Reducibility of n-ary Semigroups: from Quasitriviality Towards Idempotency

Miguel Couceiro, Jimmy Devillet, Jean-Luc Marichal, Pierre Mathonet

REDUCIBILITY OF n-ARY SEMIGROUPS: FROM QUASITRIVIALITY TOWARDS IDEMPOTENCY

MIGUEL COUCEIRO, JIMMY DEVILLET, JEAN-LUC MARICHAL, AND PIERRE MATHONET

ABSTRACT. Let X be a nonempty set. Denote by $\mathcal{F}_k^n$ the class of associative operations $F\colon X^n \to X$ satisfying the condition $F(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}$ whenever at least k of the elements $x_1,\dots,x_n$ are equal to each other. The elements of $\mathcal{F}_1^n$ are said to be quasitrivial and those of $\mathcal{F}_n^n$ are said to be idempotent. We show that $\mathcal{F}_1^n = \cdots = \mathcal{F}_{n-2}^n \subseteq \mathcal{F}_{n-1}^n \subseteq \mathcal{F}_n^n$ and we give conditions on the set X for the last inclusions to be strict. The class $\mathcal{F}_1^n$ was recently characterized by Couceiro and Devillet [2], who showed that its elements are reducible to binary associative operations. However, some elements of $\mathcal{F}_n^n$ are not reducible. In this paper, we characterize the class $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ and show that its elements are reducible. We give a full description of the corresponding reductions and show how each of them is built from a quasitrivial semigroup and an Abelian group whose exponent divides $n-1$.

1. INTRODUCTION

Let X be a nonempty set, let $|X|$ be its cardinality, and let $n \geq 2$ be an integer. An n-ary operation $F\colon X^n \to X$ is said to be *associative* if

$$F(x_1,\dots,x_{i-1},F(x_i,\dots,x_{i+n-1}),x_{i+n},\dots,x_{2n-1}) = F(x_1,\dots,x_i,F(x_{i+1},\dots,x_{i+n}),x_{i+n+1},\dots,x_{2n-1}),$$

for all $x_1,\dots,x_{2n-1} \in X$ and all $1 \leq i \leq n-1$. The pair (X,F) is then called an *n-ary semigroup*. This notion is due to Dörnte [8] and has led to the concept of n-ary group, which was first studied by Post [13].

In [7] the authors investigated associative n-ary operations that are determined by binary associative operations. An n-ary operation $F\colon X^n \to X$ is said to be *reducible to* an associative binary operation $G\colon X^2 \to X$ if there are $G^m\colon X^{m+1} \to X$ ($m = 1,\dots,n-1$) such that $G^{n-1} = F$, $G^1 = G$, and

$$G^m(x_1,\dots,x_{m+1}) = G^{m-1}(x_1,\dots,x_{m-1},G(x_m,x_{m+1})), \qquad m \geq 2.$$

The pair (X,F) is then said to be the *n-ary extension* of (X,G). In that case, we also say that F is the n-ary extension of G.

Also, an n-ary operation $F\colon X^n \to X$ is said to be

- *idempotent* if $F(x,\dots,x) = x$ for all $x \in X$,
- *quasitrivial* [1, 11] (or *conservative* [14]) if $F(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}$ for all $x_1,\dots,x_n \in X$.

Date: June 16, 2020.

2010 *Mathematics Subject Classification.* Primary 20M10, 20N15; Secondary 16B99, 20K25.

Key words and phrases. Semigroup, polyadic semigroup, Abelian group, reducibility, quasitriviality, idempotency.

Clearly, any quasitrivial n-ary operation is idempotent. As we will illustrate below, the converse is not true, even for associative operations.

The quest for conditions under which an associative n-ary operation is reducible to an associative binary operation gained an increasing interest since the pioneering work of Post [13] (see, e.g., [1, 2, 4, 7, 9, 10]). A necessary and sufficient condition for reducibility was given by Dudek and Mukhin [7] using the concept of *neutral element*. Recall that an element $e \in X$ is said to be *neutral* for $F\colon X^n \to X$ if

$$F((k-1)\cdot e, x, (n-k)\cdot e) = x, \qquad x \in X,\ k \in \{1,\dots,n\}. \tag{1}$$

Here and throughout, for any $k \in \{0,\dots,n\}$ and any $x \in X$, the notation $k \cdot x$ stands for the k-tuple $x,\dots,x$. For instance, we have

$$F(3\cdot x, 0\cdot y, 2\cdot z) = F(x,x,x,z,z).$$

Throughout this paper we also denote the set of neutral elements for an operation $F\colon X^n \to X$ by E_F. Recall that for any binary operation $G\colon X^2 \to X$ we have $|E_G| \le 1$.

Dudek and Mukhin [7, Lemma 1] proved that if an associative operation $F\colon X^n \to X$ has a neutral element e, then it is reducible to the associative operation $G_e\colon X^2 \to X$ defined by

$$G_e(x,y) = F(x,(n-2)\cdot e, y), \qquad x,y \in X. \tag{2}$$

Furthermore, it was recently observed [2, Corollary 2.3] that all the quasitrivial associative n-ary operations are reducible to associative binary operations. However, there are associative operations that are neither quasitrivial nor reducible to any binary operation; for instance, the associative and idempotent ternary operation $F\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x,y,z) = x - y + z$ (see, e.g., [16] or more recently [12]).

The observations above show that it is natural to seek conditions under which an idempotent n-ary semigroup is reducible to a semigroup. To this extent, we will investigate certain subclasses of idempotent n-ary semigroups that contain the quasitrivial ones. In this direction, we will consider classes where the condition

$$F(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}$$

holds on at least some subsets of X^n. More precisely, for a set $S \subseteq \{1,\dots,n\}$, let

$$D_S^n = \{(x_1,\dots,x_n) \in X^n :\ \forall i,j \in S, x_i = x_j\},$$

and, for every $k \in \{1,\dots,n\}$, let

$$D_k^n = \bigcup_{\substack{S\subseteq\{1,\dots,n\}\\ |S|\ge k}} D_S^n = \bigcup_{\substack{S\subseteq\{1,\dots,n\}\\ |S|= k}} D_S^n.$$

Thus, the set D_k^n consists of those tuples of X^n for which at least k components are equal to each other. In particular, $D_1^n = X^n$ and $D_n^n = \{(x,\dots,x) : x \in X\}$.

For every $k \in \{1,\dots,n\}$, denote by $\mathcal{F}_k^n$ the class of those associative n-ary operations $F\colon X^n \to X$ that satisfy

$$F(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}, \quad \text{whenever } (x_1,\dots,x_n) \in D_k^n.$$

We say that these operations are *quasitrivial on* D_k^n.

Thus defined, $\mathcal{F}_1^n$ is exactly the class of quasitrivial associative n-ary operations and $\mathcal{F}_n^n$ is exactly the class of idempotent associative n-ary operations. It follows directly from the definition of the classes $\mathcal{F}_k^n$ that $\mathcal{F}_1^n = \mathcal{F}_2^n = \cdots = \mathcal{F}_n^n$ if $|X| \le 2$. Therefore, throughout

the rest of this paper we assume that $|X| \geq 3$. Since the sets D_k^n are nested in the sense that $D_{k+1}^n \subseteq D_k^n$ for $1 \leq k \leq n-1$, the classes $\mathcal{F}_k^n$ clearly form a filtration, that is,

$$\mathcal{F}_1^n \subseteq \mathcal{F}_2^n \subseteq \cdots \subseteq \mathcal{F}_n^n.$$

Quite surprisingly, we have the following result, which shows that this filtration actually reduces to three nested classes only.

Proposition 1.1. *For every $n \geq 3$, we have $\mathcal{F}_1^n = \mathcal{F}_{n-2}^n$.*

The proof is deferred until Section 2, and so are the proofs of the other results in this introduction.

