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ON REGULARIZABLE BIRATIONAL MAPS

JULIE DÉSERTI

ABSTRACT. BEDFORD asked if there exists a birational self map f of the complex projective plane such that for any automorphism A of the complex projective plane $A\circ f$ is not conjugate to an automorphism. In [3] BLANC gave such a f of degree 6 and asked if there exists an example of smaller degree. In this article we give an example of degree 5.

1. INTRODUCTION

Denote by $\mathrm{Bir}(\mathbb{P}^k_\mathbb{C})$ the group of all birational self maps of $\mathbb{P}^k_\mathbb{C}$, also called the *k-dimensional* CREMONA *group*. Let $\mathrm{Bir}_d(\mathbb{P}^k_\mathbb{C})$ be the algebraic variety of all birational self maps of $\mathbb{P}^k_\mathbb{C}$ of degree d. When $k=2$ and $d\geq 2$ these varieties have many distinct components, of various dimensions ([6, 2]). The group $\mathrm{Aut}(\mathbb{P}^k_\mathbb{C})=\mathrm{PGL}(k+1,\mathbb{C})$ acts by left translations, by right translations, and by conjugacy on $\mathrm{Bir}_d(\mathbb{P}^k_\mathbb{C})$. Since this group is connected, these actions preserve each connected component.

A birational map $f\colon\mathbb{P}^k_\mathbb{C}\dashrightarrow\mathbb{P}^k_\mathbb{C}$ is *regularizable* if there there exist a smooth projective variety V and a birational map $g\colon V\dashrightarrow\mathbb{P}^k_\mathbb{C}$ such that $g^{-1}\circ f\circ g$ is an automorphism of V. To any element f of $\mathrm{Bir}(\mathbb{P}^k_\mathbb{C})$ we associate the set $\mathrm{Reg}(f)$ defined by

$$\mathrm{Reg}(f):=\big\{A\in\mathrm{Aut}(\mathbb{P}^k_\mathbb{C})\mid A\circ f\ \text{ is regularizable}\big\}.$$

DOLGACHEV asked whether there exists a birational self map of $\mathbb{P}^k_\mathbb{C}$ of degree >1 such that $\mathrm{Reg}(f)=\mathrm{Aut}(\mathbb{P}^k_\mathbb{C})$. In [5] we give a negative answer to this question. More precisely we prove

Theorem 1.1 ([5])**.** *Let f be a birational self map of $\mathbb{P}^k_\mathbb{C}$ of degree $d\geq 2$.*

The set of automorphisms A of $\mathbb{P}^k_\mathbb{C}$ such that $\deg\big((A\circ f)^n\big)\neq\big(\deg(A\circ f)\big)^n$ *for some $n>0$ is a countable union of proper* ZARISKI *closed subsets of* $\mathrm{PGL}(k+1,\mathbb{C})$.

In particular there exists an automorphism A of $\mathbb{P}^k_\mathbb{C}$ such that $A\circ f$ is not regularizable.

BEDFORD asked: does there exist a birational map f of $\mathbb{P}^k_\mathbb{C}$ such that $\mathrm{Reg}(f)=\emptyset$? We will focus on the case $k=2$. According to [1, 7] if $\deg f=2$, then $\mathrm{Reg}(f)\neq\emptyset$. What about birational

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maps of degree 3 ? BLANC proves that the set

$$\big\{f\in \mathrm{Bir}_3(\mathbb{P}^2_{\mathbb{C}})\,|\,\mathrm{Reg}(f)\neq\emptyset,\ \lim_{n\to+\infty}(\deg(f^n))^{1/n}>1\big\}$$

is dense in $\mathrm{Bir}_3(\mathbb{P}^2_{\mathbb{C}})$ and that its complement has codimension 1 (*see* [3]). BLANC also gives a positive answer to BEDFORD question in dimension 2: if $\chi\colon\mathbb{P}^2_{\mathbb{C}}\dashrightarrow\mathbb{P}^2_{\mathbb{C}}$ is the birational map given by

$$\chi\colon(x:y:z)\dashrightarrow\big(xz^5+(yz^2+x^3)^2:yz^5+x^3z^3:z^6\big)$$

then $\mathrm{Reg}(\chi)=\emptyset$.

