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The Cosmological Constant as a Zero Action Boundary

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ABSTRACT

The measured cosmological constant Λ is usually interpreted as Dark Energy (DE) or modified gravity (MG). Here we propose instead that Λ corresponds to a boundary term in the action of classical General Relativity. The action is zero for a perfect fluid solution and this fixes Λ to the average density ρ and pressure p inside a primordial causal boundary set by inflation: $\Lambda = 4\pi G < \rho + 3p >$. This explains both why the observed value of Λ is related to the matter density today and also why other contributions to Λ , such as DE or MG, do not produce cosmic expansion. Cosmic acceleration results from the repulsive boundary force that occurs when the expansion reaches the causal horizon. This universe is similar to the Λ CDM universe, except on the largest observable scales, where we expect departures from homogeneity/isotropy, such as variations in cosmological parameters indicated by recent observations.

Key words: Cosmology: dark energy, cosmic background radiation, cosmological parameters, early Universe, inflation

1 INTRODUCTION

Causality is a key property for any physical explanation. General Relativity (GR) is built as a covariant theory but this does not warrant a causal structure (see e.g. Howard 2010). Standard solutions to the Friedmann-Lemaître-Robertson-Walker (FLRW) metric lack a proper causal structure: the energy-density and the expansion rate are the same everywhere, in comoving coordinates, no matter how distant. This is not physically possible and has no explanation.

In the FLRW Universe, the Hubble Horizon r_H is defined as $r_H = c/H(t)$, where $H(t) = \dot{a}/a$ is the expansion rate and $a = a(t)$ is the scale factor. The corresponding comoving coordinate is $\chi_H = r_H/a$. In normal circumstances r_H has to grow with time because gravity is attractive and slows down the expansion rate $H(t)$. Scales larger than r_H can't evolve because the time a perturbation takes to travel that distance is shorter than the expansion time. This means that $r > r_H$ scales are "frozen out" (structure can not evolve) and are causally disconnected from the rest. Thus, χ_H represents a dynamical causal horizon that is evolving.

This does not mean that we can't observe scales larger than χ_H . The comoving particle horizon χ , defined as the distance traveled by light since the beginning of time, is the integral of the Hubble Horizon: $\chi = c \int dt/a(t) = \int d \ln a \chi_H$ and is therefore larger than χ_H . Thus we can see parts of our Universe which are causally disconnected from its evolution. This happens with CMB observations, where the largest angular scales are sometimes called "super-horizon". In fact, if we look back in time χ_H becomes smaller and smaller and all observable patches of the Universe become causally disconnected! This is a very important inconsistency of the FLRW Universe if we seek a causal explanation.

These contradictions can be addressed with inflation. According to inflation (Starobinskiĭ 1979; Guth 1981; Linde 1982; Albrecht & Steinhardt 1982) a small primordial (quantum) patch of size r_s , which is causally disconnected from the rest of space-time, starts a DeSitter

phase of exponential expansion during which the energy density ρ_i and Hubble rate H_i were constant ($8\pi G\rho_i = 3H_i^2$). The causal horizon χ_s is identified with the particle horizon during inflation: $\chi_s = c/(a_i H_i)$ or the Hubble horizon when inflation begins. This is the largest causally connected scale at the beginning of inflation a_i . This initial patch grows exponentially in physical units, $r_s = a\chi_s$, while the corresponding comoving scale $\chi_s = c/(a_i H_i)$ and Hubble horizon $r_H = c/H_i$ remain constant. After 60-70 e-folds r_s becomes cosmological in size while χ_H is microscopically small. This is the beginning of the standard FLRW model, where the initial Hubble horizon χ_H is negligible as if time had just started. When inflation ends, ρ_i converts into ρ and p (reheating) and χ_H starts to grow inside the primordial causal horizon χ_s .

Fig.1 illustrates this causal structure. Models of inflation expect that χ_s is larger than the particle horizon χ_0 today so that $a_s \gg 1$. The second inflation today is then attributed to DE or some other exotic explanation but not to χ_s . In our model we will show instead that $a_s \simeq 1$ and $\chi_0 \simeq \chi_s$, so the second inflation today is a consequence of the first one. In such scenario, we can observe causally disconnected regions ($\chi > \chi_s$ in Fig.1) and this should result in cosmic anisotropy and inhomogeneities. Either way, inflation mostly solves the horizon, homogeneity and flatness problems (see e.g. Dodelson 2003; Liddle 1999) because we encounter a homogeneous universe that was causally frozen before.

How can we distinguish if Λ is related to χ_s , to DE or to some other change in GR? Observationally, the boundary explanation predicts anisotropies and inhomogeneities on scales corresponding to χ_s . Theoretically, we will show that the value of Λ is determined by χ_s using the zero action principle. The least action principle tells us that arbitrary variations around the true field solutions produce no changes to the action. The action is a functional that assigns a number to each possible metric evolution. The action on-shell, S^{on-sh} , is the value that corresponds to the actual solution of the field equations. We will show that S^{on-sh} only depends on the value at the boundary of

space-time. We are interested here in asymptotically flat space-time (Minkowski space): for a universe of finite age, we expect no effect at scales larger than the true (primordial) dynamical causal horizon (created by inflation). In such case, we will show that $S^{on-sh} = 0$. We call this new property for the Universe the zero action principle. This fixes Λ to the average density ρ and pressure p inside the primordial causal boundary set by inflation: $\Lambda = 4\pi G < \rho + 3p >$.

To explain cosmic acceleration, scientists have thought of alternative ways (such as DE, MG or massive gravity) that can contribute to Λ or suppress gravity. But each new alternative can't explain why the other contributions can be neglected. The zero action principle implies that other contributions to Λ do not produce cosmic expansion because its value is renormalized by the boundary. So the contribution of vacuum energy ρ_{vac} to Λ could be as large as the Planck mass ($\rho_{vac} \simeq M_{Pl}^4$) and still produce no observational effects. This provides a true solution to the Λ problem.

That Λ is related to some cosmic scale, rather than quantum vacuum is hinted by its observed value of $(\Lambda/8\pi G)^{-1/2}$ which is close to $\chi_s \simeq c/H_0$ (this is sometimes called the coincidence problem). The holographic principle (Susskind 1995; Maartens 2004) provides a way to connect Planck and $\Lambda^{-1/2}$ scales via string theories. But note how in comoving coordinates χ_s is both scales at the same time! The primordial causal horizon can start as a Planck scale and become cosmological c/H_0 through cosmic expansion (Gaztañaga 2020).

