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Multiverse and the Society

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Abstract

This article follows the philosophical line of mathematical pluralism to make two points: that if there is a view that mathematics is a basis for sciences including social sciences, and if there is a pluralism in mathematics, then this implies a model of the society as a pluralist one. Along the way to this conclusion we argue why mathematical logic is a good part of mathematics to use in modelling and we, in particular, explain the use of set theory in this context. [1]

1 Introduction

Mathematicians of the XVIIIth century, starting with the celebrated Carl Friedrich Gauss, were the first in the Western world to question the uniqueness of mathematical interpretation. Thinking of geometry on the surface of the globe, Gauss realised that it exhibits a non-Euclidean geometry, consistent with all but the fifth postulate of Euclid. Due to its departure from the then established view that the whole of mathematics is based on the axioms of Euclid, the core of mathematics, this discovery was quite shocking to Gauss. So shocking that he did not publish it. [2] Yet, there is no way to stop

[1]The author would like to thank the Institut d'Histoire et de Philosophie des Sciences et des Techniques CNRS-Université Paris 1 Panthéon-Sorbonne, Paris, where she is an Associate Member, the Department of Mathematics at the University of East Anglia in Norwich, where she holds the title of a Visiting Professor and the Institute of Mathematics at the Czech Academy of Sciences in Prague, where she was employed during a large part of the preparation of the manuscript. This research was supported by the GAČR project EXPRO 20-31529X and RVO: 67985840 and funding from the European's Union Horizon 2020 research and innovation programme under the Maria Skłodowska-Curie grant agreement No. 1010232.

[2]In fact, Gauss was not the only one. In 1733, Giovanni Girolamo Saccheri [29] and in 1766 Jean-Henri Lambert, separately found counter-examples to the parallel postulate (see

history and the fact was re-discovered independently by János Bolyai in 1832 [6] and Nikolaï Lobatchewsky in 1837 [25]. Speaking of this and the similarly dramatic discovery of non-commutative algebra, Paul Dirac has said in [9] "Non-Euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of the general facts of the physical world".

Following this line of reasoning, this article contributes to the demonstration of the usefulness of mathematical logic and set theory as modelling tools. Mathematical logic is a part of mathematics that some contemporaries may consider as "purely fictions of mind and pastimes for logical thinkers", while in fact it has "often been found to be very necessary for the description of the general facts of the physical world". We concentrate on an aspect of mathematical logic that is considered even more abstract than the others, set theory, and on its application as a model of what is even more applied than the physical world studied by Dirac: the society. Set theory is considered so abstract for reasons similar to the ones that led the non-Euclidean geometry to be considered too abstract in its own time: it shakes the conviction that there is only one universe of mathematics, as much as geometry shook the conviction in the universality of the Euclidean thought. Both geometry and set theory are examples of the phenomenon of mathematical pluralism, where mathematicians have to find ways to grasp the situation in which absolute truths are a rarity rather than a norm that they were once thought to be. The reader is probably ready to accept that plurality, given that mathematicians have been studying it for several centuries. We would like to use that good will of the reader to explain a logical consequence: if we accept that mathematics is a basis for sciences, including social sciences, and if there is a pluralism in mathematics, then this implies that models for the society should include pluralism.

The article will indeed lead us to a pluralist model of the society, while using set theory as foundations. It will not add any knowledge to set theory, nor to the discussion of the place of set theory within science. It will rather use our experience in set theory and the multitude of its universes to try to understand something about the plurality in the human society.[3]

Let us end this introduction by a rather temporal statement, which we however find of such importance that it would be entirely impossible to think

[18]). Saccheri's also did not quite know how to treat this discovery and it only appears implicitly in his final work, while Lambert discretely added it to the second edition of his book [22].

[3]In this text the noun *human* is used for members of any gender identity and the pronoun *he* is used as a generic human pronoun as well as the masculine pronoun.

about the world abstractly at this moment without noticing its context. Our world has just gone through the pandemic of the Covid-19 virus, which had put our civilisation to its knees. While fighting the virus, we had to revisit our relationship to the Nature. Even though the scientists have found a vaccine, the origin of the pandemic is still not understood. At this moment, it is not clear to what kind of world we are waking up. Dangerous separatists and isolationist thoughts have appeared under the cover of the protection against the virus. For all of these reasons, it is obviously the time to revisit the question of our relationship with the Nature and the society. In mathematics and in philosophy, we cannot stay silent. Within mathematics, this does not only apply to mathematicians who write differential equations and use statistics. In set theory, if we are serious about the philosophical content of mathematics and about the foundations of mathematics on the one hand, and about the connection between mathematics and Nature on the other, we need to see what contribution our ideas can have on that unknown world awakening.

We shall use the modelling power of set theory to propose a philosophical model of that new world, defending the view that the existence of a viable pluralist model of the society follows by an application of a very fundamental logical method, that of set theory. That conclusion feeds into our philosophical position which is that the the acceptance of the plurality of opinion is crucial both in mathematics and in the society.

