Multistate intermittency on the route to chaos of a semiconductor laser subjected to optical feedback from a long external cavity

Daeyoung Choi, Michael J Wishon, C. y Chang, D. S Citrin, A. Locquet

► To cite this version:

Daeyoung Choi, Michael J Wishon, C. y Chang, D. S Citrin, A. Locquet. Multistate intermittency on the route to chaos of a semiconductor laser subjected to optical feedback from a long external cavity. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 2018, 28 (1), pp.011102. 10.1063/1.5013332 . hal-03008301

HAL Id: hal-03008301

<https://hal.science/hal-03008301>

Submitted on 16 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Multistate intermittency on the route to chaos of a semiconductor laser subjected to optical feedback from a long external cavity

Daeyoung Choi, Michael J. Wishon, C. Y. Chang, D. S. Citrin, and A. Locquet

Citation: *Chaos* **28**, 011102 (2018);

View online: <https://doi.org/10.1063/1.5013332>

View Table of Contents: <http://aip.scitation.org/toc/cha/28/1>

Published by the [American Institute of Physics](#)

Welcome to a

Smarter Search

PHYSICS
TODAY

with the redesigned
Physics Today Buyer's Guide

Find the tools you're looking for today!

Multistate intermittency on the route to chaos of a semiconductor laser subjected to optical feedback from a long external cavity

Daeyoung Choi,^{1,2,a)} Michael J. Wishon,^{1,2} C. Y. Chang,^{1,3} D. S. Citrin,^{1,2,b)} and A. Locquet^{1,2,c)}

¹Georgia Tech-CNRS UMI 2958, Georgia Tech Lorraine, 2 Rue Marconi, F-57070 Metz, France

²School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0250, USA

³School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332-0250, USA

(Received 13 November 2017; accepted 3 January 2018; published online 17 January 2018)

We observe experimentally two regimes of intermittency on the route to chaos of a semiconductor laser subjected to optical feedback from a long external cavity as the feedback level is increased. The first regime encountered corresponds to multistate intermittency involving two or three states composed of several combinations of periodic, quasiperiodic, and subharmonic dynamics. The second regime is observed for larger feedback levels and involves intermittency between period-doubled and chaotic regimes. This latter type of intermittency displays statistical properties similar to those of on-off intermittency. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5013332>

Irregular switching between dynamical regimes, or, intermittency, is a ubiquitous phenomenon observed in many nonlinear systems when a parameter of the system approaches a critical point. External-cavity lasers (ECLs) exploit an external mirror to provide time-delayed optical feedback into the laser diode. Their dynamics can be high-dimensional and highly multistable; thus, ECLs are interesting testbeds to observe and understand transitions between attractors and the involved timescales. Despite the existence of several reports on intermittency in ECLs, there is still a lack of knowledge in the important case in which the laser is biased well above its threshold. We observe in this case two distinct intermittency regimes on the route to chaos, as the feedback strength is increased. The first regime corresponds to multistate intermittency involving two or three states composed of several combinations of periodic, quasiperiodic, and subharmonic dynamics. The second regime is observed for larger feedback levels and involves intermittency between period-doubled and chaotic regimes.

with a saddle-node bifurcation, type II with a Hopf bifurcation, and III with a period-doubling (PD) bifurcation. Subsequently, other types of intermittency have been revealed and discussed. Crisis-induced intermittency originates from a crisis in which chaotic attractors undergo sudden widening or merging.³ Multistate intermittency switching between more than two dynamical states can occur due to the presence of external modulation or noise.^{4–7} Unlike previous types, on-off intermittency is characterized by a dynamic bifurcation point which induces laminar (steady) and bursting (turbulent) phases described as off and on states.^{8–10} On-off intermittency has recently been reported in numerous physical systems.^{11–20} Interestingly, on-off intermittency has two universal statistical properties: the distribution of laminar phases shows a $-3/2$ power-law scaling for short laminar phases and the mean duration of laminar phases *versus* a control parameter has an exponential decay with slope -1 on a logarithmic scale near the onset of intermittency.

Intermittency has been observed and explored in various types of lasers including Nd:YAG,²¹ CO₂,^{22–24} fiber,^{6,7,25,26} optically pumped ring lasers,^{27,28} and semiconductor lasers.^{29–31} In particular, intermittency in semiconductor lasers subjected to optical feedback from an external mirror, or external-cavity lasers (ECLs) has attracted considerable interest.