Remark 1.2. Note that $\chi=(x+y^2,y)\circ(x,y+x^3)$ in the affine chart $z=1$. Indeed BLANC example can be generalized as follows: the birational map given in the affine chart $z=1$ by

$$\chi_{n,p}=(x+y^n,y)\circ(x,y+x^p)=\big(x+(y+x^p)^n,y+x^p\big)$$

satisfies $\mathrm{Reg}(\chi_{n,p})=\emptyset$ (*see* §3).

Then BLANC asked: does there exist $f\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ such that deg <6 and $\mathrm{Reg}(f)=\emptyset$? The following statement gives a positive answer to this question:

Theorem A. *If* $\varphi\colon\mathbb{P}^2_{\mathbb{C}}\dashrightarrow\mathbb{P}^2_{\mathbb{C}}$ *is the birational map given by*

$$\varphi\colon(x:y:z)\dashrightarrow\big(y^3(z^2-xy):z^5-y^5-xyz^3:y^2z(z^2-xy)\big),$$

then $\mathrm{Reg}(\varphi)=\emptyset$.

Acknowledgements. I would like to thank Serge CANTAT for many interesting discussions.

2. PROOF OF THEOREM A

Let S be a smooth projective surface. Let $\phi\colon S\dashrightarrow S$ be a birational map. This map admits a resolution

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S \arrow[rr, dashed, "\phi"'] & & S \end{tikzcd}$$

where $\pi_1\colon Z\to S$ and $\pi_2\colon Z\to S$ are finite sequences of blow-ups. The resolution is *minimal* if and only if no (-1)-curve of Z is contracted by both π_1 and π_2. The *base-points* $\mathrm{Base}(\phi)$ of ϕ are the points blown-up by π_1, which can be points of S or infinitely near points. The proper base-points of ϕ are called *indeterminacy points* of ϕ and form a set denoted $\mathrm{Ind}(\phi)$. Finally we denote by $\mathrm{Exc}(\phi)$ the set of curves contracted by ϕ.

Denote by $\mathfrak{b}(\phi)$ the number of base-points of ϕ; note that $\mathfrak{b}(\phi)$ is equal to the difference of the ranks of $\mathrm{Pic}(Z)$ and $\mathrm{Pic}(S)$ and thus equal to $\mathfrak{b}(\phi^{-1})$. Let us introduce the *dynamical number of the base-points of* ϕ. Since $\mathfrak{b}(\phi\circ\psi)\leq\mathfrak{b}(\phi)+\mathfrak{b}(\psi)$ for any birational self map ψ of S, $\mu(\phi)$ is a non-negative real number. As $\mathfrak{b}(\phi)=\mathfrak{b}(\phi^{-1})$ one gets $\mu(\phi^k)=|k\mu(\phi)|$ for any $k\in\mathbb{Z}$. Furthermore if Z is a smooth projective surface and $\psi\colon S\dashrightarrow Z$ a birational map, then for all $n\in\mathbb{Z}$

$$-2\mathfrak{b}(\psi)+\mathfrak{b}(\phi^n)\leq\mathfrak{b}(\psi\circ\varphi^n\circ\psi^{-1})\leq 2\mathfrak{b}(\psi)+\mathfrak{b}(\phi^n);$$

hence $\mu(\phi)=\mu(\psi\circ\phi\circ\psi^{-1})$. One can thus state the following result:

Lemma 2.1 ([4]). *The dynamical number of base-points is an invariant of conjugation. In particular if ϕ is a regularizable birational self map of a smooth projective surface, then $\mu(\phi)=0$.*

A base-point p of ϕ is a *persistent base-point* if there exists an integer N such that for any $k\geq N$

$$\begin{cases} p\in\mathrm{Base}(\phi^k)\\ p\not\in\mathrm{Base}(\phi^{-k})\end{cases}$$