2 Notation and assumptions

In this paper, by Set Theory we understand the standard Zermelo-Fraenkel axiomatisation of set theory given in the first order classical logic. This is known as ZF. The Axiom of Choice could be added to this to obtain ZFC, as it is known since Kurt Gödel's work in [15] that if ZF is consistent so is ZFC. However, we have no use of the Axiom of Choice in what are interested in here and hence we shall skip it.

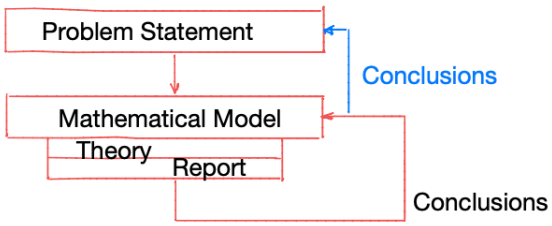
There are several other worthy formulations of set theory that have been studied for philosophical or mathematical reasons (see [13] for more on this matter), but ZF(C) is the mainstream one and the one to which all the results cited in the paper refer to. It is part of the standard toolbox of a mathematical logician to know that nothing essential would change in the philosophical conclusion of this material by either changing the axioms or even by changing the foundations from set theory to some other ones, as a trace of Gödel's Incompleteness [14] would catch up with us one way or another (see, for example, [13]).

The reader not familiar with mathematical logic may simply think of ZF(C) as a significant representative of the methods that mathematicians use to obtain mathematical foundations. The reader not familiar with the concept of mathematical foundations may simplify further and remember that any theory in human experience is based on some foundational truths, some unquestionable statements that are a base for all reasoning within the theory. Mathematical foundations are what mathematicians would like to declare their foundational truths. The question of what exactly mathematical foundations are is not a simple question and no simple answer is possible; taking set theory as foundations is a very good and very common partial answer.

3 On the methodology of mathematical modelling

As stated, we wish to develop a model of the human society based on our knowledge of mathematical logic and then use the conclusions about the model to make some conclusions about the society. In other words, we develop a mathematical model. At this point, it is worthwhile to discuss what this means exactly, addressing both the advantages and the limitations of this process.

The steps involved in mathematical modelling are rather standard, let us for example use the description from [26]. We start with a Problem Statement, provide a Mathematical Model, including elements of Theory and their Report with the Problem. We draw conclusions about the Mathematical Model and hope that the fit with the Problem is sufficient to be able to carry the conclusions back to the Problem Statement. Then we make conclusions about the Problem Statement.



While some parts of this diagram are purely logical, we point out that the fit of the model with the problem is a leap of faith. This in some cases can be defended by heuristic information and experiments, for example in biology, and in others, as is the case here, by a philosophical position. In all honesty, the art of mathematical modelling is a bit of a mixture of experiments and

the philosophy and believing a model is never more than that: believing a model. [4]

3.1 Set theory as a modelling tool

There are many examples where mathematical logic has made considerable advances in modelling social situations. Perhaps the most striking recent example is Jouko Väänänen's Dependence Logic [33]. This logic, although of great independent interest in pure abstract logic, for its original treatment of dependence, independence and of teams, has also found numerous applications including some in social sciences, such as an analysis of Arrow's theorem ([1]) (of which more will be said in §6) done by Eric Pacuit and Fan Yang ([27]). Game theory, which is in close connection with logic through game-theoretic definitions of semantics, was in part developed as a theory of social organisation in the tumult of the Second World War, as is documented in the book [23] by the well known historian of economic thought, Robert Leonard. In fact, much earlier in mathematics, we can mention Condorcet's Jury Theorem and Condorcet's Paradox by Nicolas de Condorcet in [8], as one of the first attempts to model democratic processes by mathematics.

We shall stay within the area of mathematical logic, specifically set theory. Set theory has proved its worth as a modeller in many mathematical contexts, making contributions to a diversity of fields, within logic, such as model theory, and in a broader way in mathematics, such as in topology, measure theory, analysis, algebra and combinatorics, and further, in computer sciences. There are whole fields of mathematics whose name starts with the words 'set-theoretic', such as the set-theoretic topology, and there are innumerable books on various aspects of mathematics whose title is 'Set-Theoretic Methods in ...', for example [5] in control theory, [32] in the theory of automata and [12] in algebra, to cite just a few. Moreover, set theory has played a major foundational role in mathematics, in that it is understood by many mathematicians as *the* foundations of mathematics and by all as *a* possible foundation of mathematics. This success speaks to the modelling power of set theory. Simply, a great variety of mathematical concepts are sets, with some additional properties. This is indeed what made the reputation of set theory in the foundations of mathematics and incited David Hilbert to say, perhaps somewhat too optimistically, that 'nobody will take us out of the paradise created by Cantor' [17].

