Complex and homomorphic chromatic number of signed planar simple graphs

Reza Naserasr, Lan Anh Pham

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Abstract

We introduce the notion of complex chromatic number of signed graphs as follows: given the set $\mathbb{C}_{k,l} = \{\pm 1, \pm 2, \ldots, \pm k\} \cup \{\pm 1i, \pm 2i, \ldots, \pm li\}$, where $i = \sqrt{-1}$, a signed graph (G, σ) is said to be (k, l)-colorable if there exists a mapping c of vertices of G to $\mathbb{C}_{k,l}$ such that for every edge xy of G we have

$$c(x)c(y) \neq \sigma(xy)|c(x)^2|.$$

The *complex chromatic number* of a signed graph (G, σ), denoted $\chi_{com}(G, \sigma)$, is defined to be the smallest order of $\mathbb{C}_{k,l}$ such that (G, σ) admits a (k, l)-coloring.

In this work, after providing an equivalent definition in the language of homomorphisms of signed graphs, we show that there are signed planar simple graphs which are not 4-colorable. That is to say: there is a signed planar simple graph which is neither $(2, 0)$-colorable, nor $(1, 1)$-colorable, nor $(0, 2)$-colorable. That every signed planar simple graph is $(2, 0)$-colorable was the subject of a conjecture by Máčajová, Raspaud and Škoviera which was recently disproved by Kardoš and Narboni using a dual notion. We provide a direct approach and a short proof. That every signed planar simple graph is $(1, 1)$-colorable is a recent conjecture of Jiang and Zhu which we disprove in this work. Noting that $(0, 2)$-coloring of (G, σ) is the same as $(2, 0)$-coloring of $(G, -\sigma)$, this proves the existence of a signed planar simple graph whose complex chromatic number is larger than 4.

Further developing the homomorphism approach, and as an analogue of the 5-color theorem, we find three minimal signed graphs each on three vertices, without a $K_1^{\pm}$ (a vertex with both a positive and a negative loop) and each having the property that admits a homomorphism from every signed planar simple graph. Finally we identify several other problems of high interest in colorings and homomorphisms of signed planar simple graphs.

Keywords: Signed graph; graph homomorphism, proper coloring

1 Introduction

A *signed graph* (G, σ) is a graph (allowing multi edges and loops) together with an assignment σ of signs (i.e. $+$ or $-$) to the edges of G. The sign of a structure in (G, σ)

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(such as a subgraph, a walk, etc) is the product of the signs of all of its edges considering multiplicity. Of particular importance are the signs of cycles and closed walks which are invariant under the switching operation defined next.

A *switching* of a signed graph at a vertex x is to switch signs of all edges incident with x. A switching of (G, σ) is a collection of switchings at each of the elements of a given set X of vertices (which is equivalent to switching of signs of all edges in the edge-cut formed by X and its complement). A *digon* in a signed graph is a cycle of length 2 where one edge of this cycle is positive and the other edge is negative. A *loop* of a signed graph is an edge such that its two endpoints are the same. Thus, in particular, a switching at a vertex incident with a loop e is considered as switchings at both ends, and, hence, does not change the sign of e. We use $(G, +)$ (respectively, $(G, -)$) to denoted a signed graph on G where all edges are positive (negative). A signed graph (G, σ) which is switching equivalent to $(G, +)$ (respectively, $(G, -)$) is called *balanced* (respectively, *antibalanced*).

Extending the notion of proper colorings of graphs, a notion of (proper) coloring of signed graphs was introduced by T. Zaslavsky in [11]. That is a coloring c of vertices where colors are (nonzero) integers such that $c(x) \neq \sigma(xy)c(y)$ for each edge xy. Various directions of study and extensions of this notion of proper coloring of signed graphs have been recently introduced. Here we propose the following extension which has been developed as a natural optimization problem from the study of homomorphisms of signed graphs.

Definition 1. Given the set $\mathbb{C}_{k,l} = \{\pm 1, \pm 2, \ldots, \pm k\} \cup \{\pm 1i, \pm 2i, \ldots, \pm li\}$, where $i = \sqrt{-1}$, a signed graph (G, σ) is said to be (k, l)-colorable if there exists a mapping c of the vertices of G to $\mathbb{C}_{k+l}$ such that for every edge xy of G we have

$$c(x)c(y) \neq \sigma(xy)|c(x)^2|.$$

Observe that a vertex with both a positive loop and a negative loop admits no complex coloring. This is analogue of the fact that graphs with loop do not admit a proper coloring. For a signed graph (G, σ) which does not contain such a vertex, the complex chromatic number of it, denoted $\chi_{com}(G, \sigma)$, is defined to be the smallest order of a $\mathbb{C}_{k,l}$ such that (G, σ) admits a (k, l)-coloring. After defining the notion of homomorphism of graphs and signed graphs in the next section, we will see that this is the most natural extension of chromatic number of graphs to chromatic number of signed graphs from a homomorphism point of view: both are about finding the smallest homomorphic image in absence of a trivial mapping.

We remark two important properties of the complex chromatic number of signed graphs. First is that it is independent of a switching. More precisely, if ϕ is a (k, l)-coloring of (G, σ) and (G, σ') is obtained from (G, σ) after a switching at a vertex x, then by changing the color of x to $-\phi(x)$, while keeping all other colors the same, we have a (k, l)-coloring of (G, σ'). The second is the following observation:

Observation 2. *A signed graph (G, σ) is (k, l)-colorable if and only if $(G, -\sigma)$ is (l, k)-colorable.*

More precisely, if ϕ is a (k, l)-coloring of (G, σ), then $i\phi$ is an (l, k)-coloring of $(G, -\sigma)$.

2 Homomorphisms and coloring

Given two signed graphs (G, σ) and (H, π), a homomorphism of (G, σ) to (H, π) is a mapping f which maps the vertices and the edges of G to the vertices and the edges of H (respectively) with the property that all incidences are preserved and, moreover, signs of closed walks are preserved as well. An *edge-sign preserving homomorphism* of (G, σ) to (H, π) is a mapping f which maps the vertices and the edges of G to that of H such that preserves incidences, adjacencies and signs of edges. The following theorem shows a strong connection between these two notions (see [7] for a proof and more on homomorphisms of signed graphs).

