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Guillaume Delaplace\*, Romain Jeantet, Richard Grenville, Gérard Cuvelier and Karine Loubiere

# How dimensional analysis allows to go beyond Metzner–Otto concept for non-Newtonian fluids

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**Abstract:** The concept of Metzner and Otto was initially developed for correlating power measurements in stirred vessels for shear-thinning fluids in the laminar regime with regard to those obtained for Newtonian liquids. To get this overlap, Metzner and Otto postulated and determined an “effective shear rate” which was proportional to the rotational speed of the impeller. Although it was not based on a strong theoretical background, it was rapidly admitted as a practical engineering approach and was extended for seeking out a “Newtonian correspondence” with non-Newtonian results (i.e. different classes of fluids). This was applied in a variety of tank processes even for predicting heat transfer or mixing time, which stretches far away from the frame initially envisaged by Metzner and Otto themselves. This paper aimed to show how dimensional analysis offers a theoretically founded framework to address this issue without the experimental determination of effective quantities. This work also aimed to enlarge the underlying questions to any process in which a variable material property exists and impacts the process. For that purpose, the pending questions of Metzner and Otto

concept were first reminded (i.e. dependence of the Metzner–Otto constant to rheological parameters, physical meaning of the effective shear rate, etc.). Then, the theoretical background underlying the dimensional analysis was described and applied to the case of variable material properties (including non-Newtonian fluids), by introducing in particular the concept of material similarity. Finally, two examples were proposed to demonstrate how the rigorous framework associated with the dimensional analysis is a powerful method to exceed the concept of Metzner and Otto and can be adapted beyond the Ostwald–de Waele power law model to a wide range of non-Newtonian fluids in various processes, without being restricted to batch reactor and laminar regime.

**Keywords:** agitation process; dimensional analysis; Metzner–Otto concept; non-Newtonian fluids.

## 1 Introduction

Industrial processes involve transformation of matter (fluid, emulsion, suspension, etc.), using various kinds of equipment. In many cases, one (or several) physical properties of the matter change between the inlet and the outlet of the equipment and/or there is a spatial and/or temporal distribution of these properties within the equipment insofar as they are dependent on various scalar fields (composition, temperature, etc.). This dependence has a significant effect on the process. For instance, the variation of apparent viscosity with shear rate for non-Newtonian fluids significantly influences the velocity field within the equipment compared to that obtained when viscosity is constant (Newtonian fluids); this can be illustrated by considering the flow of fluids presenting a yield stress in a stirred tank: caverns are formed in the vicinity of the impeller which does not occur during the agitation of Newtonian fluids.

The concept introduced by Metzner and Otto (1957) initially aimed to provide engineering solutions for predicting the power demands when mixing shear-thinning fluids adopting an Ostwald–de Waele power law model

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**\*Corresponding author: Guillaume Delaplace**, Université Lille, CNRS, INRAE, Centrale Lille, UMR 8207 – Unité Matériaux et Transformations (UMET), F-59000, Lille, France, E-mail: [guillaume.delaplace@inrae.fr](mailto:guillaume.delaplace@inrae.fr). <https://orcid.org/0000-0001-7618-3243>

**Romain Jeantet**, STLO, INRAE, Institut Agro, 35042, Rennes, France, E-mail: [romain.jeantet@agrocampus-ouest.fr](mailto:romain.jeantet@agrocampus-ouest.fr). <https://orcid.org/0000-0001-5405-5056>

**Richard Grenville**, Philadelphia Mixing Solutions, Ltd., 1221, East Main Street, Palmyra, PA, 17078, USA, E-mail: [rgrenville@philamixers.com](mailto:rgrenville@philamixers.com)

**Gérard Cuvelier**, UMR Ingénierie Procédés Aliments, AgroParisTech, INRA, Université Paris-Saclay, 1 avenue des Olympiades, 91300, Massy, France, E-mail: [Gerard.Cuvelier@AgroParisTech.fr](mailto:Gerard.Cuvelier@AgroParisTech.fr)

**Karine Loubiere**, Laboratoire de Génie Chimique, Université de Toulouse, CNRS, INPT, UPS, Toulouse, France, E-mail: [karine.loubiere@ensiacet.fr](mailto:karine.loubiere@ensiacet.fr). <https://orcid.org/0000-0001-6245-2844>

under laminar regime in agitated vessel equipped with Rushton turbine. The practical engineering approach proposed by these authors has overcome the initially envisaged applications and had a major impact over the last 60 years for considering non-Newtonian fluids in a variety of processes operated in stirred vessels, even for correlating power requirement, heat transfer or mixing time in the laminar regime.

While Newtonian fluids are characterized by a constant value of viscosity, the apparent viscosity ( $\mu_a$ ) of a non-Newtonian purely viscous fluid (i.e. a non-Newtonian fluid with any or negligible elastic properties and any time-dependent behaviour), varies from point-to-point in the system due to its shear-dependence. This fact requires to evaluate an average apparent viscosity in order to be able to design such mixing processes. Since neither the flow field nor the shear-rate distribution is known a priori for non-Newtonian fluids, the basic idea developed by Metzner and Otto (1957) consisted in introducing a so-called effective shear rate to calculate an average apparent viscosity (also named the effective viscosity). By definition, this effective viscosity is approximate in such way that, for a given mixing system, the master curves for Newtonian and non-Newtonian liquids become superimposed.

Based on these considerations, Metzner and Otto (1957) proposed that, in the laminar regime, the effective shear rate in the region of the impeller,  $\dot{\gamma}_{\text{eff}}$ , shall be linearly linked to the rotational speed of the impeller,  $N$ , as:

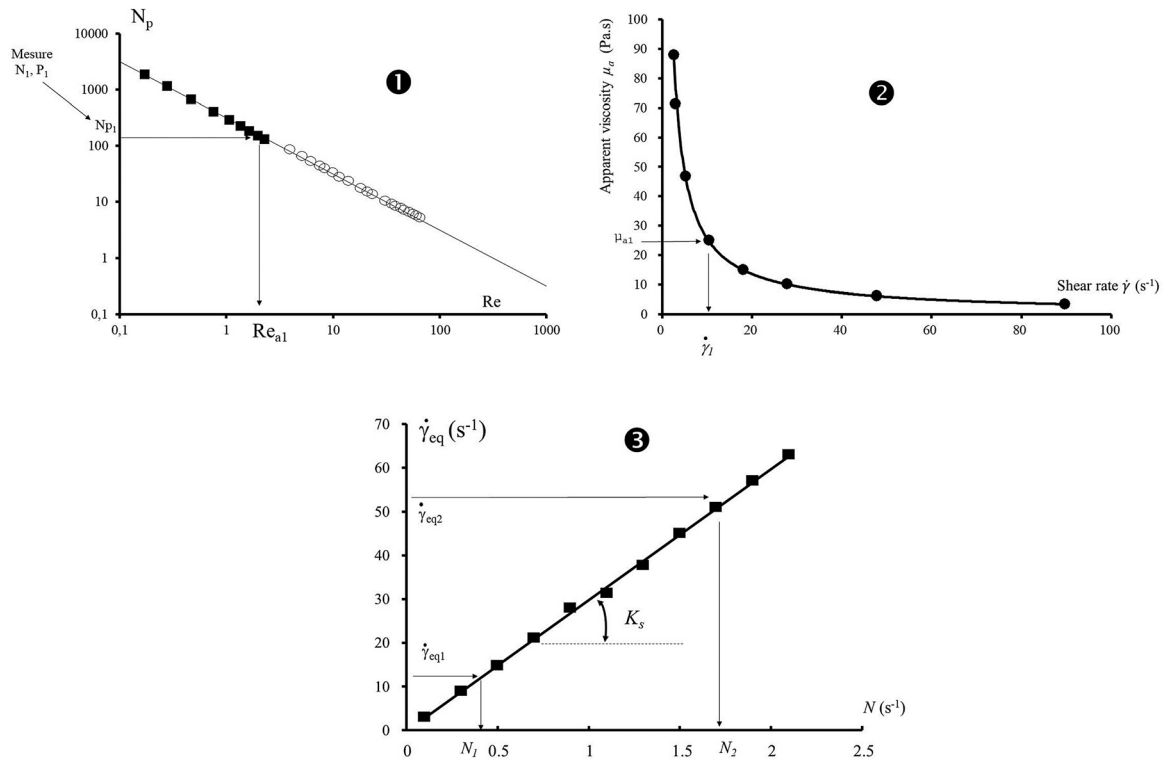
$$\dot{\gamma}_{\text{eff}} = K_S N \quad (1)$$

where  $K_S$  is the so-called Metzner–Otto constant, which depends only on the mixing system (tank and agitator) geometry.

This latter is estimated, for a given mixing system, using power consumption measurements for Newtonian and non-Newtonian fluids, coupled with rheological curve. Figure 1 illustrates the main steps for determining the Metzner–Otto constant  $K_S$ . At a specified rotational speed  $N$ , measurement of the power  $P$  required for agitating a Newtonian fluid of density,  $\rho$ , with an impeller of diameter,  $D$  (e.g. by torque measurement) enables calculation of the power number,  $N_p$ , for the non-Newtonian fluid.

$$N_p = \frac{P}{\rho N^3 D^5} \quad (2)$$

The corresponding apparent Reynolds number,  $Re_a$ , can be extracted from the Newtonian curve (step ❶). Thereafter, an apparent viscosity,  $\mu_a$  ( $\dot{\gamma}_{\text{eff}}$ ) can be deduced from the definition of an apparent Reynolds number:



**Figure 1:** The main steps for determining the Metzner and Otto constant  $K_S$ .

$$Re_a = \frac{\rho ND^2}{\mu_a(\dot{\gamma}_{\text{eff}})} \quad (3)$$

The effective shear rate,  $\dot{\gamma}_{\text{eff}}$ , corresponding to this apparent viscosity is then determined from the rheological curve for the non-Newtonian fluid (step ②). Finally, the Metzner–Otto constant  $K_S$  is determined from the slope of the straight line connecting the dots ( $N, \dot{\gamma}_{\text{eff}}$ ), as expressed in Equation (1) (step ③). Typically, the Metzner–Otto constant ranges from 5 to 80, depending on the type of agitation system (Paul et al. 2004).

Notwithstanding the huge simplification inherent in Equation (1), this approach has revealed success in establishing satisfactory correlations between Newtonian and shear-thinning power law fluid characteristics for different types of turbines (Doraiswamy et al. 1994). This trick has therefore gained acceptance in the scientific community, and stretched out more and more from the initial frame considered by Metzner and Otto (1957) (i.e. out of laminar regime, with axial and close clearance agitators which are different than radial impellers such as turbine, with varying classes of rheological complex fluids presenting elastic properties or yield stress, for correlating more than power consumption data with Newtonian fluids, etc.).

Indeed, the sole dependency of  $K_S$  on the geometrical parameters was initially stated by Metzner and Otto when correlating power consumption of Rushton turbine with shear-thinning (adopting an Ostwald–de Waele power law model) and Newtonian fluids. The authors also pointed out that this approach using effective shear rate, was only valid within the laminar region ( $Re_a < 20$  when the flow index  $n$  for shear-thinning fluids ranged between about 0.25 and 0.45, and  $Re_a < 10$  when  $n = 1$ ).

For instance, it is worth to report that the application of this concept for close clearance impellers (e.g. ribbons, anchors) differs from that of turbine impellers. Indeed, turbine impellers do not feel the presence of the tank wall as the ratio of agitator to tank diameters  $D/T$  is close to one-third, whereas it is about 0.9 for close clearance impellers. In addition, the flow pattern induced by the agitator is axial, differing from the radial one of turbines (Chhabra and Richardson 2008; Paul et al. 2004). Similar remarks can be done when measurements are made out of the laminar regime, since rigorously shear rate can only be defined for this regime. At the onset of transitional regime and in turbulent regime, fluctuating components of velocities appear, changing the nature of the relation between the shear rate to the power input, as underlined by Doraiswamy et al. (1994).