When an ECL is biased slightly above the threshold current J_{th} , intermittency between continuous-wave operation and low-frequency fluctuations (LFFs) may be observed.^{31–36} Spontaneous-emission noise is believed to play a main role in causing the intermittency in this regime. The average time $\langle\tau\rangle$ between subsequent LFF periods versus injection current J follows a slope of -1 on a log-log scale, which was seemingly considered as type-II intermittency;³⁷ however, Sukow *et al.* showed numerically that this intermittency cannot be of type II because it is not consistent with type-II theory, which predicts $\langle\tau\rangle \rightarrow \infty$ when $J \rightarrow J_{th}$.³⁵ Later, this intermittency was described in Ref. 31 as on-off intermittency based on power-law properties.

I. INTRODUCTION

Intermittency in a dynamical system is characterized by the erratic alternation between regular (laminar) and irregular (turbulent) phases.¹ Intermittency has been widely observed in nonlinear systems as a parameter approaches a critical value corresponding to a bifurcation point or a crisis. Different types of intermittent behaviors are associated with different dynamical instabilities. Pomeau and Manneville² introduced three intermittency mechanisms involving a fixed bifurcation point in the Poincaré map of a continuous-time system. They named three types of intermittency I, II, and III based on the nature of the bifurcation. Type I is associated

^{a)}Electronic mail: daeyoung@gatech.edu

^{b)}Electronic mail: david.citrin@ece.gatech.edu

^{c)}Electronic mail: alexandre@gatech.edu

At higher injection current, a recent study by Bosco *et al.*³⁹ utilized a short external-cavity laser (ECL), implemented by a photonic integrated circuit and found crisis-induced intermittency between PD dynamics and fully developed coherence collapse (CC), which is interpreted as being caused by the switching between separated attractors born from different external-cavity modes (ECMs) in phase space. Later, it was shown that switching times between the regimes could be used as an entropy source for random number generation.⁴⁰ However, a bigger picture for intermittency of ECLs operated at larger current levels that do not lead to LFF, in the long-cavity case, is still missing. In previous studies,^{38,41} we drew experimental bifurcation diagrams (a typical bifurcation diagram is represented in Fig. 1) of an ECL biased well-above J_{th} as a function of feedback level η , which display different dynamical regimes, including limit cycles (LC), quasiperiodicity (QP), subharmonics (SH), and PD oscillations, as well as CC. We noted that intermittency is generally observed on the way to CC, regardless of the initial condition used to draw the bifurcation diagrams.³⁸ In this paper, we will demonstrate that two distinct intermittency regimes can be encountered by an ECL on the way to CC: an intermittency between PD and CC regimes, similarly to what has been reported in the short-cavity case³⁹ but also multi-state intermittency involving several different dynamical regimes.

Since the external-cavity length L plays an important role in determining the route to chaos,⁴² we study how the evolution of the various dynamical regimes observed during intermittency depends on both η and L .

Specifically, we investigate certain ranges of feedback level η where two regimes of intermittency are observed in the route to chaos as pointed out in Fig. 1. First, multistate intermittency is observed in region δ of Fig. 1. We statistically analyze the coexistence of various regimes, including LC, QP, SH, and PD regimes. For cavities that are not too long ($L < 60$ cm), multistate intermittency gives way to a stable PD regime (ϵ in Fig. 1) after an increase in feedback. As feedback is increased even further, intermittency between PD and chaotic regimes (ζ in Fig. 1) is observed before

FIG. 1. Example of an experimental bifurcation diagram of an external-cavity laser as a function of feedback strength.³⁸ We observe successively (δ) multistate intermittency and (ζ) intermittency between period-doubled and coherence collapse regimes.

finally leading to chaos. We find that the statistics of the latter intermittency is comparable to that of on-off intermittency.

