Toeplitz band matrices with small random perturbations

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JOHANNES SJÖSTRAND AND MARTIN VOGEL

In memory of Hans Duistermaat

Abstract. We study the spectra of $N \times N$ Toeplitz band matrices perturbed by small complex Gaussian random matrices, in the regime $N \gg 1$. We prove a probabilistic Weyl law, which provides an precise asymptotic formula for the number of eigenvalues in certain domains, which may depend on N, with probability sub-exponentially (in N) close to 1. We show that most eigenvalues of the perturbed Toeplitz matrix are at a distance of at most $\mathcal{O}(N^{-1+\varepsilon})$, for all $\varepsilon > 0$, to the curve in the complex plane given by the symbol of the unperturbed Toeplitz matrix.

Contents

1. Introduction

The aim of this article is to study the eigenvalue distribution of $N \times N$ Toeplitz band matrices subject to small random perturbations. In particular we are interested in obtaining a precise asymptotic formula for the number of eigenvalues of the perturbed matrix in suitable domains as $N \to \infty$.

There has been a great recent interest in this subject, in particular when the unperturbed matrix is non-normal. For such matrices we do not have in general a good control over the resolvent and its norm may be very large even far away from the spectrum. Consequently the eigenvalues of such matrices may be very sensitive to small perturbations. This phenomenon is sometimes called pseudospectral effect or spectral instability, see Section 2.3 for more details. In view of this spectral instability it is very natural to study the distribution of the eigenvalues of such non-normal matrices subjet to small random perturbations. In this article we focus on the case when the unperturbed matrix is given by a Toeplitz matrix with a finite number of bands, see (1.1).

In the recent work [GuWoZe14] the authors considered the case of a large Jordan block matrix and showed that the empirical spectral measure, that is the eigenvalue counting measure, of the Jordan block subject to a perturbation by a complex Gaussian matrix with a coupling constant which is polynomially small in N (with minimal decay rate of order $N^{-\gamma}$, $\gamma > 1/2$), converges weakly in probability to the uniform measure on the unit circle

In [Wo16], using a replacement principle developed in [TaVuKr10], it was shown that the result of [GuWoZe14] holds for perturbations given by complex random matrices whose entries

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are independent and identically distributed random complex random variables with expectation 0 and variance 1 and a coupling constant $\delta = N^{-\gamma}$, with $\gamma > 2$.

The result of [GuWoZe14] on Jordan block matrices was also generalised in [BaPaZe18a, BaPaZe18b]. There the authors prove that the empirical spectral measure of finitely banded Toeplitz matrices subject to a random perturbation converges in probability to the push forward measure of the uniform measure on the unit circle under the the symbol of the unperturbed matrix, see (1.3). See also Figure 1 for a numerical illustration. In [BaPaZe18a, BaPaZe18b], they also proved that this convergence holds for not only for Gaussian perturbations but also for a large class of random perturbations, which satisfy a certain anti-concentration condition on the smallest singular value and a second moment condition on its entries.

In this work we follow the line of [Sj19, SjVo16], and consider perturbations with complex Gaussian random matrices with coupling constants $\leq N^{-4}/C$ (with $C > 0$ sufficiently large) allowed to be sub-exponentially small in N (and not only polynomially small as in previous works), see (2.2). We prove a precise asymptotic formula for the number of eigenvalues in certain domains, valid with probability close to one, see Theorem 2.1. Our results improve those of previous works:

> They provide more detailed information on the spectral distribution than just the convergence of the empirical measure.
> The domains in which we count the eigenvalues may be small and depend on N (see Theorem 6.5) and this gives information about the local behaviour of the eigenvalues.

From Theorem 2.1 we infer the convergence of the empirical spectral measure, see Corollary 2.2.

1.1. **Toeplitz band matrices.**

Let $N_\pm \geq 0$ be in $\mathbb{N}$, such that either $N_+ \neq 0$ or $N_- \neq 0$, and consider the operator

$$p(\tau) \stackrel{\mathrm{def}}{=} \sum_{j=-N_-}^{N_+} a_j \tau^j, \quad a_{-N_-}, a_{-N_-+1}, \dots, a_{N_+} \in \mathbb{C}, \ a_{\pm N_\pm} \neq 0, \tag{1.1}$$

acting on $\ell^2(\mathbb{Z})$, or more generally on functions $\psi : \mathbb{Z} \to \mathbb{C}$, where

$$(\tau u)(k) \stackrel{\mathrm{def}}{=} u(k-1), \tag{1.2}$$

defines the translation to the right by one unit. We shall work on $\mathbb{Z}$, on an interval in $\mathbb{Z}$ and on $\mathbb{Z}/M\mathbb{Z}$, for some $\mathbb{N} \ni M \geq 1$. The symbol of $\tau = \exp(-iD_x)$ is $1/\zeta$, with $\zeta = e^{i\xi}$. Therefore, the symbol of the operator (1.1) is given by the meromorphic function

$$\mathbb{C} \ni \zeta \mapsto p(1/\zeta) = \sum_{j=-N_-}^{N_+} a_j \zeta^{-j}. \tag{1.3}$$

We obtain a *Toeplitz band matrix* from the operator $p(\tau)$ by restricting it to the finite dimensional space $\mathbb{C}^N$. Indeed, we let $N \geq 1$ and identify $\mathbb{C}^N$ with $\ell^2([1,N])$, $[1,N] = \{1, 2, .., N\}$, and also with $\ell^2_{[1,N]}(\mathbb{Z})$ (the space of all $u \in \ell^2(\mathbb{Z})$ with support in $[1,N]$). Then, we consider the $N \times N$ *Toeplitz band matrix*

$$P_N \stackrel{\mathrm{def}}{=} 1_{[1,N]}\, p(\tau)\, 1_{[1,N]}, \tag{1.4}$$

acting on $\mathbb{C}^N \simeq \ell^2_{[1,N]}(\mathbb{Z})$.