We observe that the class $\mathcal{F}_1^n = \mathcal{F}_2^n = \cdots = \mathcal{F}_{n-2}^n$ was characterized by Couceiro and Devillet [2] who showed that all its elements are reducible. More precisely, the following result summarizes [2, Corollary 3.8] and [2, Corollary 3.11].

Proposition 1.2. *If $F\colon X^n \to X$ is an associative quasitrivial operation, then $|E_F| \leq 2$ and F has either one or two binary reductions. Furthermore, the binary reductions depend on E_F as follows.*

(a) *If $E_F = \varnothing$, then the operation $G\colon X^2 \to X$ defined for every $x, y \in X$ by*

$$G(x,y) = F((n-1)\cdot x, y) = F(x, (n-1)\cdot y)$$

is the only binary reduction of F, and G is quasitrivial.

(b) *If $E_F = \{e\}$, then the operation $G_e\colon X^2 \to X$ defined for every $x, y \in X$ by*

$$G_e(x,y) = F(x, (n-2)\cdot e, y) = F((n-1)\cdot x, y) = F(x, (n-1)\cdot y)$$

is the only binary reduction of F, and G_e is quasitrivial.

(c) *If $E_F = \{e_1, e_2\}$ (with $e_1 \neq e_2$), then the operations $G_{e_1}, G_{e_2}\colon X^2 \to X$ defined for every $x, y \in X$ by*

$$G_{e_1}(x,y) = F(x, (n-2)\cdot e_1, y) \quad \text{and} \quad G_{e_2}(x,y) = F(x, (n-2)\cdot e_2, y)$$

are the only binary reductions of F (and $G_{e_1} \neq G_{e_2}$). Neither of G_{e_1} and G_{e_2} is quasitrivial and, in this case, the identity $F((n-1)\cdot x, y) = F(x, (n-1)\cdot y)$ does not hold.

Proposition 1.2 is of particular interest since the class of associative and quasitrivial binary operations was characterized by Länger in [11, Theorem 1].

In this paper, we provide a characterization of the class $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$. We show that all of its elements are also reducible to binary associative operations. We give a full description of the possible reductions of the operations in this class.

Let us begin with the particular case when all the elements in X are neutral. Recall that a group (X, G) with neutral element e has *bounded exponent* if there exists an integer $m \geq 1$ such that $G^{m-1}(m\cdot x) = e$ for any $x \in X$ (with the usual convention that $G^0(x) = x$ for every $x \in X$). In that case, the *exponent* of the group is the smallest integer having this property. The following result provides a description of the class of n-ary semigroups containing only neutral elements. It was stated without proof in [6, p. 2] in the framework of n-ary groups, but it can be easily extended to n-ary semigroups by using [6, Corollary 4]. For the sake of completeness, we provide a direct proof that basically uses [7, Lemma 1].

Theorem 1.3. *Let $F\colon X^n \to X$ ($n \geq 3$) be an associative operation. Then $E_F = X$ if and only if (X, F) is the n-ary extension of an Abelian group whose exponent divides $n-1$.*

Abelian groups having bounded exponent play a central role in this first result, but also in the next theorems. We recall that Prüfer and Baer (see, e.g., [15, Corollary 10.37]) showed that if an Abelian group has bounded exponent, then it is isomorphic to a direct sum of cyclic groups. Hence, the exponent of an Abelian group divides $n-1$ if and only if the Abelian group is isomorphic to a direct sum of cyclic groups whose orders divide $n-1$.

Theorem 1.3 also highlights the fact that an n-ary associative operation may have several reductions, associated with distinct neutral elements. For instance, the ternary sum on $\mathbb{Z}_2$ has two neutral elements, namely 0 and 1. It is reducible to the operations $G_0, G_1 \colon \mathbb{Z}_2^2 \to \mathbb{Z}_2$ defined by $G_0(x,y) = x + y \pmod 2$ and $G_1(x,y) = x + y + 1 \pmod 2$, respectively. We can also easily see that the semigroups $(\mathbb{Z}_2, G_0)$ and $(\mathbb{Z}_2, G_1)$ are isomorphic. In fact, this result can be generalized to other underlying sets: all reductions obtained in this way are isomorphic, as stated in the following result.

Proposition 1.4. *Let $F \colon X^n \to X$ ($n \geq 3$) be an associative operation such that $E_F \neq \varnothing$. Then every reduction of F is of the form G_e for some $e \in E_F$. Moreover, if $e_1, e_2 \in E_F$, then (X, G_{e_1}) and (X, G_{e_2}) are isomorphic.*

In order to state one of the main results of this paper, we shall make use of the following classes of operations. Recall that an element $a \in X$ is said to be an *annihilator* for $F \colon X^n \to X$ if $F(x_1, \ldots, x_n) = a$ whenever $a \in \{x_1, \ldots, x_n\}$.

Definition 1.5. For every integer $m \geq 1$, let $\mathcal{H}_m$ be the class of binary operations $G \colon X^2 \to X$ such that there exists a subset $Y \subseteq X$ with $|Y| \geq 3$ for which the following assertions hold.

(a) $(Y, G|_{Y^2})$ is an Abelian group whose exponent divides m.
(b) $G|_{(X \setminus Y)^2}$ is associative and quasitrivial.
(c) Any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\} \cup Y)^2}$.

Note that $\mathcal{H}_1 = \varnothing$. As we will see, all operations in $\mathcal{H}_m$ are associative, and the set Y is unique. In fact, the family of classes $\mathcal{H}_m$ is the key for the characterization of the classes $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$.

Theorem 1.6. *Every $G \in \mathcal{H}_m$ is associative ($m \geq 1$). If $G \in \mathcal{H}_{n-1}$, then its n-ary extension $F = G^{n-1}$ is in $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$. Conversely, for every $F \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ we have that $|E_F| \geq 3$, and the reductions of F are exactly the operations G_e for $e \in E_F$ and they lie in $\mathcal{H}_{n-1}$.*

As an immediate corollary we solve the reducibility problem for operations in $\mathcal{F}_{n-1}^n$.

Corollary 1.7. *Every operation in $\mathcal{F}_{n-1}^n$ is reducible to a binary associative operation.*

Theorem 1.6 is of particular interest as it enables us to easily construct n-ary operations in $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$. For instance, for any integers $n \geq 3$ and $p \geq 1$, the operation of the cyclic group $(\mathbb{Z}_n, +)$ is in $\mathcal{H}_{np}$, and thus the operation associated with its $(np+1)$-ary extension is in $\mathcal{F}_{np}^{np+1} \setminus \mathcal{F}_1^{np+1}$.

To give another example, consider the chain $(X, \leq) = (\{1,2,3,4,5\}, \leq)$ together with the operation $G \colon X^2 \to X$ defined by the following conditions:

- $(\{1,2,3\}, G|_{\{1,2,3\}^2})$ is isomorphic to $(\mathbb{Z}_3, +)$,
- $G|_{\{4,5\}^2} = \vee|_{\{4,5\}^2}$, where $\vee \colon X^2 \to X$ is the maximum operation for $\leq$,
- for any $x \in \{1,2,3\}$, $G(x,4) = G(4,x) = 4$ and $G(x,5) = G(5,x) = 5$.

Then we have $G \in \mathcal{H}_{3p}$ for any integer $p \geq 1$ and so G^{3p} is in $\mathcal{F}_{3p}^{3p+1} \setminus \mathcal{F}_1^{3p+1}$.

Now we give a reformulation of Theorem 1.6 that is not based on binary reductions.