Theorem 3. *A signed graph (G, σ) admits a homomorphism to a signed graph (H, π) (i.e $(G, \sigma) \to (H, \pi)$) if and only if there is a switching σ' of σ and an edge-sign preserving homomorphism of (G, σ') to (H, π).*

This shows that the notion of homomorphisms of signed graphs extends the notion of homomorphisms of graphs because a homomorphism of $(G, +)$ to $(H, +)$ is the same as a homomorphism of G to H. One may easily observe that the chromatic number of a graph without a loop is the number of the vertices in a smallest homomorphic image without a loop. Let $(K_1^{\pm})$ be the signed graph on one vertex whose edge set consists of one positive and one negative loop. Observing that every signed graph admits a homomorphism to $(K_1^{\pm})$, and that this is the only minimal signed graph with this property, we consider the following natural definitions:

Given a signed graph which does not contain $(K_1^{\pm})$, we define the homomorphism chromatic number of a signed graph (G, σ), denoted $\chi_{hom}(G, \sigma)$, to be the smallest number of the vertices in a homomorphic image of (G, σ) which does not contain $(K_1^{\pm})$. We now show that the homomorphism chromatic number and complex chromatic are almost the same concepts.

Theorem 4. *For any signed graph without a $(K_1^{\pm})$ we have $\chi_{com}(G, \sigma) = 2\chi_{hom}(G, \sigma)$.*

Proof. Let $K_{k-,l+}$ be the complete complex signed graph on a set $X \cup Y$ of $k + l$ vertices where $X = \{x_1, \ldots, x_k\}$ is a set of order k and $Y = \{y_1, \ldots, y_l\}$ is a set of order l. Edge set of $K_{k-,l+}$ consists of: a negative loop at every vertex of X, a positive loop at every vertex of Y and a digon connecting any pair of distinct vertices.

We claim that a (k, l)-coloring of (G, σ) is equivalent to a homomorphism of (G, σ) to $K_{k-,l+}$. Given a (k, l)-coloring ϕ of (G, σ), if a vertex v colored $+p$, then we map v to the vertex x_p, if a vertex u is colored $-p$, then we first apply a switching at u and then map it to x_p. Similarly, we map the vertices colored qi or $-qi$ to y_q. It is easily verified that this is a one to one correspondence between (k, l)-colorings of (G, σ) and homomorphisms (G, σ) to $K_{k-,l+}$.

The claim of the theorem now follows: Suppose $\chi_{com}G = 2(k + l)$ and that (G, σ) admits a (k, l)-coloring. Then $K_{k-,l+}$ is a signed graph on $k + l$ vertices with no $(K_1^{\pm})$ to which (G, σ) admits a homomorphism, therefore, $\chi_{com}(G, \sigma) = 2(k + l) \geq 2\chi_{hom}(G, \sigma)$.

On the other hand, let (H, π) be a homomorphic image of (G, σ) with the smallest possible number of vertices and assume $|v(H)| = p$. Then, observing that parallel edges of a same sign are of no importance, one may add edges to (H, π) without creating a $(K_1^{\pm})$ until it becomes a $K_{k-,l+}$ for a choice of k and l (thus $p = k + l$). Then the given homomorphism of (G,σ) to (H,π) can now be viewed as a homomorphism of (G,σ) to $K_{k-,l+}$, proving that (G,σ) admits a (k,l)-coloring. Hence, $\chi_{com}(G,\sigma) \leq 2(k+l) = 2\chi_{hom}(G,\sigma)$. □

In particular, a signed graph without $(K_1^{\pm})$ has complex chromatic number at most 4 if and only if it admits a homomorphism to one of the three signed graphs of Figure 1. (In Figures of this work a blue or solid edge represents a positive edge and a red or dotted edge represents a negative edge). More precisely, the digon with two negative loops D^- (left) corresponds to $(2,0)$-coloring, the digon with two positive loops D^+ (middle) corresponds to $(0,2)$-coloring and the one with one positive and one negative loop $D^{\pm}$ (right) corresponds to $(1,1)$-coloring.

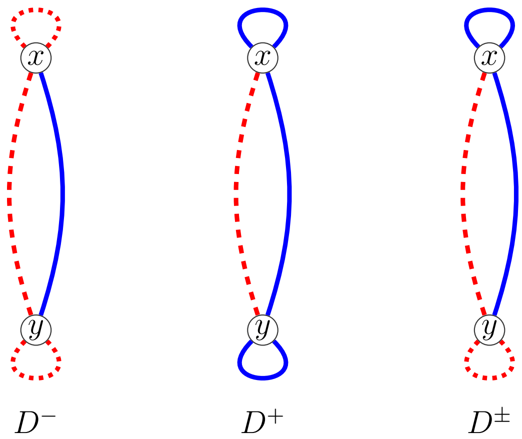


Figure 1: The homomorphism targets for complex 4-coloring

A comprehensive study of the complex chromatic number then is under way in a joint work with Weiqiang Yu. In this work, we have a look at maximum possible complex chromatic number of signed planar simple graphs. That all signed planar simple graphs are $(2,0)$-colorable was the subject of a conjecture by Máčajová, Raspaud and Škoviera [5]. Kardănd J. Narboni [3], based on the notion of dual, provided an edge-coloring interpretation of this conjecture, and then, using examples of non-hamiltonian cubic birdgeless planar graphs, disproved the conjecture. Here, we give a direct approach to the problem. Our proof becomes simpler because of the choice of the signature on the gadgets and it leads to families of counterexamples.

That every signed planar simple graph is $(1,1)$-colorable is a recent conjecture of Jiang and Zhu [2]. Indeed they have the first step toward introducing the notion of complex coloring of signed graphs. We disprove this conjecture by building a counterexample using similar techniques.

Combined together, these results can be viewed as an existence of a signed planar simple graph whose vertices cannot be partitioned into two parts, each part inducing either a balanced subgraph or an antibalanced subgraph.

Thus, turning our attention to signed graphs on three vertices, we identify three minimal signed graphs on at most three vertices to which every signed planar simple graph admits a homomorphism. This answer the question of "whether a given signed graph (B,π) on at most three vertices bounds the class of signed planar simple graphs" for all but essentially one choice of (B,π) (see Figure 15).

One of the main tools in our constructions is the dual of what is known as the Tutte fragment. Considering the classic relation between vertex-coloring and edge-coloring of planar graphs (first shown by Tait in 1890), using the dual of Tutte fragment in coloring problems is not a surprise. For a recent use of this gadget, we refer to [4] where the gadget is referred to as the Wenger graph. We note that a strong connection between $(2,0)$-coloring and special types of 4-list coloring is established in [12].

3 The gadgets

The construction of the counterexamples is based on several gadgets. We first provide small gadgets to force three vertices of a facial triangle to be multicolored.