Let p be a point of S or a point infinitely near S such that $p\not\in\mathrm{Base}(\phi)$. Consider a minimal resolution of ϕ

$$\begin{tikzcd} & Z \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ S \arrow[rr, dashed, "\phi"'] & & S \end{tikzcd}$$

Because p is not a base-point of ϕ it corresponds via π_1 to a point of Z or infinitely near; using π_2 we view this point on S again maybe infinitely near and denote it $\phi^\bullet(p)$. For instance if $S=\mathbb{P}^2_\mathbb{C}$, $p=(1:0:0)$ and f is the birational self map of $\mathbb{P}^2_\mathbb{C}$ given by

$$(z_0:z_1:z_2)\dashrightarrow(z_1z_2+z_0^2:z_0z_2:z_2^2)$$

the point $f^\bullet(p)$ is not equal to $p=f(p)$ but is infinitely near to it. Note that if ψ is a birational self map of S and p is a point of S such that $p\not\in\mathrm{Base}(\phi)$, $\phi(p)\not\in\mathrm{Base}(\psi)$, then $(\psi\circ\phi)^\bullet(p)=\psi^\bullet(\phi^\bullet(p))$. One can put an equivalence relation on the set of points of S or infinitely near S: the point p is *equivalent* to the point q if there exists an integer k such that $(\phi^k)^\bullet(p)=q$; in particular $p\not\in\mathrm{Base}(\phi^k)$ and $q\not\in\mathrm{Base}(\phi^{-k})$. Remark that the equivalence class is the generalization of set of orbits for birational maps.

Let us give the relationship between the dynamical number of base-points and the equivalence classes of persistent base-points:

Proposition 2.2 ([4]). *Let S be a smooth projective surface. Let ϕ be a birational self map of S.*

Then $\mu(\phi)$ coincides with the number of equivalence classes of persistent base-points of ϕ. In particular $\mu(\phi)$ is an integer.

This interpretation of the dynamical number of base-points allows to prove the following result that gives a characterization of regularizable birational maps:

Theorem 2.3 ([4]). *Let ϕ be a birational self map of a smooth projective surface. Then ϕ is regularizable if and only if $\mu(\phi)=0$.*

The birational map

$$\varphi\colon (x:y:z) \dashrightarrow \big(y^3(z^2-xy) : z^5-y^5-xyz^3 : y^2z(z^2-xy)\big)$$

blows down the conic $\mathcal{C}$ given by $z^2-xy=0$ onto the point $p=(0:1:0)$ and the line L_y defined by $y=0$ onto p. Furthermore φ has only one point of indeterminacy which is $q=(1:0:0)=L_y\cap\mathcal{C}$. The inverse of φ is the map

$$\varphi^{-1}\colon (x:y:z) \dashrightarrow \big(x^2yz^2-z^5+x^5 : x^2(x^2y-z^3) : xz(x^2y-z^3)\big)$$

which blows down $\mathcal{C}'$ given by $x^2y-z^3=0$ onto q and the line L_x defined by $x=0$ onto q. Moreover $\mathrm{Ind}(\varphi^{-1})=\mathcal{C}'\cap L_x=\{p\}$.

If A is an automorphism of $\mathbb{P}^2_{\mathbb{C}}$ let us set $\varphi_A=A\circ\varphi$. We will prove the two following statements:

Lemma 2.4. *The positive orbit of any point $p_i^{(1)}\in Base(\varphi_A^{-1})$ is infinite.*

Lemma 2.5. *The negative orbit of any point $q_i\in Base(\varphi_A)$ is infinite.*

Lemmas 2.4 and 2.5 imply that $\mu(\varphi_A)=0$; Theorem A thus follows from Theorem 2.3. We will now prove Lemmas 2.4 and 2.5.