II. EXPERIMENTAL SETUP

The experiments are conducted with an ECL as shown in Fig. 2. The laser diode used is a single-longitudinal-mode multi-quantum well InGaAsP distributed feedback (DFB) laser emitting at 1550 nm and its threshold current is 9.27 mA at 298.2 K. The optical intensity $I(t)$ is obtained with an oscilloscope and a photodetector with 12 GHz of bandwidth. The optical spectrum is measured by an optical spectrum analyzer with 10 MHz resolution (Aragon Photonics BOSA 200C). The current J and temperature are stabilized to an accuracy of 0.01 mA and 0.01 K. The position of the mirror (M) relative to the laser determines the cavity length L ; η is proportional to the optical intensity fed back into the laser diode and is varied via the relative angle between a quarter-wave plate (QWP) and linear polarizer in the feedback loop. The QWP angle can be mechanically controlled with 0.01° precision and is normalized to $\eta_{max} = 1$ when $\sim 20\%$ of the optical power is fed back onto the collimating lens.

III. MULTISTATE INTERMITTENCY

As the feedback level η ramps up, the ECL exhibits a complex route to fully developed CC as shown in the bifurcation diagram of Fig. 1. In the beginning, the ECL shows continuous-wave dynamics (α of Fig. 1), followed by QP (β of Fig. 1) and LC (γ of Fig. 1) oscillations. Before the ECL enters the PD regime (ϵ of Fig. 1), the first intermittency region encountered (δ of Fig. 1) corresponds to a transitional region taking the laser from LC to PD oscillations. This is the intermittency alluded to Refs. 38 and 41. Contrary to what might be expected, we find that the intermittency does not only involve the LC observed in regime δ and PD observed in regime ϵ in Fig. 1, but also additional QP and SH regimes. The complex scenario described hereafter stems from the use of a long cavity, which leads to a highly multistable system in which numerous attractors and corresponding regimes can

FIG. 2. Experimental setup. Light from the laser diode (LD) is split by a beam splitter (BS). One optical path provides time-delayed optical feedback, which is adjusted for feedback strength η and external cavity length L . The other path is used for acquisition of the optical intensity $I(t)$. TC: temperature controller, CS: current supply, CL: collimating lens, LP: linear polarizer, QWP: quarter-wave plate, M: mirror, OI: optical isolator, P: photodetector, OSC: oscilloscope, and PC: personal computer.

play a role in the dynamics. Figures 3(a) and 3(b) illustrate the switching between LC and QP, and SH and QP regimes, respectively. We tested different ways of identifying the various regimes. As described by in Ref. 7, using amplitude threshold in time series may not be accurate for certain cases. The authors used wavelet spectrum energies to better distinguish several dynamical behaviors. The amplitude method is also not applicable in our case since amplitude levels of different dynamics are very similar except for CC. Although we find that the results are substantially the same when using the wavelet spectrum method, due to the size of the oscilloscope's buffer and slow speed processing for a large amount of data, we identify the characteristics of a given phase based on the RF spectrum obtained by taking the Fourier transform of the intensity time series $I(t)$ since the main peaks are clearly different in the RF spectra from one regime to another as shown in Fig. 4. The LC regime [Fig. 4(a)] is characterized by a main peak close to the relaxation-oscillation frequency f_{RO} , while the QP regime [Fig. 4(b)] has, in addition to f_{RO} , RF

sidebands separated by the external-cavity free-spectral range $f_L = c/(2L)$ where c is the speed of the light. The PD regime [Fig. 4(d)] is manifested by two main RF peaks f_{RO} and $f_{RO}/2$. The CC regime [Fig. 4(e)] exhibits rapid chaotic fluctuations and can be identified by its broad RF spectrum. The previous regimes and the associated spectral characteristics are extensively described in Refs. 38 and 41. An additional regime, named subharmonic (SH), is observed here, involving three main frequencies, f_1 , f_2 , and f_{RO} such that $f_{RO} = f_1 + f_2$. The frequencies f_1 and f_2 are subharmonic frequencies evident in the RF spectrum and are mainly located at the odd fractions of f_{RO} : $f_1/f_{RO} = (\frac{1}{3}, \frac{1}{5}, \text{ or } \frac{2}{7})$, and thus, $f_2/f_{RO} = (\frac{2}{3}, \frac{4}{5}, \text{ or } \frac{5}{7})$, respectively. Figure 4(c) depicts one example of SH dynamics when $f_{RO} = 6.3$ GHz ($f_1 = 2.1$ GHz, $f_2 = 4.2$ GHz).

