A User's Guide to the Cornish Fisher Expansion

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Abstract

Using the Cornish Fisher expansion is a relatively easy and parsimonious way of dealing with non-normality in asset price or return distributions, in such fields as insurance asset liability management or portfolio optimization with assets such as derivatives. It also allows to implement portfolio optimization with a risk measure more sophisticated than variance, such as Value-at-Risk or Conditional Value-at-Risk

The use of Cornish Fisher expansion should avoid two pitfalls: (i) exiting the domain of validity of the formula; (ii) confusing the skewness and kurtosis parameters of the formula with the actual skewness and kurtosis of the distribution.

This paper provides guidelines for a proper use of the Cornish Fisher expansion.

Keywords: risk, value at risk, conditional value at risk, Variance, volatility, skewness, kurtosis, portfolio optimization, asset liability management, non Gaussian distribution

JEL Classification: C02, C51, G11, G32

The purpose of this last revision is to correct for a typo in the last line of the computation of the fourth moment (no impact on table in Appendix 2), to mention another way of computing the moments through Hermite polynomials, to mention a method for reversing the relationship between the skewness and kurtosis parameters and the actual skewness and kurtosis, and to refer to an application of the Cornish Expansion to option pricing and the smile curve.

1 – Introduction

Non normality is a fact of life as far as the distributions of asset prices or returns are concerned. The presence of skewness and kurtosis affects the perception and measure of risk and the framework of risk-return optimisation.

The Cornish Fisher expansion (CF) is a by-product of considerations on the "Moments and Cumulants in the Specification of Distributions", by E. A. Cornish and R.A. Fisher (1937), revived to provide an easy and parsimonious way to take into consideration higher moments in the distribution of assets prices and returns, in such fields as asset liability management when liabilities are non normal (insurance claims for example), or portfolio optimization with a measure of risk more sophisticated than Variance, such as value at risk (VaR) or conditional value at risk (CVaR, or Expected Shortfall).

The Cornish Fisher expansion is not the only method to generate non Gaussian random variables: possible substitutes are the Edgeworth expansion, the Gram-Charlier expansion (Leon, Mencia and Sentana, 2009), processes with jumps, etc.

The Cornish Fisher expansion in particular provides a simple relation between the skewness and kurtosis parameters and the value at risk and conditional value at risk, and thus facilitates the implementation of mean-VaR or mean-CVaR optimizations, as well as risk measurement and risk control of portfolios (Cao, Harris and Jian, 2010; Fabozzi, Rachev and Stoyanov, 2012). It may be useful in option pricing and the explanation of the smile curve (Aboura and Maillard, 2016).

The use of CF should however avoid two pitfalls: (i) exiting the domain of validity of the formula; (ii) confusing the skewness and kurtosis parameters of the formula with the actual skewness and kurtosis of the distribution.

The first point has been documented and ways to remedy the possible narrowness of the domain of validity have been proposed (Chernozhukov, Fernandez-Val and Galichon, 2007). However, expressed in actual skewness and kurtosis, the area of the domain seems to give sufficient room for manoeuvre in most circumstances. The second point, the distinction between skewness and kurtosis parameters and actual values does not seem to have received sufficient attention, as the gap may be quite huge even for observable skewness and kurtosis.

2 – The Cornish Fisher expansion methodology

The Cornish Fisher expansion (CF) is a way to transform a standard Gaussian random variable *z* into a non Gaussian *Z* random variable.

$$z \approx N(0,1) \quad E(z)=0 \quad E(z^2)=1 \quad E(z^3)=0 \quad E(z^4)=3$$

$$Z = z + (z^2-1)\frac{S}{6} + (z^3-3z)\frac{K}{24} - (2z^3-5z)\frac{S^2}{36}$$

It may be convenient to rewrite the CF expansion as such:

$$k = \frac{K}{24} \quad s = \frac{S}{6}$$

$$Z = z + (z^2-1)s + (z^3-3z)k - (2z^3-5z)s^2$$

$$Z = z^3(k-2s^2) + z^2 s + z(1-3k+5s^2) - s = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

$$a_0 = -s \quad a_1 = 1-3k+5s^2 \quad a_2 = s \quad a_3 = k-2s^2$$

K is a kurtosis parameter, or rather an excess kurtosis parameter (in excess of 3, which corresponds to a Gaussian distribution). *S* is a skewness parameter. However, as will be apparent below, the actual kurtosis and skewness of the transformed distribution differ, significantly as soon as *K* and *S* can no longer be considered as infinitesimal, from those parameters.

3 – Domain of validity of the transformation

The transformation has to be bijective. Otherwise, the order in the quantiles of the distribution would not be conserved. That requires that:

$$\frac{dZ}{dz} > 0 \quad \forall z$$

$$\frac{dZ}{dz} = 3z^2(k-2s^2) + 2zs + 1 - 3k + 5s^2$$

This is a second degree polynomial, who is positive for high values and therefore is positive for any value if it has no root (or just one), i.e. if its discriminant is negative.

$$\Delta' = s^2 - 3(k - 2s^2)(1 - 3k + 5s^2) \leq 0$$
$$s^2 - 3k + 6s^2 + 9k^2 - 18ks^2 - 15ks^2 + 30s^4 \leq 0$$
$$9k^2 - (3 + 33s^2)k + 30s^4 + 7s^2 \leq 0$$

This implies that k sit between its two roots, if they exist. For that, the polynomial in k should have a positive discriminant.

$$\Delta = (3 + 33s^2)^2 - 36(30s^4 + 7s^2) \geq 0$$
$$(1 + 11s^2)^2 - 4(30s^4 + 7s^2) \geq 0$$
$$1 + 22s^2 + 121s^4 - 120s^4 - 28s^2 \geq 0$$
$$s^4 - 6s^2 + 1 \geq 0$$
$$u^2 - 6u + 1 \geq 0 \qquad u = s^2$$

u should sit below or above the roots.

$$u' = 3 - \sqrt{9 - 1} = 3 - \sqrt{8}$$
$$u'' = 3 + \sqrt{8}$$
$$|s| \leq \sqrt{3 - \sqrt{8}} = \sqrt{2} - 1 \quad or \quad |s| \geq \sqrt{3 + \sqrt{8}} = \sqrt{2} + 1$$
$$|S| \leq 6\left(\sqrt{2} - 1\right) \cong 2.485 \quad or \quad |S| \geq 6\left(\sqrt{2} + 1\right) \cong 14.485$$

