in \mathcal{N}^{\perp}(\mathcal{L})$. Then, we can write

$$
\begin{aligned}
\mathrm{III}_2 &= \frac{1}{\varepsilon}\nabla_x \cdot \left(g_\varepsilon, v\Big(\frac{|v|^2}{3} - 1\Big)\sqrt{\mu}\right)_{L^2_v} \\
&= \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot \left(g_\varepsilon, v\Big(\frac{|v|^2}{2} - \frac{5}{2}\Big)\sqrt{\mu}\right)_{L^2_v} + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot (g_\varepsilon, v\sqrt{\mu})_{L^2_v} \\
&= \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot (g_\varepsilon, B(v)\sqrt{\mu})_{L^2_v} + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon \\
&= \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot \Big(\hat{B}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\Big)_{L^2_v} + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon.
\end{aligned}
\tag{75}
$$

Collecting the above relations, we deduce that

$$
\partial_t\theta_\varepsilon + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot \Big(\hat{B}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\Big)_{L^2_v} = 0, \tag{76}
$$

hence the third equation of (67) holds. □

## 4.2 Limits from the global energy estimate

Based on Theorem 1.1, the Cauchy problem (3) admits a global solution $g_\varepsilon$ belonging to $L^\infty([0,\infty); H^3_x L^2_v)$ which is subject to the global energy estimate (12), namely, there is a positive constant, independent of $\varepsilon$, such that

$$\sup_{t\geq 0} \|g_\varepsilon(t)\|^2_{H^3_x L^2_v} \leq C \tag{77}$$

and

$$\int_0^\infty \|g_{\varepsilon,2}(t)\|^2_{\mathcal{X}^3(\mathbb{T}^3_x\times\mathbb{R}^3_v)} \mathrm{d}t \leq C\varepsilon^2. \tag{78}$$

From the energy bound (77), there exists $g \in L^\infty([0,\infty); H^3_x L^2_v)$, such that

$$g_\varepsilon \rightharpoonup g \ \text{ as } \ \varepsilon \longrightarrow 0 \tag{79}$$

where the convergence is weak-$\star$ in $L^\infty([0,\infty); H^3_x L^2_v)$ (the limits may hold for some subsequence). From the energy dissipation bound (78), we have

$$\{I-\Pi_0\}g_\varepsilon \longrightarrow 0 \ \text{ strongly in } \ L^2([0,\infty); H^3_x L^2_v) \tag{80}$$

as $\varepsilon \longrightarrow 0$. We thereby deduce from combining the first convergence in (79) and (80) that

$$\{I-\Pi_0\}g = 0. \tag{81}$$

Indeed, we have $\{I-\Pi_0\}g_\varepsilon$ converges in the sense of distributions to $\{I-\Pi_0\}g$ as $\varepsilon$ tends to zero. By uniqueness of the limit, we obtain the proof. Then, it immediately gives that there are $(\rho, u, \theta) \in L^\infty([0,\infty); H^3(\mathbb{T}^3_x))$ such that

$$g = \rho(x)\sqrt{\mu} + u(x)\cdot v\sqrt{\mu} + \theta(x)\Big(\frac{|v|^2}{2} - \frac{3}{2}\Big)\sqrt{\mu}. \tag{82}$$

Via the definitions of $\rho_\varepsilon, u_\varepsilon$ and $\theta_\varepsilon$ in (66) and the uniform energy bound (77), we obtain

$$\sup_{t\geq 0}\Big(\|\rho_\varepsilon\|^2_{H^3(\mathbb{T}^3_x)} + \|u_\varepsilon\|^2_{H^3(\mathbb{T}^3_x)} + \|\theta_\varepsilon\|^2_{H^3(\mathbb{T}^3_x)}\Big) \lesssim \sup_{t\geq 0}\|g_\varepsilon\|^2_{H^3_x L^2_v} \leq C. \tag{83}$$

We thereby deduce the following convergences from the convergence of (79) and the limit function $g(t,x,v)$ given in (82) that

$$\begin{aligned}
\rho_\varepsilon &= (g_\varepsilon, \sqrt{\mu})_{L^2_v} \longrightarrow (g, \sqrt{\mu})_{L^2_v} = \rho,\\
u_\varepsilon &= (g_\varepsilon, v\sqrt{\mu})_{L^2_v} \longrightarrow (g, v\sqrt{\mu})_{L^2_v} = u,\\
\theta_\varepsilon &= \Big(g_\varepsilon, \Big(\frac{|v|^2}{3} - 1\Big)\sqrt{\mu}\Big)_{L^2_v} \longrightarrow \Big(g, \Big(\frac{|v|^2}{3} - 1\Big)\sqrt{\mu}\Big)_{L^2_v} = \theta,
\end{aligned} \tag{84}$$

weakly-$\star$ in $L^\infty([0,\infty); H^3(\mathbb{T}^3_x))$.

## 4.3 Convergences to limiting equations

In this subsection, we will derive the incompressible Navier-Stokes-Fourier system (13) from the conservation laws (67) in Lemma 4.1 and the convergences obtained in the previous subsection.

**Incompressibility and Boussinesq relation:** From the first equation of (67) in Lemma 4.1 and the energy uniform bound (77), it is easy to deduce

$$\nabla_x \cdot u_\varepsilon = -\varepsilon \partial_t \rho_\varepsilon \longrightarrow 0 \tag{85}$$

in the sense of distributions as $\varepsilon \longrightarrow 0$. By using the convergence (84), we have $\nabla_x \cdot u_\varepsilon$ converges in the sense of distributions to $\nabla_x \cdot u$ as $\varepsilon$ tends to zero. By uniqueness of the limit, we obtain

$$\nabla_x \cdot u = 0. \tag{86}$$

Via the second equation of (67), we have

$$\nabla_x(\rho_\varepsilon + \theta_\varepsilon) = -\varepsilon \partial_t u_\varepsilon - \nabla_x \cdot \Big(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\Big)_{L^2_v}. \tag{87}$$

Notice that

$$\nabla_x \cdot \Big(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\Big)_{L^2_v} = \nabla_x \cdot (A(v)\sqrt{\mu}, \{I - \Pi_0\}g_\varepsilon)_{L^2_v}$$

where the self-adjointness of $\mathcal{L}$ and the fact that $A\sqrt{\mu} \in \mathcal{N}^\perp(\mathcal{L})$ are utilized. Then we derive from the Hölder inequality, and the uniform energy dissipation bound (78) that

$$\begin{aligned}
\int_0^\infty \|\nabla_x \cdot \Big(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\Big)_{L^2_v}\|^2_{H^2_x} \mathrm{d}t &= \int_0^\infty \|\,(A(v)\sqrt{\mu}, \nabla_x\{I - \Pi_0\}g_\varepsilon)_{L^2_v}\,\|^2_{H^2_x} \mathrm{d}t \\
&\lesssim \int_0^\infty \|\{I - \Pi_0\}g_\varepsilon\|^2_{H^3_x L^2_v} \mathrm{d}t \\
&\lesssim \int_0^\infty \mathcal{D}^2(t) \mathrm{d}t \\
&\leq C\varepsilon^2,
\end{aligned}$$

then, $\nabla_x \cdot \Big(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\Big)_{L^2_v} \longrightarrow 0$ strongly in $L^2([0,\infty); H^2(\mathbb{T}^3_x))$. Consequently, it is easy to deduce that

$$\nabla_x(\rho_\varepsilon + \theta_\varepsilon) \longrightarrow 0 \tag{88}$$

in the sense of distributions as $\varepsilon \longrightarrow 0$. By using the convergence (84), we have $\nabla_x(\rho_\varepsilon + \theta_\varepsilon)$ converges in the sense of distributions to $\nabla_x(\rho + \theta)$ as $\varepsilon$ tends to zero. By uniqueness of the limit, we obtain the Boussinesq relation