Specifically, we investigate the statistics of each phase for moderate η , from LC to CC, spanning over regimes δ , ϵ , and ζ in Fig. 1. We record 1000 samples at each η at 40 GS/s from the oscilloscope (corresponding to 25 ns duration per sample, which is sufficient for classification) and classify the

FIG. 3. Snapshots of the time series of $I(t)$ when dynamical behavior change (a) from LC to QP, (b) from SH to QP, and (c) from CC to PD. Figures (a) and (b) correspond to region δ of Fig. 1, while (c) corresponds to region ζ . Insets are close-up of each dynamical behavior. $I(t)$ is sampled at 40 GS/s. $L = 30$ cm, $J = 21.15$ mA.

FIG. 4. RF spectrum of dynamical behaviors observed in intermittency (a) LC, (b) QP, (c) SH, (d) PD, and (e) CC. $J = 21.15$ mA, $L = 30$ cm.

dynamic type based on its RF spectrum and obtain the probabilistic dominance of each phase. We estimate that the accuracy of detecting each dynamics is $\sim 98\%$. Of note, we repeated the measurements presented in Fig. 5 multiple times and observed consistent results, allowing to exclude random thermal or mechanical fluctuations as being the origin of the observed phenomena. As shown in Fig. 5 for an external-cavity length $L = 30$ cm, we observe multistate intermittency involving 2 to 3 varying dynamical regimes. As η increases, the dominant phase progresses from LC, QP, SH, PD, to CC. LC dominates for small η , but its dominance decreases as first QP and then intermittent switching between LC, QP, and SH is prominently observed in a range around $\eta \sim 0.55$ [Fig. 5(a)]. SH occurs more frequently at moderate η after LC and QP become less dominant. Thus, we reveal that the transition from LC to PD, observed in Refs. 38 and 41 and corresponding to region δ in Fig. 1, is in fact a complex regime of multistate intermittency. Two additional regimes, QP and SH, play a mediating role in the transition between the LC and DP regimes and highlight the multistability of the ECL. As L increases, SH takes dominance over DP on

FIG. 5. Probability of observing each dynamical type for (a) $L = 30$ cm, (b) $L = 55$ cm. $J = 21.15$ mA.

the route to CC [Fig. 5(b)]. For $L > 60$ cm, we suspect that a variant of the route to chaos is achieved as PD is almost unobserved. The value of L is also related to the occurrence of intermittency between PD and CC, which is the subject of the following Sec. IV. Similarly to what was done in Ref. 6, determining the critical exponents of the various coexisting dynamical regimes as the feedback level is increased would be of great interest. However, the regime of multistate intermittency involves the coexistence of short timescales (sub-ns oscillations) with significantly longer durations of the dynamical phases (up to ms). This leads to the experimental constraint of recording very long time series at very high sampling rates (in order to be able to distinguish between the regimes), vastly exceeding the memory depth of our oscilloscope. The determination of these characteristic exponents will be the subject of future research.

IV. INTERMITTENCY BETWEEN PD AND CC REGIMES

When η is increased above the levels leading to multistate intermittency, the PD regime becomes stable, while QP and SH lose their dominance. This corresponds to region ϵ in Fig. 1. As η reaches the critical value η_c , we observe another intermittency corresponding to region ζ of Fig. 1 before entering fully developed CC (θ in Fig. 1). This type of intermittency is qualitatively similar to what was observed in Ref. 39 for the short-cavity case but was not identified in our previous publications^{38,41} on the bifurcation diagrams of the long-cavity case. An example of such an intermittency between PD and CC is shown in Fig. 3(c).

We have studied the statistics of the transition from PD, which we consider as a laminar phase, to CC, which we name the turbulent phase since PD has periodic behavior and CC has higher amplitude pulsations. For this intermittency between PD and CC regimes, it is possible to measure the duration of laminar phases (τ) through a simple threshold method as amplitudes show distinct differences. The sampling rate of the oscilloscope is adjusted, ranging from 20 MS/s to 40 GS/s as η increases. The oscilloscope can stream, in real time, 12×10^6 samples at rates (from 20 MS/s to 5 GS/s) and 2×10^6 samples at rates (from 10 GS/s to 40 GS/s). We are able to measure the laminar phase length for a broad range of time scales ($100 \text{ ns} < \tau < 0.6 \text{ s}$). The threshold is chosen empirically and the laminar phase is indicated by the absence of maxima above the threshold since the local maxima of the PD are lower than that of CC.