We shall test the modelling power of set theory by going one step further,

[4] Those who claim otherwise, waging equations and statistics in their own defence, need only to consider the remarkable lack of success that their approach has had in the fight against the Covid-19 virus. Modelling is hard and nobody has a monopoly.

to social sciences. Set theory has famously entered into humanities through the work of Alain Badiou, including the book [2] which gives a mathematical ontology through set theory and uses the axioms of ZFC to identify the relationship of a being to the Nature, to the God, to the State. As the name of the book suggests, Badiou discusses the greatest philosophical questions using set theory, which he continues to do in further volumes of work [3] and [4]. Of course, the four pillars of Badiou's work being Art, Love, Politics, and Science, he certainly uses his set-theoretical ontology to a great variety of social ideas. Our ambition is more modest and rather concrete: it will evolve around the idea of membership as a founding concept in set theory, but also in the Society. Therefore, if we accept to define the Society using the idea of membership as a defining feature, as we argue in the following, then it is reasonable to model that same society as a model of some sort of set theory. As mentioned above, we shall work with the axioms of ZF. We shall show, firstly, that the axioms of ZF(C) set theory have a natural interpretation in a model of the Society, and that hence, the Society can be modelled using the ideas we have about the universes of set theory. We shall call the resulting model of the Society *the societal interpretation.*

4 Formulating the societal interpretation

The basic object in set theory is, evidently, a set. It is not a concept that we define, rather, we take it as given and we build set theory by studying the relation of membership between sets. This relation is guided by the Axioms of ZF(C) and deductions from the axioms, which are done using the rules of deduction in first order logic. The resulting theory is a rich domain of mathematics and it is made even richer by the fact that many objects in mathematics can be modelled using sets and the membership relation: the set of natural numbers, the set of real numbers, the set of all lines in a plane and so on. As much as many mathematical concepts are sets, even more *social* concepts are sets: the set of members of a family, the set of students in a class, the set of members of a political party, followers of a religion or the set of citizens of a country. Indeed, when introducing the concept of a set to a new audience, one would usually first go through this type of example, before even entering into the examples coming from mathematics. Furthermore, the basic relation of set theory is that of belonging to a set and we support the idea that this is also a basic relation in the society. While we might find that the idea proposed by Robert Park [28] that 'the individual is a product of group life', that is, the identity of a member of a society is determined by which sets he belongs to, is somewhat extreme, it would be

difficult to disagree with the converse, that is

the identity of any society is determined by its members.

This is exactly one of the basic Axioms of Set Theory, the *Axiom of Extensionality*. Next, in Section 4.1 we explain how the rest of the axioms of ZF can be interpreted in the Society. The reader not familiar with the axioms of ZF can find them listed in the Appendix. We shall not attempt to duplicate the work already done by Badiou [2] in interpreting ZF(C) set-theoretic axioms as ontology. Rather, we shall give a self-contained and very concrete interpretation of the axioms in the context that interests us.

4.1 ZF in the societal interpretation

To interpret a list of axioms in a logical context we need to understand the domain where the variables appearing in the axioms are allowed to range. This domain for us will be the Society. Therefore, the societal interpretation will have one Society, the society of all humans on the planet and many societies, which represent smaller groups of humans. Humans alone, in families, in their work place, in their street, in their city, in their political party, on their Facebook page. All of these examples of societies will be represented as sets.

In the introduction to this section, we have already discussed the Axiom of Extensionality in the societal interpretation. Several following axioms in the list of ZF have a very natural interpretation in the model, as we now discuss.

The *Axiom of Pairing* says that for every two sets, there is a set which contains exactly the two of them. This would simply mean that for any two societies, one can form a third which consists of exactly the two we started with, which conforms with observation. The *Union Set Axiom* says that if we have a set Z of sets, there is one set which contains exactly the elements that belong to some element of Z. Therefore, if we have a society of societies, say the Russell group of universities in the UK, each university consisting of students in that university then we can form the union of all students of all Russell group universities. The *Power Set Axiom* says that if we have a set A, then we can form a set which contains exactly the subsets of A. For example, we can form a set which consists of all possible social subgroups of the students of all Russell group universities. The *Foundation Axiom*, which is given a very prominent role in [2], has a rather natural interpretation, saying that in any society, one can go back to the root to break the society into its constituent members which have no societies as members.

4.2 The Axiom of Infinity and relativisation

Now we arrive to the Axiom of Infinity, which might pose a problem. Namely, if we put ourselves at the external point of view, clearly the number of all inhabitants of the planet is finite, and so the number of all societies is finite. The *Axiom of Infinity* states that there is an infinite set. Do we have a contradiction?

No, we do not have a contradiction, and this is the main point. To understand this, we need to introduce the idea of the relativisation, an idea which plays a major role in set theory since the work of Gödel on the constructible universe, in [15], but which, stated in different terms has already played a role in the work of Bolyai [6] and Lobatchewsky [25], on the models of geometry. Let us explain that idea abstractly.

