Quantitative fluid approximation in transport theory: a unified approach

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ABSTRACT. We propose a unified method for the large space-time scaling limit of *linear* collisional kinetic equations in the whole space. The limit is of *fractional* diffusion type for heavy tail equilibria with slow enough decay, and of diffusive type otherwise. The proof is constructive and the fractional/standard diffusion matrix is obtained. The equilibria satisfy a *generalised* weighted mass condition and can have infinite mass. The method combines energy estimates and quantitative spectral methods to construct a 'fluid mode'. The method is applied to scattering models (without assuming detailed balance conditions), Fokker-Planck operators and Lévy-Fokker-Planck operators. It proves a series of new results, including the fractional diffusive limit for Fokker-Planck operators in any dimension, for which the characterization of the diffusion coefficient was not known, for Lévy-Fokker-Planck operators with general equilibria, and in cases where the equilibrium has infinite mass. It also unifies and generalises the results of ten previous papers with a quantitative method, and our estimates on the fluid approximation error seem novel in these cases.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

The study of *transport processes*, i.e. linear collisional kinetic equations, has its theoretical roots in the mean-free path argument of Maxwell [35] and the kinetic theory of gases of Maxwell and Boltzmann [36, 10]. A linear version of the Maxwell-Boltzmann equation can be written for the movement of a tagged particle within a rarefied gas, but the study of such transport processes was given a crucial new impetus in the twentieth century with:

(1) the *radiative transfer theory* [42], where the kinetic distribution models the flux of photons that are transported in the plasma making up the internal layers of the sun,
(2) the *nuclear reactor theory* (see [47], the collection [7] and in particular its fifth chapter [48]) where the kinetic distribution models the neutrons transported and scattered inside the reactor, whose flux is used to initiate and maintain the chain reaction,
(3) the *semi-conductor theory* [33] where the kinetic distribution models the flow of charge carriers in semiconductors, i.e. the evolution of the position-momentum distribution of negatively charged conduction electrons or of positively charged holes, which are responsible for the current flow in semiconductor crystals.

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The main mathematical object of study in *transport theory* is the linear equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f \tag{1.1}$$

on the time-dependent density of particles $f = f(t,x,v) \geq 0$ over $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, for $t \geq 0$. The left hand side accounts for free motion and the right hand side accounts for the interaction with a background, for instance scatterers, with an operator $\mathcal{L}$ that only acts on the kinetic variable v. Several forms are possible. In nuclear reactor, radiative transfer and semi-conductor theories it is common to consider *scattering operators*, sometimes also called *linear Boltzmann operators*, which write

$$\mathcal{L} f(v) = \left(\int_{\mathbb{R}^d} b(v,v') f(v') \, \mathrm{d}v' \right) \mathcal{M}(v) - \nu(v) f(v) \tag{1.2}$$

$$\text{given the } \textit{collision frequency} \quad \nu(v) := \int_{\mathbb{R}^d} b(v,v') \mathcal{M}(v') \, \mathrm{d}v',$$

some *collisional kernel* $b = b(v,v')$ and an *equilibrium distribution* $\mathcal{M}(v)$. In astrophysics and sometimes in semi-conductor theory, one also considers *Fokker-Planck operators* which write

$$\mathcal{L} f := \nabla_v \cdot \left(\mathcal{M} \nabla_v \left(\frac{f}{\mathcal{M}} \right) \right). \tag{1.3}$$

Finally, as a simplified model of long-range collisional interactions in a gas of charged particles, we also consider *Lévy-Fokker-Planck operators* (given $s \in (0,1)$):

$$\begin{cases} \mathcal{L}(f) = \Delta_v^s f + \nabla_v \cdot (U \, f) & \text{with } U(v) = U(|v|) \text{ radially symmetric so that} \\ \Delta_v^s \mathcal{M} + \nabla_v \cdot (U \, \mathcal{M}) = 0. \end{cases} \tag{1.4}$$

Denoting $\mathcal{F}$ the Fourier transform, the fractional Laplacian is defined as

$$\Delta_v^s f(v) := -\mathcal{F}^{-1} \left[|\iota|^{2s} \mathcal{F} f(\iota) \right](v). \tag{1.5}$$

These three operators are discussed respectively in Sections 6-7-8. Extensions, such as Fokker-Planck operators with non-gradient force, are discussed in Section 9.

The equation (1.1) is too intricate for many applications. When the relevant time and space scales of observation are much larger than the mean free time and mean free path, it is thus natural to search for a simplified regime. The so-called *diffusion theory* was born out of this endeavour, and in the words of Wigner [48], 'this [diffusion] theory gives the spatial variation of the [neutron transport] flux quite accurately in regions well removed from interfaces'. We also refer to [47, Chap. IX] for the diffusion theory of monoenergetic neutrons, to [42, Chap. III.2] for the so-called *Eddington approximation* in radiative transfer theory, and to [12, Chap. 2] for a modern mathematical review. Note that anomalous diffusions and Levy flights are observed by biologists and physicists [3, 45, 5, 34, 43].

We rewrite the equation (1.1) by changing the unknown to $h := \frac{f}{\mathcal{M}}$:

$$\partial_t h + v \cdot \nabla_v h = Lh \quad \text{where} \quad Lh := \mathcal{M}^{-1} \mathcal{L}(\mathcal{M} h). \tag{1.6}$$

This change of unknown is convenient since asymptotic estimates compare f with the equilibrium $\mathcal{M}$. Consider the complex Hilbert spaces $L^2(\mathbb{R}^d; \mathcal{M} \, \mathrm{d}v) =: L^2_v(\mathcal{M})$ and $L^2(\mathbb{R}^d \times \mathbb{R}^d; \mathcal{M} \, \mathrm{d}x \, \mathrm{d}v) =: L^2_{x,v}(\mathcal{M})$ and denote $\|h\|_k := \|(1 + |\cdot|^2)^{\frac{k}{2}} h\|_{L^2(\mathcal{M})}$ (the integration variable(s) will be emphasized when there is ambiguity). We omit the index when $k = 0$. The scalar product $\langle \cdot, \cdot \rangle$ refers to $L^2_v(\mathcal{M})$ or $L^2_{x,v}(\mathcal{M})$ depending on context.

We assume, for some $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$ and $\lambda \in \mathbb{R}^*_+$:

Hypothesis 1 (Equilibria)**.** *The equilibrium $\mathcal{M}$ takes one of the following two forms.*

(i) *Either it is given by*

$$\mathcal{M}(v) = c_{\alpha,\beta} \langle v \rangle^{-(d+\alpha)} \textit{ with } c_{\alpha,\beta} := \left(\int_{\mathbb{R}^d} \langle v \rangle^{-d-\alpha-\beta} \, \mathrm{d}v \right)^{-1} \textit{ and } \langle v \rangle := \sqrt{1 + |v|^2}. \tag{1.7}$$

(ii) *Or it is a smooth positive radially symmetric function decaying faster than any polynomial. The latter case is denoted by '$\alpha = +\infty$' in the sequel.*

Note that the normalisation implies the following **generalised mass condition**

$$\int_{\mathbb{R}^d} \mathcal{M}_\beta(v)\,\mathrm{d}v = 1 \quad \text{with} \quad \mathcal{M}_\beta := \lfloor\cdot\rceil^{-\beta}\mathcal{M}. \tag{1.8}$$

We present our main results assuming that the equilibrium $\mathcal{M}$ is given by the exact formula (1.7) in the case of a polynomial decay because it leads to a neater treatment. However, as discussed in Section 9, our results remain true with an equilibrium $\mathcal{M}$ that is not an explicit power-law or even symmetric or centered, but only comparable to $\lfloor\cdot\rceil^{-(d+\alpha)}$ (see equation (9.1) and Subsections 9.1-9.3); this requires a few technical changes in the proofs that we present separately in this last section so as not to clutter the paper.

Hypothesis 2 (Weighted coercivity)**.** *The operator L is linear, independent of time t and space x, commutes with rotations in v, is closed densely defined on $\mathrm{Dom}(L) \subset L^2_v(\mathcal{M})$ and satisfies $L(1) = L^*(1) = 0$, where L^* is the $L^2_v(\mathcal{M})$-adjoint. Finally $\tilde{L} := \lfloor\cdot\rceil^{\frac{\beta}{2}} L(\lfloor\cdot\rceil^{\frac{\beta}{2}}\cdot)$ is closed densely defined on $\mathrm{Dom}(\tilde{L}) \subset L^2_v(\mathcal{M})$, with the spectral gap estimate*

$$\forall\, g \in \mathrm{Dom}(\tilde{L}), \quad g \perp \lfloor\cdot\rceil^{-\frac{\beta}{2}}, \quad -\mathrm{Re}\,\langle \tilde{L}g, g\rangle \geq \lambda\, \|g\|^2.$$

The latter means, translating back to L,

$$\forall\, h \in \mathrm{Dom}(L), \quad -\mathrm{Re}\,\langle Lh, h\rangle \geq \lambda\, \|h - \mathcal{P}h\|^2_{-\beta} \quad \textit{with} \quad \mathcal{P}h := \left(\int_{\mathbb{R}^d} h(v')\mathcal{M}_\beta(v')\,\mathrm{d}v'\right).$$

The assumption that $\mathcal{L}$ commutes with rotation in v is convenient (and satisfied for most physical models), but in fact only $\mathcal{M}(v) = \mathcal{M}(-v)$ is really used in the proof. The latter could in turn be relaxed at the price of a few technical changes in the proofs discussed in Section 9.

Hypothesis 3 (Amplitude of collisions at large velocities)**.** *Given $0 \leq \chi \leq 1$ a smooth function that is 1 on $B(0,1)$ and 0 outside $B(0,2)$, and $\chi_R = \chi(\frac{\cdot}{R})$ for $R \geq 1$, one has*

$$\|L(\chi_R)\|_\beta \lesssim R^{-\frac{\alpha+\beta}{2}}.$$

Our first result, on the basis of the three previous hypothesis, is a quantitative construction of a branch of 'fluid eigenmode' in the asymptotic of large time and small spatial frequencies, i.e. a unique eigenvalue branching from zero for $\tilde{L}^* + i\eta\lfloor v\rceil^\beta(v\cdot\sigma)$ for small η (see Figure 1):

Lemma 1.1 (Construction of the fluid mode)**.** *Given Hypothesis 1–2–3, there are $\eta_0 > 0$ and $r_0 \in (0,\lambda)$, explicit in terms of the constants in these hypothesis, such that for any $\eta \in (0,\eta_0)$ and any $\sigma \in \mathbb{S}^{d-1}$, there is a unique solution $\phi_\eta = \phi_\eta(v) \in L^2_v(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ and $\mu(\eta) \in (0, r_0)$ to*

$$-L^*\phi_\eta - i\eta(v\cdot\sigma)\phi_\eta = \mu(\eta)\lfloor v\rceil^{-\beta}\phi_\eta \quad \textit{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v)\,\mathcal{M}_\beta(v)\,\mathrm{d}v = 1.$$

Moreover, the branch $(\phi_\eta, \mu(\eta))$ connects to $(1,0)$ as $\eta \to 0$, with

$$\|\phi_\eta - 1\|_{-\beta} \lesssim \mu(\eta)^{\frac{1}{2}} \quad \textit{and} \quad \mu(\eta) \lesssim \Theta(\eta), \tag{1.9}$$

where the function Θ is defined by

$$\Theta(\eta) := \begin{cases} \eta^2 & \textit{when } \alpha > 2+\beta, \\ \eta^2|\ln(\eta)| & \textit{when } \alpha = 2+\beta, \\ \eta^{\frac{\alpha+\beta}{1+\beta}} & \textit{when } -\beta < \alpha < 2+\beta. \end{cases} \tag{1.10}$$

Note that Θ is well-defined in the case $\alpha \in (-\beta, 2+\beta)$ since $(1+\beta) > (\alpha+\beta)/2 > 0$. In this lemma and in the rest of the paper the dependency in σ is kept implicit rather than explicit in order to lighten notation. In fact, ϕ_η also depends on σ, but $\mu(\eta)$ does not if L is invariant by rotations in v. To identify the macroscopic limit with quantitative rates and constants, it is necessary to estimate the leading order of $\mu(\eta)$, and this requires estimates on the eigenvector, which is our last hypothesis. We denote $|u|_\eta := (\eta^{\frac{2}{1+\beta}} + |u|^2)^{\frac{1}{2}}$.

Hypothesis 4 (Scaling of the fluid mode)**.** *We make different assumptions depending on α:*

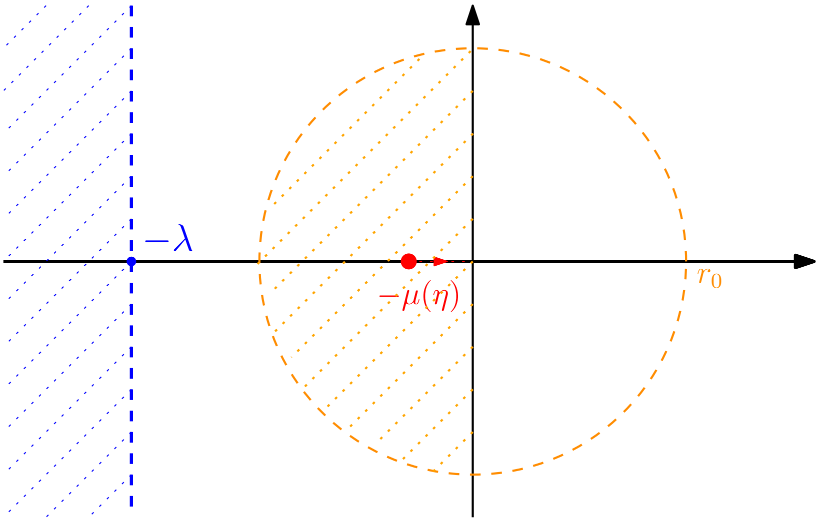

FIGURE 1. The blue dashed zone on the left of $\mathrm{Re}\, z = -\lambda$ corresponds to the spectral gap estimates on $\tilde{L}^* + i\eta\lfloor v\rceil^\beta(v\cdot\sigma)$ for $g \perp \lfloor\cdot\rceil^{-\frac{\beta}{2}}$ (Hypothesis 2). The orange dashed zone is where Lemmas 1.1-1.2 construct a unique real eigenvalue $-\mu(\eta) \sim -\mu_0\Theta(\eta)$ of the latter operator, that goes to zero as $\eta \to 0$.

(i) *Case $\alpha > 2+\beta$: The fluid mode ϕ_η constructed in Lemma 1.1 satisfies*

$$\forall\, \ell < \alpha, \quad \|\phi_\eta\|_\ell \lesssim_\ell 1.$$

(ii) *Case $\alpha = 2+\beta$: The rescaled fluid mode $\Phi_\eta := \phi_\eta(\eta^{-\frac{1}{1+\beta}}\cdot)$ is converging in $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash 0)$ as $\eta \to 0$ to a limit Φ and satisfies the pointwise controls*

$$\forall\, \eta \in (0,\eta_1),\ \forall\, u \in \mathbb{R}^d, \quad \begin{cases} |\Phi_\eta(u)| \lesssim |u|_\eta^{C\mu(\eta)}, \\ |\mathrm{Im}\,(\Phi_\eta(u))| \lesssim |u|_\eta^{\min(\alpha,1)+\beta-C\mu(\eta)}. \end{cases} \tag{1.11}$$

for some $\eta_1 \in (0,\eta_0)$ and $C > 0$, and there are $\mathfrak{a}(\eta) \to 0$ and $\Omega \in L^1(\mathbb{S}^{d-1})$ such that

$$\begin{cases} \left| \displaystyle\int_{1 \ge |u| \ge \eta^{\frac{1}{1+\beta}}} (u\cdot\sigma)\Big[\mathrm{Im}\,\Phi_\eta(u) - \mathrm{Im}\,\Phi(u)\Big] |u|_\eta^{-d-\alpha}\,\mathrm{d}u \right| \le \mathfrak{a}(\eta)|\ln(\eta)|, \\ \forall\, \sigma' \in \mathbb{S}^{d-1}, \quad \dfrac{\mathrm{Im}\,\Phi\,(\lambda\sigma')}{\lambda^{1+\beta}} \xrightarrow[\lambda\to 0]{\lambda\neq 0} \Omega(\sigma'). \end{cases}$$

(iii) *Case $\alpha \in (\beta, 2+\beta]$: The rescaled fluid mode Φ_η is converging in $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash 0)$ as $\eta \to 0$ to a limit Φ and satisfies the pointwise controls (1.11).*

(iv) *Case $\alpha \in (-\beta, \beta]$: The rescaled fluid mode Φ_η is converging in $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash 0)$ as $\eta \to 0$ to a limit Φ and satisfies the pointwise controls (1.11), together with the additional control*

$$\int_{|u|\ge 1} |\Phi_\eta(u)|^2\, |u|_\eta^{-d-\alpha+\beta}\,\mathrm{d}u \lesssim 1. \tag{1.12}$$

Note that in (1.11), $|u|_\eta^{C\mu(\eta)} \sim 1$ as $\eta \to 0$ in the region $|u| \lesssim \eta^{\frac{1}{1+\beta}}$. The second part of point (ii) above is subtle and made necessary by the fact that the case $\alpha = 2+\beta$ is borderline between two different regimes (standard diffusion vs. fractional diffusion) as well as borderline between two different scalings for obtaining the diffusion coefficient (fluid mode in variable v vs. fluid mode in the rescaled variable $u = \eta^{-\frac{1}{1+\beta}}v$).

With these four hypothesis we can characterise the precise scaling of the fluid eigenvalue:

Lemma 1.2 (Scaling of the fluid eigenvalue)**.** *Assume Hypothesis 1–2–3–4. The eigenvalue $\mu(\eta)$ constructed in Lemma 1.1 satisfies (with convergence rate explicit in terms of the constants,* *error terms and convergence rates in the hypothesis)*

$$\mu(\eta) \sim_{\eta\to 0} \mu_0 \Theta(\eta), \tag{1.13}$$

where the constant $\mu_0 > 0$ *is determined as follows:*

$$\begin{cases} \mu_0 := \displaystyle\int_{\mathbb{R}^d} (v\cdot\sigma)\, F(v)\mathcal{M}(v)\,\mathrm{d}v \quad \textit{when } \alpha > 2+\beta, \\ \qquad \textit{where } \ F = \displaystyle\lim_{\eta\to 0} \frac{\operatorname{Im}\phi_\eta}{\eta} \ \textit{is solution to } LF = -(v\cdot\sigma) \ \textit{ and } \ \int_{\mathbb{R}^d} F(v)\,\mathcal{M}_\beta(v)\,\mathrm{d}v = 0, \\ \mu_0 := \dfrac{c_{2+\beta,\beta}}{1+\beta} \displaystyle\int_{\mathbb{S}^{d-1}} (\sigma\cdot\sigma')\Omega(\sigma')\,\mathrm{d}\sigma' \quad \textit{when } \alpha = 2+\beta, \\ \qquad \textit{where } \ \Omega(u) = \displaystyle\lim_{\lambda\to 0,\ \lambda\neq 0} \frac{\operatorname{Im}\Phi(\lambda u)}{\lambda^{1+\beta}} \quad \textit{and} \quad \Phi = \lim_{\eta\to 0}\Phi_\eta = \lim_{\eta\to 0}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}}\cdot\right), \\ \mu_0 := c_{\alpha,\beta}\displaystyle\int_{\mathbb{R}^d} (u\cdot\sigma)\operatorname{Im}\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u \quad \textit{when } \alpha\in(-\beta, 2+\beta). \end{cases}$$

Note how in the previous statement, when $\alpha > 2+\beta$, the function F used in the previous works on standard diffusive limit (usually with $\beta = 0$) is recovered here as a limit of our fluid mode; this allows our proof to track the convergence rate. Define the *diffusion exponent*

$$\zeta = \zeta(\alpha,\beta) := \begin{cases} 2 & \text{when } \alpha\in[2+\beta,+\infty] \\ \dfrac{\alpha_+ + \beta}{1+\beta} & \text{when } \alpha\in(-\beta, 2+\beta), \end{cases} \tag{1.14}$$

with $\alpha_+ := \max(\alpha, 0)$, and the *scaling function*

$$\theta(\varepsilon) := \begin{cases} \varepsilon^\zeta & \text{when } \alpha\in(-\beta,+\infty]\setminus\{0, 2+\beta\}, \\ \varepsilon^2|\ln\varepsilon| & \text{when } \alpha = 2+\beta, \\ \dfrac{\varepsilon^{\frac{\beta}{1+\beta}}}{|\ln\varepsilon|} & \text{when } \alpha = 0. \end{cases} \tag{1.15}$$

Note that the threshold $\alpha = 2+\beta$ between standard and fractional diffusion corresponds to whether or not $\mathcal{M}_\beta$ has finite variance. We finally derive the *diffusion coefficient*:

Lemma 1.3 (Diffusion coefficient)**.** *Assume Hypothesis 1–2–3–4. Then the following limit holds (with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypothesis)*

$$\kappa := \lim_{\eta\to 0}\frac{\mu(\eta)|\xi|^{-\zeta}}{\theta(\varepsilon)\,\langle 1,\phi_\eta\rangle} = \mu_0 \times \begin{cases} \|\mathcal{M}\|^{-1}_{L^1(\mathbb{R}^d)} & \textit{when } \alpha > 0, \\ \dfrac{1+\beta}{|\mathbb{S}^{d-1}|} & \textit{when } \alpha = 0, \\ \left[c_{\alpha,\beta}\displaystyle\int_{\mathbb{R}^d}\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u\right]^{-1} & \textit{when } \alpha\in(-\beta,0). \end{cases} \tag{1.16}$$

The diffusion coefficient thus emerges from the ratio between integral quantities that reveal the comparison between physical scales:

$$\kappa := \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} (v \cdot \sigma)\, F(v) \mathcal{M}(v)\, \mathrm{d}v}{\|\mathcal{M}\|_{L^1(\mathbb{R}^d)}} & \boxed{\text{when } \alpha > 2+\beta} \\[2ex] \dfrac{1}{1+\beta} \dfrac{\displaystyle\int_{\mathbb{S}^{d-1}} (\sigma \cdot \sigma') \Omega(\sigma')\, \mathrm{d}\sigma'}{\displaystyle\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha}\, \mathrm{d}v} & \boxed{\text{when } \alpha = 2+\beta} \\[2ex] \dfrac{\displaystyle\int_{\mathbb{R}^d} (u \cdot \sigma)\, \mathrm{Im}\, \Phi(u) |u|^{-d-\alpha}\, \mathrm{d}u}{\displaystyle\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha}\, \mathrm{d}v} & \boxed{\text{when } \alpha \in (0, 2+\beta)} \\[2ex] \dfrac{1+\beta}{|\mathbb{S}^{d-1}|} \dfrac{\displaystyle\int_{\mathbb{R}^d} (u \cdot \sigma)\, \mathrm{Im}\, \Phi(u) |u|^{-d-\alpha}\, \mathrm{d}u}{\displaystyle\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha-\beta}\, \mathrm{d}v} & \boxed{\text{when } \alpha = 0} \\[2ex] \dfrac{\displaystyle\int_{\mathbb{R}^d} (u \cdot \sigma)\, \mathrm{Im}\, \Phi(u) |u|^{-d-\alpha}\, \mathrm{d}u}{\displaystyle\int_{\mathbb{R}^d} \Phi(u) |u|^{-d-\alpha}\, \mathrm{d}u} & \boxed{\text{when } \alpha \in (-\beta, 0)} \end{cases} \tag{1.17}$$

where we recall, for the legibility of this catalogue of formula:

$$F = \lim_{\eta \to 0} \frac{\mathrm{Im}\, \phi_\eta}{\eta}, \qquad \Phi = \lim_{\eta \to 0} \Phi_\eta = \lim_{\eta \to 0} \phi_\eta \left(\eta^{-\frac{1}{1+\beta}} \cdot \right), \qquad \Omega(u) = \lim_{\lambda \to 0,\ \lambda \neq 0} \frac{\mathrm{Im}\, \Phi(\lambda u)}{\lambda^{1+\beta}},$$

and (note that $\alpha > 2+\beta$ in this case) F is also the unique solution to $LF = -(v \cdot \sigma)$ with $\int_{\mathbb{R}^d} F(v) \lfloor v \rceil^{-d-\alpha-\beta}\, \mathrm{d}v = 0$. For legibility again, we wrote, in the cases $\alpha \in (-\beta, 2+\beta]$, the formula for κ with $\mathcal{M}$ given by (1.7), and we refer to Section 9 for more general $\mathcal{M}$.

The proof of Lemma 1.3 is done in Section 5; it requires the estimating of $\langle 1, \phi_\eta \rangle$, which is done in Lemma 5.1. The limit rescaled fluid mode Φ can also be defined as the solution to

$$\mathbb{L}^* \Phi = i (u \cdot \sigma) |u|^\beta \Phi \quad \text{ with } \quad \Phi(0) = 1,$$

when the rescaling of the operator L^* in the new variable $u = v \eta^{\frac{1}{1+\beta}}$ has a limit $\mathbb{L}^*$.