As η is increased, the duration τ of laminar phases tends to decrease until intermittency gives way to CC. Based on the collection, 1000 durations of laminar phase (PD), as η increased from η_c for various cavity lengths L (25, 35, 45, and 50 cm), the mean laminar length $\langle \tau \rangle$ versus η on a logarithmic scale is plotted as Fig. 6. Some of the laminar phases right after the onset of the intermittency are longer than 0.6 s, which we cannot detect due to our experimental limitations and are thus ignored to avoid bad statistics. Even with the experimental limitation, we still have a large enough testing range to gather meaningful statistics on the intermittent regime. The average duration $\langle \tau \rangle$ of the laminar phase over η is consistent with a line of slope -1 on a logarithmic scale

FIG. 6. Mean duration $\langle \tau \rangle$ of the laminar phases (PD) as a function of η from the onset of intermittency (η_c) between PD and CC. The increment of η corresponds to a 0.1° change in the angle of the quarter-wave plate. The black line has a slope -1 and η_c varies with L . $J = 21.15$ mA.

near the onset of intermittency [$0 < \log(\eta - \eta_c) < 1.6$]. In contrast, slopes ranging from 0.52 to 4.60, depending on operating parameters, were observed in Ref. 39 for the short-cavity case. A deviation from a power-law relation at sufficiently large values of η from the onset in Fig. 6 is seen. We expect that the solutions of external cavity modes on the PD regime are more easily destabilized as feedback increases, resulting in shorter laminar phases. Experimental statistical results on the average laminar phases in various types of lasers show behavior to that we observe.^{24,31} In short, our tentative attribution of the exponential decay for a larger controlling parameter in the statistics of the laminar phase durations in Fig. 6 appears to be consistent with the assignment of this effect in other works. The range of fall-off tends to shorten with increasing L as the CC regime is entered for smaller η . It must be noted that the quality of the fit in Fig. 6 depends on what range of η close to the critical value is chosen because the ECL enters CC at smaller η for longer L . For $L \geq 60$ cm, the PD regime is hardly observed and SH takes over until the ECL enters CC.

In Fig. 7, we study the probability distribution of the duration of the laminar phases near the onset of intermittency. It appears to be well-matched to a line of slope $-3/2$ on a logarithmic scale. Interestingly, these two power-law characteristics are usually observed in on-off intermittency.^{10,11,21,24,30,31}

FIG. 7. Probability distribution of the laminar phases near the onset of intermittency; The line indicates a power law fit with the slope $-3/2$. The shorter laminar phases are well fitted by the line. $\eta = 0.6$, $L = 30$ cm, $J = 21.15$ mA.

In our system, external noise is not applied; nonetheless, statistical properties similar to those of on-off intermittency are observed when the route to chaos is interrupted by PD dynamics (i.e., when $L < 60$ cm). As predicted by theory,^{9,10} modulating a parameter or adding different amplitudes of noise to a system can change the crossover point in which exponential fall-off starts on a log-log plot, while the power-law properties near the onset of intermittency are still the same. These features show similarities to our experimental results from varying L . Specifically, we observe the same critical exponent and probability distribution of periodic phases, while the crossover point occurs with smaller η for longer L . This earlier fall-off effect with longer L indicates that intermittent behavior becomes less robust as the cavity length increases.

In Fig. 8, we compare the optical spectra in the PD and CC regimes for $L = 30$ cm. The main peak of the PD regime in the optical spectrum is positioned at a negative mode (ECM -11) (Fig. 8), which highlights the fact that the PD regime remains stable around what is probably the maximum gain mode.⁴³ As the ECL enters CC, we observe from Fig. 8 that the dominant ECMs are close to those ECMs that are active in the PD regime. This proximity in phase space between attractors corroborates the interpretation that the chaotic attractor of the CC regime is born through a crisis with a nearby PD attractor, similarly to what was suggested in Ref. 39 for the short-cavity case on the basis of the Lang and Kobayashi model. The crisis nature of the transition from the PD and CC regime is confirmed in a recent study from our group.⁴⁴ In this work, the study of the optical phase reveals that the PD attractor is suddenly destroyed and gives way to a chaotic behavior.

The influence of the external-cavity length L was studied by changing L in 10 cm increments from 20 cm. We find that intermittency between PD and CC is observed systematically when $L < 60$ cm. However, when $L \geq 60$ cm, the second regime of intermittency between PD and CC is not observed. In this case, the regime of multistate intermittency leads directly to CC through a final phase which is the intermittency between SH and CC regimes. Longer L leads to ECMs that are closer in phase space and more easily destabilized, possibly explaining the instability of the PD regime for large L .