Suppose that we have two sets M_0 and M_1 in which we have interpreted some mathematical language $\mathcal{L}$. An important observation to make is that the sentences of $\mathcal{L}$ might have different truth values in M_0 and M_1. For concreteness, let us consider as M_0 the set of $\mathbb{N}$ of natural numbers, as M_1 the set of $\mathbb{R}$ of the real numbers and, for the language in question, the language $\mathcal{L} = \{\leq\}$, which we interpret as the usual orderings on the sets $\mathbb{N}$ and $\mathbb{R}$. Then to every abstractly formed sentence in $\mathcal{L}$, say

$$(\exists x)(\forall y)\, x \leq y$$

we may associate the truth value that this sentence has in the models M_0 and M_1, and this truth value will not only depend on the sentence but also on the model in which the sentence is interpreted. For example, if we interpret the above formula in $(\mathbb{N}, \leq)$, then it is true, and if we interpret it in $(\mathbb{R}, \leq)$, then it is false. This is so because $(\mathbb{N}, \leq)$ has the least element and $(\mathbb{R}, \leq)$ does not. To fomalise this idea we shall introduce some notation and a formal definition, following the presentation in our book [10], where the reader may find more details on the topic.

Let us recall how formulas of first order logic are constructed, concentrating simply on the language $\mathcal{L} = \{\in\}$ of set theory. The basic, *atomic* formulas, are of the form '$x = y$' or '$x \in y$'. The short definition of the set of first order formulas in this language is that this is the collection of well formed formulas that we obtain from the atomic formulas by applying negation $\neg$, the conjunction $\wedge$ and the existential quantifier $\exists$ any finite number of times and in any order desired[5]. Well formed formulas are those strings of symbols that obey the grammar rules for the formation of formulas, for

[5]Note that the other connectives $\vee$, $\Longrightarrow$ and $\Longleftrightarrow$, as well as the quantifier $\forall$, can be derived from the ones given above

example $(\exists x)\varphi(x)$ is well formed, but $(\exists x)$ is not. Sentences are those formulas in which all variables are bound by some quantifier. For example, $(\exists x)(\forall y)\, x \leq y$ is a sentence, but $(\exists x)\, x \leq y$ is not.

We need the following notation:

Notation 4.1 *We write $(\exists x \in M)\varphi(x)$ as a shorthand for $(\exists x)(\varphi(x) \wedge x \in M)$ and similarly we use the shorthand $(\forall x \in M)\varphi(x)$.*

Definition 4.2 *Suppose that $(M, \in)$ is a model of the language $\mathcal{L}$ of set theory and φ a formula of $\mathcal{L}$. We define the* relativisation φ^M *of φ to M by induction on the complexity of φ:*

- *If φ is $x = y$ or $x \in y$, then $\varphi^M = \varphi$.*
- *$(\varphi \wedge \psi)^M = \big(\varphi^M \wedge \psi^M\big)$, and similarly for $\vee$ and $\neg$.*
- *If φ is $(\exists x)\psi(x)$ then φ^M is $(\exists x \in M)\psi^M(x)$, and similarly for $\forall$.*

For example, the relativisation of the formula $(\exists x)\big(x^2 = -1\big)$ to $\mathbb{C}$ is $(\exists x \in \mathbb{C})\big(x^2 = -1\big)$, which is true, while its relativisation to $\mathbb{R}$ is $(\exists x \in \mathbb{R})\big(x^2 = -1\big)$, which is not true. In general, note that the definition of $\models$ is set up so that $M \models \varphi$ iff φ^M is true.

The example above shows that there can be a large difference between a formula and its relativisation. If we imagine two models of the same theory, say $M \subseteq N$ such that M is a part of N, then we can consider the formulas relativised to M and N as two different points of view on the truth in M. For a given sentence φ, we may think of the *internal* point of view of M as being represented by φ^M, and of the *external* point of view, known to N and represented by φ^N. The two statement φ^M and φ^N come from the same formula, but are not the same and they do not necessarily have the same truth value.

Let us go back to the point that started this discussion, the Axiom of Infinity. The formalisation of this axiom in ZF (see the Appendix) basically posits the existence of a set that is closed under the successor operation, defined by $S(a) = a \cup \{a\}$. This operation is not particularly interesting in the society and it would be rather futile to take this as a definition of what infinite means in everyday life. That is, we are not interested in that external view of an infinite society. Rather, we are interested to model the society as it appears to an internal observer, an inhabitant of the planet. In this we take a line of thought that is presented in applied mathematics, where 'infinite' stands for a finite object of such large finite size that for all practical purposes, this size may be infinite. For example, in modelling waves in deep