Observe that in any $\{2,0\}$-coloring, the vertices of $(K_3,-)$ can all be colored with a same color. However, as a face of a signed planar graph, we may prevent this by completing the face to a configuration of Figure 2.

Lemma 5. *In any $\{2,0\}$-coloring of the graph of Figure 2, the three vertices x, y and z cannot all receive a same color.*

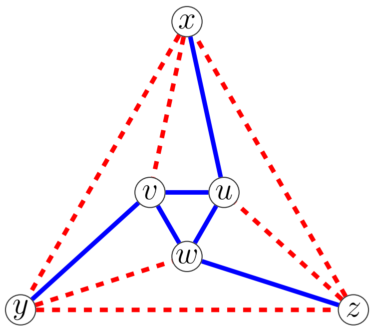


Figure 2: $(2,0)$-coloring a negative triangle

Proof. If $\varphi(x)=\varphi(y)=\varphi(z)=1$ (or $\varphi(x)=\varphi(y)=\varphi(z)=-1$), then non of u, v and w can be colored $+1$ or -1. Thus only two colors ± 2 can be used on them, but they must receive pairwise distinct color, a contradiction. □

Meanwhile, in any $\{1,1\}$-coloring, not only the vertices of $(K_3,-)$ can all be colored with a same color (e.g.,$+1$) but also the vertices of $(K_3,+)$ can all be colored with a same color (e.g.,$+i$). But similar to the previous lemma, as a face of a signed planar graph, we can prevent this by completing the face to the corresponding configuration of Figure 3 or Figure 4.

Lemma 6. *In any $\{1,1\}$-coloring of the graph of Figure 3 (or Figure 4), the three vertices x, y and z cannot all receive a same color.*

Proof. We will prove that for any $(1,1)$-coloring, the three vertices x, y and z of the graph of Figure 3 cannot all receive a same color. The proof for the graph Figure 4 is proceeded similarly.

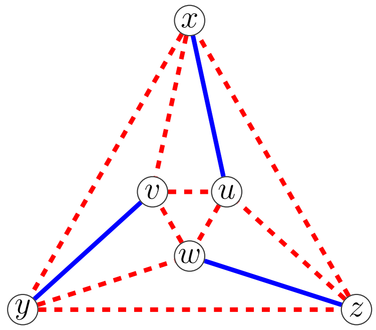


Figure 3: $(1, 1)$-coloring a negative triangle

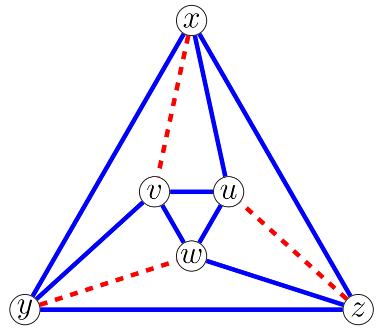


Figure 4: $(1, 1)$-coloring a positive triangle

Since xyz is the negative triangle, x, y, z cannot all receive a same color $a \in \{i, -i\}$. If $\varphi(x) = \varphi(y) = \varphi(z) = 1$ (or $\varphi(x) = \varphi(y) = \varphi(z) = -1$), then non of u, v and w can be colored $+1$ or -1. Thus only two colors $\pm i$ can be used on them, but uvw is a negative triangle, a contradiction. $\square$

Remark:

(1) In any $(2, 0)$-coloring, considering possible switchings of the signed graph of Figure 2, a general statement is to say that vertices x, y and z cannot all receive colors with a same absolute value. It is immediate from the definition that in any $(2, 0)$-coloring, three vertices of any positive facial triangle cannot all receive colors with a same absolute value;

(2) Applying the statement of Lemma 6 to possible switchings of a triangle, we may conclude that in any $(1, 1)$-coloring of a facial triangle, one may prevent the use of colors $\{a, -a\}$ on all three vertices.

Thus from here on, whether we are working on a $(2, 0)$-coloring or a $(1, 1)$-coloring, we may assume that the vertices of a facial triangle are not all colored by two colors a and $-a$. We will refer to this as the *triangle property.*

Next we build our main gadgets that are based on Wenger graph.

Lemma 7. *In a* $(2, 0)$*-coloring* c *of the graph of Figure 5 which satisfies the triangle property, we cannot have* $c(u) = -c(v)$*.*

Proof. Observe that there are two negative faces in this graph, they are: ux_1x_2 and vx_3x_4. Toward a contradiction, assume c is $(2, 0)$-coloring of this graph where on each of these two negative faces both absolutes values 1 and 2 are used and suppose, without loss of generality, $c(u) = 1$ and $c(v) = -1$. We then consider list of available colors at each vertex. At z and t the only available colors are ± 2, at x_i, for $i = 1, 4, 5$, the list of available colors is $\{-1, \pm 2\}$ and at x_i, $i = 2, 3$, the list of available colors is $\{+1, \pm 2\}$.

Currently all colors are available at w, but we claim that its color cannot be ± 1. If w is colored -1 then the positive triangle x_4x_5t must be properly colored using only ± 2, and if w is colored $+1$, then the positive triangle x_2x_3z will face the same problem.

Thus w has to be colored by ± 2. As non of the two colors are used yet, and by symmetry among the two colors, assume w is colored -2. Then on the 5-cycle $x_1x_2x_3x_4x_5$ each vertex has a list of two available colors: $+2$ together with either $+1$ or -1. Furthermore,

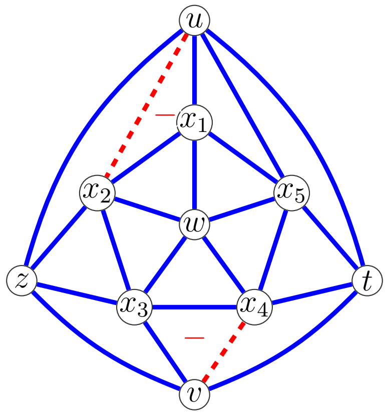


Figure 5: Gadget to forbid opposite colors on two vertices

on the edge x_1x_2 color $+2$ must be used (at least) once because of the negative face ux_1x_2 (recall that u is colored 1). Similarly, because of the negative face on the edge x_3x_4, the color $+2$ must be used once. Thus colors ± 1 cannot appear on an edge of the 5-cycle, and, therefore, coloring $|c|$ induces a proper 2-coloring of this 5-cycle, a contradiction. □

Remark. We presented a simplest form of a signature and used symmetric presentation in our proof. However, one can modify this gadget and still have a similar property. For example, one may change the sign of one of the edges tx_4 or tx_5 and then add the gadget of Figure 2 inside x_4x_5t to get a same result. Similarly, if we change the sign of ux_1 and apply triangle property to the face ux_1x_5 instead of ux_1x_2, then we have a similar property.