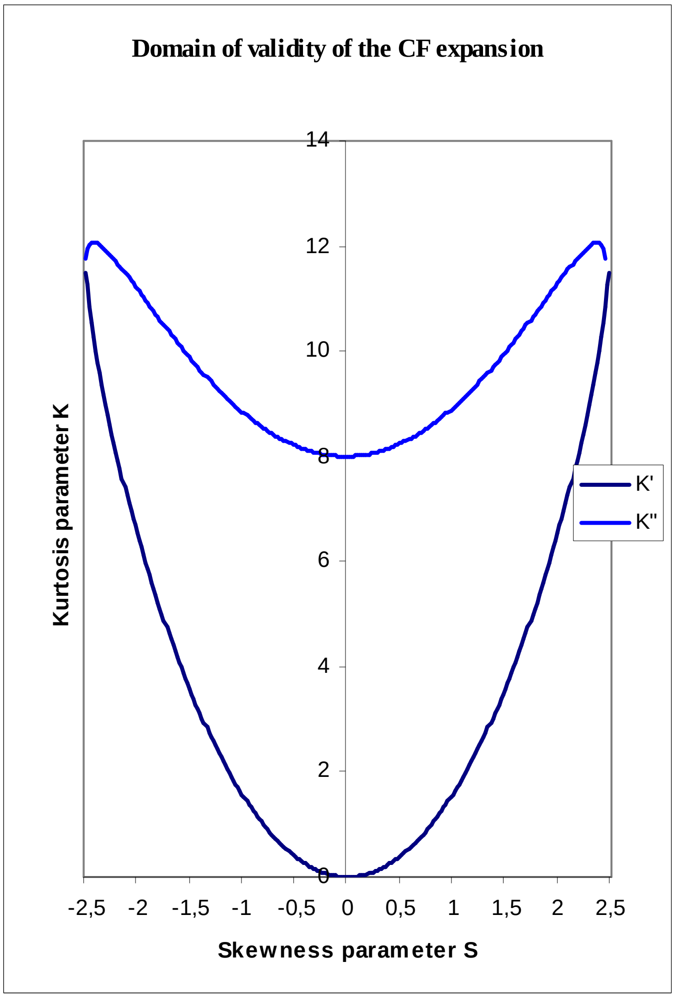
It is naturally the first area, with the skewness parameter below 2.485 in absolute terms, which is useful.

$$9k^2 - (3 + 33s^2)k + 30s^4 + 7s^2 \leq 0$$
$$k' = \frac{3 + 33s^2 - \sqrt{9s^4 - 54s^2 + 9}}{18} = \frac{1 + 11s^2 - \sqrt{s^4 - 6s^2 + 1}}{6}$$
$$k'' = \frac{1 + 11s^2 + \sqrt{s^4 - 6s^2 + 1}}{6}$$

It is obvious that the excess kurtosis parameter will always be positive. For a skewness parameter equal to 0, the excess kurtosis parameter should sit between 0 and 8. When the skewness parameter increases in absolute terms, the range of possible values for the excess kurtosis parameter moves upwards.

Note that the frontiers are symmetrical in the skewness parameter.

Chart 1



4 – The actual moments

Computing the moments of the distribution resulting from the CF transformation is both simple in theory and awful in practice[2] (see Appendix 1). The result is:

$$M1 = 0$$

$$M2 = 1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2$$

$$M3 = S - \frac{76}{216}S^3 + \frac{85}{1296}S^5 + \frac{1}{4}KS - \frac{13}{144}KS^3 + \frac{1}{32}K^2S$$

$$M4 = 3 + K + \frac{7}{16}K^2 + \frac{3}{32}K^3 + \frac{31}{3072}K^4 - \frac{7}{216}S^4 - \frac{25}{486}S^6 + \frac{21665}{559872}S^8$$
$$- \frac{7}{12}KS^2 + \frac{113}{432}KS^4 - \frac{5155}{46656}KS^6 - \frac{7}{24}K^2S^2 + \frac{2455}{20736}K^2S^4 - \frac{65}{1152}K^3S^2$$

[2] The computation is maybe a bit less cumbersome and tedious using the Hermitian polynomials recognizable in the Cornish-Fisher expansion (Maillard, 2018).

This leads to the actual values of skewness and (excess) kurtosis:

$$\hat{S} = \frac{M3}{M2^{1.5}} = \frac{S - \frac{76}{216}S^3 + \frac{85}{1296}S^5 + \frac{1}{4}KS - \frac{13}{144}KS^3 + \frac{1}{32}K^2S}{\left(1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2\right)^{1.5}}$$

$$\hat{K} = \frac{M4}{M2^2} - 3 = \frac{\begin{bmatrix} 3 + K + \frac{7}{16}K^2 + \frac{3}{32}K^3 + \frac{31}{3072}K^4 - \frac{7}{216}S^4 - \frac{25}{486}S^6 + \frac{21665}{559872}S^8 - \frac{7}{12}KS^2 \\ + \frac{113}{432}KS^4 - \frac{5155}{46656}KS^6 - \frac{7}{24}K^2S^2 + \frac{2455}{20736}K^2S^4 - \frac{65}{1152}K^3S^2 \end{bmatrix}}{\left(1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2\right)^2} - 3$$

The dependency of skewness and kurtosis upon skewness and kurtosis parameters is not straightforward, and in general cannot be assessed but numerically.

