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On the Correspondence between Risk Measures on the Space of Infinite Sequences and their Acceptance Sets

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Abstract

In this paper, we present the correspondence between natural risk statistics on the space of infinite and bounded sequences of real numbers and their associated acceptance sets.

Keywords: natural risk statistics, acceptance set, bounded sequence, comonotone convexity

1 Introduction

In [2], Artzner and al.(1999), proposed the notion of coherent risk measures for the first time, and associated it with the notion of acceptance sets. So on, the risk measure families that was presented: Insurance risk measures [8].(1997), convex risk measures [5].(2002), and natural risk statistics [6].(2006), still be concerned with defining its corresponding acceptance sets. In [3].(2010), the definition of a natural risk statistics was extended from $\mathbb{R}^n$ to $l^\infty(\mathbb{R})$ the space of infinite and bounded sequences of real numbers. We will show that there

is a correspondence between the set of natural risk statistics on the space $l^\infty(\mathbb{R}) = \{X = (x_i)_{i=1,2,3,\dots} : x_i \in \mathbb{R}, \|X\|_\infty = \sup_{i=1,2,3,\dots} |x_i| < \infty\}$ and the set of acceptance sets in the same space. If ρ is a natural risk statistics on l^∞ , the acceptance set corresponding to ρ is $A_\rho = \{X \in l^\infty : \rho(X) \leq 0\}$. Given $A \subset l^\infty$, as an acceptance set, the natural risk statistics corresponding to A , is the map giving every position $X \in l^\infty$ the minimal riskless amount that has to be combined with X in order to obtain an acceptable position, formally we have $\rho_A(X) = \inf\{m \in \mathbb{R} : X - m\mathbf{1} \in A\}$.

2 Preliminary Notes

In this section, we recall the definition of a risk measure and an acceptance set, on the space $L_B(\Omega, \Gamma)$ of bounded and measurable random variables [2,5].

Definition 2.1 Let $\rho : L_B(\Omega, \Gamma) \rightarrow \mathbb{R}$, the function ρ is a risk measure, if:

- 1- ρ is invariant by translation: $\forall X \in L_B(\Omega, \Gamma), \forall c \in \mathbb{R} \rho(X + c) = \rho(X) - c$;
- 2- ρ is monotone: $\forall X, Y \in L_B(\Omega, \Gamma)$ if $X \leq Y$, then $\rho(Y) \leq \rho(X)$.
- ρ is convex if:
- 3- $\forall \lambda \in [0, 1], \forall X, Y \in L_B(\Omega, \Gamma) \rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$.
- ρ is coherent if:
- 4- ρ is positively homogeneous: $\forall \lambda \geq 0, \forall X \in L_B(\Omega, \Gamma) \rho(\lambda X) = \lambda \rho(X)$;
- 5- ρ is subadditive: $\forall X, Y \in L_B(\Omega, \Gamma)$, we have $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Definition 2.2 Let $A \subset L_B(\Omega, \Gamma)$, A is said to be an acceptance set, if:

- 1- $A \neq \emptyset$;
- 2- A is convex;
- 3- $\forall X \in A, \forall Y \in L_B(\Omega, \Gamma)$, if $X \leq Y$, then $Y \in A$.

The next theorem, endows us with a corresponding relationship between convex risk measures on $L_B(\Omega, \Gamma)$ and the sets of acceptable positions [2].

Theorem 2.3 ([2]) Let $A \subset L_B(\Omega, \Gamma)$ be an acceptance set and ρ a convex risk measure on $L_B(\Omega, \Gamma)$, then:

- 1- $\rho_A = \rho$;
- 2- A_ρ is an acceptance set;
- 3- $\forall X \in A, \forall Y \in L_B(\Omega, \Gamma)$, the set $\{\lambda \in [0, 1] : \lambda X + (1 - \lambda)Y \in A\}$ is closed in $[0, 1]$.

Now, we present the definition of a natural risk statistics on l^∞ , as it was extended in [3], from $\mathbb{R}^n$ to l^∞ .