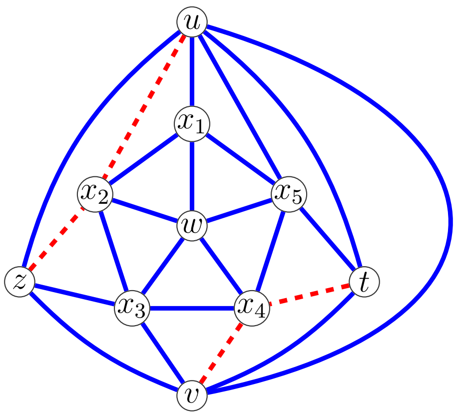


Figure 6: Signed graph H_1

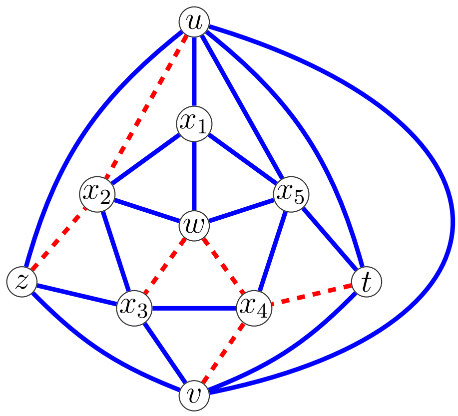


Figure 7: Signed graph H_2

In any (k,l)-coloring c of (G,σ), and for a color $a \in \mathbb{C}_{k,l}$ we define an *a-monochromatic subgraph* to be the subgraph induced by the vertices that are colored a or $-a$.

A crucial but straightforward fact is that:

Observation 8. *In any (k,l)-coloring c of (G,σ) an a-monochromatic subgraph cannot contain a negative even closed walk.*

Lemma 9. *Given a $(1,1)$-coloring c of the signed graph H_1 of Figure 6 satisfying triangle property, if an a-monochromatic subgraph (H',σ) contains both u and v, then it contains a positive 5-cycle with the edge uv as part of it.*

Proof. Consider a $(1,1)$-coloring c of the graph H_1 which satisfies the triangle property. Furthermore, assume $c(u), c(v) \in A = \{a, -a\}$ and let $B = \{b, -b\}$ be the other two colors in $\mathbb{C}_{1,1}$. Applying triangle property on the faces utv and uzv, we have $c(t) \in B$ and $c(z) \in B$.

Depending on whether $c(w) \in A$ or $c(w) \in B$ we consider two case:

- $c(w) \in A$. The triangle zx_2x_3 implies that either $c(x_2) \in A$ or $c(x_3) \in A$.

  Case $c(x_2) \in A$. As ux_2wx_5 is a negative even cycle in which three of its vertices are in a-monochromatic subgraph, $c(x_5) \in B$. If $c(x_4) \in A$, then ux_2wx_4v is a positive 5-cycle in the a-monochromatic subgraph. Thus $c(x_4) \in B$, but then tx_4x_5 is a b-monochromatic facial triangle which violates triangle property.

  Case $c(x_2) \in B$. Then $c(x_3) \in A$ because of the triangle zx_2x_3. Hence $c(x_4) \in B$ because of the triangle wx_3x_4, and $c(x_5) \in B$ because of the positive 5-cycle ux_5wx_3v. But then tx_4x_5 violates the triangle property.

- $c(w) \in B$. The triangle ux_1x_2 implies that either $c(x_1) \in B$ or $c(x_2) \in B$.

  Case $c(x_1) \in B$. Then $c(x_2), c(x_5) \in A$ because of the triangles wx_1x_2 and wx_1x_5. Now considering the negative even cycle ux_2x_3v, and by Observation 8, we have $c(x_3) \in B$ and similarly, because of the negative 4-cycle ux_5x_4v, we have $c(x_4) \in B$, but then wx_3x_4 violates the triangle condition.

  Case $c(x_2) \in B$. Then $c(x_1), c(x_3) \in A$ because of the triangles wx_1x_2 and wx_2x_3. This in turn implies that $c(x_4), c(x_5) \in B$ because of the triangles ux_1x_5 and vx_3x_4. But then the triangle tx_4x_5 violates the triangle property.

This concludes the claim of the lemma. □

The gadget H_2 of the Figure 7 holds a similar property, as its proof is quite similar to the proof of the previous one, we omit the details.

Lemma 10. *Given a $(1,1)$-coloring c of the signed graph H_2 of Figure 7 satisfying triangle property, if an a-momochromatic subgraph (H', σ) contains both u and v, then it contains a negative 5-cycle with uv as part of it.*

4 Complex chromatic number of signed planar simple graphs

Using the gadgets we have introduced we may now build signed planar simple graphs that are not $(2,0)$-colorable, $(1,1)$ and $(2,0)$-colorable. Putting disjoint copies of them together we get a signed planar simple graph whose complex chromatic number is 6 (or 5 depending on if we allow 0 as a color or not).

4.1 A simple planar signed graph which is not $(2,0)$-coloring

Using the gadget in Lemma 7, one can transform a given planar signed graph which has no loop, but may contain digons, and accepts no $(2,0)$-coloring into a simple planar signed graph which is not $(2,0)$-colorable.

Definition 11. Let (G, σ) be a planar signed (multi) graph with no loops. Define $F(G, \sigma)$ to be the following simple signed graph. For each pair u, v of vertices, if they are adjacent with more than one edge of a same sign, delete all but one of them. If u and v are adjacent with one positive and one negative edge, then delete the negative uv-edge, and add to the graph a copy F_{uv} of the graph of Figure 5, where each of the two negative triangles are completed by (a switched copy of) the configuration of Figure 2.

Using Lemma 7, we immediately have the following.

Proposition 12. *Given a planar signed (multi) graph (G, σ) with no loop, (G, σ) is $(2, 0)$-colorable if and only if the signed planar simple graph $F(G, \sigma)$ is $(2, 0)$-colorable.*

Thus, to build a planar simple graph which is not $(2, 0)$-colorable, it would be enough to build one with no loop, but allowing digons, which is not $(2, 0)$-colorable. The easiest and smallest such a construction is to replace each edge of the triangle by a digon (see Figure 8). The simple planar signed graph obtained from this three vertices graph by replacing each negative edge with a gadget of the Figure 5 has 45 vertices. However, there is a face of size 6 in this graph where all edges are positive. One may identify three vertices of this cycle to get an example of simple planar signed graph on 43 vertices which is not $(2, 0)$-colorable.