The translation operator τ on $\ell^2(\mathbb{Z})$ is unitary, i.e. $\tau^* = \tau^{-1}$, so one can easily see that $p(\tau)$ is a normal operator, meaning that it commutes with its adjoint. The Fourier transform shows that the spectrum of $p(\tau)$ (1.1) acting on $\ell^2(\mathbb{Z})$ is purely absolutely continuous and given by

$$\mathrm{Spec}(p(\tau)) = p(S^1). \tag{1.5}$$

The restriction $P_{\mathbb{N}} = p(\tau)|_{\ell^2(\mathbb{N})}$ of $p(\tau)$ to $\ell^2(\mathbb{N})$, is in general no longer normal, except for specific choices of N_+, N_- and the coefficients a_j. The essential spectrum of the Toeplitz operator $P_{\mathbb{N}}$

(1.1) is still given by $p(S^1)$. However, we gain additional pointspectrum in all loops of $p(S^1)$ with non-zero winding number, i.e.

$$\mathrm{Spec}(P_{\mathbb{N}}) = p(S^1) \cup \{z \in \mathbb{C}; \mathrm{ind}_{p(S^1)}(z) \neq 0\}. \tag{1.6}$$

Here, by a result of Krein [BöSi99, Theorem 1.15] (see also Proposition 3.11 below) the winding number of $p(S^1)$ around the point $z \notin p(S^1)$ is related to the Fredholm index of $P_{\mathbb{N}} - z$:

$$\mathrm{Ind}(P_{\mathbb{N}} - z) = -\mathrm{ind}_{p(S^1)}(z). \tag{1.7}$$

For every $\epsilon > 0$, the spectrum of the finite Toeplitz matrix P_N (1.4) satisfies

$$\mathrm{Spec}(P_N) \subset \mathrm{Spec}(P_{\mathbb{N}}) + D(0, \epsilon) \tag{1.8}$$

for $N > 0$ sufficiently large, where $D(z, r)$ denotes the open disc of radius r, centered at z. The limit of $\mathrm{Spec}(P_N)$ as $N \to \infty$ is contained in a union of analytic arcs inside $\mathrm{Spec}(P_{\mathbb{N}})$, see [BöSi99, Theorem 5.28]. This phenomenon can also be observed in the numerical simulations presented in Figures 1, 2.

However, we will show that after a small random perturbation of P_N, most of the eigenvalues of the perturbed operator will be very close to the curve $p(S^1)$, see Figures 1, 2.

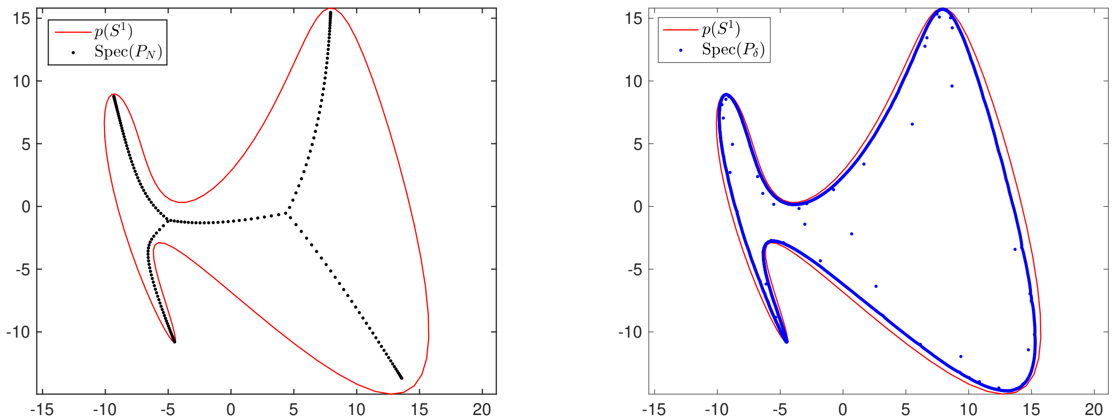

Figure 1. The pictures on the left hand side shows the spectrum of the Toeplitz matrix P_N, with $N = 100$, given by the symbol $p(1/\zeta) = 2i\zeta^{-1} + \zeta^2 + \frac{7}{10}\zeta^3$ and the right hand side shows the spectrum of a random perturbation P_δ, as in (1.9) below, with coupling constant $\delta = 10^{-14}$ and dimension $N = 1000$. The red line shows the symbol curve $p(S^1)$.

1.2. **Adding a small random perturbation.** Let $(\mathcal{M}, \mathcal{A}, \mathbb{P})$ denote a probability space and let $\mathcal{H}_N(\mathbb{C}^{N\times N}, \|\cdot\|_{\mathrm{HS}})$ denote the space of $N \times N$ complex valued matrices equipped with the Hilbert-Schmidt norm. Consider the random matrix

$$\mathcal{M} \ni \omega \mapsto Q_\omega \stackrel{\mathrm{def}}{=} Q_\omega(N) \stackrel{\mathrm{def}}{=} (q_{j,k}(\omega))_{1\leq j,k\leq N} \in \mathcal{H}_N$$

with Gaussian law

$$(Q_\omega)_*(d\mathbb{P}) = \pi^{-N^2} \mathrm{e}^{-\|Q\|^2_{\mathrm{HS}}} L(dQ),$$

where L denotes the Lebesgue measure on $\mathbb{C}^{N\times N}$. We are interested in the spectrum of the random perturbations of the matrix $P_N^0 = P_N$:

$$P_N^\delta \stackrel{\mathrm{def}}{=} P_N^0 + \delta Q_\omega, \quad 0 \leq \delta \ll 1. \tag{1.9}$$

Notice that the entries $q_{j,k}(\omega)$ of Q_ω are independent and identically distributed complex Gaussian random variables with expectation 0, and variance 1.

We recall that the probability distribution of a complex Gaussian random variable $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, 1)$, defined on the probability space $(\mathcal{M}, \mathcal{A}, \mathbb{P})$, is given by

$$\alpha_*(d\mathbb{P}) = \pi^{-1} \mathrm{e}^{-|\alpha|^2} L(d\alpha),$$

where $L(d\alpha)$ denotes the Lebesgue measure on $\mathbb{C}$. If $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$, then

$$\mathbb{E}[\alpha] = 0, \quad \mathbb{E}[|\alpha|^2] = 1.$$

In this paper we consider the Gaussian case for the sake of simplicity. However, we believe that our method can be adapted to the case of more general complex valued random matrices. The main difficulty lies in showing that the logarithm of the determinant of a certain matrix valued stochastic process is not too small with probability close to 1 (see Proposition 5.3 below).