Theorem 1.8. *If $F \in \mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1$, then, setting $Y = E_F$, we have that $|Y| \geq 3$ and the following assertions hold.*

(a) *$(Y, F|_{Y^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$.*
(b) *$F|_{(X\setminus Y)^n}$ is associative, quasitrivial, and has at most one neutral element.*
(c) *For all $x_1, \ldots, x_n \in X$ and $i \in \{1, \ldots, n-1\}$ such that $\{x_i, x_{i+1}\} \cap (X \setminus Y) = \{x\}$ we have*

$$F(x_1, \ldots, x_n) = F(x_1, \ldots, x_{i-1}, x, x, x_{i+2}, \ldots, x_n).$$

Conversely, if an operation F satisfies these conditions for some $Y \subseteq X$ with $|Y| \geq 3$, then $F \in \mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1$ and $E_F = Y$.

Proposition 1.1 shows that all operations in $\mathcal{F}^n_{n-2}$ are quasitrivial. The examples we just presented show that there are operations in $\mathcal{F}^n_{n-1}$ that are not quasitrivial, for some $n \geq 3$ and some sets X. Theorem 1.6 enables us to provide necessary and sufficient conditions on the set X for such operations to exist.

Definition 1.9. For any integer $m \geq 2$, let c_m denote the cardinality of the smallest Abelian group with at least three elements whose exponent divides m.

Proposition 1.10. *For every $n \geq 3$, we have $\mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1 \neq \varnothing$ if and only if $|X| \geq c_{n-1}$.*

Corollary 1.11. *For any integer $n \geq 3$, let p be the least odd prime divisor of $n-1$ if $n-1$ is not a power of 2; otherwise, set $p = 4$. The following assertions hold.*

(a) *If n is even, then $\mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1 \neq \varnothing$ if and only if $|X| \geq p$.*
(b) *If n is odd, then $\mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1 \neq \varnothing$ if and only if $|X| \geq \min(4, p)$.*

Finally, we observe that if $(X, \leq)$ is a semilattice that is not a chain, then the n-ary operation $F\colon X^n \to X$ defined by $F(x_1, \ldots, x_n) = x_1 \vee \ldots \vee x_n$ is in $\mathcal{F}^n_n$. However, it is not in $\mathcal{F}^n_{n-1}$ since $F((n-1)\cdot x, y) \notin \{x, y\}$ whenever x and y are not comparable, i.e, $x \vee y \notin \{x, y\}$. Since such a semilattice structure exists on every set X such that $|X| \geq 3$, we obtain the following result.

Proposition 1.12. *For every $n \geq 2$, we have $\mathcal{F}^n_n \setminus \mathcal{F}^n_{n-1} \neq \varnothing$ if and only if $|X| \geq 3$.*

In Section 2 we give the proofs of the results above, using some more technical statements that may have interest on their own. In Section 3 we introduce and investigate an alternative hierarchy of subclasses of idempotent operations. We end the paper by some concluding remarks in Section 4.

2. Technicalities and proofs of the main results

Let us begin with Proposition 1.1, which essentially follows from the very definition of the classes $\mathcal{F}^n_k$.

Proof of Proposition 1.1. We only need to prove that $\mathcal{F}^n_{n-2} \subseteq \mathcal{F}^n_1$, and so we can assume that $n \geq 4$. Let $F \in \mathcal{F}^n_{n-2}$ and let us show by induction that for every $k \in \{1, \ldots, n\}$ we have

$$F(k\cdot x_1, x_{k+1}, \ldots, x_n) \in \{x_1, x_{k+1}, \ldots, x_n\}, \qquad x_1, x_{k+1}, \ldots, x_n \in X. \tag{3}$$

By the definition of $\mathcal{F}^n_{n-2}$, condition (3) holds for any $k \in \{n-2, n-1, n\}$. Let us now assume that it holds for some $k \in \{2, \ldots, n\}$ and let us show that it still holds for $k-1$. Using associativity and idempotency, we have

$$\begin{aligned} F((k-1)\cdot x_1, x_k, \ldots, x_n) &= F(F(n\cdot x_1), (k-2)\cdot x_1, x_k, \ldots, x_n) \\ &= F(k\cdot x_1, F((n-2)\cdot x_1, x_k, x_{k+1}), \ldots, x_n). \end{aligned}$$

By the induction hypothesis, the latter expression lies in $\{x_1, x_k, \dots, x_n\}$.

Thus, Equation (3) holds for every $k \in \{1, \dots, n\}$. Using it for $k = 1$, we obtain that F is quasitrivial. □

In Theorem 1.3 and Proposition 1.4 we deal with neutral elements. We first state and prove some intermediate results concerning such elements. The following two lemmas were stated and proved in [5, Theorem 3] for n-ary groups. We provide a proof of the first one that does not use the n-ary group structure but basically uses [7, Lemma 1], and we give a slightly different proof for the second one in the framework of n-ary semigroups.

Lemma 2.1. *Let $F\colon X^n \to X$ be an associative operation and let $e \in E_F$. Then, for any $x_1, \dots, x_{n-1} \in X$ we have*

$$F(x_1, \dots, x_{n-1}, e) = F(x_1, \dots, e, x_{n-1}) = \cdots = F(e, x_1, \dots, x_{n-1}).$$

Moreover, for any $x \in X$ the restriction $F|_{(\{x\} \cup E_F)^n}$ is symmetric.

Proof. Let $x_1, \dots, x_{n-1} \in X$ and let G_e be the reduction of F defined by (2). For $i \in \{1, \dots, n-1\}$ we have $G_e(x_i, e) = x_i = G_e(e, x_i)$, which proves the first part of the statement for $n = 2$. For $n \geq 3$ we have

$$F(x_1, \dots, x_i, e, x_{i+1}, \dots, x_{n-1}) = G_e^{n-2}(x_1, \dots, x_{i-1}, G_e(x_i, e), x_{i+1}, \dots, x_{n-1}),$$

and the first part of the statement follows from the fact that each x_i commutes with e in G_e. The second part is a direct consequence of the first part. □

Lemma 2.2. *Let $F\colon X^n \to X$ be an associative operation such that $E_F \neq \varnothing$. Then F preserves E_F, i.e., $F(E_F^n) \subseteq E_F$.*

Proof. Let $e_1, \dots, e_n \in E_F$ and let us show that $F(e_1, \dots, e_n) \in E_F$. By Lemma 2.1 and associativity of F, for any $x \in X$ we have

$$\begin{aligned} &F((n-1)\cdot F(e_1, \dots, e_n), x) \\ &\quad = F(F(e_1, (n-1)\cdot e_2), F(e_1, (n-1)\cdot e_3), \dots, F(e_1, (n-1)\cdot e_n), x) \\ &\quad = F((n-1)\cdot e_1, x) = x. \end{aligned}$$

Similarly, for any $x \in X$ we can show that

$$F(i\cdot F(e_1, \dots, e_n), x, (n-i-1)\cdot F(e_1, \dots, e_n)) = x, \qquad i \in \{0, \dots, n-2\}.$$

Thus $F(e_1, \dots, e_n)$ satisfies (1), i.e., $F(e_1, \dots, e_n) \in E_F$. □

Combining Lemmas 2.1 and 2.2, we immediately derive the following result.

Corollary 2.3. *If (X, F) is an n-ary monoid, then $(E_F, F|_{E_F^n})$ is a symmetric n-ary monoid.*

Proof of Theorem 1.3. (Sufficiency) Obvious.

(Necessity) Suppose that $X = E_F$. Let $e \in E_F$ and let $G_e\colon X^2 \to X$ be the corresponding reduction of F defined by (2). Recall that e is the (unique) neutral element of G_e by (2). By Corollary 2.3, we have that F is symmetric. Thus, we have that G_e also is symmetric. Moreover, since G_e is a binary reduction of F and $E_F = X$, it follows that

$$G_e(G_e^{n-2}((n-1)\cdot x), y) = y = G_e(y, G_e^{n-2}((n-1)\cdot x)), \qquad x, y \in X,$$

which shows that $G_e^{n-2}((n-1)\cdot x) \in E_{G_e}$ for any $x \in X$. However, since $E_{G_e} = \{e\}$, we have that $G_e^{n-2}((n-1)\cdot x) = e$ for any $x \in X$. Thus, (X, G_e) is an Abelian group whose exponent divides $n-1$. □

The following result follows immediately from Theorem 1.3.