One may build an example with fewer vertices using the signed graph on the right side of Figure 9. After minimizing the facial cycle of the resulting graph, we have examples on 39 vertices. This is the same order as the smallest graph given in [3], but we note that first of all there is no proof given for the smaller example in [3], and, secondly, that our construction gives several non isomorphic copies, depending on two factors: The possible signature we choose on our gadget, and the choices we have for a vertex of a digon to be the vertex u of the gadget or the vertex v of it.

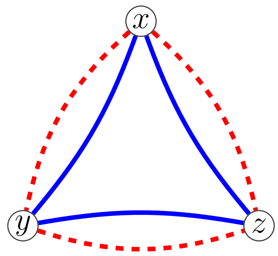


Figure 8: Smallest not $(2, 0)$-colorable graph without a loop

Proposition 13. *Let B be the signed graph on the right of Figure 9 where inside each of the two negative faces vx_3x_4 and vx_1x_5 are completed by a configuration of Figure 2. Then B is not $(2, 0)$-colorable.*

Proof. By contradiction, suppose B has a $(2, 0)$-coloring c. Observe that in a $(2, 0)$-coloring of a digon two colors of different absolute values must be used. Using this fact, and with a focus on the subgraph presented on the left, we show that u and v cannot receive colors of a same absolute value. By the symmetry of colors, it is enough to consider two cases:

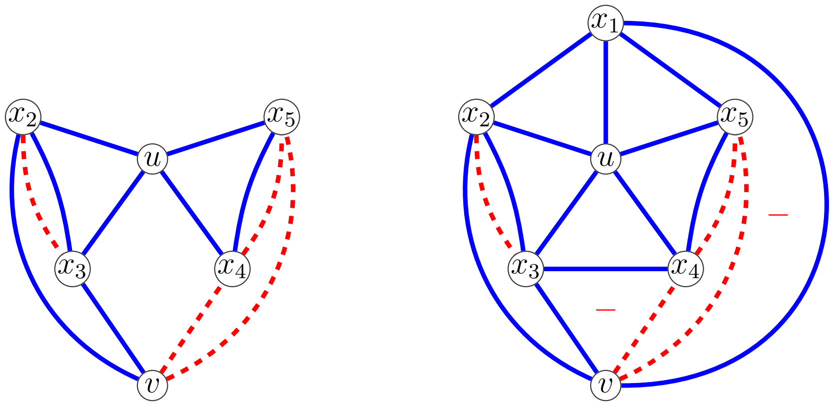


Figure 9: The graph B

- $c(u) = c(v) = 1$. In this case, x_4 and x_5 can only use colors ± 2, but they form a digon.
- $c(u) = 1$ and $c(v) = -1$. In this case, x_2 and x_3 can only use colors ± 2, but they also form a digon.

Thus, without loss of generality, we may assume $c(u) = 1$ and $c(v) = 2$. We now claim that on each of the (positive) edges of the 5-cycle $x_1x_2x_3x_4x_5$ the coloring c must use two different absolute values, but this would result in a 2-coloring of this odd-cycle, which is not possible. Observe that all these five vertices are adjacent to u by a positive edge thus non of them can be colored 1. Hence, to prove our claim, it would be enough to show that non of the five edges receive colors $+2$ and -2. This is the case for edges x_2x_3 and x_4x_5 because they are part of a digon. Edges x_1x_5 and x_3x_4 each together with v form a configuration of Figure 2, and as $c(v) = 2$, this is the claim of Lemma 5. For the last edge, x_1x_2, since each end is adjacent to v by a positive edge, there can be no color $+2$ on these two vertices. □

4.2 A simple planar signed graph which is not $(1, 1)$-coloring

Let H be the signed planar graph obtained from disjoint copies of H_1 of Figure 6 and H_2 of Figure 7 by identifying the vertices labeled u together, vertices labeled v together and edge uv together. The result then is a signed planar simple graph that has the following property.

Lemma 14. *In any $(1, 1)$-coloring c of the signed graph H, either $c(u) \in \{+1, -1\}$ and $c(v) \in \{i, -i\}$ or $c(v) \in \{+1, -1\}$ and $c(u) \in \{i, -i\}$.*

Proof. Otherwise, $c(u), c(v) \in \{a, -a\}$ for $a = 1$ or $a = i$. Then by Lemma 9, we have an a-monochromatic positive 5-cycle in H_1 part of H which contains uv and by Lemma 10, we have an a-monochromatic negative 5-cycle in H_2 part of H which contains uv. Combined together, and after removing the edge uv we get an a-monochromatic negative 8-cycle, contradicting Obervation 8. □

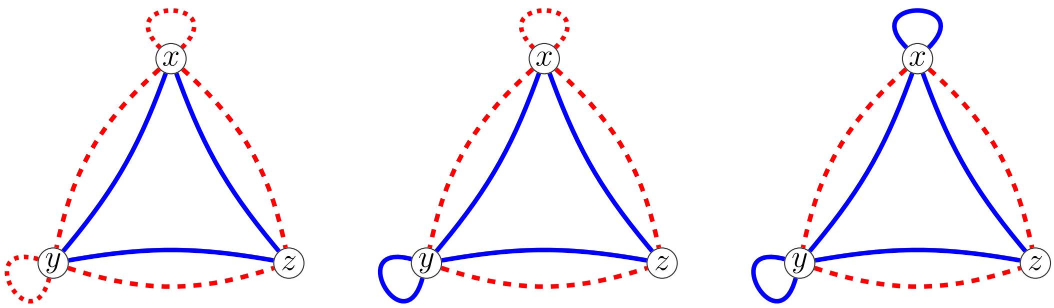
A signed planar simple graph which is not $(1, 1)$-colorable is now built by replacing the three edges of a triangle each with a copy of H. To calculate the order of the graph that is constructed, we note that for the gadget H_i $(i = 1, 2)$ to work we need eleven of its faces to have the triangle property. Thus each H_i has 43 vertices, and H has 84 vertices. The final graph then has 249 vertices. One can get an example with a few more vertices by identifying some vertics, but it would be of interest to build a substantially smaller example.

5 Minimal homomorphism bounds

It follows from the definition of the complex coloring that given a (k, l)-partial coloring c of a signed graph (G, σ), the color $c(v)$ of a neighbor v of a vertex u forbids only one color on u. Thus, it follows that if G is d-degenerate for $d < 2(k+l)$, then (G, σ) admits a (k, l)-coloring. Thus, in particular, every signed planar simple graph admits a $(3, 0)$-coloring, a $(2, 1)$-coloring, a $(1, 2)$-coloring and a $(0, 3)$-coloring.