2. Main results

We will provide precise eigenvalue asympotics for the eigenvalues of P_N^δ in certain domains which show that most eigenvalues of P_N^δ are close to the curve $p(S^1)$ with probability sub-exponentially (in N) close to 1, see Theorem 2.1 below. We also prove eigenvalue asymptotics in *thin* N-dependent domains in scales up to order $N^{-1+\varepsilon}$, for every $\varepsilon > 0$. This shows in particular that for every $\varepsilon > 0$, with probability sub-exponentially (in N) close to 1, most eigenvalues can be found at a distance $\leq \mathcal{O}(N^{-1+\varepsilon})$ from $p(S^1)$, see Theorem 6.5 for the precise statement.

Our results also provide an upper bound on the number of eigenvalues of P_N^δ which remain far from the curve $p(S^1)$. Finally, we will show that our results on the eigenvalue asymptotics of P_N^δ imply the almost sure weak convergence of the empirical measure of eigenvalues of P_N^δ to the uniform measure on $p(S^1)$, see Corollary 2.2. This corresponds to the leading term of our asymptotic result.

2.1. **Eigenvalue asymptotics in fixed smooth domains.** Let $\Omega \Subset \mathbb{C}$ be an open simply connected set with smooth boundary $\partial\Omega$ which is independent of N. We suppose that

(Ω1) $\partial\Omega$ intersects $p(S^1)$ in at most finitely many points;
(Ω2) the points of intersection are non-degenerate, i.e.

$$\partial_\zeta p \neq 0 \text{ on } p^{-1}(\partial\Omega \cap p(S^1)); \tag{2.1}$$

(Ω3) $\partial\Omega$ intersects $p(S^1)$ transversally, in the following precise sense : for each $z_0 \in \partial\Omega \cap p(S^1)$ let $\gamma_k \subset p(S^1)$, $k = 1, \dots, n$ denote the mutually distinct segments of $p(S^1)$ passing through z_0, i.e. each γ_k is given by the image of a small neighborhood in S^1 of a point in $p^{-1}(z_0) \cap S^1$. Then γ_k and $\partial\Omega$ intersect transversally at z_0.

We then have the following result:

Theorem 2.1. *Let p be as in (1.1), set $M = N_+ + N_-$ and let P_N^δ be as in (1.9). Let Ω be as above, satisfying conditions (Ω1)–(Ω3) and pick a $\varepsilon_0 \in]0,1[$. There exists a constant $C > 0$, such that, for $N > 1$ sufficiently large, if*

$$C\mathrm{e}^{-N^{\varepsilon_0}/(2M)} \leq \delta \leq \frac{N^{-4}}{C}, \tag{2.2}$$

then we have that

$$\left| \#(\mathrm{Spec}(P_N^\delta) \cap \Omega) - \frac{N}{2\pi} \int_{p^{-1}(\Omega \cap p(S^1))} L_{S^1}(d\theta) \right| \leq \mathcal{O}(N^{\varepsilon_0} \log N). \tag{2.3}$$

with probability

$$\geq 1 - \mathcal{O}(\log N) \left(\mathrm{e}^{-N^2} + \delta^{-M} \mathrm{e}^{-\frac{1}{2}N^{\varepsilon_0}} \right). \tag{2.4}$$

Let us give some remarks on this result. The e^{-N^2} term in the estimate (2.4) is an artifact from the proof where we restrict to the event that $\|Q_\omega\|_{\mathrm{HS}} \leq CN$ which occurs with probability $\geq 1 - \mathrm{e}^{-N^2}$, see (2.6).

The factor N^{ε_0} in the error estimate in (2.3) is a consequence of our aim to show that (2.3) holds with probability which is sub-exponentially close to 1. However, it is clear from the proof, see Proposition 5.3, that if we were to settle for a probability $\geq 1 - N^{-\kappa}$, for every $\kappa > 0$, then we can ameliorate the error estimate in (2.3) to $\mathcal{O}((\log N)^2)$.

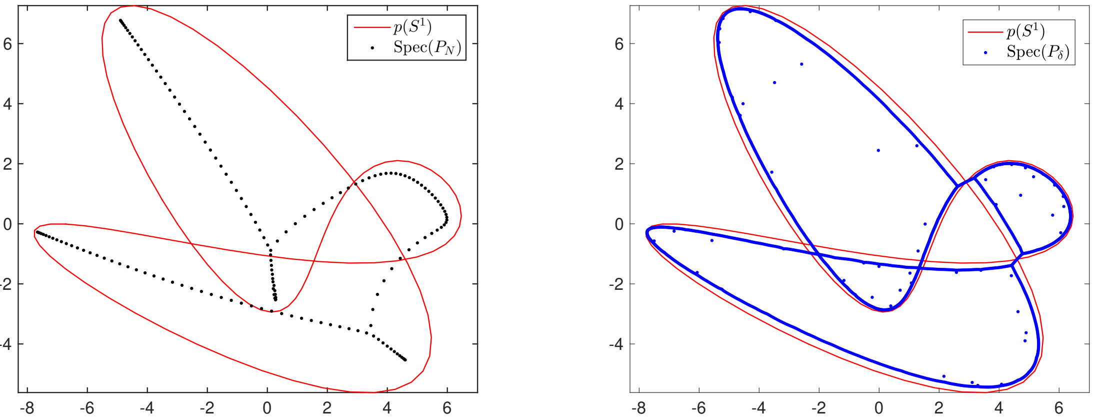

Figure 2. The pictures on the left hand side shows the spectrum of the Toeplitz matrix P_N, with $N = 100$, given by the symbol $p(1/\zeta) = 2\zeta^{-3} - \zeta^{-2} + 2i\zeta^{-1} - 4\zeta^2 - 2i\zeta^3$ and the right hand side shows the spectrum of a random perturbation P_δ, as in (1.9), with coupling constant $\delta = 10^{-14}$ and $N = 1000$. The red line shows the symbol curve $p(S^1)$.