Given a solution f in $L^\infty_t([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ to equation (1.1) we denote

$$f_\varepsilon(t,x,v) := f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v \right) \in L^\infty_t\left([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}) \right) \text{ and } r_\varepsilon(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v) \lfloor v \rceil^{-\beta}\, \mathrm{d}v,$$

where $\varepsilon > 0$, and $\theta(\varepsilon)$ is defined in (1.15). The equation satisfied by f_ε is

$$\theta(\varepsilon) \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \mathcal{L} f_\varepsilon. \tag{1.18}$$

Theorem 1.4 (Unified second fluid approximation, see Figure 2). *Assume Hypothesis 1-2-3-4, and consider $f \in L^\infty_t([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ solving (1.1) in the weak sense with initially*

$$\left\| \frac{f_\varepsilon}{\mathcal{M}}(0,\cdot,\cdot) - r_\varepsilon(0,\cdot) \right\|_{-\beta} \lesssim \theta(\varepsilon) \quad \text{and} \quad \lim_{\varepsilon \to 0} r_\varepsilon(0,\cdot) := r(0,\cdot) \text{ in } H^{-\zeta}(\mathbb{R}^d). \tag{1.19}$$

Then for any $T > 0$,

$$\left\| \frac{f_\varepsilon}{\mathcal{M}} - r \right\|_{L^2_t([0,T]; H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon \to 0]{} 0$$

when $\alpha > \beta$ and

$$\left\| \left| \ln \frac{2|\nabla_x|}{1+|\nabla_x|} \right| \left(\frac{f_\varepsilon}{\mathcal{M}} - r \right) \right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon \to 0]{} 0$$

when $\alpha = \beta$ and

$$\left\| |\nabla_x|^{\frac{\beta-|\alpha|}{2(1+\beta)}} \lfloor \nabla_x \rceil^{-\frac{\beta-|\alpha|}{2(1+\beta)}} \left(\frac{f_\varepsilon}{\mathcal{M}} - r \right) \right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon \to 0]{} 0$$

when $\alpha \in (-\beta, \beta)$, where $r = r(t,x)$ solves

$$\partial_t r = \kappa \, \Delta_x^{\frac{\zeta}{2}} r, \quad t > 0, \qquad \textit{with initial data } r(0,\cdot) \textit{ defined in } (1.19).$$

The rates of convergence are estimated in terms of T, the constants, error terms and convergence rates in Hypothesis 1–2–3–4, and the initial convergence (1.19). Apart from the error in the initial convergence (that depends on the initial data), the rate we obtain is polynomial for $\alpha \in (-\beta, +\infty) \setminus \{0, 2+\beta\}$ and logarithmic for $\alpha \in \{0, 2+\beta\}$.

Note that when (1.19) is not satisfied at $t = 0$, the energy estimate implies that it is for any later time $\tau > 0$, and we could still deduce the fluid approximation for $t \in [\tau, T]$ (at the price of an initial time layer). We however chose to keep the assumption (1.19) since we are interested in tracking precisely the rate of convergence.

This theorem is the core contribution of the paper, and is used to obtain new results on concrete models (see the corollaries below). Together with Lemmas 1.1–1.2–1.3, it reveals the relevant macroscopic scales for a large class of operators in any dimension and provides a unified theoretical framework to answer questions of the last decades on the topic. The diffusive limit is reduced to a spectral problem –the construction of the fluid mode– that we solve in a general setting. The proof is constructive and the key constants governing the macroscopic behaviours are derived. The fractional Laplacian in the space variable is defined as in (1.5), and $r(t,x)$ is the limit (in the topology of the above theorem) of the weighted velocity average

$$r_\varepsilon(t,x) = \int_{\mathbb{R}^d} f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v \right) \lfloor v \rceil^{-\beta} \, \mathrm{d}v.$$

When $\alpha > 0$, the density $\rho_\varepsilon(t,x) := \int_{\mathbb{R}^d} f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v \right) \mathrm{d}v$ exists and also converges to $r(t,x)$.

We now apply the previous abstract theorem to particular models:

Corollary 1.5 (Scattering equation)**.** *Assume that $\mathcal{L}$ is the scattering operator (1.2) with $b \in \mathcal{C}^1$ and $\mathcal{M}$ satisfying Hypothesis 1 and that, for some constant $\nu_0 > 0$ and $\beta > -\alpha$*

$$(1.20) \qquad \begin{cases} \forall\, v \in \mathbb{R}^d, & \lfloor v \rceil^{-\beta} \lesssim \nu(v) \lesssim \lfloor v \rceil^{-\beta} \\ \forall\, v \in \mathbb{R}^d \setminus \{0\}, & \lambda^\beta \nu(\lambda v) \sim_{\lambda \to \infty} \nu_0 |v|^{-\beta} \\ \forall\, v \in \mathbb{R}^d, & \|b(v,\cdot)\|_\beta + \|b(\cdot, v)\|_\beta \lesssim \lfloor v \rceil^{-\beta}. \end{cases}$$

This includes $b(v,v') = \lfloor v \rceil^{-\beta} \lfloor v' \rceil^{-\beta}$ for any $\alpha + \beta > 0$, and $b(v,v') = \lfloor v - v' \rceil^{-\beta}$ when $\beta < 0$ and $\alpha + \beta > 0$ or when $\beta \geq 0$ and $\alpha > 3\beta$. Then Theorem 1.4 applies with α, β given in Hypothesis 1 and (1.20). This proves the diffusive limit for solutions to (1.18) with quantitative rate, diffusion exponent $\zeta = \frac{\alpha_+ + \beta}{1+\beta}$, scaling function (1.15) and diffusion coefficient (1.17).

In fact the constants can be computed explicitly since

$$\begin{cases} F(u) = \nu(v)^{-1} (v \cdot \sigma) & \text{when } \alpha > 2+\beta, \\ \Omega(u) = \nu_0^{-1} |u|^\beta (u \cdot \sigma) & \text{when } \alpha = 2+\beta, \\ \Phi(u) = \dfrac{\nu_0}{\nu_0 - i|u|^\beta (u \cdot \sigma)} & \text{when } \alpha \in (-\beta, 2+\beta), \end{cases}$$

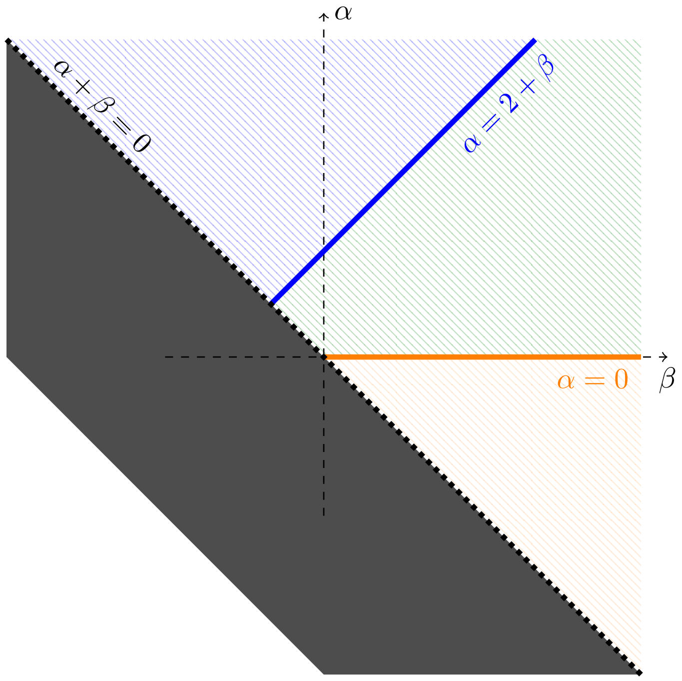

FIGURE 2. Summary of the results in the (α,β) plane. Admissible parameters are in half-plane $\alpha+\beta>0$. The blue hatched area leads to $\theta(\varepsilon)=\varepsilon^2$ and a standard diffusive limit with symbol $\kappa|\xi|^2$. The blue line is the set of parameters yielding the anomalous scaling $\theta(\varepsilon)=\varepsilon^2|\ln(\varepsilon)|$ but still a standard diffusive limit with symbol $\kappa|\xi|^2$. The green hatched area results into the fractional scaling $\theta(\varepsilon)=\varepsilon^{\frac{\alpha+\beta}{1+\beta}}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\alpha+\beta}{1+\beta}}$. The orange bold line yields the fractional scaling $\theta(\varepsilon)=\varepsilon^{\frac{\beta}{1+\beta}}|\ln(\varepsilon)|^{-1}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\beta}{1+\beta}}$. Finally, the orange hatched area yields the fractional scaling $\theta(\varepsilon)=\varepsilon^{\frac{\beta}{1+\beta}}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\beta}{1+\beta}}$.

resulting in the diffusion coefficient

$$\kappa := \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} (v\cdot\sigma)^2\,\nu(v)^{-1}\lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v}{\displaystyle\int_{\mathbb{R}^d}\lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha\in(2+\beta,+\infty)} \\[2ex] \dfrac{1}{\nu_0(1+\beta)}\,\dfrac{\int_{\mathbb{S}^{d-1}}(\sigma\cdot\sigma')^2\,\mathrm{d}\sigma'}{\displaystyle\int_{\mathbb{R}^d}\lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha=2+\beta} \\[2ex] \dfrac{\displaystyle\int_{\mathbb{R}^d}\frac{\nu_0|u|^\beta(u\cdot\sigma)^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{\mathrm{d}u}{|u|^{d+\alpha}}}{\displaystyle\int_{\mathbb{R}^d}\lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha\in(0,2+\beta)} \\[2ex] \dfrac{(1+\beta)}{|\mathbb{S}^{d-1}|}\,\dfrac{\displaystyle\int_{\mathbb{R}^d}\frac{\nu_0|u|^\beta(u\cdot\sigma)^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{\mathrm{d}u}{|u|^{d}}}{\displaystyle\int_{\mathbb{R}^d}\lfloor v\rceil^{-d-\beta}\,\mathrm{d}v} & \boxed{\text{when } \alpha=0} \\[2ex] \dfrac{\displaystyle\int_{\mathbb{R}^d}\frac{\nu_0|u|^\beta(u\cdot\sigma)^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{\mathrm{d}u}{|u|^{d+\alpha}}}{\displaystyle\int_{\mathbb{R}^d}\frac{\nu_0^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{\mathrm{d}u}{|u|^{d+\alpha}}} & \boxed{\text{when } \alpha\in(-\beta,0)} \end{cases}$$

as well as $\kappa := \|\mathcal{M}\|^{-1}_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} (v\cdot\sigma)^2 \nu^{-1} \mathcal{M}(v)\,\mathrm{d}v$ is the case "$\alpha = +\infty$". This recovers and unifies all results from [6, 20, 37, 38] (except for the case of space-dependent collision kernels in [20]) and extend them to new cases such as $\alpha \in (-\beta, 0)$ (infinite mass). The convergence rate is also new. Our approach bears partial similarities with, but differs from, the Hilbert expansions in [6] and [20], the moment method in [37] and the Fourier-Laplace calculation in [38].

Corollary 1.6 (Kinetic Fokker-Planck equation)**.** *Assume that $\mathcal{L}$ is the Fokker-Planck operator* (1.3) *with $\mathcal{M}$ satisfying Hypothesis 1 with $\alpha > -2$ if $d \geq 2$ and $\alpha > -1$ if $d = 1$. Then Theorem 1.4 applies with α given in Hypothesis 1 and $\beta = 2$. This proves the diffusive limit for solutions to* (1.18) *with quantitative rate, diffusion exponent $\zeta = \min\left(2, \frac{\alpha_+ + 2}{3}\right)$, scaling function* (1.15) *and diffusion coefficient* (1.17)*.*

Note that the constants may be precised using that Φ solves the Schrödinger-type equation

$$-|u|^2 \Delta_u \Phi + (d+\alpha) u \cdot \nabla_u \Phi - i(u\cdot\sigma)|u|^2 \Phi = 0 \quad \text{with the normalisation } \Phi(0) = 1.$$

In particular in the case $\alpha = 2 + \beta = 4$, the function Ω solves

$$-|u|^2 \Delta_u \Omega + (d+\alpha) u \cdot \nabla_u \Omega = (u\cdot\sigma)|u|^2 \quad \text{with } \Omega(0) = 0 \quad \Longrightarrow \quad \Omega(u) := \frac{|u|^2 (u\cdot\sigma)}{d+8}.$$

This recovers and unifies all results from [15, 39, 26, 25, 32] and obtains the first derivation of the diffusion coefficient in dimension higher than 1, as well as for equilibria with infinite mass when $\alpha \in (-2, 0)$ if $d \geq 2$ and $\alpha \in (-1, 0)$ if $d = 1$. The convergence rate is also new.

Corollary 1.7 (Kinetic Lévy-Fokker-Planck equation)**.** *Assume that $\mathcal{L}$ is the Lévy-Fokker-Planck operator* (1.4) *with parameter $s \in (\frac{1}{2}, 1)$ and with $\mathcal{M}$ satisfying Hypothesis 1 with $\alpha > s$. Then Theorem 1.4 applies with $\beta := 2s - \alpha$. This proves the diffusive limit for solutions to* (1.18) *with quantitative rate and diffusion exponent*

$$\zeta = \begin{cases} 2 & \textit{when } \alpha \geq 1 + s \\ \dfrac{2s}{1 + 2s - \alpha} & \textit{when } \alpha \in (s, 1+s), \end{cases}$$

and scaling function (1.15) *and diffusion coefficient* (1.17)*.*

The formula (1.17) for the diffusion coefficient may be precised with (see Section 8)

$$\Phi(u) := \exp\left(i \frac{2s c_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u\cdot\sigma)}{1+\beta} \right), \qquad \Omega(u) := \frac{2s c_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u\cdot\sigma)}{1+\beta}.$$

This gives, in particular,

$$\kappa := \begin{cases} \dfrac{2s c_{\alpha,0}^2}{c_{\alpha,\beta}(1+\beta)^2} \displaystyle\int_{\mathbb{S}^{d-1}} (\sigma'\cdot\sigma)^2 \,\mathrm{d}\sigma' & \boxed{\text{when } \alpha = 1+s} \\[2ex] \dfrac{c_{\alpha,0}}{1+\beta} \left(\dfrac{2s c_{\alpha,0}}{c_{\alpha,\beta}(1+\beta)} \right)^{\frac{\alpha-1}{1+\beta}} \displaystyle\int_{\mathbb{R}^d} (w\cdot\sigma) \sin(w\cdot\sigma) \frac{dw}{|w|^{d+\frac{\alpha+\beta}{1+\beta}}} & \boxed{\text{when } \alpha \in (s, 1+s)} \end{cases}$$

This recovers and extends the qualitative results in [1, 18] to general equilibria, with quantitative error estimates and characterizations of the diffusion coefficient. In the latter papers, the moment method initiated by Mellet is used to derive a fractional limit in the case $\beta = 0$. It raises several interesting questions: (1) can our approach be extended to $s \in (0, \frac{1}{2})$? (this seems to be a technical difficulty), (2) is the fractional diffusive limit possible for infinite mass equilibria? (i.e. $\alpha < 0$), (3) can the connexion between the kinetic Lévy-Fokker-Planck equation with $\alpha = 2s$ (for which the $\mathcal{L}$ is the generator of a Lévy process) and the standard kinetic Fokker-Planck equation with Gaussian equilibrium be clarified as $s \to 1$? (our diffusion constant κ above diverges as $s \to 1$ so the two limits in $\varepsilon \to 0$ and $s \to 1$ do not commute which calls for further investigation).

Let us summarise our contributions. Theorem 1.4 and Corollaries 1.5–1.6–1.7 recover the results of [1, 6, 15, 20, 26, 25, 32, 37, 38, 39] with a shorter and unified constructive method and prove new results for (1) Lévy-Fokker-Planck operators, (2) scattering operators with decaying collision kernel and infinite mass equilibria and importantly (3) Fokker-Planck operators in any dimension (for which the characterization of the diffusion coefficient was not known) as well as for infinite mass equilibria. The quantitative error in this fluid approximation seems to also be novel for all equations considered. Note finally that like the abstract Theorem 1.4, the Corollaries 1.5–1.6–1.7 are stated with the exact equilibrium of Hypothesis 1, but can be extended to more general equilibria, see Section 9. Moreover, it would be interesting to try and apply this method in other settings such as [4, 27] (radiative transfer theory), [11, 21, 29] (rarefied gas in a region between two parallel plates), [16, 17] (bounded domains), [2] (scattering with external acceleration field), [41] (models for chemotaxis) and [30] (adding a local conservation of momentum).

The method of the present paper extends to the fractional diffusive limit the approach pioneered in [40, 23] of constructing exact dispersion laws in the regime of parabolic time-space scaling and small eigenvalues; this extension is inspired by the recent one-dimensional result [32] and in particular we use and generalise the idea of rescaling velocities to obtain a non-trivial dispersion law in the latter paper. In comparison with [32], the main novelty of the present paper is a quantitative spectral method for constructing the branch of fluid eigenvalue: in [32] it was done by a one-dimensional argument connecting two infinite series on $\mathbb{R}_-$ and $\mathbb{R}_+$ (and it was done by fixed points in the simpler case of classical diffusive limit in the older works [40, 23]). We also provide the first quantitative error estimates, and prove the first result of fractional diffusive limit when the equilibrium has infinite mass.

Let us now compare our paper with the previous recent works by probabilists [26, 25]. In probabilistic terms, we try to describe particles moving in the full d-dimensional space along $\mathrm{d}X_t = V_t \,\mathrm{d}t$ with velocities V_t following a reversible process with invariant measure of the form given in (1). The velocity process is typically of scattering type, or Langevin type with drift and Brownian or non-Gaussian Lévy-type noises. We show in (1.4) that the rescaled process $\varepsilon X_{\theta(\varepsilon)^{-1}t}$ converges, with explicit rates and multiplicative constants, towards a Brownian motion when $\alpha \geq 2+\beta$, and towards a radially symmetric ζ-stable process when $\alpha \in (-\beta, \alpha+\beta)$. In spite of using quite different languages, the common point between [26, 25] and the present paper is the use of a scaling in velocity, which corresponds to applying some power function to the random variable in the probabilistic viewpoint and corresponds to the study of the rescaled fluid mode $\Phi_\eta := \phi_\eta(\eta^{-\frac{1}{1+\beta}}\cdot)$ in our study. Note however that in the case of equilibria with infinite mass $\alpha \in (-\beta, 0)$, the scaling considered in [25] is different from ours and more akin to a large deviation limit; it does not correspond to a fractional diffusion scaling and this explains why the authors obtain a kinetic (rather than fluid) limit equation with a Bessel process in velocity. Note also that the arguments in [26, 25] seem qualitative so the eigenvalue problem we study to compute the limit diffusion coefficient has no clear counterpart.

The rest of the paper is structured as follows. Section 2 is devoted to the proof of Theorem 1.4 assuming Lemmas 1.1, 1.2 and 1.3. We then prove Lemma 1.1 (construction of the fluid mode) in Section 3, Lemma 1.2 (scaling of the fluid mode) in Section 4, and Lemma 1.3 (derivation of the diffusion coefficient) in Section 5. Sections 6-7-8 prove the abstract hypothesis on the three concrete models; one argument of independent interest is a tightness estimate for the Schrödinger-type equation satisfied by the rescaled fluid mode in the cases of Fokker-Planck operators, see Lemma 7.3. Finally, Section 9 briefly discusses extensions of our results to more general equilibrium distributions and operators.

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2. Proof of Theorem 1.4 (convergence)

In this section we assume Lemmas 1.1, 1.2 and 1.3 and prove Theorem 1.4. Consider equation (1.6) and the rescaling

$$h_\varepsilon(t,x,v) := h\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right) = \frac{f_\varepsilon(t,x,v)}{\mathcal{M}(v)} = \frac{f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right)}{\mathcal{M}(v)}.$$

It satisfies the equation

$$\theta(\varepsilon)\partial_t h_\varepsilon + \varepsilon v \cdot \nabla_x h_\varepsilon = L h_\varepsilon. \tag{2.1}$$

2.1. The energy estimate.

Integrate (2.1) against $h_\varepsilon \mathcal{M}$ in t, x, v, and take the real part:

$$\begin{aligned}\frac{\theta(\varepsilon)}{2}\|h_\varepsilon(t)\|^2 &= \frac{\theta(\varepsilon)}{2}\|h_\varepsilon(0)\|^2 + \int_0^t \operatorname{Re}\left\langle L h_\varepsilon(\tau), h_\varepsilon(\tau)\right\rangle \mathrm{d}\tau \\ &\le \frac{\theta(\varepsilon)}{2}\|h_\varepsilon(0)\|^2 - \lambda \int_0^t \|h_\varepsilon(\tau) - r_\varepsilon(\tau)\|_{-\beta}^2 \,\mathrm{d}\tau\end{aligned}$$

where we have used Hypothesis 2 and

$$r_\varepsilon(t,x) := \int_{\mathbb{R}^d} h_\varepsilon(t,x,v)\mathcal{M}_\beta(v)\,\mathrm{d}v.$$

This proves

$$\forall\, t \ge 0, \quad \|h_\varepsilon(t)\|^2 \le \|h_\varepsilon(0)\|^2 \quad \text{and} \quad \int_0^t \|h_\varepsilon(\tau) - r_\varepsilon(\tau)\|_{-\beta}^2\,\mathrm{d}\tau \le \frac{\theta(\varepsilon)}{2\lambda}\|h_\varepsilon(0)\|^2. \tag{2.2}$$

2.2. Framework of the calculations.

Denote ξ the Fourier variable of x, and Fourier-transform equation (2.1) in x to get on $\hat{h}_\varepsilon(t,\xi,v)$

$$\theta(\varepsilon)\partial_t \hat{h}_\varepsilon = L\hat{h}_\varepsilon + i\varepsilon(v\cdot\xi)\hat{h}_\varepsilon. \tag{2.3}$$

Note that (2.2) and the Plancherel theorem imply $\hat{h}_\varepsilon \in L^\infty_t(\mathbb{R}^+; L^2_{\xi,v}(\mathcal{M}))$ and

$$\left\|\hat{h}_\varepsilon - \hat{r}_\varepsilon\right\|_{L^2_t(\mathbb{R}^+; L^2_{\xi,v}(\mathcal{M}_\beta))} \lesssim \theta(\varepsilon)^{\frac{1}{2}}. \tag{2.4}$$

Denote $\xi =: |\xi|\sigma$ and $\eta := \varepsilon|\xi|$. Test (2.3) against $\mathcal{M}\phi_\eta$ with ϕ_η constructed in Lemma 1.1:

$$\begin{aligned}\theta(\varepsilon)\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \hat{h}_\varepsilon, \phi_\eta\right\rangle &= \left\langle L\hat{h}_\varepsilon + i\varepsilon(v\cdot\xi)\hat{h}_\varepsilon, \phi_\eta\right\rangle = \left\langle \hat{h}_\varepsilon, L^*(\phi_\eta) + i\varepsilon(v\cdot\xi)\phi_\eta\right\rangle \\ &= -\mu(\eta)\left\langle \hat{h}_\varepsilon, \lfloor v\rceil^{-\beta}\phi_\eta\right\rangle.\end{aligned} \tag{2.5}$$

We then split the integrals as follows:

$$\left\langle \hat{h}_\varepsilon, \phi_\eta\right\rangle = \hat{r}_\varepsilon\langle 1, \phi_\eta\rangle + \left\langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, \phi_\eta\right\rangle =: \langle 1, \phi_\eta\rangle\left[\hat{r}_\varepsilon - E_1\right]$$

$$\left\langle \hat{h}_\varepsilon, \lfloor v\rceil^{-\beta}\phi_\eta\right\rangle = \hat{r}_\varepsilon\left\langle 1, \lfloor v\rceil^{-\beta}\phi_\eta\right\rangle + \left\langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, \lfloor v\rceil^{-\beta}\phi_\eta\right\rangle =: \langle 1, \phi_\eta\rangle\frac{\theta(\varepsilon)}{\mu(\eta)}\left[\kappa_\eta\hat{r}_\varepsilon - E_2\right]$$

with the definitions (using the normalisation $\langle 1, \lfloor v\rceil^{-\beta}\phi_\eta\rangle = 1$)

$$\begin{aligned}\kappa_\eta &:= \frac{\mu(\eta)\left\langle 1, \lfloor v\rceil^{-\beta}\phi_\eta\right\rangle}{\theta(\varepsilon)\langle 1, \phi_\eta\rangle} = \frac{\mu(\eta)}{\theta(\varepsilon)\langle 1, \phi_\eta\rangle}, \\ E_1 &:= -\frac{\left\langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, \phi_\eta\right\rangle}{\langle 1, \phi_\eta\rangle} \quad \text{and} \quad E_2 := -\frac{\mu(\eta)\left\langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, \lfloor v\rceil^{-\beta}\phi_\eta\right\rangle}{\theta(\varepsilon)\langle 1, \phi_\eta\rangle}.\end{aligned}$$

Consequently, equation (2.5) writes

$$\partial_t\hat{r}_\varepsilon + \kappa_\eta\hat{r}_\varepsilon = \partial_t E_1 + E_2.$$

We then want to pass to the limit $\varepsilon \to 0$ (hence $\eta \to 0$ for each frequency ξ).