It is generally accepted that GR is not a complete theory, both because of its singularities and also because of its apparent incompatibilities with Quantum Field Theory. There are some approaches that relate Λ to infrared boundary conditions within Quantum Gravity (see e.g. Stenflo 2020; Banks & Fischler 2018; He & Cao 2015; Huang et al. 2012; Wudka 1987 and references therein), but they do not seem to bear a direct connection to the classical GR approach presented here. A more direct connection exist with the discussion in §16 of Padmanabhan (2010), which relates boundary conditions in the action to singularities in the metric (see Appendix C).

Observationally, there is no motivation to modify gravity (MG) on cosmological scales other than to understand Λ . MG models have been proposed to replace GR using $f(R)$ theories (Nojiri et al. 2017). These models also require screening mechanisms to avoid Solar System and Astrophysical tests of gravity (Brax 2013). On cosmological scales, MG models are equivalent to DE models (Joyce et al. 2016). But neither approach solves the cosmological constant or coincidence problems. The value of Λ is added and fixed by hand and there is no explanation as to why its value is so small and so close to the matter density today. As we can also add and fix Λ by hand in GR, these MG and DE models seem an unnecessary complication.

These ideas are similar to the ones in Gaztañaga (2020), but they are cast here in terms of S^{on-sh} . Section 2 presents the zero action principle. Section 3 introduces Λ as a boundary term. Section 4 explores implications for cosmological observations. Appendix C shows how we live inside a Black Hole whose horizon corresponds to r_s (Gaztanaga 2021). Appendix D presents a proposal to start our Universe based on $S^{on-sh} = 0$. We use natural units of $c = 1$, but show c explicitly when useful. We use sign conventions as in Padmanabhan (2010) with Greek (Latin) indexes for 4D (3D).

2 ZERO ACTION PRINCIPLE

Consider the Einstein-Hilbert action (Hilbert 1915):

$$S = \int_M dM \left[\frac{R}{16\pi G} + \mathcal{L}_m \right], \quad (1)$$

Figure 1. Comoving distances in units of c/H_0 as a function of cosmic time a (scale factor). The Hubble horizon $\chi_H = c/(aH)$, blue continuous line, is compared to the particle horizon $\chi = c \int dt/a(t)$ after inflation, dashed line (dotted line includes the χ_s contribution). The primordial causal horizon $\chi_s = c/(a_i H_i)$ (dot-dashed line) is the particle horizon during inflation. Larger scales (green shading) are causally disconnected, smaller scales (yellow shading) are dynamically frozen. After inflation both χ_H and χ grow and by CMB times ($a \simeq 10^{-3}$) we can observe frozen "super-horizon" scales ($\chi > \chi_H$). At time a_s the expansion reaches χ_s . This generates a second inflation: our Universe is dynamically trapped inside χ_s .

where $dM = \sqrt{-g} d^4x$ is the invariant volume element, M is the 4D spacetime manifold, $R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu}$ the Ricci scalar curvature and $\mathcal{L}_m$ the Lagrangian of the energy-matter content. We can obtain Einstein's field equations for the metric field $g_{\mu\nu}$ from this action by requiring S to be stationary $\delta S = 0$ under arbitrary variations of the metric $\delta g^{\mu\nu}$. The solution is (Einstein 1916):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \equiv -\frac{16\pi G}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}. \quad (2)$$

Note that Eq.2 requires that boundary terms vanish (e.g. see §95 in Landau & Lifshitz 1971, Carroll 2004, Padmanabhan 2010). In the notation of Hawking & Horowitz (1996) the boundary term is:

$$S = \int_M \left[\frac{R}{16\pi G} + \mathcal{L}_m \right] + \frac{1}{8\pi} \oint_{\partial M} K \quad (3)$$

where K is the trace of the extrinsic curvature at the boundary. We can add terms to the action that do not depend on dynamical fields, as they do not change the field equations. As mentioned in the introduction we are interested in asymptotically flat (Minkowski) space-times. We can then define the action as:

$$S = \int_M \left[\frac{R}{16\pi G} + \mathcal{L}_m \right] + \frac{1}{8\pi} \oint_{\partial M} (K - K_0), \quad (4)$$

where K_0 is the trace of the extrinsic curvature of the boundary embedded into Minkowski space. This will cancel the boundary term. We will show next that the action on-shell S^{on-sh} (i.e. the action for the solution of the field equations) for Eq.1 is also a boundary term for a perfect fluid. As we require boundary terms to vanish, we have $S^{on-sh} = 0$. We call this the zero action principle.

2.1 Perfect fluid

For a perfect isotropic fluid with $\bar{p} = p(t, r)$ isotropic pressure and $\bar{\rho} = \rho(t, r)$ energy-matter density:

$$T_{\mu\nu} = (\bar{\rho} + \bar{p}) u_\mu u_\nu + \bar{p} g_{\mu\nu}, \quad (5)$$

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity ($u_\mu u^\mu = -1$). For a generic barotropic fluid with equation of state w and elastic compression $\Pi(\rho) = \int d\rho p(\rho)/\rho^2 = w \ln \rho$ we have $\bar{p} = w\rho$ and $\bar{\rho} = \rho[1 + w \ln \rho]$. This fluid is in general made of several components, each with a different equation of state. We describe the metric for an observer comoving with the fluid (so that the spatial velocity is $u^i = 0$). We then have:

$$\begin{aligned} T_0^0 &= -\bar{\rho} & ; & & T_1^1 = T_2^2 = T_3^3 = \bar{p}, \\ T &\equiv T_\mu^\mu = 3\bar{p} - \bar{\rho}, \end{aligned} \quad (6)$$

and using Eq.2:

$$\begin{aligned} R_0^0 &= -4\pi G(\bar{\rho} + 3\bar{p}) & ; & & R_1^1 = R_2^2 = R_3^3 = -4\pi G(\bar{p} - \bar{\rho}), \\ R &= -8\pi G T = 8\pi G(\bar{\rho} - 3\bar{p}). \end{aligned} \quad (7)$$

The Lagrangian on-shell for a perfect fluid is $\mathcal{L}_m^{on-sh} = -\bar{\rho}$ (Minazoli & Harko 2012) so that Eq.1 on-shell becomes:

$$S^{on-sh} = -\frac{1}{2} \int_M dM [\bar{\rho} + 3\bar{p}] = \int_M dM \frac{R_0^0}{8\pi G} \equiv -m/2 \quad (8)$$

where m is the relativistic mass inside the manifold. Note that this result also holds for anisotropic stress energy with heat flux. In this case we can have different pressure components $T_1^1 \neq T_2^2 \neq T_3^3$ and $\bar{p}$ above just corresponds to the mean value.

