Surfaces with planar curvature lines: discretization, generation and application to the rationalization of curved architectural envelopes

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Surfaces with planar curvature lines: discretization, generation and application to the rationalization of curved architectural envelopes

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Abstract

Motivated by architectural applications, we propose a method to generate circular and conical meshes with planar curvature lines in both directions. The method is based on the discretization of the Gauss map of surfaces with planar curvatures lines. It allows generation of meshes in real-time via two planar guide curves. The resulting meshes can be used as a geometric base to build gridshells with flat panels, torsion free-nodes, node offset and planar arches. A particular technological application is for gridshells built with curved members: those can be built with planar piecewise-circular beams, and identical nodes if beams have circular cross-section.

Keywords: fabrication-aware design, architectural geometry, circular meshes, facade, pre-rationalization, cyclidic net

1 Introduction

The fabrication of doubly-curved building envelopes has a been the source of significant engineering challenges in the past few decades. These challenges have led to the emergence of a new field of research, that is often referred to as construction-aware design, fabrication-aware-design, shape rationalization or architectural geometry (Pottmann 2013). For steel-glass envelopes, research has focused on methods to generate quad meshes with properties such as planar faces, torsion-free nodes, and offset, as these yield strong benefits over triangular meshes: lower node complexity (Liu et al. 2006), better light transmission (Glymph et al. 2004), reduced waste in panel fabrication, and possibility to build multi-layer systems. A special attention was devoted to circular and conical meshes, which combine all these properties together.

Methods to obtain these properties can be split in two main categories: bottom-up and top-down approaches. In bottom-up methods, the fabrication constraints are taken into account at the time of generation of the shape. The most popular method in this family are scale-trans meshes and rotational meshes (Glymph et al. 2004). In these methods, subspaces of planar quad meshes can be intuitively explored by controlling two guiding curves. Some more recent research has shown that many other types of surfaces and properties can be attained from this generation method (Mesnil et al. 2017; Douthe et al. 2017; Mesnil et al. 2018) . (Yang et al. 2011), (Jiang et al. 2014) and (Deng et al. 2015), on the other hand, propose optimization algorithms to deform a mesh under geometric constraints. In top-down approaches, also referred to as post-rationalization methods, the shape is first designed without consideration of construction properties. An optimization process is applied afterwards to improve the constructability. Examples include (Liu et al. 2006) , (Zadravec, Schiftner, and Wallner 2010).

Bottom-up methods are very convenient design tools to explore meshes for a known geometric constraint, but they require a knowledge of the fabrication method at an early stage of the project (Austern, Capeluto, and Grobman 2018). On the other hand, top-down methods can be used in a later phase of the project, but there is no guarantee that they will yield a mesh that fulfills all the desired constraints. For example, optimizing a quad mesh to make its face planar and have offsets will tend to align the edges with the principal curvature directions (Liu et al. 2006). This process can alter significantly the appearance of the mesh, and might poorly fulfill the desired constraints, especially if the mesh topology is not compatible with the one of the curvature lines. As a rule of thumb, the more constrains are desired, the less likely post-rationalization is going to succeed.

In this paper, we investigate the generation of circular and conical quadrangular meshes for which all lines are planar. More precisely, we mean that the edge polylines of any quad strip are planar. We will describe these meshes as having planar curvature lines, even though this term is usually reserved for surfaces. This work is related to the one presented in (Mesnil et al. 2018) that looked at circular meshes for which curvature lines are planar in only one direction. The planarity property can be used to simplify beam fabrication and for aesthetic aspect. An example is the Schubert Marine Band Shell in Minnesota (USA) showed in Figure 1 (Schober 2015). Given these strong geometric constraints, the generation of these meshes is badly suited for post-rationalization. We therefore propose a bottom-up method, which uses two guiding curves as an input. Using the fact that circular meshes are discrete equivalents of surfaces parametrized by curvature lines (Bobenko 2008), the method is based on the discretization of surfaces with planar curvature lines. The method enables a real-time exploration of the entire design space.

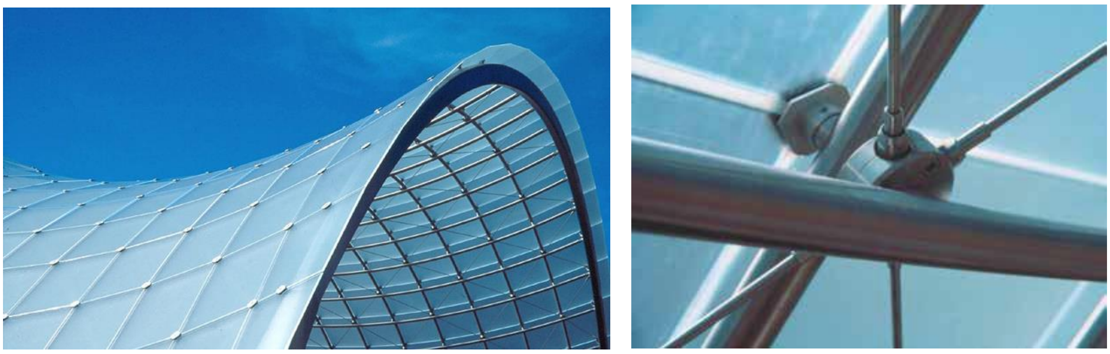

Figure 1 – Schubert Marine Club Band Shell. Left: Overview. Right: connection detail (©Brian Gulick) (pictures courtesy of James Carpenter Design Associates)

The second section of this paper introduces surfaces with planar curvature lines and our proposed discretization by circular or conical meshes. In section 3, we present a generation method for these meshes with two guiding curves. In section 4, we show examples of meshes generated by our method. Section 5 discusses the strengths and limitations of the method. Finally, in section 6, we propose a technological application of the method to the design steel-glass gridshells with interesting fabrication properties.

The main contributions of this paper are:

- A discretization of surfaces with planar curvature lines by circular or conical meshes with planar lines. These meshes have planar quadrangular faces and offset properties;
- A bottom-up generation method for these meshes. This method gives access to the entire design space;
- An exploration of the form potential of surfaces with planar curvature lines;
- A method to generate C_1 surfaces with planar curvature lines using cyclide nets;

- A technique to align meshes with boundary planes;
- An extension of the generation method to surfaces in which curvature lines are not planar, but piecewise planar;
- A method to generate networks of planar piecewise circular curves that intersect at 90°, that can be covered with planar quads and that have node offsets. One example is Dupin cylides;
- An application to the rationalization of steel-glass gridshells.

2 Discretization of surfaces with planar curvature lines

In this section we present relevant properties of surfaces with planar curvature lines and then propose a circular discretization.

2.1 Smooth surfaces with planar curvature lines

2.1.1 Early work

Surfaces with planar curvature lines were studied for the first time by Monge at the beginning of the 19^{th} century. In (Monge 1805), which is a pioneering work for the use of differentials for the study of surfaces, Monge studied the particular case of surfaces for which family of curvature lines lie on parallel planes: the so-called molding surfaces. Later on, Bonnet (Bonnet 1853) derived the general differential equations ruling the geometry of these surfaces. Other notable contributions to the understanding of these surfaces were made by Adam (Adam 1893), who studied the isothermic surfaces with planar curvature lines, and by Darboux and Bianchi (Darboux 1896; Bianchi 1894). Darboux (Darboux 1896) discovered that surfaces with planar curvature lines are the envelope of the radical planes of two families of spheres whose centers lie on focal conics (i.e. conics lying in orthogonal planes such that the apex of one is the focal point of the other) and whose radii are arbitrary functions (the radical plane of two intersecting spheres is the plane containing the intersection circle, see (Coxeter and Greitzer 1967) for the case of non-intersecting spheres). Figure 2 shows a surface constructed using this property with spheres of constant radii centered on two parabolas, thus yielding an Enneper surface (Berger and Gostiaux 1992).

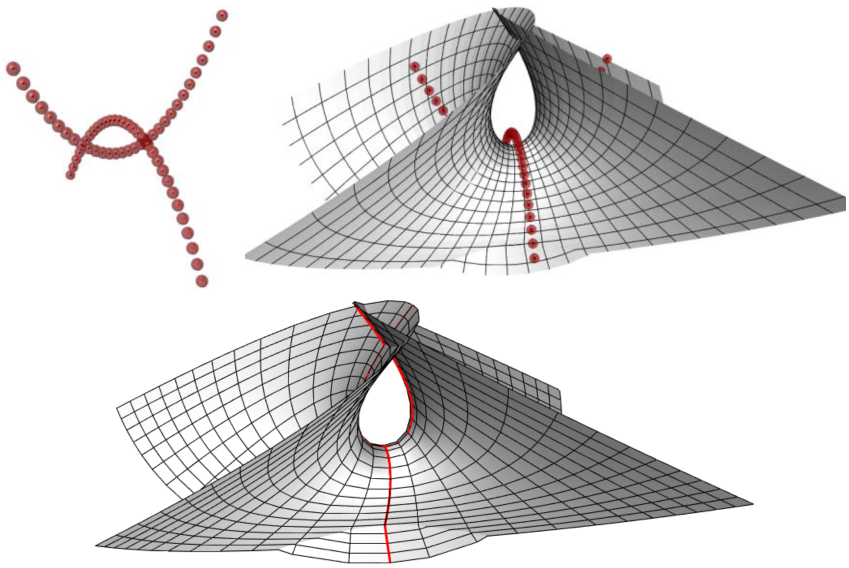

Figure 2 – A surface with planar curvature lines. Top: generated from the envelope of radical planes of two families of spheres centered on parabolas. Bottom: the same surface generated by our method

2.1.2 Gauss map geometry

On any point of a smooth surface, the unit normal vector can be described by a point on the unit sphere, S^2. The spherical surface described by all the normals is called the *Gauss map*. We shall now have a close look at the very particular structure of the Gauss map of surfaces with planar curvature lines.