Consequently, and even applied in conditions far from that initially considered by these authors, the Metzner and

Otto's concept has raised some criticism (Chhabra and Richardson 2008), and some studies have progressively reported the limits of concept by (i) mentioning its lack of genericity for gathering results obtained with different classes of rheological liquids or out of laminar regime, (ii) questioning on the underlying physical meaning of effective shear rate and its volumetric region, and (iii) discussing the best way to properly identify  $K_S$  value. Namely:

- Although the constant  $K_S$  was originally postulated to depend only on the geometrical parameters of the agitation system (tank and impeller), it was shown for shear-thinning fluids obeying Ostwald–de Waele power law model that  $K_S$  could be as well a function of the rheological parameters of the fluid. In particular, considerable confusion existed in the literature regarding the latter dependence of the constant  $K_S$  in the case of helical ribbon impellers with or without a screw (Chhabra 2003). For this close clearance impeller, on the one hand, Hall and Godfrey (1970), Nagata et al. (1971), Rieger and Novak (1973), Shamlou and Edwards (1985) and Takahashi et al. (1984) have all concluded that  $K_S$  was only a function of the agitation system geometry and it was independent of the fluid rheology. On the other hand, the works of Brito-de la Fuente (1992), Brito-de la Fuente et al. (1991, 1992, 1997, 1998), Cheng et al. (1995), Carreau et al. (1993), Leuliet et al. (1991, 1992), Netusil and Rieger (1993), Cheng and Carreau (1994) and Yap et al. (1979) have clearly suggested that  $K_S$  increased with the flow behaviour index of the power law fluids. Strictly speaking, one could expect it to only depend on the rheology of the liquid simply because the nature of the flow field (three dimensional and unsteady) is determined by the rheology of the medium and the annular gap.
- No guarantee is yet given on the validity of the Metzner–Otto concept for non-Newtonian fluids other than shear-thinning fluids describing Ostwald–de Waele power law model. In particular, for purely viscous fluids, major uncertainties exist for two subcategories, yield stress fluids (Anne-Archard et al. 2006; Bertrand et al. 1996) and shear-thickening fluids (Delaplace et al. 2000a), as well as for fluids having elastic properties such as viscoelastic fluids (Jahangiri 2008; Yap et al. 1979).
- Some works attempted to extend the use of Metzner–Otto concept outside the laminar regime for which it was originally and strictly established. For helical ribbon impellers, it was experimentally shown that the correlation worked poorly in the transition and turbulent regions since the local shear rates were higher than

the viscous shear rates predicted near the impeller by the correlation (Cheng et al. 1994). Similarly, Jahangiri (2008) investigated the local shear rate for helical ribbon impeller using laser Doppler anemometry with viscoelastic liquids. He proposed to correlate the variation of the local shear rate against the impeller speed by a power equation, i.e.  $\dot{\gamma}_{\text{eff}} = K_S N^{b'}$  ( $b' > 1$ ) in the transition region, i.e.  $70 < \text{Re}_a < 6700$ . Kelly and Gigas (2003) performed numerical simulations (CFD) to identify the effective shear rate in the transitional regime with an axial flow agitator. They also confirmed numerically that the Metzner–Otto method led to under-predict the average shear rate. Furthermore, Sanchez Perez et al. (2006) analysed again the data of Kelly and Gigas (2003) to propose a correlation in turbulent flow for both Newtonian and shear-thinning power law fluids; the average shear rate was shown to depend on agitator rotational speed raised to exponent  $3/(1 + n)$ , with  $n$  standing for the flow behaviour index of the shear-thinning fluids.

- Recent advances in Computational Fluid Dynamics and experimental velocimetry measurements (LDA, PIV, and other particle tracking methods) gave access to local shear-rate fields in the flow domain. Taking advantages on them, some authors attempted to localize the shape of the volumetric region where the effective shear rate takes place for various mixing systems, and to find an appropriate volume enclosing the agitator for extracting effective shear-rate values (Delaplace et al. 2000b; Gabelle et al. 2013; Jahangiri 2008; Kelley and Gigas 2003; Ramirez-Muñoz et al. 2017; Shekhar and Jayanti 2003; Wu et al. 2006; Zhang et al. 2008). These authors addressed questions concerning the physical meaning of the effective shear rate  $\dot{\gamma}_{\text{eff}}$  derived from the Metzner–Otto concept. In addition, they aimed to propose explicit criteria for the calculation of an effective shear rate corresponding to a given agitator rotational speed and to be able to determine  $K_S$ , without requiring any experimental approach or any numerical simulation with Newtonian fluids (Ramirez-Muñoz et al. 2017). From data analysis, it appeared that it didn't have a unique criterion to extract the effective shear rate from a given mixing system. Unfortunately, each criterion seemed very specific to the mixing system geometry and, to a certain extent, to the post-processing ability of the software used for carrying out the numerical simulation. For instance, the area-weighted average viscosity around the impeller was taken as the effective viscosity for helical ribbon mixing system (Zhang et al. 2008), while Shekhar and Jayanti (2003) proposed to take the

circumference-averaging shear rate at mid-height of the agitator. For axial and radial flow impellers, it was common practice to calculate the effective shear rate from the flow rate passing through a plane within the impeller diameter periphery (Wu et al. 2006). Even if the shear rate could be estimated or measured accurately, there is still a long way to go before defining a priori criteria for identifying the effective shear rate of a non-conventional and unknown mixing system. Note that before these numerical studies, many authors have attempted to adapt the Metzner–Otto concept without challenging the relevancy of an effective shear rate. They proposed modified ways for determining the constant  $K_S$  (Brito de la Fuente 1997, 1998; Rieger and Novak, 1973) and more generalized correlations linking  $K_S$  and rheological parameters.

- Finally, the question whether the introduction of such an effective constant  $K_S$  is well found from the standpoint of dimensional analysis has been raised. In particular, as discussed by some authors (Böhme and Stenger 1988; Pawlowski 2005), the effective shear rate is not appropriate by itself to describe the complexity of any rheological behaviour. The issue to use the same power curve to describe power numbers with Newtonian and all types of non-Newtonian fluids must be addressed. Indeed, it is not obvious that the configuration (i.e. the set of dimensionless numbers describing power number variation) should be identical for all the types of fluids. In addition, the shift method which is applied to match the Newtonian and non-Newtonian power curves should be always feasible.

In accordance with these issues, the aim of this review paper is to show how a consistent dimensional analysis makes it possible to address transport phenomena problems in agitated tanks with non-Newtonian fluids, without resorting to effective physical quantities. Before addressing this point, the pending questions of Metzner and Otto's concept will be illustrated through discussing experimental results and analytical cases (Section 2). In Section 3, the theoretical background underlying the dimensional analysis will be briefly reminded and its extending to the cases of variable material properties (including non-Newtonian fluids) will be presented. Section 4 will demonstrate how the framework associated with the dimensional analysis enables to exceed the use of effective quantities as required by the concept of Metzner and Otto, and how is adapted for considering the deviations provided by non-Newtonian properties for a wide range of fluids in various processes, without being restricted to batch reactor and laminar regime.

## 2 Concept of Metzner–Otto: pending questions

This section aims at illustrating the limits inherent to the Metzner–Otto concept. It will be first demonstrated, both theoretically (coaxial cylinders) and experimentally (helical ribbon type impellers) and for shear-thinning fluids adopting an Ostwald–de Waele power law model, that  $K_s$  is not a purely geometrical parameter depending only of the mixing system as assumed by Metzner and Otto (1957).

Then, it will be illustrated how the case of heat transfer with Newtonian fluids is analogous to the non-Newtonian flow under isothermal conditions in a stirred vessel considered here. Indeed, a spatial distribution of viscosity exists in both cases (respectively, induced by a temperature distribution and a shear-rate distribution within the flow domain), and the common aim is to account for the effect of viscosity distribution on the output of the system (respectively, heat transfer coefficient and power consumption). This will allow us to discuss the need of determining effective quantities in order to model the process.

### 2.1 Dependence of $K_s$ with flow index in the case of coaxial cylinders mixing shear-thinning fluids

Let's consider a shear-thinning fluid which rheological behaviour is described by a power law, as:

$$\mu_a = k \cdot |\dot{\gamma}|^{n-1} \quad (4)$$

where  $k$  is the consistency and  $n$  is the flow index ranging from zero to one ( $0 < n < 1$ ). This fluid is contained in the annular gap between the two coaxial cylinders with the inner cylinder rotating. In this case, the velocity profile can be theoretically predicted and derived to obtain the shear-rate profile. It can be shown that the shear rate  $\dot{\gamma}_{cc}$  for laminar regime depends on the radial location  $r$  from the centre of the vessel (Bird et al. 2001):

$$\dot{\gamma}_{cc}(r) = \frac{4 \cdot \pi \cdot N}{n} \cdot \left(\frac{T}{2}\right)^{\frac{2}{n}} \left(\frac{1}{r}\right)^{\frac{2}{n}} \cdot \left(\frac{1}{\left(\frac{T}{D}\right)^{\frac{2}{n}} - 1}\right) \quad (5)$$

where  $D$  is the diameter of the impeller (bob) and  $T$  is the diameter of the tank (cup). Assuming an axisymmetric flow,  $\dot{\gamma}_{cc}(r)$  becomes independent of the angular coordinate.

Equation (5) shows clearly that the values of analytical local shear-rate  $\dot{\gamma}_{cc}(r)$  depend on both the flow behaviour

index  $n$ , on  $T/D$  diameter ratio of the outer to inner cylinders and on radius,  $r$ . In this context, it is very unlikely that an effective shear rate,  $\dot{\gamma}_{eff}$  which is supposed to be “a representative shear-rate values”, would be independent of the flow index. This finding clearly points out that assuming that  $K_s$  is a pure geometric constant, independent of liquid rheology, is not plausible.

Note that, even for the case of coaxial cylinders, for which the local value of shear rate can be theoretically predicted, it is not possible to obtain a mathematical equation for  $K_s$  without additional hypothesis. As a matter of fact, the expression of  $\mu_{eff}$  is required as shown below.

$$\begin{aligned} \mu_{eff} = \mu_a(\dot{\gamma}_{eff}) &= k \cdot \left(|\dot{\gamma}_{eff}|\right)^{n-1} = k \cdot (K_s \cdot N)^{n-1} \\ \Rightarrow K_s &= \frac{1}{N} \left(\frac{\mu_{eff}}{k}\right)^{1/(n-1)} \end{aligned} \quad (6)$$

Unfortunately, even if Metzner and Otto stipulated the existence of an average shear rate,  $\dot{\gamma}_{eff}$  around the impeller, no indications were given in their original publication (Metzner and Otto 1957) on where the exact location could be and what was the geometric shape of this area, as pointed out by Ramirez-Muñoz et al. (2017). Therefore, it is very difficult to precise how  $\mu_{eff} = \mu_a(\dot{\gamma}_{eff})$  could be deduced from the knowledge of local value of apparent viscosity,  $\mu_a(\dot{\gamma})$  and its distribution within the flow domain. Consequently, several options are possible:

It can be imagined for example that:

$$\mu_{eff} = \langle \mu_a(\dot{\gamma}) \rangle \quad (7)$$

where  $\langle \mu_a(\dot{\gamma}) \rangle$  refers to a value of apparent viscosity  $\mu_a(\dot{\gamma})$  averaged on a part (close to the wall of the agitator or on the tank for example) or on the whole volume of the flow domain. For instance, Zhang et al. (2008), for a helical ribbon agitator, computed the apparent viscosity around the impeller and used Equation (6) to identify effective shear rate. Based on this, it is very easy to determine  $K_s$  using Equation (1), directly from non-Newtonian simulations.

Alternatively, it can be also supposed that:

$$\mu_{eff} \cdot \dot{\gamma}_{eff}^2 = \langle \mu_a(\dot{\gamma}) \cdot \dot{\gamma}^2 \rangle \quad (8)$$

where  $\langle \mu_a(\dot{\gamma}) \cdot \dot{\gamma}^2 \rangle$  refers to an average value of local dissipated power  $\mu_a(\dot{\gamma}) \cdot \dot{\gamma}^2$  computed on a part or on the whole volume of the flow domain. Indeed, it is widely admitted that the power consumption supplied by the agitator is exclusively consumed as viscous dissipation of energy inside the tank.

Equation (8) can be rewritten

$$\mu_{eff} \cdot (K_s \cdot N)^2 = \langle \mu_a(\dot{\gamma}) \cdot \dot{\gamma}^2 \rangle \quad (9)$$

Depending on the chosen option, Equation (7) or Equation (8), the analytical equation of  $\mu_{eff} = \mu_a(\dot{\gamma}_{eff})$

differs, and  $K_s$  deduced from Equation (6) can then take different expressions.

For the case of coaxial cylinder developed here, it is also reasonable to postulate that the effective viscosity  $\mu_{\text{eff}}$  corresponds to the average apparent viscosity  $\langle \mu_a(\dot{\gamma}) \rangle$  of the power law fluid inside the annular gap (noted  $\bar{\mu}_a$ ).