2.2 Gaussian flux and Zero Action

Consider the geodesic acceleration $\mathbf{g}^\mu = (\mathbf{g}^0, \mathbf{g}^i) = (\mathbf{g}^0, \vec{\mathbf{g}})$ defined from the geodesic deviation equation (Padmanabhan 2010):

$$\mathbf{g}^\mu \equiv \frac{D^2 \mathbf{v}^\mu}{D\tau^2} = R_{\alpha\beta\gamma}^\mu u^\alpha u^\beta \mathbf{v}^\gamma, \quad (9)$$

where $\mathbf{v}^\mu$ is the separation vector between neighbouring geodesics and u^α is the tangent vector to the geodesic. For an observer following the trajectory of the geodesic $u^\alpha = (1, 0)$ and $\mathbf{g}^\alpha = (0, \vec{\mathbf{g}})$:

$$\mathbf{g}^i = R_{00\gamma}^i \mathbf{v}^\gamma. \quad (10)$$

and we can choose the separation vector $\mathbf{v}^\mu$ to be the spatial coordinate. The spatial divergence of $\vec{\mathbf{g}}$ is then:

$$\vec{\nabla} \cdot \vec{\mathbf{g}} = R_0^0 = -4\pi G(\bar{\rho} + 3\bar{p}), \quad (11)$$

where for the second equality we have used Eq.7. This equation does not require any approximation and is always valid for a comoving observer (see Eq.6.105 in Padmanabhan 2010). Newtonian gravity is reproduced for the case of non-relativistic matter ($\bar{p} = 0$). The covariant version of Eq.11 is the relativistic version of Poisson's Equation (see also Gaztañaga 2020):

$$\nabla_\mu \mathbf{g}^\mu = R_0^0 = -4\pi G(\bar{\rho} + 3\bar{p}). \quad (12)$$

The solution to these equations is given by an integral over the usual propagators or retarded Green functions which account for causality and indicates that the action on-shell has to be estimated on the light-cone. From Eq.12 and Eq.8 we can calculate the value of the action on-shell using Stoke's theorem to convert the 4D M integral into a 3D ∂M boundary integral:

$$S^{on-sh} = \int_M dM \frac{R_0^0}{8\pi G} = \int_M dM \frac{\nabla_\mu \mathbf{g}^\mu}{8\pi G} = \oint_{\partial M} \frac{dV_\mu \mathbf{g}^\mu}{8\pi G}, \quad (13)$$

where V_μ is a 4D vector whose components are 3D volume elements normal to ∂M (e.g. $dV_0 \sim dx dy dz$). So the action on-shell is just the relativistic version of the Gauss flux, which results in a boundary

term or a total charge. Traditionally, we take such boundary terms to be zero at infinity (see Eq.4.7.8 in Weinberg 1972) because there is no causal connection at infinity. In other words: we expect space-time to be asymptotically Minkowski because particles should be free at spatial infinity for a finite time of evolution. This means that boundary terms should be zero and therefore $S^{on-sh} = 0$.

To see this explicitly, consider the boundary ∂M as two spacelike surfaces at $t = t_1$ and $t = t_2$ and one timelike surface at spatial infinity. The usual assumption is that the fields do not contribute at spatial infinity. This is the case for asymptotically flat space-times. So Eq.13 reduces to two volume dV_0 integrals at constant t over $\mathcal{G}^0$. But $\mathcal{G}^0 = 0$ for a comoving observer and we then have $S^{on-sh} = 0$.

We also need $S^{on-sh} = 0$ because boundary terms must vanish to reproduce Eq.2 from Eq.1. In Appendix A we present a different derivation for $S^{on-sh} = 0$ as a Minkowski boundary condition. In Appendix B we estimate S^{om-sh} explicitly for the FLRW metric and in Appendix C we show how it is possible to combine the FLRW with Minkowski boundary conditions. But note that Eq.13 does not assumed a FLRW metric or homogeneity.

3 THE Λ TERM AS A BOUNDARY

As first pointed out by Einstein, we can add a fundamental constant, that here we call Λ_F , to the action in Eq.1 (vacuum energy or Dark Energy are included in the energy content via $\mathcal{L}_m$ or $T_{\mu\nu}$):

$$S = \int_M dM \left[\frac{R - 2\Lambda_F}{16\pi G} + \mathcal{L}_m \right]. \quad (14)$$

We can fulfill the zero action condition $S^{on-sh} = 0$ (see above) by subtracting a constant term $\mathcal{B}$ from the action:

$$S = \int_M dM \left[\frac{R - 2\Lambda_F}{16\pi G} + \mathcal{L}_m \right] - \mathcal{B}. \quad (15)$$

This is sometimes called “background subtraction” (another way to write Eq.4). The boundary density $\mathcal{B}/M \equiv \Lambda_{\mathcal{B}}/8\pi G$ plays the role of a new cosmological constant $\Lambda_{\mathcal{B}}$, so we can write Eq.14 as:

$$S = \int_M dM \left[\frac{R - 2\Lambda}{16\pi G} + \mathcal{L}_m \right], \quad (16)$$

where $\Lambda \equiv \Lambda_F + \Lambda_{\mathcal{B}}$. So the value of Λ_F is renormalized by the boundary term $\Lambda_{\mathcal{B}}$, so that its actual value is irrelevant. The new field equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (17)$$

so that Eq.7 transforms into:

$$\begin{aligned} R_0^0 &= \Lambda - 4\pi G(\bar{\rho} + 3\bar{p}) & ; & & R_1^1 = R_2^2 = R_3^3 = \Lambda - 4\pi G(\bar{p} - \bar{\rho}) \\ R &= 4\Lambda - 8\pi G T = 4\Lambda + 8\pi G(\bar{\rho} - 3\bar{p}). \end{aligned} \quad (18)$$

The geodesic equation Eq.12 changes to:

$$\nabla_\mu \mathbf{g}^\mu = R_0^0 = \Lambda - 4\pi G(\bar{\rho} + 3\bar{p}). \quad (19)$$

We can see here how $\Lambda > 0$ corresponds to a repulsive force which opposes gravity, given by the G term. This is the analog of the classical Casimir effect, which induces geometrical forces as a result of a boundary condition (e.g. Elizalde & Romeo 1991; Griffiths & Ho 2001; Boyer 2003). Replacing Eq.18 back in the action of Eq.16, the condition that $S^{on-sh} = 0$ fixes Λ to:

$$\Lambda = 4\pi G < \bar{\rho} + 3\bar{p} > \equiv \frac{4\pi G}{M} \int_M dM (\bar{\rho} + 3\bar{p}). \quad (20)$$

Thus $\Lambda = < R_0^0 >$ is the action density on-shell (for $\Lambda = 0$). As the action on-shell does not evolve (see Appendix A) and the primordial causal volume $M_\S$ is fixed (see Fig. 1 and §3.2), Λ is also constant.