Definition 2.4 ([3]) A function $\rho : l^\infty \rightarrow \mathbb{R}$ is a natural risk statistics, if:

- 1- ρ is invariant by translation and positively homogeneous:

$\rho(\lambda X + c\mathbf{1}) = \rho(\lambda x_1 + c, \lambda x_2 + c, \lambda x_3 + c, \dots) = \lambda \rho(x_1, x_2, x_3, \dots) + c = \lambda \rho(X) + c$
 $\forall \lambda \geq 0$ and $\forall c \in \mathbb{R}$, where $\mathbf{1} = (1, 1, 1, \dots) \in l^\infty$;

2- ρ is monotone: If $X \leq Y$ (i.e., $x_1 \leq y_1, x_2 \leq y_2, x_3 \leq y_3, \dots$), then

$\rho(X) = \rho(x_1, x_2, x_3, \dots) \leq \rho(Y) = \rho(y_1, y_2, y_3, \dots)$;

3- ρ is comonotone subadditive:

If X and Y are comonotonic (i.e., $(x_i - x_j)(y_i - y_j) \geq 0$ for any $j \neq i$), then
 $\rho(X + Y) = \rho(x_1 + y_1, x_2 + y_2, \dots) \leq \rho(x_1, x_2, \dots) + \rho(y_1, y_2, \dots) = \rho(X) + \rho(Y)$;

4- ρ is invariant under permutation: for any permutation $\pi = (i_1, i_2, i_3, \dots)$ of $\mathbb{N}^*$, we have $\rho(X) = \rho(x_1, x_2, x_3, \dots) = \rho(x_{i_1}, x_{i_2}, x_{i_3}, \dots) = \rho(\pi(X))$.

Note that every natural risk statistics ρ is normalized (i.e., $\rho(0) = 0$) and Lipschitz continuous with respect to the usual supremum norm of l^∞ , then it is continuous.

3 Main Results

The next definition is the analogue in l^∞ , of the definition 2.2 [2,5].

Definition 3.1 Let $A \subset l^\infty$, A is said to be an acceptance set, if:

1- A contains the set $l_-^\infty = \{X \in l^\infty : x_i \leq 0, i = 1, 2, 3, \dots\}$;

2- $A \cap l_{++}^\infty = \{X \in l^\infty : x_i > 0, i = 1, 2, 3, \dots\} = \emptyset$;

3- A is comonotone convex: If X and Y are comonotonic, then
 $(\lambda X + (1 - \lambda)Y) \in A, \forall \lambda \in [0, 1]$;

4- A is positively homogeneous: $\lambda X \in A, \forall X \in A$ and $\forall \lambda \geq 0$;

5- $\forall X, Y \in l^\infty$, If $X \leq Y$ and $Y \in A$, then $X \in A$;

6- $\pi(X) \in A \subset l^\infty$, for every permutation $\pi(X)$ of $X \in A$.

Remark 3.2 The difference in inequalities and signs between this definition and the definition 2.2 in [2] and [5], is that here we consider that X is a loss, as in [3], contrary to [2] and [5] where X is considered as a profit.

Now, we are in a position to state the correspondence theorem, between a natural risk statistics on l^∞ and its associated acceptance set.

Theorem 3.3 1- If ρ is a natural risk statistics on l^∞ , then
the set $A_\rho = \{X \in l^\infty : \rho(X) \leq 0\}$ is a closed acceptance set in the topology of the usual supremum norm of l^∞ .

2- If a set $A \subset l^\infty$ is an acceptance set, then

$\rho_A(X) = \inf\{m \in \mathbb{R} : X - m\mathbf{1} \in A\}$ is a natural risk statistics on l^∞ .

Proof. 1 • $\forall X \in l_-^\infty$, we have $x_i \leq 0, i = 1, 2, 3, \dots$, so $X \leq 0$, the monotonicity of ρ implies that $\rho(X) \leq \rho(0) = 0$, then $X \in A_\rho$. Therefore, $l_-^\infty \subset A_\rho$.