Hence, it can be expressed for a coaxial cylinder:

$$\mu_{\text{eff}} = \mu_a(\dot{\gamma}_{\text{eff}}) = \bar{\mu}_a = \frac{1}{V} \iiint \mu_a(r) r dr d\theta dz \quad (10)$$

with:

$$\mu_a(r) = k. (\dot{\gamma}_{cc}(r))^{n-1} \quad (11)$$

Assuming that there are no clear bottom side effects for the coaxial cylinders and no dependency with the angular position, the integration of Equation (10) after its combination with Equations (11) and (5) and using Equation (6), leads to the following analytical expression for  $K_s$ :

$$K_s = \frac{4\pi}{n} \frac{1}{(S^{2/n} - 1)} \left[ n \frac{S^2}{(S^2 - 1)} \frac{(S^{(\frac{2}{n})} - 1)}{S^{(\frac{2}{n})}} \right]^{1/(n-1)} \quad (12)$$

with:

$$S = \frac{T}{D} \quad (13)$$

As expected, Equation (12) highlights that  $K_s$  depends on both the flow behaviour index  $n$  and on the geometry of the system. The dependence of  $K_s$  with  $n$  is more pronounced when the gap size between the two coaxial cylinders is large ( $S > 1.1$ ) and when shear-thinning properties are strong ( $n < 0.4$ ). This finding clearly points out that

assuming that  $K_s$  is a pure geometric constant, independent of liquid rheology, is not founded.

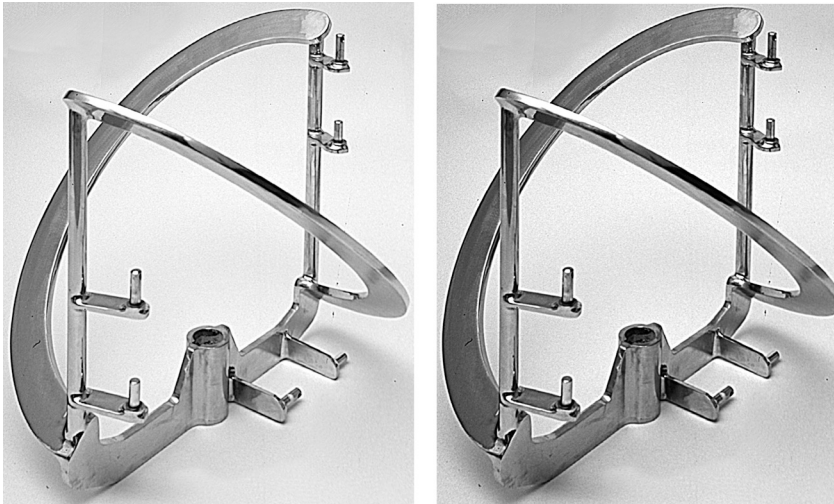
Based on this theoretical example related to a particular mixing system, it is not illogic that for other mixing systems equipped with different agitators, a dependence of Metzner and Otto's constant with flow behaviour index could be found (see Section 2.2).

## 2.2 Experimental evidence of the dependence of $K_s$ with flow index for non-coaxial mixing systems

In this section, the dependence of  $K_s$  with flow index  $n$  will be illustrated by means of various examples involving mixing systems based on helical ribbon-type impellers and anchors. In particular, it will be highlighted, by analysing experimental results of literature studies, that the concept of Metzner and Otto is not rigorously relevant for non-Newtonian fluids other than shear-thinning fluids (i.e. shear-thickening, Bingham, and viscoelastic fluids).

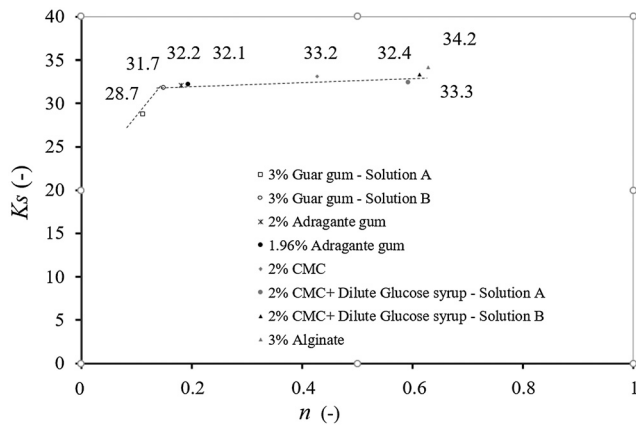
### 2.2.1 Shear-thinning fluids

Many authors have determined the values of  $K_s$  for different close-clearance impellers. Yap et al. (1979), Brito-de la Fuente et al. (1991, 1992, 1997, 1998), Leuliet et al. (1992), Carreau et al. (1993), Cheng and Carreau (1994) and Cheng et al. (1995) clearly demonstrated that the value of  $K_s$  increased for helical ribbon impellers (with or without a screw) with the flow index of shear-thinning fluids. In the same way, Delaplace et al. (2000a) established that the



**Figure 2:** Picture of the anchor (left) and Paravisc®-Ekato (right) mixing systems studied in Delaplace (1998).





**Figure 3:** Variation of the Metzner–Otto constant,  $K_s$ , with the flow behaviour index of shear-thinning fluids,  $n$ , for the Paravisc®-Ekato system (extracted from Delaplace et al. 2000a).

value of  $K_s$  for a non-conventional helical agitator, namely the Paravisc®-Ekato system (Figure 2), decreased for low values of flow index ( $n < 0.4$ , i.e. strong shear-thinning properties), as indicated in Figure 3.

Using a Couette flow analogy in which the helical ribbon impeller was replaced by an equivalent cylinder, Delaplace et al. (2006) developed an approximate analytical model for predicting power consumption when mixing shear-thinning fluids adopting an Ostwald–de Waele power law model, with helical ribbon and helical screw ribbon impellers in the laminar regime. Even if this simple model was not perfect (i.e. neglecting the drag of the blades, and openness of the structure), it showed again that the values of  $K_s$  depended on the flow index  $n$ . Namely  $K_s$  increased with the flow index  $n$  and tended asymptotically towards a finite value as  $n$  was close to unity (i.e. the Newtonian fluid behaviour). On the other hand, the dependency of  $K_s$  to flow index was very pronounced when  $n < 0.4$  and  $S > 1.1$ . Delaplace et al. (2006) showed that the weak variation of flow index handled by various authors could explain the controversy about this dependency. Indeed, Hall and Godfrey (1970), Nagata et al. (1971), Rieger and Novak (1973), Takahashi et al. (1984), Shamlou and Edwards (1985) and Zhang et al. (2008) concluded that  $K_s$  was only a function of the mixer geometry and independent of the fluid rheology.

## 2.2.2 Shear-thickening fluids

Delaplace et al. (2000a) showed experimentally that the concept of Metzner and Otto could not be extended to shear-thickening fluids (Equation (4) with  $n > 1$ ). Indeed, due to a partial “solidification” of the fluid around the

impeller, the values of  $K_s$  obtained for weakly shear-thinning fluids were no more adapted to predict the power consumption with shear-thickening fluids. For Paravisc®-Ekato system, these authors demonstrated that the representative shear rate in the vessel was 4 or 5 times higher than for shear-thinning fluids ( $K_s = 136$  instead of 32). Tanguy et al. (1996) carried out numerical simulations of the flow for shear-thickening and shear-thinning fluids in mixing vessels equipped with an anchor, computed power consumption and then identified  $K_s$ . They also concluded that the values of  $K_s$  for shear-thickening fluids were not similar and were superior to those obtained for shear-thinning fluids. They showed that  $K_s$  varied according to the relationship  $K_s = 21.3 + 5.8 n$ , for the flow indexes  $n$  ranged from 0.3 to 1.7, namely from  $K_s = 23$  for  $n = 0.3$  to  $K_s = 32$  for  $n = 1.7$ .

## 2.2.3 Bingham fluids

When the power consumption was measured in anchor and Paravisc®-Ekato mixing systems in presence of yield stress fluids, Delaplace (1998) showed that  $K_s$  was substantially different from the average value determined in presence of shear-thinning fluids ( $K_s = 28.7$  instead of  $K_s = 32.2$  for Paravisc®-Ekato system and  $K_s = 22.3$  instead of  $K_s = 29.6$  for the anchor-type system). Bertrand et al. (1996) investigated numerically the mixing of yield stress fluids with an anchor impeller in the laminar regime using a three-dimensional finite element method. They observed that the  $K_s$  slightly varied over a wide range of Bingham numbers, thus highlighting the effect of yield stress on the flow ( $21.1 < K_s < 23.8$  for  $Bi \leq 7500$ ). Finally, the mixing of yield stress fluids with a Rushton turbine was also investigated numerically in the laminar regime by Torrez and André (1999). According to this study,  $K_s$  was found to be a function of fluids properties and they reported a variation from 7.3 to 9.6.

These findings pointed out that at least it is impossible to assign a single value to  $K_s$  for two subcategories of purely viscous fluids (yield stress and shear thickening). The main reason is the following:

- by dimensional analysis, one can easily demonstrate that the value of  $K_s$  in viscoplastic fluids depends on the value of the Bingham number containing the yield stress and on the mixing system geometry (tank + agitator); so, the value of  $K_s$  would be independent on yield stress only when there are no cavern regions in the tank.
- For the shear-thickening fluids, the shear rate, and consequently the apparent viscosity, is maximum in the vicinity of the agitator; hence, the agitator

dissipates a high amount of power in this almost solidified material. However, the material shows more fluid-like characteristics elsewhere far away from the agitator and the tank wall.

In conclusion, when comparing these two fluids to the case of shear-thinning fluids, the flow domain in the tank is not really independent of the rheological liquid properties, which is likely to explain the observed variation of  $K_s$  within a single class of fluid. Probably, strong shear-thinning fluids correspond almost to the same case, since such liquids with low values of flow index can be assimilated to yield stress fluids. It is worth mentioning that all these phenomena are very sensitive to the geometry of the mixing system (radial, axial or close clearance).

#### 2.2.4 Viscoelastic fluids

Carreau et al. (1992, 1993) and Cheng et al. (1994, 1995) studied the power consumption when mixing fluids with non-negligible elastic properties using helical ribbon agitators. They pointed out that the fluid's elasticity increased significantly the power requirement, and observed deviations from the generalized Newtonian power curve in the laminar regime at smaller Reynolds numbers for viscoelastic fluids. Moreover, for some elastic solutions (0.4% CMC in glycerol/water), no constant value of  $K_s$  could be obtained within the experimental range.

Similar trends were also observed for power consumption by Yap et al. (1979) and Jahangiri (2008) with ribbon impellers, and Özcan-Taskin et Nienow (1995) with Rushton and InterMIGs impellers. For Rushton turbines, which generate very different flows from those of close clearance impellers, it has also been reported that the power demands may be either higher or less than those of Newtonian fluids, and this depends on the Reynolds and Deborah or Weissenberg numbers (Oliver et Nienow, 1984). Oliver et Nienow (1984) also pointed out that the data were poorly reproducible even for different runs under the same conditions.

As underlined by Chhabra et al. (2008), the current experimental results are not numerous but suggest that the extent to viscoelastic properties is strongly dependent on the mixing system and operating conditions. This suggestion is not illogical since the relative contributions of viscous (liquid behaviour) and elastic properties (solid behaviour) are dependent on the frequency of the mechanical solicitation in oscillatory rheology. In this context, it is likely that the constant  $K_s$  shows additional dependency on viscoelastic parameters like a suitably defined Deborah, or Weissenberg or elasticity number.

Such a correlation between  $K_s$  and the elasticity number of the viscoelastic liquids has been proposed by Jahangiri (2008).

## 2.3 Analogy with heat transfer

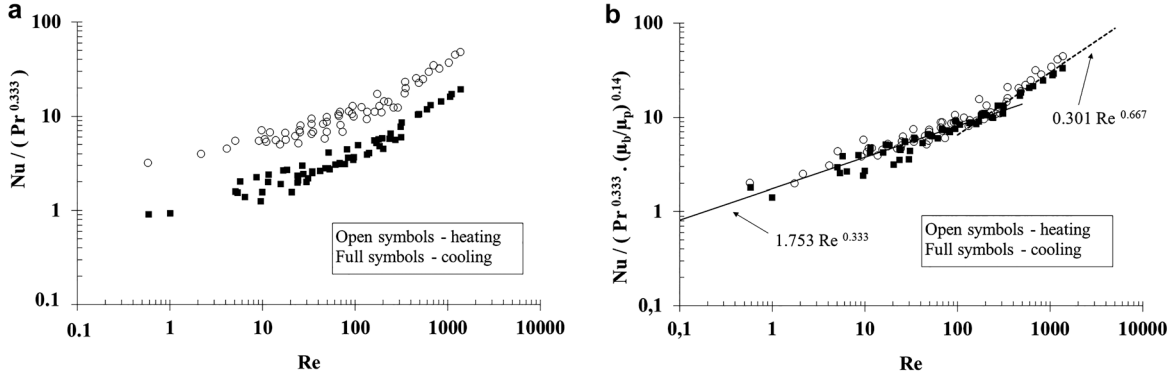
The concept of Metzner and Otto allows evaluating the power consumption of shear-thinning fluids, using the power curve established for Newtonian fluids with the idea of *matching viscosities* (see Figure 1). It requires to identify a kind of shift factor, linked to the Metzner–Otto constant  $K_s$ , so as to superimpose the data obtained (at least) for shear-thinning fluids with the Newtonian one. Let's remind that the term *matching viscosities* refers to the assumption that an apparent viscosity  $\mu_a(\dot{\gamma}_{\text{eff}})$  calculated at an effective shear rate  $\dot{\gamma}_{\text{eff}}$  could be considered for evaluating the power consumption of non-Newtonian fluids.