Let’s just remark that:

1) When the skewness and kurtosis parameters are “small”, the actual skewness and kurtosis coincide.

$$\hat{S} \approx S \quad \hat{K} \approx K$$

2) For a skewness parameter equal to zero, skewness is equal to zero and kurtosis is:

$$\hat{K} = \frac{\left[3 + K + \frac{7}{16}K^2 + \frac{3}{32}K^3 + \frac{31}{3072}K^4\right]}{\left(1 + \frac{1}{96}K^2\right)^2} - 3$$

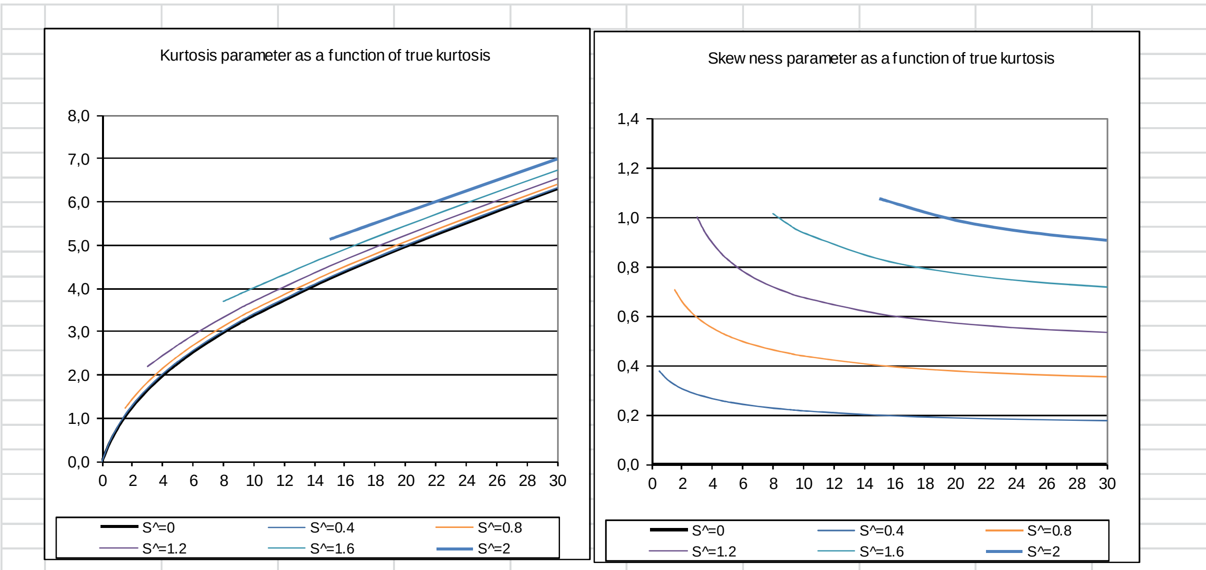
5 – Controlling for skewness and kurtosis

The actual skewness and kurtosis both depend, in a complicated manner, on both the skewness and kurtosis parameters. To properly use the CF expansion to adapt a distribution to a required skewness and kurtosis (whether or not based on historical values), one should reverse those relations.

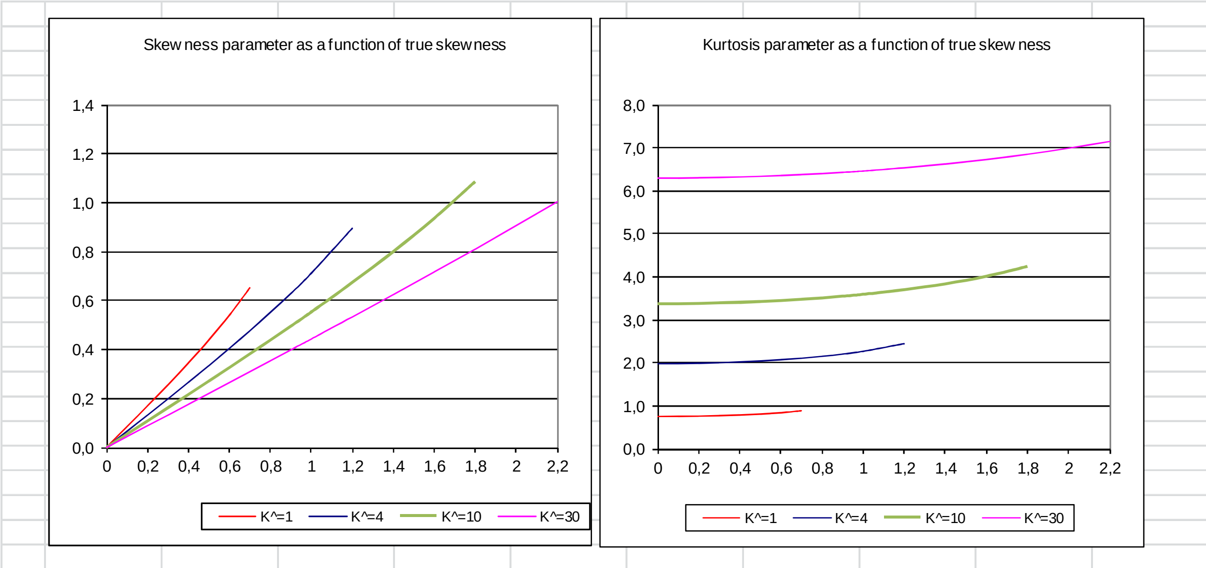
$$\hat{K} = f(K,S) \quad \hat{S} = g(K,S)$$
$$K = \varphi(\hat{K},\hat{S}) \quad S = \psi(\hat{K},\hat{S})$$

This does not seem possible analytically, and it not even obvious to prove that the dependency is bijective. Though arduous, the problem may be solved numerically, and a table has been computed (see Appendix 2)[3]. One actually finds monotonous dependencies (in the range of excess kurtosis up to 30, which is sufficiently broad for practical applications), as plotted below.