$$\nabla_x(\rho + \theta) = 0. \tag{89}$$

Now, we will show that Boussinesq relation can be strengthened. Using (4) and (66), we have that

$$\int_{\mathbb{T}^3} (\rho_\varepsilon + \theta_\varepsilon) \mathrm{d}x = 0.$$

Furthermore, using also (84), we obtain

$$\int_{\mathbb{T}^3}(\rho_\varepsilon+\theta_\varepsilon)\mathrm{d}x \longrightarrow \int_{\mathbb{T}^3}(\rho+\theta)\mathrm{d}x \quad \text{in } \mathcal{D}'_t,$$

from which we deduce that

$$\int_{\mathbb{T}^3}(\rho+\theta)\mathrm{d}x=0, \quad \text{for a.e. } t>0.$$

Finally (89) yields the strengthened form

$$\rho+\theta=0. \tag{90}$$

**Convergence of $\frac{3}{5}\theta_\varepsilon-\frac{2}{5}\rho_\varepsilon$:** Before doing this, we introduce the following Aubin-Lions-Simon Theorem, a fundamental result of compactness in the study of nonlinear evolution problems, which can be found in Theorem II.5.16 in [5].

**Lemma 4.2** (Aubin-Lions-Simon Theorem)**.** *Let $B_0\subset B_1\subset B_2$ be three Banach spaces. We assume that the embedding of $B_1$ in $B_2$ is continuous ans that the embedding of $B_0$ in $B_1$ is compact. Let $p,r$ be such that $1\le p,r\le+\infty$. For $T>0$, we define*

$$E_{p,r}=\{u\in L^p(0,T;B_0),\partial_t u\in L^r(0,T;B_2)\}.$$

1) *If $p<+\infty$, the embedding of $E_{p,r}$ in $L^p(0,T;B_1)$ is compact.*

2) *If $p=+\infty$ and if $r>1$, the embedding of $E_{p,r}$ in $C(0,T;B_1)$ is compact.*

On the one hand, the third equation of (67) multiplied by $\frac{3}{5}$ minus $\frac{2}{5}$ times of the first equation of (67) gives

$$\partial_t\Big(\frac{3}{5}\theta_\varepsilon-\frac{2}{5}\rho_\varepsilon\Big)+\frac{2}{5}\frac{1}{\varepsilon}\nabla_x\cdot\Big(\hat{B}(v)\sqrt{\mu},\mathcal{L}(g_\varepsilon)\Big)_{L^2_v}=0. \tag{91}$$

We thus have that

$$\begin{aligned}\Big\|\partial_t\Big(\frac{3}{5}\theta_\varepsilon-\frac{2}{5}\rho_\varepsilon\Big)\Big\|_{H^2_x}&=\Big\|\frac{2}{5}\frac{1}{\varepsilon}\nabla_x\cdot(B(v)\sqrt{\mu},\{I-\Pi_0\}g_\varepsilon)_{L^2_v}\Big\|_{H^2_x}\\&\lesssim\frac{1}{\varepsilon}\|B(v)\sqrt{\mu}\|_{L^2_v}\|\{I-\Pi_0\}g_\varepsilon\|_{H^3_xL^2_v}\\&\lesssim\frac{1}{\varepsilon}\|\{I-\Pi_0\}g_\varepsilon\|_{H^3_xL^2_v}\end{aligned}$$

which immediately implies from the uniform energy dissipation bound (78) that

$$\Big\|\partial_t\Big(\frac{3}{5}\theta_\varepsilon-\frac{2}{5}\rho_\varepsilon\Big)\Big\|_{L^2(0,T;H^2_x)}\lesssim\Big(\int_0^\infty\frac{1}{\varepsilon^2}\mathcal{D}^2(t)\mathrm{d}t\Big)^{1/2}\le C \tag{92}$$

for any $T>0$ and $0<\varepsilon\le1$. On the other hand, we easily have from (83) that

$$\Big\|\Big(\frac{3}{5}\theta_\varepsilon-\frac{2}{5}\rho_\varepsilon\Big)\Big\|_{L^\infty(0,T;H^3_x)}\le C \tag{93}$$

for all $T>0$ and $0<\varepsilon\le1$. One notices that

$$H^3_x\hookrightarrow H^2_x\hookrightarrow H^2_x, \tag{94}$$

where the embedding of $H_x^3$ in $H_x^2$ is compact and the embedding of $H_x^2$ in $H_x^2$ is naturally continuous. Then, from Aubin-Lions-Simon Theorem in Lemma 4.2, the bounds (92), (93) and the embeddings (94), we deduce that there is a $\tilde{\theta} \in C(\mathbb{R}^+; H^2(\mathbb{T}_x^3)) \cap L^\infty(\mathbb{R}^+; H^3(\mathbb{T}_x^3))$ such that

$$\left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right) \longrightarrow \tilde{\theta}$$

strongly in $C(\mathbb{R}^+; H^2(\mathbb{T}_x^3))$ as $\varepsilon \longrightarrow 0$. On the other hand, we have that $(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon)$ converges in the sense of distributions to $(\frac{3}{5}\theta - \frac{2}{5}\rho)$ as $\varepsilon$ tends to zero. By uniqueness of the limit, we obtain that $(\frac{3}{5}\theta - \frac{2}{5}\rho) = \tilde{\theta}$. Now we can write $\theta = (\frac{3}{5}\theta - \frac{2}{5}\rho) + \frac{2}{5}(\rho + \theta)$, which gives us $\tilde{\theta} = \theta$, since $\rho + \theta = 0$ according to the relation (90). As a result,

$$\left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right) \longrightarrow \theta \tag{95}$$

strongly in $C(\mathbb{R}^+; H^2(\mathbb{T}_x^3))$ as $\varepsilon \longrightarrow 0$, where $\theta \in C(\mathbb{R}^+; H^2(\mathbb{T}_x^3)) \cap L^\infty(\mathbb{R}^+; H^3(\mathbb{T}_x^3))$.

**Convergence of $\mathcal{P}u_\varepsilon$:** Here $\mathcal{P}$ is the Leray projection operator given by

$$\mathcal{P} = \mathcal{I} - \nabla_x \Delta_x^{-1} \nabla_x \cdot \tag{96}$$

where $\mathcal{I}$ is the identical mapping. Taking $\mathcal{P}$ on the second equation of (67) gives

$$\partial_t \mathcal{P}u_\varepsilon + \frac{1}{\varepsilon}\mathcal{P}\nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\right)_{L_v^2} = 0, \tag{97}$$

where we used $\mathcal{P}\nabla_x(\rho_\varepsilon + \theta_\varepsilon) = 0$. We thus have that

$$\begin{aligned}
\|\partial_t \mathcal{P}u_\varepsilon\|_{H_x^2} &= \left\| \frac{1}{\varepsilon}\mathcal{P}\nabla_x \cdot (A(v)\sqrt{\mu}, \{I - \Pi_0\}g_\varepsilon)_{L_v^2} \right\|_{H_x^2} \\
&\lesssim \left\| \frac{1}{\varepsilon}\nabla_x \cdot (A(v)\sqrt{\mu}, \{I - \Pi_0\}g_\varepsilon)_{L_v^2} \right\|_{H_x^2} \\
&\lesssim \frac{1}{\varepsilon}\|A(v)\sqrt{\mu}\|_{L_v^2} \|\{I - \Pi_0\}g_\varepsilon\|_{H_x^3 L_v^2} \\
&\lesssim \frac{1}{\varepsilon}\|\{I - \Pi_0\}g_\varepsilon\|_{H_x^3 L_v^2}
\end{aligned}$$

which immediately implies from the uniform energy dissipation bound (78) that

$$\|\partial_t \mathcal{P}u_\varepsilon\|_{L^2(0,T;H_x^2)} \lesssim \left(\int_0^\infty \frac{1}{\varepsilon^2}\mathcal{D}^2(t)\mathrm{d}t\right)^{1/2} \le C \tag{98}$$

for any $T > 0$ and $0 < \varepsilon \le 1$. We easily have from (83) that

$$\|\mathcal{P}u_\varepsilon\|_{L^\infty(0,T;H_x^3)} \lesssim \|u_\varepsilon\|_{L^\infty(0,T;H_x^3)} \le C. \tag{99}$$