The condition in Eq. 20 looks similar to that in Sobral-Blanco & Lombriser (2020); Lombriser (2019), who found $\Lambda/4\pi G = < T > = < 3\bar{p} - \bar{p} >$ (see also Kaloper et al. 2016; Carroll & Remmen 2017). Eq. 20 was first obtained by Gaztañaga (2020) starting from the relativistic version of Poisson equation Eq. 19. Our new derivation here shows how Λ emerges as a boundary condition to the action and not as a fundamental change to gravity (i.e. Λ_F above). A further interpretation as a BH Universe can be seen in Gaztanaga (2021) and Appendix C.

3.1 Vacuum energy and DE do not gravitate

It is well known that the effects of Λ in Eq. 17 are equivalent to that of vacuum energy, which has constant energy density and equation of state: $\omega = -1$. Replacing ρ_{vac} and p_{vac} in $T_{\mu\nu}$ by

$$\rho_\Lambda \equiv \rho_{vac} + \frac{\Lambda}{8\pi G}; \quad p_\Lambda \equiv p_{vac} - \frac{\Lambda}{8\pi G} = -\rho_\Lambda, \quad (21)$$

we can make Eq. 17 look the same as Eq. 2. Equivalently we can define a new effective energy-momentum tensor $\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}$ (Peebles & Ratra 2003) so that Eq. 17 looks the same as Eq. 2. So it looks as if we could not separate the effects of Λ from ρ_{vac} . But both quantities are related via Eq. 20, which results in a cancellation of terms in ρ_Λ . If we combine ρ_{vac} with matter ρ_m (with $p_m = 0$) and radiation ρ_R (with $p_R = \rho_R/3$), Eq. 20 gives:

$$\Lambda = 4\pi G (< \rho_m + 2\rho_R > - 2\rho_{vac}). \quad (22)$$

so that Eq. 21 results in:

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} + \rho_{vac} = < \rho_m/2 + \rho_R > \quad (23)$$

This shows that vacuum energy cancels out and can not affect the observed value of ρ_Λ or cosmic expansion. Eq. 23 also explains the coincidence problem (Peebles & Ratra 2003; Carroll 2004) connecting the measured values for ρ_m and $\rho_\Lambda \simeq 2.3\rho_m$.

Another way to see this is to combine Eq. 18 and Eq. 20:

$$R_0^0 = -4\pi G [(\rho - < \rho >) + 3(p - < p >)], \quad (24)$$

which explicitly shows that constant ρ or p components to the energy-momentum do not produce cosmic acceleration, giving $R_0^0 = 0$ (recall that $R_0^0 = \vec{\nabla} \vec{g} = 3\ddot{a}/a$). The contribution of DE with equation of state $p_{DE}(a) = \omega \rho_{DE}(a)$, is:

$$R_0^0 = -4\pi G (1 + 3\omega) \rho_{DE} [a^{-3(1+\omega)} - < a^{-3(1+\omega)} >], \quad (25)$$

where ρ_{DE} is the value today. So, for ω close to $\omega \simeq -1$, DE produces negligible cosmic acceleration or expansion. This removes the original motivation to have DE, as it represents an unnecessary complication of the model (Occam's razor). Vacuum energy or DE violate the strong ($\omega > -1/3$) and weak ($\omega > 0$) energy conditions in GR (e.g. see Visser 1997 and references therein). As shown above, in our interpretation where Λ is a boundary term given by Eq. 20, vacuum energy does not gravitate and DE is not needed, so these conditions are not necessarily violated.

In Unimodular Gravity, a modification of classical GR, Λ appears as a Lagrange multiplier or, equivalently, the determinant of the metric is fixed. In this situation Λ does not gravitate (see Anderson 1971; Smolin 2009; Garcia-Bellido et al. 2011; Percacci 2018 and references therein). This is different from what we find here, in classical GR, where Λ gravitates but it cancels ρ_{vac} , DE or Λ_F .

3.2 Causal Boundary and inflation

Without inflation the different parts of the observable Universe are causally disconnected. According to inflation, our Universe is inside a large causal horizon $\chi_\S$, which could be much larger than our observable Universe today (so that $\chi_\S \gg \chi_0$). In such case we expect the flux to be zero at $\chi_\S$, i.e. $S^{on-sh} = 0$ in Eq. 13, because sources produce no flux at distances that are out of causal reach. During inflation, the dynamics of the Universe are frozen out. After inflation our new Hubble horizon χ_H is very small $\chi_H \ll \chi_\S$ so we can assume that the Universe inside $\chi_\S$ follows the (flat) FLRW metric (see Fig. 1). For the FLRW Universe, we have $\rho_\Lambda \simeq \rho_m/(2 < a^3 >)$ (see Appendix B). The average $< a^3 >$ is over the comoving χ in the light-cone. If χ is large compare to $\chi_0 \equiv \chi(a=1)$ then $\rho_\Lambda \rightarrow 0$ as $a \rightarrow \infty$. This is contrary to current observations which find: $\rho_\Lambda \simeq 2.3\rho_m$, which corresponds to $< a^3 > \simeq 0.22$ and therefore $\chi_\S \simeq \chi_0$. So current measurements of $\rho_\Lambda \neq 0$ are the smoking gun for primordial inflation with $\chi_\S \simeq \chi_0$.