The set of base points of φ is

$$\mathrm{Base}(\varphi)=\big\{q,\,q_1,\,q_2,\,q_3,\,q_4,\,q_5,\,q_6,\,q_7,\,q_8\big\}$$

and the set of base points of φ^{-1} is

$$\mathrm{Base}(\varphi^{-1})=\big\{p,\,p_1,\,p_2,\,p_3,\,p_4,\,p_5,\,p_6,\,p_7,\,p_8\big\}.$$

We have

$$\mathrm{Base}(\varphi_A)=\mathrm{Base}(\varphi), \qquad\qquad \mathrm{Exc}(\varphi_A)=\mathrm{Exc}(\varphi).$$

However

$$\mathrm{Base}(\varphi_A^{-1})=\big\{p^{(1)},\,p_1^{(1)},\,p_2^{(1)},\,p_3^{(1)},\,p_4^{(1)},\,p_5^{(1)},\,p_6^{(1)},\,p_7^{(1)},\,p_8^{(1)}\big\}$$

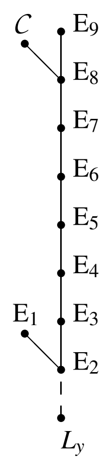
where $p^{(1)}=A(p)$ and $p_j^{(1)}=A(p_j)$. Moreover

$$\mathrm{Exc}(\varphi_A^{-1})=\big\{A(L_x),A(\mathcal{C}')\big\}.$$

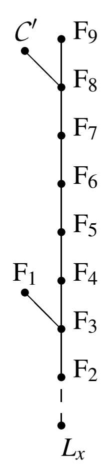
The map φ_A (resp. φ_A^{-1}) has only one proper base point, and all its base points are in tower, that is q_i (resp. p_i) is infinitely near to q_{i-1} (resp. p_{i-1}) for $i=2,\,\ldots,\,8$. We denote by $\pi_1\colon S\to\mathbb{P}^2_{\mathbb{C}}$ (resp. $\pi_2\colon S\to\mathbb{P}^2_{\mathbb{C}}$) the blow-up of the 8 base points of φ_A (resp. φ_A^{-1}). We have

$$\begin{tikzcd}
 & S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^2_{\mathbb{C}} \arrow[rr, dashed, "\varphi_A"'] & & \mathbb{P}^2_{\mathbb{C}},
\end{tikzcd}$$

We still denote by L_y and $\mathcal{C}$ (resp. L_x and $\mathcal{C}'$) the strict transform of L_y and $\mathcal{C}$ (resp. L_x and $\mathcal{C}'$). Let $\mathrm{E}_i \subset V_1$ (resp. $\mathrm{F}_i \subset V_2$) be the strict transform of the curve obtained by blowing up q_i (resp. p_i). The configuration of the curves $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_8$, $\mathcal{C}$ and L_y on S is

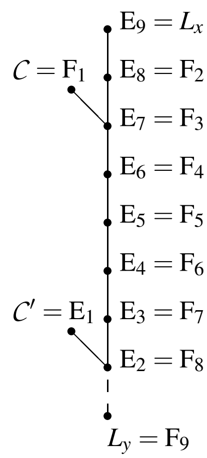


where two curves are connected by an edge if their intersection is positive. We will denote by $\mathcal{T}'$ this tree. The configuration of the curves $\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_8$, $\mathcal{C}'$ and L_x on S is:



Let us denote by $\mathcal{T}$ this tree.

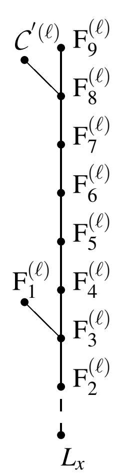
Because of the order of the curves contracted by π_2 we get equalities between $\mathcal{C}$, E_1, E_2, $\ldots$, E_9 and $\mathcal{C}'$, F_1, F_2, $\ldots$, F_9 according to the following figure:



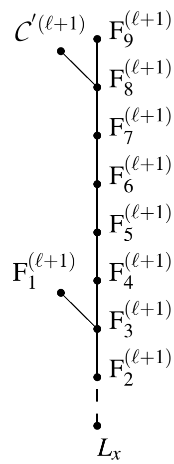
Furthermore φ sends

L_y to F_9	E_1 to $\mathcal{C}'$	E_2 to F_8	E_3 to F_7
E_4 to F_6	E_5 to F_5	E_6 to F_4	E_7 to F_3
E_8 to F_2	E_9 to L_x	$\mathcal{C}$ to F_1	