We provide a more detailed version of this result in Theorem 6.5 below. There, we present a Weyl law in probability for the eigenvalues of P_N^δ in *thin* N-dependent domains Ω_N with, roughly speaking, a width $\geq CN^{-1+}$, and whose boundary is uniformly Lipschitz. See Theorem 6.5 below for more details.

2.2. **Convergence of the empirical measure and related results.** Another way to see the limiting behavior of the spectrum of P_N^δ (1.9) is to study the limits of the *empirical measure* of the eigenvalues of P_N^δ, defined by

$$\xi_N \overset{\text{def}}{=} \frac{1}{N} \sum_{\lambda \in \mathrm{Spec}(P_N^\delta)} \delta_\lambda \tag{2.5}$$

where the eigenvalues are counted including multiplicity and δ_λ denotes the Dirac measure at $\lambda \in \mathbb{C}$. The Markov inequality implies that

$$\mathbb{P}[\|Q_\omega\|_{\mathrm{HS}} \leq CN] \geq 1 - \mathrm{e}^{-N^2}, \tag{2.6}$$

for $C > 0$ large enough. The operator norm of P_N (1.4) satisfies

$$\|P_N\| \leq \|p\|_{L^\infty(S^1)}.$$

If $\delta \leq N^{-1}$, then the Borel-Cantelli Theorem shows that, almost surely, ξ_N has compact support for $N > 0$ sufficiently large.

From Theorem 2.1 we will deduce that, almost surely, ξ_N converges weakly to the uniform distribution on $p(S^1)$.

Corollary 2.2. *Let $\varepsilon_0 \in]0,1[$, let p be as in (1.1) and write $M = N_+ + N_-$. Then, there exists a constant $C > 0$ such that if (2.2) holds,*

$$C\mathrm{e}^{-N^{\varepsilon_0}/(2M)} \leq \delta \leq \frac{N^{-4}}{C},$$

then, almost surely,

$$\xi_N \rightharpoonup p_*\left(\frac{1}{2\pi} L_{S^1}\right), \quad N \to \infty, \tag{2.7}$$

weakly, where L_{S^1} denotes the Lebesgue measure on S^1.

Our strategy to prove the precise eigenvalue asymptotics presented in Theorems 2.1 and 6.5 also provides an alternative proof of the above result via the convergence of the associated logarithmic potentials, see Section 7.

Similar results to Corollary 2.2 have been proven in a various settings. In the recent work [BaPaZe18a], the authors consider a general sequence of deterministic complex $N \times N$ matrices M_N perturbed by complex Gaussian random matrices $Q_\omega = Q_\omega(N)$, as in (1.9). They study the empirical measure ξ_N of the eigenvalues of $\mathcal{M}_N := M_N + N^{-\gamma}Q_\omega$, $\gamma > 1/2$, defined as in (2.5). The authors show that the *Logarithmic potential* $L_{\xi_N}(z)$, $z \in \mathbb{C}$, (see Section 7 below for a definition) associated with ξ_N, asymptotically coincides with a deterministic function $g_N(z)$ in probability at each point z, for which the number of singular values of $(M_N - z\mathrm{Id})$ smaller than $N^{-\gamma+1/2+\delta_N}$, $0 < \delta_N = o(1)$ as $N \to \infty$, is of order $o(N(\log N)^{-1})$ as $N \to \infty$. Since the weak convergence of the random measure ξ_N can be deduced from the point wise convergence of the Logarithmic potential $L_{\xi_N}(z)$ (see Section 7 below for details and references), this result shows that studying the weak convergence of the empirical measure ξ_N can be reduced to deterministic calculation involving only the unperturbed matrix M_N.

Moreover, in [BaPaZe18a, BaPaZe18b], the authors consider the special case of M_N being given by a band Toeplitz matrix, i.e. $M_N = P_N$ with p as in (1.1). In this case they show that the convergence (2.7) holds weakly in probability for a coupling constant $\delta = N^{-\gamma}$, with $\gamma > 1/2$. Furthermore, they prove a version of this theorem for Toeplitz matrices with non-constant coefficients in the bands, see [BaPaZe18a, Theorem 1.3, Theorem 4.1]. Their methods are quite different from ours. They compute directly the $\log|\det \mathcal{M}_N - z|$ by relating it to $\log|\det M_N(z)|$, where $M_N(z)$ is a truncation of $M_N - z$, where the smallest singular values of $M_N - z$ have been excluded. The level of truncation however depends on the strength of the coupling constant and it necessitates a very detailed analysis of the small singular values of $M_N - z$.

In the earlier work [GuWoZe14], the authors prove that the convergence (2.7) holds weakly in probability for the Jordan bloc matrix P_N with $p(\tau) = \tau^{-1}$ (1.1) and a perturbation given by a complex Gaussian random matrix whose entries are independent complex Gaussian random variables whose variances vanishes (not necessarily at the same speed) polynomially fast, with minimal decay of order $N^{-1/2+}$.

In [Wo16], using a replacement principle developed in [TaVuKr10], it was shown that the result of [GuWoZe14] holds for perturbations given by complex random matrices whose entries are independent and identically distributed random complex random variables with expectation 0 and variance 1 and a coupling constant $\delta = N^{-\gamma}$, with $\gamma > 2$.

In [DaHa09], the authors showed that in the case of large Jordan block matrix $p(\tau) = \tau^{-1}$, most eigenvalues of the perturbed matrix P_N^δ lie in the annulus

$$\{z \in \mathbb{C}; (\delta N)^{1/N}\mathrm{e}^{-\sigma} \leq |z| \leq (\delta N)^{1/N}\},$$

for any fixed $\sigma > 0$, with probability $\geq 1 - \mathcal{O}(N^{-2})$. Moreover, the authors show that there are at most $\mathcal{O}(\sigma^{-1}\log N)$ eigenvalues of P_N^δ outside this annulus, with probability $\geq 1 - \mathcal{O}(N^{-2})$.

A version of Theorem 2.1, concerning the special cases of large Jordan block matrices $p(\tau) = \tau^{-1}$ and large bi-diagonal matrices $p(\tau) = a\tau + b\tau^{-1}$, $a, b \in \mathbb{C}$, have been proven in [Sj19, SjVo16].