4 LATE TIME COSMIC ACCELERATION

If we assume that the Universe is isotropic and homogeneous inside $\chi_\S$ we have the FLRW metric in comoving coordinates (t, χ) :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 [d\chi^2 + S_k^2(\chi) dA^2], \quad (26)$$

where $dA^2 = \cos^2(\theta)d\phi^2 + d\theta^2$ is the solid angle element. For a closed (or open) Universe of radius χ_k we have $S_k = \chi_k \sin[\chi/\chi_k]$ (or $\chi_k \sinh[\chi/\chi_k]$). For the flat case $\chi_k \rightarrow \infty$ so that $S_k = \chi$. The FLRW Universe in Eq. 26, filled with matter $\rho_m \sim a^{-3}$ ($p_m = 0$), radiation $\rho_R \sim a^{-4}$ ($p_R = \rho_R/3$) and ρ_Λ has a expansion H and acceleration rate q (e.g. Peebles & Ratra 2003; Weinberg 2008):

$$3H^2 = 8\pi G(\rho_m + \rho_R + \rho_\Lambda) - 3/(a\chi_k)^2 \quad (27)$$

$$3qH^2 \equiv 3\ddot{a}/a = R_0^0 = \Lambda - 4\pi G(\rho + 3p) \quad (28)$$

where $\Omega_\chi \equiv \frac{\rho_\chi}{\rho_c}$ and $\rho_c \equiv \frac{3H^2}{8\pi G}$. The observed positive value for the cosmic acceleration $q \simeq 0.55$ indicates that $\rho_\Lambda > 0$ and corresponds to $\Omega_\Lambda \simeq 2.3\Omega_m \simeq 0.7$ today for the flat case $\Omega_k \simeq 0$. These values produce χ_H as shown numerically in Fig. 1. The Hubble radius corresponding to the causal horizon is $r_\S = \Omega_\Lambda^{-1/2} \simeq 1.2$, in units of c/H_0 , which corresponds to a Black Hole with $m \simeq 5.8 \times 10^{22} M_\odot$ (see Eq. C5). In terms of the FLRW frame (t, χ) what is relevant is the corresponding comoving particle horizon coordinate $\chi_\S$ (see Fig. 1). We can estimate $\chi_\S$ from the average in Eq. 23 with the information we have from observations after inflation (see Fig. 1). Inflation creates a large causal primordial horizon $\chi_\S$ inside which the Universe is homogeneous and nearly flat. The particle horizon after inflation is χ in Eq. 26, $dt = a(t)d\chi$, so we can write:

$$\chi(a) = \int_{t_e}^t \frac{dt}{a(t)} = \int_{a_e}^a \frac{d \ln a}{aH(a)} = \chi_0 - \bar{\chi}(a), \quad (29)$$

where t_e and a_e represent the time and scale factor when inflation ends. The particle horizon today is $\chi_0 \equiv \chi(a=1) \simeq 3\chi_H$ and $\bar{\chi}(a) = \int_a^1 da/(a^2 H)$ is the radial lookback time coordinate so that $d_A(\bar{\chi}) = S_k(\bar{\chi})$ is the comoving angular diameter distance. We can now estimate the average in Eq. 23 in the light-cone to $\chi_\S$:

$$\rho_\Lambda = \frac{\int_0^{\chi_\S} d\chi S_k^2(\chi) (1 - \cos \frac{\chi_\S}{\chi}) a^3 (\rho_m a^{-3} + 2\rho_R a^{-4})}{2 \int_0^{\chi_\S} d\chi S_k^2(\chi) (1 - \cos \frac{\chi_\S}{\chi}) a^3}, \quad (30)$$

Figure 2. Distances (in units of c/H_0) as a function of Ω_Λ . The Hubble radius of the causal horizon $r_H = \Omega_\Lambda^{-1/2} c/H_0$ (dotted line) is compared with χ_0 , the comoving particle horizon today (dashed line), which increases with Ω_Λ . The comoving primordial causal horizon χ_s (dot-dashed line) is a combination of the later two and has a minimum size of $\chi_s \approx 3$. The continuous line shows χ_s in units of χ_0 , which is always decreasing. A large boundary, $\chi_s > \chi_0$, produces $\Omega_\Lambda \rightarrow 0$, while $\chi_s < \chi_0$ gives $\Omega_\Lambda \rightarrow 1$.

where $a = a(\chi)$ is the light-cone crossing corresponding to the inverse of $\chi(a)$ in Eq.29, which also depends on ρ_Λ via the Hubble rate H . The term: $(1 - \cos \frac{\chi_s}{\chi})$ comes from the limits of χ_s on the solid angle dA and was not included in [Gaztañaga \(2020\)](#), which explains the small differences in the final results. To do this integral, it is easier to change variables to a :

$$\Omega_\Lambda = \frac{\int_0^{a_s} \frac{ada}{H} S_k^2(\chi) (1 - \cos \frac{\chi_s}{\chi}) (\Omega_m a^{-3} + 2\Omega_R a^{-4})}{2 \int_0^{a_s} \frac{ada}{H} S_k^2(\chi) (1 - \cos \frac{\chi_s}{\chi})}, \quad (31)$$

where $\chi = \chi(a)$ in Eq.29 and $a_s \equiv a(\chi_s)$. We have divided by ρ_c to express the result in terms of adimensional ratios Ω . For each value of χ_s (and Ω_k , with $\Omega_R \approx 4.2E-5$) we can find numerically the corresponding Ω_Λ . The result is shown in Fig.2. For $\Omega_\Lambda \approx 0.7 \pm 0.1$ and $\Omega_k \approx 0$ we find:

$$\chi_s = (3.34 \pm 0.18) \frac{c}{H_0} \quad \text{and} \quad a_s \equiv a(\chi_s) \approx 1.08 \pm 0.16, \quad (32)$$

compared to: $\chi_0 = 3.26c/H_0$. Because $\chi_s < \pi\chi_0$ we can actually measure χ_s on the sky. At the CMB, χ_s corresponds to an angle: $\theta_s(z_{\text{CMB}}) \equiv \chi_s/d_A \approx 60$ deg. This boundary could therefore result in CMB anomalies or tensions in measurements. Because χ_s in Eq.32 is closed to χ_0 , we could wonder if inflation lasted enough time to make $\Omega_k \approx 0$ today. In fact, $\Omega_k < 0$ is not ruled out by observations ([Di Valentino et al. 2020](#)). For $\Omega_k = -0.06$ and $\Omega_\Lambda \approx 0.7 \pm 0.1$ we find $\chi_s = (3.39 \pm 0.27) \frac{c}{H_0}$ compared to $\chi_0 = 3.31$ and $\theta_s(z_{\text{CMB}}) \approx 68$ deg. Fig.3 shows the angle on the sky $\theta_s(z) = \chi_s/d_A(z)$ corresponding to a transverse χ_s .

Because $d_A(z_{\text{CMB}}) \approx \chi_s$, we expect the cosmological parameters in our local universe to be slightly different from the ones that we see in the CMB, as they come from different primordial causal patches. This could explain tensions in measurements of H_0 and other cosmological parameters. The full CMB sky covers transverse scales up to $\approx \pi d_A \approx \pi\chi_s$, so that we are able to measure a few separate causal horizons on the CMB sky ([Fosalba & Gaztañaga 2020](#)).