water, Jean-Marc Vanden-Broeck [35] uses the expression 'infinite depth' to describe that the depth being very large, the effect of the bottom of the sea can be neglected. The well known set-theorist Đuro Kurepa in one of his popular presentations of his subject, used to describe his favourite object of study: a Souslin tree (an object of size $\aleph_1$), as a tree of life of a microorganism which starts dividing and at regular intervals has to face a choice of which units will be allowed to continue (the limit levels of the tree), raising the question of any unit surviving to the end ([21]). From the point of view of the microorganism, going to the next limit level might very likely represent a life span, or even a life span of a whole generation. This microorganism cannot imagine what comes after, all the relevant information is already encapsulated in the several steps ahead. We may interpret 'infinity' as being closed under a very large number of repetitions of a given operation, like measuring 1 mm of the depth of water in [35] or a division of a microorganism in Kurepa's example.

Let us then try to interpret Infinity in the societal interpretation. We shall do this by the notion of *distance to the influence*. To start, observe that there are some small societies, such as the nuclear family of an individual, or him and his two best friends. In both of these societies this individual has a very direct input and engagement. Then, there are larger and larger societies he might belong to and we might observe that the direct influence of him as an individual decreases. We could in fact model the relation of an individual a to the Society or to a society, by considering the membership graph generated by the individual: a belongs to a society A and it directly influences the decisions in A. However, A belongs to the society $\mathcal{A}$, where a also has an influence, but only an indirect one, passing by A. The membership distance of a to A is 1, while the membership distance of a to $\mathcal{A}$ is 2. And so on. We may notice that the influence that the individual as a unit has in a society decreases with the membership distance. And although from the external point of view, there is no truly infinite membership distance on the planet, from the internal point of view, there are membership distances which are so large that for all practical purposes, they are infinite. The distance of a typical Greek citizen λ to the Eurogroup, which in 2015 is deciding if Greece will be forced to exit the Euro or will have to increase the austerity measures which would perhaps cost the citizen his livelihood, is infinite for all practical purposes. [6]

[6]Of course, this example is somewhat subjective, which is why we emphasise that like in any model where large finite numbers are treated as infinite, there is an element of fine-tuning the model to the purpose. Usually, the rule followed in applied mathematics is that a quantity that is being treated as infinite is indeed infinite in some reasonably taken limit. In our case that limit would be the limit taken over the growing number of

We posit that large societies and legal entities such as countries, but even smaller entities such as large cities or regions are best modelled as infinite sets. In fact, we claim that this is the point of view taken in everyday practice. For example, in what is the most popular understanding of democracy at the moment, it consists of asking a large number of citizens their opinion on a multiple choice question (which usually only has two choices) in elections, and then deciding by simple majority. Obviously, this idea is much more fruitful in elections in which there is a large number of participants-elections in which only a small number of citizens vote are usually cancelled. This point that was expressed mathematically by Condorcet in his Jury Theorem [8] and it basically says that the probability that a jury will make a decision that best represents the majority increases as the size of the jury goes to infinity. Believing in democratic elections by simple majority in large societies is basically confirming that everyday practice treats these societies as infinite, as we shall do in the societal interpretation.

4.3 The remaining axioms

It remains to interpret the Axiom Schemas of Comprehension and of Replacement.

Comprehension states that if we have a set y and a property φ of its elements, then the collection of all members of y that satisfy φ is also a set. In the Society, this would correspond to forming subsocieties, that is isolating those members of a given society that have a certain property. This, of course, conforms with the everyday practice, where we might want to address the set of all citizens of Paris that live on the left side of the river Seine, or the set of all of Mr X's children that are older than 13. Comprehension is actually a scheme, since it gives a separate axiom for every formula φ.

Similarly, Replacement is a scheme. It says that if we have a formula φ which acts as a function, in the sense that to every argument it assigns exactly one value, then we can collect all these values in one set. In everyday life, this would correspond to the practice of choosing something from every member of a society. When every adult Estonian creates an income tax return declaring the flat income tax of 21% of their income, the resulting set of all the returns collected is guaranteed to exist by the Axiom of Replacement. So is the set of all exact moments of birth of all inhabitants of the Earth, or the set of all first books read of the citizens who have ever read a book.

inhabitants of the planet.

5 Plurality of models of set theory

A careful reader might have already noticed a curios property of the societal interpretation. It is not unique. There are many societies ! This is a phenomenon well studied in set theory. There are many models. In this part we shall describe how set theorists view the models or universes of set theory, with the plan to use the knowledge of these universes in §6 to make conclusions about the societal interpretation.