Corollary 2.4. *If (X,F) is an n-ary monoid, then $(E_F, F|_{E_F^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$.*

Proof of Proposition 1.4. The first part of the statement follows from [2, Proposition 3.3]. Moreover, the associativity of F and the definition of neutral elements ensure that the map $\psi\colon X \to X$ defined by

$$\psi(x) = F(e_2, x, (n-2)\cdot e_1)$$

is a bijection and that $\psi^{-1}(x) = F((n-2)\cdot e_2, x, e_1)$. We then have

$$\begin{aligned}
&G_{e_2}(\psi(x),\psi(y))\\
&= F(F(e_2,x,(n-2)\cdot e_1),(n-2)\cdot e_2,F(e_2,y,(n-2)\cdot e_1))\\
&= F(F(e_2,x,(n-2)\cdot e_1),F((n-1)\cdot e_2,y),(n-2)\cdot e_1)\\
&= F(F(e_2,x,(n-2)\cdot e_1),y,(n-2)\cdot e_1)\\
&= F(e_2,F(x,(n-2)\cdot e_1,y),(n-2)\cdot e_1)\\
&= \psi(G_{e_1}(x,y)),
\end{aligned}$$

which completes the proof. □

Let us now prove Theorem 1.6. To this extent, we first state and prove some intermediate results. We have the following remarkable lemma, which characterizes the existence of a pair of neutral elements for $F \in \mathcal{F}_{n-1}^n$ by means of two identities.

Lemma 2.5. *Let $F \in \mathcal{F}_{n-1}^n$ and let $a,b \in X$ such that $a \neq b$. Then $a,b \in E_F$ if and only if $F((n-1)\cdot a, b) = b$ and $F(a,(n-1)\cdot b) = a$.*

Proof. (Necessity) Obvious.

(Sufficiency) For any $x \in X$, we have

$$\begin{aligned}
F((n-1)\cdot a, x) &= F((n-2)\cdot a, F(a,(n-1)\cdot b), x)\\
&= F(F((n-1)\cdot a, b),(n-2)\cdot b, x) = F((n-1)\cdot b, x),
\end{aligned}$$

which implies that $F((n-1)\cdot a, x) = F((n-1)\cdot b, x) = x$ for any $x \in X$. Indeed, for $x \in \{a,b\}$ this relation follows from idempotency, and for $x \notin \{a,b\}$ we have

$$F((n-1)\cdot a, x) = F((n-1)\cdot b, x) \in \{a,x\} \cap \{b,x\} = \{x\},$$

due to the definition of $\mathcal{F}_{n-1}^n$. Similarly, we get $F(x,(n-1)\cdot a) = x = F(x,(n-1)\cdot b)$ for any $x \in X$. It follows from these relations, together with associativity of F, that for any $k \in \{1,\dots,n-2\}$, the maps $\psi_k, \xi_k\colon X \to X$ defined by

$$\begin{aligned}
\psi_k(x) &= F(k\cdot a, x, (n-k-1)\cdot a)\\
\xi_k(x) &= F(k\cdot b, x, (n-k-1)\cdot b)
\end{aligned}$$

are bijections with inverse maps ψ_{n-k-1} and ξ_{n-k-1}, respectively. It then follows that, for any $k \in \{1,\dots,n-2\}$, we have $F(k\cdot a, x, (n-k-1)\cdot a) = \psi_k(x) = x$ for every $x \in X$. Indeed, for $x = a$, this relation follows from idempotency, and for $x \neq a$, we have $\psi_k(x) \in \{a,x\}$ and $\psi_k(x) \neq a$. Similarly, we can show that $F(k\cdot b, x, (n-k-1)\cdot b) = \xi_k(x) = x$ for every $x \in X$, which shows that $a,b \in E_F$. □

Given an associative operation $F\colon X^n \to X$, we can define the sequence $(F^q)_{q\geq 1}$ of $(qn-q+1)$-ary associative operations inductively by the rules $F^1 = F$ and

$$F^q(x_1,\dots,x_{qn-q+1}) = F^{q-1}(x_1,\dots,x_{(q-1)n-q+1}, F(x_{(q-1)n-q+2},\dots,x_{qn-q+1})),$$

for any integer $q \geq 2$ and any $x_1, \ldots, x_{qn-q+1} \in X$.

The following proposition shows that every n-tuple that violates the quasitriviality condition for $F \in \mathcal{F}_{n-1}^n$ belongs to E_F^n.

Proposition 2.6. *Let $F \in \mathcal{F}_{n-1}^n$. For any $a_1, \ldots, a_n \in X$ such that $F(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$, we have that $a_1, \ldots, a_n, F(a_1, \ldots, a_n) \in E_F$. Moreover, $F|_{(X \setminus E_F)^n}$ is quasitrivial.*

Proof. The case $n = 2$ is trivial. So assume that $n \geq 3$. Let us prove by induction on $k \in \{1, \ldots, n-1\}$ that for every $a_1, a_2, \ldots, a_{k+1} \in X$ the condition

$$F((n-k) \cdot a_1, a_2, \ldots, a_{k+1}) \notin \{a_1, \ldots, a_{k+1}\}$$

implies $a_1, \ldots, a_{k+1} \in E_F$. For $k = 1$, there is nothing to prove. We thus assume that the result holds true for a given $k \in \{1, \ldots, n-2\}$ and we show that it still holds for $k+1$. Now, consider elements $a_1, \ldots, a_{k+2}$ such that

$$F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \notin \{a_1, \ldots, a_{k+2}\}. \tag{4}$$

We first prove that $a_1, a_2 \in E_F$.

If $a_1 = a_2$, then $a_1, \ldots, a_{k+2} \in E_F$ by the induction hypothesis.

If $a_1 \neq a_2$, then we prove that $F((n-1) \cdot a_1, a_2) = a_2$ and $F(a_1, (n-1) \cdot a_2) = a_1$, which show that $a_1, a_2 \in E_F$ by Lemma 2.5.

- For the sake of a contradiction, assume first that $F((n-1) \cdot a_1, a_2) = a_1$. Then, for $\ell \geq 1$ we have

$$\begin{aligned} &F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \\ &\quad = \quad F^{\ell+1}(((n-k-1) + \ell(n-2)) \cdot a_1, (\ell+1) \cdot a_2, \ldots, a_{k+2}). \end{aligned} \tag{5}$$

  Choosing $\ell = n-k-1$ and using idempotency of F, we obtain

$$F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \;=\; F^2((n-1) \cdot a_1, (n-k) \cdot a_2, a_3, \ldots, a_{k+2}).$$

  Since the left-hand side of this equation does not lie in $\{a_1, \ldots, a_{k+2}\}$ by (4), we obtain

$$F((n-k) \cdot a_2, a_3, \ldots, a_{k+2}) \notin \{a_1, \ldots, a_{k+2}\}.$$

  By the induction hypothesis, we have $a_2, \ldots, a_{k+2} \in E_F$. Then choosing $\ell = n-2$ in (5) and using idempotency and the fact that $a_2 \in E_F$, we obtain

$$\begin{aligned} &F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \\ &\quad = \quad F^{n-1}(((n-k-1) + (n-2)^2) \cdot a_1, (n-1) \cdot a_2, \ldots, a_{k+2}) \\ &\quad = \quad F^2((n-k) \cdot a_1, (n-1) \cdot a_2, a_3, \ldots, a_{k+2}) \\ &\quad = \quad F((n-k) \cdot a_1, a_3, \ldots, a_{k+2}). \end{aligned}$$

  By the induction hypothesis, we have $a_1 \in E_F$. We then have $F((n-1) \cdot a_1, a_2) = a_2 \neq a_1$, a contradiction.