Figure 3. Angle on the sky $\theta_s(z) = \chi_s/d_A(z)$ corresponding to a (transverse) primordial causal boundary χ_s , for different Ω_k and Ω_Λ . At $z \approx 2$ about half of the sky is causally disconnected. The lack of structure at $\theta > 60$ deg. in the CMB ($z \approx 10^3$) favours $\Omega_k \lesssim 0$ and $\Omega_\Lambda \lesssim 0.7$. We should look for other tracers of χ_s at lower z .

5 DISCUSSION AND CONCLUSIONS

We have argued that our universe not only has "zero energy" (i.e. critical density) but also zero action on-shell. This generates a boundary term for the action which fixes the cosmological constant to $\Lambda = 4\pi G < \rho + 3p >$, i.e. Eq.20. As a consequence vacuum energy, DE or a fundamental Λ_F do not gravitate (see §3.1) which provides a true solution to the cosmological problem. It predicts that the observed value of ρ_Λ is related to the mean matter and radiation content inside the boundary $\chi < \chi_s$, see Eq.23 and 30. In our interpretation, Λ is not a dark energy component or a fundamental constant Λ_F ([Weinberg 1989](#); [Huterer & Turner 1999](#); [Carroll et al. 1992](#)), but just a boundary term (see §3). Measurements of ρ_Λ allow us to estimate χ_s (see Eq.32 and Fig.2). For $\chi_s > \chi_0$ we have $\Omega_\Lambda \approx 0$, which is contrary to observations. While $\Omega_\Lambda \approx 1$ indicates that $\chi_s \approx \chi_0$. This can be used to constrain models of inflation ([Gaztañaga 2020](#)). In contrast to the Hubble horizon, $\chi_H = c/(aH)$, which increases with a , the primordial boundary χ_s remains constant throughout cosmic evolution (see Fig.1). The Λ boundary corresponds to a singularity in the metric of our Universe. We are dynamically trapped inside χ_s , like observers inside a Black Hole (BH). In Appendix C we show how χ_s corresponds to the BH horizon, which explains how we can have a Minkowski boundary and a FLRW metric at the same time ([Gaztanaga 2021](#)). Cosmic acceleration occurs when our particle horizon approaches this horizon. Thus late time cosmic acceleration is the smoking gun of primordial inflation.

Each causally disconnected patch (or horizon) that emerges from inflation evolves like a separate Universe (see Fig.4 and Appendix D). Our past evolution is not influenced by other horizons because $\chi_H < \chi_s$. This is why the zero action principle applies to each separate horizon. But we have a window to observe other horizons because our observable Universe (or past particle horizon) in Eq.29 is larger than the Hubble horizon: $\chi_0 \approx 3\chi_H$ (see dashed line in Fig.1) and $\chi_0 \approx \chi_s$ so that $\pi\chi_0 \approx \pi d_A > \chi_s$. Properties might not be homogeneous across nearby primordial patches which could result in cosmic anisotropies. One could imagine that this could lead to large scale super-horizon fluctuations, like in the Grishchuk-Zel'dovich effect on the CMB quadrupole ([Grishchuk & Zeldovich 1978](#)). But recall from Fig.1 that the so called super-horizon scales are frozen scales

Figure 4. Illustration of the causal structure of the Universe after inflation. Each circle (with continuous lines) represent an independent Universe: a causally disconnected horizon with different cosmological parameters and values of $\chi_\S$ (and ρ_Λ). Our galaxy is depicted at the center of the dashed concentric circles which represent the growth evolution of our particle horizon χ_0 after inflation ends. A maximum size is reached when $\chi_0 \approx \chi_\S$. Further evolution freezes out as our Hubble horizon shrinks again because the expansion becomes dominated by ρ_Λ (see Fig.1).

smaller than $\chi_\S$, the horizon during inflation. Each causal horizon is by definition independent so the spectrum of fluctuations vanish on causal horizon scales. This is in agreement with the measured CMB quadrupole, which is smaller, and not larger, than expected. Solutions in separate primordial horizons should be matched into a larger manifold solution (Sanghai & Clifton 2015; Gaztanaga 2021). Continuity across contiguous regions then forces any uncorrelated differences to be small. This could explain why the mean CMB temperature can be similar across nearby patches in the sky.

That the primordial causal boundary scale, $\chi_\S$, is similar in size to our observable Universe today, χ_0 , solves the coincidence problem (Peebles & Ratra 2003; Carroll 2004) of why is $\rho_\Lambda \approx 2.3\rho_m$ or why $(\Lambda/8\pi G)^{-1/2}$ is close to c/H_0 . But it rises new questions: why $\chi_\S \approx \chi_0$? Fig.2 shows how very different values of Ω_Λ have similar $\chi_\S/\chi_0$ values. So may be this is not such a big coincidence. In terms of anthropic reasoning, at earlier times the Universe is dominated by radiation and there are no stars or galaxies to host observers. Moreover, $\chi_0 \approx \chi_\S$, has the largest possible Hubble radius (see Fig.1) with the highest chances to host observers like us. Ultimately, the reason for $\chi_\S \approx \chi_0$ resides in the details of the initial conditions (see also Appendix D).

We should be able to find observational evidence of such a primordial boundary $\chi_\S$ in our sky (see Fig.3). Because separate horizons are not causally connected, we expect to see no statistical correlations in the sky on angular scales $\theta > \theta_\S$. This seems in agreement with the anomalous lack of correlations (respect to Λ CDM model) in the CMB temperature correlations $w(\theta)$ for $\theta \gtrsim 60^\circ$. Such anomaly has been used to predict $\Omega_\Lambda \approx 0.7$ for a flat Universe with independence of any other measurements (Gaztañaga 2020). But note that the value of Ω_Λ that we measure locally, could be different from the one affecting the faraway distance where CMB photons were emitted. Tensions between measurements of cosmological parameters (or fundamental constants) from very different redshifts or different regions of the sky (at high $z > 2$) could be related to such in-homogeneities.

In a recent analysis Fosalba & Gaztañaga (2020) found strong evidence for such horizons in the coherent variation of cosmological parameter in the CMB sky. We should look closer for such differences to further validate or falsify these ideas.

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DATA AVAILABILITY STATEMENT

The data and codes used in this article will be shared on request.

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APPENDIX A: AOS THEOREM

We next enunciate and proof what we call the Action on-shell (AoS) Theorem, which basically says that the action on-shell is fixed by its boundary. This is reminiscent of the holographic principle (Susskind 1995; Maartens 2004). Indeed the Einstein–Hilbert action in Eq.1 possesses such holographic relation between the surface and bulk terms (see Eq.15.45 in Padmanabhan 2010).