Let us study the positive orbits of the base points of $\mathrm{Base}(\varphi_A^{-1})$. Set $p^{(k)} = \varphi_A^k(p^{(1)})$ and $p_i^{(k)} = \varphi_A^k(p_i^{(1)})$. As soon as $p^{(\ell)}$ belongs to $\mathbb{P}^2_{\mathbb{C}} \smallsetminus \{L_y, \mathcal{C}\}$, φ_A sends the tree



onto the tree

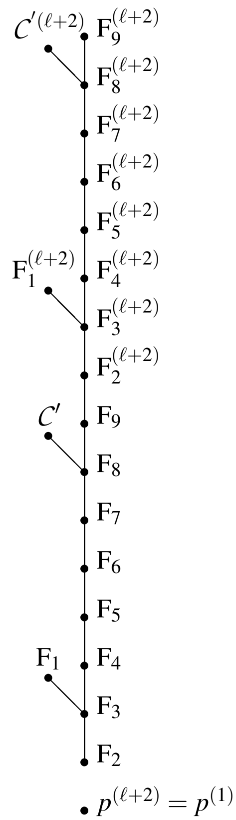


We will denote by $\mathcal{T}^{(\ell)}$ (resp. $\mathcal{T}'^{(\ell)}$) the tree above $p^{(\ell)}$ (resp. $q^{(\ell)}$) and we will say that φ_A sends $(p^{(\ell)},\mathcal{T}^{(\ell)})$ onto $(p^{(\ell+1)},\mathcal{T}^{(\ell+1)})$.

The orbit of $p^{(1)}$ is finite if there exists an integer ℓ such that

- either $p^{(\ell+1)}$ lies on $L_y\smallsetminus\{q\}$
- or $p^{(\ell+1)}$ belongs to $\mathcal{C}\smallsetminus\{q\}$;
- or $p^{(\ell+1)}=q$.

Proof of Lemma 2.4. (1) Let us first assume that $p^{(\ell+1)}$ lies on $L_y\smallsetminus\{q\}$ or on $\mathcal{C}\smallsetminus\{q\}$. Then φ_A sends $(p^{(\ell)},\mathcal{T}^{(\ell)})$ onto $(p^{(\ell+1)},\mathcal{T}^{(\ell+1)})$ and $(p^{(\ell+1)},\mathcal{T}^{(\ell+1)})$ onto

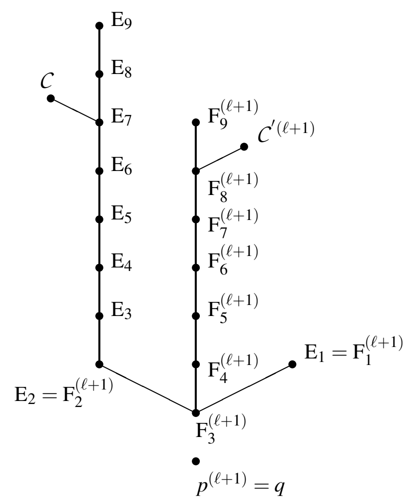


(2) Suppose finally that $p^{(\ell+1)} = q$. Since φ_A is a local diffeomorphism at $p^{(\ell)}$ the map φ_A sends $(p^{(\ell)}, \mathcal{T}^{(\ell)})$ onto $(q = p^{(\ell+1)}, \mathcal{T}')$.