2.3. **Estimating κ_η, $\partial_t E_1$ and E_2.** Lemma 1.1 yields

$$\lim_{\varepsilon\to 0} \frac{\mu(\eta)}{\theta(\varepsilon)} = \mu_0 |\xi|^\zeta \quad \text{with} \quad \zeta := \frac{\alpha_+ + \beta}{1+\beta},$$

with constructive rate, for each frequency $\xi \in \mathbb{R}^d$ (note that in the cases $\alpha = 0$ or $\alpha = 2+\beta$, the error in the convergence includes a loss of frequency weight $|\ln|\xi||$). Lemma 1.3 implies

$$\lim_{\eta\to 0} \kappa_\eta = \kappa|\xi|^\zeta = \mu_0|\xi|^\zeta \times \begin{cases} \|\mathcal{M}\|^{-1}_{L^1(\mathbb{R}^d)} & \text{when } \alpha > 0, \\ \dfrac{1+\beta}{|\mathbb{S}^{d-1}|} & \text{when } \alpha = 0, \\ \left[c_{\alpha,\beta} \displaystyle\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha}\, \mathrm{d}u \right]^{-1} & \text{when } \alpha \in (-\beta, 0), \end{cases} \tag{2.6}$$

with constructive convergence rate. Observe that the previous estimates also imply

$$\frac{\mu(\eta)}{\theta(\varepsilon)|\langle 1, \phi_\eta\rangle|} \lesssim |\xi|^\zeta.$$

To estimate E_2, write

$$\left| \left\langle \hat h_\varepsilon - \hat r_\varepsilon, \lfloor \cdot \rceil^{-\beta} \phi_\eta \right\rangle \right| \lesssim \left\| \hat h_\varepsilon - \hat r_\varepsilon \right\|_{-\beta}$$

where we have used $\|\phi_\eta\|_{-\beta} = 1$. All in all, we get, using again Lemmas 1.3 and 5.1,

$$|E_2| \lesssim \frac{\mu(\eta)}{\theta(\varepsilon)|\langle 1, \phi_\eta\rangle|} \left\| \hat h_\varepsilon - \hat r_\varepsilon \right\|_{-\beta} \lesssim \left\| \hat h_\varepsilon - \hat r_\varepsilon \right\|_{-\beta} |\xi|^\zeta.$$

To estimate E_1, compute first

$$\left| \left\langle \hat h_\varepsilon - \hat r_\varepsilon, \phi_\eta \right\rangle \right| \le \left\| \hat h_\varepsilon - \hat r_\varepsilon \right\|_{-\beta} \|\phi_\eta\|_\beta,$$

to get

$$|E_1| \lesssim \frac{\|\phi_\eta\|_\beta}{|\langle 1, \phi_\eta\rangle|} \left\| \hat h_\varepsilon - \hat r_\varepsilon \right\|_{-\beta}.$$

One then estimates $\|\phi_\eta\|_\beta$. When $\alpha > \beta$, it is bounded by construction, and when $\alpha \le \beta$,

$$\|\phi_\eta\|^2_\beta = \eta^{\frac{\alpha-\beta}{1+\beta}} \int_{\mathbb{R}^d} |\Phi_\eta(u)|^2 \, |u|_\eta^{-d-\alpha+\beta}\, \mathrm{d}u.$$

Using the pointwise bound (1.11) and the moment bound (1.12) from Hypothesis 4, the latter integral exists and is uniformly bounded in η for $\alpha \in (-\beta, \beta)$ and is bounded by $|\ln\eta|$ when $\alpha = \beta$. Thus we get, using Lemma 5.1 to estimate $\langle 1, \phi_\eta\rangle$ again,

$$|E_1| \lesssim \theta(\varepsilon)^{-\frac12} \left\| \hat h_\varepsilon - \hat r_\varepsilon \right\|_{-\beta} \times \begin{cases} \theta(\varepsilon)^{\frac12}, & \text{when } \alpha > \beta, \\ \varepsilon^{\frac{\beta}{1+\beta}} |\ln(\varepsilon|\xi|)| & \text{when } \alpha = \beta, \\ \varepsilon^{\frac{\alpha}{1+\beta}} |\xi|^{\frac{\alpha-\beta}{2(1+\beta)}} & \text{when } \alpha \in (0, \beta), \\ |\ln(\varepsilon)|^{-1} (\ln|\xi|)^{-1} |\xi|^{-\frac{\beta}{2(1+\beta)}} & \text{when } \alpha = 0, \\ \varepsilon^{-\frac{\alpha}{2(1+\beta)}} |\xi|^{-\frac{\alpha+\beta}{2(1+\beta)}} & \text{when } \alpha \in (-\beta, 0). \end{cases}$$

We then define $r := r(t,x)$ solution to $\partial_t r + \kappa|\xi|^\zeta r = 0$ with initial data $r(0,\cdot)$ defined in (1.19) and deduce that $\omega_\varepsilon := \hat r_\varepsilon - \hat r$ satisfies

$$\partial_t \omega_\varepsilon + \kappa|\xi|^\zeta \omega_\varepsilon = \partial_t E_1 + E_2 + (\kappa - \kappa_\eta)\, \hat r_\varepsilon$$

which implies

$$\begin{aligned}\omega_\varepsilon(t,\xi) &= e^{-\kappa|\xi|^\zeta t}\omega_\varepsilon(0,\xi) + \int_0^t e^{-\kappa|\xi|^\zeta(t-s)}\left[\partial_t E_1(s,\xi) + E_2(s,\xi) + (\kappa-\kappa_\eta)\,\hat r_\varepsilon(s,\xi)\right] \mathrm{d}s \\ &= \omega_\varepsilon(0,\xi)e^{-\kappa|\xi|^\zeta t} + E_1(t,\xi) - e^{-\kappa|\xi|^\zeta t}E_1(0,\xi) \\ &\quad + \int_0^t e^{-\kappa|\xi|^\zeta(t-s)}\left[\kappa|\xi|^\zeta E_1(s,\xi) + E_2(s,\xi) + (\kappa-\kappa_\eta)\,\hat r_\varepsilon(s,\xi)\right] \mathrm{d}s.\end{aligned}$$

Define then

$$W(\xi) := \lfloor\xi\rceil^{-\zeta} \times \begin{cases} 1 \quad \text{when} \quad \alpha > \beta, \\ \left|\ln\frac{2|\xi|}{1+|\xi|}\right|^{-1} \quad \text{when} \quad \alpha=\beta, \\ |\xi|^{\frac{\beta-|\alpha|}{2(1+\beta)}}\lfloor\xi\rceil^{-\frac{\beta-|\alpha|}{2(1+\beta)}} \quad \text{when} \quad \alpha\in(-\beta,\beta).\end{cases}$$

and integrate in $L^2_\xi(W)$ and then in $L^2_t([0,T])$ to get, using again (2.4) as well as (1.19), $\|\omega_\varepsilon\|_{L^2_t([0,T];L^2_\xi(W))}$ which concludes the proof.

3. Proof of Lemma 1.1 (construction of the fluid mode)

In this section we prove Lemma 1.1, assuming Hypothesis 1–2–3. Denote

$$\tilde L^*_\eta\psi := \lfloor v\rceil^{\frac{\beta}{2}}L^*_\eta\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right) = \lfloor v\rceil^{\frac{\beta}{2}}L^*\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right) + i\eta\lfloor v\rceil^{\beta}(v\cdot\sigma)\psi.$$

As before, the dependency in σ is omitted from the subscripts for readability.

3.1. **Existence of the resolvent.** We first prove that when $r\in(r_0',r_0)$ with $0<r_0'<r_0<\lambda$ and η *small* enough and $z\in S(0,r)$ (circle with radius r in $\mathbb{C}$), the operator $\tilde L^*_\eta - z$ has a bounded inverse in $L^2_v(\mathcal{M})$, and the bound is uniform in $z\in S(0,r)$.

Given $G\in L^2(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ and $z\in S(0,r)$, consider *a priori* a solution $F\in L^2(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ to

$$-L^*F - i\eta(v\cdot\sigma)F - z\lfloor v\rceil^{-\beta}F = \lfloor v\rceil^{-\beta}G. \tag{3.1}$$

Recall the decomposition

$$F = \mathcal{P}F + \mathcal{P}^\perp F := m[F] + \mathcal{P}^\perp F \quad \text{with} \quad m[F] := \int_{\mathbb{R}^d} F(v)\mathcal{M}_\beta(v)\,\mathrm{d}v, \tag{3.2}$$

which is orthogonal for the scalar product associated with $\|\cdot\|_{-\beta}$. Integrate (3.1) against $\bar F\mathcal{M}$ and take the real part to get, using Hypothesis 2,

$$\begin{aligned} & \lambda\left\|\mathcal{P}^\perp F\right\|^2_{-\beta} - r\|F\|^2_{-\beta} \le \|G\|_{-\beta}\|F\|_{-\beta} \\ \Longrightarrow \quad & (\lambda-r)\left\|\mathcal{P}^\perp F\right\|^2_{-\beta} \le \|G\|_{-\beta}\|F\|_{-\beta} + r\,|m(F)|^2 \\ \Longrightarrow \quad & (\lambda-r_0)\left\|\mathcal{P}^\perp F\right\|^2_{-\beta} \le \frac{\lambda-r_0}{2}\|F\|^2_{-\beta} + \frac{1}{2(\lambda-r_0)}\|G\|^2_{-\beta} + r\,|m(F)|^2 \\ \Longrightarrow \quad & \frac{\lambda-r_0}{2}\left\|\mathcal{P}^\perp F\right\|^2_{-\beta} \le \left(\frac{\lambda-r_0}{2}+r\right)|m(F)|^2 + \frac{1}{2(\lambda-r_0)}\|G\|^2_{-\beta}\end{aligned}$$

which implies finally

$$\left\|\mathcal{P}^\perp F\right\|^2_{-\beta} \le \frac{1}{(\lambda-r_0)^2}\|G\|^2_{-\beta} + \left(1+\frac{2r}{\lambda-r_0}\right)|m(F)|^2. \tag{3.3}$$

Consider then a function $0\le\chi\le1$ smooth radially symmetric and such that $\chi\equiv1$ on $B(0,1)$ and $\chi\equiv0$ outside $B(0,2)$, and denote $\chi_R(v) := \chi(\frac{v}{R})$ for $R>0$. Integrate (3.1) against $\chi_R\mathcal{M}$:

$$-\langle L^*F,\chi_R\rangle - i\eta\int_{\mathbb{R}^d}(v\cdot\sigma)F(v)\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v - zm_R[F] = m_R[G] \tag{3.4}$$

where we denote the truncated average

$$m_R[F] := \int_{\mathbb{R}^d} F(v)\chi_R(v)\mathcal{M}_\beta(v)\,\mathrm{d}v.$$

Using the decomposition (3.2), $L^*1 = 0$ and Hypothesis 3:

$$|\langle L^* F, \chi_R\rangle| = \left|\left\langle L^*\left(\mathcal{P}^\perp F\right), \chi_R\right\rangle\right| \leq \|L(\chi_R)\|_\beta \left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}} \left\|\mathcal{P}^\perp F\right\|_{-\beta}. \tag{3.5}$$

Observe also that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} (v\cdot\sigma) F(v)\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v\right| &= \left|\int_{\mathbb{R}^d} (v\cdot\sigma)\left[\mathcal{P}^\perp F(v)\right]\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v\right| \\ &\leq \left(\int_{|v|\leq 2R} (v\cdot\sigma)^2 \lfloor v\rceil^\beta \mathcal{M}(v)\,dv\right)^{\frac{1}{2}} \left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \ell(R)\left\|\mathcal{P}^\perp F\right\|_{-\beta} \end{aligned} \tag{3.6}$$

with

$$\ell(R) = \left(\int_{|v|\leq 2R} (v\cdot\sigma)^2 \lfloor v\rceil^\beta \mathcal{M}(v)\,\mathrm{d}v\right)^{\frac{1}{2}} \lesssim \begin{cases} 1 & \text{when } \alpha > 2+\beta, \\ \sqrt{\ln(R)} & \text{when } \alpha = 2+\beta, \\ R^{1-\frac{\alpha-\beta}{2}} & \text{when } \alpha < 2+\beta. \end{cases} \tag{3.7}$$

Combining (3.4)–(3.5)–(3.6) yields the following estimate on the truncated average:

$$|m_R[F]| \leq \frac{1}{r}\left[\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}}\right]\left\|\mathcal{P}^\perp F\right\|_{-\beta} + \frac{1}{r}\|G\|_{-\beta}. \tag{3.8}$$

We next estimate the difference between $m[F]$ and $m_R[F]$:

$$\begin{aligned}|m[F] - m_R[F]| &\leq \int_{\mathbb{R}^d} |F|\,|1-\chi_R|\,\mathcal{M}_\beta(v)\,\mathrm{d}v \\ &\leq \left(\int_{\mathbb{R}^d} |1-\chi_R|^2\,\mathcal{M}_\beta\,\mathrm{d}v\right)^{\frac{1}{2}} \|F\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}}\|F\|_{-\beta},\end{aligned}$$

which implies for R large enough

$$|m[F]|^2 \lesssim |m_R[F]|^2 + R^{-(\alpha+\beta)}\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2. \tag{3.9}$$

Finally, taking the square of (3.8), and using the last estimate, we deduce

$$\begin{aligned}|m[F]|^2 &\lesssim \left(\frac{1}{r}\left[\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}}\right]\right)^2 \left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 + R^{-(\alpha+\beta)}\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 + \frac{1}{r^2}\|G\|_{-\beta}^2, \\ &\lesssim \frac{1}{r^2}\left(\eta^2\ell(R)^2 + R^{-(\alpha+\beta)}\right)\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 + \frac{1}{r^2}\|G\|_{-\beta}^2.\end{aligned} \tag{3.10}$$

Choosing $R = \eta^{-\frac{1}{1+\beta}}$ (with η small enough so that R is large enough in the previous calculations) implies $\eta^2\ell(R)^2 \sim R^{-(\alpha+\beta)}$ and (Θ was defined in (1.10))

$$\left[\eta^2\ell(R)^2 + R^{-(\alpha+\beta)}\right] \lesssim \Theta(\eta).$$

Combining (3.3)-(3.10) yields

$$\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \leq \left[\frac{1}{(\lambda-r_0)^2} + \frac{C}{r^2} + \frac{2C}{r(\lambda-r_0)}\right]\|G\|_{-\beta}^2 + C\left(1+\frac{2r}{\lambda-r_0}\right)\frac{\Theta(\eta)}{r^2}\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2$$

for $C>0$ uniform in $r\in(0,r_0)$ and η small enough. For η small enough in terms of r_0 and r_0'

$$\forall\, r\in(r_0', r_0), \quad C\left(1+\frac{2r}{\lambda-r_0}\right)\frac{\Theta(\eta)}{r^2} \leq \frac{1}{2},$$

and we deduce that

$$\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \le 2\left[\frac{1}{(\lambda-r_0)^2}+\frac{C}{r^2}+\frac{2C}{r(\lambda-r_0)}\right]\|G\|_{-\beta}^2 \lesssim_{r_0,r_0'} \|G\|_{-\beta}^2.$$

Plugged into (3.10), it implies $|m[F]| \lesssim_{r_0,r_0'} \|G\|_{-\beta}$, and finally

$$\|F\|_{-\beta} \lesssim_{r_0,r_0'} \|G\|_{-\beta}\,. \tag{3.11}$$

With the latter a priori estimate at hand, we now construct a solution to (3.1). The latter equation re-writes

$$-\tilde L^*\tilde F - i\eta(v\cdot\sigma)\lfloor v\rceil^\beta \tilde F - z\tilde F = \lfloor v\rceil^{-\frac{\beta}{2}} G \in L^2_v(\mathcal{M}), \tag{3.12}$$

with $\tilde F := \lfloor\cdot\rceil^{-\frac{\beta}{2}}F$. Since (Hypothesis 2) $\tilde L^*$ generates a contraction semigroup in $L^2_v(\mathcal{M})$, it is a standard result (see [24, Theorem II.3.15]) that $\tilde L^*$ is maximal dissipative. Therefore, given any $M\ge 1$, the operator $\tilde L^*_{\eta,M} := \tilde L^* + i\eta(v\cdot\sigma)\lfloor v\rceil^\beta \chi_M(v)$ is maximal dissipative (perturbation by a bounded purely imaginary multiplicative operator). Observe that the previous a priori estimate (3.11) holds for $\tilde L^*_{\eta,M}$ by the same calculation, and uniformly as $M\to+\infty$. This implies first that for each $M\ge1$ and $z\in S(0,r)$, there is $\tilde F_M \in L^2_v(\mathcal{M})$ that solves $-\tilde L^*_M \tilde F_M - z\tilde F_M = \lfloor v\rceil^{-\frac{\beta}{2}}G$, and second that $\tilde F_M$ is uniformly bounded in $L^2_v(\mathcal{M})$ as $M\to\infty$. Taking a subsequence weakly converging to some $\tilde F\in L^2_v(\mathcal{M})$ as $M\to\infty$ gives a solution to (3.12) and thus to (3.1).

3.2. **The spectral projections.** We can therefore define the spectral projections

$$\Pi_{r,\eta} := \frac{1}{2i\pi}\int_{S(0,r)} \left[\tilde L^*_\eta - z\right]^{-1} \mathrm{d}z$$

for $r\in(r_0',r_0)$, the interval of the previous subsection. In the next subsections, we first estimate the difference of the projections $\Pi_{r,\eta}$ and $\Pi_{r,0}$ when acting on $\psi_0 := \lfloor v\rceil^{-\frac{\beta}{2}}$ (the kernel of $\tilde L_0$) and projected on $\mathrm{Span}(\psi_0)$, which is enough to show that $\Pi_{r,\eta}$ is non-zero for r and η small enough and thus proves the existence of an eigenvalue. Second, on the basis of this first scalar estimate, we prove that $|||\Pi_{r,\eta}-\Pi_{r,0}|||\to 0$ as $\eta\to0$, which implies that the dimensions of these two projections are the same for r and η small enough. This implies the uniqueness of the eigenvalue and quantitative convergence estimates on it as $\eta\to0$.

3.3. **Preparation for the first scalar estimate.** Recall $\psi_0 = \lfloor\cdot\rceil^{-\frac{\beta}{2}}$, then

$$\begin{aligned}\Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0 &= \frac{1}{2i\pi}\int_{S(0,r)}\left[\tilde L^*_\eta - z\right]^{-1}\left[\tilde L^*_0 - \tilde L_\eta\right]\left[\tilde L^*_0 - z\right]^{-1}\psi_0\,\mathrm{d}z\\ &= -\frac{\eta}{2\pi}\int_{S(0,r)}\left[\tilde L^*_\eta - z\right]^{-1}\left\{(v\cdot\sigma)\lfloor v\rceil^\beta\left[\tilde L^*_0 - z\right]^{-1}\psi_0\right\}\mathrm{d}z\\ &= \frac{\eta}{2\pi}\int_{S(0,r)}\lfloor v\rceil^{-\frac{\beta}{2}}F\,\frac{\mathrm{d}z}{z}\end{aligned}$$

where we have used

$$\left(\tilde L^*_0 - z\right)^{-1}\psi_0 = \left[\lfloor\cdot\rceil^{\frac{\beta}{2}}L\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\cdot\right) - z\right]^{-1}\psi_0 = -\frac{1}{z}\psi_0$$

and we have defined F through

$$\left[\tilde L^*_\eta - z\right]^{-1}\left[v'\mapsto (v'\cdot\sigma)\lfloor v'\rceil^{\frac{\beta}{2}}\right](v) =: \lfloor v\rceil^{-\frac{\beta}{2}}F(v),$$

that is

$$-L^*F - i\eta(v\cdot\sigma)F - z\lfloor v\rceil^{-\beta}F = (v\cdot\sigma) \tag{3.13}$$

(the dependency of F on η, z and σ is omitted for readability).

Note that since $\Pi_{r,0}\psi_0 = \psi_0$ and

$$\int_{\mathbb{R}^d}\Pi_{r,0}\psi_0(v)\lfloor v\rceil^{-\frac{\beta}{2}}\mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\lfloor v\rceil^{-\beta}\mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\mathcal{M}_\beta(v)\,\mathrm{d}v = 1,$$

to prove the existence of an eigenvalue, it is enough to prove that for r and η small enough

$$A_{r,\eta} := \left| \int_{\mathbb{R}^d} (\Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0) \lfloor v \rceil^{-\frac{\beta}{2}} \mathcal{M}(v) \,\mathrm{d}v \right| < 1.$$

Using the decomposition (3.2) one gets

$$A_{r,\eta} = \left| \frac{\eta}{2\pi} \int_{S(0,r)} \frac{m[F]}{z} \,\mathrm{d}z \right|. \tag{3.14}$$

The next three steps are devoted to estimating $m[F]$.

3.4. **Localised average estimate.** Integrate (3.13) against $\chi_R \mathcal{M}$: the right hand side vanishes since $\mathcal{M}$ and χ_R are even and one gets

$$-\langle L^* F, \chi_R \rangle - i\eta \int_{\mathbb{R}^d} (v \cdot \sigma) F(v) \chi_R(v) \mathcal{M}(v) \,\mathrm{d}v - z m_R[F] = 0.$$

Using the same argument as for (3.5) and (3.6), we get

$$|m_R[F]| \le \frac{1}{r} \left[\eta \ell(R) + R^{-\frac{\alpha+\beta}{2}} \right] \left\| \mathcal{P}^\perp F \right\|_{-\beta} \tag{3.15}$$

and using (3.9) we deduce, for R large enough,

$$|m[F]|^2 \lesssim \frac{1}{r^2} \left(\eta^2 \ell(R)^2 + R^{-(\alpha+\beta)} \right) \left\| \mathcal{P}^\perp F \right\|^2_{-\beta} \lesssim \frac{\Theta(\eta)}{r^2} \left\| \mathcal{P}^\perp F \right\|^2_{-\beta}$$

with the choice $R = \eta^{-\frac{1}{1+\beta}}$.

3.5. L^2 **estimate.** Re-organise (3.13) as

$$-L^* F - i\eta (v \cdot \sigma) \left(F - \frac{1}{i\eta} \right) = z \lfloor v \rceil^{-\beta} F,$$

integrate it against

$$\overline{\left(F - \frac{1}{i\eta} \right) \mathcal{M}}$$

and take the real part to obtain

$$-\operatorname{Re} \left\langle L^* F, \left(F - \frac{1}{i\eta} \right) \right\rangle = \operatorname{Re} \left(z \int_{\mathbb{R}^d} \lfloor v \rceil^{-\beta} F \overline{\left(F - \frac{1}{i\eta} \right)} \mathcal{M} \,\mathrm{d}v \right).$$

The left hand side satisfies (using $L1 = 0$ and Hypothesis 2)

$$-\operatorname{Re} \left\langle L^* F, \left(F - \frac{1}{i\eta} \right) \right\rangle = -\operatorname{Re} \langle L^* F, F \rangle \ge \lambda \left\| \mathcal{P}^\perp F \right\|^2_{-\beta},$$

and the right hand side is bounded by

$$\operatorname{Re} \left(z \int_{\mathbb{R}^d} \lfloor v \rceil^{-\beta} F \overline{\left(F - \frac{1}{i\eta} \right)} \mathcal{M} \,\mathrm{d}v \right) \le r \|F\|^2_{-\beta} + \frac{r}{\eta} |m[F]|.$$

This results in the estimate (using again the orthogonal decomposition)

$$\begin{aligned} \lambda \left\| \mathcal{P}^\perp F \right\|^2_{-\beta} &\le r \|F\|^2_{-\beta} + \frac{r}{\eta} |m[F]| \\ &\le r |m[F]|^2 + r \left\| \mathcal{P}^\perp F \right\|^2_{-\beta} + \frac{r}{\eta} |m[F]|, \end{aligned}$$

and thus since $r < r_0 < \lambda$ stays away from λ ($r_0 \in (0, \lambda)$), we get

$$\left\| \mathcal{P}^\perp F \right\|^2_{-\beta} \lesssim r |m[F]|^2 + \frac{r}{\eta} |m[F]|. \tag{3.16}$$

3.6. **Synthesis and the first scalar estimate.** The two previous steps lead to

$$\begin{cases} |m(F)|^2 \lesssim \dfrac{\Theta(\eta)}{r^2} \left\|\mathcal{P}^\perp F\right\|_{-\beta}^2, \\ \left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \lesssim r\,|m[F]|^2 + \dfrac{r}{\eta}\,|m[F]|\,. \end{cases}$$

Plugging the second estimate into the first one, we obtain

$$|m(F)| \lesssim \frac{\Theta(\eta)}{r}\,|m[F]| + \frac{\Theta(\eta)}{\eta r}. \tag{3.17}$$

Given any $r \in [\mathrm{R}_0\Theta(\eta), r_0)$ with R_0 *large* enough and η *small* enough, we have $r^{-1}\Theta(\eta)$ small enough so that

$$|m(F)| \lesssim \frac{\Theta(\eta)}{\eta r}. \tag{3.18}$$

Plugging the latter into (3.14) finally yields

$$A_{r,\eta} \lesssim \frac{\eta}{2\pi}\int_{S(0,r)} \frac{\Theta(\eta)}{\eta r^2}\,\mathrm{d}z \lesssim \frac{\Theta(\eta)}{r},$$

which is as small as wanted for $r \in (\mathrm{R}_0\Theta(\eta), r_0)$ with R_0 *large* enough and η *small* enough. This concludes the proof of this scalar estimate. Note that so far we have proved the existence of the resolvent only for $z \in S(0,r)$ with $r \in (r_0', r_0)$, however it will prove useful to record here that the a priori estimates are still valid for even smaller r's.