However, it should be noted that this way of introducing an effective physical quantity (a shear rate for instance) does not apply only to a spatial distribution of viscosity caused by a non-Newtonian rheology. As an example, in the case of significantly thermal-sensitive products, the fluid viscosity may also vary within a heat-exchanger during thermal processing. Such a variation in viscosity with temperature,  $\mu(\theta)$ , modifies the heat transfer coefficient and hence the overall performance of the process when compared to a fluid whose thermal-dependence can be neglected. Another strategy is commonly used to solve this issues, namely to integrate in the modelling the influence of such viscosity gradient for gathering the experimental points on a single heat transfer master curve. It mainly consists in introducing a viscosity number  $Vi$  elevated at a given exponent:

$$Vi = \frac{\mu_b}{\mu_w} \quad (14)$$

where  $\mu_b$  and  $\mu_w$  are, respectively, the viscosity of the fluid taken at two different bulk and wall surface temperatures, indicating the existence of a viscosity gradient. For example, in the well-known correlations of Nagata et al. (1972) or Ishibashi et al. (1979) established in stirred tanks equipped with a double jacket and helical ribbon stirrers, or the one of Sieder and Tate (1936) in cylindrical pipe, the viscosity number is raised to the power of 0.14.

In other words, and considering the variation of viscosity within the equipment, a single master curve can be obtained either by defining a constant (for instance  $K_s$ ) or by introducing a ratio representative of the viscosity gradient (for instance the viscosity number  $Vi$ ). By far, the second method is easier since it doesn't require the prior



**Figure 4:** Heat transfer of different temperature-sensitive fluids in an agitated vessel equipped with a non-standard helical ribbon impeller transfer: Illustration of the shift effect induced by the introduction of the viscosity number  $Vi$  defined in Equation (14):  $Nu/Pr^{0.333}$  versus  $Re$  (left) against  $Nu/(Pr^{0.333} \cdot Vi^{0.14})$  versus  $Re$  (right). These figures were plotted from the data published in the study by Delaplace et al. (2001).

knowledge of Metzner–Otto constant. Figure 4 illustrates this idea in the case of heat transfer with Newtonian fluids in an agitated vessel. Considering an experimental database related to the cooling and heating of different temperature-sensitive fluids in an agitated vessel equipped with a non-standard helical ribbon impeller, Delaplace et al. (2001) demonstrated that introducing the  $Vi$  number enabled to shift all the points over a single master curve.

## 2.4 Effective shear rate $\dot{\gamma}_{eff}$

To conclude Section 2, the physical meaning of the concept of effective shear rate involved in the Metzner–Otto concept is in itself questionable. Let's go back to the work of Delaplace et al. (2000b) in which a CFD approach was carried out to simulate the shear-rate field of Newtonian viscous fluids agitated in a round-bottomed vessel equipped with a Paravisc®-Ekato system (Figure 5).

The total power consumption  $P$  in the vessel was numerically obtained by summing the energy consumed in each volume control throughout the vessel:

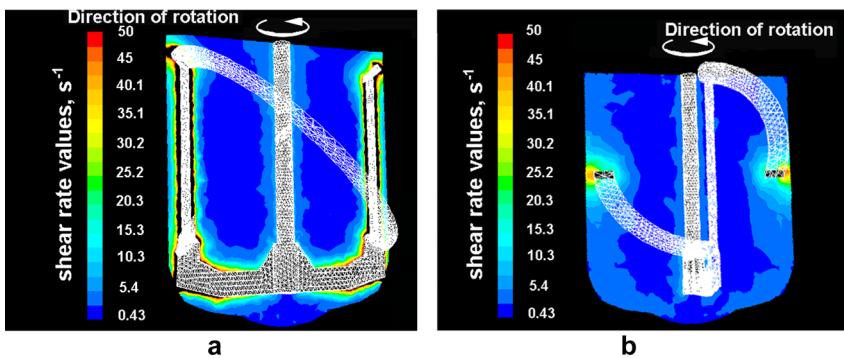
$$P = \mu \sum_i \dot{\gamma}_i^2 V_i \quad (15)$$

where  $\mu$  is the Newtonian viscosity of the fluids,  $\dot{\gamma}_i$  is the shear rate of the fluid contained in cell  $i$  and  $V_i$  is the volume of cell  $i$ . For each cell  $i$ ,  $\dot{\gamma}$  was deduced from the second invariant of the rate of the deformation tensor  $\bar{\Delta}$ :

$$\dot{\gamma} = \sqrt{\frac{1}{2} \bar{\Delta} : \bar{\Delta}} \quad (16)$$

The reliability of the numerical method was ascertained by comparing numerical and experimental power consumptions for the laminar regime with Newtonian Fluids ( $0.1 \leq Re \leq 60$ ). Less than 1% of standard deviation was obtained on the product  $N_p \cdot Re$ , equal to 315 for experimental measurements and to 312 for numerical simulations, respectively.

Even considering the lack of complete similarity between the two systems, it is obvious that, for the impeller rotational speed and the Newtonian fluid under test (Figure 5), the local values of shear rate appearing in the vertical planes are far away ( $0.4$  up to  $50 \text{ s}^{-1}$ ) from the average value of shear rate expected using the concept of Metzner and Otto ( $\dot{\gamma}_{eff} = 32.2 \text{ N} \approx 19 \text{ s}^{-1}$ ); moreover, the



**Figure 5:** Numerical shear-rate fields obtained in a round-bottomed vessel equipped with a Paravisc®-Ekato system in the case of a viscous Newtonian fluid ( $\mu = 2.82 \text{ Pa}\cdot\text{s}$ ) and for a rotational speed  $N$  of  $0.583 \text{ s}^{-1}$ : (a) in the vertical plane which contains the vertical arms of the anchor, and (b) in a vertical plane located at  $90^\circ$  from the vertical plane which contains the vertical arms of the anchor (extracted from Delaplace et al. 2000b).

part of the flow domain exhibiting the effective shear rate is much reduced and located around the blade of the Paravisc®-Ekato system, in agreement with the work of Zhang et al. (2008).

Keeping in mind that a Newtonian fluid is a shear-thinning liquid for which  $n = 1$ , this example can be generalized: the shape of the volumetric region where the effective shear rate takes place for such a mixing system remains difficult to localize, as well as the appropriate volume enclosing the agitator for extracting effective shear-rate values. The problem thus consists of defining an unambiguous and universal numerical criterion to obtain an effective shear rate so as to deduce  $K_s$  (with the help of Equation (1)) for an unknown mixing system, only by numerical simulations and without experiments, as proposed for a Rushton turbine by Ramirez-Munoz et al. (2017).

### 3 Dimensional analysis: state of the art on the theoretical frame and current practice

The method of Metzner and Otto has been developed with the view of estimating the viscosity that the agitator “feels” when operating in the laminar regime so that the dissipated power can be calculated. Consequently, it is a trick used to consider the gradient of viscosity in the tank, when the viscosity is no longer independent of the shear rate as for Newtonian fluids. A representative average shear rate is then evaluated for non-Newtonian fluids and enables to use the generalized form of the conventional power-number–Reynolds-number plot for the Newtonian ones. The great advantage is that everyone can intuitively understand this concept, without necessarily questioning about the theoretical background. Even if this concept has been settled on the sole basis of experiments with shear-thinning fluids (adopting an Ostwald–de Waele power law model) under laminar regime, the present authors examine the compatibility of this method with a modelling approach based on dimensional analysis. Some of the pending questions are: Is the use of representative average shear rate allowed from a dimensional analysis point of view? Is it always possible that the process relationship describing Newtonian results can superimpose with those obtained for different categories of non-Newtonian fluids, where apparent viscosity of the agitated medium can no longer be regarded as constant in the tank?

In summary, Metzner and Otto (1957) started to provide a partial answer to the question. Presently, the theoretical framework exists to rigorously consider the variability of material physical properties for modelling processes by dimensional analysis (i.e. for establishing the process relationship and/or defining the conditions for complete similarity between different scales). It requires to enlarge the number of internal measures (i.e. dimensionless numbers) which are necessary for describing the process and to make a subsequent change of the mathematical relationship predicting the evolution of the target variable. However, it is rarely used and the concept of Metzner and Otto is usually preferred, although it is not based on a theoretical frame.

Even widely applied in the case of constant physical properties, the implementation of dimensional analysis for a variable physical property unfortunately remains almost unexplored, mainly because it is not straightforward. Indeed, since the material’s physical property is not constant, a *material function* (e.g. a rheological law) should be introduced and considered for establishing the list of all the relevant physical quantities characterizing the process. The pioneering works of Pawlowski (1967; 1971; 1991), partially translated in English (Pawlowski 1969; Zlokarnik 2006), defined the theoretical framework and the rules required to build an unbiased list of the parameters influencing the process (including all the ones describing the variability of the physical property); they also showed why this framework complied with the principles of the theory of similarity and could thus be applied whatever the physical property considered. Similar ideas about the number of physical quantities required to achieve “material similarity” with respect to the different categories of the non-Newtonian fluids were also proposed by Astarita between the late 1960s and 1970s (Astarita 1967, 1979). Unfortunately, these works were not sufficiently disseminated to be widely applied by the scientific community (Delaplace et al. 2009; Hassan et al. 2012; Pawlowski 2005; Zlokarnik 2006). More recently, Delaplace et al. (2014, 2015) brought back to light this theoretical framework and proposed deeper explanations and new insights as well as various applications to practical cases. In this section, the key points of this theoretical framework will be emphasized with the aim at providing all information to apply this chemical engineering tool within a unique paper to the reader. Then, the efficiency of this latter to solve problems involving non-Newtonian fluid agitation without the use of Metzner–Otto concept will be demonstrated in Section 4.

### 3.1 Dimensionless material function: a key function influencing process relationship that is required for evaluating material similarity of material

Let's note that  $s(p)$  is the dimensional material function describing the variation of the physical property of the material,  $s$ , with the variable  $p$ , and  $H(v)$  is the associated dimensionless material function defined by:

$$v \rightarrow H(v) = \frac{s(p)}{s(p_0)} \quad (17)$$

where  $s(p_0)$  is the value of the dimensional material function at any reference abscissa, noted  $p_0$ , and  $v$  is the argument of the dimensionless material function  $H$ , which has to be dimensionless.

In reference to the terminology used by Pawlowski (1991) and maintained later by Zlokarnik (2006), the term "material function" refers to a variable physical property of the material involved in a process. This latter may be a fluid (or a mixture of fluids), a phase (or several phases), an agro-food product, a chemical medium and so on.

For example, let's consider a shear-thinning fluid for which the apparent viscosity,  $\mu_a$ , varies with the shear rate,  $\dot{\gamma}$ , according to Equation (4). In this case,  $p = \dot{\gamma}$  and  $s = \mu_a$ , and the associated dimensional material function is

$$H(v) = \frac{\mu_a}{\mu_0} = \frac{k \cdot \dot{\gamma}^{n-1}}{k \cdot \dot{\gamma}_0^{n-1}} = \left( \frac{\dot{\gamma}}{\dot{\gamma}_0} \right)^{n-1} \quad (18)$$

where  $\mu_0$  is the apparent viscosity calculated at any reference shear rate,  $\dot{\gamma}_0$ . In Equation (18),  $v$  refers to the argument of the function  $H$  and is defined as:

$$v = \frac{\dot{\gamma}}{\dot{\gamma}_0} \quad (19)$$

This gives

$$H(v) = v^{n-1} \quad (20)$$

$v$  is the dimensionless input variable to the function  $H$  (argument of the function) and is obtained when the variable  $p$  of the function  $s(p)$  is made dimensionless.

