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The stability of the fractional Volterra integro-differential equation by means of Ψ -Hilfer operator revisited

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Abstract

In this note, we prove the Ulam-Hyers stability of a fractional Volterra integral equation. Our results modify the Theorem 4 of [9] and present a corrected proof with a modified approximation.

Keywords: Fixed point theorem, fractional Volterra integral equation, Ulam-Hyers stability, Ψ -Hilfer fractional derivative.

1 Introduction and Preliminaries

Consider continuous map $\omega : [0, P] \rightarrow \Omega$ which Ω is a Banach space. In this paper, we apply a fixed point theorem (FPT) to study Ulam-Hyers stability (UH-stability) of the solution of the fractional Volterra integral equation

$$\omega(\tau) = \zeta(\tau, \omega(\tau)) + \mathcal{I}_{0+}^{\varrho; \Psi} \mathcal{K}(\tau, \sigma, \omega(\sigma)) d\sigma, \quad (1.1)$$

where $\mathcal{I}_{0+}^{\varrho; \Psi}$ is the Ψ -Riemann-Liouville fractional integral defined by

$$\mathcal{I}_{0+}^{\varrho; \Psi} \omega(\tau) = \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \omega(\sigma) d\sigma, \quad (1.2)$$

where $\mathcal{Q}_\Psi^\varrho(\tau, \sigma) := \Psi'(\sigma)(\Psi(\tau) - \Psi(\sigma))^{\varrho-1}$, [1]–[7], with $0 < \varrho < 1$, and $\zeta : [0, P] \times \Omega \rightarrow \Omega$, $\mathcal{K} : [0, P]^2 \times \Omega \rightarrow \Omega$.

We work on the weighted space $C_{1-\xi; \Psi}[0, P]$ of continuous mappings ω which defined by [7, 9] and with norm $\|\cdot\|_{C_{1-\xi; \Psi}[0, P]}$.

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For a continuous map $\omega(\tau)$, satisfying

$$\left\| \zeta(\tau, \omega(\tau)) + \mathcal{I}_{0+}^{\varrho; \Psi} \mathcal{K}(\tau, \sigma, \omega(\sigma)) d\sigma - \omega(\tau) \right\|_{C_{1-\xi; \Psi}} \leq \psi(\tau),$$

with $\psi(\tau) > 0$ and $0 \leq \tau \leq P$, let we can find a solution $\nu(\tau)$ of (1.1) such that

$$\|\omega(\tau) - \nu(\tau)\|_{C_{1-\xi; \Psi}[0, P]} \leq K\psi(\tau),$$

for some $K > 0$, so (1.1) has the UH-stability. Note that (1.1) can be considered as integral equation

$$\omega(\tau) = \zeta(\tau, \omega(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^t \mathcal{Q}_{\Psi}^{\varrho}(\tau, \sigma) \mathcal{K}(\tau, \sigma, \omega(\sigma)) d\sigma. \quad (1.3)$$

In Proposition 1 of [9] the authors proved that if $(\Omega, \|\cdot\|_{C_{1-\xi; \Psi}})$ is a normed space, then the function $\delta : \Omega \times \Omega \rightarrow \mathbb{R}$, $\delta(\omega, \nu) = \|\omega - \nu\|_{C_{1-\xi; \Psi}}$ is a metric on Ω . By [10] every normed space induce a metric by equality $\delta(\omega, \nu) = \|\omega - \nu\|$, then Proposition 1 of [9] is obvious and no need any proof.

The next theorem introduced by Diaz and Margolis [11] and used by Cădariu and Radu for studying the concept of stability of functional equations [12, 13, 14].

Theorem 1.1 (Alternative FPT). *Let (Γ, δ) be a complete $[0, \infty]$ -valued metric space. Assume that $\Lambda : \Gamma \times \Gamma$ is a strictly contractive operator with the Lipschitz constant $\mathcal{L} < 1$. If there exists a nonnegative integer k such that $\delta(\Lambda^{k+1}\omega, \Lambda^k\omega) < 1$, for some $\omega \in \Gamma$. Thus*

1. $\{\Lambda^n\omega\}$ converges to a FP ω^* of Λ .
2. $\Lambda\omega^* = \omega^*$, also ω^* is the unique in

$$\Gamma^* = \{\nu \in \Gamma; \delta(\Lambda^k\omega, \nu) < \infty\}.$$

3. If $\nu \in \Gamma^*$, $\delta(\nu, \omega^*) \leq \frac{1}{1-\mathcal{L}} d(\Lambda\nu, \nu)$.

2 Ulam-Hyers stability

In the proof of Theorem 4 of [9] the authors defined the mapping, $\Lambda : \mathcal{M} \rightarrow \mathcal{M}$, by