3.7. **Estimating the full norm of the difference of projections at ψ_0.** Combining (3.16) and (3.18) yields

$$\left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \frac{1}{r^{\frac12}\eta}\Theta(\eta) + \frac{1}{\eta}\Theta(\eta)^{\frac12}.$$

This implies

$$\begin{aligned} \|\Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0\| &\lesssim \frac{\eta}{2\pi}\int_{S(0,r)} \frac{1}{r}\|F\|_{-\beta}\,\mathrm{d}z \\ &\lesssim \frac{\eta}{2\pi}\int_{S(0,r)} \frac{1}{r}\left(|m[F]| + \left\|\mathcal{P}^\perp F\right\|_{-\beta}\right)\mathrm{d}z \\ &\lesssim \frac{1}{r}\Theta(\eta) + \frac{1}{r^{\frac12}}\Theta(\eta) + \Theta(\eta)^{\frac12} \end{aligned} \tag{3.19}$$

which is as small as wanted for $r \in (\mathrm{R}_0\Theta(\eta), r_0)$ with R_0 *large* enough and η *small* enough.

3.8. **Estimating the full norm of the difference of projections.** Take now any $\psi \in L^2(\mathcal{M})$. Then $\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi \in L^2(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ and the following decomposition holds

$$\psi = \lfloor v\rceil^{-\frac{\beta}{2}} m\left[\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right] + \lfloor v\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right).$$

As a consequence,

$$\|(\Pi_{r,\eta} - \Pi_{r,0})\psi\| \le \left|m\left[\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right]\right|\,\|(\Pi_{r,\eta} - \Pi_{r,0})\psi_0\| + \left\|(\Pi_{r,\eta} - \Pi_{r,0})\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right\|.$$

The first term in the right hand side is estimated by (3.19). We estimate the second term in the right hand side by the triangle inequality:

$$\begin{aligned} &\left\|(\Pi_{r,\eta} - \Pi_{r,0})\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right\| \\ &\qquad \le \left\|\Pi_{r,\eta}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right\| + \left\|\Pi_{r,0}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right\| \end{aligned}$$

and now consider each term separately. Start with

$$\Pi_{r,\eta}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right] = \frac{1}{2i\pi}\int_{S(0,r)}\left[\tilde{L}_\eta - z\right]^{-1}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\mathrm{d}z$$

$$(3.20)\qquad = \frac{1}{2i\pi}\int_{S(0,r)}\lfloor v\rceil^{-\frac{\beta}{2}} F\,\mathrm{d}z$$

where F satisfies this time (as before we omit writing the dependency in η, z, σ)

$$(3.21)\qquad -L^*F - i\eta(v\cdot\sigma)F - z\lfloor v\rceil^{-\beta}F = \lfloor\cdot\rceil^{-\beta}\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right).$$

First, test (3.21) on $\overline{G}\mathcal{M}$, take the real part and use $m[\mathcal{P}^{\perp}(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi)] = 0$:

$$\begin{aligned}\lambda\left\|\mathcal{P}^{\perp}F\right\|_{-\beta}^2 &\le (\mathrm{Re}\ z)\,\|F\|_{-\beta}^2 + \mathrm{Re}\left\langle\lfloor\cdot\rceil^{-\beta}\mathcal{P}^{\perp}(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi), F\right\rangle\\ &= (\mathrm{Re}\ z)\,\|F\|_{-\beta}^2 + \mathrm{Re}\left\langle\lfloor\cdot\rceil^{-\beta}\mathcal{P}^{\perp}(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi), \mathcal{P}^{\perp}F\right\rangle\\ &\le r|m[F]|^2 + r\|\mathcal{P}^{\perp}F\|_{-\beta}^2 + \left\|\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right\|_{-\beta}\left\|\mathcal{P}^{\perp}F\right\|_{-\beta},\end{aligned}$$

which implies, since $r < r_0$ stays away from λ ($r_0 \in (0,\lambda)$),

$$(3.22)\qquad \left\|\mathcal{P}^{\perp}F\right\|_{-\beta}^2 \lesssim r\ |m[F]|^2 + \left\|\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right\|_{-\beta}^2 \lesssim r\ |m[F]|^2 + \|\psi\|^2.$$

We now estimate $m[F]$. Integrate (3.21) against $\chi_R\mathcal{M}$ with $R = \eta^{-\frac{1}{1+\beta}}$

$$-\langle L^*F, \chi_R\rangle - i\eta\int_{\mathbb{R}^d}(v\cdot\sigma)F(v)\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v - zm_R[F] = m_R\left[\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right].$$

Using the same arguments as in Subsections 3.1 and 3.4 we obtain

$$|m_R[F]| \le \frac{\sqrt{\Theta(\eta)}}{r}\left\|\mathcal{P}^{\perp}F\right\|_{-\beta} + \frac{1}{r}\left|m_R\left[\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right|.$$

Since $m[\mathcal{P}^{\perp}(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi)] = 0$, we can estimate $|m_R[\mathcal{P}^{\perp}(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi)]|$ as follows:

$$\begin{aligned}\left|m_R\left[\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right| &= \left|m\left[\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right] - m_R\left[\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right|\\ &\lesssim R^{-\frac{\alpha+\beta}{2}}\left\|\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}}\|\psi\| \lesssim \sqrt{\Theta(\eta)}\|\psi\|.\end{aligned}$$

From this, we deduce

$$|m_R[F]| \le \frac{\sqrt{\Theta(\eta)}}{r}\left(\left\|\mathcal{P}^{\perp}F\right\|_{-\beta} + \|\psi\|\right),$$

and using

$$|m[F]|^2 \lesssim |m_R[F]|^2 + R^{-(\alpha+\beta)}\left\|\mathcal{P}^{\perp}F\right\|_{-\beta}^2 \le |m_R(F)|^2 + \Theta(\eta)\left\|\mathcal{P}^{\perp}G\right\|_{-\beta}^2$$

we finally get

$$(3.23)\quad |m[F]|^2 \lesssim \frac{\Theta(\eta)}{r^2}\left(\left\|\mathcal{P}^{\perp}F\right\|_{-\beta}^2 + \|\psi\|^2\right) + \Theta(\eta)\left\|\mathcal{P}^{\perp}F\right\|_{-\beta}^2 \lesssim \frac{\Theta(\eta)}{r^2}\left(\left\|\mathcal{P}^{\perp}F\right\|_{-\beta}^2 + \|\psi\|^2\right).$$

Combining (3.22)–(3.23) implies for $r \in [\mathrm{R}_0\Theta(\eta), r_0)$ with R_0 *large* enough and η *small* enough

$$|m[F]|^2 \lesssim \frac{\Theta(\eta)}{r^2}\|\psi\|^2 \quad \text{and thus} \quad \left\|\mathcal{P}^{\perp}F\right\|_{-\beta}^2 \lesssim \frac{\Theta(\eta)}{r}\|\psi\|^2 + \|\psi\|^2.$$

Plugging the latter estimates into (3.20) yields

$$\left\|\Pi_{r,\eta}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^{\perp}\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right\| \le r\|F\|_{-\beta} \lesssim r|m[F]| + r\|\mathcal{P}^{\perp}F\|_{-\beta} \lesssim \Theta(\eta)^{\frac{1}{2}}\|\psi\| + r\|\psi\|.$$

We now come to the estimate of

$$\Pi_{r,0}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right] = \frac{1}{2i\pi}\int_{S(0,r)}\left[\tilde{L}_0 - z\right]^{-1}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\mathrm{d}z$$
$$= \frac{1}{2i\pi}\int_{S(0,r)}\lfloor v\rceil^{-\frac{\beta}{2}}F\,\mathrm{d}z$$

where F satisfies this time

$$-L^*F - z\lfloor v\rceil^{-\beta}F = \lfloor v\rceil^{-\beta}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right). \tag{3.24}$$

Integrating this equation against $\mathcal{M}$ implies $m[F]=0$ since $\langle L^*F,1\rangle = m[\mathcal{P}^\perp(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi)] = 0$ and $z\neq 0$. Hypothesis 2 then implies since $r<r_0<\lambda$ is away from λ:

$$\|F\|_{-\beta} = \left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \|\psi\|,$$

and thus

$$\left\|\Pi_{r,0}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right\| \lesssim r\|\psi\|.$$

The conclusion is that for any $\psi\in L^2(\mathcal{M})$,

$$\|(\Pi_{r,\eta}-\Pi_{r,0})\,\psi\| \lesssim \Theta(\eta)^{\frac{1}{2}}\|\psi\| + r\|\psi\|$$

which means that (combining all the previous conditions), for $r\in(\mathrm{R}_0\Theta(\eta),r_1)$ with $r_1\in(0,r_0)$ *small* enough (*independently* of η) and R_0 *large* enough (*independently* of η) and η *small* enough, the operator norm

$$\|\Pi_{r,\eta}-\Pi_{r,0}\|_{L^2(\mathcal{M})\to L^2(\mathcal{M})} < 1.$$

It implies that, for any $r\in(r_0',r_1)$ and η small enough, the projections $\Pi_{r,\eta}$ and $\Pi_{r,0}$ both exist thanks to Subsection 3.1 and their dimensions are the same, i.e. 1, which proves uniqueness of the eigenvalue within $B(0,r_1)$. This in turn shows that the eigenvalue is real: if $(\psi_\eta,-\mu(\eta))$ is an eigenpair of $\tilde{L}_\eta$ with $\mu(\eta)\in B(0,r_1)$, then so is $(\overline{\psi_\eta(-\cdot)},-\overline{\mu(\eta)})$. Since $\tilde{L}_\eta\le 0$ and 0 is not eigenvalue for $\eta\neq 0$, this proves that $\mu(\eta)>0$.

3.9. **Estimate on the branch as $\eta\to 0$.** Since the projection has dimension one, there are no other eigenvalues in a disc of radius r_1 independent of $\eta\to 0$. We can therefore vary and decrease the radius until it touches the eigenvalue, and since our estimates above are uniform in $r\in(\mathrm{R}_0\Theta(\eta),r_1)$ with $r_1\in(0,r_0)$ *small* enough (*independently* of η) and R_0 *large* enough (*independently* of η), we deduce that this eigenvalue in fact belongs to the disc with radius $\mathrm{R}_0\Theta(\eta)$, i.e. $\mu(\eta)\lesssim\Theta(\eta)$. Moreover denoting $\phi_\eta := \lfloor\cdot\rceil^{\frac{\beta}{2}}\psi_\eta$ and normalizing the eigenvector as

$$\int_{\mathbb{R}^d}\psi_\eta(v)\lfloor v\rceil^{-\frac{\beta}{2}}\mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\phi_\eta(v)\lfloor v\rceil^{-\beta}\mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\phi_\eta(v)\mathcal{M}_\beta(v)\,\mathrm{d}v = m\left[\phi_\eta\right] = 1,$$

then integrating the equation against $\overline{\psi_\eta}\mathcal{M}$, taking the real part and using Hypothesis 2:

$$\lambda\left\|\psi_\eta-\psi_0\right\|^2 \le \mu(\eta)\left\|\psi_\eta\right\|^2 \lesssim \mu(\eta)\left\|\psi_\eta-\psi_0\right\|^2 + \mu(\eta)$$

where we have used $\|\psi_0\|=1$. Hence for η small enough we deduce

$$\|\phi_\eta - 1\|_{-\beta} = \|\psi_\eta-\psi_0\| \lesssim \mu(\eta)^{\frac{1}{2}}.$$

This concludes the proof of Lemma 1.1.

4. Proof of Lemma 1.2 (scaling of the eigenvalue)

In this section we prove Lemma 1.2, assuming all Hypothesis 1–2–3–4. Consider the unique eigenpair $(\phi_\eta,\mu(\eta))$ that satisfies $\mu(\eta)\in B(0,r_1)$ and

$$-L^*\phi_\eta - i\eta(v\cdot\sigma)\phi_\eta = \mu(\eta)\lfloor v\rceil^{-\beta}\phi_\eta \quad\text{and}\quad \int_{\mathbb{R}^d}\phi_\eta(v)\mathcal{M}_\beta(v)\,\mathrm{d}v = 1. \tag{4.1}$$

4.1. **Proof in the case $\alpha > 2+\beta$.** The function $F_\eta := \frac{\operatorname{Im}\phi_\eta}{\eta}$ satisfies

$$-L^* F_\eta - \mu(\eta)\lfloor v \rceil^{-\beta} F_\eta = (v\cdot\sigma)\operatorname{Re}\phi_\eta \quad \text{and} \quad \int_{\mathbb{R}^d} F_\eta(v)\mathcal{M}_\beta \,\mathrm{d}v = 0.$$

Since (Hypothesis 2) $\tilde{L}^*$ is invertible on the $L^2_v(\mathcal{M})$-orthogonal of $\lfloor\cdot\rceil^{-\frac{\beta}{2}}$, and $\lfloor\cdot\rceil^{\frac{\beta}{2}}\mathcal{M} \in L^2_v(\mathcal{M})$ when $\alpha > 2+\beta$, we can define then $F \in L^2_v(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ solution to

$$-L^* F = (v\cdot\sigma) \quad \text{with} \quad \int_{\mathbb{R}^d} F(v)\mathcal{M}_\beta \,\mathrm{d}v = 0.$$

The difference $F_\eta - F$ satisfies

$$-L^*\left(F_\eta - F\right) - \mu(\eta)\lfloor v\rceil^{-\beta}\left(F_\eta - F\right) = (v\cdot\sigma)\left[\operatorname{Re}\phi_\eta - 1\right] + \mu(\eta)\lfloor v\rceil^{-\beta} F.$$

Integrate the latter against $(F_\eta - F)\mathcal{M}$ and use Hypothesis 2:

$$\begin{aligned}\left[\lambda - \mu(\eta)\right]\|F_\eta - F\|^2_{-\beta} &\le \int_{\mathbb{R}^d} (v\cdot\sigma)\left(\operatorname{Re}\phi_\eta - 1\right)\left(F_\eta - F\right)\mathcal{M}\,\mathrm{d}v + \mu(\eta)\int_{\mathbb{R}^d} F\left(F_\eta - F\right)\mathcal{M}_\beta\,\mathrm{d}v \\ &\le \|\operatorname{Re}\phi_\eta - 1\|_{2+\beta}\|F_\eta - F\|_{-\beta} + \mu(\eta)\|F\|_{-\beta}\|F_\eta - F\|_{-\beta}.\end{aligned}$$

Write for any $\ell \in (2+\beta, \alpha)$

$$\|\operatorname{Re}\phi_\eta - 1\|_{2+\beta} \le \|\operatorname{Re}\phi_\eta - 1\|^{\zeta}_{-\beta}\|\operatorname{Re}\phi_\eta - 1\|^{1-\zeta}_{\ell} \le \mu(\eta)^{\frac{\zeta}{2}}\|\operatorname{Re}\phi_\eta - 1\|^{1-\zeta}_{\ell}$$

with $\mathsf{z} = \frac{\ell-(2+\beta)}{\ell+\beta} > 0$, then Hypothesis 4-(i) implies

$$\|\operatorname{Re}\phi_\eta - 1\|_{2+\beta} \lesssim \mu(\eta)^{\frac{\mathsf{z}}{2}} \to 0 \quad \text{as} \quad \eta \to 0,$$

and thus, since $\alpha > 1$ (combining $\alpha > 2+\beta$ and $\alpha+\beta > 0$)

$$\|F\|_{-\beta} \lesssim \|(v\cdot\sigma)\| \lesssim 1,$$

we deduce

$$\|F_\eta - F\|_{-\beta} \lesssim \mu(\eta)^{\frac{\mathsf{z}}{2}} \to 0 \quad \text{as} \quad \eta \to 0.$$

Finally,

$$\begin{aligned}\left|\frac{\mu(\eta)}{\eta^2} - \int_{\mathbb{R}^d}(v\cdot\sigma)F(v)\mathcal{M}(v)\,\mathrm{d}v\right| &\le \left|\int_{\mathbb{R}^d}(v\cdot\sigma)(F_\eta(v) - F(v))\mathcal{M}(v)\,\mathrm{d}v\right| \\ &\lesssim \|1\|_{2+\beta}\|F_\eta - F\|_{-\beta} \lesssim \mu(\eta)^{\frac{\mathsf{z}}{2}} \to 0 \text{ as } \eta \to 0,\end{aligned}$$

which identifies the limit of $\mu(\eta)$ and the rate.

4.2. **Proof in the case $\alpha < 2+\beta$.** Take $0 \le \chi \le 1$ a smooth test function that is 1 on $B(0,R_0)$ and zero outside $B(0,2R_0)$. Integrate (4.1) against $\Theta(\eta)^{-1}\chi(\cdot\eta^{\frac{1}{1+\beta}})\mathcal{M}$ and take the real part:

$$\begin{aligned}(4.2)\qquad &\frac{\mu(\eta)}{\Theta(\eta)} + \frac{1}{\Theta(\eta)}\eta\left\langle (v\cdot\sigma)\operatorname{Im}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right\rangle \\ &= -\frac{\mu(\eta)}{\Theta(\eta)}\left(\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right\rangle - 1\right) - \frac{1}{\Theta(\eta)}\left\langle L^*(\operatorname{Re}\phi_\eta - 1), \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right\rangle \\ &= -\frac{\mu(\eta)}{\Theta(\eta)}\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle - \frac{1}{\Theta(\eta)}\left\langle \operatorname{Re}\phi_\eta - 1, L\left(\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right)\right\rangle.\end{aligned}$$

The first term in the right hand side is controlled by

$$\left|\frac{\mu(\eta)}{\Theta(\eta)}\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle\right| \lesssim \left|\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle\right| \lesssim R_0^{-\frac{\alpha+\beta}{2(1+\beta)}}\eta^{\frac{\alpha+\beta}{2(1+\beta)}}$$

and the second term is controlled by

$$\begin{aligned}\left|\frac{1}{\Theta(\eta)}\left\langle \operatorname{Re}\phi_\eta - 1, L\left(\chi(\cdot\eta^{\frac{1}{1+\beta}})\right)\right\rangle\right| &\leq \frac{1}{\Theta(\eta)}\|\phi_\eta - 1\|_{-\beta}\left\|L\left[\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right]\right\|_\beta \\ &\lesssim \Theta(\eta)^{-\frac{1}{2}}\left\|L\left[\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right]\right\|_\beta \\ &\lesssim \Theta(\eta)^{-\frac{1}{2}}\eta^{\frac{\alpha+\beta}{2(1+\beta)}}R_0^{-\frac{\alpha+\beta}{2(1+\beta)}} \lesssim R_0^{-\frac{\alpha+\beta}{2(1+\beta)}}.\end{aligned}$$

The second term in the left hand side satisfies

$$\frac{\eta}{\Theta(\eta)}\left\langle (v\cdot\sigma)\operatorname{Im}\phi_\eta, \chi(\cdot\eta^{\frac{1}{1+\beta}})\right\rangle = c_{\alpha,\beta}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u$$

and we deduce

$$\left|\frac{\mu(\eta)}{\Theta(\eta)} + c_{\alpha,\beta}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\right| \lesssim R_0^{-\frac{\alpha+\beta}{2}}.$$

Then observe that assumption (1.11) in Hypothesis 4 implies the uniform integrability of the integrand on the support of χ and the convergence of the integral as $\eta \to 0$ for a given χ.

All in all we have the double limit

$$\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u \xrightarrow[R_0\to\infty]{\eta\to 0} \int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi|u|^{-d-\alpha}\,\mathrm{d}u.$$

This double limit thus proves that $\frac{\mu(\eta)}{\Theta(\eta)}$ converges and

$$\lim_{\eta\to 0}\frac{\mu(\eta)}{\Theta(\eta)} = c_{\alpha,\beta}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi|u|^{-d-\alpha}\,\mathrm{d}u.$$

4.3. **Proof in the case** $\alpha = 2+\beta$**.** Take $0 \leq \chi \leq 1$ a smooth test function that is 1 on $B(0,1)$ and zero outside $B(0,2)$. Consider again (4.2) (with now $\Theta(\eta) = \eta^2|\ln\eta|$) and estimate

$$\left|\frac{\mu(\eta)}{\Theta(\eta)}\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle\right| \lesssim \left|\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle\right| \lesssim \eta$$

and

$$\begin{aligned}\left|\frac{1}{\eta^2|\ln\eta|}\left\langle \operatorname{Re}\phi_\eta - 1, L\left(\chi(\cdot\eta^{\frac{1}{1+\beta}})\right)\right\rangle\right| &\leq \frac{1}{\eta^2|\ln\eta|}\|\phi_\eta - 1\|_{-\beta}\left\|L\left[\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right]\right\|_\beta \\ &\lesssim \frac{1}{\eta|\ln(\eta)|^{\frac{1}{2}}}\left\|L\left[\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right]\right\|_\beta \lesssim \frac{1}{|\ln(\eta)|^{\frac{1}{2}}}.\end{aligned}$$

We have also

$$\frac{1}{\eta|\ln\eta|}\left\langle (v\cdot\sigma)\operatorname{Im}\phi_\eta, \chi\left(\cdot\eta^{-\frac{1}{1+\beta}}\right)\right\rangle = \frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u$$

which gives

$$\left|\frac{\mu(\eta)}{\Theta(\eta)} + \frac{c_{\alpha,\beta}}{|\ln(\eta)|}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\right| \lesssim \frac{1}{|\ln(\eta)|^{\frac{1}{2}}} + \eta. \tag{4.3}$$

Let us decompose

$$\begin{aligned}&\frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u \\ &= \frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{|u|\leq\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\operatorname{Im}\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u + \frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{|u|\geq\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\operatorname{Im}\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u.\end{aligned}$$

The first term is bounded by

$$\frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{|u|\le\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\,\mathrm{Im}\,\Phi_\eta(u)|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u=\frac{c_{\alpha,\beta}}{\eta|\ln\eta|}\int_{|v|\le1}(v\cdot\sigma)\,\mathrm{Im}\,\phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v$$
$$\lesssim\frac{\Theta(\eta)^{\frac12}}{\eta|\ln\eta|}\lesssim\frac{1}{|\ln(\eta)|^{\frac12}}.$$

We approximate, using the second part of Hypothesis 4-(ii)

$$\frac{1}{|\ln(\eta)|}\left|\int_{|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\Big[\mathrm{Im}\,\Phi_\eta(u)-\mathrm{Im}\,\Phi(u)\Big]|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\right|\lesssim\mathfrak{a}(\eta).$$

Define,

$$N(\eta):=\int_{|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|^{-d-\alpha}\chi(u)\,\mathrm{d}u.$$

Observe that since $|\mathrm{Im}\,\Phi(u)|\lesssim|u|^{2+\beta}$, and $\alpha=2+\beta$,

$$\left|N(\eta)-\int_{|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\right|\le\int_{2\ge|u|\ge\eta^{\frac{1}{1+\beta}}}|u|^{2+\beta}\left||u|_\eta^{-d-\alpha}-|u|^{-d-\alpha}\right|\mathrm{d}u$$
$$\le\int_{1\le|v|\le2\eta^{-\frac{1}{1+\beta}}}|v|^{-d}\left||v|^{d+\alpha}\lfloor v\rceil^{-d-\alpha}-1\right|\mathrm{d}v$$
$$\lesssim1,$$

since $\left||v|^{d+\alpha}\lfloor v\rceil^{-d-\alpha}-1\right|\sim_{v\to\infty}\frac{d+\alpha}{2}\frac{1}{1+|v|^2}$. We get, using the second part of Hypothesis 4-(iv),

$$-\eta N'(\eta)\sim\frac{1}{1+\beta}\int_{\sigma'\in\mathbb{S}^{d-1}}(\sigma\cdot\sigma')\,\frac{\mathrm{Im}\,\Phi\left(\eta^{\frac{1}{1+\beta}}\sigma'\right)}{\eta}\,\mathrm{d}\sigma'\sim\frac{1}{1+\beta}\int_{\sigma'\in\mathbb{S}^{d-1}}(\sigma\cdot\sigma')\,\Omega(\sigma')\,\mathrm{d}\sigma'.$$

Apply then L'Hôpital's rule to deduce

$$\lim_{\eta\to0}\frac{N(\eta)}{|\ln\eta|}=\frac{1}{1+\beta}\int_{\sigma'\in\mathbb{S}^{d-1}}(\sigma\cdot\sigma')\,\Omega(\sigma')d\sigma'.$$

We conclude by taking $\eta\to0$ in (4.3).