Now, let's remind that the dimensional analysis is an efficient approach to build the set of dimensionless numbers  $\{\pi_i\}$  (also called internal measures) that impact the target dimensionless number,  $\pi_{target}$ , as recently remembered by Delaplace et al. (2014, 2015). The set of the independent dimensionless numbers associated to the causes are linked by a mathematical relationship to the

target dimensionless number. This mathematical relationship is called the *process relationship*, and is generally determined by carrying out experiments in a laboratory pilot. As an example, if the power consumption ( $P$ ) of Newtonian fluids (density  $\rho$  and viscosity  $\mu$ ) in a tank mixed by an impeller (diameter  $D$ ) at constant rotational speed  $N$  is taken as target variable, dimensional analysis will lead to (Delaplace et al. 2014, 2015):

$$N_p = \frac{P}{\rho N^3 D^5} = F\left(\text{Re} = \frac{\rho N D^2}{\mu}; \text{Fr} = \frac{D N^2}{g}\right) \quad (21)$$

where  $N_p$ ,  $\text{Re}$  and  $\text{Fr}$  are the dimensionless power, Reynolds and Froude numbers, respectively. The process relationship in laminar regime deduced from experiments under isothermal condition and without vortex is:

$$N_p = \frac{P}{\rho N^3 D^5} = \frac{K_p}{\text{Re}} \quad (22)$$

with  $K_p$  as a constant parameter depending on the mixing system geometry.

In the case of handling materials with variable physical properties, Pawlowski (1991) demonstrated that the process relationship depends on the dimensionless material function  $H$  (and not on its dimensional formulation  $s$ ). Consequently, a representation of the dimensionless argument and parameters composing this dimensionless material function  $H$  becomes fundamental. This dependence constitutes the pillar on which the concept of *material similarity* is based. Indeed, two materials are assumed similar if they can be represented by a single and unique dimensionless material function  $H$ , regardless the concordance of their dimensional material function.

Such a definition has a straightforward practical consequence. When experiments are carried out on a laboratory-scale equipment at an ambient temperature with a view to size a thermal process at high temperatures, the search for suitable model materials is a key issue, which is often difficult to overcome. The fact that the model material does not need to exhibit the same dimensional material function  $s$ , but only meet the same dimensionless material function  $H$  as the material used in the thermal process on a 1/1 scale, makes the problem easier.

Another consequence is that the identity of an operating point on two scales means conserving all of the internal measures associated with the variability of the material's physical property (noted  $\{\pi_m\}$ ) and all of the other internal measures  $\{\pi_i\}$  responsible for the causes influencing the target internal measure. This implies that before applying a process relationship established for a material with variable physical property, it has to be ensured that the investigated material has the same dimensionless material function that

the material used to establish process relationship. It means that all of the internal measures associated with the variability of the material's physical property (noted  $\{\pi_m\}$ ) should be conserved. Finally, one should keep in mind that the latter theoretical background supposes implicitly that the dimensional material function (for instance, the rheological law) is perfectly known as well as the related parameters. In practice, this is unfortunately not so trivial, especially with fluids presenting complex rheological properties, and thus required to provide strong efforts to be able to define a well-representative function with a good precision.

The last consequence of this functional dependency of the process relationship with the dimensionless material function for non-Newtonian fluids is that obtaining a unique process relationship for a given category of fluids (e.g. shear-thinning fluids adopting an Ostwald–de Waele power law) will make necessary:

- to identify the internal measures composing the dimensionless rheological function,
- and to enlarge the relevant list of physical quantities used to perform dimensional analysis by additional parameters (either dimensional or dimensionless) derived from examination of standard dimensionless material function.

The standard dimensionless material function is in fact a specific dimensionless material function deduced by applying a specific non-dimensionalization method from the rheological law (i.e. the dimensional material function). In the next section, we will see the requirement to adopt for this specific non-dimensionalization method.

### 3.2 Standard dimensionless material function: a pre-requisite to comply with the principles of the theory of similarity

As illustrated by Delaplace et al. (2014, 2015), and in addition to the mathematical expression of the dimensional material function  $s(p)$ , the graphical representation of the dimensionless material function depends on the reference abscissa,  $p_0$  and on the definition of the argument,  $v$ , that is to say on the non-dimensionalization method used.

Pawlowski (1991) demonstrated that all the non-dimensionalization methods didn't comply with the principles of the theory of similarity. The latter states that only the identity of the “operating points” on both scales (i.e. the equality of the numerical values of all the

dimensionless numbers describing the configuration of the system) is required to ensure the identity of the target dimensionless number. For materials with variable properties, this identity is more difficult to obtain as it also concerns the internal measures associated with the variability of the material's physical property (noted  $\{\pi_m\}$ ).

In brief, the theory of similarity requests only identity of dimensional numbers associated to the two scales (laboratory-scale and industrial-scale) but doesn't limit the choice of dimensional physical quantities to obtain this identity. Unfortunately, Pawlowski (1991) and Delaplace et al. (2014, 2015), showed that depending of the non-dimensionalization methods, some supplementary relationships between dimensional physical quantities should be required, thus implying that the latter could not be freely selected. The existence or the non-existence of these constraints is linked to the non-dimensionalization method of the material function. So, in order to fulfil the principles of the theory of similarity, precaution should be taken with regard to the non-dimensionalization method employed: it is highly recommended to apply a specific non-dimensionalization method, called the *standard non-dimensionalization method* (noted SNDM).

Let's define the SNDM. One should consider first  $p_0$  the reference abscissa and  $(p_0 ; s(p_0))$  the reference point so that  $s(p_0) \neq 0$  and  $\left(\frac{ds}{dp}\right)_{p=p_0} \neq 0$ . Whatever the material function  $s(p)$ , the SNDM is based on the definition of the argument of the dimensionless material function such as:

$$u = a_0 (p - p_0) \quad (23)$$

Note that the letter  $u$  is used to distinguish it from an argument  $v$  defined by any other method than the standard non-dimensionalization method.  $a_0$  is called the coefficient of physical property  $s$  relative to variable  $p$  and is expressed as follows:

$$a_0 = \frac{1}{s(p_0)} \left( \frac{ds}{dp} \right)_{p=p_0} \quad (24)$$

$a_0$  is then the slope of the dimensional material function calculated in  $p = p_0$  and divided by  $s(p_0)$ . By noting  $w$  the *standard dimensionless material function* (noted SDMF and obtained after applying the SNDM), one obtains:

$$H(v) = \frac{s(p)}{s(p_0)} = w(u) \quad \text{if and only if} \quad v = u = a_0 (p - p_0) \quad (25)$$

This non-dimensionalization method ensures that in the vicinity of the reference abscissa ( $u = 0$ ), the dimensionless material function  $w(u)$  and its derivative  $w'(u)$  are

constant and equal to 1 whatever the form of  $s(p)$ , that is to say that they are independent from the dimensional physical quantities which influence the system. Such a method enables to overcome the problem of non-compliance with the principles of the theory of similarity. As highlighted by Pawlowski (1991), the SDMF originated from a Taylor series of the dimensional material function in the vicinity of the reference abscissa.

To give a clearer view on how to apply the SNDM, let's go back to the example of shear-thinning fluids ( $0 < n < 1$ ). We have previously established the expression of the dimensionless material function,  $H$ , in relation to the shear rate (Equation (18)) using an argument  $v$  (Eq (19)). Now, to calculate the associated standard dimensionless material function,  $w(u)$ , one should consider the definition of the argument  $u$  proposed in Eqs. (23) and (24); this leads to:

$$u = \frac{1}{\mu_a(\dot{\gamma}_0)} \left( \frac{d\mu_a \dot{\gamma}}{d\dot{\gamma}} \right) (\dot{\gamma} - \dot{\gamma}_0) = \frac{\dot{\gamma} - \dot{\gamma}_0}{\dot{\gamma}_0} (n - 1) \quad (26)$$

From Equation (26), one can deduce:

$$\dot{\gamma} = \dot{\gamma}_0 \left( \frac{u}{n-1} + 1 \right) \quad (27)$$

Combining Equation (27) with Equation (25) leads to the expression of the standard dimensionless material function,  $w(u)$ , associated with shear-thinning fluids:

$$w(u) = \frac{\mu_a}{\mu_0} = \left( \frac{\dot{\gamma}}{\dot{\gamma}_0} \right)^{n-1} = \left( \frac{u}{n-1} + 1 \right)^{n-1} \quad (28)$$

Tables 1 and 2 summarize the argument  $u$  related to the standard dimensionless material function and standard dimensionless material function,  $w(u)$ , for common purely viscous non-Newtonian fluids (Ostwald–de Waele power law, Bingham, Herschel–Bulkley and Williamson–Cross fluids). In the next section, a particular standard dimensionless material function will be introduced for which the number of parameters issued from the standard

dimensionless function which should be added in the relevant list for dimensional analysis can be shortened.

### 3.3 Invariant standard dimensionless material function

It is important to keep in mind that, as defined in Equation (25), the curve representing the SDMF remains dependent on the chosen reference abscissa,  $p_0$ . Nevertheless, some classes of SDMF do not depend on it: they are called *invariant* standard dimensionless material function (noted ISDMF) and denoted  $\varphi$ , in order to distinguish them from the non-invariant ones ( $w$ ). It can be mathematically proven (Delaplace et al. 2014, 2015; Pawlowski 1991) that only two families of SDMF possesses these invariant properties, which are described by the following equations:

$$\varphi(u) = (1 + \beta u)^{1/\beta} \quad (29)$$

where  $\beta$  is a constant so that  $\beta \neq 0$ , and

$$\varphi(u) = \exp(u) \quad \text{if } \beta = 0. \quad (30)$$

Therefore, whatever the reference abscissa chosen, the graphical representation of the reference-invariant standard dimensionless material function,  $\varphi(u)$  is identical.

The SDMF associated with shear-thinning fluids is for example invariant as Equation (28) has the form of Equation (29) with  $\beta = \frac{1}{(n-1)}$ .

Finally, one can show that a necessary and sufficient condition to be satisfied so that the SDMF does not depend on the reference abscissa is that its dimensional material function takes the form of:

$$s(p) = (A + Bp)^C \quad (31)$$

or,

**Table 1:** Expressions for common non-Newtonian fluids of the dimensional material function and the argument related to the standard dimensionless material function.

Non-Newtonian fluid	Dimensional material function $\mu_a(\dot{\gamma})$	Argument $u$
Ostwald de Waele	$k \cdot \dot{\gamma}^{n-1}$ with $0 < n < 1$	$\left( \frac{\dot{\gamma} - \dot{\gamma}_0}{\dot{\gamma}_0} \right) (n - 1)$
Bingham	$\frac{\tau_y}{\dot{\gamma}} + \mu_p$	$Bi \cdot \left[ 1 - \frac{\dot{\gamma}}{\dot{\gamma}_0} \right]$
Herschel–Bulkley	$\frac{\tau_y}{\dot{\gamma}} + k \cdot \dot{\gamma}^{n-1}$	$(-Bi + (n - 1) \cdot [1 - Bi]) \cdot \left[ \frac{\dot{\gamma}}{\dot{\gamma}_0} - 1 \right]$
Williamson–Cross	$\frac{\mu_w}{1 + (t_w \cdot \dot{\gamma})^{1-n}}$	$\frac{(n-1) \cdot (t_w \cdot \dot{\gamma}_0)^{1-n}}{1 + (t_w \cdot \dot{\gamma}_0)^{1-n}} \cdot \left[ \frac{\dot{\gamma}}{\dot{\gamma}_0} - 1 \right]$

**Table 2:** Expressions for common non-Newtonian fluids of the standard dimensionless material function; here,  $Bi = \frac{\tau_y}{\mu_0 \dot{\gamma}_0}$ .

Non-Newtonian fluid	Standard dimensionless material function
Ostwald de Waele	$\left(\frac{u}{n-1} + 1\right)^{n-1}$
Bingham	$1 + \frac{u}{(1-\frac{Bi}{n})}$
Herschel-Bulkley	$\frac{Bi}{n(1-Bi)^{-1}+1} + (1-Bi) \left[ \frac{u}{n(1-Bi)^{-1}+1} + 1 \right]$
Williamson-Cross	$\frac{1+(t_w \dot{\gamma}_0)^{1-n}}{1+(t_w \dot{\gamma}_0)^{1-n} \left[ 1 + \frac{u}{n-1} \left( 1 + \frac{1}{(t_w \dot{\gamma}_0)^{1-n}} \right) \right]^{1-n}}$

$$s(p) = \exp(A + Bp) \quad (32)$$

where A, B and C are the constants. This result is convenient as it is often easier to consider the mathematical expression of the dimensional material function, rather than its dimensionless formulation.

Such reference-invariant material functions have a straightforward practical consequence. In the case of reference-invariant material functions, the reference abscissa  $p_0$  must not be added to the initial relevance list of physical quantities influencing the target variables.