2.3. **Spectral instability.** In general, the spectra of non-selfadjoint operators can be highly unstable under small perturbations due to the lack of good control over the norm of the resolvent. This phenomenon, sometimes referred to as *pseudospectral effect* or *spectral instability*, can be observed in the case of non-normal Toeplitz matrices P_N (1.4), as illustrated in Figures 1 and 2. To quantify the zone of spectral instability in the complex Plane, one defines the ε-*pseudospectrum* of a linear operator P acting on some complex Hilbert space $\mathcal{H}$ as follows: for $\varepsilon > 0$ set

$$\mathrm{Spec}_\varepsilon(P) \stackrel{\mathrm{def}}{=} \{z \in \mathbb{C}; \|(P - z)^{-1}\| > \varepsilon^{-1}\}. \tag{2.8}$$

The points $z \in \mathbb{C}$ in the ε-pseudospectrum of P are precisely the points $z \in \mathbb{C}$ for whom there exists a bounded linear operator Q acting on $\mathcal{H}$ with $\|Q\| \leq 1$, such that $z \in \mathrm{Spec}(P + \varepsilon Q)$, see [EmTr05, Da07] for a detailed exposition.

For the Toeplitz band matrices P_N, we have that any fixed point in $\mathbb{C}\backslash p(S^1)$ with

$$z \notin \{0, +\infty\} \quad \text{and} \quad z \neq a_0, \text{ when } N_+ \text{ or } N_- = 0, \tag{2.9}$$

which is contained in the pointspectrum of $P_{\mathbb{N}}$ (1.6) is contained in the $C\mathrm{e}^{-N/C}$-pseudospectrum of P_N. Recall from (1.6) that the pointspectrum of $P_{\mathbb{N}}$ in $\mathbb{C}\backslash p(S^1)$ is given by the points z around which the curve $p(S^1)$ has a non-zero winding number $\mathrm{ind}_{p(S^1)}(z) \neq 0$. In fact, provided that we avoid the special cases (2.9), we have that

- if $\mathrm{ind}_{p(S^1)}(z) < 0$, then the Fredholm index of $P_{\mathbb{N}} - z$ satisfies
$$\mathrm{Ind}(P_{\mathbb{N}} - z) = \dim\ker(P_{\mathbb{N}} - z) = -\mathrm{ind}_{p(S^1)}(z);$$
- if $\mathrm{ind}_{p(S^1)}(z) > 0$,
$$\mathrm{Ind}(P_{\mathbb{N}} - z) = -\dim\ker(P_{\mathbb{N}} - z)^* = -\mathrm{ind}_{p(S^1)}(z),$$

see Propositions 3.10 and 3.11. Moreover, these kernels are spanned by exponentially decaying functions, see the discussion in Section 3.4. In the first case, restricting such a function $u \in \ker(P_{\mathbb{N}} - z)$ to the interval $[1, N]$ yields an approximate solution to the equation $(P_N - z)u = 0$, sometimes called a *quasimode*. More precisely, setting $e_+ = \|\mathbf{1}_{[1,N]}u\|^{-1}\mathbf{1}_{[1,N]}u$, we get that

$$(P_N - z)e_+ = \mathcal{O}(\mathrm{e}^{-N/C}).$$

Similarly, we get in the second case an $e_- \in \ell^2([1, N])$, $\|e_-\| = 1$, with

$$(P_N - z)^* e_- = \mathcal{O}(\mathrm{e}^{-N/C}).$$

These exponentially precise quasimodes show that any fixed z with $\mathrm{ind}_{p(S^1)}(z) \neq 0$ satisfying (2.9), is contained in the $C\mathrm{e}^{-N/C}$-pseudospectrum of P_N.

On the other hand, for any compact set $\Omega \Subset \mathbb{C}\backslash p(S^1)$, with $z \in \Omega$ satisfying (2.9) and

$$\mathrm{ind}_{p(S^1)}(z) = 0,$$

we have that that for $N > 0$ sufficiently large $\|(P_N - z)^{-1}\| = \mathcal{O}(1)$ uniformly for $z \in \Omega$, see Proposition 3.13. Hence, outside the spectrum of $P_{\mathbb{N}}$ (1.6) is a zone of spectral stability for P_N. This explains why the eigenvalues of P_N^δ can (with high probability) only be found in a small neighborhood of the spectrum of $P_{\mathbb{N}}$.

However, only analysing the pseudospectrum does not yield any information on *where* the eigenvalues of P_N^δ can be found. Theorem 2.1, shows that with probability very close to one, all but $\mathcal{O}(N^{\varepsilon_0}\log N)$ many eigenvalues of P_N^δ can be found close to the curve $p(S^1)$. Theorem 6.5 below shows that still probability very close to one, most eigenvalues of P_N^δ are at a distance of $\leq N^{-1+\varepsilon}$, for every $\varepsilon > 0$, from $p(S^1)$, see (6.54) for the precise error estimate.

It would be interesting to perform a precise analysis of the boundary of the ε-pseudospectrum of P_N to see whether the eigenvalues of P_N^δ accumulate there, as in the case of small random perturbations of semiclassical differential operators in [Vo16].

2.4. **Outline of the proof.** The overall strategy of the proof is based on a *Grushin* reduction. In Section 4 we review the basic idea of such a reduction and we set up a Grushin problem $\mathcal{P}_N$ by considering the operator $p(\tau)$ (1.1) on the discrete circle $\mathbb{Z}/\widetilde{N}\mathbb{Z}$, $\widetilde{N} = N + N_- + N_+$,

$$\mathcal{P}_N = p(\tau) : \ell^2(\mathbb{Z}/\widetilde{N}\mathbb{Z}) \to \ell^2(\mathbb{Z}/\widetilde{N}\mathbb{Z}),$$

which can be used to describe the eigenvalues of the unperturbed operator P_N. In Section 3 we provide a general discussion of band Toeplitz matrices and their Fredholm properties. However, for this paper only Sections 3.2 and 3.3 are of immediate importance as we discuss properties of $p(\tau)$ on $\mathbb{Z}/\widetilde{N}\mathbb{Z}$.