5. Proof of Lemma 1.3 (the diffusion coefficient)

Using Lemma 1.2 and the definition (1.15) of θ, Lemma 1.3 follows from:

Lemma 5.1. *Assume Hypothesis 1–2–3–4. Then, the following convergence holds*

$$\langle1,\phi_\eta\rangle\sim_{\eta\to0}\begin{cases}\|\mathcal{M}\|_{L^1(\mathbb{R}^d)} & \text{when }\alpha>0\\[1ex] \dfrac{|\mathbb{S}^{d-1}|}{1+\beta}|\ln(\eta)| & \text{when }\alpha=0\\[1ex] \eta^{\frac{\alpha}{1+\beta}}c_{\alpha,\beta}\displaystyle\int_{\mathbb{R}^d}\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u & \text{when }\alpha\in(-\beta,0).\end{cases}$$

with explicit convergence rate.

Proof. When $\alpha>0$, the integral

$$\frac{c_{\alpha,\beta}}{c_{\alpha,0}}=\langle1,1\rangle=\int_{\mathbb{R}^d}\mathcal{M}\,\mathrm{d}v<+\infty$$

is well defined and, choosing $\ell\in(0,\alpha)$,

$$|\langle1,\phi_\eta\rangle-\langle1,1\rangle|\le|\langle1,\phi_\eta-1\rangle|\le\|1\|_{\min(\ell,\beta)}\,\|\phi_\eta-1\|_{-\min(\ell,\beta)}$$
$$\lesssim\|\phi_\eta-1\|_{-\beta}^a\,\|\phi_\eta-1\|_0^{1-a}\lesssim\mu(\eta)^{\frac{a}{2}}$$

with $a = \min(\frac{\ell}{\beta}, 1) \in (0,1]$, which shows $\langle 1, \phi_\eta \rangle \sim \langle 1, 1\rangle$ with explicit rate, and thus

$$\langle 1, \phi_\eta \rangle \xrightarrow{\eta \to 0} \langle 1, 1 \rangle = \frac{c_{\alpha,\beta}}{c_{\alpha,0}} = \|\mathcal{M}\|_{L^1(\mathbb{R}^d)} \qquad \text{when } \alpha > 0.$$

In the case $\alpha = 0$,

$$\int_{\mathbb{R}^d} \phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v = \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v + \int_{|v| \ge \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v.$$

The second term is estimated by

$$\begin{aligned}\int_{|v| \ge \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v &= c_{0,\beta} \int_{|u|\ge 1} \Phi_\eta(u)|u|_\eta^{-d}\,\mathrm{d}u \\ &= c_{0,\beta} \left(\int_{|u|\ge 1} |\Phi_\eta(u)|^2 |u|_\eta^{-d+\beta}\,\mathrm{d}u \right)^{\frac{1}{2}} \left(\int_{|u|\ge 1} |u|_\eta^{-d-\beta}\,\mathrm{d}u \right)^{\frac{1}{2}} \lesssim 1,\end{aligned}$$

using the moment bounds (1.12). The first term is decomposed into

$$\int_{|v| \le \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v = \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} (\phi_\eta(v) - 1)\,\mathcal{M}(v)\,\mathrm{d}v + \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} \mathcal{M}(v)\,\mathrm{d}v.$$

Since

$$\begin{aligned}\left| \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} (\phi_\eta(v) - 1)\,\mathcal{M}(v)\,\mathrm{d}v \right| &\le \|\phi_\eta(v) - 1\|_{-\beta} \left\| \mathbf{1}_{|\cdot| \le \eta^{-\frac{1}{1+\beta}}} \right\|_\beta \\ &\lesssim \mu(\eta)^{\frac{1}{2}} \eta^{-\frac{\beta}{2(1+\beta)}} \lesssim 1,\end{aligned}$$

we deduce

$$\int_{\mathbb{R}^d} \phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v \sim \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} \mathcal{M}(v)\,\mathrm{d}v \sim c_{0,\beta} \frac{|\mathbb{S}^{d-1}|}{1+\beta} |\ln \eta|$$

with explicit error term.

We finally consider, in the case $\alpha \in (-\beta, 0)$, the convergence of the integral

$$\eta^{-\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v \xrightarrow{\eta \to 0} c_{\alpha,\beta} \int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u.$$

Observe that the left hand side is

$$\eta^{-\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v = c_{\alpha,\beta} \int_{\mathbb{R}^d} \Phi_\eta(u)|u|_\eta^{-d-\alpha}\,\mathrm{d}u.$$

The bound (1.11) implies that the integrand is uniformly integrable near zero:

$$\int_{|u| \le \epsilon} \Phi_\eta(u)|u|_\eta^{-d-\alpha}\,\mathrm{d}u \xrightarrow{\epsilon \to 0} 0$$

uniformly as $\eta \to 0$. On the region $|u| \ge R_0$ the integral bound (1.12) implies

$$\int_{|u| \ge R_0} |\Phi_\eta(u)|\, |u|_\eta^{-d-\alpha}\,\mathrm{d}u \xrightarrow{R_0 \to \infty} 0$$

uniformly as $\eta \to 0$. We finally use the L^2-convergence $\Phi_\eta \to \Phi$ on $\{\epsilon \le |u| \le R_0\}$. ☐

6. Proof of the hypothesis for scattering equations

In this section we consider the scattering operator

$$\begin{cases} \mathcal{L}f = \displaystyle\int_{\mathbb{R}^d} b(\cdot, v') \left[f(v')\,\mathcal{M}(\cdot) - f(\cdot)\,\mathcal{M}(v') \right] \mathrm{d}v', \\ Lh = \displaystyle\int_{\mathbb{R}^d} b(\cdot, v')\,\mathcal{M}(v') \left[h(v') - h(\cdot) \right] \mathrm{d}v'. \end{cases}$$

We assume that b is C^1, that the operator conserves the local mass

$$\int_{\mathbb{R}^d} \big(\mathrm{b}(v,v') - \mathrm{b}(v',v)\big)\, \mathcal{M}(v')\,\mathrm{d}v' = 0$$

and that the *collision kernel* b and *collision frequency*

$$\nu(v) := \int_{\mathbb{R}^d} b(v,v')\,\mathcal{M}(v')\,\mathrm{d}v'$$

satisfy, for some constant $\nu_0 > 0$,

$$\lfloor v\rceil^{-\beta} \lesssim \nu(v) \lesssim \lfloor v\rceil^{-\beta}, \qquad \lambda^\beta \nu(\lambda u) \sim_{\lambda\to\infty} \nu_0 |u|^{-\beta} \quad \text{ and } \quad \|b(v,\cdot)\|_\beta + \|b(\cdot,v)\|_\beta \lesssim \lfloor v\rceil^{-\beta}.$$

This includes $b(v,v') = \lfloor v\rceil^{-\beta}\lfloor v'\rceil^{-\beta}$ for any $\alpha+\beta>0$, and $b(v,v') = \lfloor v-v'\rceil^{-\beta}$ when $\beta<0$ and $\alpha+\beta>0$ or when $\beta\geq 0$ and $\alpha > 3\beta$.

6.1. **Proof of Hypothesis 2.** Hypothesis 2 is standard and proved for instance in [20].

6.2. **Proof of Hypothesis 3.** We perform the following calculations:

$$\begin{aligned}
\|L(\chi_R)\|_\beta^2 &= \int_{\mathbb{R}^d} \lfloor v\rceil^\beta |L(\chi_R)|^2 \mathcal{M}(v)\,\mathrm{d}v \\
&\leq \int_{\mathbb{R}^d} \lfloor v\rceil^\beta \nu(v) \int_{\mathbb{R}^d} |\chi_R(v) - \chi_R(v')|^2\, \mathrm{b}(v,v')\,\mathcal{M}(v')\,\mathcal{M}(v)\,\mathrm{d}v'\,\mathrm{d}v \\
&\lesssim \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\chi_R(v) - \chi_R(v')|^2\, \mathrm{b}(v,v')\,\mathcal{M}(v)\,\mathcal{M}(v')\,\mathrm{d}v\,\mathrm{d}v' \\
&\lesssim \iint_{\{|v|<R\}\times\mathbb{R}^d} \cdots + \iint_{\{|v|>R\}\times\mathbb{R}^d} \cdots \\
&\lesssim \iint_{\mathbb{R}^d\times\{|v'|>R\}} \mathrm{b}(v,v')\,\mathcal{M}(v)\,\mathcal{M}(v')\,\mathrm{d}v\,\mathrm{d}v' + \iint_{\{|v|>R\}\times\mathbb{R}^d} \mathrm{b}(v,v')\,\mathcal{M}(v)\,\mathcal{M}(v')\,\mathrm{d}v'\,\mathrm{d}v \\
&\lesssim \|\chi_R^c \mathcal{M}\|_{-\beta} \lesssim R^{-\frac{(\alpha+\beta)}{2}} \xrightarrow[R\to\infty]{} 0.
\end{aligned}$$

6.3. **Proof of Hypothesis 4.** The eigenvalue problem writes

$$\begin{aligned}
&\left(L^{*,+}\phi_\eta\right)(v) := \int_{\mathbb{R}^d} b(v',v)\mathcal{M}(v')\phi_\eta(v')\,\mathrm{d}v' = \Big(\nu(v) - i\eta(v\cdot\sigma) - \mu(\eta)\lfloor v\rceil^{-\beta}\Big)\,\phi_\eta(v) \\
&\int_{\mathbb{R}^d} \mathcal{M}_\beta(v')\phi_\eta(v')\,\mathrm{d}v' = 1.
\end{aligned}$$

Observe first that Hypothesis 2 implies

$$\|\phi_\eta - 1\|_{-\beta}^2 \leq \mu(\eta)\,\|\phi_\eta\|_{-\beta}$$

and thus, for η small enough

$$\|\phi_\eta\|_{-\beta}^2 \leq \frac{1}{\lambda - \mu(\eta)}$$

is uniformly bounded as $\eta\to 0$. Observe second that

$$\left|L^{*,+}(\phi_\eta)(v)\right| \leq \|b(\cdot,v)\|_\beta\,\|\phi_\eta\|_{-\beta} \lesssim \lfloor v\rceil^{-\beta}$$

which yields, for η small enough,

$$|\phi_\eta(v)| \lesssim \frac{\lfloor v\rceil^{-\beta}}{\Big[\big(\nu(v) - \mu(\eta)\lfloor v\rceil^{-\beta}\big)^2 + \eta^2(v\cdot\sigma)^2\Big]^{\frac12}} \lesssim \frac{1}{\lfloor v\rceil^\beta \nu(v) - \mu(\eta)} \lesssim 1,$$

i.e. ϕ_η is uniformly bounded in $L^\infty(\mathbb{R}^d)$ as $\eta\to 0$, and Hypothesis 4-(i) when $\alpha > 2+\beta$ follows.

The rescaled eigenvector Φ_η satisfies

$$\Phi_\eta(u) := \phi_\eta\left(\eta^{-\frac{1}{1+\beta}}u\right) = \frac{\eta^{\frac{\beta}{1+\beta}}L^{*,+}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}}u\right)}{\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}}.$$

We turn to the case $\alpha \le 2+\beta$. Estimate (1.11) in Hypothesis 4 follows from Φ_η uniformly bounded and for η small and $|u| \le 1$ (using $|L^{*,+}(\phi_\eta)(v)| \lesssim \lfloor v\rceil^{-\beta}$),

$$|\mathrm{Im}\,\Phi_\eta(u)| = \left|\frac{(u\cdot\sigma)\left[\eta^{\frac{\beta}{1+\beta}}L^{*,+}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}}u\right)\right]}{\left(\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right) - \mu(\eta)|u|_\eta^{-\beta}\right)^2 + (u\cdot\sigma)^2}\right| \lesssim \frac{|u\cdot\sigma|}{\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right)} \lesssim |u|_\eta^{1+\beta}.$$

The integral moment bound (1.12) in Hypothesis 4-(iv) follows from (for small η and large u and using again $|L^{*,+}(\phi_\eta)(v)| \lesssim \lfloor v\rceil^{-\beta}$)

$$|\Phi_\eta(u)| \lesssim \frac{1}{1+|u|^\beta|u\cdot\sigma|}$$

which implies

$$\|\Phi_\eta\|_\beta^2 \lesssim \int_0^\pi\int_1^{+\infty}\frac{r^{-1-\alpha+\beta}}{1+r^{2+2\beta}\cos\theta^2}\,\mathrm{d}r\,\mathrm{d}\theta < +\infty.$$

To prove the remaining points we use $L^*1 = 0$ to write

$$\begin{aligned}\Phi_\eta(u) - \frac{\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right)}{\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}} &= \frac{\eta^{\frac{\beta}{1+\beta}}L^{*,+}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}}u\right) - \eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right)}{\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}} \\ &= \frac{\eta^{\frac{\beta}{1+\beta}}L^{*,+}\left(\phi_\eta - 1\right)\left(\eta^{-\frac{1}{1+\beta}}u\right)}{\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}}.\end{aligned}$$

Since then

$$\left|\eta^{\frac{\beta}{1+\beta}}L^{*,+}\left(\phi_\eta - 1\right)\left(\eta^{-\frac{1}{1+\beta}}u\right)\right| \lesssim \eta^{\frac{\beta}{1+\beta}}\left\lfloor\eta^{-\frac{1}{1+\beta}}u\right\rceil^{-\beta}\|\phi_\eta - 1\|_{-\beta} \lesssim \sqrt{\mu(\eta)}\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right),$$

we deduce

$$\left\|\frac{\Phi_\eta}{\Phi_{\eta,0}} - 1\right\|_{L^\infty(\mathbb{R}^d)} \lesssim \sqrt{\mu(\eta)} \xrightarrow[\eta\to 0]{} 0$$

with the simpler function

$$\Phi_{\eta,0}(u) := \frac{\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right)}{\eta^{\frac{\beta}{1+\beta}}\nu\left(\eta^{-\frac{1}{1+\beta}}u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}}.$$

To prove the convergence of Φ_η it is thus enough to check the convergence of $\Phi_{\eta,0}$:

$$\lim_{\eta\to 0}\Phi_\eta(u) = \lim_{\eta\to 0}\Phi_{\eta,0}(u) = \frac{\nu_0}{\nu_0 - i|u|^\beta(u\cdot\sigma)} := \Phi(u)$$

$$\Omega(u) = \lim_{\lambda\to 0,\ \lambda\neq 0}\lambda^{-(1+\beta)}\frac{\nu_0|\lambda u|^\beta(\lambda u\cdot\sigma)}{\nu_0^2 + |\lambda u|^{2\beta}(\lambda u\cdot\sigma)^2} = \nu_0^{-1}|u|^\beta(u\cdot\sigma),$$

and the corresponding diffusion coefficients are given in the statement of Corollary 1.5.

Moreover, since

$$\mathrm{Im}\,\Phi_{\eta,0}(u) := \frac{|u|_\eta^\beta\nu_\eta(u)|u|_\eta^\beta(u\cdot\sigma)}{\left(|u|_\eta^\beta\nu_\eta(u) - \mu(\eta)\right)^2 + |u|_\eta^{2\beta}(u\cdot\sigma)^2},$$

and

$$\mathrm{Im}\, \Phi(u) := \frac{\nu_0 |u|^\beta (u\cdot\sigma)}{\nu_0^2 + |u|^{2\beta}(u\cdot\sigma)^2},$$

we deduce

$$\begin{aligned}\frac{\mathrm{Im}\,\Phi(u)}{\mathrm{Im}\,\Phi_{\eta,0}(u)} &= \frac{\nu_0}{|u|_\eta^\beta \nu_\eta(u)} \frac{|u|^\beta}{|u|_\eta^\beta} \frac{\left(|u|_\eta^\beta \nu_\eta(u) - \mu(\eta)\right)^2 + |u|_\eta^{2\beta}(u\cdot\sigma)^2}{\nu_0^2 + |u|^{2\beta}(u\cdot\sigma)^2} \\ &= (1+o(1))\frac{|u|^\beta}{|u|_\eta^\beta}.\end{aligned}$$

Since $|\mathrm{Im}\,\Phi_\eta - \mathrm{Im}\,\Phi_{\eta,0}| \le \sqrt{\mu(\eta)}\, |\mathrm{Im}\,\Phi_{\eta,0}|$,

$$\frac{\mathrm{Im}\,\Phi(u)}{\mathrm{Im}\,\Phi_\eta(u)} = \frac{\mathrm{Im}\,\Phi(u)}{\mathrm{Im}\,\Phi_{\eta,0}(u)} \frac{\mathrm{Im}\,\Phi_{\eta,0}(u)}{\mathrm{Im}\,\Phi(u)} = (1+o(1))\frac{|u|^\beta}{|u|_\eta^\beta}.$$

and this gives the second part of Hypothesis 4-(ii).

7. Proof of the hypothesis for kinetic Fokker-Planck equations

In this section we consider the operator

$$\mathcal{L}(f) := \nabla_v \cdot \left(\mathcal{M} \nabla_v \left(\frac{f}{\mathcal{M}} \right) \right)$$

where $\mathcal{M}$ is given as in Hypothesis 1. In this section the constant $\beta = 2$, and the operator $\mathcal{L}$ and L are self-adjoint respectively in $L^2_v(\mathcal{M}^{-1})$ and $L^2_v(\mathcal{M})$.

7.1. Proof of Hypothesis 2.

This hypothesis reads in the case of Fokker-Planck operators:

$$\int_{\mathbb{R}^d} |\nabla_v h|^2 \, \mathcal{M}(v) \, \mathrm{d}v \ge \lambda \, \|h - \mathcal{P}h\|_{-2} \ \text{ with } \mathcal{P}h := \int_{\mathbb{R}^d} h(v') \langle v' \rangle^{-2} \mathcal{M}(v') \, \mathrm{d}v'.$$

for some $\lambda > 0$ (recall that $\int \langle \cdot \rangle^{-2} \mathcal{M} = 1$ as per Hypothesis 1). It is a form of the so-called *Hardy-Poincaré inequality*, see for instance [8, equation (1)] where references are collected for proving it for $d \ge 3$ and $\alpha > -2$, [22, Corollary 1] and [9, Appendix A] where it is proved in all dimensions $d \ge 1$ under the condition $d + \alpha > 0$ (for instance the "α" in [22, Corollary 1] corresponds to our "$-(d+\alpha)$"). Note that the case when $d \ge 3$ and $\alpha \in (-d,-2)$ would correspond in [22, 9] to situations where the Hardy-Poincaré inequality holds without the need of the zero-average condition; this case is however excluded by our assumption $\alpha + \beta > 0$.

7.2. Proof of Hypothesis 3.

It is proved via the following computation

$$\begin{aligned}\|L(\chi_R)\|_2^2 &= \int_{\mathbb{R}^d} |\nabla \cdot (\mathcal{M} \nabla_v \chi_R)|^2 \, \lfloor v \rceil^2 \, \frac{\mathrm{d}v}{\mathcal{M}} \\ &= \int_{\mathbb{R}^d} \left| \Delta \chi_R(v) + \frac{\nabla_v \mathcal{M}(v)}{\mathcal{M}(v)} \cdot \nabla \chi_R(v) \right|^2 \lfloor v \rceil^2 \, \mathcal{M}(v) \, \mathrm{d}v \\ &= \int_{B_{2R} \setminus B_R} \left| \Delta \chi_R + \frac{\nabla_v \mathcal{M}}{\mathcal{M}} \cdot \nabla \chi_R \right|^2 \lfloor v \rceil^2 \, \mathcal{M} \, \mathrm{d}v \\ &= \int_{B_{2R} \setminus B_R} \lfloor v \rceil^{-2} \, \mathcal{M} \, \mathrm{d}v \lesssim_\chi R^{-(2+\alpha)} = R^{-(\beta+\alpha)}.\end{aligned}$$

7.3. Proof of Hypothesis 4.

The equation satisfied by Φ_η is

$$-|u|_\eta^2 \Delta_u \Phi_\eta + (d+\alpha) u \cdot \nabla_u \Phi_\eta - i(u\cdot\sigma)|u|_\eta^2 \Phi_\eta = \mu(\eta) \Phi_\eta. \tag{7.1}$$

We first prove the pointwise bounds, i.e. (1.11) in Hypothesis 4. We start with pointwise estimates of the non-rescaled eigenfunction.

Lemma 7.1. *The unique solution to*

$$-L\phi_\eta - i\eta(v\cdot\sigma)\phi_\eta = \mu(\eta)\lfloor v\rceil^{-2}\phi_\eta \quad \textit{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v)\,\lfloor v\rceil^{-2}\,\mathcal{M}(v)\,\mathrm{d}v = 1$$

satisfies for any $R \geq 1$

$$\|\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R 1 \quad \textit{and} \quad \|\operatorname{Im}\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R \max(\eta,\mu(\eta))$$

with constants depending only on R but uniform in $\eta \to 0$.

Proof of Lemma 7.1. As for the scattering equation, Hypothesis 2 implies, for η small enough

$$\|\phi_\eta - 1\|_{-2}^2 \leq \mu(\eta)\,\|\phi_\eta\|_{-2} \quad \Rightarrow \quad \|\phi_\eta\|_{-2}^2 \leq \frac{1}{\lambda-\mu(\eta)} \lesssim 1.$$

The elliptic regularity of the operator $L = \Delta - (d+\alpha)\langle v\rangle^{-2} v\cdot\nabla_v$, with uniform ellipticity constant, then classically implies that

$$\|\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R 1.$$

Since $\mathcal{P}\phi_\eta = 1$ in the decomposition $\phi_\eta = \mathcal{P}\phi_\eta + \phi_\eta^\perp$, one deduces

$$\|\operatorname{Im}\phi_\eta\|_{-2} \leq \left\|\phi_\eta^\perp\right\|_{-2} \lesssim \mu(\eta)$$

and the imaginary part satisfies the equation

$$-L(\operatorname{Im}\phi_\eta) - \mu(\eta)\lfloor v\rceil^{-2}\operatorname{Im}\phi_\eta = \eta(v\cdot\sigma)\operatorname{Re}\phi_\eta.$$

Therefore the elliptic regularity combined with the integral bound on $\operatorname{Im}\phi_\eta$ and the bound $\|\eta(v\cdot\sigma)\operatorname{Re}\phi_\eta\|_{L^2(B(0,2))} \lesssim \eta$ on the right hand side implies that

$$\|\operatorname{Im}\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R \max(\eta,\mu(\eta))$$

which concludes the proof. $\square$

The following lemma proves (1.11).