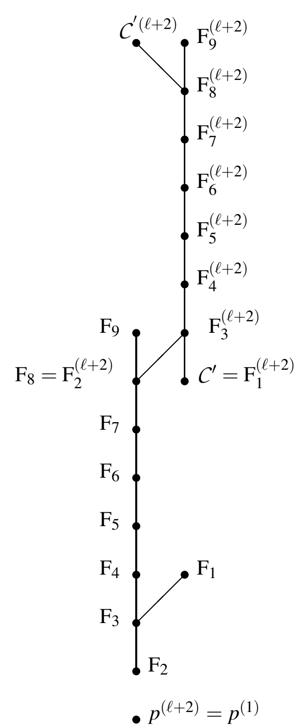
The curve $F_1^{(\ell)}$ has to be sent onto E_1 since $F_1^{(\ell)}$ is the exceptional divisor obtained from the first blow up of $p^{(\ell)}$. Then

- either $F_2^{(\ell)}$ is sent onto E_2,
- or not.

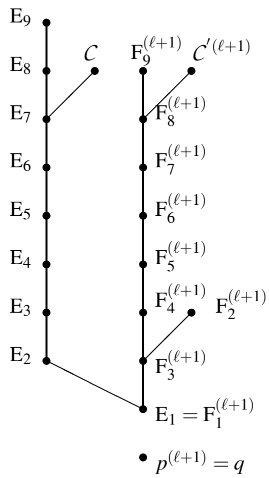
If $F_2^{(\ell)}$ is sent onto E_2, then φ_A sends the tree



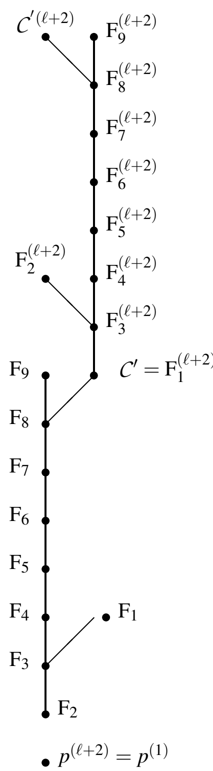
onto the tree



If $F_2^{(\ell)}$ is not sent onto E_2, then φ_A sends the tree



onto the tree



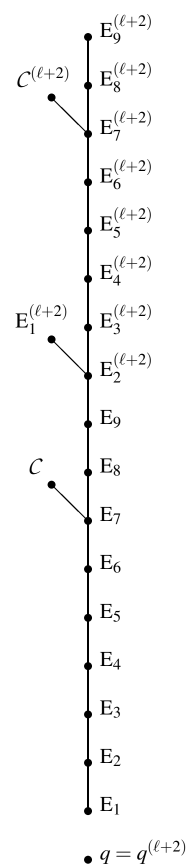
□

Similarly the study of the positive orbits of the base points of $\mathrm{Base}(\varphi_A^{-1})$ allows to prove Lemma 2.5. Let us denote by $\{q^{(i)}\}$ the orbit of q under the action of φ_A^{-1}. As soon as $q^{(\ell)}$ belongs to $\mathbb{P}^2_{\mathbb{C}} \smallsetminus \{A(L_x), A(\mathcal{C})\}$ the map φ_A^{-1} sends $(q^{(\ell)}, \mathcal{T}'^{(\ell)})$ onto $(q^{(\ell+1)}, \mathcal{T}'^{(\ell+1)})$.

The orbit of q is finite if one of the following holds

- $q^{(\ell+1)}$ lies on $A(L_x) \smallsetminus \{p^{(1)}\}$;
- $q^{(\ell+1)}$ belongs to $A(\mathcal{C}) \smallsetminus \{p^{(1)}\}$;
- $q^{(\ell+1)} = p^{(1)}$.

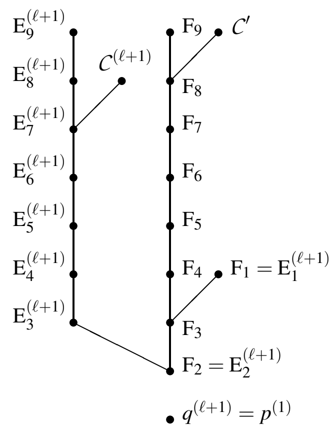
Proof of Lemma 2.5. (1) Let us first assume that $q^{(\ell+1)}$ belongs to $A(L_x) \smallsetminus \{p^{(1)}\}$ or to $A(\mathcal{C}) \smallsetminus \{p^{(1)}\}$. Then φ_A^{-1} sends $(q^{(\ell)}, \mathcal{T}'^{(\ell)})$ onto $(q^{(\ell+1)}, \mathcal{T}'^{(\ell+1)})$ and $(q^{(\ell+1)}, \mathcal{T}'^{(\ell+1)})$ onto the tree