The theorem of Joachimsthal states that if the intersection curve of two surfaces is a curvature line for both surfaces, then the angle between the two surfaces along the curve is constant, which means that the angle between the normal vectors of the two surfaces is constant along the curve. Since any planar curve is a curvature line of its plane, the angle between the normal of the surface and the plane of the curvature curve is constant. The Gauss map of this curve is therefore constrained to a cone: it is an arc of circle. Curvature lines always intersect at a right angle, and so do their Gauss maps. As a result, the Gauss map of a surface with planar curvature lines is a system of orthogonal arcs of circle on the unit sphere. (Eisenhart 1909) showed that a system of orthogonal circles on the sphere has necessarily the following geometric properties:

- Circles can be decomposed in two families. For each family, all the planes of the circles intersect on a common axis (thus forming a so-called *pencil* of planes).
- The axes of the two families are *polar reciprocal*. This property is shown in Figure 3 and can be explained as follows. Let us assume that the position of one axis L_1 is known and is such that it intersects the sphere (centered at O) at two points A and B. In the plane (OAB), the lines tangent to the sphere at A and B intersect at a point M_2. The polar reciprocal line to L_1 is the line passing through M_2 orthogonally to the plane (OAB).

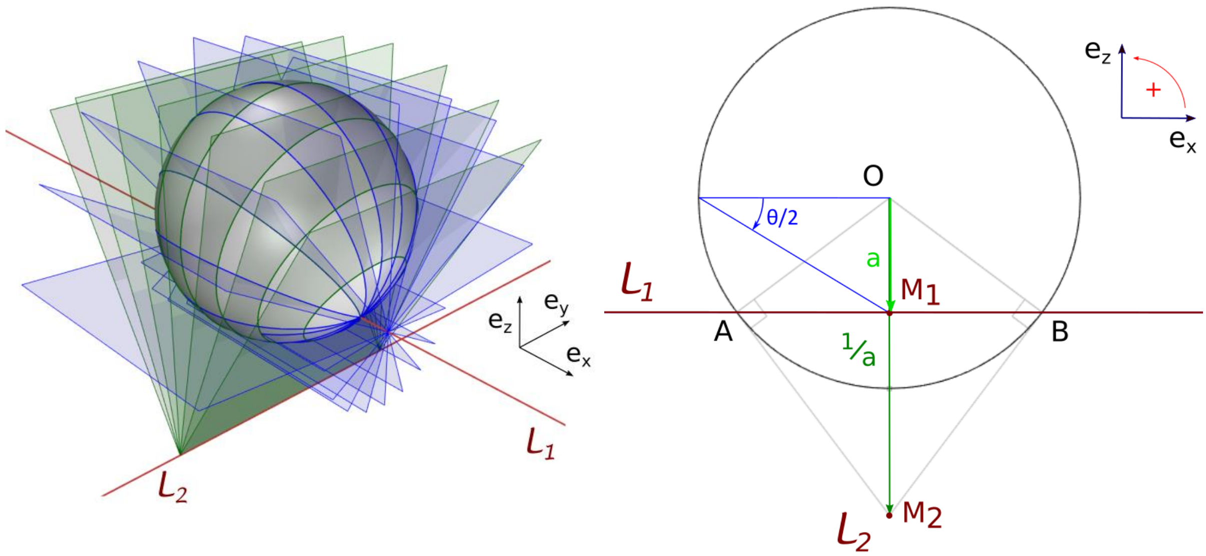

Figure 3 – Geometry of orthogonal circles on a sphere and associated polar reciprocal axes

Considering a unit sphere centered at O = (0,0,0), the equation of these axes may be written as:

$$L_1 = \{\, M_1 + \lambda\, e_x \ ,\lambda \in R \,\}$$

$$L_2 = \{\, M_2 + \mu\, e_y \ ,\mu \in R \,\}$$

Where: $$M_1 = (0,0,a) \ \ , \ \ M_2 = \left(0,0,\frac{1}{a}\right) \ , \ \ a \in R$$

The position of the two axes determines the structure of the Gauss map, and there is only one degree of freedom (the constant *a*) for positioning these axes relatively to each other. Figure 4 shows how the structure of the Gauss map varies for values of *a* ranging from 0 to 1. It can be observed that only two orthogonal circles (in thick lines) do not vary, these circles are great circles of the sphere. The parameter *a* has the following symmetries (Figure 3):

- If *a* is replaced by *1/a*, the heights of the two axes are exchanged, which is equivalent to applying a rotation of 90° around e_z.
- Replacing *a* by *–a* is equivalent to applying a symmetry about plane (O, e_x, e_z).
- The cases a → + ∞ and a → - ∞ yield the same axes positions: L_2 is the axis (O,e_x), and L_1 is at infinity.

In order to manipulate a variable that highlights these symmetries, we introduce a reparametrization of the variable *a* by the variable θ defined as follows:

$$a = \tan\left(\frac{\theta}{2}\right)$$

The symmetries of the problems are then reflected as follows:

- Exchanging the heights of the two axis is done by taking the complementary angle $\pi - \theta$;
- Mirroring the axis positions about (O, e_x, e_z) is done by negating the angle;
- The identical cases a → + ∞ and a → - ∞ correspond to $\theta = +\pi$ and $\theta = -\pi$, which are equal angles (modulo 2π). θ does not take infinite values, and is therefore easier to handle in a programming environment.

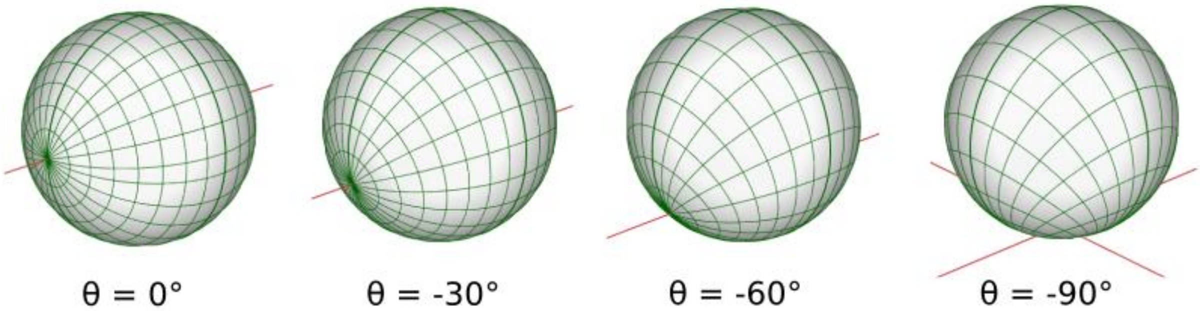

Figure 4 – Structure of the Gauss map of a surface with planar curvature lines parametrized by curvature lines, for different distances between polar reciprocal axes

2.1.3 Developable surfaces

The Gauss map of developable surfaces with planar curvature lines is degenerated into a single curve. Therefore the structure shown in Figure 4 is of little help to understand their structure. However, their geometry is well known. First, we can observe that the rulings of a developable surface form its first family of curvature lines, these are obviously planar. For the 2nd family of curvature lines, we have to look separately at the three types of smooth developable surfaces: Cylinders, cones, and envelopes of the tangent lines to a 3D curve. The second family of curvature lines is always planar for cylinders and for right cones with a circular base. The more complex 3rd type was studied by Bonnet (Bonnet 1853). He proved that surfaces generated by the tangent to a general helix (i.e. a curve which tangent has a constant angle with a given axis) are the only developable surfaces of this type with planar curvature lines. An example is shown in Figure 5.

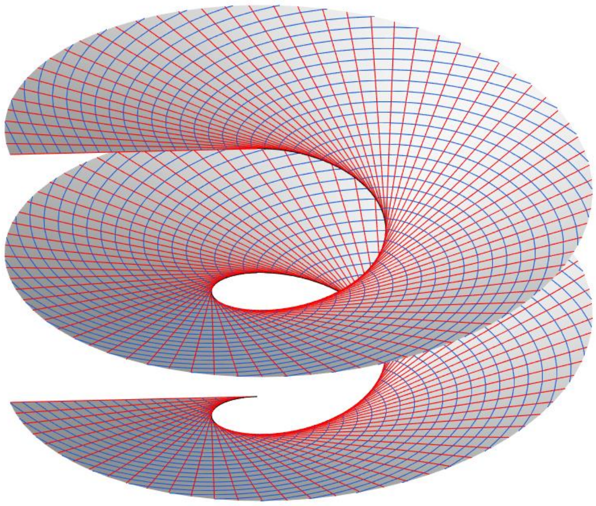

Figure 5 – Tangents to a general helix form a developable surface with planar curvature lines (shown in blue and red).

2.2 Circular quad mesh with planar lines

Meshes for which each face is inscribed in a circle are called *circular meshes*. Quad circular meshes are interesting for architectural gridshell design, as they have planar faces, torsion-free nodes and vertex offset. The offset allows to easily give a depth to the mesh for materialization and structural purposes. Circular meshes can be seen as a discretization of surfaces parametrized by curvature lines (Pottmann and Wallner 2006). In this section, we will introduce a discretization of surfaces with planar curvature lines by circular meshes with planar lines.

2.2.1 Gauss map discretization

Our discretization is based on a discretization of the Gauss map. It turns out that the orthogonal circles discussed in section 2.1.2 and shown in Figure 4 naturally yield a circular mesh with vertices on the unit sphere. We can summarize this result as follows:

Proposition 1
The intersection points of two orthogonal families of circles on a sphere define the vertices of a circular mesh with planar lines.

Proof

We start by the case where the two reciprocal axes are tangent to the sphere (the case $\Theta = \pm 90$). We use the stereographic projection of the sphere centered at the point of tangency with the two axes. This transformation projects the sphere on a plane, as shown on Figure 6 (left). The circles all pass through the center of the projection, and are therefore mapped to straight lines on the plane. The stereographic projection conserves intersection angles: Since the circles intersect at 90°, their projections also intersect at a right angle. As a result, the projection of two pairs of circles are straight lines that intersect at 90°, thus forming a rectangle, which is a circular quad. Since the projection and its inverse map circles to circles, we conclude that the four intersection points on the sphere are cocyclic.

In the case where the axes are not tangent to the sphere (Figure 6, right), we consider the stereographic projection from an intersection point of a reciprocal axis with the sphere. This projection transforms the circle passing through the pole into straight lines (which are concurrent with the reciprocal axis). The circles from the other family are mapped to circles, which are orthogonal to the straight lines since the projection is conformal. As a result, the projection of four intersection points are the intersection of two concentric circles and two of their rays, and are therefore cocyclic.