- Assume now that $F(a_1, (n-1) \cdot a_2) = a_2$. Then, for $\ell \geq 1$ we have

$$\begin{aligned} &F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \\ &\quad = \quad F^{\ell+1}((n-k-1+\ell) \cdot a_1, (\ell(n-2)+1) \cdot a_2, \ldots, a_{k+2}). \end{aligned}$$

  For $\ell = k$, using idempotency and the fact that $k(n-2)+1 = n-k+(k-1)(n-1)$, we obtain

$$\begin{aligned} &F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \\ &\quad = \quad F^2((n-1) \cdot a_1, (n-k) \cdot a_2, a_3, \ldots, a_{k+2}). \end{aligned}$$

Thus, $F((n-k)\cdot a_2, a_3, \ldots, a_{k+2}) \notin \{a_1, \ldots, a_{k+2}\}$. By the induction hypothesis, we have $a_2, \ldots, a_{k+2} \in E_F$. It follows that $F(a_1, (n-1)\cdot a_2) = a_1 \neq a_2$, a contradiction.

Now, since $a_2 \in E_F$, it commutes with all other arguments of F by Lemma 2.1. Also, by (4) we have

$$F((n-k-1)\cdot a_1, a_3, \ldots, a_{k+2}, a_2) \notin \{a_1, \ldots, a_{k+2}\},$$

and thus $a_3 \in E_F$. Repeating this argument, we have that $a_1, \ldots, a_{k+2} \in E_F$.

It follows from the induction that if $F(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$, then $a_1, \ldots, a_n \in E_F$. Finally we have $F(a_1, \ldots, a_n) \in E_F$ by Lemma 2.2. The second part is straightforward. □

Proposition 1.2 shows that a quasitrivial n-ary semigroup cannot have more than two neutral elements. The next result shows that an operation in $\mathcal{F}^n_{n-1}$ is quasitrivial whenever it has at most two neutral elements.

Corollary 2.7. *An operation $F \in \mathcal{F}^n_{n-1}$ is quasitrivial if and only if $|E_F| \leq 2$.*

Proof. (Necessity) This follows from Proposition 1.2.

(Sufficiency) Suppose that F is not quasitrivial, i.e., there exist $a_1, \ldots, a_n \in X$ such that $F(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$. Since F is idempotent, we must have $|\{a_1, \ldots, a_n\}| \geq 2$ and so $|\{a_1, \ldots, a_n, F(a_1, \ldots, a_n)\}| \geq 3$. We also have $\{a_1, \ldots, a_n, F(a_1, \ldots, a_n)\} \subseteq E_F$ by Proposition 2.6. Therefore we have $|E_F| \geq 3$. □

Proposition 2.8. *Let $F \in \mathcal{F}^n_{n-1}$ and suppose that $|E_F| \geq 3$. Then, any element $x \in X \setminus E_F$ is an annihilator of $F|_{(\{x\} \cup E_F)^n}$. Moreover, $F|_{(X \setminus E_F)^n}$ is quasitrivial and has at most one neutral element.*

Proof. Let $x \in X \setminus E_F$ and $e \in E_F$ and let us show that $F(k\cdot x, (n-k)\cdot e) = x$ for any $k \in \{1, \ldots, n-1\}$. If $k = 1$, then this equality follows from the definition of a neutral element. Now, suppose that there exists $k \in \{2, \ldots, n-1\}$ such that $F(k\cdot x, (n-k)\cdot e) \neq x$. Since $x \in X \setminus E_F$, by Proposition 2.6 we must have $F(k\cdot x, (n-k)\cdot e) = e$. But then, using the associativity of F, we get

$$\begin{aligned} F((n-1)\cdot x, e) &= F((n-1)\cdot x, F(k\cdot x, (n-k)\cdot e)) \\ &= F(k\cdot x, (n-k)\cdot e) = e, \end{aligned}$$

and we conclude by Lemma 2.5 that $x \in E_F$, which contradicts our assumption. Thus, we have

$$F(k\cdot x, (n-k)\cdot e) = x, \qquad k \in \{1, \ldots, n-1\}. \tag{6}$$

Now, let us show that $F(k\cdot x, e_{k+1}, \ldots, e_n) = x$ for any $k \in \{1, \ldots, n-1\}$ and any $e_{k+1}, \ldots, e_n \in E_F$. To this extent, we only need to show that

$$F(k\cdot x, e_{k+1}, \ldots, e_n) = F((k+1)\cdot x, e_{k+2}, \ldots, e_n),$$

for any $k \in \{1, \ldots, n-1\}$ and any $e_{k+1}, \ldots, e_n \in E_F$. So, let $k \in \{1, \ldots, n-1\}$ and $e_{k+1}, \ldots, e_n \in E_F$. Using (6) and the associativity of F we get

$$\begin{aligned} F(k\cdot x, e_{k+1}, \ldots, e_n) &= F((k-1)\cdot x, F(2\cdot x, (n-2)\cdot e_{k+1}), e_{k+1}, \ldots, e_n) \\ &= F(k\cdot x, F(x, (n-1)\cdot e_{k+1}), e_{k+2}, \ldots, e_n) \\ &= F((k+1)\cdot x, e_{k+2}, \ldots, e_n), \end{aligned}$$

which completes the proof by idempotency of F and Lemma 2.1. For the second part of the proposition, we observe that $F|_{(X\setminus E_F)^n}$ is quasitrivial by Proposition 2.6. Also, using (6) and the associativity of F, for any $x, y \in X \setminus E_F$ and any $e \in E_F$ we obtain

$$\begin{aligned} F((n-1)\cdot x, y) &= F((n-1)\cdot x, F(e,(n-1)\cdot y)) \\ &= F(F((n-1)\cdot x, e),(n-1)\cdot y) \;=\; F(x,(n-1)\cdot y), \end{aligned}$$

which shows that $F|_{(X\setminus E_F)^n}$ cannot have more than one neutral element. □

Proof of Theorem 1.6. In order to show that every $G \in \mathcal{H}_m$ is associative, we have to compare the expressions $G(G(x_1,x_2),x_3)$ and $G(x_1,G(x_2,x_3))$ for all x_1,x_2,x_3 in X. Clearly these expressions are equal if all their arguments are either in Y or in $X \setminus Y$ since the restriction of G to these subsets is associative. If one argument, say x_i, is in $X \setminus Y$ and the others are in Y, then both expressions are equal to x_i by Property (c) of Definition 1.5. For the same reason, if the arguments x_i, x_j are in $X \setminus Y$ and the the third one in Y, then both expressions are equal to $G(x_i,x_j)$.

Now, we consider $G \in \mathcal{H}_{n-1}$ and define $F = G^{n-1}$. Then we have $E_F = Y$. Indeed, conditions (a) and (c) of Definition 1.5 imply directly that $Y \subseteq E_F$. Moreover if $x \notin Y$, then still by condition (c) we have $F((n-1)\cdot x, y) = x \neq y$ for $y \in Y$, so $x \notin E_F$.

Next, we show that $F(k\cdot x, y, (n-k-1)\cdot x) \in \{x,y\}$ for every $x, y \in X$. If $x \in Y$, x is a neutral element, so this expression is equal to y. If $x \in X \setminus Y$, then either $y \in Y$ and this expression is equal to x (by condition (c)), or $y \in X \setminus Y$, and this expression is in $\{x,y\}$ (by condition (b)). Finally, $F \notin \mathcal{F}_1^n$ by Corollary 2.7, since $|E_F| = |Y| \geq 3$.