Chart 2



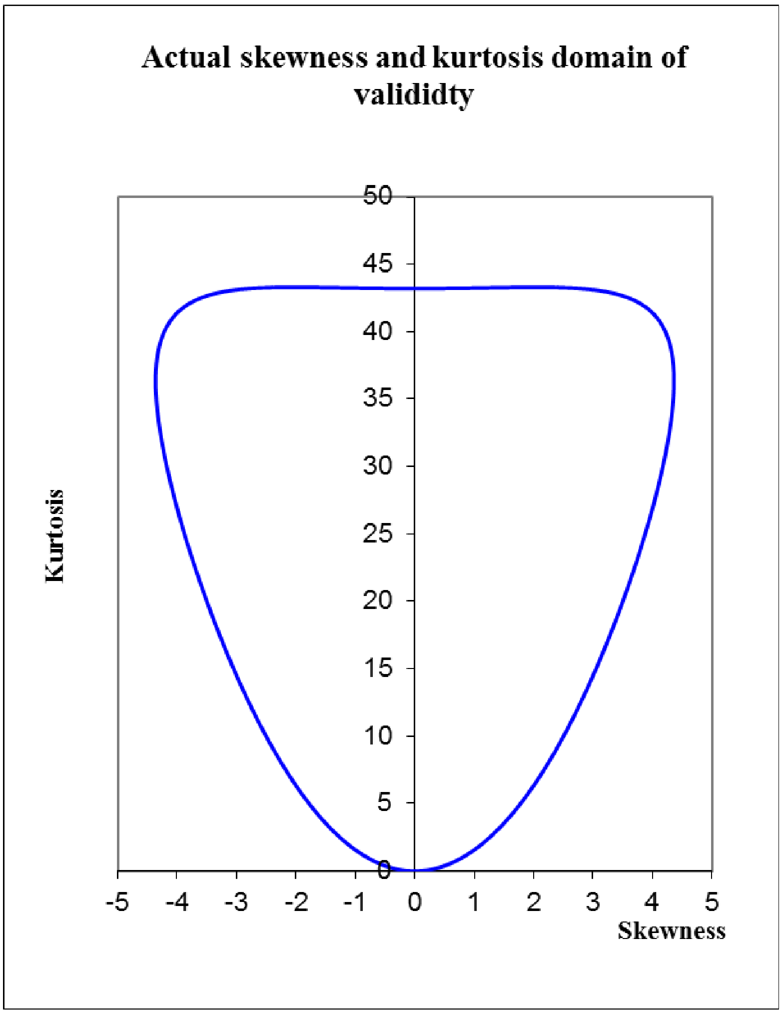
[3] The inversion of the relationship through the response surface methodology, giving and approximate analytical expression, has been addressed in Amédée-Manesme, Barthélémy and Maillard (2018).



The border problem is easier to solve and one may obtain a relationship between actual skewness and actual kurtosis corresponding to the borders of the domain of validity of the parameters. Below is a plot of this relationship[4].

[4] This chart corrects an earlier version which was affected by a coding bug. The author is grateful to Luca Patruno, from the University of Bologna, for signaling the point. His field is civil engineering, which indicates that the interest for the Cornish-Fisher methodology exceeds the domain of finance and insurance…

Chart 3



6 – The link with VaR and CVaR

The CF transformation provides an easy way to express value-at-risk and conditional value-at-risk risk measures as a function of the skewness and kurtosis parameters. Given targeted values for (actual) skewness and kurtosis, one should therefore compute parameters K and S and use them as input in the following formulae.

For a Gaussian distribution, value-at-risk (centred and reduced) at confidence level 1-α is:

$$VaR_{1-\alpha} = v_\alpha = -z_\alpha = -N^{-1}(\alpha)$$

$$CVaR_{1-\alpha} = -\frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}} z e^{-\frac{z^2}{2}} dz = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z_\alpha} e^{-\frac{z^2}{2}} d\left(-\frac{z^2}{2}\right) = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}\left[e^{-\frac{z^2}{2}}\right]_{-\infty}^{z_\alpha} = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}} e^{-\frac{z_\alpha^2}{2}} = y_\alpha$$

For instance, for 1 - α = 1%, VaR is 2.326 and CVaR 2.665.

For the transformed distribution:

$$VaR_{1-\alpha} = V_\alpha = -Z_\alpha = -a_0 - a_1 z_\alpha - a_2 z_\alpha^2 - a_3 z_\alpha^3 = s(1-v_\alpha^2) + (1-3k+5s^2)v_\alpha + (k-2s^2)v_\alpha^3$$
$$= v_\alpha + (1-v_\alpha^2)\frac{S}{6} + (5v_\alpha - 2v_\alpha^3)\frac{S^2}{36} + (v_\alpha^3 - 3v_\alpha)\frac{K}{24}$$

That is a simple expression involving the skewness and kurtosis parameters and the VaR value at the same threshold for a Gaussian distribution.

$$CVaR_{1-\alpha} = Y_\alpha = -\frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3\right)e^{-\frac{z^2}{2}}dz = a_0 A_0 + a_1 A_1 + a_2 A_2 + a_3 A_3$$

$$A_0 = -\frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = -\frac{1}{\alpha}N(z_\alpha) = -1$$

$$A_1 = -\frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}ze^{-\frac{z^2}{2}}dz = \frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}d\left(-\frac{z^2}{2}\right) = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{z_\alpha^2}{2}} = y_\alpha$$

$$A_2 = -\frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}z^2 e^{-\frac{z^2}{2}}dz = \frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}ze^{-\frac{z^2}{2}}d\left(-\frac{z^2}{2}\right) = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}\left[ze^{-\frac{z^2}{2}}\right]_{-\infty}^{z_\alpha} - \frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$
$$= \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}z_\alpha e^{-\frac{z_\alpha^2}{2}} - 1 = z_\alpha A_1 - 1 = -v_\alpha y_\alpha - 1$$

$$A_3 = -\frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}z^3 e^{-\frac{z^2}{2}}dz = \frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}z^2 e^{-\frac{z^2}{2}}d\left(-\frac{z^2}{2}\right) = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}\left[z^2 e^{-\frac{z^2}{2}}\right]_{-\infty}^{z_\alpha} - \frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}2zdz$$
$$= \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}z_\alpha^2 e^{-\frac{z_\alpha^2}{2}} + 2A_1 = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}\left(z_\alpha^2 + 2\right)e^{-\frac{z_\alpha^2}{2}} = \left(z_\alpha^2 + 2\right)A_1 = y_\alpha(v_\alpha^2 + 2)$$

$$CVaR_{1-\alpha} = Y_\alpha = -\frac{1}{\alpha}\int_{-\infty}^{z_\alpha}\frac{1}{\sqrt{2\pi}}\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3\right)e^{-\frac{z^2}{2}}dz = a_0 A_0 + a_1 A_1 + a_2 A_2 + a_3 A_3$$
$$Y_\alpha = s(1 - v_\alpha y_\alpha - 1) + (1-3k+5s^2)y_\alpha + (k-2s^2)y_\alpha(v_\alpha^2 + 2)$$
$$Y_\alpha = y_\alpha - sv_\alpha y_\alpha + s^2(5y_\alpha - 4y_\alpha - 2y_\alpha v_\alpha^2) + k(-3y_\alpha + 2y_\alpha + y_\alpha v_\alpha^2)$$
$$Y_\alpha = y_\alpha - sv_\alpha y_\alpha + s^2(y_\alpha - 2y_\alpha v_\alpha^2) + k(-y_\alpha + y_\alpha v_\alpha^2)$$
$$CvaR_{1-\alpha} = Y_\alpha = y_\alpha - \frac{S}{6}v_\alpha y_\alpha + \frac{S^2}{36}(y_\alpha - 2y_\alpha v_\alpha^2) + \frac{K}{24}(-y_\alpha + y_\alpha v_\alpha^2)$$

That expression involves the skewness and kurtosis parameters, and the VaR and CVaR values at the same threshold for a Gaussian distribution.