Then, from Aubin-Lions-Simon Theorem in Lemma 4.2, the bounds (98), (99) and the embeddings (94), we deduce that there is a $\tilde{u} \in C(\mathbb{R}^+; H^2(\mathbb{T}_x^3)) \cap L^\infty(\mathbb{R}^+; H^3(\mathbb{T}_x^3))$ such that

$$\mathcal{P}u_\varepsilon \longrightarrow \tilde{u}$$

strongly in $C(\mathbb{R}^+; H^2(\mathbb{T}_x^3))$ as $\varepsilon \longrightarrow 0$. On the other hand, we have $\mathcal{P}u_\varepsilon$ converges in the sense of distributions to $\mathcal{P}u$ as $\varepsilon$ tends to zero. By uniqueness of the limit, we obtain that $\tilde{u} = \mathcal{P}u$. Using that where we used $\nabla_x \cdot u = 0$ according to the incompressibility relation (86), we get

$$\mathcal{P}u_\varepsilon \longrightarrow u \tag{100}$$

strongly in $C(\mathbb{R}^+; H^2(\mathbb{T}_x^3))$ as $\varepsilon \longrightarrow 0$, where $u \in C(\mathbb{R}^+; H^2(\mathbb{T}_x^3)) \cap L^\infty(\mathbb{R}^+; H^3(\mathbb{T}_x^3))$. Regarding the convergence of $\mathcal{P}^\perp$. We have

$$\mathcal{P}^\perp u_\varepsilon \longrightarrow 0 \tag{101}$$

weakly-$\star$ in $L^\infty([0,\infty); H^3(\mathbb{T}_x^3))$. Indeed, on the one hand we have

$$\|\mathcal{P}^\perp u_\varepsilon\|_{L^\infty(0,T;H_x^3)} \lesssim \|u_\varepsilon\|_{L^\infty(0,T;H_x^3)} \leq C. \tag{102}$$

On the other hand, we have $\mathcal{P}^\perp u_\varepsilon$ converges in the sense of distributions to 0 as $\varepsilon$ tends to zero, hence the relation (101) is true.
**Equation of $u$ and $\theta$:** We first calculate the term

$$\left(\hat{E}(v)\sqrt{M}, \frac{1}{\varepsilon}\mathcal{L}g_\varepsilon\right)_{L_v^2}$$

where $\hat{E} = \hat{A}$ or $\hat{B}$. Following the standard formal derivations of fluid dynamic limits of Boltzmann equation (see [4] for instance), we obtain

$$\left(\hat{A}(v)\sqrt{\mu}, \frac{1}{\varepsilon}\mathcal{L}g_\varepsilon\right)_{L_v^2} = u_\varepsilon \otimes u_\varepsilon - \frac{|u_\varepsilon|^2}{3}\mathbb{I}_3 - \nu\Sigma(u_\varepsilon) - R_{\varepsilon,A}, \tag{103}$$

and

$$\left(\hat{B}(v)\sqrt{\mu}, \frac{1}{\varepsilon}\mathcal{L}g_\varepsilon\right)_{L_v^2} = \frac{5}{2}u_\varepsilon\theta_\varepsilon - \frac{5}{2}\kappa\nabla_x\theta_\varepsilon - R_{\varepsilon,B}, \tag{104}$$

where

$$\Sigma(u_\varepsilon) := \nabla_x u_\varepsilon + \nabla_x u_\varepsilon^\top - \frac{2}{3}\nabla_x \cdot u_\varepsilon \mathbb{I}_3, \tag{105}$$

$$\nu := \frac{1}{10}\left(\sqrt{\mu}A, \sqrt{\mu}\hat{A}\right)_{L_v^2}, \tag{106}$$

$$\kappa := \frac{2}{15}\left(\sqrt{\mu}B, \sqrt{\mu}\hat{B}\right)_{L_v^2}. \tag{107}$$

For $E =$ A or B, $R_{\varepsilon,E}$ are of the form

$$\begin{aligned} R_{\varepsilon,E} &= \varepsilon\left(\hat{E}(v)\sqrt{\mu}, \partial_t g_\varepsilon\right)_{L_v^2} + \left(\hat{E}(v)\sqrt{\mu}, v\cdot\nabla_x\{I-\Pi_0\}g_\varepsilon\right)_{L_v^2} \\ &+ \left(\hat{E}(v)\sqrt{\mu}, \Gamma(\{I-\Pi_0\}g_\varepsilon, \{I-\Pi_0\})g_\varepsilon\right)_{L_v^2} \\ &+ \left(\hat{E}(v)\sqrt{\mu}, \Gamma(\Pi_0 g_\varepsilon, \{I-\Pi_0\}g_\varepsilon)\right)_{L_v^2} \\ &+ \left(\hat{E}(v)\sqrt{\mu}, \Gamma(\{I-\Pi_0\}g_\varepsilon, \Pi_0 g_\varepsilon)\right)_{L_v^2}. \end{aligned} \tag{108}$$

For the vector field $u_\varepsilon$, we decompose $u_\varepsilon = \mathcal{P}u_\varepsilon + \mathcal{P}^\perp u_\varepsilon$. Then, plugging the relation (103) into the equation (97), we have

$$\partial_t \mathcal{P}u_\varepsilon + \mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}u_\varepsilon) - \nu\Delta_x \mathcal{P}u_\varepsilon = R_{\varepsilon,u} \tag{109}$$

where,

$$R_{\varepsilon,u} = \mathcal{P}\nabla_x \cdot R_{\varepsilon,A} - \mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon). \tag{110}$$

Noticing that $\theta_\varepsilon = \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right) + \frac{2}{5}(\rho_\varepsilon + \theta_\varepsilon)$, we substitue the relation (104) into the equation (91) and then obtain

$$\partial_t\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big) + \nabla_x \cdot \Big[\mathcal{P}u_\varepsilon\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big] - \kappa\Delta_x\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big) = R_{\varepsilon,\theta} \tag{111}$$

where

$$\begin{aligned} R_{\varepsilon,\theta} = & \frac{2}{5}\nabla_x \cdot R_{\varepsilon,B} - \frac{2}{5}\nabla_x \cdot [\mathcal{P}u_\varepsilon(\rho_\varepsilon + \theta_\varepsilon)] - \nabla_x \cdot \Big[\mathcal{P}^\perp u_\varepsilon\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big] \\ & - \frac{2}{5}\nabla_x \cdot \Big[\mathcal{P}^\perp u_\varepsilon(\rho_\varepsilon + \theta_\varepsilon)\Big] + \frac{2}{5}\kappa\Delta_x(\rho_\varepsilon + \theta_\varepsilon). \end{aligned} \tag{112}$$