Now we prove the converse statement and consider $F \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$. Setting $Y = E_F$ we have $|Y| \geq 3$ by Corollary 2.7. By Proposition 1.4, every reduction of F is of the form G_e for some $e \in E_F$.

Finally, we show that $G_e \in \mathcal{H}_{n-1}$. We have that $(Y, G_e|_{Y^2})$ is an Abelian group whose exponent divides $n-1$ by Corollary 2.4 and Proposition 1.4. Also, we have that $G_e|_{(X\setminus Y)^2}$ is quasitrivial by Propositions 1.2 and 2.8. Finally, we have that any $x \in X \setminus Y$ is an annihilator for $G_e|_{(\{x\}\cup Y)^2}$ by Proposition 2.8. □

Proof of Corollary 1.7. This follows from Proposition 1.2 and Theorem 1.6. □

Remark 1. In the proof of Corollary 1.7 we used [2, Corollary 3.11] which is based on results obtained by Ackerman [1]. In the appendix we provide an alternative proof of Corollary 1.7 that does not make use of [2, Corollary 3.11].

Proof of Theorem 1.8. If $F \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$, then by Theorem 1.6 we have $|E_F| \geq 3$ and for every $e \in E_F$, G_e is in $\mathcal{H}_{n-1}$. Then $G_e|_{Y^2}$ is a reduction of $F|_{Y^n}$ and (a) holds true. Also $G_e|_{(X\setminus Y)^2}$ is a quasitrivial reduction of $F|_{(X\setminus Y)^n}$, so (b) holds true by Proposition 1.2. Finally, if x_i, x_{i+1} satisfy the conditions of (c), we have $G_e(x_i,x_{i+1}) = x = G_e(x,x)$, so that (c) holds true.

Let us now assume that an operation F satisfies conditions (a), (b), and (c). By (a), there exists an Abelian group (Y, G_Y) whose exponent divides $n-1$ such that $(Y, F|_{Y^n})$ is the n-ary extension of (Y, G_Y). We denote by e the neutral element of G_Y. We also define the operation $G\colon X^2 \to X$ by $G(x,y) = F(x,(n-2)\cdot e, y)$ for every $x, y \in X$. We now show that G is in $\mathcal{H}_{n-1}$. It is easy to see that $G|_{Y^2} = G_Y$. Then, by condition (c), $G|_{(X\setminus Y)^2}(x,y) = F((n-1)\cdot x, y)$, so $G|_{(X\setminus Y)^2}$ is the unique quasitrivial reduction of $F|_{(X\setminus Y)^n}$ (see Proposition 1.2). Finally, condition (c) also implies that any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\}\cup Y)^2}$. Then by Theorem 1.6, G is associative and we have $G^{n-1} \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$. We conclude the proof by showing that $G^{n-1} = F$. To this aim we compare

$G^{n-1}(x_1,\dots,x_n)$ and $F(x_1,\dots,x_n)$ for every $(x_1,\dots,x_n)\in X^n$. We already showed that both expressions coincide if $(x_1,\dots,x_n)$ belongs to Y^n or $(X\setminus Y)^n$. Otherwise, let us denote by $\sigma_1,\dots,\sigma_r$ the integers such that $\{i : x_i\in X\setminus Y\}=\{\sigma_1,\dots,\sigma_r\}$ and $\sigma_1<\cdots<\sigma_r$. By condition (c) there exist integers $a_1,\dots,a_r$ such that

$$F(x_1,\dots,x_n)=F(a_1\cdot x_{\sigma_1},\dots,a_r\cdot x_{\sigma_r}).$$

This expression is equal to $G^{r-1}(x_{\sigma_1},\dots,x_{\sigma_r})$ because $G|_{(X\setminus Y)^2}$ is a quasitrivial reduction of $F|_{(X\setminus Y)^n}$. Using condition (c) in Definition 1.5 for $G\in\mathcal{H}_{n-1}$ we get that this expression is equal to $G^{n-1}(x_1,\dots,x_n)$. □

Proof of Proposition 1.10. If $\mathcal{F}^n_{n-1}\setminus\mathcal{F}^n_1\neq\varnothing$, then Theorem 1.6 implies that there is a subset $Y\subseteq X$ and an Abelian group (Y,G) whose exponent divides $n-1$ and $|Y|\geq 3$. This shows that $|X|\geq|Y|\geq c_{n-1}$. Conversely, assume that $|X|\geq c_{n-1}$. Then we choose a subset $Y\subseteq X$ such that $|Y|=c_{n-1}\geq 3$ and we endow Y with an operation G_Y such that (Y,G_Y) is an Abelian group whose exponent divides $n-1$.

Let us consider the operation $G\colon X^2\to X$ defined by the conditions that any $x\in X\setminus Y$ is an annihilator for $G|_{(\{x\}\cup Y)^2}$, that $G|_{Y^2}=G_Y$, and that $G(x,y)=y$ for any $x,y\in X\setminus Y$. Then we have $G\in\mathcal{H}_{n-1}$ and so $G^{n-1}\in\mathcal{F}^n_{n-1}\setminus\mathcal{F}^n_1$ by Theorem 1.6, which concludes the proof. □

Proof of Corollary 1.11. By Proposition 1.10 it is sufficient to compute c_{n-1} in the two cases.

(a) The cyclic group of order p is an Abelian group with at least three elements whose exponent divides $n-1$, hence $c_{n-1}\leq p$. On the other hand, let (Y,G) be any Abelian group with at least three elements whose exponent m divides $n-1$. Let q be a prime divisor of m; then q divides $n-1$, hence q is odd. From the definition of the exponent it follows that Y contains an element of order q, thus $|Y|\geq q$. Since q divides $n-1$, we have $q\geq p$ by the minimality of p. Therefore, $|Y|\geq q\geq p$, which shows that $c_{n-1}\geq p$.

(b) If $p=3$, then we can take the group $\mathbb{Z}_p$ as in the previous case; if $p\geq 5$, then we can take the group $\mathbb{Z}_2^2$ (with exponent 2 dividing $n-1$) in order to see that $c_{n-1}\leq\min(4,p)$. Conversely, let (Y,G) be any Abelian group with at least three elements and with exponent m such that m divides $n-1$. If m has an odd prime divisor q, then we can conclude that $|Y|\geq q\geq p\geq\min(4,p)$ just as in (a). If m has no odd prime divisors, then m is a power of 2, and then $|Y|$ is even, which together with $|Y|\geq 3$ implies that $|Y|\geq 4\geq\min(4,p)$. Thus, we conclude that $c_{n-1}\geq\min(4,p)$. □

3. An alternative hierarchy

For any integer $k\geq 1$, let S^n_k be the set of n-tuples $(x_1,\dots,x_n)\in X^n$ such that $|\{x_1,\dots,x_n\}|\leq k$. Of course, we have $D^n_k\subseteq S^n_{n-k+1}$ for $k\in\{1,\dots,n\}$. Also, we have $S^n_k\subseteq S^n_{k+1}$ for $k\in\{1,\dots,n-1\}$. Now, denote by $\mathcal{G}^n_k$ the class of those associative n-ary operations $F\colon X^n\to X$ satisfying

$$F(x_1,\dots,x_n)\in\{x_1,\dots,x_n\},\qquad (x_1,\dots,x_n)\in S^n_k.$$

We say that these operations are *quasitrivial on* S^n_k.

It is not difficult to see that if $F\in\mathcal{G}^n_k$, then $F\in\mathcal{F}^n_{n-k+1}$. Actually, we have $\mathcal{G}^n_1=\mathcal{F}^n_n$ and $\mathcal{G}^n_n=\mathcal{F}^n_1$. These are the only classes when $n=2$, and thus we assume throughout this section that $n\geq 3$. Due to Proposition 1.1, we have that $\mathcal{G}^n_n=\cdots=\mathcal{G}^n_3$ is exactly the class

of quasitrivial associative n-ary operations, and hence we only need to consider operations in $\mathcal{G}_2^n$. The counterpart of Theorem 1.6 can then be stated as follows.