We can rewrite:

$$CVaR_{1-\alpha} = Y_\alpha = y_\alpha \left[1 - v_\alpha \frac{S}{6} + (1 - 2v_\alpha^2) \frac{S^2}{36} + (-1 + v_\alpha^2) \frac{K}{24} \right] = y_\alpha \left[1 + m_\alpha S + p_\alpha S^2 + q_\alpha K \right]$$

The expression within brackets is a multiplier of the Gaussian distribution risk measure taking into account the skewness and kurtosis of the distribution (through the parameters).

Here are finally the corresponding parameters for usual thresholds.

α	m_α	p_α	$q\alpha$
0,1%	-0,5150	-0,5028	0,3562
0,5%	-0,4293	-0,3408	0,2348
1,0%	-0,3877	-0,2729	0,1838
5,0%	-0,2741	-0,1225	0,0711
10,0%	-0,2136	-0,0635	0,0268

Remember that those coefficients apply to the skewness and kurtosis parameters and not to the actual skewness and kurtosis.

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Appendix 1

Computation of the 1st, 2nd, 3rd and 4th moments

Let's first note that:

$$E(z)=E(z^3)=E(z^5)=E(z^7)=E(z^9)=E(z^{11})=0$$
$$E(z^2)=1 \quad E(z^4)=3 \quad E(z^6)=15 \quad E(z^8)=105 \quad E(z^{10})=945 \quad E(z^{12})=10395$$

First moment

$$Z=a_0+a_1z+a_2z^2+a_3z^3$$
$$a_0=-s \quad a_1=1-3k+5s^2 \quad a_2=s \quad a_3=k-2s^2$$
$$M1=E(Z)=a_0+a_2=-s+s=0$$

Second moment

$$Z^2=b_0+b_1z+b_2z^2+b_3z^3+b_4z^4+b_5z^5+b_6z^6$$
$$b_0=a_0^2=s^2$$
$$b_1=2a_0a_1=-2s+6ks-10s^3$$
$$b_2=2a_0a_2+a_1^2=-2s^2+(1-3k+5s^2)^2=-2s^2+25s^4+9k^2+1+10s^2-6k-30ks^2$$
$$=1-6k+8s^2+9k^2-30ks^2+25s^4$$
$$b_3=2a_0a_3+2a_1a_2=-2s(k-2s^2)+2s-6ks+10s^3=2s-8ks+14s^3$$
$$b_4=2a_1a_3+a_2^2=s^2+2k-4s^2-6k^2+12ks^2+10ks^2-20s^4=2k-3s^2-6k^2+22ks^2-20s^4$$
$$b_5=2a_2a_3=2ks-4s^3$$
$$b_6=a_3^2=k^2+4s^4-4ks^2$$
$$M2=E(Z^2)=b_0+b_2+3b_4+15b_6=s^2+1-6k+8s^2+9k^2-30ks^2+25s^4+6k-9s^2-18k^2$$
$$+66ks^2-60s^4$$
$$+15k^2+60s^4-60ks^2=1+6k^2-24ks^2+25s^4$$
$$M2=1+\frac{1}{96}K^2+\frac{25}{1296}S^4-\frac{1}{36}KS^2$$

Third moment

$$Z^3 = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + c_7 z^7 + c_8 z^8 + c_9 z^9$$
$$Z^3 = ZZ^2$$
$$c_0 = a_0 b_0 = -ss^2 = -s^3$$
$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = -s(1-6k+8s^2+9k^2-30ks^2+25s^4)+(1-3k+5s^2)(-2s+6ks-10s^3)+ss^2$$
$$= -s+6ks-8s^3-9sk^2+30ks^3-25s^5-2s+6ks-10s^3+6ks-18sk^2+30ks^3-10s^3+30ks^3-50s^5+s^3$$
$$= -3s+18ks-27s^3-27sk^2+90ks^3-75s^5$$
$$c_4 = a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 = -s(2k-3s^2-6k^2+22ks^2-20s^4)+(1-3k+5s^2)(2s-8ks+14s^3)$$
$$+s(1-6k+8s^2+9k^2-30ks^2+25s^4)+(k-2s^2)(-2s+6ks-10s^3)$$
$$= -2ks+3s^3+6sk^2-22ks^3+20s^5+2s-8ks+14s^3-6ks+24sk^2-42ks^3+10s^3-40ks^3+70s^5$$
$$+s-6ks+8s^3+9sk^2-30ks^3+25s^5-2ks+6sk^2-10ks^3+4s^3-12ks^3+20s^5$$
$$= 3s-24ks+39s^3+45sk^2-156ks^3+135s^5$$

$$c_6 = a_0 b_6 + a_1 b_5 + a_2 b_4 + a_3 b_3 = s(2k-3s^2-6k^2+22ks^2-20s^4-k^2-4s^4+4ks^2)$$
$$+(1-3k+5s^2)(2ks-4s^3)+(k-2s^2)(2s-8ks+14s^3)$$
$$= 2ks-3s^3-7sk^2+26ks^3-24s^5+2ks-6sk^2+10ks^3-4s^3+12ks^3-20s^5+2ks-8sk^2$$
$$+14ks^3-4s^3+16ks^3-28s^5$$
$$= 6ks-11s^3-21sk^2+78ks^3-72s^5$$
$$c_8 = a_2 b_6 + a_3 b_5 = s(k^2+4s^4-4ks^2)+(k-2s^2)(2ks-4s^3)$$
$$= sk^2+4s^5-4ks^3+2sk^2-4ks^3-4ks^3+8s^5 = 3sk^2-12ks^3+12s^5$$
$$M3 = c_0 + c_2 + 3c_4 + 15c_6 + 105c_8$$
$$M3 = 6s+36ks-76s^3+108sk^2-468ks^3+510s^5$$
$$M3 = S + \frac{1}{4}SK - \frac{76}{216}S^3 + \frac{1}{32}SK^2 - \frac{13}{144}KS^3 + \frac{85}{1296}S^5$$