These claims can be regarded as analogue of the 6-color theorem for planar graphs and can be strengthened to what one might think of as analogue of the 5-color theorem for planar graphs. We will give a short proof using a strong result of Borodin that every planar graph is acyclically 5-colorable. But we will point out in Section 6 at that one may prove the results directly using techniques similar to the known proofs of the 5-color theorem.

Given a signed planar simple graph (G, σ), we consider an acyclic 5-coloring of G and let V_1, V_2, V_3, V_4 and V_5 be the color classes. Since $V_1 \cup V_2$ induces a forest, one may apply a switch on this subset of vertices to have all induced edges of a same sign (of our choice) and then map all vertices to a vertex with one loop. After doing the same to $V_3 \cup V_4$, one may map vertices in V_5 to a single vertex (with no loop). This implies that each of the three graphs of Figure 10 is a homomorphism bound for the class of signed planar simple graphs, or even for the larger class of signed graphs whose underlying graph admits a 5-acyclic coloring.



T^{--} T^{+-} T^{++}

Figure 10: Three minimal homomorphism bounds of signed planar simple graphs

We claim here that each of these three signed graphs of Figure 10 is minimal homomorphism bound for the class $\mathcal{SP}$ of signed planar simple graphs.

To see that, observe that if we remove one of the vertices, we either get one of the graphs of Figure 1 or a subgraph of one of them which we have already shown cannot bound $\mathcal{SP}$. If in T^{--} or T^{++} we delete any of the edges that is not a loop, or delete an edge incident with z in T^{+-}, the result then maps to one of the three graphs of Figure 1 and we have already seen that they cannot bound the class of all signed planar simple graphs. Thus, up to symmetries and isomorphism, we have two cases to consider: Removing one of the loops, thus we have the graph T^- of Figure 11, we will build a signed planar simple graph (G, σ) which does not map to T^- and we note that if we have a positive loop instead then $(G, -\sigma)$ works. The other case is T' (depicted in Figure 12) which is obtained from T^{+-} by removing one of the two xy edges. Note that the two resulting signed graphs are switching equivalent. We will also build an example of signed planar simple graph that does not map to T'.

5.1 Mapping to T^-

Let T^- be the signed graph obtained from the signed graph on the left of Figure 10 by removing the loop on the vertex y. It is not difficult to show that a signed graph (G, σ) maps to T^- if and only if (G, σ) admits a coloring by the elements of $\mathbb{Z}_4$ satisfying $c(x) - \sigma(xy)c(y) \neq 0$ where the operation are taken $\pmod 4$. Disproving a conjecture of Kang and Steffen, Zhu [12] built a signed planar simple graph which does not admits a $\mathbb{Z}_4$-coloring. However, since we have already built the main gadgets, and to highlight the homomorphism properties of T^-, we give the following construction.

Observe that if a signed graph (G, σ) maps to T^-, then $V(G)$ is partitioned into two sets V_1 and V_2 such that V_1 (set of the vertices that are mapped to x) induces an antibalanced signed graph and V_2 (set of the vertices that are mapped to y or z) induces a signed bipartite graph.

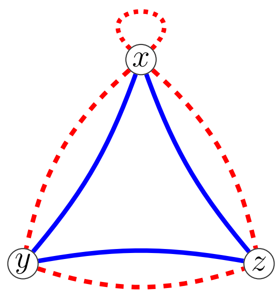


Figure 11: T^- does not bound $\mathcal{SP}$

Thus to build a signed planar simple graph which does not map to T^- it would be enough to build one for which in any 2-coloring of vertices we have:

- either a monochromatic positive odd cycle, or
- a monochromatic negative odd cycle in each color, or
- a monochromatic negative even cycle in each color.

Toward building an example which satisfies the property, we may again use the gadget of Figure 2 to assume that in every 2-coloring of our signed planar simple graphs the triangle property holds, otherwise we would have a monochromatic positive triangle, but T^- has no positive loop.

We then consider the following variation of Lemma 9 which is easily verified by modifying the proof:

Lemma 15. *Given 2-coloring c of the signed graph H_1 of Figure 6 satisfying triangle property, if c induces no monochromatic positive odd cycle and if $c(u) = c(v) = a$, then the color a induces a monochromatic negative even cycle.*

We may now build a signed planar simple graph (G, σ) which does not map to T^-. Start with a planar embedding of $(K_4, +)$ and for each edge xy of it add a copy of H_1 where x is identified with u, y is identified with v and xy is replaced by the edge uv. Let c be a 2-coloring of (G, σ). We claim that one of the three conditions will hold. If three vertices of the original K_4 are colored the same, then we already have a positive triangle. Else, of the four vertices two are colored a and the other two are colored b. Applying Lemma 15 on the copy of H_1 corresponding to two vertices colored a, we get a negative even cycle all whose vertices are colored a. Applying it on the other two vertices, we get a negative even cycle all whose vertices are colored b. This completes the proof of our claim.

5.2 Mapping to T'

Let T' be the singed graph obtained from T^{+-} by removing the negative xy-edge, see Figure 12 for a depiction. We will use the fact that remaining xy-edge in T' is a positive edge and build a signed planar simple graph (G, σ) which dones not map to it. We note that a switching at x would change sign of xy-edge to negative and thus our example would also not map to $T^{+-} - e$ if e was the positive xy-edge.

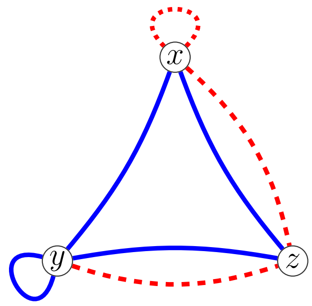


Figure 12: $T^{+-} - (xy)^-$ does not bound $\mathcal{SP}$

The following observation is the key tool in building a signed planar simple graph which does not map to T'.