Theorem 3.1. *If n is odd and $G \in \mathcal{H}_2$, then its n-ary extension $F = G^{n-1}$ is in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$. Conversely, for every $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$ we have $|E_F| \geq 3$, n is odd, the reductions of F are exactly the operations G_e for $e \in E_F$, and they lie in $\mathcal{H}_2$.*

Proof. If n is odd and $G \in \mathcal{H}_2$, then $n-1$ is even, and so $G \in \mathcal{H}_{n-1}$. Therefore by Theorem 1.6, $F = G^{n-1}$ is in $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n = \mathcal{F}_{n-1}^n \setminus \mathcal{G}_n^n$. We have shown in the proof of Theorem 1.6 that $E_F = Y$. In order to show $F \in \mathcal{G}_2^n$, we need to show that if $x_1, \ldots, x_n \in \{x, y\}$, then $F(x_1, \ldots, x_n) \in \{x, y\}$. If x or y is in $X \setminus Y$, this follows from Proposition 2.8. If $\{x, y\} \subseteq Y$, then if k arguments are equal to x and $n-k$ are equal to y, $F(x_1, \ldots, x_n) = F(k \cdot x, (n-k) \cdot y)$ because $(Y, G|_{Y^2})$ is an Abelian group. Since n is odd, the parity of k and of $n-k$ are different. Since $(Y, G|_{Y^2})$ has exponent 2, this expression is equal to x (resp. y) when k is odd (resp. even).

Conversely, if $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n \subseteq \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$, then by Theorem 1.6, we have $|E_F| \geq 3$, all the reductions of F are exactly the operations G_e for $e \in E_F$ and they lie in $\mathcal{H}_{n-1}$. In particular, for any $e \in E_F$, we have that (E_F, G_e) is an Abelian group whose exponent divides $n-1$. However, since the neutral element is the only idempotent element of a group and since $G_e(e', e') \in \{e, e'\}$ for any $e, e' \in E_F$, it follows that $G_e(e', e') = e$ for any $e, e' \in E_F$, i.e., for any $e \in E_F$ we have that (E_F, G_e) is a group of exponent 2. Therefore, we conclude that $(E_F, F|_{E_F^n})$ is the n-ary extension of an Abelian group of exponent 2. Also, since 2 divides $n-1$ we conclude that n is odd. □

Theorem 3.1 is particularly interesting as it enables us to construct easily n-ary operations in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$. For instance, consider the set $X = \{1, 2, 3, 4, 5, 6\}$ together with the operation $G\colon X^2 \to X$ defined by the following conditions:

- $(\{1, 2, 3, 4\}, G|_{\{1,2,3,4\}^2})$ is isomorphic to $(\mathbb{Z}_2^2, +)$,
- $G|_{\{5,6\}^2} = \pi_1|_{\{5,6\}^2}$, where $\pi_1\colon X^2 \to X$ is defined by $\pi_1(x, y) = x$ for any $x, y \in X$,
- for any $x \in \{1, 2, 3, 4\}$, $G(x, 5) = G(5, x) = 5$ and $G(x, 6) = G(6, x) = 6$.

Then for any integer $p \geq 1$, we have that the operation associated with any $(2p+1)$-ary extension of $(\{1, 2, 3, 4, 5, 6\}, G)$ is in $\mathcal{G}_2^{2p+1} \setminus \mathcal{G}_{2p+1}^{2p+1}$ by Theorem 3.1.

We now state a reformulation of Theorem 3.1 that does not make use of binary reductions. We omit the proof of this result as it is a straightforward adaptation of the proof of Theorem 1.8.

Theorem 3.2. *If an operation F is in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$, then n is odd and setting $Y = E_F$ we have $|Y| \geq 3$ and the following assertions hold.*

(a) *$(Y, F|_{Y^n})$ is the n-ary extension of an Abelian group of exponent 2.*
(b) *$F|_{(X \setminus Y)^n}$ is associative, quasitrivial, and has at most one neutral element.*
(c) *For all $x_1, \ldots, x_n \in X$ and $i \in \{1, \ldots, n-1\}$ such that $\{x_i, x_{i+1}\} \cap (X \setminus Y) = \{x\}$ we have*

$$F(x_1, \ldots, x_n) = F(x_1, \ldots, x_{i-1}, x, x, x_{i+2}, \ldots, x_n).$$

Conversely, if n is odd and F is an operation that satisfies these conditions for some $Y \subseteq X$ such that $|Y| \geq 3$, then $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$ and $E_F = Y$.

We end this section with the counterpart of Proposition 1.10 and Corollary 1.11 for operations in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$.

Corollary 3.3. *We have $\mathcal{G}_2^n \setminus \mathcal{G}_n^n \neq \varnothing$ if and only if n is odd and $|X| \geq 4$.*

Proof. (Necessity) By Theorem 3.1, we have that n is odd and there exists a subset $Y \subseteq X$ and an Abelian group (Y, G) of exponent 2 such that $|Y| \geq 3$. Since (Y, G) is of exponent 2 we have $|X| \geq |Y| \geq 4$.

(Sufficiency) Let $Y \subseteq X$ such that $|Y| = 4$. We can endow Y with an operation G_Y such that (Y, G_Y) is an Abelian group of exponent 2 that is isomorphic to $(\mathbb{Z}_2^2, +)$. Let us consider the operation $G\colon X^2 \to X$ defined by the following conditions:

- $G|_{Y^2} = G_Y$.
- $G(x, y) = y$ for any $x, y \in X \setminus Y$.
- Any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\} \cup Y)^2}$.

It is not difficult to see that $G \in \mathcal{H}_2$ (see Definition 1.5). Thus, we have $G^{n-1} \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$ by Theorem 3.1, which concludes the proof. □

4. Concluding remarks

In this paper we characterized the class $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$, i.e., the class of those associative operations $F\colon X^n \to X$ that are not quasitrivial but satisfy the condition $F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ whenever at least $n-1$ of the elements $x_1, \ldots, x_n$ are equal to each other (Theorems 1.6 and 1.8). These characterizations enabled us to obtain necessary and sufficient conditions on the cardinality of X so that $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \varnothing$. Moreover, we proved that any operation in $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ is reducible to a binary associative operation (Corollary 1.7). Finally, we characterized the class $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$, i.e., the class of those associative operations $F\colon X^n \to X$ that are not quasitrivial but satisfy the condition $F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ whenever $|\{x_1, \ldots, x_n\}| \leq 2$ (Theorems 3.1 and 3.2). As a byproduct of these characterizations, we obtained necessary and sufficient conditions on the cardinality of X for which $\mathcal{G}_2^n \setminus \mathcal{G}_n^n \neq \varnothing$.

The main results of this paper thus characterize several relevant subclasses of associative and idempotent n-ary operations. However, the characterization of the class $\mathcal{F}_n^n$ of associative and idempotent n-ary operations still eludes us. This and related enumeration results [2, 3] constitute a topic of current research.

Acknowledgments

The authors would like to thank the anonymous reviewers for their insightful remarks that helped improving the current paper. They are especially grateful for Proposition 1.10 and Corollary 1.11. The second author is supported by the Luxembourg National Research Fund under the project PRIDE 15/10949314/GSM.

Appendix A. Alternative proof of Corollary 1.7

We provide an alternative proof of Corollary 1.7 that does not use [2, Corollary 2.3]. To this extent, we first prove the following general result.