Lemma 7.2. *There is $\eta_1 \in (0,\eta_0)$ small enough and A and C large enough so that*

$$\forall\,\eta\in(0,\eta_1),\ \forall\,u\in\mathbb{R}^d, \qquad |\Phi_\eta(u)| \lesssim |u|_\eta^{C\mu(\eta)}, \quad |\operatorname{Im}\Phi_\eta(u)| \lesssim_\delta |u|_\eta^{\min(2+\alpha,3)-C\mu(\eta)}.$$

Proof of Lemma 7.2. Multiply (7.1) by $\frac{\Phi_\eta}{|\Phi_\eta|}$ and take the real part,

$$-|u|_\eta^2\operatorname{Re}\left(\frac{\Phi_\eta}{|\Phi_\eta|}\Delta_u\Phi_\eta\right) + (d+\alpha)u\cdot\operatorname{Re}\left(\frac{\Phi_\eta}{|\Phi_\eta|}\nabla_u\Phi_\eta\right) = \mu(\eta)|\Phi_\eta|.$$

Since

$$\nabla_u|\Phi_\eta| = \operatorname{Re}\left(\frac{\Phi_\eta}{|\Phi_\eta|}\nabla_u\Phi_\eta\right), \qquad \Delta_u|\Phi_\eta| \geq \operatorname{Re}\left(\frac{\Phi_\eta}{|\Phi_\eta|}\Delta_u\Phi_\eta\right),$$

one gets

$$-|u|_\eta^2\Delta_u|\Phi_\eta| + (d+\alpha)u\cdot\nabla_u|\Phi_\eta| - \mu(\eta)|\Phi_\eta| \leq 0.$$

Then observe that the real function $F(u) = |u|_\eta^{C\mu(\eta)}$ satisfies for $|u| \geq A\eta^{\frac13}$ with A large:

$$\begin{aligned}
&-|u|_\eta^2\Delta_u F + (d+\alpha)u\cdot\nabla_u F - \mu(\eta)F\\
&\geq \mu(\eta)F\left[-Cd\frac{|u|_\eta^2}{|u|^2} - C(C\mu(\eta)-2)\frac{|u|_\eta^2}{|u|^2} + C(d+\alpha) - 1\right]\\
&\geq \mu(\eta)F\left[-Cd(1+\varepsilon) - C(C\mu(\eta)-2)(1+\varepsilon) + C(d+\alpha) - 1\right]\\
&\geq \mu(\eta)F\left[C(2+\alpha) - 1 - \varepsilon C(d-2) - C^2\mu(\eta)(1+\varepsilon)\right]
\end{aligned}$$

where we have used that $\frac{|u|_\eta^2}{|u|^2} \leq 1+\varepsilon$ with ε small for $|u| \geq A\eta^{\frac13}$ when A large enough. The right hand side is thus positive for C large enough and ε and η small enough, since $2+\alpha > 0$:

$$\forall\,|u| \geq A\eta^{\frac13}, \quad -|u|_\eta^2\Delta_u F + (d+\alpha)u\cdot\nabla_u F - \mu(\eta)F \geq 0$$

i.e. F is a super-solution in this region. Moreover the previous lemma shows that

$$\sup_{|u|\le A\eta^{\frac13}} |\Phi_\eta(u)| \le \|\phi_\eta\|_{L^\infty(B(0,A))} \lesssim_A 1$$

and we can therefore compare Φ_η and F on the ball $|u|\le A\eta^{\frac13}$ with a bound uniform in η. The maximum principle thus implies that $|\Phi_\eta| \lesssim |u|_\eta^{C\mu(\eta)}$ for all $|u|\ge A\eta^{\frac13}$ with a bound uniform in η. Finally, since $\eta^{C\mu(\eta)}\sim 1$ as $\eta\to 0$, this bound extends to any $u\in\mathbb{R}^d$ up to enlarging the comparison constant (independently of $\eta\to 0$).

Take then the imaginary part of equation (7.1)

$$-|u|_\eta^2\Delta_u \operatorname{Im}\Phi_\eta + (d+\alpha)u\cdot\nabla_u \operatorname{Im}\Phi_\eta - \mu(\eta)\operatorname{Im}\Phi_\eta = (u\cdot\sigma)|u|_\eta^2 \operatorname{Re}\Phi_\eta,$$

multiply by $\frac{\operatorname{Im}\Phi_\eta}{|\operatorname{Im}\Phi_\eta|}$ and use the previous estimate to get for $|u|\ge A\eta^{\frac13}$

$$-|u|_\eta^2\Delta_u|\operatorname{Im}(\Phi_\eta)| + (d+\alpha)u\cdot\nabla_u|\operatorname{Im}(\Phi_\eta)| - \mu(\eta)|\operatorname{Im}(\Phi_\eta)| \le |u|_\eta^{3+C\mu(\eta)}.$$

Define $G(u) := |u|_\eta^{\zeta'}$ with $\zeta' := \min(2+\alpha,3) - C\mu(\eta)$ and compute for $|u|\in[A\eta^{\frac13},1]$:

$$-|u|_\eta^2\Delta_u G + (d+\alpha)u\cdot\nabla_u G - \mu(\eta)G \quad \ge G\left[\zeta'\left(2+\alpha-\zeta'\right) - \mu(\eta) - \mathcal{O}\left(A^{-2}\right)\right] \gtrsim_a |u|_\eta^{3+C\mu(\eta)}$$

for C large enough and η small enough. The maximum principle then shows again that $|\operatorname{Im}\Phi_\eta| \lesssim |u|_\eta^{\zeta'}$ on $|u|\in[A\eta^{\frac13},1]$ by comparing $\operatorname{Im}\Phi_\eta$ and G on $|u| = A\eta^{\frac13}$ thanks to the second inequality in the previous lemma. Again the bound extends to any $|u|\le A\eta^{\frac13}$ using the second inequality in the previous lemma, since $\max(\eta,\mu(\eta))\lesssim \eta^{\frac{\zeta'}{3}}$ uniformly as $\eta\to 0$ (examining separately the cases $\alpha\in(-2,1]$ and $\alpha\in(1,4)$). ☐

The next lemma allows to prove the integral moment estimate (1.12) in Hypothesis 4-(iv).

Lemma 7.3. *There is $\zeta>0$ such that for any $q\ge -2$ and $G,\Phi\in L^2(\lfloor u\rceil^{q-d-\alpha})$ such that*

$$-|u|_\eta^2\Delta_u\Phi + (d+\alpha)u\cdot\nabla_u\Phi_\eta - i(u\cdot\sigma)|u|_\eta^2\Phi = G, \tag{7.2}$$

the following gain of decay at infinity holds

$$\int_{\mathbb{R}^d} |\Phi(u)|^2\lfloor u\rceil^{q+\zeta-d-\alpha}\,\mathrm{d}u \lesssim_{q,\zeta} \int_{\mathbb{R}^d}|G(u)|^2\lfloor u\rceil^{q-d-\alpha}\,\mathrm{d}u + \int_{\mathbb{R}^d}|\Phi(u)|^2\lfloor u\rceil^{q-d-\alpha}\,\mathrm{d}u.$$

Proof of Lemma 7.3. Consider a real-valued smooth function $\chi_0(u)$ that is zero on $|u|\le\frac12$ and equal to 1 on $|u|\ge 1$, and integrate (7.2) against $\overline{\Phi}\chi_0^2|u|_\eta^{q-d-\alpha}$ and take the real part to get

$$\begin{aligned}\int_{|u|\ge 1}|u|_\eta^2\left|\nabla\Phi_\eta(u)\right|^2|u|_\eta^{q-d-\alpha}\,\mathrm{d}u &\lesssim \int_{\mathbb{R}^d}\left(\chi_0^2\left|\Phi_\eta\right|^2 + \left|\Delta\left(\chi_0^2\right)\right||u|_\eta^2\left|\Phi_\eta\right|^2 + \chi_0^2\left|G\right|^2\right)|u|_\eta^{q-d-\alpha}\,\mathrm{d}u\\ &\lesssim \int_{|u|\ge 1}\left(\left|\Phi_\eta\right|^2 + |G|^2\right)|u|_\eta^{q-d-\alpha}\,\mathrm{d}u.\end{aligned}$$

Integrate then (7.2) against $\overline{\Phi}(u\cdot\sigma)\chi_1^2|u|_\eta^{q-d-\alpha-1}$ where χ_1 is a real-valued smooth function that is zero on $|u|\le 1$ and equal to 1 on $|u|\ge 1$, and take the imaginary part to get

$$\begin{aligned}\int_{\mathbb{R}^d}(u\cdot\sigma)^2\chi_1(u)^2\left|\Phi_\eta(u)\right|^2|u|_\eta^{1+q-d-\alpha}\,\mathrm{d}u &\lesssim \int_{|u|\ge 1}|u|_\eta^2\left|\nabla\Phi_\eta(u)\right|^2|u|_\eta^{q-d-\alpha}\,\mathrm{d}u\\ &\quad + \int_{|u|\ge 1}\left(\left|\Phi_\eta\right|^2 + |G|^2\right)|u|_\eta^{q-d-\alpha}\,\mathrm{d}u\\ &\lesssim \int_{|u|\ge 1}\left(\left|\Phi_\eta\right|^2 + |G|^2\right)|u|_\eta^{q-d-\alpha}\,\mathrm{d}u\end{aligned}$$

where we have used the previous real part estimate in the last line. This yields

$$\int_{\mathbb{R}^d}(u\cdot\sigma)^2|u|\left|\Phi_\eta(u)\right|^2\lfloor u\rceil^{q-d-\alpha}\,\mathrm{d}u \lesssim \int_{\mathbb{R}^d}\left(\left|\Phi_\eta\right|^2 + |G|^2\right)\lfloor u\rceil^{q-d-\alpha}\,\mathrm{d}u.$$

This first estimate improves the decay at infinity outside a cone around $u \perp \sigma$. We now use the ellipticity of the equation to control this latter region. The operator writes $\mathsf{L}_\eta = -|u|_\eta^{d+\alpha} \nabla_u \cdot [|u|_\eta^{-d-\alpha} \nabla_u]$ and we deduce by simple commutator estimates that

$$\int_{|u|\geq 2} \left| \nabla_u \left(\Phi_\eta(u) |u|^{\frac{q-d-\alpha+2}{2}} \right) \right|^2 \mathrm{d}u \lesssim \int_{|u|\geq 1} \left(|\Phi_\eta|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u.$$

Consider first the case $d > 2$. The Caffarelli-Kohn-Nirenberg inequality yields

$$\left\| \Phi_\eta(u) |u|^{\frac{q-d-\alpha+2}{2}} 1_{|u|\geq 2} \right\|^2_{L^{\frac{2d}{d-2}}} \lesssim \int_{|u|\geq 1} \left(|\Phi_\eta|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u.$$

Consider now the cone $\mathcal{C} := \{ |\frac{u}{|u|} \cdot \sigma| \leq \lfloor u \rceil^{-\frac{\delta}{2}}, |u| \geq 2 \}$ for some $\delta > 0$, and a gain of weight $\lfloor u \rceil^\zeta$ for some $\zeta > 0$ to be precised later. The Hölder inequality then yields

$$\int_{\mathcal{C}} |\Phi_\eta(u)|^2 |u|^{q-d-\alpha+\zeta} \, \mathrm{d}u \leq \left\| \Phi_\eta(u) |u|^{\frac{q-d-\alpha+2}{2}} 1_{|u|\geq 2} \right\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \left(\int_{\mathcal{C}} |u|^{\frac{d(\zeta-2)}{2}} \, \mathrm{d}u \right)^{\frac{2}{d}}.$$

The extra volume integral may be estimated as follows using spherical coordinates,

$$\int_{\mathcal{C}} |u|^{\frac{d(\zeta-2)}{2}} \, \mathrm{d}u \lesssim \int_1^{+\infty} r^{d-1} \langle r \rangle^{\frac{d(\zeta-2)}{2}} \langle r \rangle^{-\frac{\delta}{2}} \, \mathrm{d}r \lesssim \int_1^{+\infty} r^{-1+\frac{d\zeta}{2}-\frac{\delta}{2}} \, \mathrm{d}r$$

which is finite as soon as $\zeta < \frac{\delta}{d}$ (which defines and restricts δ). Outside the cone we use the first estimate:

$$\int_{\mathcal{C}^c \cap \{|u|\geq 2\}} |\Phi_\eta|^2 |u|^{q-d-\alpha+\zeta} \, \mathrm{d}u \lesssim \int_{\mathbb{R}^d} |\Phi_\eta|^2 (u\cdot\sigma)^2 |u|^{-2+\delta+q-d-\alpha+\zeta} \, \mathrm{d}u$$

which is controlled as soon as $\zeta \leq 3 - \delta$. The constraints are compatible for $\zeta \in (0, \frac{3}{d+1})$.

In the case $d = 1$, the gain of decay is immediate from the first estimate alone. In the case $d = 2$, we follow a similar argument but replace the Caffarelli-Kohn-Nirenberg inequality with the Onofri inequality:

$$\left\| \Phi_\eta(u) |u|^{\frac{q-d-\alpha+2}{2}} 1_{|u|\geq 2} \right\|^2_{L^{2p}} \lesssim_p \int_{|u|\geq 1} \left(|\Phi_\eta|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u$$

for any $p < \infty$. The Hölder inequality then gives

$$\int_{\mathcal{C}} |\Phi_\eta(u)|^2 |u|^{q-d-\alpha+\zeta} \, \mathrm{d}u \leq \left\| \Phi_\eta(u) |u|^{\frac{q-d-\alpha+2}{2}} 1_{|u|\geq 2} \right\|^2_{L^{2p}(\mathbb{R}^d)} \left(\int_{\mathcal{C}} |u|^{q(\zeta-2)} \, \mathrm{d}u \right)^{\frac{1}{q}}$$

where $q = \frac{1}{1-1/p}$ is the exponent conjugate to p. The conclusion follows as before by taking p large enough. □

We now apply this latter lemma to obtain the moment bound (1.12). Observe first that the pointwise bound $|\Phi_\eta(u)| \lesssim |u|^{C\mu(\eta)}$ proved in the previous lemma implies that

$$\int_{|u|\geq 1} |\Phi_\eta(u)|^2 |u|^{q-d-\alpha} \, \mathrm{d}u < +\infty$$

for $q = -2$ and η small enough with bound uniform in η, since $\alpha + 2 > 0$. On this basis, we then apply repeatedly the latter lemma with $G = \mu(\eta)\Phi_\eta$ to obtain that Φ_η decays faster than any polynomial at infinity, with constants uniform in η (note that ζ is independent of q in the lemma). Finally the convergence $\Phi_\eta \to \Phi$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d \backslash \{0\})$ follows easily from the bounds established above and the convergence of the coefficients of the equation satisfied by Φ_η: one can prove that $\eta \mapsto \Phi_\eta$ is Cauchy in L^2 on any such compact set as $\eta \to 0$, and such convergence has a polynomial rate and is uniform on any compact set in $\mathbb{R}^d \setminus \{0\}$.

We now prove the second part of Hypothesis 4-(ii). The equation for $W_\eta := \Phi_\eta - \Phi$ is

$$\begin{aligned} &- |u|_\eta^2 \Delta_u W_\eta + (d+\alpha) u \cdot \nabla_u \, W_\eta + i(u\cdot\sigma)|u|_\eta^2 W_\eta - \mu(\eta) \, W_\eta \\ &= \eta^{\frac{2}{3}} (d+\alpha) \frac{u}{|u|^2} \cdot \nabla_u \Phi + \mu(\eta)\Phi, \end{aligned}$$

We derive a bound on $\nabla_u \Phi$. For this, differentiate the limit equation for Φ,

$$\begin{aligned}&-|u|^2 \Delta\left(\nabla \Phi\right) + (d+\alpha) u \cdot \nabla\left(\nabla \Phi\right) + (d+\alpha)\nabla\Phi + 2(\Delta\Phi) u \\ &= i(u\cdot\sigma)|u|^2 \nabla\Phi + i\nabla\left((u\cdot\sigma)|u|^2\right)\Phi.\end{aligned}$$

Test against $\frac{\nabla\Phi}{|\nabla\Phi|}$, use

$$\begin{aligned}&\nabla_u |\nabla \Phi| = \mathrm{Re}\left(\frac{\Phi}{|\Phi|}\nabla_u\Phi\right), \qquad \mathrm{Re}\left(\frac{\nabla\Phi}{|\nabla\Phi|}\Delta_u\left(\nabla\Phi\right)\right) \le \Delta_u\left(|\nabla\Phi|\right),\\ &u\cdot \mathrm{Re}\left(\Delta\Phi\frac{\nabla\Phi}{|\nabla\Phi|}\right) = u\cdot\nabla\left(|\nabla\Phi|\right), \qquad \left|\nabla\left((u\cdot\sigma)|u|^2\right)\right| \le 3|u|^2,\end{aligned}$$

and take the real part to get

$$-|u|^2\Delta\left(|\nabla\Phi|\right) + (d+\alpha-2)u\cdot\nabla\left(|\nabla\Phi|\right) + (d+\alpha)\left|\nabla\Phi\right| \le 3|u|^2\|\Phi\|_\infty.$$

The maximum principle then yields that $|\nabla\Phi| \lesssim |u|^2$. As a consequence,

$$-|u|_\eta^2\Delta_u|W_\eta| + (d+\alpha)u\cdot\nabla_u|W_\eta| - \mu(\eta)|W_\eta| \lesssim \eta^{\frac{2}{3}}|u| + \mu(\eta) \lesssim \eta^{\frac{2}{3}}|u|_\eta,$$

since Φ is bounded uniformly. From this, one deduces $|W_\eta| \lesssim \eta^{\frac{2}{3}}|u|_\eta$. Since,

$$\begin{aligned}&-|u|_\eta^2\Delta_u \,\mathrm{Im}\, W_\eta + (d+\alpha)u\cdot\nabla_u \,\mathrm{Im}\, W_\eta - \mu(\eta)\,\mathrm{Im}\, W_\eta\\ &= \eta^{\frac{2}{3}}\Delta_u\,\mathrm{Im}\,\Phi + \mu(\eta)\,\mathrm{Im}\,\Phi + (u\cdot\sigma)\left(|u|_\eta^2\,\mathrm{Re}\,\Phi_\eta - |u|^2\,\mathrm{Re}\,\Phi\right),\\ &= \eta^{\frac{2}{3}}\Delta_u\,\mathrm{Im}\,\Phi + \mu(\eta)\,\mathrm{Im}\,\Phi + (u\cdot\sigma)\left(|u|_\eta^2\,\mathrm{Re}\,W_\eta + \eta^{\frac{2}{3}}\,\mathrm{Re}\,\Phi\right),\\ &= \eta^{\frac{2}{3}}(d+\alpha)\frac{u}{|u|^2}\cdot\nabla_u\,\mathrm{Im}\,\Phi + \mu(\eta)\,\mathrm{Im}\,\Phi + (u\cdot\sigma)|u|_\eta^2\,\mathrm{Re}\,W_\eta,\end{aligned}$$

we bootstrap the bound on $\nabla_u\Phi$ to a bound on $\nabla_u\,\mathrm{Im}\,\Phi$, the imaginary part of $\nabla\Phi$. Since

$$-|u|^2\Delta\left(|\mathrm{Im}\,\nabla\Phi|\right) + (d+\alpha-2)u\cdot\nabla\left(|\mathrm{Im}\,\nabla\Phi|\right) + (d+\alpha)\left|\mathrm{Im}\,\nabla\Phi\right| \lesssim |u|^4 + |u|^5,$$

we obtain $|\mathrm{Im}\,\nabla\Phi(u)| \lesssim |u|^4$ on $B(0,1)$. From this, we get

$$\begin{aligned}&-|u|_\eta^2\Delta_u\left|\mathrm{Im}\,W_\eta\right| + (d+\alpha)u\cdot\nabla_u\left|\mathrm{Im}\,W_\eta\right| - \mu(\eta)\left|\mathrm{Im}\,W_\eta\right|\\ &\lesssim \eta^{\frac{2}{3}}|u|^3 + \mu(\eta)|u|^3 + |u\cdot\sigma||u|_\eta^2\eta^{\frac{2}{3}}|u|_\eta \lesssim \eta^{\frac{2}{3}}|u|_\eta^3 + |u\cdot\sigma||u|_\eta^2\eta^{\frac{2}{3}}|u|_\eta\end{aligned}$$

and thus

$$\frac{|\,\mathrm{Im}\,W_\eta(u)|}{|u|_\eta^{3-C\mu(\eta)}} \xrightarrow{\eta\to 0} 0$$

on $B(0,1)$, which implies the hypothesis since then

$$\begin{aligned}&\frac{1}{|\ln(\eta)|}\left|\int_{1\ge|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\Big[\,\mathrm{Im}\,\Phi_\eta(u) - \mathrm{Im}\,\Phi(u)\Big]|u|_\eta^{-d-\alpha}\,\mathrm{d}u\right|\\ &\le \frac{o_\eta(1)}{|\ln(\eta)|}\left|\int_{1\ge|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)|u|_\eta^{3-C\mu(\eta)}|u|_\eta^{-d-\alpha}\,\mathrm{d}u\right|\\ &\le \frac{o_\eta(1)}{|\ln(\eta)|}\left|\int_{1\ge|u|\ge\eta^{\frac{1}{1+\beta}}}|u|_\eta^{-d-C\mu(\eta)}\,\mathrm{d}u\right| \xrightarrow{\eta\to0} 0.\end{aligned}$$

8. Proof of the hypothesis for kinetic Lévy-Fokker-Planck equations

In this section, we consider, given $s\in(\frac{1}{2},1)$ and $\mathcal{M}$ is given by Hypothesis 1, the operator

$$\mathcal{L}(f) = \Delta_v^s f + \nabla_v\cdot(U\,f)\,.$$

The fractional Laplacian is defined as in (1.5) but we use the equivalent definition

$$\Delta_v^s f(v) := -C_{d,s}\int_{\mathbb{R}^d}\frac{f(v)-f(v')}{|v-v'|^{d+2s}}\,\mathrm{d}v' \quad \text{with} \quad C_{d,s} := \frac{4^s\Gamma\left(\frac{d}{2}+s\right)}{\pi^{\frac{d}{2}}|\Gamma(-s)|}.$$

The drift force U solves

$$\Delta_v^s \mathcal{M} + \nabla_v \cdot (U\,\mathcal{M}) = 0.$$

We restrict ourselves to $\alpha > s$. It is proved in [13] that a radial solution U to the previous equation satisfies $U(v) = \mathbb{U}(v)\lfloor v \rceil^{-\beta} v$ with $\beta := 2s - \alpha$ and $\mathbb{U}$ a uniformly positive function bounded from above. The operator L is

$$Lh = \mathcal{M}^{-1}\Delta_v^s(\mathcal{M}h) + \mathcal{M}^{-1}\nabla_v \cdot (U\,\mathcal{M}h) = \mathcal{M}^{-1}\Big[\Delta_v^s(\mathcal{M}h) - (\Delta_v^s\mathcal{M})\,h\Big] + U \cdot \nabla_v h.$$

8.1. **Proof of Hypothesis 2.** This hypothesis is implied by the fractional Hardy-Poincaré inequalities proved in [13] and earlier in [46]:

Proposition 8.1 ([13, 46]). *Let $d \ge 1$, $s \in (0,1)$, $\alpha > s$ and $\beta := 2s - \alpha$. Then there is $\lambda > 0$ (depending on s) such that*

$$-\operatorname{Re}\,\langle Lh, h\rangle \ge \lambda \|h - \mathcal{P}h\|_{-\beta}^2.$$

Proof. Compute

$$\begin{aligned}
-\operatorname{Re}\,\langle Lh, h\rangle &= -\operatorname{Re}\int_{\mathbb{R}^d}\Big[\Delta_v^s(\mathcal{M}h) + \nabla_v\cdot(U\,\mathcal{M}h)\Big]\overline{h}\,\mathrm{d}v' \\
&= -\operatorname{Re}\int_{\mathbb{R}^d}\left(\Delta_v^s\overline{h}\right)h\,\mathcal{M}\,\mathrm{d}v + \operatorname{Re}\int_{\mathbb{R}^d}\frac{1}{2}U\cdot\nabla_v\left(|h|^2\right)\mathcal{M}\,\mathrm{d}v \\
&= -\operatorname{Re}\int_{\mathbb{R}^d}\left(\Delta_v^s\overline{h}\right)h\,\mathcal{M}\,\mathrm{d}v - \operatorname{Re}\int_{\mathbb{R}^d}\frac{1}{2}\nabla_v\cdot(U\,\mathcal{M})\,|h|^2\,\mathrm{d}v \\
&= -\operatorname{Re}\int_{\mathbb{R}^d}\left(\Delta_v^s\overline{h}\right)h\,\mathcal{M}\,\mathrm{d}v + \operatorname{Re}\int_{\mathbb{R}^d}\frac{1}{2}\left(\Delta_v^s\mathcal{M}\right)|h|^2\,\mathrm{d}v \\
&= \frac{C_{d,s}}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|h-h'|^2}{|v-v'|^{d+2s}}\,\mathcal{M}\,\mathrm{d}v\,\mathrm{d}v'
\end{aligned}$$

and thus

$$-\operatorname{Re}\,\langle Lh, h\rangle = \frac{C_{d,s}}{4}\int_{\mathbb{R}^d}\frac{|h-h'|^2}{|v-v'|^{d+2s}}\left(\mathcal{M}+\mathcal{M}'\right)\mathrm{d}v\,\mathrm{d}v'.$$

Note that there is $\kappa > 0$ such that

$$\forall\,(v,v') \in \mathbb{R}^d\times\mathbb{R}^d, \quad \lfloor v\rceil^{-\beta}\lfloor v'\rceil^{-\beta}\,\mathcal{M}\,\mathcal{M}' \le \kappa\,\frac{\mathcal{M}+\mathcal{M}'}{|v-v'|^{d+2s}},$$

by matching the asymptotics at large v and v'. Hence we get that

$$-\operatorname{Re}\,\langle Lh, h\rangle \gtrsim \int_{\mathbb{R}^d}|h-h'|^2\lfloor v\rceil^{-\beta}\lfloor v'\rceil^{-\beta}\,\mathcal{M}\,\mathcal{M}'\,\mathrm{d}v\,\mathrm{d}v' \gtrsim \|h-\mathcal{P}h\|_{-\beta}^2,$$

where we used in the last line the classical coercivity for scattering operator discussed above. □

Note that in the previous coercivity inequality $\lambda(s) \to 0$ as $s \to 1$ since $C_{d,s} \to 0$ as $s \to 1$. This explains why the coercivity weight $\beta = 2$ of the Fokker-Planck operator differs from the coercivity weight $\beta = 2 - \alpha$ of the Lévy-Fokker-Planck operator when $s \to 1$. In fact when $\alpha = 2s$ and $s \to 1$ the correct formal limit is the Fokker-Planck operator with Gaussian equilibrium, in view of the general theory of Lévy processes, for which $\beta = 0$ is indeed the limit of $\beta = 2 - \alpha = 2 - 2s$ as $s \to 1$.