Fourth moment

First note that:

$$Z^4 = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + d_5 z^5 + d_6 z^6$$
$$+ d_7 z^7 + d_8 z^8 + d_9 z^9 + d_{10} z^{10} + d_{11} z^{11} + d_{12} z^{12}$$
$$Z^3 = Z^2 Z^2$$
$$d_0 = b_0^2$$
$$d_2 = b_1^2 + 2b_0 b_2$$
$$d_4 = b_2^2 + 2b_0 b_4 + 2b_1 b_3$$
$$d_6 = b_3^2 + 2b_0 b_6 + 2b_1 b_5 + 2b_2 b_4$$
$$d_8 = b_4^2 + 2b_2 b_6 + 2b_3 b_5$$
$$d_{10} = b_5^2 + 2b_4 b_6$$
$$d_{12} = b_6^2$$

$$b_0^2 = (s^2)^2 = s^4$$
$$d_0 = s^4$$

$$b_1^2 = (-2s + 6ks - 10s^3)^2 = 4s^2 + 36k^2s^2 + 100s^6 - 24ks^2 + 40s^4 - 120ks^4$$
$$= 4s^2 + 40s^4 + 100s^6 - 24ks^2 - 120ks^4 + 36k^2s^2$$
$$b_0b_2 = s^2(1 - 6k + 8s^2 + 9k^2 - 30ks^2 + 25s^4) = s^2 - 6ks^2 + 8s^4 + 9k^2s^2 - 30ks^4 + 25s^6$$
$$= s^2 + 8s^4 + 25s^6 - 6ks^2 - 30ks^4 + 9k^2s^2$$
$$d_2 = b_1^2 + 2b_0b_2 = 6s^2 + 56s^4 + 150s^6 - 36ks^2 - 180ks^4 + 54k^2s^2$$
$$b_2^2 = (1 - 6k + 8s^2 + 9k^2 - 30ks^2 + 25s^4)^2 = 1 + 36k^2 + 64s^4 + 81k^4 + 900k^2s^4 + 625s^8$$
$$-12k + 16s^2 + 18k^2 - 60ks^2 + 50s^4 - 96ks^2 - 108k^3 + 360k^2s^2 - 300ks^4 + 144k^2s^2 - 480ks^4$$
$$+ 400s^6 - 540k^3s^2 + 450k^2s^4 - 1500ks^6$$
$$= 1 - 12k + 54k^2 - 108k^3 + 81k^4 + 16s^2 + 114s^4 + 400s^6 + 625s^8 - 156ks^2 - 780ks^4 - 1500ks^6$$
$$+ 504k^2s^2 + 1350k^2s^4 - 540k^3s^2$$
$$b_0b_4 = s^2(2k - 3s^2 - 6k^2 + 22ks^2 - 20s^4) = 2ks^2 - 3s^4 - 6k^2s^2 + 22ks^4 - 20s^6$$
$$= -3s^4 - 20s^6 + 2ks^2 + 22ks^4 - 6k^2s^2$$
$$b_1b_3 = (-2s + 6ks - 10s^3)(2s - 8ks + 14s^3) = -4s^2 + 16ks^2 - 28s^4 + 12ks^2 - 48k^2s^2 + 84ks^4 - 20s^4$$
$$+ 80ks^4 - 140s^6 = -4s^2 - 48s^4 - 140s^6 + 28ks^2 + 164ks^4 - 48k^2s^2$$
$$d_4 = b_2^2 + 2b_0b_4 + 2b_1b_3 = 1 - 12k + 54k^2 - 108k^3 + 81k^4 + (16 - 8)s^2 + (114 - 6 - 96)s^4$$
$$+ (400 - 40 - 280)s^6 + 625s^8 + (-156 + 4 + 56)ks^2 + (-780 + 44 + 328)ks^4 - 1500ks^6$$
$$+ (504 - 12 - 96)k^2s^2 + 1350k^2s^4 - 540k^3s^2$$
$$d_4 = 1 - 12k + 54k^2 - 108k^3 + 81k^4 + 8s^2 + 12s^4 + 80s^6 + 625s^8 - 96ks^2 - 408ks^4 - 1500ks^6$$
$$+ 396k^2s^2 + 1350k^2s^4 - 540k^3s^2$$

$b_3^2 = (2s - 8ks + 14s^3)^2 = 4s^2 + 64k^2s^2 + 196s^6 - 32ks^2 + 56s^4 - 224ks^4$

$= 4s^2 + 56s^4 + 196s^6 - 32ks^2 - 224ks^4 + 64k^2s^2$

$b_0b_6 = s^2(k^2 + 4s^4 - 4ks^2) = 4s^6 - 4ks^4 + k^2s^2$

$b_1b_5 = (-2s + 6ks - 10s^3)(2ks - 4s^3) = -4ks^2 + 8s^4 + 12k^2s^2 - 24ks^4 - 20ks^4 + 40s^6$

$= 8s^4 + 40s^6 - 4ks^2 - 44ks^4 + 12k^2s^2$

$b_2b_4 = (1 - 6k + 8s^2 + 9k^2 - 30ks^2 + 25s^4)(2k - 3s^2 - 6k^2 + 22ks^2 - 20s^4)$

$= 2k - 3s^2 - 6k^2 + 22ks^2 - 20s^4 - 12k^2 + 18ks^2 + 36k^3 - 132k^2s^2 + 120ks^4$

$+ 16ks^2 - 24s^4 - 48k^2s^2 + 176ks^4 - 160s^6 + 18k^3 - 27k^2s^2 - 54k^4 + 198k^3s^2 - 180k^2s^4$

$- 60k^2s^2 + 90ks^4 + 180k^3s^2 - 660k^2s^4 + 600ks^6 + 50ks^4 - 75s^6 - 150k^2s^4 + 550ks^6 - 500s^8$