$$\begin{aligned} \Lambda(\omega(\tau)) &= \mathcal{I}_{0+}^{\varrho; \Psi} \left[\zeta(\tau, \omega(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^t \mathcal{Q}_{\Psi}^{\varrho}(\tau, \sigma) \mathcal{K}(\zeta, \sigma, \omega(\sigma)) d\sigma \right] \\ &= \mathcal{I}_{0+}^{\varrho; \Psi} [\zeta(\tau, \omega(\tau))] + \mathcal{I}_{0+}^{\varrho; \Psi} \left[\frac{1}{\Gamma(\varrho)} \int_0^t \mathcal{Q}_{\Psi}^{\varrho}(\tau, \sigma) \mathcal{K}(\zeta, \sigma, \omega(\sigma)) d\sigma \right], \end{aligned}$$

which is not suitable for fractional integral equations. More exactly, since by Theorem 1.1 the fixed point of Λ is the unique solution of (1.1) and the operator Λ helps to find a solution for integral equation (1.1), so could not be an integral operator as above, then the authors of [9] could not find an accurate approximation. In this section we present a correct proof by defining a suitable function $\Lambda : \mathcal{M} \rightarrow \mathcal{M}$.

Theorem 2.1. *Assume that Ω is a Banach space and $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$, and P are positive constants for which $0 < \mathcal{L}_1 + \mathcal{L}_2 \mathcal{L} < 1$. Let $\zeta : [0, P] \times \Omega \rightarrow \Omega$, $\mathcal{K} : [0, P]^2 \times \Omega \rightarrow \Omega$ and continuous map $\psi : [0, P] \rightarrow (0, \infty)$ satisfying*

$$\|\zeta(\tau, \omega) - \zeta(\tau, \nu)\|_{C_{1-\xi; \Psi}} \leq \mathcal{L}_1 \|\omega - \nu\|_{C_{1-\xi; \Psi}[0, P]}, \quad (2.1)$$

$$\|\mathcal{K}(\tau, \sigma, \omega(\sigma)) - \mathcal{K}(\tau, \sigma, \nu(\sigma))\|_{C_{1-\xi; \Psi}} \leq \mathcal{L}_2 \|\omega - \nu\|_{C_{1-\xi; \Psi}[0, P]}, \quad (2.2)$$

and

$$\frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \psi(\sigma) d\sigma \leq \mathcal{L} \psi(\tau), \quad (2.3)$$

for all $\tau, \sigma \in [0, P]$, and $\omega, \nu \in \Omega$. If the continuous map $\eta : [0, P] \rightarrow \Omega$ satisfying

$$\left\| \eta(\tau) - \zeta(\tau, \eta(\tau)) - \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \mathcal{K}(\tau, \sigma, \eta(\sigma)) d\sigma \right\|_{C_{1-\xi; \Psi}[0, P]} \leq \psi(\tau), \quad (2.4)$$

$0 \leq \tau \leq P$, thus we can find a unique continuous map $\eta_0 : [0, P] \rightarrow \Omega$ such that for every $\tau \in [0, P]$,

$$\eta_0(\tau) = \zeta(\tau, \eta_0(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \mathcal{K}(\tau, \sigma, \eta_0(\sigma)) d\sigma, \quad (2.5)$$

and

$$\|\eta(\tau) - \eta_0(\tau)\|_{C_{1-\xi; \Psi}} \leq \frac{1}{1 - [\mathcal{L}_1 + \mathcal{L}_2 \mathcal{L}]} \psi(\tau). \quad (2.6)$$

Proof. Set $\mathcal{M} = \{\omega : [0, P] \rightarrow \Omega, \omega \text{ is continuous}\}$ and define a mapping $\delta : \mathcal{M}^2 \rightarrow [0, \infty]$ by

$$\delta(\omega, \nu) = \bigwedge \{ \mathcal{C} \geq 0 : \|\omega(\sigma) - \nu(\sigma)\|_{C_{1-\xi; \Psi}} \leq \mathcal{C} \psi(\sigma), \ 0 \leq \sigma \leq P \}.$$

In [9] and [12, 13, 14] the authors proved that $(\mathcal{M}, \delta)$ is a complete $[0, \infty]$ -valued metric space. Consider $\Lambda : \mathcal{M} \rightarrow \mathcal{M}$, such that

$$\Lambda(\omega(\tau)) = \zeta(\tau, \omega(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \mathcal{K}(\tau, \sigma, \omega(\sigma)) d\sigma.$$

Suppose $\omega, \nu \in \Omega$, $\mathcal{C}_{\omega\nu} \in [0, \infty]$, and $\delta(\omega, \nu) \leq \mathcal{C}_{\omega\nu}$. So, for every $\tau \in [0, P]$,

$$\|\omega(\tau) - \nu(\tau)\|_{C_{1-\xi;\Psi}} < \mathcal{C}_{\omega\nu}\phi(t).$$