Lemma 16. *A signed positive triangle admits an edge-sign preserving homomorphism to the subgraph of T' induced by $\{x, y\}$ if and only if all its edges are positive.*

Proof. That is because there are two essentially different signatures on a positive triangle. Either all the edges are positive in which case we must map two of the vertices to y and the third is free to map to x or y. Or there are two negative edges, in this case as the only negative edge on the subgraph induced by x and y is the negative loop on x, all vertices must be mapped to x, but that would create a positive loop at x. $\square$

To build a signed planar simple graph which does not map to T', we first consider the signed outer planar graph (G, σ) of Figure 13. We show that:

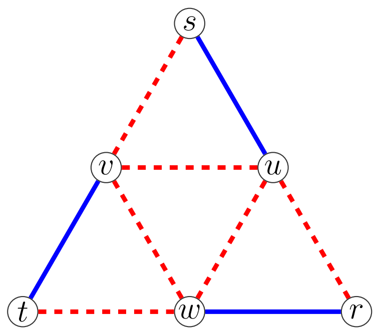


Figure 13: $(G, \sigma) \not\to T' - z$

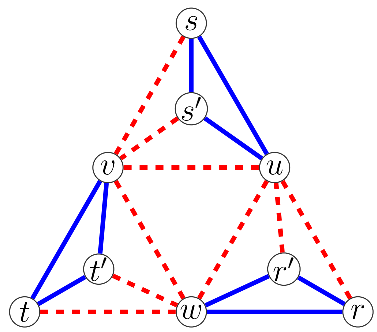


Figure 14: $(G, \sigma)^*$

Lemma 17. *The signed outer planar graph (G, σ) of Figure 13 does not map to $T' - z$.*

Proof. Assume to the contrary that (G, σ) maps to $T' - z$. Then there is a switching σ' of σ under which (G, σ') admits an edge-sign preserving mapping to $T' - z$. In such a signature, by Lemma 16, each of the positive triangles suv, tvw and rwu must consist of only positive edges, but then the triangle vuw is a positive triangle in (G, σ'), however as it is a negative triangle in (G, σ), σ' cannot be a switching of σ. □

In other words, in any mapping of (G, σ) to T' at least one vertex should be mapped to z. Let $(G, \sigma)^*$ be the signed graph obtained from (G, σ) by adding a twin copy s', t', r' of s, t, r (respectively) and adding edges ss', tt' and rr', see Figure 14. Observe that, in a mapping of $(G, \sigma)^*$ of T', of the pair s, s' at most one can be mapped to z. If non of u, v, w is mapped to z, then we will have a copy of (G, σ) which is mapped to $T' - z$, contradicting Lemma 17.

To complete the construction, we may now consider a planar embedding of $(K_4, -)$ and for each edge uv of it, add two new vertices s_{uv} and s'_{uv}. Then connect both of them to one of u and v with positive edges, and to the other with negative edges. Finally, connect s_{uv} to s'_{uv} (of any sign, but say with a positive sign to have isomorphic copies of Figure 14). Observe that, on each triangle of $(K_4, -)$ we have built a copy of $(G, \sigma)^*$. We claim that this signed planar simple graph does not map to T'. That is because at most one of the four vertices of K_4 can map to z, hence, there is a triangle of $(K_4, -)$ non of whose vertices is mapped to z, the subgraph isomorphic to $(G, \sigma)^*$ built on this triangle then would lead to a contradiction. The example then has 16 vertices.

5.3 Remaining three vertices signed graphs

Based on the previous discussion for all but two singed graphs on three vertex we know if they bound the class of signed planar simple graphs. These two are depicted in Figure 15 and Figure 16. Here, strengthening constructions of Section 4, we show that the signed graph T_2 on the right of the figure also does not bound the class $\mathcal{SP}$, leaving only one unsolved case among 3 vertices signed graphs.

Theorem 18. *There exists a signed planar simple graph which does not map to the signed graph T_2 of Figure 16.*

To prove this theorem, we will view a T_2-coloring of a signed graph (G, σ) as a 2-coloring, using colors c_1 and c_2 where vertices mapped to x are colored c_1 and vertices

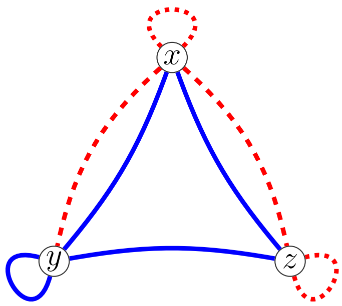


Figure 15: Does T_1 bound $\mathcal{SP}$?

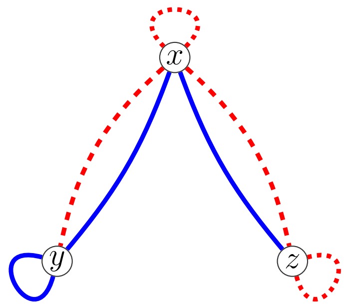


Figure 16: T_2 does not bound $\mathcal{SP}$?

mapped to y or z are colored c_2. This coloring has the property that vertcies colored c_1 induce an antibalanced subgraph and vertices colored c_2 induce a subgraph each of whose connected components is either balanced or antibalanced. It is not hard to check that existence of such a 2-coloring of vertices of (G, σ) is equivalent to mapping (G, σ) to T_2. Such a 2-coloring c can be equivalently regarded as a 2-coloring where color class c_1 induces neither positive odd cycle, nor a negative even cycle and color class c_2 does not induce a negative even closed walk.

Let (G, σ) be a planar graph and assume that it admits a T_2-coloring. Let c be the corresponding 2-coloring. Then applying the gadget of Figure 4 on each of the positive facial triangle of (G, σ) we either build a sign planar simple graph which does not map to T_2, or we are assured that no facial positive triangle of (G, σ) is monochromatic under c. To obtain a similar conclusion of facial negative triangles we need an extension of our gadgets depicted in Figure 17.

Lemma 19. *In a 2-coloring c of the graph of Figure 17 which corresponds to a T_2-coloring of this graph, vertices u, v and w cannot be monochromatic.*

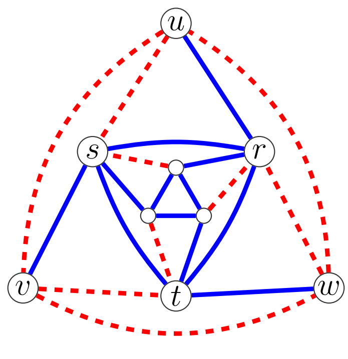


Figure 17: T_2-coloring a negative triangle

Proof. We consider the two cases separately. Suppose u, v and w are colored c_1 (i.e., they are mapped to vertex x in T_2-coloring). Then, since uvs is a positive triangle, s, and similarly r and t, are colored c_2. But then we have a monochromatic positive triangle on which we have the construction of Figure 4.