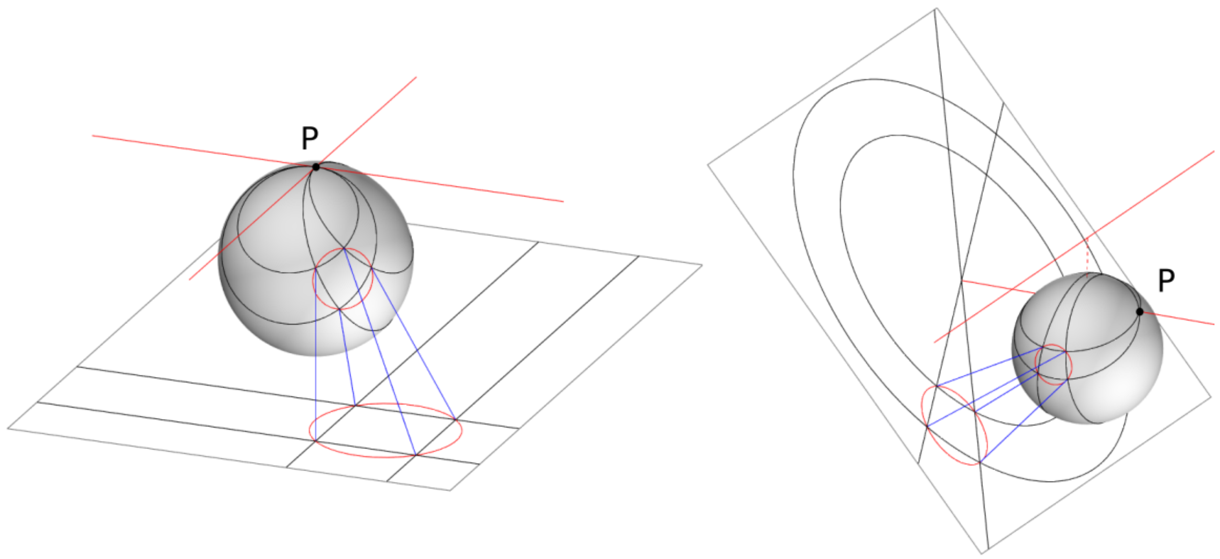

Figure 6 - Stereographic projection of orthogonal circles from a pole P. Left: case where Θ = ±90°. Right: general case

2.2.2 Property of the discrete Gauss map

Two orthogonal families of circles on the sphere have the following property (Eisenhart 1909): a cone tangent to the sphere along a circle of one family has its apex on the reciprocal axis of that family. An example is visible in Figure 7 (cone shown in orange). This property can be interpreted as a property of the Gauss map of a smooth surface with planar curvature lines: along the Gauss map of the maximum (resp. minimum) curvature line, the tangent vectors in the minimum (resp. maximum) curvature direction belong to a cone. Our discrete Gauss map model happens to have a discrete equivalent of this property:

Proposition 2

If a strip of quads is inscribed in a sphere and if its longitudinal edges are coplanar, then its transversal edges belong to an oblique circular cone with apex on the reciprocal axis.

Such a cone is visible in blue in Figure 7 (in blue). This rather abstract property will be used in section 4.6 to prove an important property of our discrete model.

Proof

Let us consider a strip of quads with the structure discussed in section 2.2.1, as shown in Figure 7. Vertices of the strip belong to two circles C_1 and C_2. We pick two vertices Q_1 and Q_2 of the mesh. The tangents to circles C_1 and C_2 at Q_1 and Q_2 belong to a same cone (in orange) with apex on the reciprocal axis L_2. They are therefore coplanar. As a result, Q_1Q_2 is a ruling of the developable surface connecting C_1 and C_2. The developable surfaces connecting two cospherical circles are quadratic cones (Glaeser, Stachel, and Odehnal 2016). Consequently, the transversal edge Q_1Q_2 is a ruling of one of these quadratic cones (in blue). Since all the rulings passing through vertices of the strip also intersect the reciprocal axis L_1, the apex of this cone is necessarily on L_1.

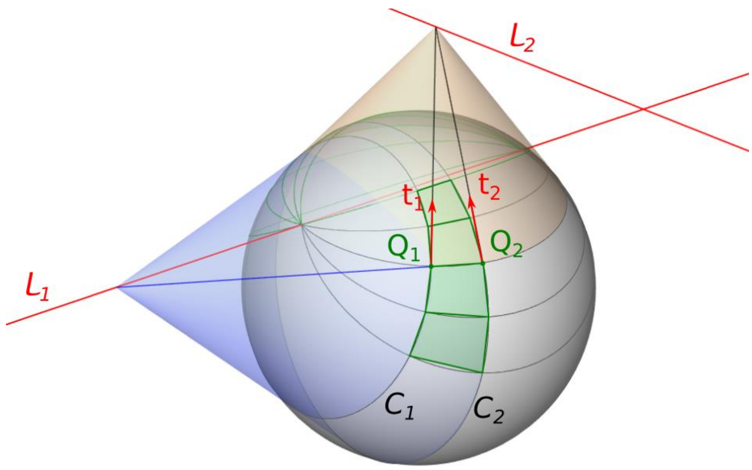

Figure 7 – Edges of a strip of quad on the discrete Gauss map belong to a cone with apex on a reciprocal axis.

2.2.3 Surface discretization

The previous sections shows a natural discretization of the Gauss map of any surface with planar curvature lines. Smooth surfaces are related to their Gauss map by a Combescure transform: at any given point of the surface, the tangent vectors in principal curvature directions are parallel to the corresponding tangent vectors on the Gauss map. Combescure transforms can be discretized using the notion of mesh parallelism. Two meshes are said to be parallel if their corresponding edges are parallel. They are then said to be related by a discrete Combescure transform (Pottmann et al. 2007). A discrete Combescure transform conserves the planarity of a face and the vertex angles, it therefore conserves the circularity of meshes.

Using these notions, a surface with planar curvature lines can be discretized as follows:

- Compute its Gauss map parametrized according to principal curvature directions, and discretize it into a circular mesh;
- Discretize the Combescure transform that maps the smooth Gauss map to the surface, and apply it to the discrete Gauss mesh.

In Section 3, we will address how this fact can be used for generation purposes.

2.3 Conical quad mesh

Conical meshes is a family of quad meshes introduced in (Liu et al. 2006) with planar faces and torsion-free nodes. Whereas circular meshes have node-offsets, conical meshes have face-offsets: we can build a parallel mesh such that corresponding faces are at a constant distance. (Pottmann and Wallner 2006) showed the existence a duality between circular and conical meshes: from any circular mesh with rectangular combinatorics, one can construct a two-parameter family of conical meshes. The method consists in building at each node a plane normal to the node axis. These planes form the faces of the dual conical mesh, and the intersection lines of these planes form its edges. It turns that this construction conserves the planarity of lines, thus allowing to generate conical meshes with planar lines:

Proposition 3
The conical dual of a circular mesh with planar curvature lines also has planar lines.

Proof:
We will show this property by building the conical dual of a circular Gauss mesh with planar lines (following the construction of (Pottmann and Wallner 2006)), and by showing that it also has planar

lines. General conical meshes with planar lines can then be obtained by applying Combescure transforms to this conical Gauss image.
Let us consider two families of orthogonal circles on the sphere. The cones that are tangent to the unit sphere along the circles of one family have their apex on the axis of the other family. On Figure 8, we show the apexes B_1 and B_2 of the cones tangent to the circles C_1 and C_2 (the cone tangent to C_2 is shown in blue). The blue lines are the tangents to the unit sphere along C_3 and C_4 at the intersection points of the circles. We then build the conical dual (shown in green) to the circular quad formed by the four circles. The faces of this dual mesh are the tangent planes to the sphere at the circle intersection points. These planes also correspond to tangent planes to the cones (tangency occurs along the blue lines). As such, their intersection line passes through the cones apex. Therefore, the edges e_1 and e_2 of the conical dual are aligned with the points B_1 and B_2. e_1 and e_2 are each in a plane that contains axis (B_1B_2). Since these two edges have a point in common, these two planes are identical. If we were to build the next edge e_3, it would also belong to that same plane. We conclude that the conical dual has planar lines. We finally note that each quad of this conical dual is tangent to the unit sphere, and can therefore be interpreted as a conical Gauss image in the sense of (Pottmann and Wallner 2006).

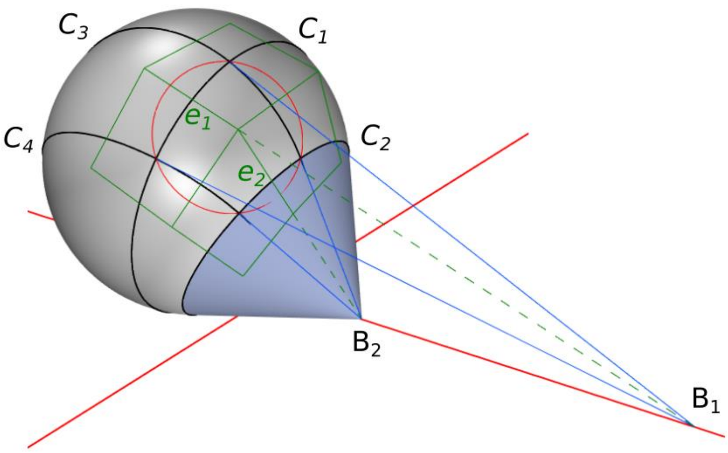

Figure 8 – The conical dual mesh of a circular mesh on the sphere with planar curvature lines also has planar curvature lines

3 Generation method

We propose here a method to construct a circular mesh with planar curvature lines from two input guide curves. Generating a surface from two curves is a method appreciated by designers (Mesnil et al. 2017). It provides an intuitive control on the shape. Sweep surfaces and translation surfaces are for example two widely used methods to generate surfaces or meshes on CAD programs. In our method, the guide curves will correspond to curvature lines of the surface. As such, they need to be planar and to intersect at 90°.

3.1 Control with characteristic guide curves

In this section, we will explain the generation method in the special case where:

- The planes of the two guide curves are orthogonal, and
- The initial tangent vector of one curve is orthogonal to the plane containing the other curve.

These properties can be easily achieved as follows: Draw one curve in the plane Oxz with initial tangent (O, e_x), and a second curve in plane Oyz with initial tangent (O,e_y). With these properties, the guide curves will correspond to curvature lines of the surface for which the normal vector is coplanar with the curvature line. Such lines are said to be characteristic of the surface, as they stand out visually (see for example Figure 2 and Figure 21), and are the most intuitive to control. We note that the orthogonality conditions on the initial tangent vectors can be easily achieved working with B-spline guide curves, for which the initial tangent is aligned with the first two control points.