Proposition A.1. *Let $F \in \mathcal{F}_n^n$. The following assertions are equivalent.*

(i) *F is reducible to an associative and idempotent operation $G\colon X^2 \to X$.*
(ii) *$F((n-1)\cdot x, y) = F(x, (n-1)\cdot y)$ for any $x, y \in X$.*

Proof. The implication (i) ⇒ (ii) is straightforward. Now, let us show that (ii) implies (i). So, suppose that

$$F((n-1)\cdot x, y) = F(x, (n-1)\cdot y) \qquad x, y \in X, \tag{7}$$

and consider the operation $G\colon X^2 \to X$ defined by $G(x,y) = F((n-1)\cdot x, y)$ for any $x, y \in X$. It is not difficult to see that G is associative and idempotent. Now, let $x_1, \ldots, x_n \in X$ and let us show that $G^{n-1}(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)$. Using repeatedly (7) and the idempotency of F we obtain

$$\begin{aligned} G^{n-1}(x_1, \ldots, x_n) &= F^{n-1}((n-1)\cdot x_1, (n-1)\cdot x_2, \ldots, (n-1)\cdot x_{n-1}, x_n) \\ &= F^{n-1}((2n-3)\cdot x_1, x_2, (n-1)\cdot x_3, \ldots, (n-1)\cdot x_{n-1}, x_n) \\ &= \cdots \\ &= F^{n-1}(((n-2)(n-1)+1)\cdot x_1, x_2, x_3, \ldots, x_{n-1}, x_n) \\ &= F(x_1, \ldots, x_n), \end{aligned}$$

which shows that F is reducible to G. □

Remark 2. Let $\leq$ be a total ordering on X. An operation $F\colon X^n \to X$ is said to be $\leq$-*preserving* if $F(x_1, \ldots, x_n) \leq F(x'_1, \ldots, x'_n)$ whenever $x_i \leq x'_i$ for any $i \in \{1, \ldots, n\}$. One of the main results of Kiss and Somlai [9, Theorem 4.8] is that every $\leq$-preserving operation $F \in \mathcal{F}_n^n$ is reducible to an associative, idempotent, and $\leq$-preserving binary operation. To this extent, they first show [9, Lemma 4.1] that any $\leq$-preserving operation $F \in \mathcal{F}_n^n$ satisfies

$$F((n-1)\cdot x, y) = F(x, (n-1)\cdot y) \qquad x, y \in X.$$

Thus, we conclude that [9, Theorem 4.8] is an immediate consequence of [9, Lemma 4.1] and Proposition A.1 above.

The following result is the key for the alternative proof of Corollary 1.7.

Proposition A.2. *Let $F \in \mathcal{F}_{n-1}^n$. The following assertions are equivalent.*

(i) *F is reducible to an associative and quasitrivial operation $G\colon X^2 \to X$.*
(ii) *F is reducible to an associative and idempotent operation $G\colon X^2 \to X$.*
(iii) *$F((n-1)\cdot x, y) = F(x, (n-1)\cdot y)$ for any $x, y \in X$.*
(iv) *$|E_F| \leq 1$.*

Proof. The equivalence (i) ⇔ (ii) and the implication (iii) ⇒ (iv) are straightforward. Also, the equivalence (ii) ⇔ (iii) follows from Proposition A.1. Now, let us show that (iv) implies (iii). So, suppose that $|E_F| \leq 1$ and suppose to the contrary that there exist $x, y \in X$ with $x \neq y$ such that $F((n-1)\cdot x, y) \neq F(x, (n-1)\cdot y)$. We have two cases to consider. If $F((n-1)\cdot x, y) = y$ and $F(x, (n-1)\cdot y) = x$, then by Lemma 2.5 we have that $x, y \in E_F$, which contradicts our assumption on E_F. Otherwise, if $F((n-1)\cdot x, y) = x$ and $F(x, (n-1)\cdot y) = y$, then we have

$$\begin{aligned} x = F((n-1)\cdot x, y) &= F((n-1)\cdot x, F(n\cdot y)) \\ &= F(F((n-1)\cdot x, y), (n-1)\cdot y) = F(x, (n-1)\cdot y) = y, \end{aligned}$$

which contradicts the fact that $x \neq y$. □

Proof of Corollary 1.7. This follows from Proposition A.2 and [7, Lemma 1]. □

REFERENCES

[1] N. L. Ackerman. A characterization of quasitrivial n-semigroups. To appear in *Algebra Universalis*. http://people.math.harvard.edu/%7enate/papers/

[2] M. Couceiro and J. Devillet. Every quasitrivial n-ary semigroup is reducible to a semigroup. *Algebra Universalis*, 80(4), 19 pp., 2019.

[3] M. Couceiro, J. Devillet, and J.-L. Marichal. Quasitrivial semigroups: characterizations and enumerations. *Semigroup Forum*, 98(3), 472–498, 2019.
[4] J. Devillet, G. Kiss, and J.-L. Marichal. Characterizations of quasitrivial symmetric nondecreasing associative operations. *Semigroup Forum*, 98(1), 154–171, 2019.
[5] W. A. Dudek. Idempotents in n-ary semigroups. *Southeast Asian Bulletin of Mathematics*, 25:97–104, 2001.
[6] W. A. Dudek and K. Głazek. Around the Hosszu-Głuskin theorem for n-ary groups. *Discrete Mathematics*, 308:4861–4876, 2008.
[7] W. A. Dudek and V. V. Mukhin. On n-ary semigroups with adjoint neutral element. *Quasigroups and Related Systems*, 14:163–168, 2006.
[8] W. Dörnte. Untersuchungen über einen verallgemeinerten Gruppenbegriff. *Math. Z.*, 29:1–19, 1928.
[9] G. Kiss and G. Somlai. Associative idempotent nondecreasing functions are reducible. *Semigroup Forum*, 98(1), 140–153, 2019.
[10] E. Lehtonen and F. Starke. On associative operations on commutative integral domains. *Semigroup Forum*, 100, 910–915, 2020.
[11] H. Länger. The free algebra in the variety generated by quasi-trivial semigroups. *Semigroup Forum*, 20(1), 151–156, 1980.
[12] J.-L. Marichal and P. Mathonet. A description of n-ary semigroups polynomial-derived from integral domains. *Semigroup Forum* 83:241–249, 2011.
[13] E. L. Post. Polyadic groups, *Trans. Amer. Math. Soc.*, 48:208–350, 1940.
[14] M. Pouzet, I. G. Rosenberg, and M. G. Stone. A projection property. *Algebra Universalis*, 36(2):159–184, 1996.
[15] J. J. Rotman, *An Introduction to the Theory of Groups*. 4th Ed. Springer-Verlag, New York, USA, 1995.
[16] Á. Szendrei, *Clones in Universal Algebra*. Séminaire de Mathématiques Supérieures, vol. 99. Les Presses de l'Université de Montréal, Montréal, 1986.

UNIVERSITÉ DE LORRAINE, CNRS, INRIA NANCY G.E., LORIA, F-54000 NANCY, FRANCE
Email address: miguel.couceiro[at]{inria,loria}.fr

UNIVERSITY OF LUXEMBOURG, DEPARTMENT OF MATHEMATICS, MAISON DU NOMBRE, 6, AVENUE DE LA FONTE, L-4364 ESCH-SUR-ALZETTE, LUXEMBOURG
Email address: jimmy.devillet[at]uni.lu

UNIVERSITY OF LUXEMBOURG, DEPARTMENT OF MATHEMATICS, MAISON DU NOMBRE, 6, AVENUE DE LA FONTE, L-4364 ESCH-SUR-ALZETTE, LUXEMBOURG
Email address: jean-luc.marichal[at]uni.lu

UNIVERSITY OF LIÈGE, DEPARTMENT OF MATHEMATICS, ALLÉE DE LA DÉCOUVERTE, 12 - B37, B-4000 LIÈGE, BELGIUM
Email address: p.mathonet[at]uliege.be