8.2. **Proof of Hypothesis 3.** We estimate

$$\|L(\chi_R)\|_\beta = \left\|\mathcal{M}^{-1}\Big[\Delta_v^s(\mathcal{M}\chi_R) - (\Delta_v^s\mathcal{M})\,\chi_R\Big] + U\cdot\nabla_v\chi_R\right\|_\beta$$

in several steps. Write first

$$\begin{aligned}
\|U\cdot\nabla_v\chi_R\|_\beta^2 &= \int_{\mathbb{R}^d}|U\cdot\nabla_v\chi_R|^2\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\left|\mathbb{U}(v)\lfloor v\rceil^{-\beta}v\cdot\nabla_v\chi_R\right|^2\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v \\
&\le \|\mathbb{U}\|_\infty\int_{\mathbb{R}^d}|v\cdot\nabla_v\chi_R|^2\,\mathcal{M}_\beta(v)\,\mathrm{d}v \lesssim R^{-\alpha-\beta}.
\end{aligned}$$

Then split the other term into

$$\begin{aligned}&\left\|\mathcal{M}^{-1}\Big[\Delta_v^s\left(\mathcal{M}\chi_R\right)-\left(\Delta_v^s\mathcal{M}\right)\chi_R\Big]\right\|_\beta^2\\&=\left\|\mathcal{M}^{-1}\Big[\Delta_v^s\left(\mathcal{M}\chi_R\right)-\left(\Delta_v^s\mathcal{M}\right)\chi_R\Big]\mathbf{1}_{|v|\le R}\right\|_\beta^2+\left\|\mathcal{M}^{-1}\Big[\Delta_v^s\left(\mathcal{M}\chi_R\right)-\left(\Delta_v^s\mathcal{M}\right)\chi_R\Big]\mathbf{1}_{|v|\ge R}\right\|_\beta^2.\end{aligned}$$

When $|v|\le R$, write $v=Rw$ with $|w|\le 1$ and observe that $\chi(w)=\chi(w')$ when $|w'|\le 1$ to get,

$$\begin{aligned}\left|\left[\Delta_v^s\left(\mathcal{M}\chi_R\right)-\left(\Delta_v^s\mathcal{M}\right)\chi_R\right](Rw)\right|&=C_{d,s}\left|\int_{|w'|\ge 1}\frac{\chi(w)-\chi(w')}{R^{d+a}\left|w-w'\right|^{d+2s}}\mathcal{M}(Rw')\,R^d\,\mathrm{d}w'\right|\\&\lesssim\int_{|w'|\ge 1}\frac{|\chi(w)-\chi(w')|}{R^{d+a+\alpha}\left|w-w'\right|^{d+2s}}\frac{\mathrm{d}w'}{\left|w'\right|^{d+\alpha}}\lesssim R^{-(d+2s+\alpha)},\end{aligned}$$

which yields (using $\beta=2s-\alpha$ and $\alpha>0$)

$$\left\|\mathcal{M}^{-1}\Big[\Delta_v^s\left(\mathcal{M}\chi_R\right)-\left(\Delta_v^s\mathcal{M}\right)\chi_R\Big]\mathbf{1}_{|v|\le R}\right\|_\beta^2\lesssim R^{-4s+\beta-\alpha}\lesssim R^{-\alpha-\beta}.$$

When $|v|\ge R$, we write

$$\Big[\Delta_v^s\left(\mathcal{M}\chi_R\right)-\left(\Delta_v^s\mathcal{M}\right)\chi_R\Big](v)=\int_{\mathbb{R}^d}\frac{\chi_R(v)-\chi_R(v')}{\left|v-v'\right|^{d+2s}}\mathcal{M}(v')\,\mathrm{d}v'=\int_{|v-v'|\le\frac{|v|}{2}}\cdots+\int_{|v-v'|\ge\frac{|v|}{2}}\cdots$$

Start with the first integral in the right hand side:

$$\begin{aligned}&\left|\int_{|v-v'|\le\frac{|v|}{2}}\frac{\chi_R(v)-\chi_R(v')}{\left|v-v'\right|^{d+2s}}\mathcal{M}(v')\,\mathrm{d}v'\right|\\&\lesssim\int_{|v-v'|\le\frac{|v|}{2}}\frac{\sup_{B\left(v,\frac{|v|}{2}\right)}\left|\nabla_{v'}^2\left[\left(\chi_R(v)-\chi_R(v')\right)\mathcal{M}(v')\right]\right|}{\left|v-v'\right|^{d+2s-2}}\,\mathrm{d}v'\\&\lesssim|v|^{2-2s}\sup_{B\left(v,\frac{|v|}{2}\right)}\left|\nabla_{v'}^2\left[\left(\chi_R(v)-\chi_R(v')\right)\mathcal{M}(v')\right]\right|.\end{aligned}$$

One has

$$\begin{aligned}&\sup_{B\left(v,\frac{|v|}{2}\right)}\left|D_{v'}^2\left((\chi_R(v)-\chi_R(v'))\mathcal{M}(v')\right)\right|\\&\lesssim R^{-2}|v|^{-d-\alpha}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi''\left(\frac{v'}{R}\right)\right|+R^{-1}|v|^{-d-\alpha-1}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi'\left(\frac{v'}{R}\right)\right|+|v|^{-d-\alpha-2}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left|\int_{|v-v'|\le\frac{|v|}{2}}\frac{\chi_R(v)-\chi_R(v')}{\left|v-v'\right|^{d+2s}}\mathcal{M}(v')\,dv'\right|\\&\lesssim|v|^{2-2s}\left[R^{-2}|v|^{-d-\alpha}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi''\left(\frac{v'}{R}\right)\right|+R^{-1}|v|^{-d-\alpha-1}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi'\left(\frac{v'}{R}\right)\right|+|v|^{-d-\alpha-2}\right]\\&\lesssim|v|^{-d-\alpha-2s}\left[\frac{|v|^2}{R^2}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi''\left(\frac{v'}{R}\right)\right|+\frac{|v|}{R}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi'\left(\frac{v'}{R}\right)\right|+1\right]\lesssim|v|^{-d-\alpha-2s},\end{aligned}$$

where we have used that χ' ans χ'' have compact support and $|v|\le 2|v'|$ in this region.

Focus now on the second integral (using $\alpha>0$)

$$\left|\int_{|v-v'|\ge\frac{|v|}{2}}\frac{\chi_R(v)-\chi_R(v')}{\left|v-v'\right|^{d+2s}}\mathcal{M}(v')\,\mathrm{d}v'\right|\le\int_{|v-v'|\ge\frac{|v|}{2}}\frac{|\chi_R(v)-\chi_R(v')|}{\left|v-v'\right|^{d+2s}}\mathcal{M}(v')\,\mathrm{d}v'\lesssim|v|^{-d-2s}.$$

As a conclusion,

$$\left\|\mathcal{M}^{-1}\left[\Delta_v^s\left(\mathcal{M}\chi_R\right)-\left(\Delta_v^s\mathcal{M}\right)\chi_R\right]\mathbf{1}_{|v|\geq R}\right\|_\beta^2 \lesssim \int_{|v'|>R} |v'|^{2\alpha-4s}\lfloor v'\rceil^{\beta-d-\alpha}\,\mathrm{d}v' \lesssim R^{-\alpha-\beta},$$

since $\beta=2s-\alpha$. This concludes the proof.

8.3. **Proof of Hypothesis 4.** The adjoint of L is $L^*=\Delta_v^s-U\cdot\nabla_v$ and following exactly the same arguments as in the proof of Lemma 7.1 for the Fokker-Planck operator yields

Lemma 8.2. *The unique solution to the eigenvalue equation*

$$-L^*\phi_\eta-i\eta(v\cdot\sigma)\phi_\eta=\mu(\eta)\lfloor v\rceil^{-\beta}\phi_\eta \quad \textit{with} \quad \int_{\mathbb{R}^d}\phi_\eta(v)\lfloor v\rceil^{-\beta}\mathcal{M}(v)\,\mathrm{d}v=1$$

satisfies for any $R\geq 1$,

$$\|\phi_\eta\|_{L^\infty(B(0,R))}\lesssim_R 1 \quad \textit{and} \quad \|\operatorname{Im}\phi_\eta\|_{L^\infty(B(0,R))}\lesssim_R \max(\eta,\mu(\eta)),$$

with constants depending only on R and uniform in $\eta\to 0$.

We now come to the pointwise estimates on the rescaled eigenvector. This is when $\alpha\leq 2+\beta$, that is $\alpha\leq 1+s$. Observe indeed that when $\alpha>1+s$, the scaling is diffusive, and the diffusion coefficient is obtained by solving

$$\Delta_v^s(\mathcal{M}F)+\nabla_v\cdot(U\,\mathcal{M}F)=-(v\cdot\sigma)\mathcal{M}(v),$$

with $\int_{\mathbb{R}^d}F(v)\,\mathcal{M}_\beta(v)\,\mathrm{d}v=0$.

Lemma 8.3. *Assume that $s\in(\frac12,1)$. There is $\eta_1\in(0,\eta_0)$ small enough and A and C large enough so that*

$$\forall\,\eta\in(0,\eta_1),\ \forall\,u\in\mathbb{R}^d,\quad |\Phi_\eta(u)|\lesssim|u|_\eta^{C\mu(\eta)} \quad \textit{and} \quad |\operatorname{Im}\Phi_\eta(u)|\lesssim|u|_\eta^{\min(1,\alpha)+\beta-C\mu(\eta)}.$$

Proof. The rescaled equation for Φ_η is, using $U_\eta(u):=\eta^{\frac{1-\beta}{1+\beta}}U(u\eta^{-\frac{1}{1+\beta}})$ and since $2s-\beta=\alpha$,

$$-\eta^{\frac{\alpha}{1+\beta}}\Delta_u^s\Phi_\eta+U_\eta(u)\cdot\nabla_u\Phi_\eta-i(u\cdot\sigma)\Phi_\eta=\mu(\eta)|u|_\eta^{-\beta}\Phi_\eta.$$

Multiply the latter equation by $\frac{\Phi_\eta}{|\Phi_\eta|}$ and take the real part:

$$-\eta^{\frac{\alpha}{1+\beta}}\operatorname{Re}\left(\frac{\Phi_\eta}{|\Phi_\eta|}\Delta_u^s\Phi_\eta\right)+U_\eta(u)\cdot\operatorname{Re}\left(\frac{\Phi_\eta}{|\Phi_\eta|}\nabla_u\Phi_\eta\right)=\mu(\eta)|u|_\eta^{-\beta}|\Phi_\eta|.$$

Using the classical Kato inequality $\Delta_u^s|\Phi_\eta|\geq\operatorname{Re}\left(\frac{\Phi_\eta}{|\Phi_\eta|}\Delta_u^s\Phi_\eta\right)$ (see [14] for the Laplacian and [19] for the fractional Laplacian), one gets

$$-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^s|\Phi_\eta|+|u|_\eta^\beta U_\eta(u)\cdot\nabla_u|\Phi_\eta|-\mu(\eta)|\Phi_\eta|\leq 0.$$

Then observe that the real function $F(u)=|u|_\eta^{C\mu(\eta)}$ satisfies for $|u|\geq A\eta^{\frac13}$:

$$\begin{aligned}
&-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^sF+|u|_\eta^\beta U_\eta(u)\cdot\nabla_uF-\mu(\eta)F\\
&=-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^sF+C\mu(\eta)|u|_\eta^{C\mu(\eta)-2}\mathbb{U}_\eta(u)\,|u|^2-\mu(\eta)|u|_\eta^{C\mu(\eta)}
\end{aligned}$$

where we have used that $U_\eta(u)=|u|_\eta^{-\beta}\mathbb{U}_\eta(u)u$ with some $\mathbb{U}_\eta$ positive bounded from below (independently of η). We now estimate $\Delta_u^s\left(|\cdot|_\eta^{C\mu(\eta)}\right)(u)$. By scaling:

$$\forall\,u\in\mathbb{R}^d,\quad \Delta_u^s\left(|\cdot|_\eta^{C\mu(\eta)}\right)(u)=\eta^{\frac{C\mu(\eta)-2s}{1+\beta}}\Delta_v^s\left(\lfloor\cdot\rceil^{C\mu(\eta)}\right)\left(u\eta^{-\frac{1}{1+\beta}}\right).$$

We then estimate $\Delta_v^s\left(\lfloor\cdot\rceil^{C\mu(\eta)}\right)$ using

$$\begin{aligned}
\Delta_v^s\left(\lfloor\cdot\rceil^{C\mu(\eta)}\right)(v)&=C_{d,s}\int_{\mathbb{R}^d}\frac{\lfloor v'\rceil^{C\mu(\eta)}-\lfloor v\rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}}\,\mathrm{d}v',\\
&=C_{d,s}\int_{|v-v'|<\frac{|v|}{2}}\frac{\lfloor v'\rceil^{C\mu(\eta)}-\lfloor v\rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}}\,\mathrm{d}v'+C_{d,s}\int_{|v-v'|>\frac{|v|}{2}}\frac{\lfloor v'\rceil^{C\mu(\eta)}-\lfloor v\rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}}\,\mathrm{d}v'.
\end{aligned}$$

To control the first term in the right hand side, use that

$$\lfloor v' \rceil^{C\mu(\eta)} - \lfloor v \rceil^{C\mu(\eta)} - \nabla_v \left(\lfloor \cdot \rceil^{C\mu(\eta)} \right) \cdot (v' - v) \lesssim C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-2} |v' - v|^2$$

to get

$$\begin{aligned} \int_{|v-v'|<\frac{\lfloor v \rceil}{2}} \frac{\lfloor v' \rceil^{C\mu(\eta)} - \lfloor v \rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}} \, \mathrm{d}v' &\lesssim \int_{|v-v'|<\frac{\lfloor v \rceil}{2}} \frac{C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-2} |v'-v|^2}{|v'-v|^{d+2s}} \, \mathrm{d}v' \\ &\lesssim C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-2} \int_{|v-v'|<\frac{\lfloor v \rceil}{2}} \frac{|v'-v|^{2-2s}}{|v'-v|^d} \, \mathrm{d}v' \\ &\lesssim C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-2s}. \end{aligned}$$

To control the second term, use that

$$\lfloor v' \rceil^{C\mu(\eta)} - \lfloor v \rceil^{C\mu(\eta)} \lesssim C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-1} |v' - v|$$

to get (using here $s > \frac{1}{2}$)

$$\begin{aligned} \int_{|v-v'|>\frac{\lfloor v \rceil}{2}} \frac{\lfloor v' \rceil^{C\mu(\eta)} - \lfloor v \rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}} \, \mathrm{d}v' &\lesssim \int_{|v-v'|>\frac{\lfloor v \rceil}{2}} \frac{C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-1} |v'-v|}{|v'-v|^{d+2s}} \, \mathrm{d}v' \\ &\lesssim C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-1} \int_{|v-v'|>\frac{\lfloor v \rceil}{2}} \frac{\mathrm{d}v'}{|v'-v|^{d+2s-1}} \\ &\lesssim C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-2s}. \end{aligned}$$

We therefore have (using the scaling)

$$\Delta_v^s \left(\lfloor \cdot \rceil^{C\mu(\eta)} \right)(v) \lesssim C\mu(\eta) \lfloor v \rceil^{C\mu(\eta)-2s} \quad \Longrightarrow \quad \Delta_u^s \left(| \cdot |_\eta^{C\mu(\eta)} \right)(u) \lesssim C\mu(\eta) |u|_\eta^{C\mu(\eta)-2s}.$$

This estimate implies, for some absolute constant $C_0 > 0$,

$$\begin{aligned} \eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s F &\le C_0 C\mu(\eta) \eta^{\frac{\alpha}{1+\beta}} |u|_\eta^{C\mu(\eta)+\beta-2s} \\ &\le C_0 C\mu(\eta) |u|_\eta^{C\mu(\eta)} \eta^{\frac{\alpha}{1+\beta}} |u|_\eta^{-\alpha} \lesssim C_0 C\mu(\eta) |u|_\eta^{C\mu(\eta)} \left(1 + A^2\right)^{-\frac{\alpha}{2}} \end{aligned}$$

in the region $|u| \ge A\eta^{\frac{1}{1+\beta}}$. As a consequence

$$\begin{aligned} &- \eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s F + \mathbb{U}_\eta(u) u \cdot \nabla_u F - \mu(\eta) F \\ &= -\eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s F + C\mu(\eta) |u|_\eta^{C\mu(\eta)-2} \mathbb{U}_\eta(u) \, |u|^2 - \mu(\eta) |u|_\eta^{C\mu(\eta)} \\ &\ge -C_0 C\mu(\eta) |u|_\eta^{C\mu(\eta)} \left(1 + A^2\right)^{-\frac{\alpha}{2}} + C\mu(\eta) |u|_\eta^{C\mu(\eta)-2} \left(\inf \mathbb{U}_\eta\right) \, |u|^2 - \mu(\eta) |u|_\eta^{C\mu(\eta)} \\ &\ge C\mu(\eta) |u|_\eta^{C\mu(\eta)} \left[-C_0 \left(1 + A^2\right)^{-\frac{\alpha}{2}} + |u|_\eta^{-2} \left(\inf \mathbb{U}_\eta\right) \, |u|^2 - C^{-1} \right] \\ &\ge C\mu(\eta) |u|_\eta^{C\mu(\eta)} \left[-C_0 \left(1 + A^2\right)^{-\frac{\alpha}{2}} + \left(1 + A^{-2}\right)^{-1} \left(\inf \mathbb{U}_\eta\right) \, - C^{-1} \right] \ge 0 \end{aligned}$$

for A and C sufficiently large, and we deduce $|\Phi_\eta| \lesssim F$ on $|u| \ge A\eta^{\frac{1}{1+\beta}}$ and, for the same reasons as for the Fokker-Planck operator, the bound extends to any $u \in \mathbb{R}^d$.

Taking now the imaginary part of the equation, one gets

$$-\eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s |\operatorname{Im} \Phi_\eta| + \mathbb{U}_\eta(u) u \cdot \nabla_u |\operatorname{Im} \Phi_\eta| - \mu(\eta) |\operatorname{Im} \Phi_\eta| \lesssim |u|_\eta^{1+\beta+C\mu(\eta)}.$$

Define then $\gamma := \min(\alpha, 1) + \beta - C\mu(\eta) \in (0, 2s)$ and the real function $G(u) := |u|_\eta^\gamma$. Note that $\gamma \in (0, 2s)$ for η small enough, which implies that $\Delta_u^s G$ makes sense. Write for $|u| \ge A\eta^{\frac{1}{3}}$

$$-\eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s G + |u|_\eta^\beta U_\eta(u) \cdot \nabla_u G - \mu(\eta) G = -\eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s G + \gamma |u|_\eta^{\gamma-2} \mathbb{U}_\eta(u) \, |u|^2 - \mu(\eta) G$$

Let us now estimate $\Delta_u^s \left(| \cdot |_\eta^\gamma \right)(u)$. Note that by scaling

$$\forall \, u \in \mathbb{R}^d, \quad \Delta_u^s \left(|u|_\eta^\gamma \right)(u) = \eta^{\frac{\gamma-2s}{1+\beta}} \Delta_v^s \left(\lfloor \cdot \rceil^\gamma \right) (u \eta^{-\frac{1}{1+\beta}}).$$

One estimates $\Delta_v^s(\lfloor\cdot\rceil^\gamma)$ using

$$\Delta_v^s(\lfloor\cdot\rceil^\gamma)(v) = C_{d,s}\int_{\mathbb{R}^d}\frac{\lfloor v'\rceil^\gamma-\lfloor v\rceil^\gamma}{|v'-v|^{d+2s}}\,\mathrm{d}v',$$
$$= C_{d,s}\int_{|v-v'|<\frac{|v|}{2}}\frac{\lfloor v'\rceil^\gamma-\lfloor v\rceil^\gamma}{|v'-v|^{d+2s}}\,\mathrm{d}v' + C_{d,s}\int_{|v-v'|>\frac{|v|}{2}}\frac{\lfloor v'\rceil^\gamma-\lfloor v\rceil^\gamma}{|v'-v|^{d+2s}}\,\mathrm{d}v'.$$

Small v's are fine since $\Delta_v^s(\lfloor\cdot\rceil^\gamma)$ is locally bounded. Continue with large v. In the first integral,

$$\int_{|v-v'|<\frac{|v|}{2}}\frac{\lfloor v'\rceil^\gamma-\lfloor v\rceil^\gamma}{|v'-v|^{d+2s}}\,\mathrm{d}v' = \int_{|v-v'|<\frac{|v|}{2}}\frac{\lfloor v'\rceil^\gamma-\lfloor v\rceil^\gamma-\nabla_v(\lfloor\cdot\rceil^\gamma)(v)(v'-v)}{|v'-v|^{d+2s}}\,\mathrm{d}v'$$
$$\lesssim \int_{|v-v'|<\frac{|v|}{2}}\frac{\sup_{z\in B(v,\frac{|v|}{2})}\left|\nabla^2(\lfloor\cdot\rceil^\gamma)(z)\right||v'-v|^2}{|v'-v|^{d+2s}}\,\mathrm{d}v' \lesssim |v|^{\gamma-2s}.$$

The second integral may be estimated from above using that $|v-v'|>\frac{|v|}{2}$ implies $|v-v'|>\frac{|v'|}{3}$,

$$\int_{|v-v'|>\frac{|v|}{2}}\frac{\lfloor v'\rceil^\gamma-\lfloor v\rceil^\gamma}{|v'-v|^{d+2s}}\,\mathrm{d}v' \lesssim \int_{|v-v'|>\frac{|v|}{2}}\frac{\lfloor v-v'\rceil^\gamma}{|v'-v|^{d+2s}}\,\mathrm{d}v' \lesssim |v|^{\gamma-2s}.$$

From this, we deduce $\Delta_u^s(|\cdot|_\eta^\gamma)(u) \lesssim \eta^{\frac{\gamma-2s}{1+\beta}}\lfloor u\eta^{-\frac{1}{1+\beta}}\rceil^{\gamma-2s} = |u|_\eta^{\gamma-2s}$ which implies

$$\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^s G \lesssim \eta^{\frac{\alpha}{1+\beta}}|u|_\eta^{\gamma+\beta-2s} \lesssim \left(1+A^2\right)^{-\frac{\alpha}{2}}|u|_\eta^\gamma$$

in the region $|u|\geq A\eta^{\frac{1}{1+\beta}}$. As a consequence, as previously,

$$-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^s G + |u|_\eta^\beta U_\eta(u)\cdot\nabla_u G - \mu(\eta)G \gtrsim |u|_\eta^\gamma$$

for A sufficiently large and we deduce $|\operatorname{Im}\Phi_\eta|\lesssim G$ on $|u|\geq A\eta^{\frac{1}{1+\beta}}$ and, for the same reasons as for the Fokker-Planck operator, the bound extends to any $u\in\mathbb{R}^d$. □

8.4. **Rescaled drift force and limit equation.** We formally discuss the behaviour of the force U_η when η goes to 0. First write the rescaled equation: setting $v = u\eta^{-\frac{1}{1+\beta}}$ gives

$$\eta^{\frac{a-\beta}{1+\beta}}\Delta_v^s\mathcal{M}_\eta + \nabla_v\cdot(U_\eta\,\mathcal{M}_\eta) = 0.$$

Observe that when $u\neq 0$,

$$\eta^{\frac{\alpha}{1+\beta}}\Delta_v^s\mathcal{M}_\eta(u) = -c_{\alpha,\beta}C_{d,s}\eta^{\frac{\alpha}{1+\beta}}\int_{\mathbb{R}^d}\frac{|u|_\eta^{-d-\alpha}-|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}}\,\mathrm{d}u'$$
$$= -c_{\alpha,\beta}C_{d,s}\eta^{\frac{\alpha}{1+\beta}}\int_{B_\varepsilon}\frac{|u|_\eta^{-d-\alpha}-|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}}\,\mathrm{d}u' - c_{\alpha,\beta}C_{d,s}\eta^{\frac{\alpha}{1+\beta}}\int_{B_\varepsilon^c}\frac{|u|_\eta^{-d-\alpha}-|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}}\,\mathrm{d}u'.$$

The second term in the right hand side goes to zero as $\eta\to 0$ since the singularity around zero has been removed from the integration domain. To deal with the first term, decompose

$$\eta^{\frac{\alpha}{1+\beta}}\int_{B_\varepsilon}\frac{|u|_\eta^{-d-\alpha}-|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}}\,\mathrm{d}u' = \eta^{\frac{\alpha}{1+\beta}}\int_{B_\varepsilon}\frac{|u|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}}\,\mathrm{d}u' - \eta^{\frac{\alpha}{1+\beta}}\int_{B_\varepsilon}\frac{|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}}\,\mathrm{d}u'.$$

The first part goes to zero if $\varepsilon<|u|$. The second part writes

$$\eta^{\frac{\alpha}{1+\beta}}\int_{B_\varepsilon}\frac{|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}}\,\mathrm{d}u' \sim_\varepsilon |u|^{-d-2s}\eta^{\frac{\alpha}{1+\beta}}\int_{B_\varepsilon}|u'|_\eta^{-d-\alpha}\,\mathrm{d}u'$$
$$= |u|^{-d-2s}\int_{B\left(0,\varepsilon\eta^{-\frac{\alpha}{1+\beta}}\right)}(1+|v|^2)^{-d-\alpha}\,\mathrm{d}v.$$

Taking η small then ε arbitrarily small yields

$$\lim_{\eta\to 0}\left(-c_{\alpha,\beta}\eta^{\frac{\alpha}{1+\beta}}\int_{B_\varepsilon}\frac{|u|_\eta^{-d-\alpha}-|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}}\,\mathrm{d}u'\right) = \frac{c_{\alpha,\beta}}{c_{\alpha,0}}\cdot\frac{1}{|u|^{d+2s}}.$$

Since $\nabla_v(|v|^{-d-2s}v) = -2s|v|^{-d-2s}$, we deduce that

$$\lim_{\eta\to 0} \eta^{\frac{1-\beta}{1+\beta}} U\left(u\eta^{-\frac{1}{1+\beta}}\right) = U_\infty(u) = \frac{c_{\alpha,\beta}}{2sc_{\alpha,0}} \frac{u}{|u|^{d+2s}} |u|^{d+\alpha} = \frac{c_{\alpha,\beta}}{2sc_{\alpha,0}} |u|^{-\beta} u.$$

This proves the scaling limit of the drift force.