Our method can be decomposed into the following steps (see Figure 9):

i) The guide curves are sampled in points. At each point, the Frenet normal vector is calculated. These vectors correspond to points of the Gauss map on S^2, they lie in two great circles.

ii) The user chooses a parameter ϴ, that structures the Gauss map as shown on Figure 4 (the Gauss map of the guide curves lie in the invariant great circles highlighted in thick). This parameter is used to construct the axes of the pencils of planes, as explained in section 2.1. The direction of the axes is given by the initial tangent vectors of the curves.

iii) For each point of the Gauss map, a plane passing through this point and containing one axis is constructed. The intersection of these planes with S^2 yields a circular mesh, as proven in section 2.2.

iv) A Combescure transform (as defined in section 2.2.3) is applied to this circular mesh to generate a mesh that fits the generatrices. Such a transformation conserves the planarity and circularity of the faces, and the planarity of the lines.

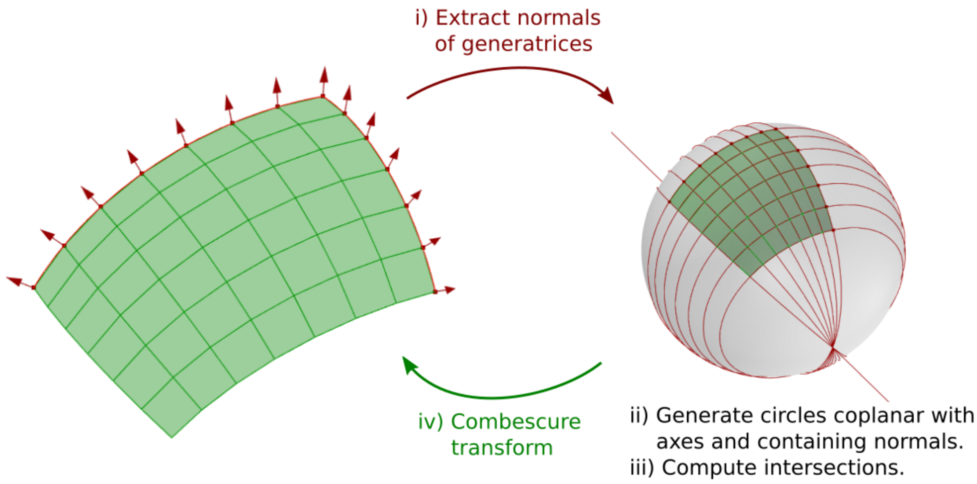

Figure 9 – Overview of the proposed method. Gauss map shown on the right

The calculation of node axes from an arbitrary circular mesh can be a tedious step. (Liu and Wang 2008) suggest for example to use SVD, and show the non-existence of the node axes for some types of circular meshes. In our method, the Gauss map is calculated, so the node axes are already determined.

3.2 Control with arbitrary guide curves

We will now show how the method can be adapted to the general case where the planes of the guide curves are not necessarily orthogonal.

As shown in Figure 10, let us consider two planar guide curves G_1 and G_2 that intersect at 90°. The initial tangents of G_1 and G_2 at the intersection point P determines the tangent plane of the surface we are going to build. We name N_P the normal vector to this plane. Since the guide curves will be curvature lines, the normals to the surface must rotate along the curve like a Darboux frame, also referred to as rotation-minimizing frames in CAD environments. We generate these normals at each point of the guide curves by propagating N_P without torsion along the guide curves. These normals describe two arcs of circles C_1 and C_2 on the unit sphere, as shown on the right of Figure 10.

The next step is to build the axis L_1, the first of the two polar reciprocal axes of the Gauss map (as explained in section 2.1.2). This axis passes through the apex A_1 of the cone tangent to S^2 along C_1. It also belongs to the plane of C_2. There is one degree of freedom left to choose the orientation of this axis. We parametrize this degree of freedom by constructing its orientation vector e_1 as follows:

$$e_1 = u + \lambda v$$

Where:

- λ is a real variable, the degree of freedom;
- u is the unit vector of the line (A_1Q_2), Q_2 being the center of circle C_2;
- v is a vector normal to u in the plane of circle C2.

For a choice of λ, the value of parameter *a* of section 2.1.2 is given by:

$$a^2 = OK_1^2 = OA_1^2 - A_1K_1^2 = OA_1^2 - \left(\frac{e_1 \cdot \overrightarrow{A_1O}}{\|e_1\|}\right)^2$$

Where K_1 is the orthogonal projection of the center O of the sphere on axis L_1.

This equation can be expressed as a polynomial of degree two in λ. So, one can choose a parameter Θ (we recall that $a=\tan(\Theta/2)$) and find the corresponding value of λ by solving this equation. A sign convention was developed to choose the root amongst the two depending on the sign of Θ; this aspect is secondary and will not be detailed here for sake of conciseness. Note that some values of Θ are impossible to reach for any choice of λ: since the axis L_1 passes through A_1 and belongs to the plane of C_2, the distance between the sphere center to axis L_1 (that is, the parameter *a*) is bounded by an upper and lower value.

Once axis L_1 is determined, its polar reciprocal axis L_2 can be constructed as explained in section 2.1.2. We can then proceed in the same way as in section 3.1 to build the Gauss map and the mesh.

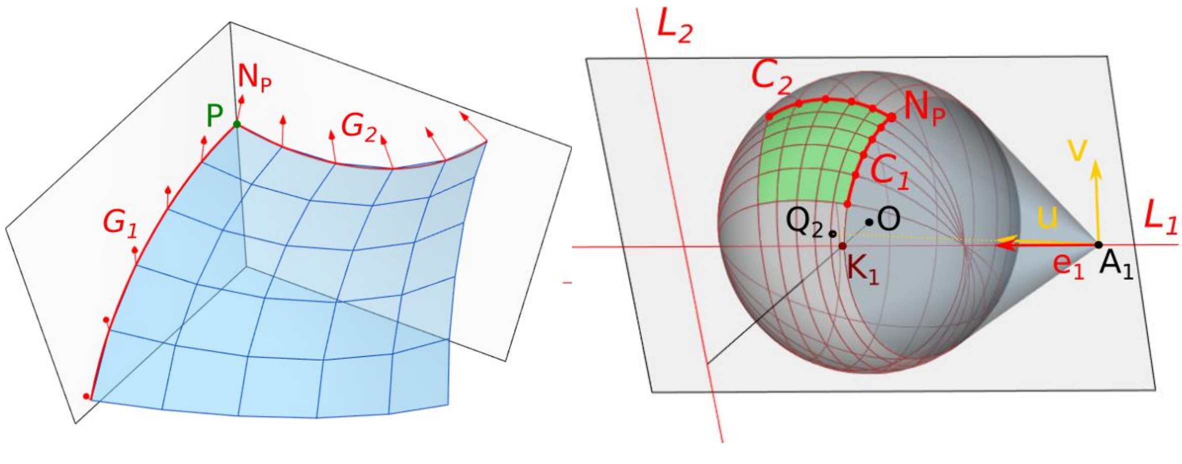

Figure 10 – Generation method with two arbitrary orthogonal planar curves

3.3 Developable surfaces

As discussed in section 2.1.3, developable surfaces may also have planar curvature lines. Since they correspond to a case where the Gauss map degenerates to a curve, the method we just presented cannot be used. However, we are going to show in this section that it is actually quite straight forward to generate a discrete circular equivalent of these surfaces for the non-trivial 3rd type of developable surfaces.

As shown in Figure 11, a discrete general helix can be defined as a polyline for which each segment has a constant angle α with a reference plane. By extending these segments, we obtain the rulings of a discrete developable surface (two adjacent rulings are coplanar since they intersect), which constitute the first family of curvature lines. The second family is obtained by intersecting these rulings with planes parallel to the reference plane. The mesh built by these curvature lines is circular. Indeed, let us consider a quad ABCD, where (AB) and (CD) intersect at a point O on the generatrix. Since (AB) and (CD) have the same slope, OA = OD and OB = OC. Therefore ABCD is an isosceles trapezoid and is inscribed in a circle.

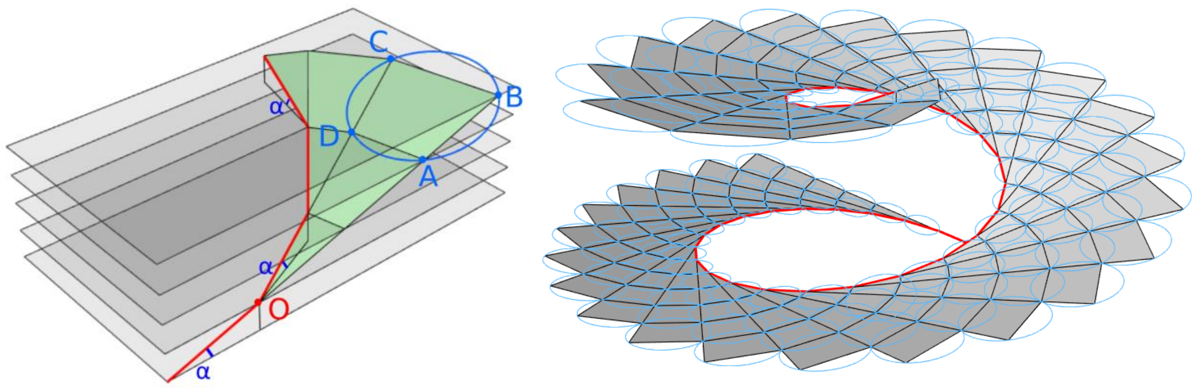

Figure 11 – Developable circular meshes with planar curvature lines.

4 Application to morphogenesis of double-curvature facades

4.1 Classic analytical surfaces

Our method can be used to generate some classical surfaces of differential geometry.

4.1.1 Surfaces of revolution

Meshes of revolution correspond to the particular case where ϴ = 0° or ±180° and one generatrix is a circle. The Gauss map is then a network of parallels and meridians on the sphere. An example is the torus shown on Figure 12.

4.1.2 Moulding surfaces

Moulding surfaces are surfaces that can be generated by sliding a planar generatrix along a planar rail curve, such that the plane of the generatrix is always perpendicular to the rail. Their potential for fabrication-aware design was investigated in (Mesnil et al. 2015). They can be obtained from our method by setting the parameter ϴ to 0° or ±180° , with any type of guide curves. If ϴ = 0°, one curve is the generatrix, the other is the rail. If ϴ = 180°, the roles of the two curves are exchanged. For both cases, the Gauss map has the same structure as for a surface or revolution. Two moulding surfaces are shown on Figure 14.