In Section 5, we will use the Grushin problem for the unperturbed operator P_N to set up a *Grushin Problem* $\mathcal{P}_N^\delta$ for the perturbed P_N^δ, resulting in an effective description of its eigenvalues

$$\log\det(P_N^\delta - z) = \log\det\mathcal{P}_N^\delta(z) + \log\det E_{-+}^\delta(z),$$

with probability $\geq 1 - \mathrm{e}^{-N^2}$. Here, $E_{-+}(z)^\delta$ is an $(N_+ + N_-) \times (N_+ + N_-)$ complex valued matrix. Furthermore, the Grushin problem shows that we have a trivial upper bound on the quantity $\log\det E_{-+}^\delta(z)$. In Section 5.3, we show that with probability very close to 1 we have a quantitative lower bound on $\log\det E_{-+}^\delta(z)$.

To obtain our main results on eigenvalue asympotics from this description we apply a general estimate [Sj10] on the number of zeros of a holomorphic function $u(z;N)$ of exponential growth. We will recall this result in Section 6.1 below, see Theorem 6.2. Roughly speaking, if the available information is

(i) an *upper bound* $\log|u(z;N)| \leq N\phi(z)$, for z near the boundary $\partial\Omega$ and ϕ a subharmonic continuous function and
(ii) a *lower bound* $\log|u(z;N)| \geq N(\phi(z)-\varepsilon_j)$, with $\varepsilon_j \geq 0$, for finitely many points $z_j = z_j(N)$, $j = 1,\dots,M(N)$, which are situated near the boundary of $\partial\Omega$,

then the number of zeros of u in Ω is given by

$$\#(u^{-1}(0)\cap\Omega) \sim \frac{N}{2\pi}\int_\Omega \Delta\phi L(dz),$$

asymptotically as $N \to +\infty$. In Section 6.2 we check that our effective description for $\log\det(P_N^\delta - z)$ satisfies the required upper bound (i), and in Section 6.3, using Section 5.3, we check the lower bound (ii).

In Section 6.4 we then use these bounds in combination with Theorem 6.2 to prove Theorem 2.1.

In Section 6.5 we provide a more general version of Theorem 2.1 for N-dependent domains. Finally, in Section 7 we give two proofs of Corollary 2.2 via the method of logarithmic potentials.

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3. A general discussion of Toeplitz band matrices

Let $z \in \mathbb{C}$ and recall (1.2). The exponential function $u : \mathbb{Z} \ni \nu \mapsto \zeta^\nu$, for $\zeta \in \mathbb{C}\backslash\{0\}$, is a solution to

$$(p(\tau) - z)u = 0, \tag{3.1}$$

if and only if

$$p(1/\zeta) - z = 0. \tag{3.2}$$

Here, we assume that

$$z \notin \{0,\infty\} \tag{3.3}$$

Suppose furthermore that

$$z \neq a_0, \text{ when } N_- = 0. \tag{3.4}$$

Then (3.2) is equivalent to the following polynomial equation

$$\sum_{j=0}^{N_+ + N_-} a_{N_+ - j}\zeta^j - z\zeta^{N_+} = 0. \tag{3.5}$$

This is a polynomial equation of degree $N_+ + N_-$ (when $N_- = 0$ we have $a_0 - z \neq 0$ by (3.4)). It has $N_- + N_+$ roots, counted with their multiplicity.

If $z \notin p(S^1)$, no root is in S^1, and we let

$$\zeta_1^+, \dots, \zeta_{m_+}^+ \quad \text{be the roots in } D(0,1) \tag{3.6}$$

and

$$\zeta_1^-, \dots, \zeta_{m_-}^- \quad \text{be the roots in } \mathbb{C}\backslash D(0,1), \tag{3.7}$$

repeated according to their multiplicity. Notice that

$$m_+ + m_- = N_+ + N_-. \tag{3.8}$$

3.1. **Remark on exponential solutions.** Let $z \in \mathbb{C}\backslash(\{0\}\cup p(S^1))$. We strengthen assumption (3.4) and assume that

$$\text{if } N_+ \text{ or } N_- = 0, \text{ then } a_0 \neq z. \tag{3.9}$$

Let $\zeta_1, \zeta_2, ..., \zeta_m \in \mathbb{C} \setminus \{0\}$ be the distinct roots of the characteristic equation (3.2):

$$p(1/\zeta) - z = 0.$$

Let $\mathrm{mult}\,(\zeta_j) \geq 1$ be the corresponding multiplicity so that

$$\sum_1^m \mathrm{mult}\,(\zeta_j) = N_+ + N_-. \tag{3.10}$$

Similarly to (3.6), (3.7), we let

$$\zeta_1^+, \dots, \zeta_{\widetilde{m}_+}^+ \quad \text{be the } \textit{distinct} \text{ roots in } D(0,1) \text{ with multiplicities } 1 \leq \mathrm{mult}(\zeta_j^+) < +\infty, \tag{3.11}$$

and

$$\zeta_1^-, \dots, \zeta_{\widetilde{m}_-}^- \quad \text{be the } \textit{distinct} \text{ roots in } \mathbb{C}\backslash D(0,1) \text{ with multiplicities } 1 \leq \mathrm{mult}(\zeta_j^-) < +\infty, \tag{3.12}$$

so that $\widetilde{m}_- + \widetilde{m}_+ = m$ in (3.10). Notice also that

$$\sum_1^{\widetilde{m}_\pm} \mathrm{mult}\,(\zeta_j^\pm) = m_\pm. \tag{3.13}$$

The functions

$$\mathbb{Z} \ni \nu \mapsto f_{\zeta,k}(\nu) := (\zeta\partial_\zeta)^k(\zeta^\nu),\ \ 0 \leq k \leq \mathrm{mult}\,(\zeta) - 1$$

are solutions to

$$(p(\tau) - z) f_{\zeta,k} = 0, \tag{3.14}$$

for $\zeta = \zeta_1, ..., \zeta_m$. In fact, if ζ is such a root, then for ω close to ζ

$$(p(\tau) - z)(\omega^\nu) = (p(1/\omega) - z)\omega^\nu = \mathcal{O}((\omega - \zeta)^{\mathrm{mult}\,(\zeta)})$$

and applying $(\omega\partial_\omega)^k$ with $0 \leq k \leq \mathrm{mult}\,(\zeta) - 1$, and then putting ω equal to ζ, we get (3.14).