FIG. 8. Optical spectrum at $L = 30$ cm, $J = 21.15$ mA. The main peak of the PD regime is located on a negative ECM before the onset of intermittency. In CC, the active ECMs appear to be around those that were active in PD. Optical power is normalized for better comparison. ECM 0 corresponds to the frequency of a solitary laser.

V. CONCLUSION

We have shown experimentally that two regimes of intermittency can be observed on the route to chaos of a semiconductor laser subjected to optical feedback from a long external cavity, as the feedback level is increased. Multistate intermittency is experimentally observed first involving combinations of two or three of the following dynamics: limit cycle, quasiperiodic, subharmonic, and period-doubled. We find that the probabilistic dominance of the dynamics varies with external cavity length and feedback level. A theoretical description for this phenomenon, however, is not yet available. For larger feedback, we find that when the period-doubled regime is visited on the way to chaos, corresponding to an external-cavity length smaller than 60 cm, intermittency is seen between the two regimes, generalizing the results of Ref. 39 to the long-cavity case. In addition, we find that the latter intermittency has universal power-law properties which are comparable to the statistical properties of on-off intermittency. For cavities longer than 60 cm, a stable period-doubled regime is not observed and longer L hinders the ECMs from staying in the maximum gain mode. Instead, intermittency between subharmonic and chaotic regimes is observed. These results highlight the complex nature of the intermittency phenomenon involved in the route to chaos, resulting from the high dimension and significant multistability of the dynamics of an external-cavity laser.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the support of the Conseil Régional du Grand Est.