The latter sub-sections have reminded how to build a standard dimensionless material function in order to identify the parameters which characterize the material function and to evaluate material similarity. The following sub-section will show how this standard dimensionless material function makes it possible to identify the parameters to add in the relevant list for building a  $\pi$ -space considering the rheological function.

### 3.4 How to build the $\pi$ -space in the case of a process involving a material with a variable physical property?

The knowledge of the SDMF makes it possible to unambiguously identify the *material configuration*, that is to say the set of dimensionless numbers (internal measures) linked to the variable material property. This configuration should then be added to the initial configuration of the system, that is, to the one that would be established if the material property could be considered as constant. Concerning the latter, one should refer to the studies by Delaplace et al. (2014, 2015) to know how to proceed and to establish it.

Now, the strategy for defining the complete space of dimensionless numbers, including the material configuration, will be described according to six main steps:

- Step no. 1: choosing the target variable (or the variable of interest) which is a characteristic of the phenomenon being studied. This should be a quantifiable measure of the performance of the process, such as a mixing time, a droplet size, a reaction conversion, etc.;
- Step no. 2: establishing the list of the relevant physical quantities influencing the target variable, except the physical property of the material  $s$ . Special caution should be taken to verify the physical independence of these quantities;
- Step no. 3: adding to this initial list the reference abscissa,  $p_0$ , except if the material function has invariance properties. At this level, it is important to understand that there is no *a priori* hypothesis governing the choice of the reference abscissa  $p_0$ ; in other words, any value of  $p_0$  can be chosen. Nevertheless, some choices are more relevant than others as they can lead to reduce the number of dimensionless numbers accounting for the variable material physical properties. Such relevant choices are illustrated in Table 3 (column II). Moving from column (I) to column (II) helps us to understand that the set of dimensionless numbers defining the material configuration can be reduced. Additional comments of Table 3 are given below it.
- Step no. 4: adding to the initial list, the material variable physical property calculated at the reference abscissa,  $s(p_0)$ ; the choice of the reference abscissa  $p_0$  is free as explained above;
- Step no. 5: determining the set of dimensionless numbers, denoted  $\{\pi_m\}$ , which appears in the expression of the dimensionless argument  $u$  of the SDMF  $w$ , except the ratio  $(p/p_0)$ . The general expression of  $\{\pi_m\}$  is:

$$\{\pi_m\} = \{a_0 p_0\} \quad (33)$$

As the analytical expression of the material function is generally known, it is possible to give an analytical expression of the coefficient  $a_0$ , and therefore of the dimensionless number  $\{a_0 p_0\}$ , directly using the expression of  $\left(\frac{ds}{dp}\right)_{p=p_0}$ . One can then observe that  $a_0 p_0$  is expressed by algebraic combinations involving one or more dimensionless numbers. Then, the set  $\{\pi_m\} = \{a_0 p_0\}$  can be replaced by this (these) dimensionless number(s). Note that, when the analytical expression of the material function is unknown, Delaplace et al. (2014, 2015) showed that the value of the dimensionless number  $a_0 p_0$  can be directly considered instead (in this case, the configuration of the material is not fully explicit).

- Step no. 6: establishing the *new* list of the relevant physical quantities and determining the complete set



**Table 3:** Physical quantities and supplementary dimensionless numbers to introduce to the relevant list (column I) and the relevant choice of the reference shear rate (column II) for common non-Newtonian fluids; here,  $\mu_0 = \mu(\dot{\gamma}_0)$ .

Non-Newtonian fluid	Column (I)	Column (II)
Ostwald de Waele	$\{\mu_0, n\}$	<ul style="list-style-type: none"> <li>– Invariant material function: free choice for <math>\dot{\gamma}_0</math></li> <li>– No change: <math>\{\mu_0, n\}</math></li> </ul>
Bingham	$\dot{\gamma}_0, \mu_0, Bi = \frac{\tau_y}{\mu_0 \cdot \dot{\gamma}_0}$	<ul style="list-style-type: none"> <li>– Relevant reference shear rate: <math>\dot{\gamma}_0 = \frac{\tau_y}{\mu_p}</math></li> <li>– Eq. of column (I) becomes <math>\left\{ \dot{\gamma}_0 = \frac{\tau_y}{\mu_p}, \mu_0 = 2\mu_p \right\}</math> with <math>Bi = 1/2</math></li> </ul>
Herschel-Bulkley	$\{\dot{\gamma}_0, \mu_0, n, Bi\}$	<ul style="list-style-type: none"> <li>– Relevant reference shear rate: <math>\dot{\gamma}_0 = \left( \frac{\tau_y}{k} \right)^{\frac{1}{n}}</math></li> <li>– Eq. of column (I) becomes <math>\left\{ \dot{\gamma}_0 = \left( \frac{\tau_y}{k} \right)^{1/n}, \mu_0 = 2 \cdot k^{\frac{1}{n}} \cdot (\tau_y)^{\frac{n-1}{n}}, n \right\}</math> with <math>Bi = 1/2</math></li> </ul>
Williamson–Cross	$\{\dot{\gamma}_0, \mu_0, n, (t_w \cdot \dot{\gamma}_0)\}$	<ul style="list-style-type: none"> <li>– Relevant reference shear rate: <math>\dot{\gamma}_0 = \frac{1}{t_w}</math></li> <li>– Eq. of column (I) becomes <math>\left\{ \dot{\gamma}_0 = \frac{1}{t_w}, \mu_0 = \frac{\mu_w}{2}, n \right\}</math> with <math>t_w \cdot \dot{\gamma}_0 = 1</math></li> </ul>

of dimensionless numbers  $\{\pi_i\}$  using on the Vaschy–Buckingham theorem.

Note that step nos. 3 and 4 have two major consequences:

- Including in the list the reference abscissa  $p_0$  for non-invariant standard material functions will automatically lead to a new dimensionless number:

$$\frac{p_0}{\{base\}} \quad (34)$$

where  $\{base\}$  represents the product of the repeated physical variables (chosen to constitute the core matrix) raised to different exponents, which has thus the same dimension than  $p_0$  (Delaplace et al. 2014, 2015).

- Whatever the material function, reference-invariant or not, adding  $s(p_0)$  in the list implies that this variable will appear in one or several dimensionless numbers. As a result, the user should not be surprised by the fact that, depending on the chosen reference abscissa  $s(p_0)$ , the dimensionless numbers in which  $s(p_0)$  is embedded will cover different ranges of numerical values, and therefore that the mathematical function linking the target dimensionless number to the other dimensionless numbers (i.e. the process relationship) can also change.

Finally, after implementing these six steps, the complete list of relevant physical quantities influencing the target

variable  $V_{target}$  can be established. To summarize, two options exist:

- If the material function is not reference-invariant:

$$\{V_{target}, V_1, V_2, V_3, \dots, V_m, p_0, s(p_0), \{\pi_m\}\} \quad (35)$$

- If the material function is reference-invariant:

$$\{V_{target}, V_1, V_2, V_3, \dots, V_m, s(p_0), \{\pi_m\}\} \quad (36)$$

where  $(V_1, V_2, V_3, \dots, V_m)$  are the physical (dimensional) quantities involved in the dimensional analysis of an identical process in which the material physical property  $s$  is constant. Using this list, the dimensional matrix can be written, and once the repeated variables (hereinafter named  $\{base\}$ ) are chosen, the complete set of dimensionless numbers  $\{\pi_i\}$  (including  $\{\pi_m\}$ ) responsible for the variation of  $V_{target}$  formed in the same way than for a material with a constant property (Delaplace et al. 2014, 2015).

### 3.5 Application to some common non-Newtonian fluids

In the previous sub-sections, the six-step framework has been detailed in order to unambiguously establish the complete space of dimensionless numbers influencing the target dimensionless number in the case of a process involving a material with a variable physical property. From this, the material configuration associated with some

common purely viscous non-Newtonian fluids (Ostwald–de Waele power law, Bingham, Herschel–Bulkley and Williamson–Cross fluids) can be defined. This is summarized in the tables shown, given for each fluid type:

- the expression of the dimensional material function  $s(p)$ , namely the rheological law linking the apparent viscosity (physical property) with the shear rate (the variable),  $\mu_a(\dot{\gamma})$ , (Table 1);
- the argument  $u$  related to the standard dimensionless material function (Table 1),
- the standard dimensionless material function,  $w(u)$ , (Table 2),
- the physical quantities and supplementary dimensionless numbers to introduce to the initial configuration of the system (Table 3),
- the relevant choice for the reference shear rate,  $\dot{\gamma}_0$ , i.e. the one enabling to reduce the material configuration and hence the configuration of the system (Table 3).

One can refer to the studies by Delaplace et al. (2014, 2015) for further details about these tables.

Table 3 particularly outlines how the material configuration related to non-Newtonian fluids is enlarged when compared to the Newtonian's one. For example, for shear-thinning fluids, one dimensionless number is added (the flow index  $n$ ) while two ( $n$  and the Bingham number  $Bi$ ) are necessary for Herschel–Bulkley fluids. From this, one can easily understand why there is any reason that the configuration associated with a Newtonian fluid, which is intrinsically reduced compared to non-Newtonian fluids, correspond to the one of these latter. As a consequence, using the concept of Metzner–Otto with the aim to apply the same  $\pi_i$ -space whatever the nature of the fluids is not founded from a dimensional, and hence theoretical point of view, as already underlined by Pawlowski (2005). Even if smaller sets of dimensional numbers are easier to explore and visualize (as reducing the configuration of a system), and are thus more attracting, this reduced configuration naturally comes at the expense of accuracy and genericity.

For viscoelastic liquids with non-negligible elastic properties (e.g. Bingham or Herschel–Bulkley fluids), the constitutive equations describing the rheological behaviour of the fluid are more complex. In this case, advanced material dimensional function should be used to account for both the elastic and viscous properties (apparent viscosity versus shear rate, first and second normal stress differences rate versus shear rate). Consequently, the parameters which would appear in the expression of the standard dimensionless material functions are supposed to be numerous, but the frame

is still theoretically valid. This point is reported by Delaplace et al. (2015).

## 4 Examples

This last section aims at demonstrating, based on two examples, how the framework associated with the dimensional analysis enables to overcome the use of the concept of Metzner and Otto. Additional examples involving variable material properties are reported in Delaplace et al. (2014).

### 4.1 Mixing of shear-thinning fluids adopting an Ostwald–de Waele power law model in a Paravisc®-Ekato system

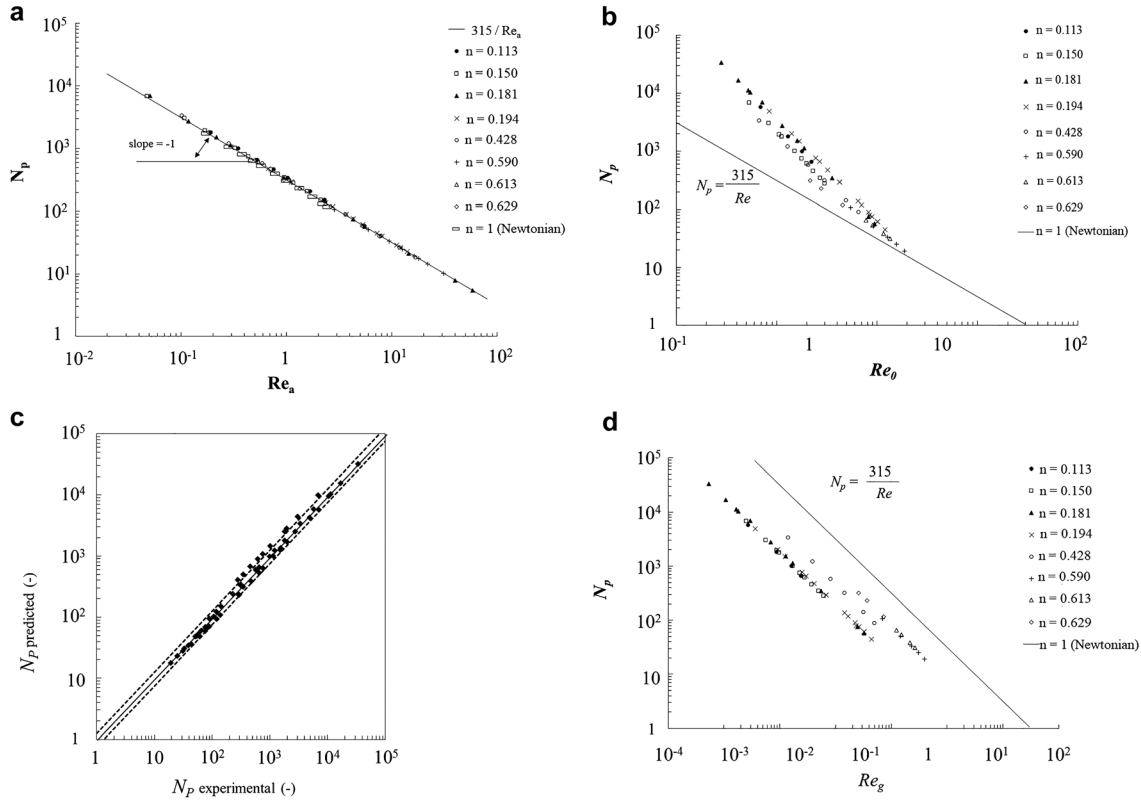
Delaplace et al. (2014) applied the previous theoretical developments in the case where shear-thinning fluids were agitated with a Paravisc®-Ekato system (Figure 2). The shear-thinning fluids were aqueous solutions of carboxy methy cellulose, alginate, guar gum and adragante gum, and their rheological behaviour could be described by a power law.