The present state of knowledge in set theory has fully acknowledged that there are some simply stated statements about sets which are not decided by ZF or ZFC. A canonical such statement is the Continuum Hypothesis CH, which posits in ZFC that every infinite subset of the real line is either bijective with the set of natural numbers or with the set of real numbers, the two being known not to admit a bijection. If there is a model of ZF at all then there is one in which ZFC+CH holds (Gödel [15]) and there is another one in which ZFC+$\neg$ CH holds (Paul Cohen [7]). We say that CH is *independent* of the axioms of set theory. In fact, there are many other such independent statements, and even if we were to axiomatise a truth value to CH, so accepting that it is true or false, there still would be independent statements, concrete examples of which are well known. One such statement is the Souslin Hypothesis which is independent of ZFC+CH and of ZFC+$\neg$CH, see books [20], [30] or [10] for expositions.

The state of affairs with CH is the tip of the iceberg. Modern set theory abounds with ways to produce new models of set theory from the existing ones by using forcing or large cardinals or both. This leads to rather opposing views of the structure of the part of mathematics described by the axioms of set theory. These are, roughly, the universe view and the multiverse view. The societal interpretation has elements of both, and in fact is the best described by an elegant combination of them in Väänänen's view from [34]. Let us describe the universe and the multiverse views.

The universe view holds that there exists the true Universe of set theory, axiomatised categorically by some axioms, leading to a unique model. These axioms, evidently, are still to be found, given the situation with CH and other independent statements. Gödel's Incompleteness theorems [14] tell us that without a radical solution such as changing the logic, there is no way out of this conundrum. There is a solution if we change the logic to the second order logic, which does not lend itself to the arguments used in the Incompleteness Theorems. Moreover, second order ZFC set theory enjoys what is known as quasi-categoricity. Namely, Ernst Zermelo [37] proved that the second order version of ZFC axioms, known as ZFC^2, has one single type of models, namely the sets V_κ for a strongly inaccessible cardinal κ. There are unfortunately

many disadvantages of second order set logic for the practical purposes in mathematics, since exactly by being abundantly expressive, it fails to satisfy any properties that make mathematical logic useful in practice. For example, the completeness or compactness theorems, or Löwenheim-Skolem theorems fail for second order logic. In fact, a theorem of Per Lindström [24] characterises the first order logic as the maximal satisfying both the compactness theorem and the Löwenheim-Skolem theorem downward to $\aleph_0$. So, if we want to have a useful, rather than merely expressive, set theory out of ZF(C), we better stuck with the classical first order version, and then the universe view needs to be defended or defeated.

This leads to the the multiverse view, which defends the idea there is no canonical Universe of set theory, rather, there are many models and it is their plurality that gives the real information on the subject. Many authors defend this view, which is well known in set theory, but got known to a wider philosophical audience through the much-cited paper [16] by Joel Hamkins. This beautifully written paper centred on forcing extensions is sometimes over-simplified in applications, which do not always account very carefully for the seeming contradiction that in order to take forcing extensions of a model in the multiverse, this model already needs to be in a larger universe. To understand this, one has to distinguish the internal and the external views. This important point is underlined in Väänänen's view from [34], where the *multiverse* consists of a multitude of universes. We are going to take this view, except that for us a model of set theory will not be called a universe but simply a model, and the multiverse of Hamkins and Väänänen will be called Universe. This terminology aligns us with Saharon Shelah's view from [31], where he explains a fine line that differentiates a pure Platonist from the one that believes in a universe but also in undecidable problems.

6 Universe and models in the societal interpretation

For the application that we are interested in here, the societal interpretation, there is an obvious Universe, namely the Society as a whole. However, like in any other context where set theory appears, there are also many smaller models of set theory, sometimes in contradiction with each other, sometimes defending the opposite axioms, but it is their multitude that gives the real information of the Universe. By applying the same logic that we have used in constructing the societal interpretation, we may interpret as models of set theory the large societal units, concretely the states-countries. These countries obey the basic laws of the Society, but they differ greatly in their interpretation of further laws. Some are communist, some are liberalist, some

are religious, some are western-democratic, and so on. Each defines its own model. We shall interpret the set-theoretic knowledge about the societal interpretation to the Society to defend the idea that it is the multitude of societies that describes the truths in the Society and that is only through their multitude that the Society can advance.

The strength of set theory is to recognise the fallacy of modelling an absolute truth. A set theorist can look at the Universe and see two models, M and N such that a model M sees certain statements as true, while N sees them as false. However, both M and N being models of set theory, they agree on the basics and can live in collaboration in the Universe of models. By no means the vision of M will negate the vision of N or vice versa; there is simply a plurality of visions. This is the main point we wish to retain regarding the societal interpretation. There are many societies in the Society and various societies may have different opinions on some statements, but the plurality of their opinions is what determines the Society.

6.1 The silent majority

The view that it is only the plurality of voices that determines the Society is in contradiction with the reality of the lives of most inhabitants of the planet, whose opinion is not heard. They pass under the various qualifications, from the 'silent majority' to some more negative descriptions. The fault for this silence is often put on the specific society that they inhabit, the lack of something or other in their country, the lack of freedom, or education, for example. While these are undeniably aspects that the Society suffers from, we would like to point out another aspect that passes like a taboo under the radars of most but the most outspoken critics.