Our final goal is now to perform the limit $\varepsilon$ to 0 in (109) and (111) in order to deduce the $u$ and $\theta$ equations in (13). We first focus on (109).
Now, we take the limit from (109) to obtain the u-equation of (13). For any $T > 0$, let a vector-valued test function $\psi(t,x) \in C^1(0,T; C_c^\infty(\mathbb{T}^3))$ with $\nabla_x \cdot \psi = 0$, $\psi(0,x) = \psi_0(x) \in C_c^\infty(\mathbb{T}^3)$ and $\psi(t,x) = 0$ for $t \geq T'$ where $T' < T$. We multiply (109) by $\psi(t,x)$ and integrate by parts over $(t,x) \in [0,T] \times \mathbb{T}^3$. Then we obtain

$$\int_0^T \int_{\mathbb{T}^3} \partial_t \mathcal{P}u_\varepsilon \cdot \psi(t,x) \mathrm{d}x\mathrm{d}t = -\underbrace{\int_{\mathbb{T}^3} \mathcal{P}u_\varepsilon(0,x) \cdot \psi(0,x)\mathrm{d}x}_{\mathrm{IV}_1} - \underbrace{\int_0^T \int_{\mathbb{T}^3} \mathcal{P}u_\varepsilon \cdot \partial_t\psi(t,x)\mathrm{d}x\mathrm{d}t}_{\mathrm{IV}_2}. \tag{113}$$

From the initial condition (14) in Theorem 1.2 and the convergence (100) , we deduce that

$$\int_{\mathbb{T}^3} \mathcal{P}u_\varepsilon \cdot \partial_t\psi(t,x)\mathrm{d}x\mathrm{d}t \to \int_{\mathbb{T}^3} u \cdot \partial_t\psi(t,x)\mathrm{d}x\mathrm{d}t \tag{114}$$

and

$$\int_{\mathbb{T}^3} \mathcal{P}u_\varepsilon(0) \cdot \psi_0(x)\mathrm{d}x \to \int_{\mathbb{T}^3} \mathcal{P}u_0 \cdot \psi_0(x)\mathrm{d}x \tag{115}$$

as $\varepsilon \to 0$. Indeed, for the term $\mathrm{IV}_2$, we have

$$\begin{aligned} \left|\int_0^T \int_{\mathbb{T}^3} (\mathcal{P}u_\varepsilon - u) \cdot \partial_t\psi(t,x)\mathrm{d}x\mathrm{d}t\right| & \lesssim \max_{t\geq 0} \|\mathcal{P}u_\varepsilon - u\|_{L_x^2} \|\partial_t\psi\|_{L^1(0,T;L_x^2)} \\ & \leq C_{\psi,T} \max_{t\geq 0} \|\mathcal{P}u_\varepsilon - u\|_{L_x^2} \longrightarrow 0. \end{aligned}$$

For the term $\mathrm{IV}_1$ we have,

$$\begin{aligned} \left|\int_{\mathbb{T}^3} (\mathcal{P}u_\varepsilon(0) - \mathcal{P}u_0)\psi_0(x)\mathrm{d}x\right| & \lesssim \|\mathcal{P}u_\varepsilon(0) - \mathcal{P}u_0\|_{L_x^2} \|\psi_0(x)\|_{L_x^2} \\ & \lesssim \|\mathcal{P}u_\varepsilon(0) - \mathcal{P}u_0\|_{L_x^2} \longrightarrow 0. \end{aligned}$$

As a consequence, we have

$$\int_0^T \int_{\mathbb{T}^3} \partial_t\mathcal{P}u_\varepsilon \cdot \psi(t,x)\mathrm{d}x\mathrm{d}t \longrightarrow -\int_{\mathbb{T}^3} \mathcal{P}u_0 \cdot \psi_0(x)\mathrm{d}x - \int_{\mathbb{T}^3} u \cdot \partial_t\psi(t,x)\mathrm{d}x\mathrm{d}t \tag{116}$$

as $\varepsilon \to 0$. Now, we will study the convergence of terms $\mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}u_\varepsilon)$ and $\nu\Delta_x\mathcal{P}u_\varepsilon$ in the following lemma.

**Lemma 4.3.** *It holds that*

$$\begin{aligned} \mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}u_\varepsilon) \to \mathcal{P}\nabla_x \cdot (u \otimes u) \quad & \textit{strongly in} \quad C(\mathbb{R}^+; H_x^1), \\ \nu \Delta_x \mathcal{P}u_\varepsilon \to \nu \Delta_x u \quad & \textit{strongly in} \quad C(\mathbb{R}^+; L_x^2), \end{aligned} \tag{117}$$

*as* $\varepsilon \to 0$, *where* $\nu$ *is defined in* (106).

*Proof.* Using that $\nabla_x \cdot \mathcal{P}u_\varepsilon = \nabla_x \cdot u = 0$, we have

$$\begin{aligned} \|\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}u_\varepsilon - u \otimes u)\|_{H_x^1} \lesssim\ & \|\nabla_x(\mathcal{P}u_\varepsilon - u) \cdot \mathcal{P}u_\varepsilon\|_{L_x^2} + \|\nabla_x u \cdot (\mathcal{P}u_\varepsilon - u)\|_{L_x^2} \\ & + \|\tilde{\nabla}_x \nabla_x(\mathcal{P}u_\varepsilon - u) \cdot \mathcal{P}u_\varepsilon\|_{L_x^2} + \|\nabla_x(\mathcal{P}u_\varepsilon - u) \cdot \nabla_x \mathcal{P}u_\varepsilon\|_{L_x^2} \\ & + \|\tilde{\nabla}_x \nabla_x u \cdot (\mathcal{P}u_\varepsilon - u)\|_{L_x^2} + \|\nabla_x u \cdot \nabla_x(\mathcal{P}u_\varepsilon - u)\|_{L_x^2} \\ \lesssim\ & \Big(\|u\|_{L_t^\infty H_x^3} + \|\mathcal{P}u_\varepsilon\|_{L_t^\infty H_x^3}\Big) \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+; H_x^2)} \\ \lesssim\ & \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+; H_x^2)} \longrightarrow 0, \end{aligned}$$

where we denote by $\tilde{\nabla}_x \Lambda$ with $\Lambda = (\lambda_{ij})_{1 \le i,j \le 3}$ the matrix defined by

$$\tilde{\nabla}_x \Lambda = \begin{pmatrix} \partial_{x_1}\lambda_{11} & \partial_{x_1}\lambda_{12} & \partial_{x_1}\lambda_{13} \\ \partial_{x_2}\lambda_{21} & \partial_{x_2}\lambda_{22} & \partial_{x_2}\lambda_{23} \\ \partial_{x_3}\lambda_{31} & \partial_{x_3}\lambda_{32} & \partial_{x_3}\lambda_{33} \end{pmatrix}.$$

In addition, we have

$$\|\mu \Delta_x(\mathcal{P}u_\varepsilon - u)\|_{C(\mathbb{R}^+; L_x^2)} \lesssim \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+; H_x^2)} \longrightarrow 0.$$

□

The limit of the last term $R_{\varepsilon,u}$ is handled in the following lemma.

**Lemma 4.4.** *In the distributional sense,*

$$R_{\varepsilon,u} \to 0 \tag{118}$$

*as* $\varepsilon \to 0$, *where* $R_{\varepsilon,u}$ *is defined in* (110).