From the rescaled equation for Φ_η, we deduce that Φ_η goes to Φ, where Φ solves,

$$\frac{c_{\alpha,\beta}}{2asc_{\alpha,0}} \frac{u}{|u|^\beta} \cdot \nabla_u \Phi - i(u\cdot\sigma)\Phi = 0 \quad \text{with} \quad \Phi(0) = 1 \quad \Longrightarrow \quad \Phi(u) := \exp\left(i \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u\cdot\sigma)}{1+\beta}\right).$$

Thus, $\Omega(u) = \lim_{\lambda\to 0,\ \lambda\neq 0} \frac{\operatorname{Im}\Phi(\lambda u)}{\lambda^{1+\beta}}$ satisfies,

$$\frac{c_{\alpha,\beta}}{2ac_{\alpha,0}} u\cdot\nabla_u\Omega = (u\cdot\sigma)|u|^\beta \quad \Longrightarrow \quad \Omega(u) := \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u\cdot\sigma)}{1+\beta}.$$

8.5. **The particular case $\alpha = 2s$.** Explicit calculations are available when $\alpha = 2s$. In this case $\beta = 0$, $U(v) = c_0 v$ for some constant $c_0 > 0$, and the eigenproblem is

$$-\Delta_v^s \phi + c_0 v\cdot\nabla_v\phi - i\eta(v\cdot\sigma)\phi = \mu(\eta)\phi.$$

Taking the Fourier transform (in the dual of the Schwarz space) gives

$$-|\xi|^{2s}\hat\phi - c_0\xi\cdot\nabla_\xi\hat\phi + \eta\sigma\cdot\nabla_\xi\hat\phi = (\mu(\eta) + c_0)\hat\phi$$

or equivalently

$$(\eta\sigma - c_0\xi)\cdot\nabla_\xi\hat\phi = \left(\mu(\eta) + c_0 + |\xi|^{2s}\right)\hat\phi.$$

The solution to this equation is given by $\hat\phi = \delta_{c_0^{-1}\eta\sigma}$ and $\mu(\eta) = -|c_0^{-1}\eta\sigma|^{2s} = c_0^{-2s}\eta^{2s}$, which yields by inverse Fourier transform $\phi_\eta(v) := \exp\left(ic_0^{-1}\eta(v\cdot\sigma)\right)$. This agrees with the expression of Φ given above, and allows to compute $c_0 = \frac{1}{2s}$.

9. Remarks and extensions

In Hypothesis 1, the equilibrium $\mathcal{M}$ is an explicit power law, and in particular is centered and even. We discuss in this section the changes required for our proofs to deal with more general $\mathcal{M}$ that are (i) characterised by *asymptotic* power-law estimates rather than exact formula, and (ii) non necessarily centered or even. This means replacing Hypothesis 1 with:

Hypothesis 1' (Equilibria)**.** *The equilibrium distribution satisfies*

$$\mathcal{M} = \lfloor\cdot\rceil^{-(d+\alpha)}\mathcal{S}(v), \tag{9.1}$$

where $\mathcal{S}$ is a slowly varying *function, $\lfloor\cdot\rceil := (1+|\cdot|^2)^{1/2}$, with the* ***generalised mass condition*** (1.8).

Slowly varying functions are non-vanishing measurable functions that satisfy $\mathcal{S}(ax) \sim \mathcal{S}(x)$ as x goes to infinity, for any $a > 0$. Examples of slowly varying functions are positive constants, functions that converge to positive constants, logarithms and iterated logarithms.

9.1. **Equilibria characterised only asymptotically.** If one considers an *even* equilibrium $\mathcal{M}$ that satisfies Hypothesis 1', the proof of Theorem 1.4 in Section 2 and the proof of Lemma 1.1 in Section 3 are essentially unchanged. The formulas for μ_0 and κ in Lemmas 1.2 and 1.3 are slightly modified, and rely on the existence of a scaling limit of $\eta^{-\frac{d+\alpha}{1+\beta}}\mathcal{M}(u\eta^{-\frac{1}{1+\beta}})$ as $\eta\to 0$, which follows from Hypothesis 1'. Everything else remains unchanged and the structures of the proofs in Sections 4 and 5 are the same. Rates of convergence will depend on the form of $\mathcal{S}$.

9.2. **More general velocity fields.** One could replace the transport operator $v \cdot \nabla_x$ by a more general $a(v) \cdot \nabla_x$, where a is odd. All our results and proofs can be extended, even though the scalings found may be changed since the scaling of $\ell(R)$ in (3.7) will be different. If $a(v)$ scales like $|v|^\delta$, redoing the computations as in Section 2 and Section 3 then one would find

$$\Theta(\eta) := \begin{cases} \eta^2 & \text{when } \alpha > 2\delta + \beta, \\ \eta^2 |\ln(\eta)| & \text{when } \alpha = 2\delta + \beta, \\ \eta^{\frac{\alpha+\beta}{\delta+\beta}} & \text{when } -\beta < \alpha < 2\delta + \beta. \end{cases}$$

An example is given by relativistic particles, for which $a(v) := c\frac{v}{\sqrt{c^2+v^2}}$, where c is the speed of light. Such transport operators are relevant to special relativity, see for instance [44] in physics and [28] in mathematics. There, $\delta = 0$, so that

$$\Theta(\eta) := \begin{cases} \eta^2 & \text{when } \alpha > \beta, \\ \eta^2 |\ln(\eta)| & \text{when } \alpha = \beta, \\ \eta^{1+\frac{\alpha}{\beta}} & \text{when } -\beta < \alpha < \beta. \end{cases}$$

9.3. **Non-centered equilibria.** When the microscopic equilibrium $\mathcal{M}(v)$ is not centered, it results in a drift in the macroscopic equation. Our approach however allows to tackle such a situation, with the following changes depending on whether this macroscopic drift is of higher, comparable or smaller order than the resulting (fractional) macroscopic diffusion. In view of Theorem 1.4 in the centered situation, we expect a macroscopic diffusion of order $\zeta(\alpha, \beta) = \min\left(2, \frac{\alpha_+ + \beta}{1+\beta}\right)$, and therefore we expect the drift to be dominant when $\alpha > 1$ and dominated when $\alpha < 1$, with a borderline case at $\alpha = 1$. Observe that $\alpha = 1$ is also the threshold for the absolute convergence of the integral $\int_{\mathbb{R}^d} (v \cdot \sigma)\mathcal{M}(v)\, \mathrm{d}v$ defining the macroscopic drift.

Consider a solution f in $L^\infty([0, +\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ to equation (1.1) and denote

$$f_\varepsilon(t, x, v) := f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon} + \frac{\bar{v}_\varepsilon t}{\theta(\varepsilon)}, v\right) \in L^\infty_t([0, +\infty); L^2_{x,v}(\mathcal{M}^{-1}))$$

where $\varepsilon > 0$ and $\theta(\varepsilon)$ is defined in (1.15), and where the *velocity corrector* $\bar{v}_\varepsilon$ is defined by

$$\bar{v}_\varepsilon := \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} v\mathcal{M}(v)\, \mathrm{d}v}{\displaystyle\int_{\mathbb{R}^d} \mathcal{M}(v)\, \mathrm{d}v} & \text{when } \alpha > 1, \\ \left(\lim_{R\to\infty} \dfrac{1}{\ln(R)} \dfrac{\displaystyle\int_{\mathbb{R}^d} v\chi_R(v)\mathcal{M}(v)\, \mathrm{d}v}{\displaystyle\int_{\mathbb{R}^d} \chi_R(v)\mathcal{M}(v)\, \mathrm{d}v}\right) \dfrac{|\ln(\varepsilon)|}{1+\beta} & \text{when } \alpha = 1, \\ 0 & \text{when } \alpha \in (-\beta, 1). \end{cases} \tag{9.2}$$

The equation satisfied by f_ε is

$$\theta(\varepsilon)\partial_t f_\varepsilon + \varepsilon(v - \bar{v}_\varepsilon) \cdot \nabla_x f_\varepsilon = \mathcal{L} f_\varepsilon. \tag{9.3}$$

With this definition of f_ε, Theorem 1.4 holds and yields the (fractional) diffusive limit of f_ε. The changes in the proofs are as follows. The arguments presented in Section 2 are essentially unchanged with a few modifications to obtain the scaling of the eigenvalue resulting from (9.3). We chose $\bar{v}_\varepsilon$ in such a way that the dominant eigenmode has the scaling obtained in Lemmas 1.2 and 1.3. The new spectral problem to be considered in the modified Lemma 1.1 is

$$-L^*\phi_\eta - i\eta\left[(v - \bar{v}_\varepsilon) \cdot \sigma\right]\phi_\eta = \mu(\eta)\lfloor v \rceil^{-\beta}\phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v)\, \mathcal{M}_\beta(v)\, \mathrm{d}v = 1.$$

Line-by-line technical modifications are needed in the proof of Lemmas 1.2 and 1.3 due to the additional drift but the procedure and method are preserved and we do not repeat the arguments. Let us just explain why we define the correction velocity $\bar{v}_\varepsilon$ in this way. The spectral projector estimate follows the same procedure, but (3.13) is replaced with

$$-L^*F - i\eta\left[(v-\bar{v}_\varepsilon)\cdot\sigma\right]F - z\lfloor v\rceil^{-\beta}F = (v-\bar{v}_\varepsilon)\cdot\sigma. \tag{9.4}$$

The L^2 estimate is unchanged and the crucial estimate (3.18) remains true as long as

$$q(R) := \int_{\mathbb{R}^d}\left[v-\bar{v}_\varepsilon\right]\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v \quad \text{at} \quad R := \eta^{-\frac{1}{1+\beta}}$$

is small compared with $\mathrm{R}_1\eta^{-1}\Theta(\eta)$, when R_1 is large enough. This implies that the influence of the drift is smaller than the size of the fluid mode, which is of order $\eta^{-1}\Theta(\eta)$. Recall that

$$\frac{\Theta(\eta)}{\eta} := \begin{cases} \eta & \text{when } \alpha > 2+\beta, \\ \eta|\ln(\eta)| & \text{when } \alpha = 2+\beta, \\ \eta^{\frac{\alpha-1}{1+\beta}} & \text{when } -\beta < \alpha < 2+\beta \end{cases} \tag{9.5}$$

One can then prove that for all $\alpha > -\beta$, one has $q(\eta^{-\frac{1}{1+\beta}}) \lesssim \eta^{\frac{\alpha-1}{1+\beta}}$, which proves that $q(\eta^{-\frac{1}{1+\beta}})$ is small compared with $\mathrm{R}_1\eta^{-1}\Theta(\eta)$ when R_1 is large enough.

9.4. **Kinetic Fokker-Planck equation with non gradient confining force.** All the results we obtain for the Fokker-Planck equation with gradient force can be extended to Fokker-Planck operators with non-gradient confining force at little expense. We chose not to present this more general setting in the core of the paper to stay consistent with the clean and simple Hypothesis 1 and to help with readability. It is however possible to consider

$$\mathcal{L}(f) = \Delta_v f + \nabla_v\cdot(U\,f) \qquad \text{where } U \text{ satisfies} \qquad \Delta_v\mathcal{M} + \nabla_v\cdot(U\,\mathcal{M}) = 0,$$

provided that quantitative bounds are available on U to ensure it is comparable to the drift in the Fokker-Planck operator. The analysis is then similar.

9.5. **About the limitation $\alpha > -1$ in the Fokker-Planck case in dimension 1.** The scaling av with $a \to 0$ shows that $d+\alpha > 0$ is necessary for the Hardy-Poincaré inequality to hold since otherwise it would imply the inequality $\|\frac{f(v)}{v}\|_{L^2(\mathbb{R})} \lesssim \|f'\|_{L^2(\mathbb{R})}$ which is false. This restriction is implied by our condition $\alpha+\beta > 0$ with $\beta = 2$ in dimension $d \geq 2$, but further restricts $\alpha > -1$ in dimension $d = 1$. The borderline case $\alpha = -1$ in dimension $d = 1$ corresponds to $d+\alpha = 0$ and a constant $\mathcal{M}$ and the Kolmogorov equation $\partial_t f + v\cdot\nabla_x f = \Delta_v f$. Given initial data $f_{\mathrm{in}}(x,v) := \rho_{\mathrm{in}}(x)\mathcal{M}(v) = \rho_{\mathrm{in}}(x)$ the solution computed in [31] is

$$\begin{cases} f(t,x,v) = \displaystyle\int_{\mathbb{R}^2} G(t, x-x'-tv', v-v')\rho_{\mathrm{in}}(x')\,\mathrm{d}x'\,\mathrm{d}v' \\ \text{with} \quad G(t,x,v) := \dfrac{\sqrt{3}}{2\pi t^2}\exp\left(-\dfrac{3}{t^3}\left|x-\dfrac{t}{2}v\right|^2 - \dfrac{|v|^2}{4t}\right). \end{cases}$$

After integrating in v against $\lfloor v\rceil^{-2}$ and rescaling, one gets

$$r_\varepsilon(t,x) = \int_{\mathbb{R}^3} G\left(t, x-x'-tv', \varepsilon^{\frac{1}{3}}v-v'\right)\rho_{\mathrm{in}}\left(\frac{x'}{\varepsilon}\right)\lfloor v\rceil^{-2}\,\mathrm{d}x'\,\mathrm{d}v'\,\mathrm{d}v.$$

Assuming that $\rho_{\mathrm{in}}\left(\frac{x}{\varepsilon}\right) \sim \tilde{\rho}_{\mathrm{in}}(x)$ at initial time, we obtain the limit $r_\varepsilon \to r$ with

$$\begin{aligned} r(t,x) &= \int_{\mathbb{R}^3} G(t, x-x'-tv', -v')\tilde{\rho}_{\mathrm{in}}(x')\lfloor v\rceil^{-2}\,\mathrm{d}x'\,\mathrm{d}v'\,\mathrm{d}v \\ &= \mathrm{cst}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} G(t,x-x',v')\,\mathrm{d}v'\right)\tilde{\rho}_{\mathrm{in}}(x')\,\mathrm{d}x' = \mathrm{cst}\int_{x'}\frac{1}{t^{\frac{3}{2}}}\exp\left(-\frac{3}{4}\frac{(x-x')}{t^3}\right)\tilde{\rho}_{\mathrm{in}}(x')\,\mathrm{d}x' \end{aligned}$$

which does not solve the fractional heat equation, but the heat equation with a change of time.

References

[1] P. Aceves-Sanchez and L. Cesbron. Fractional diffusion limit for a fractional Vlasov-Fokker-Planck equation. *SIAM J. Math. Anal.*, 51(1):469–488, 2019.

[2] P. Aceves-Sanchez and A. Mellet. Anomalous diffusion limit for a linear Boltzmann equation with external force field. *Math. Models Methods Appl. Sci.*, 27(5):845–878, 2017.

[3] G. Ariel, A. Rabani, S. Benisty, J. D. Partridge, R. M. Harshey, and A. Be'er. Swarming bacteria migrate by Lévy Walk. *Nature Communications*, 6(1):8396, Sept. 2015.

[4] C. Bardos, F. Golse, and I. Moyano. Linear Boltzmann equation and fractional diffusion. *Kinet. Relat. Models*, 11(4):1011–1036, 2018.

[5] E. Barkai, E. Aghion, and D. A. Kessler. From the area under the bessel excursion to anomalous diffusion of cold atoms. *Phys. Rev. X*, 4:021036, May 2014.

[6] N. Ben Abdallah, A. Mellet, and M. Puel. Fractional diffusion limit for collisional kinetic equations: a Hilbert expansion approach. *Kinet. Relat. Models*, 4(4):873–900, 2011.

[7] G. Birkhoff and E. P. Wigner, editors. *Nuclear Reactor Theory*, volume 11 of *Proceedings of Symposia in Applied Mathematics*. American Mathematical Society, 1961.

[8] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.-L. Vázquez. Hardy-Poincaré inequalities and applications to nonlinear diffusions. *C. R. Math. Acad. Sci. Paris*, 344(7):431–436, 2007.

[9] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J. L. Vázquez. Asymptotics of the Fast Diffusion Equation via Entropy Estimates. *Archive for Rational Mechanics and Analysis*, 191(2):347–385, Feb. 2009.

[10] L. Boltzmann. Weitere Studien über das Warmegleichgewicht unter Gasmolekülen. *Sitzungsberichte Akad. Wiss., Vienna*, 66(275), 1872. English translation: L. Boltzmann, Further studies on the thermal equilibrium of gas molecules, in S. G. Brush Kinetic Theory Vol. 2, pp.88-175, Oxford 1966.

[11] C. Börgers, C. Greengard, and E. Thomann. The diffusion limit of free molecular flow in thin plane channels. *SIAM Journal on Applied Mathematics*, 52(4):1057–1075, 1992.

[12] F. Bouchut, F. Golse, and M. Pulvirenti. *Kinetic equations and asymptotic theory*, volume 4 of *Series in Applied Mathematics (Paris)*. Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris, 2000. Edited and with a foreword by Benoît Perthame and Laurent Desvillettes.

[13] E. Bouin, J. Dolbeault, L. Lafleche, and C. Schmeiser. Fractional Hypocoercivity. ArXiv:1911.11020, 2019.

[14] H. Brezis and A. C. Ponce. Kato's inequality when Δu is a measure. *C. R. Math. Acad. Sci. Paris*, 338(8):599–604, 2004.

[15] P. Cattiaux, E. Nasreddine, and M. Puel. Diffusion limit for kinetic Fokker-Planck equation with heavy tails equilibria: the critical case. *Kinet. Relat. Models*, 12(4):727–748, 2019.

[16] L. Cesbron. Anomalous diffusion limit of kinetic equations in spatially bounded domains. *Comm. Math. Phys.*, 364(1):233–286, 2018.

[17] L. Cesbron, A. Mellet, and M. Puel. Fractional diffusion limit of a kinetic equation with diffusive boundary conditions in the upper-half space. *Arch. Ration. Mech. Anal.*, 235(2):1245–1288, 2020.

[18] L. Cesbron, A. Mellet, and K. Trivisa. Anomalous transport of particles in plasma physics. *Applied Mathematics Letters*, 25(12):2344–2348, 2012.

[19] H. Chen and L. Véron. Semilinear fractional elliptic equations with gradient nonlinearity involving measures. *J. Funct. Anal.*, 266(8):5467–5492, 2014.

[20] P. Degond, T. Goudon, and F. Poupaud. Diffusion limit for nonhomogeneous and non-micro-reversible processes. *Indiana Univ. Math. J.*, 49(3):1175–1198, 2000.

[21] C. Dogbé. Anomalous diffusion limit induced on a kinetic equation. *J. Statist. Phys.*, 100(3-4):603–632, 2000.

[22] J. Dolbeault and G. Toscani. Fast diffusion equations: Matching large time asymptotics by relative entropy methods. *Kinetic & Related Models*, 4(3):701–716, 2011.

[23] R. S. Ellis and M. A. Pinsky. The first and second fluid approximations to the linearized Boltzmann equation. *J. Math. Pures Appl. (9)*, 54:125–156, 1975.

[24] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[25] N. Fournier and C. Tardif. Anomalous diffusion for multi-dimensional critical kinetic Fokker–Planck equations. *Ann. Probab.*, 48(5):2359–2403, 2020.

[26] N. Fournier and C. Tardif. One dimensional critical kinetic Fokker-Planck equations, Bessel and stable processes. *Comm. Math. Phys.*, 381(1):143–173, 2021.

[27] U. Frisch and H. Frisch. Non-LTE transfer–III. Asymptotic expansion for small ε. *Monthly Notices of the Royal Astronomical Society*, 181(2):273–280, 1977.

[28] R. T. Glassey. *The Cauchy Problem in Kinetic Theory.* Society for Industrial and Applied Mathematics, 1996.

[29] F. Golse. Anomalous diffusion limit for the Knudsen gas. *Asymptot. Anal.*, 17(1):1–12, 1998.

[30] S. Hittmeir and S. Merino-Aceituno. Kinetic derivation of fractional Stokes and Stokes-Fourier systems. *Kinet. Relat. Models*, 9(1):105–129, 2016.

[31] A. Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)*, 35(1):116–117, 1934.

[32] G. Lebeau and M. Puel. Diffusion approximation for Fokker Planck with heavy tail equilibria: a spectral method in dimension 1. *Comm. Math. Phys.*, 366(2):709–735, 2019.
[33] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser. *Semiconductor equations*. Springer-Verlag, Vienna, 1990.
[34] S. Marksteiner, K. Ellinger, and P. Zoller. Anomalous diffusion and Lévy walks in optical lattices. *Phys. Rev. A*, 53:3409–3430, May 1996.
[35] J. C. Maxwell. On the process of diffusion of two or more kinds of moving particles among one another. *Philosophical Magazine*, 20:21–38, 1860.
[36] J. C. Maxwell. On the dynamical theory of gases. *Philosophical transactions of the Royal Society of London*, 157:49–88, 1867.
[37] A. Mellet. Fractional diffusion limit for collisional kinetic equations: A moments method. *Indiana Univ. Math. J.*, 59:1333–1360, 2010.
[38] A. Mellet, S. Mischler, and C. Mouhot. Fractional diffusion limit for collisional kinetic equations. *Archive for Rational Mechanics and Analysis*, 199(2):493–525, 2011.
[39] E. Nasreddine and M. Puel. Diffusion limit of Fokker-Planck equation with heavy tail equilibria. *ESAIM Math. Model. Numer. Anal.*, 49(1):1–17, 2015.
[40] B. Nicolaenko. Dispersion Laws for plane wave propagation. In A. Grünbaum, editor, *The Boltzmann Equation*. Courant Institute, 1971.
[41] B. Perthame, W. Sun, and M. Tang. The fractional diffusion limit of a kinetic model with biochemical pathway. *Zeitschrift für angewandte Mathematik und Physik*, 69(3):67, May 2018.
[42] G. C. Pomraning. *The Equations of Radiation Hydrodynamics*. Pergamon Press, 1973.
[43] Y. Sagi, M. Brook, I. Almog, and N. Davidson. Observation of anomalous diffusion and fractional self-similarity in one dimension. *Phys. Rev. Lett.*, 108:093002, Mar 2012.
[44] J. M. Stewart. *Non-equilibrium relativistic kinetic theory [by] John M. Stewart*. Springer-Verlag Berlin, New York, 1971.
[45] Y. Tu and G. Grinstein. How white noise generates power-law switching in bacterial flagellar motors. *Phys. Rev. Lett.*, 94:208101, May 2005.
[46] F.-Y. Wang and J. Wang. Functional inequalities for stable-like Dirichlet forms. *J. Theoret. Probab.*, 28(2):423–448, 2015.
[47] A. M. Weinberg and E. P. Wigner. *The physical theory of neutron chain reactors*. The University of Chicago Press, Chicago, Ill., 1958.
[48] E. P. Wigner. Mathematical problems of nuclear reactor theory. In *Proc. Sympos. Appl. Math., Vol. XI*, pages 89–104. American Mathematical Society, Providence, R.I., 1961.