What we claim is that the silent majority exists because there is a particular idea that has permeated the political life of this planet and which silences and marginalises everybody who is not adhering to it. Under the umbrella of making the political choices more just, we have started believing in the existence of a universal system which will make everybody's voice heard. However, we have not defined what this would mean exactly. Those who think that as soon as everybody has the right to vote implies that we have reached the ideal society, have still to define what is actually understood by this ideal society and then, defend its existence. However, in today's society a particular idea, these days called Democracy, pretends to being an absolute truth. Everyday experience shows that the latter is clearly not the case. What we claim here is that there is a mathematical proof that it cannot be the case. This Universal Democracy, as pushed on us as it may be by certain big powers, is not the absolute truth. Simply because the absolute

truth does not exist.

Even the simple belief that such an absolute truth exists has stopped the Society from accepting the plurality of its constituents and advancing towards a more fruitful international cooperation.

6.2 Universal Democracy is undefined

Universal Democracy, which has been presented as the only true way, has not ever been correctly defined. In fact, universal democracy has a lot to do with the Continuum Hypothesis. We do not seem anywhere close to having a decision on the value of the Continuum Hypothesis and we are not anywhere close to having a good definition of what the word 'democracy' might actually mean, and yet less close to what 'universal' might mean, apart from the simplistic and unsatisfactory answer of everybody having the right to vote. The question of how the votes are distributed, influenced and used, and moreover, what it is that we are voting for, is very variable.

In fact, there is a theorem which states that the question is much more difficult than just giving the votes to everybody, namely a theorem that illustrates the impossibility of having an ideal voting structure.

Theorem 6.1 (Arrow's Impossibility Theorem, [1]) *In all cases where preferences are ranked, it is impossible to formulate a social ordering without violating one of the following conditions:*

- *Nondictatorship: The wishes of multiple voters should be taken into consideration.*
- *Pareto Efficiency: Unanimous individual preferences must be respected: If every voter prefers candidate A over candidate B, candidate A should win.*
- *Independence of Irrelevant Alternatives: If a choice is removed, then the others' order should not change: If candidate A ranks ahead of candidate B, candidate A should still be ahead of candidate B, even if a third candidate, candidate C, is removed from participation.*
- *Unrestricted Domain: Voting must account for all individual preferences.*
- *Social Ordering: Each individual should be able to order the choices in any way and indicate ties.*

Arrow's Impossibility Theorem forms a basis of social choice theory. There are many theorems which indicate that simple voting systems are impossible or at least very difficult to implement; for an early example one can see Condorcet's Theorem, [8]. The experience including the two world wars and the Cold War and many other wars since, has taught us that universal democracy is an ideal which might not be easy to reach. Some think that universal democracy is to have the rule of the market, others think that it is to have the pure rule of the majority, yet others find themselves embracing various compromises between the visions of democracy. Democracy seems to live on a spectrum. At the moment of writing there are at least two big powers in the world who are located on the two opposite parts of that spectrum, both entirely convinced in their own model, and many others who remain unconvinced by any of the two. Therefore, due to a still rather vague and discorded understanding of what the word 'democracy' actually means, searching for a universal meaning of this word might not even be an aim that one would want to reach, or at least not an aim that seems to be anywhere in the remit of the Society of today.

Like CH, the Universal Democracy is a vision which does not admit discord. Like CH, for the moment we do not know how to obtain it and moreover we have proofs that we cannot do it. So, universal democracy is an Absolutely Undecidable Proposition, in the language of [34].

6.3 Back to Kant

The proponents of the Universal Democracy find that their vision is so clear that it does not even have to be defined. In their minds it goes without saying that every great philosopher has started with the Universal Democracy as given. For example, Kant is reputed to have claimed that perpetual peace could be secured by universal democracy and international cooperation in [19]. But closer inspection of [19] shows that there is no discussion of universal democracy, not even of what today is a given, a vote for everybody. Instead there is a lot of discussion of international cooperation and very concrete suggestions of how such a collaboration could be achieved. Our view is completely in line with Kant, in believing that international cooperation is a definable and achievable goal.

As for the democracy, we still can have a measure of understanding it, by observing how it behaves in various models. In our paper with Marco Panza, [11], we have argued that concentrating on resolving or not resolving CH and hitting their head on the wall has made set theorists blind to what they actually can resolve. It has made them, for a long time, neglect the parts of the cardinal hierarchy where their methods are very powerful. In [11], we

have discussed the situation of singular cardinals as an example of where ZFC has much to say, in spite of it not having anything to say on CH. It is the same with the universal democracy. Once we accept that for the moment we do not really have the means to give a just definition of this concept, then, rather than burning the planet in yet another war of ideological supremacy to show that one or another possible definition of the concept is correct, we should concentrate on what we *can* do.