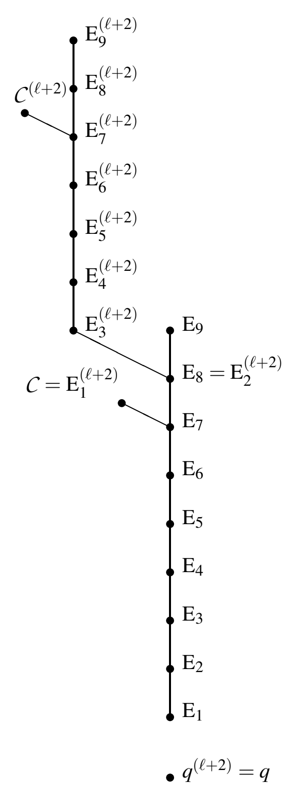
(2) Suppose finally that $q^{(\ell+1)} = p^{(1)}$. The curve $\mathrm{E}_1^{(\ell)}$ has to be sent onto F_1 since $\mathrm{E}_1^{(\ell)}$ is the exceptional divisor obtained from the first blow up of $q^{(\ell)}$. Then

- either $E_2^{(\ell)}$ is sent onto F_2,
- or not.

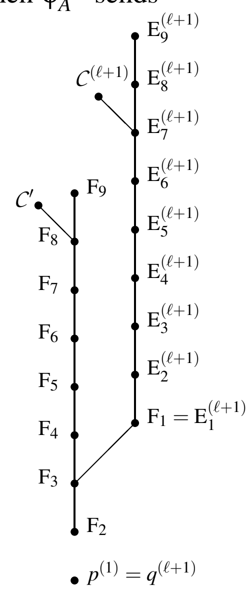
If $E_2^{(\ell)}$ is sent onto F_2, then φ_A^{-1} sends



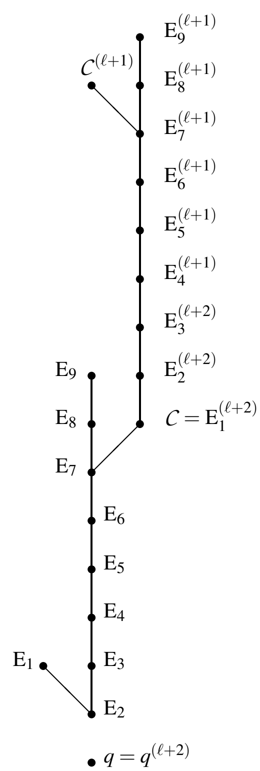
onto



If $E_2^{(\ell)}$ is not sent onto F_2, then φ_A^{-1} sends



onto the tree



□

3. BLANC EXAMPLE IN HIGHER DEGREE

Let us deal with Remark 1.2.

In [3] BLANC consider the birational map $\chi_{23} = \varphi_2 \circ \psi_3$ with

$$\varphi_2 \colon (x:y:z) \dashrightarrow (xz+y^2 : yz : z^2), \qquad \psi_3 \colon (x:y:z) \dashrightarrow (xz^2 : yz^2+x^3 : z^3).$$

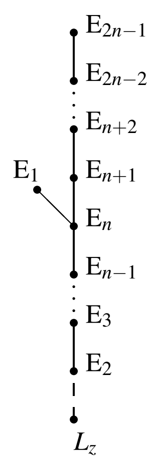
BLANC proves that for any $A \in \mathrm{PGL}(3,\mathbb{C})$

- the positive orbit of any point of $\mathrm{Base}((A\circ\chi_{23})^{-1})$ is infinite,
- the negative orbit of any point of $\mathrm{Base}(A\circ\chi_{23})$ is infinite.