4.1.3 Dupin cyclides

Dupin cyclides is a family of surfaces that can be obtained by applying an inversion to a torus. A key property of these surfaces is that their curvature lines are circles, and are therefore planar. They are often used in CAD design, as they can be parametrized by NURBS and they allow a C^1-continuous (i.e. without crease) transition between cylinders, cones, tori, spheres and planes (Pratt 1990; Zube and Krasauskas 2015).

Dupin cyclides can be generated with our method using two orthogonal circles as guide curves. Figure 12 shows three different cyclides generated from the two same circles but with different values of the parameter ϴ. The curvature lines are all inscribed in circles, similarly to the smooth Dupin cyclides. This fact is a consequence of a more general property of our meshes which will be more detailed in section 4.6.

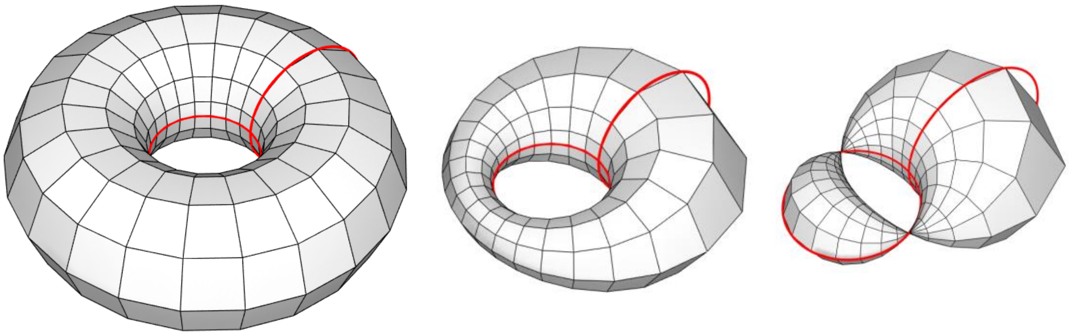

Figure 12 - Generation of discrete Dupin cyclides from two circles. Left: ϴ =0° (torus). Middle: ϴ =30°. Right: ϴ=65°.

4.2 Free-form surfaces

A large variety of surfaces can be generated by playing with the geometry of the guide curves and the parameter ϴ. An example is shown in Figure 13, where the guide curves are highlighted in red. Another example is shown in Figure 2. We can notice that our methods allow to design the same surface as one built as an envelope of radical planes of spheres with centers on conics, as explained in section 2.1.1.

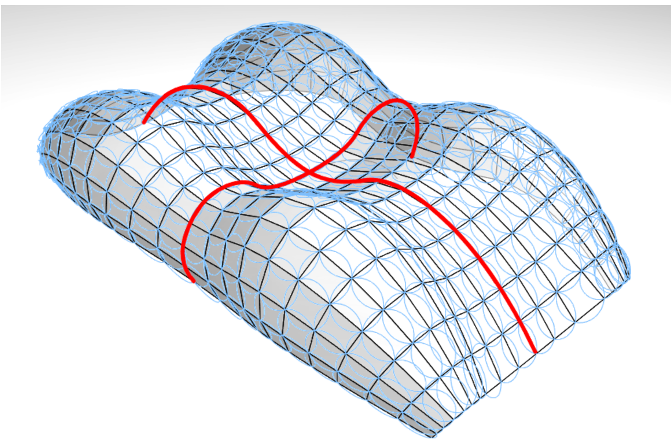

Figure 13 – Freeform circular meshes with planar curvature lines

4.3 Effect of parameter ϴ

Figure 14 illustrates the family of shapes that can be obtained by the two same curves by varying the parameter ϴ from -180 to 180°. As discussed in section 4.1, the cases ϴ =-180° and 0° correspond to moulding surfaces. Our method gives access to a full range of surfaces that can be interpreted as the result of a morphing between these two moulding surfaces. There is a discontinuity in the aspect of the surface between ϴ = 45° and 150°. When increasing ϴ beyond 45°, a portion of the mesh bulges out, the mesh self-intersects, so the mesh is less interesting for architectural purposes. However, when ϴ is further increased, fair meshes are obtained again, with a very different geometry. At ϴ = 180° the exact same mesh as with ϴ = -180° is obtained. The range of angles for which the mesh is fair and does not auto-intersect is specific to the chosen guide curves. The figure also shows the effect of varying parameter ϴ with fixed guide curves, but from side views. The rotation of the planes can be clearly observed.

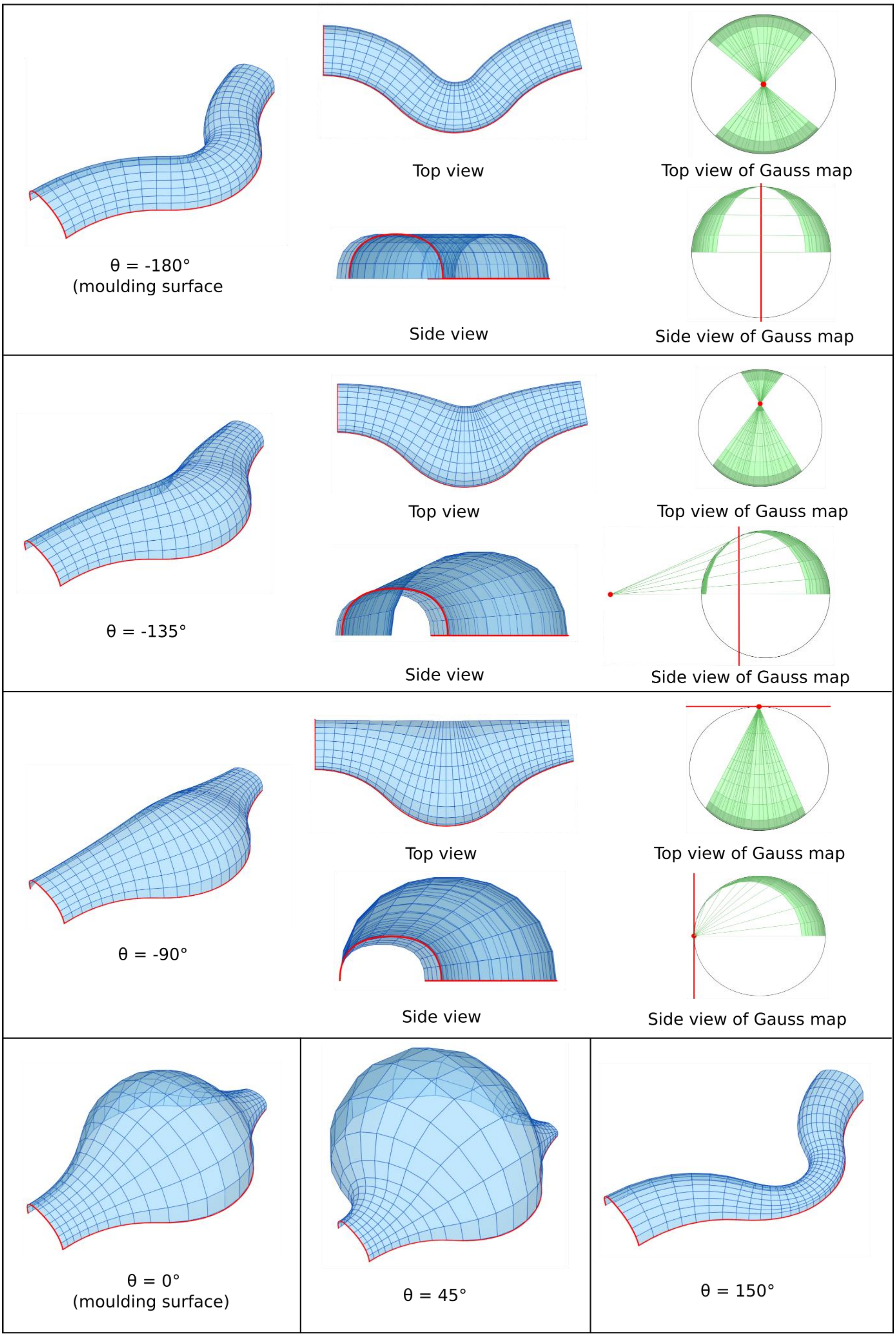

Figure 14 – Family of meshes that can be generated from two curves with different values of Θ

4.4 Smooth surfaces with planar curvature lines

Our method can be used to generate a mesh as fine as needed: the discretization steps of the generatrices can be arbitrarily small. However, in some applications, it is desirable to work with a smooth surface or smooth edges. Smooth surfaces can be obtained from our circular meshes by generating a patch of Dupin cyclide on each quad, thus forming a so-called cyclidic net (McLean 1985). Cyclidic nets are C^1-continuous surfaces (the surface and its tangent planes are continuous), and their curvature lines are composed of circular arcs. (Bobenko and Huhnen-Venedey 2012) showed that these nets can be generated on any circular mesh with the topology of a disk. These nets are used in (Bo et al. 2011) to generate the geometry of gridshells where beams are arcs of circles and nodes are identical. There are three degrees of freedom to generate such a net. They correspond to the orientation of one reference frame that defines the orientation of the curvature line tangents at one vertex.

The successive arcs of circle that compose a curvature line of a cyclidic net usually lie in different planes – the curvature lines are therefore not planar across the net. However, when the underlying circular mesh has planar lines, it is possible to construct a cyclidic net with planar curvature lines.

Let us first see how such a net can be constructed. A circular mesh with planar lines is showed on Figure 15 (only the resulting cyclidic net is shown for clarity, the vertices of this mesh are the vertices of the net). At a vertex A, we compute the normal N_A and the corresponding orthogonal circles C_{AB} and C_{AD} on the Gauss map. We then take the tangent vectors to these circles, t_1 and t_2. These two vectors define the reference frame that generates a cyclidic net with planar curvature lines.

We shall now prove that this resulting cyclidic net actually has planar curvature lines. Since t_1 is coplanar with C_{AB}, the arc AB of the net is coplanar with the plane of A, B and C. Therefore, the bi-arcs ABC and DEF of the net are planar. Their Gauss maps are contained in circles C_{AB} and C_{DE}, we name L_1 the intersection line of the planes of these two circles (this is one of the two polar reciprocal axes). We then pick a point P on arc AD, and build the arcs PQ and QR. Since the patch ABED has planar curvature lines, the plane of the Gauss map of arc PQ contains L_1. The same result holds for the Gauss map of arc QR, so the planes of these two Gauss maps are identical: the Gauss map of bi-arc PQR is planar. At any point of the bi-arc PQR, the tangent vector is tangent to the one of the Gauss map. Therefore, therefore, the bi-arc PQR is also planar. We conclude that the whole cyclidic net has planar curvature lines.