International cooperation has received much less attention than the universal democracy and when it did, it has often been overwhelmed by the latter. Many wars repeatedly came to the point of destroying the planet, run by ideologies. Bringing democracy' or 'liberating' was always somewhere in the background, and it almost always went wrong. This is because even when genuine, those messengers of the universal democracy have acted in a biased way that has been based on their belief that they have all the answers. It is painfully clear that this is not a way to proceed. Moreover, we feel that the moment is propitious to point out that no society is more worth than another, as the crisis we live in has brought up nationalist and selfish behaviour on the parts of many governments who feel that the 'natural supremacy' of their system gives them the right to make decisions and choices that badly influence livelihoods and even lives of those living in other systems. Yet, if we unite in an international effort to resolve the crisis, we might succeed.

To conclude, we suggest that the ideal of international cooperation is possible even without the fuzzy notion of universal democracy. International cooperation is exactly the lesson we can learn from believing that the Society is a multiverse. Nobody has all the answers.

7 Conclusion

It is a good moment to think of what the new world will be after the crisis we are living in now. We join the voices of those who have pointed out that the present international system is biased and that it has to change. Many mathematical competences have been used in modelling possible consequences of economic and political decisions, sometimes obtaining surprising results which show that ill defined moves may have consequences which are completely opposed to what the superficial definition might indicate. The inherent error in the theory leads to disastrous consequences, see for example the book [36] by Yanis Varoufakis, Joseph Halevi and Nicholas J. Theorcarakis who, among other competences, use game theory, to make this conclusion about economic reflexivity. Game theory is a staple of economic reflection and the inherent study of the Society. In this article we suggest a

complementary competence coming from set theory to make the same conclusion, which we study on the concrete examples of universal democracy and international cooperation.

In simple words, the conclusion of this article is short. Unify in respect of each other or perish while searching for dominance, which is nothing but an elusive and ill defined concept.

8 Appendix: Axioms of ZF

The following is the standard axiomatisation of the ZF system, as found for example in our book [10]. We express our axioms in first order logic, where the variables are taken to range over sets. It is common to assume that the domain of discourse is non-empty, so we do not need Axiom 0 from [20]. We shall use the following notation.

Notation 8.1 *(1) For any choice of $y, x_1, \ldots, x_n$ the notation*

$$(\exists! x)\ \varphi(x, y, x_1, \ldots, x_n)$$

is a replacement for

$$(\exists x)\big(\varphi(x, y, x_1, \ldots, x_n) \wedge (\forall z)(\varphi(z, y, x_1, \ldots, x_n)\implies z = x)\big).$$

(2) When writing $\varphi(x, y, x_1, \ldots, x_n)$, we mean that the free variables of φ are among $\{x, y, x_1, \ldots, x_n\}$, but do not necessarily include them all.
(3) For any set x the operation $S(x)$ is defined as $x \cup \{x\}$.[7]

Axiom of Extensionality Two sets have exactly the same elements if and only if they are the same set or

$$\forall x \forall y\ \big(\forall z(z \in x \iff z \in y)\implies x = y\big).$$

Axiom of Pairing For every two sets x, y there is a set whose elements are exactly x, y or

$$\forall x \forall y \exists z\big(x \in z \wedge y \in z \wedge \forall w(w \in z \implies w = x \vee w = y)\big).$$

Union Set Axiom $(\forall A)(\exists Z)(\forall b)\big(b \in Z \iff (\exists a \in A)(b \in a)\big)$.

Replacement Axiom Scheme For each formula $\varphi(x, y, x_1, \ldots, x_n)$ we have

$$(\forall x_1, \ldots, x_n, I)\big((\forall x \in I)(\exists! y\, \varphi(x, y, x_1, \ldots, x_n)\implies$$

$$(\exists A)(\forall z)(z \in A \iff (\exists x \in I)\varphi(x, z, x_1, \ldots, x_n))\big).$$

Comprehension Axiom Scheme For each formula $\varphi(x, y, x_1, \ldots, x_n)$ we have

$$(\forall x_1, \ldots, x_n, y)(\exists z)(\forall x)\big(x \in z \iff x \in y \wedge \varphi(x, y, x_1, \ldots, x_n)\big).$$

[7] One needs the axioms of Union and Pairing to verify that for any set x, the resulting set $S(x)$ is a set.

Axiom of Infinity There exists an infinite set, which we formalise as[8].

$$(\exists x)\big(\emptyset \in x \wedge (\forall y \in x)\, S(y) \in x\big).$$

Power Set Axiom For every x there is y whose elements are exactly those z which are a subset of x or

$$(\forall x)(\exists y)(\forall z)\ (z \subseteq x \iff z \in y).$$

Axiom of Foundation Any non-empty set is disjoint from at least one of its members or

$$\forall x\big(\neg(x = \emptyset) \implies \exists y(y \in x \wedge y \cap x = \emptyset)\big).$$

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