*Proof.* First, we have

$$\begin{aligned} & \|\mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}u_\varepsilon)\|_{L_t^\infty H_x^2} \\ & \lesssim \|\mathcal{P}u_\varepsilon\|_{L_t^\infty H_x^3} \|\mathcal{P}^\perp u_\varepsilon\|_{L_t^\infty H_x^3}, \end{aligned}$$

and by employing the convergences (100) and (101), one can obtain

$$\mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}u_\varepsilon) \to 0 \tag{119}$$

in the sense of distributions as $\varepsilon \to 0$. Let us now show that

$$\mathcal{P}\nabla_x \cdot (\mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon) \to 0 \tag{120}$$

in the sense of distributions as $\varepsilon \to 0$. To this end, we set

$$\boldsymbol{\beta}_\varepsilon := \rho_\varepsilon + \theta_\varepsilon.$$

One observes that, Equation (73) reads

$$\varepsilon\partial_t u_\varepsilon + \nabla_x \boldsymbol{\beta}_\varepsilon = -\nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\right)_{L^2_v}. \tag{121}$$

Now, we multiply the first $g_\varepsilon$-equation of (3) by $\frac{|v|^2}{2}\sqrt{\mu} \in \mathcal{N}(\mathcal{L})$ and integrate over $v \in \mathbb{R}^3$, we have

$$\varepsilon\partial_t \boldsymbol{\beta}_\varepsilon + \nabla_x \cdot \left(g_\varepsilon, v\frac{|v|^2}{3}\sqrt{\mu}\right)_{L^2_v} = 0, \tag{122}$$

where we check easily that

$$\begin{aligned}\nabla_x \cdot \left(g_\varepsilon, v\frac{|v|^2}{3}\sqrt{\mu}\right)_{L^2_v} &= \frac{2}{3}\nabla_x \cdot \left(g_\varepsilon, v\left(\frac{|v|^2}{2} - \frac{5}{2}\right)\sqrt{\mu}\right)_{L^2_v} + \frac{5}{3}\nabla_x \cdot (g_\varepsilon, v\sqrt{\mu})_{L^2_v}\\ &= \frac{2}{3}\nabla_x \cdot \left(\hat{B}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon)\right)_{L^2_v} + \frac{5}{3}\nabla_x \cdot \mathcal{P}^\perp u_\varepsilon.\end{aligned}$$

Using [25, Proposition 1.6], we can write

$$\mathcal{P}^\perp u_\varepsilon = \nabla_x \mathbf{U}_\varepsilon$$

with $\mathbf{U}_\varepsilon \in L^\infty([0,\infty); H^4(\mathbb{T}^3_x))$. After applying $\mathcal{P}^\perp$ to (121) and reformulating (122), we obtain that $\mathbf{U}_\varepsilon$ and $\boldsymbol{\beta}_\varepsilon$ satisfy

$$\begin{cases} \varepsilon\partial_t \nabla_x \mathbf{U}_\varepsilon + \nabla_x \boldsymbol{\beta}_\varepsilon = \mathbf{F}_\varepsilon \\ \varepsilon\partial_t \boldsymbol{\beta}_\varepsilon + \frac{5}{3}\Delta_x \mathbf{U}_\varepsilon = \mathbf{G}_\varepsilon \end{cases} \tag{123}$$

with

$$\mathbf{F}_\varepsilon = -\mathcal{P}^\perp\nabla_x \cdot (A(v)\sqrt{\mu}, \{I - \Pi_0\}g_\varepsilon)_{L^2_v} \quad \mathbf{G}_\varepsilon = -\frac{2}{3}\nabla_x \cdot (B(v)\sqrt{\mu}, \{I - \Pi_0\}g_\varepsilon)_{L^2_v}.$$

We get from the uniform energy dissipation bound (78) that

$$\|\mathbf{F}_\varepsilon\|_{L^2([0,\infty);L^2(\mathbb{T}^3_x))} \lesssim \varepsilon \quad \text{and} \quad \|\mathbf{G}_\varepsilon\|_{L^2([0,\infty);L^2(\mathbb{T}^3_x))} \lesssim \varepsilon$$

then $\mathbf{F}_\varepsilon$ and $\mathbf{G}_\varepsilon$ converge strongly to 0 in $L^2([0,\infty); L^2(\mathbb{T}^3_x))$ which implies that $\mathbf{F}_\varepsilon$ and $\mathbf{G}_\varepsilon$ converge strongly to 0 in $L^1_{loc}(dt; L^2_{loc}(\mathrm{d}x))$. Moreover, returning to (83), (102) we have $\mathbf{U}_\varepsilon \in L^\infty([0,\infty); H^4(\mathbb{T}^3_x))$, $\boldsymbol{\beta}_\varepsilon \in L^\infty([0,\infty); H^3(\mathbb{T}^3_x))$. Then, according to [13, Lemma 13.1] we deduce that (120) is true. From the above, we conclude that

$$\mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon) \to 0 \tag{124}$$

in the sense of distributions as $\varepsilon \to 0$. Next we prove that $E = \mathrm{A}$ or $\mathrm{B}$

$$R_{\varepsilon,E} \to 0 \tag{125}$$

in the sense of distributions as $\varepsilon \to 0$, where $R_{\varepsilon,E}$ are defined in (108). Indeed, For any $T > 0$, let a vector-valued test function $\psi(t,x) \in C^1(0,T; C_c^\infty(\mathbb{T}^3))$ , $\psi(0,x) =$

$\psi_0(x) \in C_c^\infty(\mathbb{T}^3)$ and $\psi(t,x) = 0$ for $t \geq T'$ where $T' < T$. Then, from the uniform bound (77) and the initial energy bounds given in Theorem 1.2 that

$$
\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^3} \varepsilon \left( \hat{E}(v)\sqrt{\mu}, \partial_t g_\varepsilon \right)_{L_v^2} \cdot \psi(t,x) \mathrm{d}x\mathrm{d}t \right| \\
&\lesssim \left| \int_{\mathbb{T}^3} \varepsilon \left( \hat{E}(v)\sqrt{\mu}, g_{\varepsilon,0} \right)_{L_v^2} \cdot \psi(0,x) \mathrm{d}x \right| + \left| \int_0^T \int_{\mathbb{T}^3} \varepsilon \left( \hat{E}(v)\sqrt{\mu}, g_\varepsilon \right)_{L_v^2} \cdot \partial_t \psi(t,x) \mathrm{d}x\mathrm{d}t \right| \\
&\lesssim \varepsilon \|\hat{E}(v)\sqrt{\mu}\|_{L_v^2} \Big( \|g_{\varepsilon,0}\|_{L_x^2 L_v^2} \|\psi_0\|_{L_x^2} + \|g_\varepsilon\|_{L^\infty(0,T;L_x^2 L_v^2)} \|\partial_t \psi\|_{L^1(0,T;L_x^2)} \Big) \\
&\leq C_{\psi,T}\, \varepsilon \Big( \|g_\varepsilon\|_{L^\infty(0,T;H_x^3 L_v^2)} + \|g_{\varepsilon,0}\|_{H_x^3 L_v^2} \Big) \longrightarrow 0,
\end{aligned}
\tag{126}
$$

as $\varepsilon \longrightarrow 0$, which implies that $\varepsilon \left( \hat{E}(v)\sqrt{M}, \partial_t g_\varepsilon \right)_{L_v^2} \longrightarrow 0$ in the sense of distributions as $\varepsilon \to 0$. Hölder inequality yields that

$$
\left\| \left( \hat{E}(v)\sqrt{\mu}, v \cdot \nabla_x \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \right\|_{H_x^2}^2 \lesssim \|\{I - \Pi_0\} g_\varepsilon\|_{H_x^3 L_v^2}^2
$$

which immediately derives from the uniform energy dissipation bound (78) that

$$
\left\| \left( \hat{E}(v)\sqrt{\mu}, v \cdot \nabla_x \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \right\|_{L^2(0,T;H_x^2)} \lesssim \left( \int_0^\infty \mathcal{D}^2(t) \mathrm{d}t \right)^{1/2} \leq C\, \varepsilon \longrightarrow 0.
$$