In order to establish an unbiased process relationship for power number  $N_p = \frac{P}{\rho \cdot N^3 \cdot D^5}$ , a reference shear rate  $\dot{\gamma}_0$  should be first chosen. Let's remind that the material function associated with such shear-thinning fluids is invariant (see Section 3.4). Therefore, the reference shear rate  $\dot{\gamma}_0$  should not then be listed. Nevertheless, a value for this quantity has to be chosen to calculate the other dimensionless numbers. As this choice is fully free, Delaplace et al. (2014) considered successively several reference shear rates  $\dot{\gamma}_0$ : respectively, at  $32 \text{ s}^{-1}$ ,  $4 \text{ s}^{-1}$  and  $N (\text{s}^{-1})$ . The associated Reynolds number, noted  $Re_0$ , is defined as:

$$Re_0 = \frac{\rho \cdot N \cdot D^2}{\mu_0} \quad (37)$$

where  $\mu_0$  is the apparent viscosity defined at the reference shear rate  $\dot{\gamma}_0$ .

For comparison purpose, in Figure 6a, the variation of the power number  $N_p$  is reported as a function of apparent Reynolds number  $Re_a$  defined from Equation (2) using the concept of Metzner–Otto ( $\dot{\gamma}_{\text{eff}} = K_s \cdot N$  with  $K_s = 32.2$ , Delaplace 1998). As expected, the experimental points cluster on the curve obtained with Newtonian fluids. In this case, the unified representation is not so bad since  $K_s$  is not so far away from a pure geometrical constant, for the Paravisc® mixing system, but it could be different for other agitators.



**Figure 6:** Power consumption of the mixing equipment Paravisc®-Ekato. (a) Power number  $N_p$  as a function of apparent Reynolds number  $Re_a$  defined from Equation (2) using the Metzner–Otto concept ( $\dot{\gamma}_{\text{eff}} = K_s \cdot N$ ). (b) Power number  $N_p$  as a function of Reynolds number  $Re_0$ , defined from Equation (37) with a reference shear rate equal to  $\dot{\gamma}_0 = 32 \text{ s}^{-1}$ , and of flow index  $n$ . (c) Comparison of the power number predicted and measured experimentally in the presence of shear-thinning fluids (reference shear rate  $\dot{\gamma}_0 = 32 \text{ s}^{-1}$ ); the dotted lines correspond to the error lines  $\pm 25\%$ . (d) Power number  $N_p$  as a function of Reynolds number  $Re_g$ , defined from Equation (41) with a reference shear rate equal to  $\dot{\gamma}_0 = N$ , and of flow index. Extracted from the studies by Delaplace (1998) and Delaplace et al. (2014).

On the contrary, a set of straight lines with slopes differing from  $-1$ , each being relative to a shear-thinning fluid (i.e. to a specific value of the flow index  $n$ ), is obtained when considering a Reynolds number  $Re_0$  defined from Equation (37) at  $\dot{\gamma}_0 = 32 \text{ s}^{-1}$  (Figure 6b; extracted from Delaplace et al. 2014).

Moreover, the authors observed that, even if the set of straight lines corresponding to the different shear-thinning fluids didn't merge into a single curve superimposed with the Newtonian one (when applying Metzner and Otto approach), the difficulties for identifying a process relationship fitting the experimental data are not higher. Indeed, in this case, it has been shown that power number can be fitted by following correlation

$$N_p = \frac{A_0}{(Re_0)^{2-n}} \quad (38)$$

where  $A_0$  is a second-order polynomial function, using  $(1 - n)$  as argument.

Changing the reference shear rate from  $32$  to  $4 \text{ s}^{-1}$  (figure not showed) shifted the set of straight lines towards lower Reynolds numbers as expected, since  $Re_0$  is defined from the apparent viscosity  $\mu_a(\dot{\gamma}_0)$  at the reference shear rate  $\dot{\gamma}_0$  (Equation (37)). These observations highlight that the process relationship predicting the power number as a function of  $Re_0$  has not the same analytical expression (and thus the same graphical representation) depending on the reference shear rate chosen. Although this point could appear confusing, Delaplace et al. (2014) demonstrated that:

- there exists some shift mathematical functions making possible to obtain the analytical expression of the process relationship at any  $\dot{\gamma}_{02}$  reference shear rate (leading to a Reynolds value  $Re_{02}$ ) from its knowledge at a given  $\dot{\gamma}_{01}$  value (leading to a Reynolds value  $Re_{01}$ ),
- applying these shift functions was easy and didn't need further multivariable optimization from the experimental data. Indeed, as the power number  $N_p$  is jointly

determined from both process relationships, one can write:

$$N_p = \frac{A_1}{(\text{Re}_{01})^{2-n}} = \frac{A_2}{(\text{Re}_{02})^{2-n}} \quad (39)$$

where  $A_1$ ,  $A_2$  are second-order polynomial functions (using  $(1 - n)$  as argument) at the reference shear rates  $\dot{\gamma}_{01}$  ( $4 \text{ s}^{-1}$ ) and  $\dot{\gamma}_{02}$  ( $32 \text{ s}^{-1}$ ), respectively. Therefore, knowing  $A_1$ ,  $\dot{\gamma}_{01}$  and  $\dot{\gamma}_{02}$  makes it possible to directly determine  $A_2$ :

$$N_p = \frac{A_1 \left( \frac{\dot{\gamma}_{01}}{\dot{\gamma}_{02}} \right)^{(n-1) \cdot (2-n)}}{(\text{Re}_{02})^{2-n}} \quad (40)$$

Moreover, the authors observed that the quality of the fit (i.e. the value of standard deviation) between the predicted and those experimentally measured power numbers is not affected by the method used for adjusting the experimental data, as illustrated in Figure 6c at  $\dot{\gamma}_0 = 32 \text{ s}^{-1}$ .

In other words, the Metzner and Otto approach provides no added value in terms of accuracy for prediction. It however offers the possibility to obtain a unified representation of the process relationship, which is often more convenient, if and only if the Metzner–Otto constant has been previously determined. Nevertheless, the Metzner–Otto’s constant reveals usefulness for describing non-Newtonian effects for other target variables (see Section 3 below which addresses this issue).

Finally, Delaplace et al. (2014) showed that choosing  $\dot{\gamma}_0 = N$  led to a third representation of the power number,  $N_p$  versus Reynolds number  $\text{Re}_g$ , which is defined as:

$$\text{Re}_g = \frac{\rho \cdot N^{2-n} \cdot D^2}{k} \quad (41)$$

It is important to remind that all these representations are allowed from dimensional analysis point of view, since the reference shear rate can be freely selected. In the case of  $\dot{\gamma}_0 = N$ , Figure 6d (extracted from Delaplace et al. 2014) shows that, contrary to what could be obtained using the Metzner and Otto’s approach (Figure 6a), and as in Figure 6b for  $\dot{\gamma}_0 = 32 \text{ s}^{-1}$ , the experimental points do not cluster on the curve obtained with Newtonian fluids ( $n = 1$ ). Nevertheless, in this particular case, the straight lines shifted in ordinate axis with slopes equal to  $-1$  where each line corresponds to a given power law fluid. Delaplace et al. (2014) demonstrated that in this case, the product  $N_p \cdot \text{Re}_g$  could be linked to Metzner–Otto constant ( $K_s$ ), via the flow index ( $n$ ) and the power constant ( $K_p$ ) using the following process relationship:

$$N_p \cdot \text{Re}_g = \frac{K_p}{K_s^{(1-n)}} \quad (42)$$

Beyond the choice of the reference shear rate and the graphical representation of the power curve, all these findings reveal that the process relationship for power number (associated with a given mixing system agitating without vortex, and for fluids obeying to a power law, including Newtonian fluids) is dependent of two internal measures:

- the first one is related to viscous effects and described by a Reynolds number ( $\text{Re}_0$ ), and,
- the other one is related to material function (i.e. to the fact the fluid obeyed to a power law) and described by a flow index, ( $n$ ).

Nothing guarantees that this “shift in ordinate” compared to Newtonian behaviour ( $n = 1$ ) exists:

- for fluids which does not satisfy an Ostwald–de Waele power law model and for other purely viscous fluids (with yield stress as example), the material function is more complex and requires enlarged internal measures for describing material properties (see Section 3 below which tackles this issue);
- for other target variables, as mixing times for example. Indeed, for each target variable, the impact of flow index (for shear-thinning fluids) on the process relationship is supposed to be really different.

As a consequence, out of the scope of power consumption with power law fluids, determining the Metzner and Otto constant is not useful.

## 4.2 Gas–liquid mass transfer in a mechanically stirred tank containing a shear-thinning fluid

Hassan et al. (2012) showed that it was possible, using the framework presented in Section 3, to establish an unbiased process relationship which characterized the factors controlling the volumetric gas–liquid mass transfer coefficient ( $k_L a$ ) when purely viscous fluids (without elastic properties) were aerated in a mechanically stirred tank.

These authors first measured, by different techniques, the volumetric gas–liquid mass transfer coefficient in a stirred tank in which the aeration was performed by generating bubbles using an air sparger located below a Chemineer® type turbine. Seven liquid phases were tested: three Newtonian fluids (water, aqueous glycerol solutions of glycerol) and four shear-thinning fluids (aqueous solutions of CMC and xanthan gum). In this study, two models

were tested to describe the rheological behaviour of the shear-thinning fluids:

- the Ostwald–de Waele power law model, such as expressed by Equation (4);
- the Williamson–Cross model described in Table 1 and involving three parameters  $\mu_w$ ,  $t_w$  and  $n_{cross}$ . The latter model was used as it enabled to describe the most faithfully possible shape of the rheograms over the whole range of shear rates (Newtonian at low shear rates and purely shear-thinning after; see Hassan et al. (2012) for more details).

By applying the rigorous theoretical framework related to dimensional analysis, and accounting for the experimental program, Hassan et al. (2012) could establish the reduced configuration of the system (i.e. the set of dimensionless numbers) depending on the considered rheological model. Five (Equation (43)) and six (Equation (44)) internal measures were found to be responsible for the variation of the target dimensionless number ( $k_1 a^*$ , defined by Equation (45)) when the rheological behaviour of the fluids was, respectively, described by the Ostwald–de Waele and the Williamson–Cross models. Respectively, two and three dimensionless parameters came from the analysis of standard material function, as shown in column II of Table 3.