$= 2k - 18k^2 + 54k^3 - 54k^4 - 3s^2 - 44s^4 - 235s^6 - 500s^8 + 56ks^2 + 436ks^4 + 1150ks^6 - 267k^2s^2$

$- 990k^2s^4 + 378k^3s^2$

$d_6 = 4k - 36k^2 + 108k^3 - 108k^4 - 6s^2 - 88s^4 - 470s^6 - 1000s^8 + 112ks^2 + 872ks^4 + 2300ks^6$

$- 534k^2s^2 - 1980k^2s^4 + 756k^3s^2 + 16s^4 + 80s^6 - 8ks^2 - 88ks^4 + 24k^2s^2 + 8s^6 - 8ks^4 + 2k^2s^2 +$

$4s^2 + 56s^4 + 196s^6 - 32ks^2 - 224ks^4 + 64k^2s^2$

$= 4k - 36k^2 + 108k^3 - 108k^4 - 6s^2 + 4s^2 - 88s^4 + 16s^4 + 56s^4 - 470s^6 + 80s^6 + 8s^6 + 196s^6 - 1000s^8$

$+ 112ks^2 - 8ks^2 - 32ks^2 + 872ks^4 - 88ks^4 - 8ks^4 - 224ks^4 + 2300ks^6 - 534k^2s^2 + 24k^2s^2 + 2k^2s^2$

$+ 64k^2s^2 - 1980k^2s^4 + 756k^3s^2$

$= 4k - 36k^2 + 108k^3 - 108k^4 - 2s^2 - 16s^4 - 186s^6 - 1000s^8 + 72ks^2 + 552ks^4 + 2300ks^6 - 444k^2s^2$

$- 1980k^2s^4 + 756k^3s^2$

$b_4^2 = (2k - 3s^2 - 6k^2 + 22ks^2 - 20s^4)^2 = 4k^2 + 9s^4 + 36k^4 + 484k^2s^4 + 400s^8 - 12ks^2 - 24k^3 + 88k^2s^2$

$- 80ks^4 + 36k^2s^2 - 132ks^4 + 120s^6 - 264k^3s^2 - 880ks^6 + 240k^2s^4$

$= 4k^2 - 24k^3 + 36k^4 + 9s^4 + 120s^6 + 400s^8 - 12ks^2 - 212ks^4 - 880ks^6 + 124k^2s^2 + 724k^2s^4 - 264k^3s^2$

$b_2b_6 = (k^2 + 4s^4 - 4ks^2)(1 - 6k + 8s^2 + 9k^2 - 30ks^2 + 25s^4) = k^2 - 6k^3 + 8k^2s^2 + 9k^4 - 30k^3s^2 + 25k^2s^4$

$+ 4s^4 - 24ks^4 + 32s^6 + 36k^2s^4 - 120ks^6 + 100s^8 - 4ks^2 + 24k^2s^2 - 32ks^4 - 36k^3s^2 + 120k^2s^4 - 100ks^6$

$= k^2 - 6k^3 + 9k^4 + 4s^4 + 32s^6 + 100s^8 - 4ks^2 - 56ks^4 - 220ks^6 + 32k^2s^2 + 181k^2s^4 - 66k^3s^2$

$b_3b_5 = (2ks - 4s^3)(2s - 8ks + 14s^3) = 4ks^2 - 16k^2s^2 + 28ks^4 - 8s^4 + 32ks^4 - 56s^6$

$= -8s^4 - 56s^6 + 4ks^2 + 60ks^4 - 16k^2s^2$

$d_8 = 4k^2 - 24k^3 + 36k^4 + 9s^4 + 120s^6 + 400s^8 - 12ks^2 - 212ks^4 - 880ks^6 + 124k^2s^2 + 724k^2s^4 - 264k^3s^2$

$+ 2k^2 - 12k^3 + 18k^4 + 8s^4 + 64s^6 + 200s^8 - 8ks^2 - 112ks^4 - 440ks^6 + 64k^2s^2 + 362k^2s^4 - 132k^3s^2$

$- 16s^4 - 112s^6 + 8ks^2 + 120ks^4 - 32k^2s^2$

$= 6k^2 - 36k^3 + 54k^4 + s^4 + 72s^6 + 600s^8 - 12ks^2 - 204ks^4 - 1320ks^6 + 156k^2s^2 + 1086k^2s^4 - 396k^3s^2$

$$b_5^2 = (2ks - 4s^3)^2 = 4k^2s^2 + 16s^6 - 16ks^4 = 16s^6 - 16ks^4 + 4k^2s^2$$

$$b_4b_6 = (k^2 + 4s^4 - 4ks^2)(2k - 3s^2 - 6k^2 + 22ks^2 - 20s^4) = 2k^3 - 3k^2s^2 - 6k^4 + 22k^3s^2 - 20k^2s^4$$
$$+8ks^4 - 12s^6 - 24k^2s^4 + 88ks^6 - 80s^8 - 8k^2s^2 + 12ks^4 + 24k^3s^2 - 88k^2s^4 + 80ks^6$$
$$= 2k^3 - 6k^4 - 12s^6 - 80s^8 + 20ks^4 + 168ks^6 - 11k^2s^2 - 132k^2s^4 + 46k^3s^2$$
$$d_{10} = 4k^3 - 12k^4 - 8s^6 - 160s^8 + 24ks^4 + 336ks^6 - 18k^2s^2 - 264k^2s^4 + 92k^3s^2$$

$$d_{12} = b_6^2 = (k^2 + 4s^4 - 4ks^2)^2 = k^4 + 16s^8 + 16k^2s^4 + 8k^2s^4 - 8k^3s^2 - 32ks^6$$
$$= k^4 + 16s^8 - 32ks^6 + 24k^2s^4 - 8k^3s^2$$

$$M4 = d_0 + d_2 + 3d_4 + 15d_6 + 105d_8 + 945d_{10} + 10325d_{12}$$
$$M4 = 3 + 24k + 252k^2 + 1296k^3 + 3348k^4 - 42s^4 - 2400s^6 + 64995s^8 - 504ks^2 + 8136ks^4$$
$$-123720ks^6 - 6048k^2s^2 + 88380k^2s^4 - 28080k^3s^2$$
$$M4 = 3 + K + \frac{7}{16}K^2 + \frac{3}{32}K^3 + \frac{31}{3072}K^4 - \frac{7}{216}S^4 - \frac{25}{486}S^6 + \frac{21665}{559872}S^8$$
$$-\frac{7}{12}KS^2 + \frac{113}{432}KS^4 - \frac{5155}{46656}KS^6 - \frac{7}{24}K^2S^2 + \frac{2455}{20736}K^2S^4 - \frac{65}{1152}K^3S^2$$