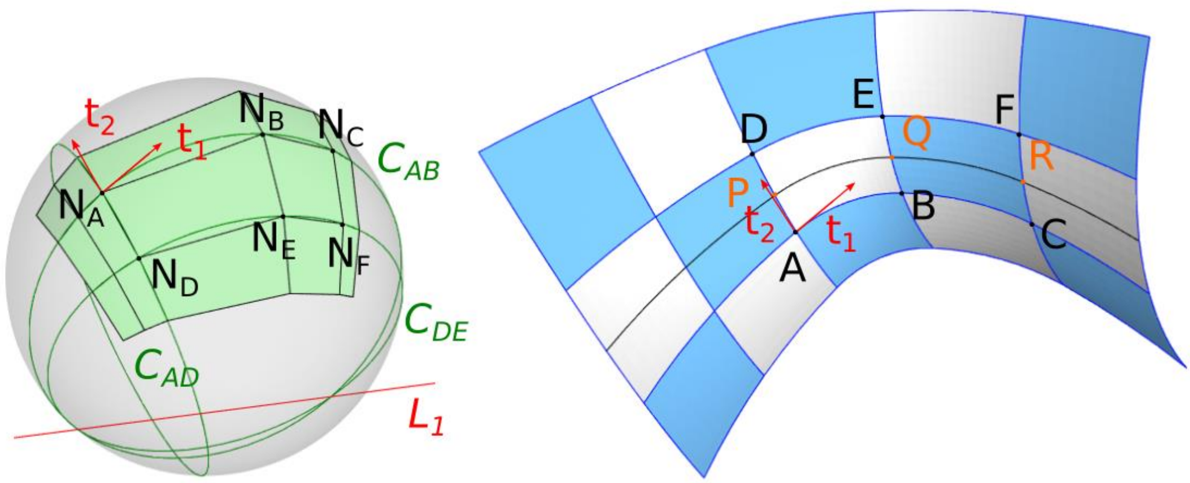

Figure 15 - Smooth surface with planar curvature lines and its Gauss map.

4.5 Aligning a mesh with boundary planes

Our method can be used to generate meshes where the edge lines are contained in a target plane. One typical application is to align a mesh with a vertical planar facade. For example, Figure 16 shows a mesh whose boundaries are contained in four vertical planes. The only requirement to obtain this alignment is that the tangent vectors at the ends of the guiding curves are horizontal.

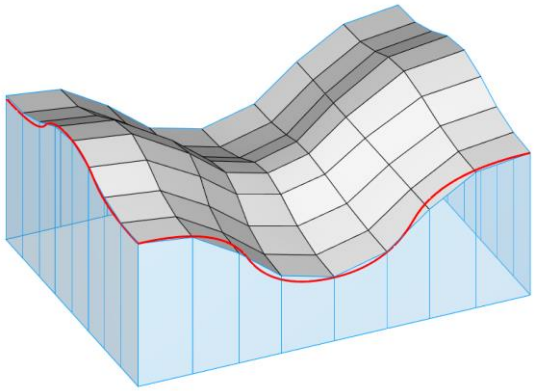

Figure 16 - Mesh aligned with four vertical planes

In a more general way, a mesh can be aligned with four planes if two intersection lines have orthogonal directions. This property is shown on Figure 17, where the two orthogonal intersection lines are shown in yellow. The following method can then be used to align the mesh with four boundary planes:

i) Pick a value for parameter Θ such that the planes P'_i intersect S^2 in a pleasing way (reminder: axis 1 is the axis of the pencil of planes P_1 and P_2, same thing for axis 2 with P_3 and P_4) ;
ii) The green arc on the Gauss map gives the normals of the generatrices ;
iii) To construct the generatrix, we can apply a parallel transformation to the green arc. We can also just draw any curve such that the normals at the tips correspond to the tips of the green arc.

The mesh shown on Figure 18 is based on a patch that is fitted on two planes that intersect at 36°. This allows successive reflections to generate a final mesh with a symmetry of order five. The mesh is aligned with the floor and the pentagonal building inside.

4.6 Piecewise circular curvature lines

(Darboux 1896) showed that if a curvature line of a surface with planar curvature lines is an arc of circle, then all the curvature lines of the same family are also circular. It turns out that our model enjoys a discrete version of this property. There are many interesting applications for fabrication purposes, as it is much simpler technologically to bend a beam if the radius of curvature is constant. This property is used for the shading structure shown in Figure 25, which is built entirely from planar bi-arcs. This property can be formalized as follows:

Proposition 4
If a polyline of a circular mesh with planar curvature lines is inscribed in an arc of circle (or an n-arc), then all the subsequent curvature lines are also inscribed in arcs of circle (or in n-arcs).

Note: In that proposition, we assume that the change of curvature radius of the n-arcs occurs at nodes, and not in an edge.

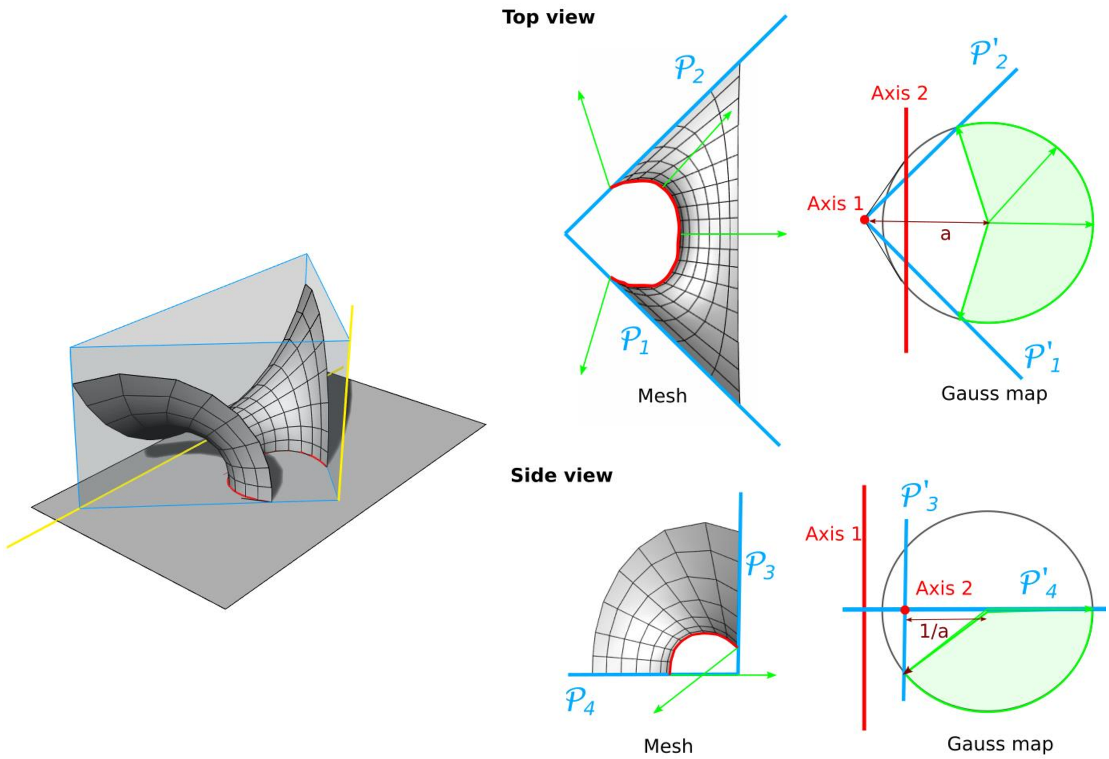

Figure 17 - Fitting four boundary planes

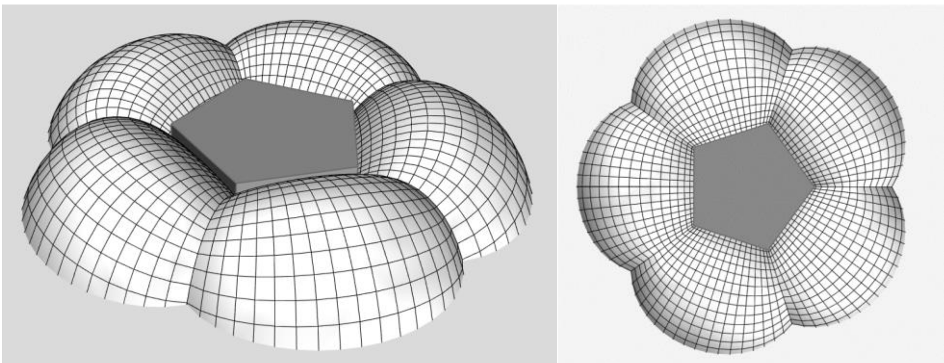

Figure 18 – Symmetric mesh built from a patch generated by fitting four boundary planes

Proof:

Referring to Figure 19, let us consider two adjacent polylines H_1 and H_2 of a circular mesh with planar lines. Let us assume that the vertices of H_1 are inscribed in a circle. The Gauss map H_1' of H_1 is then homothetic to H_1. We call H_2' the Gauss map of H_2. As proven in section 2.2, the vertices of H_1' and H_2' lie on a common cone.

Since H_1 is homothetic to H_1', and since all the edges between H_1 and H_2 are parallel to the ones between H_1' and H_2', the edges between H_1 and H_2 are also located on a cone. This cone is homothetic to the one of the Gauss map. Since the plane of H_1 (resp. H_2) is parallel to the plane of H_1' (resp H_2'), and since the vertices of H_1 are inscribed in a circle, the vertices of H_2 are also included in a circle.