- Let $X \in l_{++}^\infty$, so $X > 0$ and $\rho(X) \geq \rho(0) = 0$. If $\rho(X) > 0$, then $X \notin A_\rho$. If $\rho(X) = 0$, then $\forall Y$ such that $0 < Y \leq X < \mathbf{1}$, we have $0 \leq \rho(Y) \leq \rho(X) = 0$, thus $\rho(Y) = 0$. Let $B = \{Z \in l_{++}^\infty : X < Z < \mathbf{1}, 0 < \rho(Z), 0 < \text{Inf}(Z)\}$, by continuity of ρ we obtain that $\text{Inf}(B) = X$, then $\forall \epsilon > 0, \exists Z \in B$ such that $Z < X + \epsilon \mathbf{1}$, then $Z - \epsilon \mathbf{1} < X$, we choose $\epsilon = \text{Inf}(Z')$ where $0 < Z' < Z$, then $0 < Z' - \text{Inf}(Z')\mathbf{1} < Z - \text{Inf}(Z')\mathbf{1} < X$, hence $\rho(Z' - \text{Inf}(Z')\mathbf{1}) = 0$ and $\rho(Z') = \text{Inf}(Z')$, which is impossible for Z' non constant. Therefore, $\rho(X) > 0$, and $X \notin A_\rho$. Finally, $l_{++}^\infty \cap A_\rho = \emptyset$.
- If X and Y are comonotonic, where $X \in A_\rho$ and $Y \in A_\rho$, then $\rho(X) \leq 0$ and $\rho(Y) \leq 0$, in another side λX and $(1 - \lambda)Y$ are comonotonic $\forall \lambda \in [0, 1]$. Thus, the comonotonic subadditivity and the positive homogeneity of ρ , imply that $\rho(\lambda X + (1 - \lambda)Y) \leq \rho(\lambda X) + \rho((1 - \lambda)Y) = \lambda \rho(X) + (1 - \lambda)\rho(Y) \leq 0$, then $(\lambda X + (1 - \lambda)Y) \in A_\rho$. Hence, A_ρ is comonotonic convex.
- $\forall X \in A_\rho$, we have $\rho(X) \leq 0$. $\forall \lambda \geq 0$, the positive homogeneity of ρ implies that $\rho(\lambda X) = \lambda \rho(X) \leq 0$, then $\lambda X \in A_\rho$. Therefore, A_ρ is positively homogeneous.
- $\forall X, Y \in l^\infty$, If $X \leq Y$ and $Y \in A_\rho$, then $\rho(Y) \leq 0$. By monotonicity of ρ , we have $\rho(X) \leq \rho(Y) \leq 0$, then $X \in A_\rho$. Hence, every element of l^∞ less than Y , is in A_ρ .
- If $X \in A_\rho$, then $\rho(X) \leq 0$. Let $\pi = (i_1, i_2, i_3, \dots)$ be a permutation of $\mathbb{N}^*$. By invariance under permutation of ρ , we have $\rho(\pi(X)) = \rho(x_{i_1}, x_{i_2}, x_{i_3}, \dots) = \rho(X) = \rho(x_1, x_2, x_3, \dots) \leq 0$, then $\pi(X) \in A_\rho$. Therefore, A_ρ contains the permutations of its elements.
- If $(X^k)_{k \in \mathbb{N}^*}$ is a sequence in A_ρ , converges in the norm $\|\cdot\|_\infty$ to $X = (x_1, x_2, x_3, \dots) \in l^\infty$, we have $\rho(X^k) \leq 0, \forall k = 1, 2, 3, \dots$. If $X \notin A_\rho$, then $\rho(X) > 0$. Thus $\exists \epsilon > 0$ such that $\rho(X - \epsilon \mathbf{1}) > 0$. From the fact that $\lim_{k \rightarrow +\infty} X^k = X$, it follows that $\exists k_0 \in \mathbb{N}^*$ such that $X - \epsilon \mathbf{1} < X^{k_0}$. By monotonicity of ρ , we have $0 < \rho(X - \epsilon \mathbf{1}) \leq \rho(X^{k_0})$, which contradicts the fact that X^{k_0} is an element of A_ρ . Finally, we have $X \in A_\rho$ and A_ρ is closed in the topology of the supremum norm of l^∞ .