- ¹H. G. Schuster and W. Just, *Deterministic Chaos: An Introduction* (John Wiley & Sons, 2006).
- ²Y. Pomeau and P. Manneville, *Commun. Math. Phys.* **74**, 189 (1980).
- ³C. Grebogi, E. Ott, F. Romeiras, and J. A. Yorke, *Phys. Rev. A* **36**, 5365 (1987).
- ⁴B. K. Goswami, *Phys. Rev. E* **78**, 066208 (2008).
- ⁵S. Kraut and U. Feudel, *Phys. Rev. E* **66**, 015207 (2002).
- ⁶G. Huerta-Cuellar, A. N. Pisarchik, and Y. O. Barmenkov, *Phys. Rev. E* **78**, 035202 (2008).
- ⁷A. E. Hramov, A. A. Koronovskii, O. I. Moskalenko, M. O. Zhuravlev, R. Jaimes-Reategui, and A. N. Pisarchik, *Phys. Rev. E* **93**, 052218 (2016).
- ⁸N. Platt, E. A. Spiegel, and C. Tresser, *Phys. Rev. Lett.* **70**, 279 (1993).
- ⁹J. F. Heagy, N. Platt, and S. M. Hammel, *Phys. Rev. E* **49**, 1140 (1994).
- ¹⁰N. Platt, S. M. Hammel, and J. F. Heagy, *Phys. Rev. Lett.* **72**, 3498 (1994).
- ¹¹P. W. Hammer, N. Platt, S. M. Hammel, J. F. Heagy, and B. D. Lee, *Phys. Rev. Lett.* **73**, 1095 (1994).
- ¹²D. L. Feng, C. X. Yu, J. L. Xie, and W. X. Ding, *Phys. Rev. E* **58**, 3678 (1998).
- ¹³A. Vella, A. Setaro, B. Piccirillo, and E. Santamato, *Phys. Rev. E* **67**, 051704 (2003).
- ¹⁴J. L. Cabrera and J. G. Milton, *Phys. Rev. Lett.* **89**, 158702 (2002).
- ¹⁵F. Rödelsperger, A. Čenys, and H. Benner, *Phys. Rev. Lett.* **75**, 2594 (1995).
- ¹⁶R. Delage, Y. Takayama, and T. Biwa, *Chaos* **27**, 043111 (2017).
- ¹⁷M. Bottiglieri and C. Godano, *Phys. Rev. E* **75**, 026101 (2007).
- ¹⁸A. Hramov, A. A. Koronovskii, I. S. Midzyanovskaya, E. Sitnikova, and C. M. van Rijn, *Chaos* **16**, 043111 (2006).
- ¹⁹A. A. Koronovskii, A. E. Hramov, V. V. Grubov, O. I. Moskalenko, E. Sitnikova, and A. N. Pavlov, *Phys. Rev. E* **93**, 032220 (2016).
- ²⁰E. Sitnikova, A. E. Hramov, V. V. Grubov, A. A. Ovchinnikov, and A. A. Koronovskii, *Brain Res.* **1436**, 147 (2012).
- ²¹S.-H. Gong and C.-M. Kim, *J. Opt. Soc. Am. B* **18**, 1285 (2001).
- ²²F. Papoff, D. Dangoisse, E. Poite-Hanoteau, and P. Glorieux, *Opt. Commun.* **67**, 358 (1988).
- ²³V. N. Chizhevsky and R. Corbalán, *Phys. Rev. E* **54**, 4576 (1996).
- ²⁴G. S. Yim, Y. J. Park, C. M. Kim, and Y. S. Kim, *J. Opt. Soc. Am. B* **21**, 2112 (2004).
- ²⁵A. N. Pisarchik, R. Jaimes-Reategui, R. Sevilla-Escoboza, G. Huerta-Cuellar, and M. Taki, *Phys. Rev. Lett.* **107**, 274101 (2011).
- ²⁶A. N. Pisarchik, R. Jaimes-Reategui, R. Sevilla-Escoboza, and G. Huerta-Cuellar, *Phys. Rev. E* **86**, 056219 (2012).
- ²⁷D. Y. Tang, J. Pujol, and C. O. Weiss, *Phys. Rev. A* **44**, R35 (1991).
- ²⁸D. Y. Tang, M. Y. Li, and C. O. Weiss, *Phys. Rev. A* **46**, 676 (1992).
- ²⁹A. N. Pisarchik and V. J. Pinto-Robledo, *Phys. Rev. E* **66**, 027203 (2002).
- ³⁰S. Osborne, A. Amann, D. Bitauld, and S. O'Brien, *Phys. Rev. E* **85**, 056204 (2012).
- ³¹A. Campos-Mejía, A. N. Pisarchik, and D. A. Arroyo-Almanza, *Chaos, Solitons Fractals* **54**, 96 (2013).
- ³²T. Heil, I. Fischer, W. Elsässer, J. Mulet, and C. R. Mirasso, *Opt. Lett.* **24**, 1275 (1999).
- ³³T. Heil, I. Fischer, and W. Elsässer, *Phys. Rev. A* **58**, R2672 (1998).
- ³⁴A. Hohl, H. J. van der Linden, and R. Roy, *Opt. Lett.* **20**, 2396 (1995).
- ³⁵D. W. Sukow, J. R. Gardner, and D. J. Gauthier, *Phys. Rev. A* **56**, R3370 (1997).
- ³⁶D. Choi, M. J. Wishon, J. Barnoud, C. Y. Chang, Y. Bouazizi, A. Locquet, and D. S. Citrin, *Phys. Rev. E* **93**, 042216 (2016).
- ³⁷J. Sacher, W. Elsässer, and E. O. Göbel, *Phys. Rev. Lett.* **63**, 2224 (1989).
- ³⁸A. Locquet, B. Kim, D. Choi, N. Li, and D. S. Citrin, *Phys. Rev. A* **95**, 023801 (2017).
- ³⁹A. K. D. Bosco, Y. Akizawa, K. Kanno, A. Uchida, T. Harayama, and K. Yoshimura, *Opt. Express* **24**, 22198 (2016).
- ⁴⁰A. K. D. Bosco, N. Sato, Y. Terashima, S. Ohara, A. Uchida, T. Harayama, and M. Inubushi, *IEEE J. Sel. Top. Quantum Electron.* **PP**, 1 (2017).
- ⁴¹B. Kim, A. Locquet, D. Choi, and D. S. Citrin, *Phys. Rev. A* **91**, 061802 (2015).
- ⁴²Y. H. Kao, N. M. Wang, and H. M. Chen, *IEEE J. Quantum Electron.* **30**, 1732 (1994).
- ⁴³A. M. Levine, G. H. M. van Tartwijk, D. Lenstra, and T. Erneux, *Phys. Rev. A* **52**, R3436 (1995).
- ⁴⁴M. J. Wishon, A. Locquet, C. Chang, D. Choi, and D. Citrin, "Crisis route to chaos in semiconductor lasers subjected to external optical feedback," (unpublished).