More generally, let $\zeta_1, ..., \zeta_m \in \mathbb{C} \setminus \{0\}$ be distinct numbers and let $1 \leq m_j < \infty$, $1 \leq j \leq m$.

Proposition 3.1. *The functions $f_{\zeta_j,k} : \mathbb{Z} \to \mathbb{C}$, $1 \leq j \leq m$, $0 \leq k \leq m_j - 1$ are linearly independent. More precisely, if $K \subset \mathbb{Z}$ is an interval with $\#K = m_1 + m_2 + ... + m_m$, then ${f_{\zeta_j,k}}_{|_K}$ form a basis in $\ell^2(K)$.*

Proof. We first prove the linear independence of $f_{\zeta_j,k}$ as functions on $\mathbb{Z}$.

Lemma 3.2. *Let ζ_j, $j = 1, \dots, J$, be finitely many distinct elements of S^1. If $a_j \in \mathbb{C}$, $j = 1, \dots, J$, and $\lim\limits_{\nu \to +\infty} \sum a_j \zeta_j^\nu = 0$, then $a_j = 0$.*

Proof of Lemma 3.2. Write $\zeta_j = \mathrm{e}^{i\sigma_j}$, $\sigma_j \in \mathbb{R}$ and let $\delta_{\sigma_j} \in \mathcal{D}'(S^1)$ be the delta function centered at σ_j. Then we have

$$\lim_{\nu\to+\infty} \mathcal{F}^{-1}\left(\sum a_j \delta_{\sigma_j}\right)(\nu) = 0$$

where $\mathcal{F}^{-1}(u)(\nu) = \frac{1}{2\pi}\int_{S^1} u(x)\mathrm{e}^{ix\nu}dx$. Let $\chi \in \mathcal{C}^\infty(S^1)$, $\chi(\sigma_{j_0}) = 1$, $\chi(\sigma_j) = 0$, $j \neq j_0$. Then

$$\begin{aligned}\lim_{\nu\to+\infty} \mathcal{F}^{-1}\left(a_{j_0}\delta_{\sigma_{j_0}}\right)(\nu) &= \lim_{\nu\to+\infty} \mathcal{F}^{-1}\left(\chi\sum a_j \delta_{\sigma_j}\right)(\nu) \\ &= \lim_{\nu\to+\infty} \mathcal{F}^{-1}(\chi) * \mathcal{F}^{-1}\left(\sum a_j \delta_{\sigma_j}\right)(\nu) = 0,\end{aligned}$$

where $*$ indicates the standard convolution on $\ell^p(\mathbb{Z})$. Hence, $a_{j_0} = 0$ for any $j_0 = 1,\dots,J$. $\square$

Now consider

$$\sum_{j=1}^{m}\sum_{k=0}^{m_j-1} a_{j,k} f_{\zeta_j,k} = 0 \text{ on } \mathbb{Z}$$

and notice that

$$f_{\zeta_j,k} = (\zeta\partial_\zeta)^k(\zeta^\nu)_{\zeta=\zeta_j} = \nu^k\zeta_j^\nu.$$

Let $S = \{j; |\zeta_j| = \max_{\widetilde{j}}|\zeta_{\widetilde{j}}|\}$, $M = \max_{j\in S} m_j$ and write $\zeta_j = \mathrm{e}^{s+i\sigma_j}$. Then we get

$$\lim_{\nu\to\infty} \sum_{\substack{j\in S,\\ m_j=M}} a_{j,M-1}\,\mathrm{e}^{i\nu\sigma_j} = 0.$$

Lemma 3.2 then implies that $a_{j,k} = 0$ when $|\zeta_j| = \max_{\widetilde{j}}|\zeta_{\widetilde{j}}|$ and $k = m_j - 1$ is maximal. Repeating this procedure we get $a_{j,k} = 0$, $0 \le k \le m_j - 1$, $j \in S$. Repeating the procedure we finally get $a_{j,k} = 0$ for all j,k and we have shown that $f_{\zeta_j,k}$ are independent as functions on $\mathbb{Z}$.

Let

$$Q_\infty = \prod_1^m(\tau - 1/\zeta_j)^{m_j} = \tau^{\widetilde{m}} + a_1\tau^{\widetilde{m}-1} + \dots + a_{\widetilde{m}}, \quad \widetilde{m} = m_1 + \dots + m_m.$$

Then as in the case of $p(\tau) - z$, the functions $f_{\zeta_j,k}$, $1 \le j \le m$, $0 \le k \le m_j - 1$ satisfy

$$Q_\infty f_{\zeta_j,k} = 0.$$

Assume that a linear combination u of these functions vanishes on the interval K of length $m_1 + \dots + m_m = \widetilde{m}$. Then $Q_\infty u = 0$ on $\mathbb{Z}$, $u = 0$ on K, and we conclude that $u = 0$ on $\mathbb{Z}$. Hence $f_{\zeta_j,k}|_K$, $1 \le j \le m$, $0 \le k \le m_j - 1$ are linearly independent. $\square$

3.2. **Operators on the line and circulant matrices.** Let $S_N \stackrel{\text{def}}{=} \mathbb{Z}/N\mathbb{Z}$, for $N \in \mathbb{N}\backslash\{0\}$. In applications we will replace N by $N_+ + N_- + N$. By convention we set $S_\infty = \mathbb{Z}$.

Recall (1.1). We are interested in

$$(p(\tau) - z)u = v, \quad u, v \in \ell^2(\mathbb{Z}). \tag{3.15}$$

Let $\mathcal{F}u(\xi) = \sum_{k=-\infty}^{\infty} u(k)\mathrm{e}^{-ik\xi}$, so that $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(S^1, \frac{d\xi}{2\pi})$ is unitary. We have

$$(\mathcal{F}\tau u)(\xi) = \sum_{k=-\infty}^{\infty} u(k-1)\mathrm{e}^{-ik\xi} = \mathrm{e}^{-i\xi}(\mathcal{F}u)(\xi), \tag{3.16}$$

explaining why $e^{-i\xi} = 1/\zeta$ is the symbol of τ. Hence, application of $\mathcal{F}$ to (3.15) gives the equivalent equation

$$(p(\mathrm{e}^{-i\xi}) - z)\widehat{u} = \widehat{v}, \quad \widehat{u} = \mathcal{F}u, \quad \widehat{v} = \mathcal{F}v. \tag{3.17}$$