Using (2.1) to (2.3), we get

$$\begin{aligned} & \|\Lambda\omega(\tau) - \Lambda\nu(\tau)\|_{C_{1-\xi;\Psi}} \\ & \leq \|\zeta(\tau, \omega) - \zeta(\tau, \nu)\|_{C_{1-\xi;\Psi}} + \left\| \mathcal{I}_{0+}^{\varrho;\Psi} (\mathcal{K}(\varsigma, \sigma, \omega(\sigma)) - \mathcal{K}(\varsigma, \sigma, \nu(\sigma))) \right\|_{C_{1-\xi;\Psi}} \\ & \leq \mathcal{L}_1 \|\omega(\tau) - \nu(\tau)\|_{C_{1-\xi;\Psi}} + \mathcal{I}_{0+}^{\varrho;\Psi} \mathcal{L}_2 \|\omega(\tau) - \nu(\tau)\|_{C_{1-\xi;\Psi}} \\ & \leq \mathcal{L}_1 \mathcal{C}_{\omega\nu} \psi(\tau) + \mathcal{L}_2 \mathcal{I}_{0+}^{\varrho;\Psi} [\mathcal{C}_{\omega\nu} \psi(\sigma)] \\ & \leq [\mathcal{L}_1 + \mathcal{L}_2 \mathcal{L}] \mathcal{C}_{\omega\nu} \psi(\tau). \end{aligned}$$

Then we have

$$\delta(\Lambda\omega, \Lambda\nu) \leq [\mathcal{L}_1 + \mathcal{L}_2 \mathcal{L}] \delta(\omega, \nu), \quad (2.7)$$

which implies that the contractively property of Λ , because $0 < \mathcal{L}_1 + \mathcal{L}_2 \mathcal{L} < 1$. Since $\eta \in \mathcal{M}$ and using (2.4), we get

$$\begin{aligned} & \|\Lambda\eta(\tau) - \eta(\tau)\|_{C_{1-\xi;\Psi}} \\ & = \left\| \zeta(\tau, \omega(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \mathcal{K}(\tau, \sigma, \omega(\sigma)) d\sigma - \eta(\tau) \right\|_{C_{1-\xi;\Psi}} \\ & \leq \psi(\tau), \end{aligned}$$

and so,

$$\delta(\Lambda\eta, \eta) \leq 1 < \infty. \quad (2.8)$$

Now, using Theorem 1.1 let us, to find a unique map $\eta_0 \in \mathcal{M}^* = \{\nu \in \mathcal{M} : \delta(\Lambda\eta, \nu) < \infty\}$ such that $\Lambda\eta_0 = \eta_0$ and so

$$\eta_0(\tau) = \zeta(\tau, \eta_0(\tau)) + \mathcal{I}_{0+}^{\varrho;\Psi} [\mathcal{K}(\varsigma, \sigma, \eta_0(\sigma))]. \quad (2.9)$$

Using Theorem 1.1 and (2.8), we have

$$\delta(\eta, \eta_0) \leq \frac{1}{1 - [\mathcal{L}_1 + \mathcal{L}_2 \mathcal{L}]} d(\Lambda f, f) \leq \frac{1}{1 - [\mathcal{L}_1 + \mathcal{L}_2 \mathcal{L}]},$$

which implies that (2.6). Set

$$\beta = \frac{1}{1 - [\mathcal{L}_1 + \mathcal{L}_2 \mathcal{L}]}.$$

Consider another continuous map θ satisfying (2.5) and (2.6). Thus, $\eta \in \mathcal{M}$, $\delta(\eta, \theta) < \beta$, and

$$\theta(\tau) = \zeta(\tau, \theta(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \mathcal{K}(\tau, \sigma, \theta(\sigma)) d\sigma. \quad (2.10)$$

We prove that, $\theta \in \mathcal{M}^*$ is a FP of Λ .

By (2.10), we have $\Lambda\theta = \theta$. Now, we are ready to prove that $\delta(\Lambda\eta, \theta) < \infty$.

Using (2.10) and $\delta(\eta, \theta) < \beta$, imply that

$$\begin{aligned}
& \|\Lambda\eta(\tau) - \theta(\tau)\|_{C_{1-\xi;\Psi}} \\
&= \left\| \zeta(\tau, \eta(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \mathcal{K}(\tau, \sigma, \eta(\sigma)) d\sigma - \eta(\tau) + \eta(\tau) - \theta(\tau) \right\|_{C_{1-\xi;\Psi}} \\
&\leq \left\| \zeta(\tau, \eta(\tau)) + \frac{1}{\Gamma(\varrho)} \int_0^\tau \mathcal{Q}_\Psi^\varrho(\tau, \sigma) \mathcal{K}(\tau, \sigma, \eta(\sigma)) d\sigma - \eta(\tau) - \eta(\tau) \right\|_{C_{1-\xi;\Psi}} \\
&\quad + \|\eta(\tau) - \theta(\tau)\|_{C_{1-\xi;\Psi}} \\
&\leq (1 + \beta)\psi(\tau),
\end{aligned}$$

which implies that $\delta(\Lambda\eta, \theta) \leq 1 + \beta < \infty$. $\square$

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