This theorem applies to any stationary action principle. But we note that the proof presented here lacks the mathematical rigor of the Stokes’ Theorem which was used to derive the zero action principle in section 2.2. Our arguments below assume continuous and smooth functions and do not account for situations with singularities.

Definitions: given an action functional:

$$S[g, M] = \int_M dM \mathcal{L}_m[g], \quad (A1)$$

where M is a spacetime manifold and $dM = \sqrt{-g}d^4x$ is the invariant 4D volume and $g = g(x)$ stands for any dynamical field in our problem, including the metric $g^{\mu\nu}$. Assume that S obeys an stationary action principle, i.e. the equations of motion for g are such that arbitrary small variation δg around the solution g^* produce no changes in S , so that:

$$\delta S[g^*, M] \equiv S[g^* + \delta g, M] - S[g^*, M] = 0. \quad (A2)$$

The action on-shell is defined as $S^{on-sh} \equiv S[g^*, M]$.

Theorem: $S^{on-sh} = S[g^*, \partial M]$ where ∂M is a boundary of M .

Proof: Consider a partition of $M = M_1 + M_2$:

$$S[g^*, M] = S[g^*, M_1] + S[g^*, M_2], \quad (A3)$$

it follows from Eq.A2:

$$S[g^*, M] = S[g^* + \delta g, M_1] + S[g^* + \delta g, M_2]. \quad (A4)$$

Because δg can be arbitrary we can choose:

$$\delta g = \begin{cases} \delta g_1 & \text{for } M_1 \\ 0 & \text{for } M_2 \end{cases} \quad (A5)$$

so that:

$$S[g^*, M] = S[g^* + \delta g_1, M_1] + S[g^*, M_2], \quad (A6)$$

and comparing with Eq.A3 we find:

$$S[g^*, M_1] = S[g^* + \delta g_1, M_1], \quad (A7)$$

so that the action is also stationary as in Eq.A2 for any part of M . Consider a volume element dM centered around a coordinate (t, x) of the manifold. We can choose $M_1 = dM(t, x)$ so that each step in the integration in Eq.A1 is stationary:

$$\delta S[g^*, dM(t, x)] = 0. \quad (A8)$$

The solution $g^*(t, x)$ evolves when we move one step dM away from (t, x) . We denote this $\delta g^*(t, x)$ to distinguish it from some arbitrary variation of the field δg . Because the action for each volume element $dM(t, x)$ is stationary around an arbitrary variation of δg , it should also be stationary if we choose $\delta g = \delta g^*(t, x)$, so that:

$$S[g^*, dM(t, x)] = S[g^* + \delta g^*, dM(t, x)], \quad (A9)$$

which shows that the evolution of the field in step dM does not change the action. As this is true for each step in the action integral, we can conclude that the action integral value on-shell is the same as the one in its boundary: $S[g^*, M] = S[g^*, \partial M]$. Q.E.D.

Corollary-I: If the Universe has a Minkowski boundary, the action on-shell must be zero, $S^{on-sh} = 0$, for its whole evolution.

Proof: Because the Minkowski metric has $S^{on-sh} = 0$, we can use the AoS Theorem to conclude that $S^{on-sh} = 0$ also for the full evolution. Q.E.D.

An example of Corollary-I is the zero Gaussian flux condition presented in section 2.2. Another example is the Schwarzschild metric, which asymptotically goes to Minkowski and produces $S^{on-sh} = 0$ (if we avoid the singularity at $r = 0$). Other examples where $S^{on-sh} = 0$ are the Dirac action or the harmonic oscillator action. More generally, the kinetic term of a field theory $\mathcal{L}_m \propto \psi_i L_{ij} \psi_j$ (where L_{ij} is a linear operator) is zero on-shell because the equation of motion is $L_{ij} \psi_j = 0$. Potential terms are usually assumed to be zero at the boundaries because of causality.

Corollary-II: If the Universe started out of nothing, the action on-shell must be zero, $S^{on-sh} = 0$, for its whole evolution.

Proof: The solution for the field equation for a Universe without content $T_{\mu\nu} = 0$ is the Minkowski metric. Using Corollary-I we conclude that $S^{on-sh} = 0$ also for the full evolution. Q.E.D.

APPENDIX B: THE ACTION FOR THE FLRW METRIC

To estimate Eq.8 for a flat matter dominated FLRW metric, we choose our location as the origin of the radial coordinate and the end of inflation as the initial condition. For simplicity consider the flat matter dominated Universe in the FLRW metric in Eq.26 and Eq.27. The 4D volume element is $dM = dt \sqrt{-g} d^3x = dt a^3 4\pi \chi^2 d\chi$ where we have used: $\sqrt{-g} = a^3$. As we integrate in the radial direction we need to update time to the light cone $dt = d\chi/a$ because the solution of the field equations obey causality (on-shell). The integral then becomes integral over $\chi = \chi(a)$ or $a = a(\chi)$ (see Eq.29):

$$S_{FLRW}^{on-sh} = -\frac{1}{2} \int_0^\chi d\chi 4\pi \chi^2 a^3 (\rho_m a^{-3}) = -V \frac{\rho_m}{2}. \quad (B1)$$

where $V = 4/3\pi \chi^3$ is the 3D comoving volume and ρ_m the matter density today ($a = 1$). This integral is unbounded and diverges as we increase the comoving volume V (or the time involved). This illustrates the issue with the FLRW metric that we mentioned in the introduction. It produces boundary terms because it has a non compact geometry. This is an issue because we can not reproduce Eq.2 from Eq.1, as boundary terms don’t cancel out. In the case of a FLRW model with Λ , we can use Eq.16 to find:

$$S_{FLRW}^{on-sh} = V \left(\frac{\Lambda \langle a^3 \rangle}{8\pi G} - \frac{\rho_m}{2} \right) \quad (B2)$$

$$\langle a^3 \rangle \equiv \frac{1}{V} \int_0^\chi d\chi 4\pi \chi^2 a^3 \quad (B3)$$

We can see here how we can fix $\Lambda = 4\pi G \rho_m / \langle a^3 \rangle$ to cancel this boundary term. Note that in this case V is finite because there is a maximum value for χ (see Fig.1). If we also add ρ_{vac} we find: $\rho_\Lambda = \rho_m / (2 \langle a^3 \rangle)$ which cancels ρ_{vac} (see §3.1).