Appendix 2

Skewness and kurtosis parameters as a function of actual skewness and kurtosis

		Actual skewness $\hat{S}$																
Actual Kurtosis $\hat{K}$		0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	1,2	1,4	1,6	1,8	2	2,2
0	K	0,000																
	S	0,000																
0,5	K	0,426	0,428	0,432	0,439	0,452	0,470											
	S	0,000	0,091	0,183	0,278	0,377	0,484											
1	K	0,754	0,757	0,763	0,773	0,788	0,810	0,841	0,886									
	S	0,000	0,084	0,169	0,255	0,345	0,439	0,540	0,652									
1,5	K	1,026	1,028	1,035	1,046	1,063	1,086	1,117	1,159	1,218								
	S	0,000	0,079	0,158	0,239	0,322	0,409	0,500	0,599	0,707								
2	K	1,259	1,262	1,269	1,281	1,298	1,321	1,352	1,392	1,446	1,517	1,618						
	S	0,000	0,075	0,150	0,227	0,306	0,387	0,472	0,561	0,658	0,764	0,885						
2,5	K	1,466	1,469	1,476	1,488	1,505	1,529	1,559	1,598	1,648	1,712	1,796						
	S	0,000	0,072	0,144	0,218	0,293	0,370	0,450	0,533	0,622	0,716	0,823						
3	K	1,653	1,655	1,662	1,674	1,692	1,715	1,745	1,783	1,830	1,888	1,964	2,194					
	S	0,000	0,069	0,139	0,210	0,282	0,357	0,431	0,510	0,593	0,680	0,776	1,000					
3,5	K	1,823	1,826	1,833	1,845	1,862	1,885	1,915	1,951	1,996	2,051	2,121	2,319					
	S	0,000	0,067	0,135	0,204	0,273	0,345	0,417	0,492	0,570	0,652	0,740	0,940					
4	K	1,982	1,984	1,991	2,003	2,020	2,043	2,071	2,107	2,150	2,203	2,268	2,446					
	S	0,000	0,065	0,131	0,198	0,265	0,334	0,404	0,476	0,551	0,629	0,712	0,895					
4,5	K	2,129	2,132	2,139	2,150	2,167	2,190	2,218	2,252	2,295	2,345	2,406	2,569					
	S	0,000	0,063	0,128	0,193	0,258	0,325	0,393	0,463	0,535	0,610	0,687	0,858					
5	K	2,268	2,271	2,278	2,289	2,306	2,328	2,355	2,389	2,430	2,479	2,536	2,688					
	S	0,000	0,062	0,125	0,189	0,252	0,317	0,384	0,451	0,520	0,593	0,666	0,828					
6	K	2,525	2,528	2,534	2,546	2,562	2,583	2,610	2,642	2,680	2,727	2,780	2,916	3,109				
	S	0,000	0,060	0,121	0,182	0,242	0,304	0,367	0,431	0,496	0,564	0,633	0,781	0,947				
7	K	2,760	2,762	2,769	2,780	2,796	2,817	2,842	2,873	2,910	2,953	3,003	3,130	3,300				
	S	0,000	0,058	0,116	0,175	0,234	0,293	0,354	0,415	0,478	0,542	0,607	0,746	0,897				
8	K	2,978	2,980	2,986	2,997	3,013	3,033	3,058	3,087	3,123	3,164	3,212	3,330	3,486	3,696			
	S	0,000	0,056	0,113	0,170	0,227	0,285	0,343	0,402	0,463	0,524	0,586	0,717	0,857	1,013			
9	K	3,182	3,184	3,190	3,201	3,216	3,235	3,259	3,288	3,323	3,362	3,408	3,520	3,665	3,855			
	S	0,000	0,055	0,110	0,165	0,221	0,277	0,334	0,391	0,450	0,509	0,569	0,694	0,826	0,971			
10	K	3,374	3,376	3,383	3,393	3,408	3,427	3,450	3,478	3,511	3,550	3,593	3,701	3,837	4,012	4,243		
	S	0,000	0,054	0,108	0,162	0,216	0,271	0,326	0,382	0,438	0,495	0,554	0,674	0,800	0,936	1,086		
15	K	4,225	4,226	4,232	4,241	4,255	4,272	4,293	4,318	4,347	4,381	4,419	4,510	4,622	4,759	4,926	5,131	
	S	0,000	0,049	0,099	0,148	0,198	0,248	0,298	0,349	0,400	0,451	0,502	0,608	0,717	0,830	0,948	1,074	
20	K	4,962	4,964	4,970	4,978	4,991	5,007	5,026	5,050	5,077	5,107	5,143	5,225	5,326	5,446	5,588	5,756	5,954
	S	0,000	0,047	0,094	0,140	0,188	0,235	0,282	0,329	0,377	0,425	0,473	0,571	0,671	0,773	0,878	0,988	1,102
25	K	5,643	5,645	5,650	5,658	5,670	5,685	5,704	5,726	5,752	5,781	5,814	5,892	5,985	6,095	6,224	6,374	6,548
	S	0,000	0,046	0,090	0,135	0,181	0,226	0,271	0,317	0,363	0,409	0,455	0,548	0,643	0,739	0,837	0,937	1,041
30	K	6,295	6,297	6,302	6,310	6,321	6,336	6,354	6,375	6,400	6,428	6,460	6,534	6,623	6,728	6,849	6,989	7,148
	S	0,000	0,044	0,088	0,132	0,177	0,221	0,265	0,309	0,354	0,399	0,443	0,533	0,625	0,717	0,810	0,906	1,003

K : Kurtosis parameter (as appears in the CF transformation)

S : Skewness parameter

$\hat{K}$: Actual kurtosis

$\hat{S}$: Actual skewness