Then we have

$$
\left( \hat{E}(v)\sqrt{\mu}, v \cdot \nabla_x \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \longrightarrow 0
$$

strongly in $L^2(\mathbb{R}^+; H_x^2)$ as $\varepsilon \longrightarrow 0$. For any $T > 0$, we take any vector-valued test function $\phi(t,x) \in C_c^\infty([0,T] \times \mathbb{T}^3)$. Then, by employing the uniform bound (78) and Lemma 4.1 we get that

$$
\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^3} \left( \hat{E}(v)\sqrt{\mu}, \Gamma\{I - \Pi_0\} g_\varepsilon, \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \cdot \phi(t,x) \mathrm{d}x\mathrm{d}t \right| \\
&\lesssim \|\hat{E}(v)\sqrt{\mu}\|_{L_v^2}\, \|\phi\|_{L^\infty([0,T]\times\mathbb{T}^3)} \left( \int_0^\infty \mathcal{D}^2(t) \mathrm{d}t \right) \\
&\leq C_{\phi,T}\, \varepsilon^2 \longrightarrow 0.
\end{aligned}
$$

Thus, we know that

$$
\left( \hat{E}(v)\sqrt{\mu}, \Gamma\{I - \Pi_0\} g_\varepsilon, \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \longrightarrow 0
$$

in the sense of distributions as $\varepsilon \to 0$. Analogously, one easily derives that

$$
\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^3} \left( \hat{E}(v)\sqrt{\mu}, \Gamma(\{I - \Pi_0\} g_\varepsilon, \Pi_0 g_\varepsilon) + \Gamma(\Pi_0 g_\varepsilon, \{I - \Pi_0\} g_\varepsilon) \right)_{L_v^2} \cdot \phi(t,x) \mathrm{d}x\mathrm{d}t \right| \\
&\lesssim \|\hat{E}(v)\sqrt{\mu}\|_{L_v^2}\, \|\phi\|_{L^\infty([0,T]\times\mathbb{T}^3)}\, \|g_\varepsilon\|_{L^\infty(0,T;H_x^3 L_v^2)} \left( \int_0^\infty \mathcal{D}^2(t) \mathrm{d}t \right)^{1/2} \\
&\leq C_{\phi,T}\, \varepsilon \longrightarrow 0,
\end{aligned}
$$

which immediately implies that

$$
\left( \hat{E}(v)\sqrt{\mu}, \Gamma(\{I - \Pi_0\} g_\varepsilon, \Pi_0 g_\varepsilon) + \Gamma(\Pi_0 g_\varepsilon, \{I - \Pi_0\} g_\varepsilon) \right)_{L_v^2} \longrightarrow 0
$$

in the sense of distributions as $\varepsilon \to 0$. □

Collecting the limits (116), (117) and (118) gives that $u \in L^\infty(\mathbb{R}^+; H^3_x) \cap C(\mathbb{R}^+; H^2_x)$ satisfies

$$\partial_t u + \mathcal{P}\nabla_x \cdot (u \otimes u) - \nu \Delta_x u = 0 \tag{127}$$

with the initial data

$$u(0,x) = \mathcal{P}u_0(x). \tag{128}$$

Finally, we take the limit from (111) to the third $\theta$-equation in (13) as $\varepsilon \to 0$. For any $T > 0$, let $\xi(t,x)$ be a test function satisfying $\xi(t,x) \in C^1(0,T; C_c^\infty(\mathbb{T}^3))$ with $\xi(0,x) = \xi_0(x) \in C_c^\infty(\mathbb{T}^3)$ and $\xi(t,x) = 0$ for $t \geq T'$ where $T' < T$. We have that

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^3} \partial_t \Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)(t,x)\xi(t,x)\mathrm{d}x\mathrm{d}t = & - \underbrace{\int_0^T \int_{\mathbb{T}^3} \Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)(t,x)\partial_t\xi(t,x)\mathrm{d}x\mathrm{d}t}_{\mathbb{V}_1} \\ & - \underbrace{\int_{\mathbb{T}^3} \Big( g_{\varepsilon,0}, \Big(\frac{3}{5}\Big(\frac{|v|^2}{3} - 1\Big) - \frac{2}{5}\Big)\sqrt{\mu}\Big)_{L^2_v} \xi_0(x)\mathrm{d}x}_{\mathbb{V}_2}\,.\end{aligned} \tag{129}$$

From the initial condition (14) in Theorem 1.2 and the convergence (95) , we deduce that

$$\int_0^T \int_{\mathbb{T}^3} \Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)(t,x)\partial_t\xi(t,x)\mathrm{d}x\mathrm{d}t \longrightarrow \int_0^T \int_{\mathbb{T}^3} \theta(t,x)\partial_t\xi(t,x)\mathrm{d}x\mathrm{d}t$$

and

$$\int_{\mathbb{T}^3} \Big( g_{\varepsilon,0}, \Big(\frac{3}{5}\Big(\frac{|v|^2}{3} - 1\Big) - \frac{2}{5}\Big)\sqrt{\mu}\Big)_{L^2_v} \xi_0(x)\mathrm{d}x \to \int_{\mathbb{T}^3} \Big( g_0, \Big(\frac{3}{5}\Big(\frac{|v|^2}{3} - 1\Big) - \frac{2}{5}\Big)\sqrt{\mu}\Big)_{L^2_v} \xi_0(x)\mathrm{d}x.$$

Indeed, for the term $\mathbb{V}_1$, we have

$$\begin{aligned}\Big|\int_0^T \int_{\mathbb{T}^3} \Big(\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big) - \theta\Big) \cdot \partial_t\xi(t,x)\mathrm{d}x\mathrm{d}t\Big| & \lesssim \max_{t\geq 0} \Big\|\Big(\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big) - \theta\Big)\Big\|_{L^2_x} \|\partial_t \xi\|_{L^1(0,T;L^2_x)} \\ & \leq C_{\xi,T} \max_{t\geq 0} \Big\|\Big(\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big) - \theta\Big)\Big\|_{L^2_x} \longrightarrow 0.\end{aligned}$$

For the term $\mathbb{V}_2$ we have,

$$\Big|\int_{\mathbb{T}^3} \Big( g_{\varepsilon,0} - g_0, \Big(\frac{3}{5}\Big(\frac{|v|^2}{3} - 1\Big) - \frac{2}{5}\Big)\sqrt{\mu}\Big)_{L^2_v} \xi_0(x)\mathrm{d}x\Big| \lesssim \|g_{\varepsilon,0} - g_0\|_{L^2_{x,v}} \|\xi_0(x)\|_{L^2_x} \longrightarrow 0.$$

As a consequence, we have

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}^3} \partial_t \Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)(t,x)\xi(t,x)\mathrm{d}x\mathrm{d}t \\ &\longrightarrow - \int_0^T \int_{\mathbb{T}^3} \theta(t,x)\partial_t\xi(t,x)\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3} \Big(\frac{3}{5}\theta_0 - \frac{2}{5}\rho_0\Big)\xi_0(x)\mathrm{d}x\end{aligned} \tag{130}$$

as $\varepsilon \to 0$. Now, we will study the convergence of terms $\nabla_x \cdot \Big[\mathcal{P}u_\varepsilon\big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\big)\Big]$ and $\kappa\Delta_x\big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\big)$ in the following lemma.