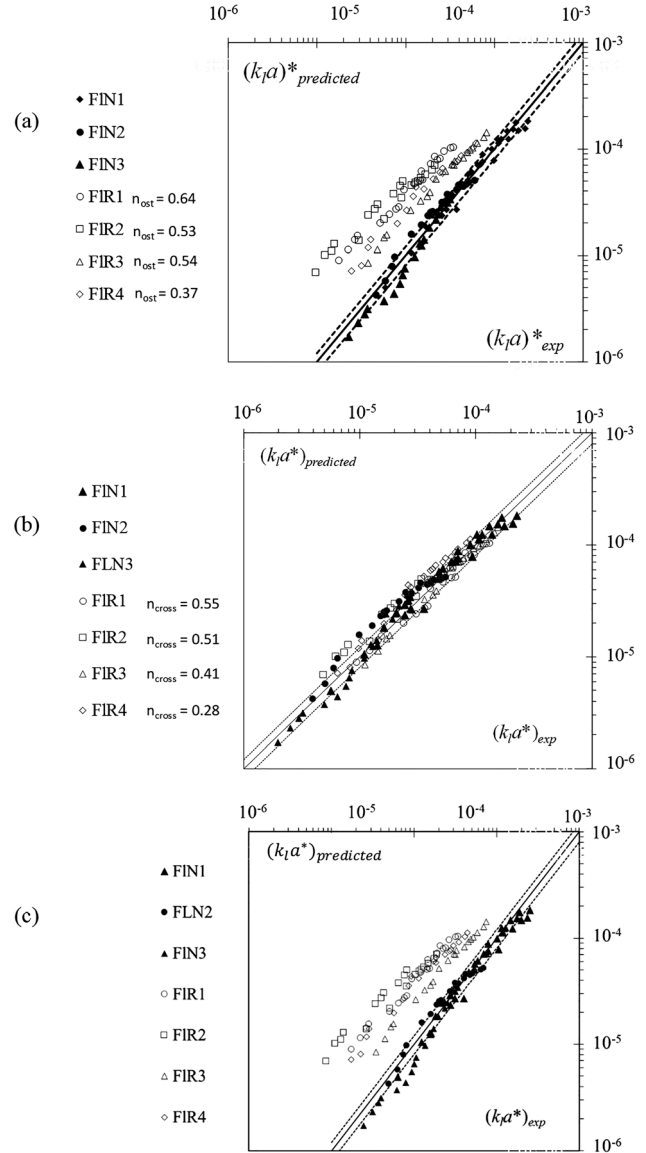
$$k_1 a^* = F_{ost} \left( U_g^* = \frac{U_g}{(v_g \cdot g)^{1/3}}; Fr = \frac{N^2 d}{g}; \mu^* = \frac{\mu_0}{\mu_g}; \sigma^* = \frac{\sigma}{(\rho_g^3 \cdot v_g^4 g)^{1/3}}; n \right) \quad (43)$$

$$k_1 a^* = F_{wc} \left( U_g^* = \frac{U_g}{(v_g \cdot g)^{1/3}}; Fr = \frac{N^2 d}{g}; \mu^* = \frac{\mu_0}{\mu_g}; \sigma^* = \frac{\sigma}{(\rho_g^3 \cdot v_g^4 g)^{1/3}}; n_{cross}; t_w^* = 1/t_w \left( \frac{v_g}{g^2} \right)^{1/3} \right) \quad (44)$$

$$k_1 a^* = k_1 a \left( \frac{v_g}{g^2} \right)^{1/3} \quad (45)$$

Note that, in the latter equations,  $U_g^*$  is the dimensionless number representing the effect of the gas flow rate. Such definition differs from the commonly used (the aeration number), but offers the advantage to separate the effects of the gas flow rate and of the rotational speed, the latter being quantified through the Froude number.

In a first step, Hassan et al. (2012) considered the Ostwald–de Waele model for all the non-Newtonian fluids and attempts to determine a process relationship  $F_{ost}$ .



**Figure 7:** Gas–liquid mass transfer in a stirred tank: comparison between the experimental dimensionless mass transfer coefficient ( $k_1 a^*$ ) and those predicted by the process relationship where shear-thinning fluids are described by (a) the Ostwald–de Waele model, (b) the Williamson–Cross model and (c) the Metzner–Otto concept (extracted from the study by Delaplace et al. 2014).

As illustrated in Figure 7a, adding a single material dimensionless number (i.e. the flow index  $n$  in Equation (43)) to take into account the shear-thinning rheology of the fluids to the initial relevance list of physical quantities influencing  $k_1 a^*$  did not enable to group all the non-Newtonian data (FIR1 to FIR4) into a single master curve obtained for Newtonian fluids (FIN1 to FIN3). A group of five straight lines, with one for each fluid, is obtained and the shift in ordinate of each straight line is not ranked according to the values of flow index. In particular, the

associated flow index is not necessarily decreasing when deviating from the Newtonian curve (i.e. when the flow index  $n$  is equal to 1).

Therefore, the material function associated with the Ostwald–de Waele power law model was found to be not adapted to describe the influence of the variability in the viscosity of the tested shear-thinning fluids. In other words, an over-simplification of the relevance list of parameters characterizing the behaviour of a non-Newtonian fluid led to a bias in the building of the set of internal measures which influenced  $k_t a^*$  and made it hard to find out a master curve gathering Newtonian and non-Newtonian fluids.

On the contrary, Hassan et al. (2012) showed that, when considering the Williamson–Cross model, the points associated with each shear-thinning fluid could be remarkably grouped together, by means of a process relationship involving two material numbers,  $n_{cross}$  and

$$t_w^* = \frac{1}{t_w^*} \left( \frac{v_g}{g^2} \right)^{1/3} \text{ such as:}$$

$$k_t a^* = 0.021 \cdot (Fr \cdot U_g^*)^{2/3} \cdot (\mu^*)^{-0.59} \cdot (\sigma^*)^{-0.25} \cdot (n_{cross})^{-2.40} \cdot (t_w^*)^{-0.17} \quad (46)$$

The accuracy of this process relationship is illustrated in Figure 7b.

To conclude, in order to definitively convince the readers, let's try to apply the Metzner–Otto concept in the present case. For that, the effective shear rate  $\dot{\gamma}_{eff}$  is calculated from Equation (1) by considering the Metzner–Otto constant  $K_s = 11.5$ , as reported by Chhabra (2003) and Paul et al. (2004) for this type of the agitating system involving a six-concave-blade disk turbine.

In Figure 7c, we plot the dimensionless mass transfer coefficient obtained by carrying out some experiments with Newtonian fluids. Then, the process relationship describing the Newtonian results was identified, leading to the following equation:

$$k_t a^* = 0.201 \cdot (Fr \cdot U_g^*)^{2/3} \cdot (\mu^*)^{-0.59} \cdot (\sigma^*)^{-0.25} \quad (47)$$

After this step, the correlation obtained with Newtonian fluids was applied to the non-Newtonian fluids using an effective viscosity. Precisely, the dimensionless number  $\mu^* = \frac{\mu_a(\dot{\gamma}_{eff})}{\mu_g}$  involved in Equation (47)) was evaluated by taking an apparent viscosity  $\mu_a(\dot{\gamma}_{eff})$  calculated at the effective shear rate and using the Ostwald–de Waele model (Equation 4). Finally, the predictions obtained by applying Metzner and Otto approach were compared to experimental measurements performed with non-Newtonian fluids in Figure 7c. It clearly pointed out that the

Metzner–Otto concept failed to gather all the data (Newtonian and shear-thinning fluids) on a single master curve. This can be explained by the fact that the Metzner–Otto concept is not valid for shear-thinning fluids which cannot be described by a power law model (i.e. a purely shear-thinning fluid).

This example illustrated that single process relationship is able to predict the dimensionless volumetric gas–liquid mass transfer coefficient in the presence of Newtonian and shear-thinning fluids could be obtained where the material configuration is correctly defined; but this requires using the rigorous theoretical framework related to dimensional analysis, which overcomes the Metzner–Otto's concept (1957).

## 5 Conclusions

In this review paper, the principle and limitations of the concept of Metzner and Otto were first reminded and illustrated through discussing experimental results and analytical cases. It was shown that the physical meaning of this concept was limited as it involved the determination of a pseudo constant,  $K_s$ , which depended on the geometrical parameters of the agitation system and sometimes on the rheological parameters of the fluid. The application of this concept reveals satisfactory to the sole shear-thinning fluids adopting an Ostwald–de Waele power law model, in a narrow range of shear-thinning index among all the non-Newtonian fluid categories, and when power consumption is the target variable and under laminar regime. Therefore, it should not be considered as a universal and rigorous approach, as often reported in the literature studies.

Then, the theoretical background underlying the modelling by dimensional analysis for variable material properties was reminded and applied to purely viscous non-Newtonian fluids. An effort of concision has been performed to gather and exemplify the theory in a concise way in order to pave the route for the practitioner. In particular, the importance of material configuration was emphasized to build a complete set of dimensionless numbers responsible for the evolution of the studied system. It was shown that using this rigorous framework made possible to predict, accurately and as well as the Metzner and Otto approach, the power when mixing Ostwald–de Waele power law fluids with a non-conventional helical agitator. On the other hand, it was demonstrated that an over-simplification of the relevance list of parameters characterizing the material function led sometimes to a

bias in the establishment of the set of dimensionless numbers which characterize the system. In this case, the effect of non-Newtonian rheological properties cannot be precisely considered by the use of effective shear rate.

Finally, the review paper pointed out that the theoretically founded framework proposed by dimensional analysis is a generic method to correlate the influence of some non-Newtonian properties on the process. It exceeds the concept of Metzner and Otto: it is not only limited to power as target variable or to laminar regime or to batch reactor, and also it is not required the determination of effective quantities. Moreover, it is a powerful and rational technique to address process scaling-up or scaling-down issues.

## Nomenclature

$a_0$	coefficient of physical property, its units depends on the unit of $s(p)$ , (see Equation (24))
$D$	diameter of the impeller, m
$g$	gravity acceleration, $\text{m s}^{-2}$
$k$	consistency index, $\text{Pa.s}^n$
$k_I a$	overall volumetric mass transfer coefficient, $\text{s}^{-1}$
$n$	flow index, (-)
$N$	rotational speed of the impeller, $\text{s}^{-1}$
$p$	dimensional variable
$P$	power consumption, W
$r$	radial position, m
$s$	dimensional material function
$T$	diameter of the tank, m
$t_w$	characteristic time related to the transition between Newtonian and purely shear-thinning behaviour for Williamson–Cross fluid, $\text{s}^{-1}$
$U_g$	gas superficial velocity, $\text{m s}^{-1}$

## Greek

$\dot{\gamma}$	shear rate, $\text{s}^{-1}$
$\mu_a$	apparent viscosity, $\text{Pa.s}$
$\mu_0$	apparent viscosity calculated at shear rate $\dot{\gamma}_0$ , $\text{Pa.s}$
$\{\pi\}$	set of dimensionless numbers (also called internal measures), (-)
$\varphi$	invariant standard dimensionless material function (noted ISDMF), defined in Equations (29–30), (-)
$\rho$	density, $\text{kg m}^{-3}$
$\tau_y$	yield stress, Pa
$\theta$	temperature, K
$u_g$	kinematic viscosity for gas phase $u_g = \frac{\rho_a}{\mu_g}$ , $\text{m}^2 \text{s}^{-1}$
$\mu_g$	dynamic viscosity for gas phase, $\text{Pa.s}$
$\mu_p$	plastic viscosity of Bingham fluid for when $\tau > \tau_y$ , $\text{Pa.s}$
$\mu_w$	viscosity of Williamson–Cross fluid for low shear rates where it behaved as a Newtonian behaviour fluid, $\text{Pa.s}$
$\sigma$	surface tension, $\text{N m}^{-1}$

## Dimensionless numbers

$Bi$	Bingham number, $Bi = \frac{\tau_y}{\mu_0 \dot{\gamma}_0}$ , (-)
$Fr$	Froude number, defined by Equation (21), (-)
$H$	dimensionless material function, (-)
$k_I a$	dimensionless overall volumetric mass transfer coefficient, defined by Equation (45), (-)
	$k_I a^* = k_I a \left( \frac{u_g}{g} \right)^{1/3}$
$K_p$	power constant dependant on the agitation system, (-)
$K_s$	Metzner–Otto constant, (-)
$N_p$	Power number, defined by Equation (2), (-)
$n$	flow index, (-)
$Nu$	Nusselt number, (-)
$Pr$	Prandtl number, (-)
$Re$	Reynolds number for a Newtonian fluid in a mixing system, internal measure of Newtonian viscosity, (-)
$Re_a$	Reynolds number for a purely viscous fluid in a mixing system, internal measure of apparent viscosity calculated at $\dot{\gamma}_{\text{eff}} = K_s.N$ (Metzner–Otto concept), (-)
$Re_g$	Reynolds number for a purely viscous fluid in a mixing system, internal measure of apparent viscosity calculated at $\dot{\gamma} = N$ , (-)
$Re_0$	Reynolds number for a purely viscous fluid in a mixing system, internal measure of apparent viscosity calculated at $\dot{\gamma}_0$ , (-)
$S$	ratio between the diameter of the impeller and the diameter of the tank, (-)
$u$	argument of the standard dimensionless material function, defined in Equation (23), (-)
$U_g$	superficial gas velocity, $\text{m s}^{-1}$
$v$	argument of the dimensionless material function, (-)
$Vi$	viscosity number (Equation (14)), (-)
$w$	standard dimensionless material function (noted SDMF), defined in Equation (25), (-)

## Subscripts

$B$	bulk
$cc$	coaxial cylinders
$cross$	model of Williamson–Cross
$eff$	effective
$g$	gas phase
$l$	liquid phase
$m$	material
$0$	at the reference abscissa
$ost$	model of Ostwald–de Waele
$w$	wall

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