2 • Firstly, $\rho_A(X) = \text{Inf}\{m \in \mathbb{R} : X - m\mathbf{1} \in A\}$ is well defined, we have $l_-^\infty \subset A$, so $A \neq \emptyset$. For $Y \in A$ $Y - \text{Inf}(Y)\mathbf{1} = Y - \text{Inf}\{y_i, i = 1, 2, 3, \dots\}\mathbf{1} \geq 0$ and $\forall X \in l^\infty$, we have $X - \text{Sup}(X)\mathbf{1} = X - \text{Sup}\{x_i, i = 1, 2, 3, \dots\}\mathbf{1} \leq 0$, so $X + (\text{Inf}(Y) - \text{Sup}(X))\mathbf{1} \leq Y$, which means that $X + (\text{Inf}(Y) - \text{Sup}(X))\mathbf{1} \in A$, then $\text{Sup}(X) - \text{Inf}(Y) \in \{m \in \mathbb{R} : X - m\mathbf{1} \in A\}$.

Thus, $\{m \in \mathbb{R} : X - m\mathbf{1} \in A\} \neq \emptyset$.

Also, $\rho_A(X) = \text{Inf}\{m \in \mathbb{R} : X - m\mathbf{1} \in A\} > -\infty$, because X is bounded $(\text{Inf}(X) - m)\mathbf{1} \leq X - m\mathbf{1}$, so if $X - m\mathbf{1} \in A$, then $(\text{Inf}(X) - m)\mathbf{1} \in A$, which means that $(\text{Inf}(X) - m)\mathbf{1} \in l_-^\infty$, then $\text{Inf}(X) \leq m$. Therefore, ρ_A exists.

- Let $X \in l^\infty$ and $c \in \mathbb{R}$, we have $\rho_A(X + c\mathbf{1}) = \text{Inf}\{m \in \mathbb{R} : X + c\mathbf{1} - m\mathbf{1} \in A\} = c + \text{Inf}\{m \in \mathbb{R} : X - m\mathbf{1} \in A\} = c + \rho_A(X)$. Let $\lambda \geq 0$. If $\lambda = 0$, then we

have $\rho_A(\lambda X) = \inf\{m \in \mathbb{R} : -m\mathbf{1} \in A\} = \lambda \rho_A(X) = 0$, because $l_{++}^\infty \cap A = \emptyset$ and $l_-^\infty \subset A$. If $\lambda > 0$, then $\rho_A(\lambda X) = \inf\{m \in \mathbb{R} : \lambda X - m\mathbf{1} \in A\}$

$= \lambda \inf\{v \in \mathbb{R} : \lambda(X - v\mathbf{1}) \in A\} = \lambda \inf\{v \in \mathbb{R} : (X - v\mathbf{1}) \in A\} = \lambda \rho_A(X)$, because A is positively homogeneous. Therefore, ρ_A is positively homogeneous and invariant by translation.

• $\forall X, Y \in l^\infty$, if $X \leq Y$ and $Y - m\mathbf{1} \in A$, then $X - m\mathbf{1} \in A$ as an element of l^∞ less than $Y - m\mathbf{1}$ and $\{m \in \mathbb{R} : Y - m\mathbf{1} \in A\} \subset \{m \in \mathbb{R} : X - m\mathbf{1} \in A\}$. Hence, $\rho_A(Y) = \inf\{m \in \mathbb{R} : Y - m\mathbf{1} \in A\} \geq \rho_A(X) = \inf\{m \in \mathbb{R} : X - m\mathbf{1} \in A\}$, which means that ρ_A is monotone.