**Lemma 4.5.** *It holds that*

$$\begin{aligned}\nabla_x \cdot \Big[\mathcal{P}u_\varepsilon\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big] &\to \nabla_x \cdot (u\theta) \quad \textit{strongly in} \quad C(\mathbb{R}^+; H^1_x),\\ \kappa\Delta_x\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big) &\to \kappa\Delta_x\theta \quad \textit{strongly in} \quad C(\mathbb{R}^+; L^2_x),\end{aligned} \tag{131}$$

*as $\varepsilon \to 0$, where $\kappa$ is defined in (107).*

*Proof.* Indeed, we have

$$\Big\|\kappa\Delta_x\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta\Big)\Big\|_{C(\mathbb{R}^+;L^2_x)} \lesssim \Big\|\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta\Big)\Big\|_{C(\mathbb{R}^+;H^2_x)} \longrightarrow 0.$$

In addition, we have

$$\begin{aligned}&\Big\|\nabla_x \cdot \Big((\mathcal{P}u_\varepsilon - u)\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big) + \nabla_x \cdot \Big(u\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta\Big)\Big)\Big\|_{H^1_x}\\ &\lesssim \Big\|(\mathcal{P}u_\varepsilon - u)\cdot\nabla_x\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big\|_{L^2_x} + \Big\|u\cdot\nabla_x\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta\Big)\Big\|_{L^2_x}\\ &+ \Big\|\nabla_x(\mathcal{P}u_\varepsilon - u)^\top\cdot\nabla_x\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big\|_{L^2_x} + \Big\|\nabla_x\Big(\nabla_x\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big)^\top(\mathcal{P}u_\varepsilon - u)\Big\|_{L^2_x}\\ &+ \Big\|\nabla_x u^\top\cdot\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta\Big)\Big\|_{L^2_x} + \Big\|\nabla_x\Big(\nabla_x\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big)^\top\cdot u\Big\|_{L^2_x}\\ &\lesssim \|u\|_{L^\infty_t H^3_x}\Big\|\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta\Big)\Big\|_{C(\mathbb{R}^+;H^2_x)} + \Big\|\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big\|_{L^\infty_t H^3_x}\|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+;H^2_x)}\\ &\lesssim \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+;H^2_x)} + \Big\|\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta\Big)\Big\|_{C(\mathbb{R}^+;H^2_x)} \longrightarrow 0.\end{aligned}$$

□

The limit of the last term $R_{\varepsilon,\theta}$ is handled in the following lemma.

**Lemma 4.6.** *In the distributional sense,*

$$R_{\varepsilon,\theta} \to 0 \tag{132}$$

*as $\varepsilon \to 0$, where $R_{\varepsilon,\theta}$ is defined in (109).*

*Proof.* First, we have

$$\frac{2}{5}\nabla_x\cdot[\mathcal{P}u_\varepsilon(\rho_\varepsilon+\theta_\varepsilon)] + \frac{2}{5}\nabla_x\cdot\Big[\mathcal{P}^\perp u_\varepsilon(\rho_\varepsilon+\theta_\varepsilon)\Big] + \nabla_x\cdot\Big[\mathcal{P}^\perp u_\varepsilon\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big] \to 0 \tag{133}$$

weakly-$\star$ in $L^\infty([0,\infty); H^2(\mathbb{T}^3_x))$ as $\varepsilon \to 0$. Indeed, we have

$$\Big\|\frac{2}{5}\nabla_x\cdot[\mathcal{P}u_\varepsilon(\rho_\varepsilon+\theta_\varepsilon)] + \frac{2}{5}\nabla_x\cdot\Big[\mathcal{P}^\perp u_\varepsilon(\rho_\varepsilon+\theta_\varepsilon)\Big] + \nabla_x\cdot\Big[\mathcal{P}^\perp u_\varepsilon\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big]\Big\|_{L^\infty_t H^2_x} \leq C$$

and by employing the convergence (100) and (101) we derive that

$$\frac{2}{5}\nabla_x\cdot[\mathcal{P}u_\varepsilon(\rho_\varepsilon+\theta_\varepsilon)] + \frac{2}{5}\nabla_x\cdot\Big[\mathcal{P}^\perp u_\varepsilon(\rho_\varepsilon+\theta_\varepsilon)\Big] + \nabla_x\cdot\Big[\mathcal{P}^\perp u_\varepsilon\Big(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\Big)\Big] \to 0 \tag{134}$$

in the sense of distributions as $\varepsilon \to 0$. Let us now show that

$$\kappa \Delta_x(\rho_\varepsilon + \theta_\varepsilon) \to 0 \tag{135}$$

weakly-$\star$ in $L^\infty([0,\infty); H^1(\mathbb{T}_x^3))$ as $\varepsilon \to 0$. We have

$$\|\Delta_x(\rho_\varepsilon + \theta_\varepsilon)\|_{L^\infty(\mathbb{R}^+;H_x^1)} \lesssim \|\rho_\varepsilon + \theta_\varepsilon\|_{L^\infty(\mathbb{R}^+;H_x^3)} \le C, \tag{136}$$

and by employing the convergence (88) we derive that

$$\nabla_x \cdot \nabla_x(\rho_\varepsilon + \theta_\varepsilon) \longrightarrow 0$$

in the sense of distributions as $\varepsilon \to 0$. Consequently, the convergences (125), (133) and (135) imply the convergence (132). □

Collecting the limits (130), (131) and (132) gives that $\theta \in L^\infty(\mathbb{R}^+; H_x^3) \cap C(\mathbb{R}^+; H_x^2)$ satisfies

$$\partial_t \theta + \nabla_x \cdot (u\theta) = \kappa \Delta_x \theta \tag{137}$$

with the initial data

$$\theta(0,x) = \frac{3}{5}\theta_0(x) - \frac{2}{5}\rho_0(x). \tag{138}$$

# A Appendix

## A.1 Study of the linear Landau equation

In this section, we show a local existence of the solution of the linear Landau equation. We use the technique introduced by Degond in [9] for the linear Fokker-Planck equation.
We consider now the linear Cauchy problem

$$\begin{cases} \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g + \frac{1}{\varepsilon^2}\mathcal{L} g = \mathrm{U}_\varepsilon \\ g(0,x,v) = g_0(x,v) \end{cases} \tag{139}$$

where $g_0(x,v)$ and $\mathrm{U}_\varepsilon(t,x,v)$ are given functions.
For simplicity denote $X := L^2([0,T]; H_x^3 H_{v,*}^1)$ and $X' := L^2([0,T]; H_x^3(H_{v,*}^1)')$ (see (8)), where $X'$ is the dual of $X$ w.r.t. $H_x^3 L_v^2$. We denote by $Y$ the set of functions defined by

$$Y = \left\{ g \in X, \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g \in X' \right\}.$$

We will construct a local solutions of the problem (139) on $Y$. In what follows, we consider $\varepsilon$ as a fixed parameter.

**Proposition A.1.** *Let $\varepsilon > 0$, we assume that $g_0 \in H_x^3 L_v^2$ and $\mathrm{U}_\varepsilon \in X'$. Then the Cauchy problem (139) admits a weak solution $g \in Y$.*

*Proof.* We take the change of unknown as follows

$$\tilde{g}(t,x,v) = e^{-\lambda t} g(t,x,v).$$

The linear Landau equation (139) for $\tilde{g} = \tilde{g}(t,x,v)$ takes the following form

$$\begin{cases} \partial_t \tilde{g} + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{g} + \frac{1}{\varepsilon^2} \mathcal{L}\tilde{g} + \lambda \tilde{g} = e^{-\lambda t} \mathrm{U}_\varepsilon := \tilde{\mathrm{U}}_\varepsilon \\ \tilde{g}(0,x,v) = g_0(x,v). \end{cases} \tag{140}$$

We introduce the following Lions Theorem, that we will use to prove the existence of solution, which can be found in [21].