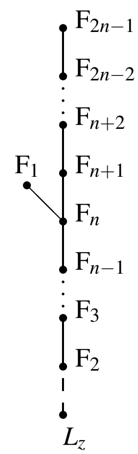
This implies that $A\circ\chi_{23}$ is not regularizable, and so $\mathrm{Reg}(\chi_{23}) = \emptyset$. It can be generalize in higher degree. Let us set

$$\varphi_n \colon (x:y:z) \dashrightarrow (xz^{n-1}+y^n : yz^{n-1} : z^n), \qquad \psi_p \colon (x:y:z) \dashrightarrow (xz^{p-1} : yz^{p-1}+x^p : z^p).$$

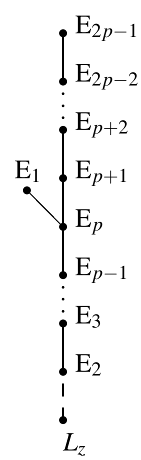
The tree of rational curves obtained by solving the indeterminacy of φ_n is



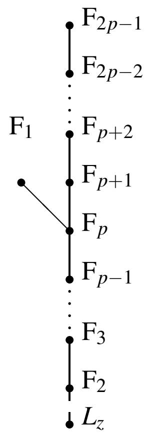
The tree of rational curves obtained by solving the indeterminacy of φ_n^{-1} is



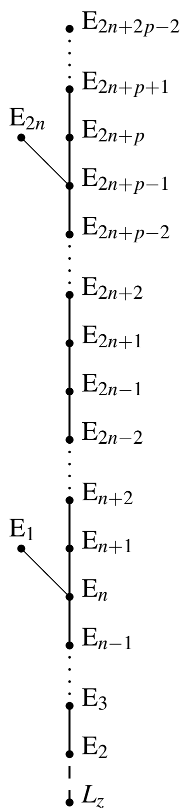
The tree of rational curves obtained by solving the indeterminacy of ψ_p is



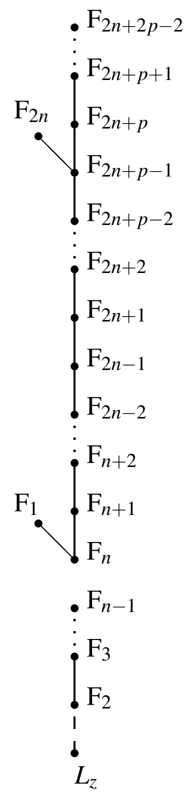
The tree of rational curves obtained by solving the indeterminacy of ψ_p^{-1} is



Let us now consider $\chi_{n,p} = \varphi_n \circ \psi_p$.
The tree of rational curves obtained by solving the indeterminacy of $\chi_{n,p}$ is



The tree of rational curves obtained by solving the indeterminacy of $\chi_{n,p}^{-1}$ is



Furthermore $\chi_{n,p}$ sends

L_z to $F_{2p+2n-2}$	E_1 to F_p	E_{2n} to F_1	$E_{2p+2n-2}$ to L_z
E_2 to $F_{2p+2n-3}$	E_3 to $F_{2p+2n-4}$	$\dots$	E_{n-1} to F_{2p+n}
E_n to F_{2p+n-1}	E_{n+1} to F_{2p+n-2}	$\dots$	E_{2n+p-2} to F_{p+1}
E_{p+2n-1} to F_p	E_{p+2n} to F_{p-1}	$\dots$	$E_{2p+2n-1}$ to F_2

As a result using [3] we can state:

Theorem 3.1. *If*

$$\varphi_n\colon (x:y:z)\mapsto (xz^{n-1}+y^n : yz^{n-1} : z^n) \qquad \psi_p\colon (x:y:z)\mapsto (xz^{p-1} : yz^{p-1}+x^p : z^p)$$

and $\chi_{n,p}=\varphi_n\circ\psi_p$*, then* $\mathrm{Reg}(\chi_{n,p})=\emptyset$.

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