APPENDIX C: THE BLACK HOLE UNIVERSE (BHU)

The idea of the causal boundary $\chi_\S$ results in a non-homogeneous universe. This makes sense for a universe of finite age because there

Figure C1. Metric elements $g_{tt}^{-1} = -g_{rr} = (1 - x^n)^{-1}$ of the Black Hole Universe (BHU) in Eq.C3 as a function of $x = r/r_s$. The BHU metric corresponds to Schwarzschild ($n = -1$) for the outside $x > 1$ and to deSitter ($n = 2$) for the inside $x < 1$, which preserves the signature for all r . The outside of deSitter metric or the inside of Schwarzschild have negative metric signature (red shading) which indicates that they are not causally connected.

are causally disconnected regions with different densities (see Fig.4). For $\chi < \chi_s$ the FLRW metric should be a good approximation, but we want an asymptotically flat metric on larger scale. How can we combine these two different metrics?

Consider the flat FLRW metric of Eq.26 in comoving coordinates (t, χ) . When ρ_Λ dominates the expansion rate: $H^2 \equiv (\dot{a}/a)^2 = 8\pi G\rho_\Lambda/3$, which results in exponential inflation: $a(t) \propto e^{Ht}$. We can change variables from comoving χ to proper $r = a(t)\chi$ distances, and introduce a new time variable $T = T(t, \chi)$ such that $\partial_t T = (1 - H^2 r^2)^{-1}$ and $\partial_\chi T = aHr \partial_t T$. In these new variables (T, r) , the FLRW metric becomes DeSitter:

$$ds^2 = -(1 - H^2 r^2) dT^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 dA^2, \quad (C1)$$

where $H^2 = 8\pi G\rho_\Lambda/3$ is constant. So in (T, r) coordinates the FLRW expansion is static (see Mitra 2012). DeSitter metric above is very similar to the Schwarzschild metric:

$$ds^2 = -(1 - 2Gm/r) dT^2 + \frac{dr^2}{1 - 2Gm/r} + r^2 dA^2, \quad (C2)$$

with an Event Horizon (EH) at $r_s = 2Gm$. Both metrics are static. DeSitter metric also has an EH at $r_s = 1/H$. But DeSitter solution corresponds to the interior $r < r_{EH}$, while the Schwarzschild metric represents the exterior $r > r_{EH}$. We can match both solutions into a Black Hole Universe (BHU) where we consider DeSitter metric as the interior of the Schwarzschild metric:

$$ds^2 = -[1 - (r/r_s)^n] dT^2 + \frac{dr^2}{1 - (r/r_s)^n} + r^2 dA^2, \quad (C3)$$

$$n = \begin{cases} 2 & \text{for } r < r_s \\ -1 & \text{for } r > r_s \end{cases}$$

which is continuous and smoothed everywhere except at the EH (see Fig.C1). This results in $r_s = 1/H = 2mG$, which gives:

$$\rho_\Lambda = 3m/(4\pi r_s^3), \quad (C4)$$

$$m = \frac{4\pi}{3} r_s^3 < \rho_m/2 + \rho_R. \quad (C5)$$

where in the second equation we have use Eq.23. The expansion of our universe is not always dominated by ρ_Λ . But Eq.C1 is still

the exact transformation of the FLRW metric in the Schwarzschild frame (T, r) , by just replacing H by $H(t)$ (Gaztanaga 2021). Thus, the FLRW metric also has an EH at $r_H \equiv 1/H(t) < r_s$. But this is an expanding horizon that grows inside r_s . As r_H approaches r_s a repulsive force (from the Λ term or the EH) forces $r_H = r_s$ (see Fig.1). So Eq.C3-C4 is valid throughout cosmic evolution (for more details on the BHU see Gaztanaga 2021).

In summary, we can combine the FLRW metric and the Schwarzschild metric into a BHU to obtain the causal structure required by the zero action principle: $S^{on-sh} = 0$.

APPENDIX D: THE APOLLONIAL MULTIVERSE

The zero action principle could also be used to build a multiverse of BHU. We can picture the start of our universe as an empty Minkowski space-time with quantum fluctuations of some field $\rho_I(a)$ with equation of state ω or just false vacuum BH (Gaztanaga 2021). Because $S^{on-sh} = 0$, this fluctuation will generate a boundary Λ term (also call a BH). We can consider these fluctuations homogeneous inside some (quantum) causal scale $a_i \chi_s$. The Hubble rate is then:

$$H^2 = \frac{8\pi G}{3} \left[\rho_\Lambda + \rho_I (a/a_i)^{-3(1+\omega)} \right], \quad (D1)$$

where a_i is some reference primordial time and from Eq.30:

$$\rho_\Lambda = \rho_I \frac{1 + 3\omega}{2} \frac{\int_0^{\chi_s} d\chi S_k^2(\chi) a^3 ((a/a_i)^{-3(1+\omega)})}{\int_0^{\chi_s} d\chi S_k^2(\chi) a^3}, \quad (D2)$$

where χ_s (and $a_s \equiv a(\chi_s)$) corresponds to the causal scale. For $a_s < a_i$ and $\omega > -1$ (or $a_s > a_i$ and $\omega < -1$) the ρ_Λ term dominates over $\rho_I (a/a_i)^{-3(1+\omega)}$. This will generate an inflationary expansion. The quantum fluctuation will be enlarged and become macroscopic. This inflation will also make the Hubble horizon smaller than χ_s (see Fig.1) so our observable Universe will eventually become quite large but trapped inside χ_s . This will happen in different locations of the initial Minkowski manifold so that we can end up with multiple disconnected and inflating Universes. Each Universe will have different χ_s and ρ_Λ and they will inflate until they touch each other creating an Apollonial fractal, as illustrated by Fig.4.

Another way to picture this, for an external observer, is that we will have a network of interacting BHU (see Appendix §C) which could also be in equilibrium with some external (Hawking) radiation or matter fields. It seems reasonable to predict that their interactions could end inflation inside some BHU (e.g. because of the dynamical collisions) or just because false vacuum decay or false vacuum rolling (Gaztanaga 2021). The dynamical energy of the expansion could transform into thermal energy, in a similar way to the reheating mechanism in traditional inflationary models.

Inflation will happen under quite general considerations according to the zero action principle. The principle will still apply to each separate Universe (with some influence of its surroundings) because the interaction with nearby horizons will be limited in time and space. Later quantum fluctuations within one universe represent negligible contribution and do not change χ_s or H .

Our universe is now entering a second inflationary phase and it is possible that we could cycle to a situation similar to our previous inflation. So repeated phases of inflation could also play a role to understand our Universe and the value of the cosmological parameters. Could this result in an eternal return? or in natural selection (Smolin 1992)? Details of this model need to be worked out.