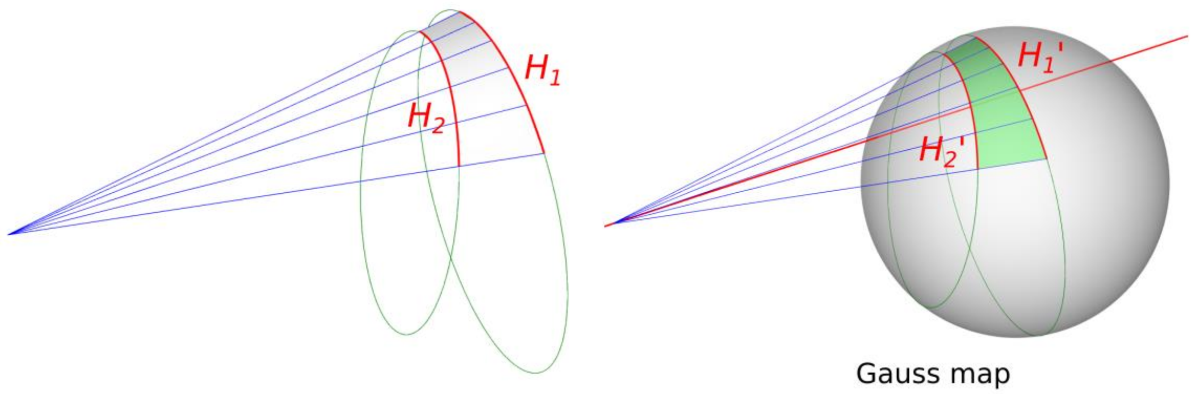

Figure 19 – If a guide curve H_1 is an arc of circle, the subsequent curvature lines (e.g. H_2) will also be inscribed in an arc of circle

4.7 Piecewise planar curvature lines

One important application of the proposed meshes is for the fabrication of gridshells. For such structures, beams never span the whole length of the building: several beam segments are spliced together. If we allow a change of the orientation of the plane of a beam at each splice, the variety of possible surfaces becomes significantly richer. One way to generate surfaces with piecewise planar curvature lines is to use cyclidic nets. As discussed in section 4.4, these nets can be built on any circular mesh with topology of a disk, and their curvature lines are piecewise circular. However, the resulting surfaces are often highly wavy. (Mesnil, Douthe, Baverel, and Léger 2017) suggest to minimize the Willmore energy of the surface by optimizing the orientation of the reference frame. However, convergence towards a desired surface aspect is not guaranteed when the net is built on an arbitrary circular mesh.

Our method can be used to design meshes in which curvature lines are piecewise planar. One way to obtain this is to generate a first patch with a given value of Θ, and to use one of its boundary as a guide curve to generate a second patch with a different value of Θ. This method is applied in the mesh shown in Figure 20. A first patch is generated from the two blue guide curves. The second patch it then generated using the two red guide curves, where one is the boundary of the previous patch. The C^1 continuity of the beams is guaranteed by the C^1 continuity of the bottom guide curves. The rest of the mesh is obtained by symmetry. Note that, in order to apply the symmetry, the second patch must end in the plane of symmetry. This fact constrains the value of Θ for the second patch. In the resulting surface, the radial curvature lines are planar, while each longitudinal curvature line lies in four different planes.

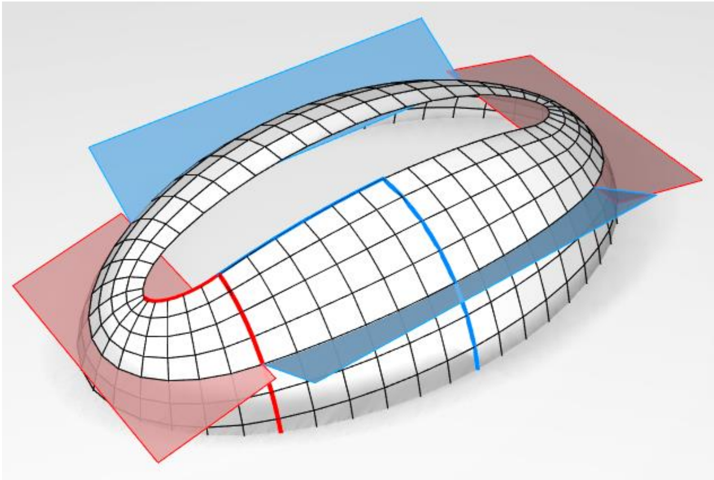

Figure 20 – Mesh with piecewise planar curvature lines: Each longitudinal line lies in four different planes

Figure 21 shows a case study in which we approximated the geometry of the Hippo Haus of the zoo of Berlin, designed by the firm Schlaich Bergermann und Partners, by a circular mesh with piecewise planar lines. The original geometry was derived from two translation surfaces, joined by a transition area. We reconstructed the actual guide curves (G_1, G_2, G_3) used on the project, their geometry being given in (Schlaich and Schober 1998). We then used our method to generate a circular mesh. Similarly to the actual design, we implemented a transition area between the two domes. As a result, there is a twist in the plane of the beams happening at the junctions between the domes and this transition area. The surface has three degrees of freedom, one parameter Θ for each patch: Θ_1, Θ_2 and Θ_3. We generated the mesh using G_1 and G_2 as guide curves. For arbitrary values of the parameters Θ, the end boundary does not match G_3. In order to match, we choose a value for parameter Θ_1, and then optimize the parameters Θ_2 and Θ_3 in order to minimize the distance between G_3 and the end boundary. G_2 and G_3 are similar to each other, as they are both parabolas, so a nearly perfect fit can be obtained. The optimization is performed with the BOBYQA algorithm (Powell 2007). In order to improve the aesthetic of the mesh, different values of Θ are also used for the portions of domes outside guide curves G_2 and G_3 (Θ_0 and Θ_4).

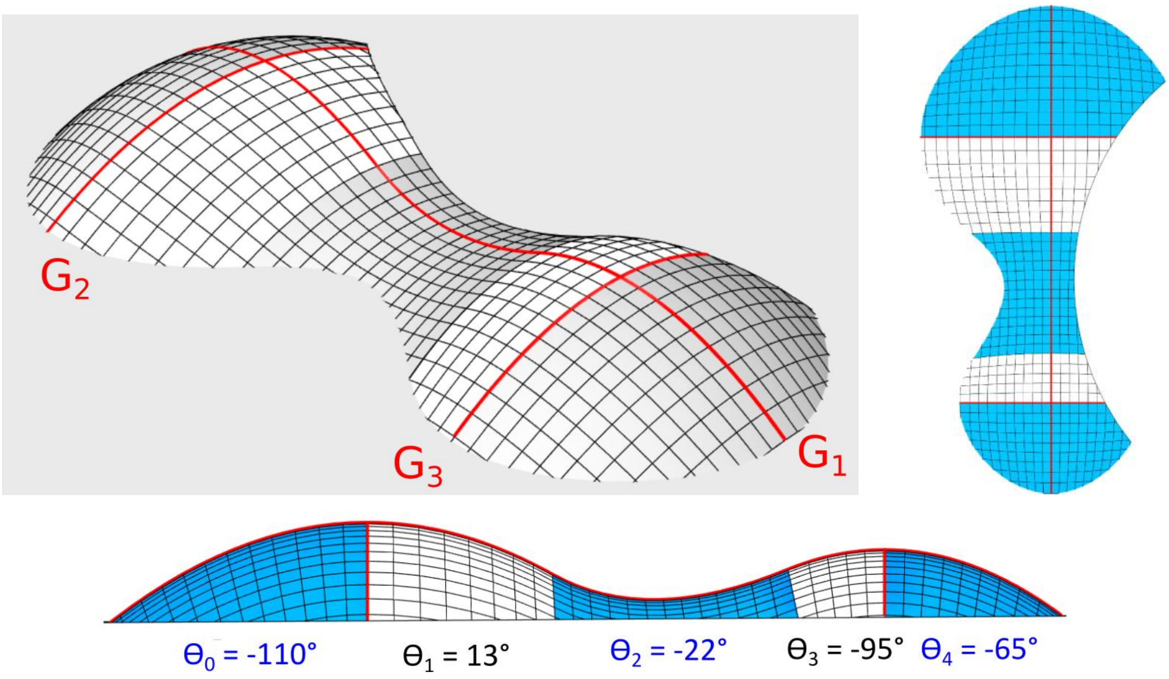

Figure 21 – A circular mesh with piecewise planar lines with a geometry close to the one of the HippoHaus in Berlin

4.8 Minimizing variation of panel size

The discretization of the guide curves allows to control the size of the mesh faces. Although this discretization can be easily controlled manually, it can also be obtained by an optimization process aiming at minimizing the variations of the size of panels. The low computation time (later discussed in section 5.1) of our method allows to perform this post-processing quickly. Figure 22 compares two meshes obtained from the same two B-spline guide curves, but discretized differently. In the first one, the guide curves are discretized with constant curve parameter increment. Because of the high variation of curvature, some faces are very elongated. In the second one, the discretization is chosen such that variance of the panel areas is minimized, using the BOBYQA algorithm. We observe that the face sizes are much more uniform.

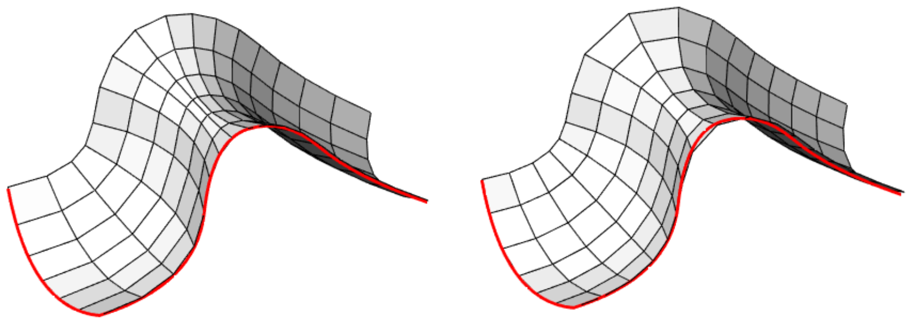

Figure 22 – Effect of guide curve discretization. Left: Constant increment of B-Spine parameter. Right: Minimizing variation of panel area

5 Discussion

5.1 Algorithmic performance

The method was implemented in the CAD software Rhinoceros™ via the plugin Grasshopper™, which offers a programming interface. In this section, we shall discuss the performance of the algorithm.

The first main step of the algorithm is to compute the Gauss map. The planes passing through the points of the Gauss map are first computed. Then, the intersection lines of these planes are constructed. Finally, the intersection of the sphere with these lines is calculated. The second main step is the computation of the Combescure transform. This is done by propagation starting from the two guide curves. Each vertex is built by calculating the intersection points between two lines. All these steps can be analytically computed and are thus highly efficient. The computing time increases linearly with the number of faces.

The performance was evaluated on a case study with 100 x 100 faces: computing time is below 75ms on a desktop with 3.5GHz processor and 16GB RAM. This low computation time allows for a real-time design exploration.