Now suppose u, v and w are colored c_2 (i.e., they are mapped to either y or z in T_2-coloring). Since they induce a connected subgraph, they all must map to a same vertex of T_2 and as they induce a negative triangle, they all must map to z. Then, by similar arguments, vertices s, r and t all must be mapped to vertex x of T_2, however they induce a positive triangle. $\square$

We now consider the gadgets H_1 and H_2 of Figure 6 and Figure 7 where each facial triangle, depending on its sign, is either completed to a (possibly a switching equivalent) copy of the signed graph of Figure 4, or Figure 17. This ensures that in any 2-coloring corresponding to a T_2-coloring of these graphs the triangle property can be enforced. The construction of a signed planar simple graph that does not map to T_2 is then completed as in Section 4.2. The signed graph H obtained from H_1 and H_2 by identifying the uv edge has property that in any 2-coloring satisfying triangle property, if u and v are coloring the same, then there is a monochromatic negative even cycle. The construction then is completed by using three copies of H where edges corresponding to uv form a triangle.

6 Concluding remarks

In this work, we have first of all introduced the notions of complex-coloring and complex-chromatic number of signed graphs which extend the notion of 0-free coloring of signed graphs introduced by T. Zaslavsky. We will show in forthcoming works that the notion is of high interests on its own, for example we show that the problem of finding the largest possible complex chromatic number on a given complete graph is strongly related to the Ramsey numbers.

We have shown that the concept is a natural optimization problem from a homomorphism point of view. And then we showed that signed planar simple graphs may need more than 4 colors. Here, we have only considered (k, l)-colorings which are extensions of 0-free colorings. One could analogously allow 0 to be in the set of colors. Then the set of the vertices colored 0 must induce an independent set of the graph. This, similar to the work of [5], would allow for the complex chromatic number to be an odd integer as well. That signed planar simple graphs are 5-colorable with this definition is already observed.

From a homomorphism point of view, we determined three minimal graphs on three vertices each of which bounds the class of signed planar simple graphs. In analogy to the coloring of planar graphs (not signed), our results say that four colors are not enough for coloring all signed planar simple graphs but five colors are enough. We used the fact that planar graphs are acyclically 5-colorable to prove that each of these three signed graphs indeed bounds the class of signed planar simple graphs, but we would like to point out that the techniques for proving the 5-color theorem works here as well and one does not really need the acyclic coloring result. We give a bit more details for mapping of to T^{--}, the other two are similar.

To map a signed graph (G, σ) to T^{--}, one must find a switching σ' of σ after which (G, σ') admits an edge-sign preserving mapping to T^{--}. However, if a vertex v is mapped to z, then a switch at v would not affect the homomorphism properties. Thus for each vertex v of (G, σ) we have five possibilities: 1. map it to z, 2. switch and map it to x (we may then say it is mapped to $-x$), 3. do not switch and map it to x (or simply map it to $+x$), 4. map it to $-y$, 5. map to $+y$. Then observe that in a partial mapping of

(G, σ) to T', each vertex that is already mapped forbids exactly one of the five options from the list of availablities at a neighboring vertex. For example, if u is mapped to z, then it forbids z from being used on any of its neighbors. If u is mapped to $-x$ and has a positive connection to v, then it forbids the option of $-x$ on v. One may then complete a proof of the fact that T^- bounds the class of signed planar simple graphs by either using the Kempe chain argument or employing Thomassen's list coloring proof.

Thus except for the signed graph T_1 of Figure 15 and the signed graph T_1', obtained from T_1 where the loop on the vertex x is positive, we have determined for each signed graph on three vertices if it bounds the class of signed planar simple graphs.

A closely related problem is the following:

Problem 20. Can the vertices of every signed planar simple graph be partitioned to two parts non of which induces a negative closed walk of even length?

It is not hard to check that the existence of such a partition for (G, σ) is equivalent to a mapping of (G, σ) to the signed graph of Figure 18. For one direction, observe that in such a mapping, the vertices mapped to x or t will form one part of the partition and those mapped to y and z form the other part. For the opposite direction, as it is observed in [7], a connected signed graph with no negative even closed walk is either balanced or antibalanced. Therefore, each connected component, if (G, σ) admits a 2-partition non of which induces a negative even closed walk, the subgraph induced on each part maps to a graph on two vertices, one having a positive loop, the other having a negative loop.

A counterexample for this problem will also work as common counterexample for the three signed graphs of Figure 1 as it contains all three as a subgraph.

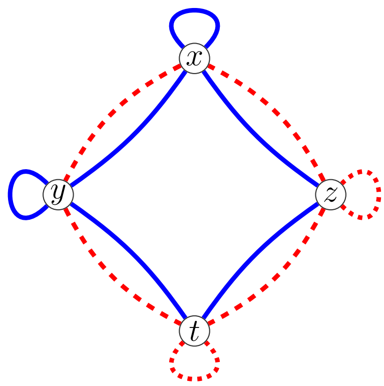


Figure 18: The configuration associated to Problem 20

Among singed graphs of order five there are two, depicted in Figure 19, that are of special interest. The one of the left is homomorphism equivalent of the Problem 7.6 in [10]. The one on the right implies an upper bound of 5 on the circular chromatic number of signed planar simple graphs in the sense of [8].

It is also natural to ask for a homomorphism bound which itself is a signed simple graph. Existence of such a bound on 80 vertices, even in the stronger sense of edge-sign preserving homomorphisms, follows from a result of Alon and Marshal [1]. This is improved to 48 in [6] and to 40 in [9]. A lower bound of 10 is also given in [6]. It is shown in [9], that, furthermore, if there is a bound on 10 vertices, it has to be the signed graph obtained from Signed Paley Graph on $\mathbb{F}_9$ by adding a universally positive vertex.

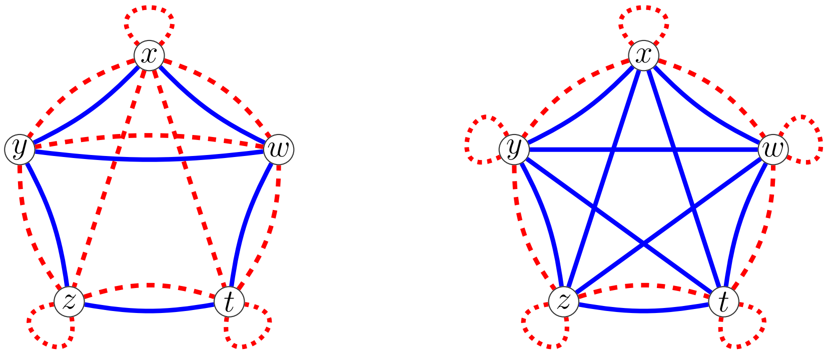


Figure 19: 5-vertices bound on $\mathcal{SP}$?

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