**Theorem A.2** (Lions Theorem)**.** *Let* $\mathbf{F}$ *be a Hilbert space, provided with a norm* $\|\cdot\|_{\mathbf{F}}$*, and an inner product* $(\cdot,\cdot)_{\mathbf{F}}$*. Let* $\mathbf{V}$ *be a subspace of* $\mathbf{F}$*, provided with a prehilbertian norm* $\|\cdot\|_{\mathbf{V}}$*, such that the injection* $\mathbf{V} \hookrightarrow \mathbf{F}$*, is continuous. We consider a bilinear form* $\mathbf{E}$

$$\begin{array}{rcccc} \mathbf{E} & : & \mathbf{F} \times \mathbf{V} & \to & \mathbb{R} \\ & & (g,\phi) & \mapsto & \mathbf{E}(g,\phi) \end{array}$$

*such that* $\mathbf{E}(\cdot,\phi)$ *is continuous on* $\mathbf{F}$ *for any fixed* $\phi \in \mathbf{V}$*, and such that*

$$|\mathbf{E}(\phi,\phi)| \geq \alpha \|\phi\|_{\mathbf{V}}^2, \quad \forall \phi \in \mathbf{V}, \quad \textit{with}\, \alpha > 0.$$

*Then, given a linear form* $\mathbf{L}$ *in* $\mathbf{V}'$*, there exists a solution* $g$ *in* $\mathbf{F}$ *of the problem*

$$\mathbf{E}(g,\phi) = \mathbf{L}(\phi), \quad \forall \phi \in \mathbf{V}. \tag{141}$$

In the remaining part of the proof, to lighten the notations, we will drop the tildes. Let $\mathbf{F} = X$ be a Hilbert space with norm

$$\|f\|_{\mathbf{X}}^2 = \int_0^T \|f\|_{\mathcal{X}^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}^2 \mathrm{d}t,$$

where $\|f\|_{\mathcal{X}^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}$ is defined in (7). Let $\mathbf{V}$ be the space $C_c^\infty([0,T[\times \mathbb{T}_x^3 \times \mathbb{R}_v^3)$ of infinitely differentiable functions, with compact support in $[0,T[\times \mathbb{T}_x^3 \times \mathbb{R}_v^3$. $\mathbf{V}$ is provided with a norm defined by

$$\|\phi\|_{\mathbf{V}}^2 = \|\phi\|_X^2 + \frac{1}{2}\|\phi(0)\|_{H_x^3 L_v^2}^2, \quad \forall \phi \in \mathbf{V}.$$

We define the operator $\mathcal{Q}$ as follows:

$$\mathcal{Q} := -\partial_t - \frac{1}{\varepsilon} v \cdot \nabla_x + \frac{1}{\varepsilon^2}\mathcal{L}, \tag{142}$$

with domain $C_c^\infty([0,T[\times \mathbb{T}_x^3 \times \mathbb{R}_v^3)$. We also define the bilinear form $\mathbf{E}$, and the linear form $\mathbf{L}$ as following

$$\begin{aligned} \mathbf{E}(g,\phi) &:= \int_0^T \Big( (g, \mathcal{Q}\phi)_{H_x^3 L_v^2} + \lambda\, (g,\phi)_{H_x^3 L_v^2} \Big) \mathrm{d}t, \\ \mathbf{L}(\phi) &:= \langle \mathrm{U}_\varepsilon, \phi \rangle_{X',X} + (g_0, \phi(0))_{H_x^3 L_v^2}. \end{aligned}$$

The mapping $\mathbf{E}(\cdot,\phi)$ is continuous on $X$: let $g \in X$, we have

$$|\mathbf{E}(g,\phi)| \leq (\|\mathcal{Q}\phi\|_X + \lambda \|\phi\|_X)\|g\|_X \leq C_\phi \|g\|_X.$$

The mapping $\mathbf{E}$ is a bilinear and coercive form on $\mathbf{V}$: let $\phi \in \mathbf{V}$, using (33), we have

$$|\mathbf{E}(\phi,\phi)| \geq \frac{1}{2}\|\phi(0)\|^2_{H^3_xL^2_v} + \frac{C_\gamma}{\varepsilon^2}\int_0^T \mathcal{D}^2(\phi)\mathrm{d}t + \lambda\int_0^T \|\phi\|^2_{H^3_xL^2_v}\mathrm{d}t,$$

by using that

$$\|\phi_1\|^2_{\mathcal{X}^3(\mathbb{T}^3_x\times\mathbb{R}^3_v)} \leq C\|\phi\|^2_{H^3_xL^2_v},$$

where $C>0$ and $\phi_1 = \Pi_0\phi$ ( see (9)), we obtain

$$\begin{aligned}|\mathbf{E}(\phi,\phi)| &\geq \frac{1}{2}\|\phi(0)\|^2_{H^3_xL^2_v} + C_\gamma\int_0^T \mathcal{D}^2(\phi)\mathrm{d}t + \frac{\lambda}{C}\int_0^T \|\phi_1\|^2_{\mathcal{X}^3(\mathbb{T}^3_x\times\mathbb{R}^3_v)}\mathrm{d}t,\\ &\geq \min\Big(1, C_\gamma, \frac{\lambda}{C}\Big)\Big(\int_0^T \|\phi\|^2_{\mathcal{X}^3(\mathbb{T}^3_x\times\mathbb{R}^3_v)}\mathrm{d}t + \frac{1}{2}\|\phi(0)\|^2_{H^3_xL^2_v}\Big),\end{aligned}$$

then

$$|\mathbf{E}(\phi,\phi)| \geq \alpha\|\phi\|^2_{\mathbf{V}}.$$

The mapping $\mathbf{L}$ is a linear continuous form on $\mathbf{V}$: let $\phi \in \mathbf{V}$, we have

$$\begin{aligned}|\mathbf{L}(\phi)| &\leq \|\mathrm{U}_\varepsilon\|_{X'}\|\phi\|_X + \|g_0\|_{H^3_xL^2_v}\|\phi(0)\|_{H^3_xL^2_v}\\ &\leq \|\mathrm{U}_\varepsilon\|_{X'}\|\phi\|_{\mathbf{V}} + \sqrt{2}\,\|g_0\|_{H^3_xL^2_v}\|\phi\|_{\mathbf{V}}\\ &\leq (\|\mathrm{U}_\varepsilon\|_{X'} + \sqrt{2}\,\|g_0\|_{H^3_xL^2_v})\|\phi\|_{\mathbf{V}},\end{aligned}$$

then

$$|\mathbf{L}(\phi)| \leq C'\|\phi\|_{\mathbf{V}}.$$

Finally thanks to Lions Theorem, there exists a solution $g \in X$ for the variational problem (141) and in particular, we deduce that

$$\partial_t g + \frac{1}{\varepsilon}v\cdot\nabla_x g = \mathrm{U}_\varepsilon - \frac{1}{\varepsilon^2}\mathcal{L}g - \lambda g \in X',$$

then $g$ belongs to $Y$. Now, to prove that $g$ satisfies the initial condition, we use the following Lemma:

**Lemma A.3.** *1) If $g$ belongs to $Y$, $g$ admits (continuous) trace values $g(0,x,v)$, $g(T,x,v)$ in $H^3_xL^2_v$.*

*2) For $g$ and $\tilde{g}$ in $Y$, we have*

$$\begin{aligned}\Big\langle \partial_t g + \frac{1}{\varepsilon}v\cdot\nabla_x g, \tilde{g}\Big\rangle_{X',X} &+ \Big\langle \partial_t \tilde{g} + \frac{1}{\varepsilon}v\cdot\nabla_x \tilde{g}, g\Big\rangle_{X',X}\\ &= (g(T),\tilde{g}(T))_{H^3_xL^2_v} - (g(0),\tilde{g}(0))_{H^3_xL^2_v}.\end{aligned} \tag{143}$$

We note that the proof of Lemma A.3 is similar to the proof of Lemma A.1 in [9] by considering $H^3_xL^2_v$ as a pivot space. Now, using (140) and (143), we obtain that the solution $g$ of the variational problem (141) satisfies

$$(g(0)-g_0,\phi(0))_{H^3_xL^2_v} = 0, \quad \forall\phi \in \mathbf{V}.$$

Then, the initial condition is satisfied in $H^3_xL^2_v$. □

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