5.2 Limitations

The following limitations apply to the shapes that can be obtained with our method. First of all, surfaces with planar curvature lines cannot have umbilics, so there cannot be singularities in the mesh. Secondly, self-intersection of the mesh can occur for some combinations of guiding curves. As shown on Figure 23, once a guiding curve (e.g. the red one) and the parameter Θ are defined, the planes in one direction are entirely defined (as shown on the right-hand figure). These planes, when not parallel, necessarily intersect. The second curve (e.g. the blue one) then needs to lie within some boundaries in order to prevent the mesh to reach these plane intersections. On Figure 23, the blue curve is chosen close to that boundaries: a pinching of the mesh can be observed.

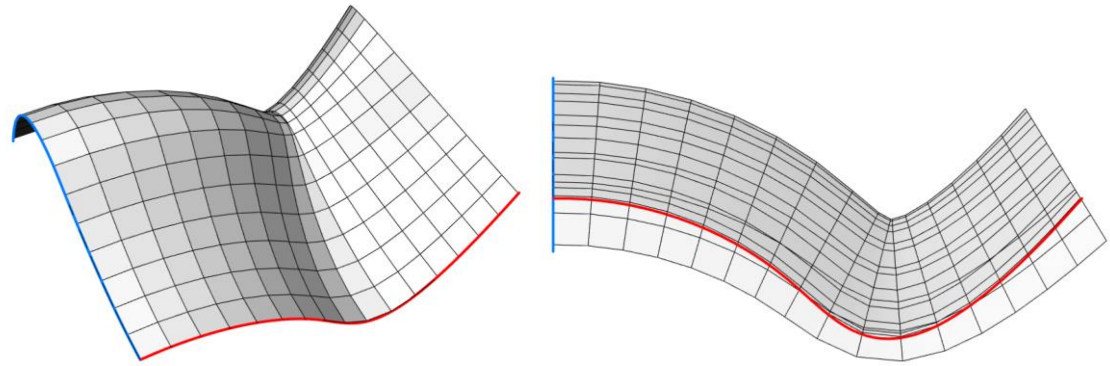

Figure 23 – Mesh pinching. Pushing up the blue guide curve would cause self-intersection

5.3 C$^\infty$ surfaces with planar curvature lines

For some applications, it might be desirable to have a C$^\infty$ surface. One could wonder if surfaces with planar curvature lines can be described by B-splines or NURBS surfaces, as these are ubiquitous in computational design. (Darboux 1896) showed that if one curvature line of a surface with planar curvature lines is polynomial, then all the subsequent curvature lines are also polynomial with the same degree. Hearing this, one might wonder if these surfaces can be modelled by B-splines, with iso-lines being curvature lines, thus forming a new family of principal patches. The answer is unfortunately negative, as shown in the following counter-example in Figure 24. A surface with parabolic curvature lines is constructed. It then approximated by a Bezier patch of degree two (for which all iso-lines are parabolas), such that boundary curves match the surface boundary exactly. We observe that the iso-lines are not aligned with the curvature lines. Nonetheless, the possibility to model surfaces with planar curvature lines by NURBS is an open question. Such a model was for example proposed for Dupin cyclides in (Zube and Krasauskas 2015).

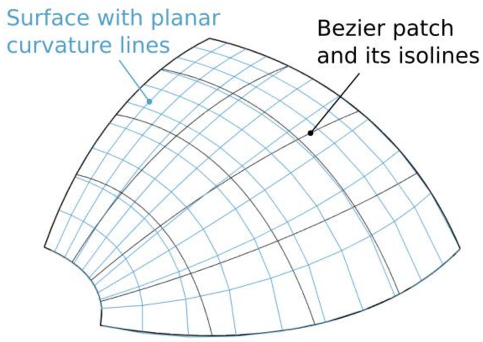

Figure 24 – Difference of parametrization between a surface with parabolic curvature lines and a 2nd order Bezier patch approximating it

6 Application to gridshell fabrication

The benefits of using circular meshes for architectural envelopes was discussed in the introduction: they offer a reference geometry that can be built with a network of beams with torsion-free nodes of constant height, that can be covered with flat panels. Our method has the additional property of having planar curvature lines. In this section, we highlight one significant application of this property to metalglass gridshells built with curved structural pipes covered with flat quadrangular glass panels. Such structures have the benefit of appearing curved from an indoor perspective, as the beams are then more noticeable than the panels, without having to resort to costly curved glass. In such structures, pipes are produced straight, and are then bent in factory, most often by a roller bender. The bending process is much more precise if the target curve is planar and piecewise circular, which is precisely the property of the curvature lines of cyclidic nets built on circular meshes with planar lines.

Figure 25 shows an application of our method to the design of a shading structure. This structure is generated from a mesh with three offsets. The top offset is used to lay a quad mesh with planar faces, which defines the geometry of glass panels. The other three layers are used to lay beams: one layer in the ortho-radial direction, and two layers in the radial direction in order to provide a high bending stiffness. All the radial beams are bi-arcs, all the ortho-radial beams are circular. This fact simplifies significantly the beam forming process, and offers the possibility to build the beams by splicing two circular arcs. Due to the vertex offset, all nodes have the same height. Furthermore, beams always cross at a right angle. Thanks to these properties, a single connector detail can be used for all connections. Interestingly, the combination of vertex-offset and piecewise circular beams yields an edge offset: the top and bottom beams are at a constant distance to each other. They are also parallel: a plane that is perpendicular to the bottom chord will be also perpendicular to the top chord. These properties can be used to design standard shear connectors between the top and bottom layers.

Such a structure can for example be fabricated with similar details as the Schubert Club Band Shell, shown in Figure 1. We note that this structure has all the fabrication properties described above, since its shape is constrained to a portion of torus, a particular surface with planar curvature lines. Our method allows to obtain a much wider design space.

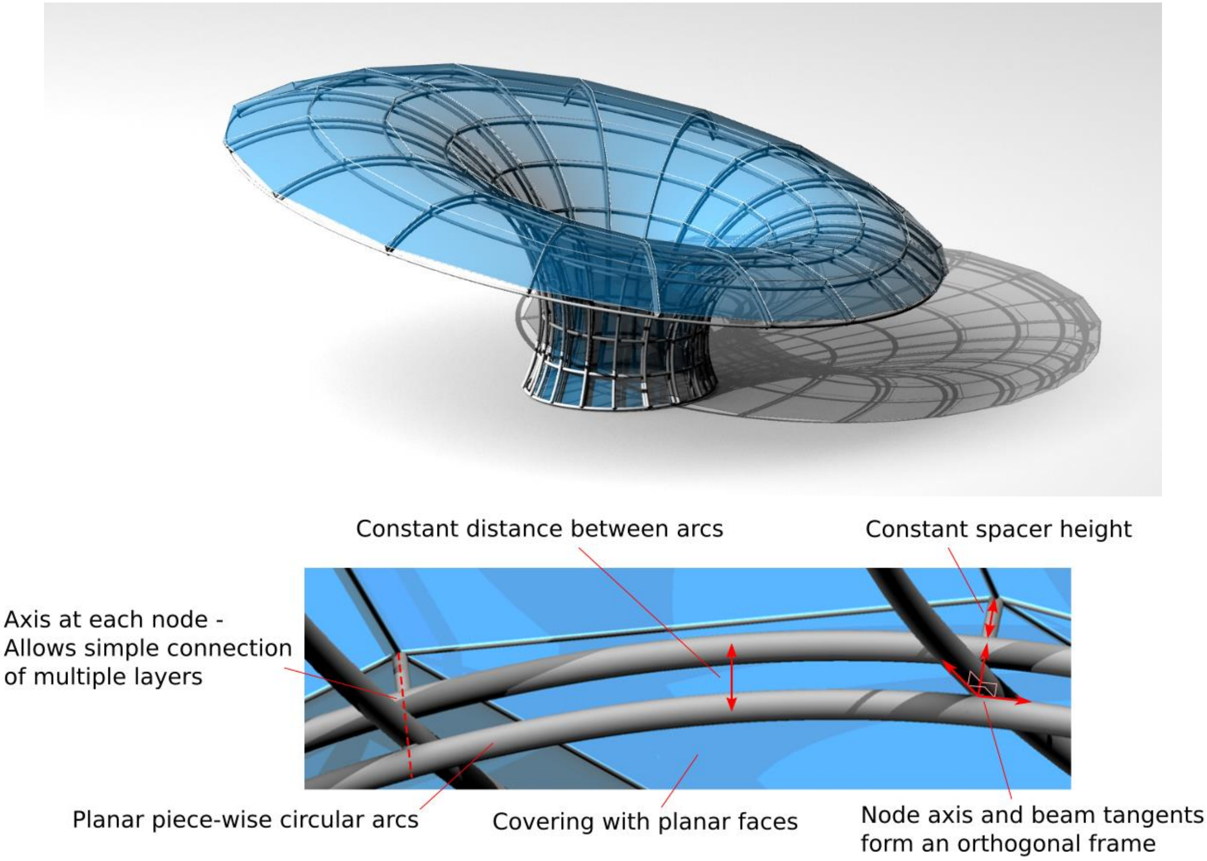

Figure 25 – Shading structure built from flat bi-arcs, covered with flat panels, and with torsion-free nodes. Top: overall view. Bottom: Summary of geometrical properties (bulky sections used on purpose to highlight properties)

Planar lines are also interesting for timber structures. Curved wooden beam are often fabricated in glued laminated timber (glulam), a process in which thin, flexible strips of wood are glued together. If a beam is not planar, each strip of wood has a geodesic curvature, and therefore needs to be split longitudinally into “sticks”: this complexifies significantly milling and gluing. Such a process had to be used for example for the fabrication of the Centre Pompidou in Metz (France) in 2010. The geometries generated with our method have planar lines, therefore they can be fabricated out of regular glulam (in which the strips need not be split into sticks).

7 Conclusion

In this paper, we presented a method to generate circular meshes, conical meshes and C^1 surfaces with planar curvature lines. The method allows to align a mesh with boundary planes and to build a network of piecewise circular beams that can be covered with flat quads. These properties are all guaranteed exactly by geometrical proofs. The proposed method enables an intuitive, robust and real-time exploration of the full design space. We also introduce a method to design a surface in which curvature lines are piecewise planar in order to get access to a wider range of surfaces. Resulting meshes are of particular interest for the fabrication of gridshells, and they have planar quad faces, node-offset, and planar lines. An application to steel-glass gridshells is detailed. For these structures, the proposed geometry rationalizes the fabrication of beams and panels, and allow a standardization of the nodes and connectors.

Acknowledgement

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