• Let $X, Y \in l^\infty$ be comonotonic. If $m, m' \in \mathbb{R}$ are such that $X - m\mathbf{1} \in A$ and $Y - m'\mathbf{1} \in A$, then from the fact that $X - m\mathbf{1}$ and $Y - m'\mathbf{1}$ are comonotonic and A is comonotone convex, it follows that $\frac{1}{2}(X - m\mathbf{1}) + \frac{1}{2}(Y - m'\mathbf{1}) \in A$. By positive homogeneity of A , we obtain $X + Y - (m + m')\mathbf{1} \in A$. Thus, $\rho_A(X + Y) \leq m + m'$, especially

$\rho_A(X + Y) \leq \inf\{m \in \mathbb{R} : X - m\mathbf{1} \in A\} + \inf\{m' \in \mathbb{R} : Y - m'\mathbf{1} \in A\}$, which means that $\rho_A(X + Y) \leq \rho_A(X) + \rho_A(Y)$. Therefore, ρ_A is comonotone subadditive.

• $\forall \pi = (i_1, i_2, i_3, \dots)$ a permutation of $\mathbb{N}^*$, $(\pi(X)) \in A$, $\forall X \in A$, so $\forall m \in \mathbb{R}$ we have $(X - m\mathbf{1}) \in A$ if and only if $(\pi(X) - m\mathbf{1}) \in A$, thus $\{m \in \mathbb{R} : X - m\mathbf{1} \in A\} = \{m \in \mathbb{R} : \pi(X) - m\mathbf{1} \in A\}$, so $\inf\{m \in \mathbb{R} : X - m\mathbf{1} \in A\} = \inf\{m \in \mathbb{R} : \pi(X) - m\mathbf{1} \in A\}$. Therefore, $\rho_A(X) = \rho_A(\pi(X))$.

The next corollary shows that the set of acceptable positions associated with the natural risk statistics defined by an acceptance set, is the closure of the set itself.

Corollary 3.4 *Let $C \subset l^\infty$ be an acceptance set, we have $A_{\rho_C} = \overline{C}$, where $\overline{C}$ is the closure of C in the topology of the supremum norm of l^∞ .*

Proof. If $X \in C$, then $\rho_C(X) \leq 0$, thus $X \in A_{\rho_C}$, which means that $C \subset A_{\rho_C}$. According to the theorem 3.3, A_{ρ_C} is a closed set in the topology of the supremum norm of l^∞ , so $\overline{C} \subset A_{\rho_C}$.

$\forall X \in A_{\rho_C}$ we have $\rho_C(X) = \inf\{m \in \mathbb{R} : X - m\mathbf{1} \in C\} \leq 0$.

If $\inf\{m \in \mathbb{R} : X - m\mathbf{1} \in C\} < 0$, then $\exists m < 0$ such that $X - m\mathbf{1} \in C$, since X is an element of l^∞ such that $X < X - m\mathbf{1}$, then $X \in C \subset \overline{C}$.

If $\inf\{m \in \mathbb{R} : X - m\mathbf{1} \in C\} = 0$, then $\exists (m_i)_{i=1,2,3,\dots}$ a sequence of real positives such that $\lim_{i \rightarrow +\infty} m_i = 0$ and $X - m_i\mathbf{1} \in C$, so $\lim_{i \rightarrow +\infty} (X - m_i\mathbf{1})^i = X$ in the supremum norm of l^∞ .

Thus, $X \in \overline{C}$ and $A_{\rho_C} \subset \overline{C}$. Finally, we have $A_{\rho_C} = \overline{C}$.

Now we explicit the natural risk statistics, associated with A_ρ .

Proposition 3.5 *If ρ is a natural risk statistics on l^∞ , then*
 $\forall X \in l^\infty \quad \rho(X) = \rho_{A_\rho}(X) = \text{Inf}\{m \in \mathbb{R} : X - m\mathbf{1} \in A_\rho\}.$

Proof. $\forall X = (x_1, x_2, x_3, \dots) \in l^\infty$, we have by translation invariance of ρ
 $\rho_{A_\rho}(X) = \text{Inf}\{m \in \mathbb{R} : X - m\mathbf{1} \in A_\rho\} = \text{Inf}\{m \in \mathbb{R} : \rho(X - m\mathbf{1}) \leq 0\}$
 $= \text{Inf}\{m \in \mathbb{R} : \rho(X) \leq m\} = \rho(X).$ So, $\rho = \rho_{A_\rho}.$

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