Thus, $\mathrm{Spec}(p(\tau)) = p(S^1)$ and if $z \notin p(S^1)$, we can invert (3.17)

$$\widehat{u}(\xi) = \frac{1}{p(\mathrm{e}^{-i\xi}) - z}\widehat{v}(\xi) \tag{3.18}$$

Applying $(\mathcal{F}^{-1}\widehat{u})(k) = \frac{1}{2\pi}\int_{S^1} \mathrm{e}^{ik\xi}\widehat{u}(\xi)d\xi$, we get

$$(p(\tau)-z)^{-1}v = K_\infty * v, \tag{3.19}$$

where

$$K_\infty(z;k) = \frac{1}{2\pi}\int_{S^1} \frac{1}{p(\mathrm{e}^{-i\xi})-z}\mathrm{e}^{ik\xi}d\xi. \tag{3.20}$$

In this formula, S^1 is identified with $\mathbb{R}/2\pi\mathbb{Z}$. Introduce $\zeta = \mathrm{e}^{i\xi}$ as the new integration variable, so that $dx = \frac{d\zeta}{i\zeta}$. Then (3.20) becomes

$$K_\infty(z;k) = \frac{1}{2\pi i}\int_{S^1} \frac{1}{p(1/\zeta)-z}\zeta^k\frac{d\zeta}{\zeta}, \tag{3.21}$$

where now S^1 is the boundary of the unit disk $D(0,1)\subset\mathbb{C}$. Recall (3.11), (3.12) and write $m_j^\pm = \mathrm{mult}(\zeta_j^\pm)$. If $k\gg 1$, we shrink the contour to 0 and get by the residue theorem

$$\begin{aligned} K_\infty(z;k) &= \sum_{j=1}^{\widetilde{m}_+} \lim_{\zeta\to\zeta_j^+} \frac{1}{(m_j^+-1)!}\frac{d^{m_j^+-1}}{d\zeta^{m_j^+-1}}\frac{\zeta^{k-1}(\zeta-\zeta_j^+)^{m_j^+}}{p(1/\zeta)-z} \\ &= \sum_{j=1}^{\widetilde{m}_+}\sum_{l=1}^{m_j^+} \binom{k-1}{l} b_{j,l}^+ (\zeta_j^+)^{k-l-1}, \quad b_{j,l}^+\in\mathbb{C}. \end{aligned} \tag{3.22}$$

If $k\ll -1$, we use (3.21), enlarge the contour to $|\zeta| = R$, $R\to\infty$, and get

$$\begin{aligned} K_\infty(z;k) &= -\sum_{j=1}^{\widetilde{m}_-} \lim_{\zeta\to\zeta_j^-} \frac{1}{(m_j^--1)!}\frac{d^{m_j^--1}}{d\zeta^{m_j^--1}}\frac{\zeta^{k-1}(\zeta-\zeta_j^-)^{m_j^-}}{p(1/\zeta)-z} \\ &= -\sum_{j=1}^{\widetilde{m}_-}\sum_{l=1}^{m_j^-} \binom{k-1}{l} b_{j,l}^- (\zeta_j^-)^{k-l-1}, \quad b_{j,l}^-\in\mathbb{C}. \end{aligned} \tag{3.23}$$

Remark 3.3. *When all roots of the polynomial (3.5) are simple, then we have by (3.6),(3.7), (3.11), (3.11) as well as (3.22), (3.23) that*

$$K_\infty(z;k) = \begin{cases} \sum_{j=1}^{m_+} \frac{1}{\partial_\zeta(p(1/\zeta))_{\zeta=\zeta_j^+}}(\zeta_j^+)^{k-1}, & \text{if } k\geq 1, \\ -\sum_{j=1}^{m_-} \frac{1}{\partial_\zeta(p(1/\zeta))_{\zeta=\zeta_j^-}}(\zeta_j^-)^{k-1}, & \text{if } k\leq -1. \end{cases} \tag{3.24}$$

Notice that $K_\infty(z;k)$ decays exponentially as $|k|\to\infty$. Hence, we can solve (3.15) for $u,v\in\ell^\infty$.

If $v\in\ell^2(S_N)$, then we can view v as an N-periodic function on $\mathbb{Z}$ and the solution u is N-periodic and given by (3.19).

Let $\Omega\subset\mathbb{Z}$ be a finite set of cardinal $\#\Omega = N$ such that

$$\begin{cases} (\Omega+jN)\cap(\Omega+kN) = \emptyset \text{ for } j\neq k \\ \bigcup_{j\in\mathbb{Z}} \Omega+jN = \mathbb{Z}. \end{cases}$$

Let $N\geq N_+ + N_- + 1$. Still when u,v are N-periodic we make (3.19) more explicit

$$\begin{aligned} u(\nu) &= \sum_{\mu\in\mathbb{Z}} K_\infty(z;\nu-\mu)v(\mu) = \sum_{j\in\mathbb{Z}}\sum_{\mu\in\Omega+jN} K_\infty(z;\nu-\mu)v(\mu) \\ &= \sum_{j\in\mathbb{Z}}\sum_{\mu\in\Omega} K_\infty(z;\nu-\mu-jN)v(\mu) = \sum_{\mu\in\Omega} K_N(z;\nu-\mu)v(\mu), \end{aligned} \tag{3.25}$$

where

$$K_N(z;\nu-\mu) = \sum_{j\in\mathbb{Z}} K_\infty(z;\nu-\mu-jN) \tag{3.26}$$

and the series converges geometrically. We check that $K_N(z;\nu+N) = K_N(z;\nu)$. Identifying $\Omega \simeq S_N$, and defining

$$P_{S_N} \stackrel{\text{def}}{=} p(\tau) : \ell^2(S_N) \to \ell^2(S_N), \tag{3.27}$$

we get

Proposition 3.4. *If $z \notin p(S^1)$, then $z \notin \operatorname{Spec}(P_{S_N})$ and the resolvent $(P_{S_N} - z)^{-1}$ is given by*

$$(P_{S_N} - z)^{-1} v(\nu) = (K_N(z) * v)(\nu) = \sum_{\mu \in S_N} K_N(z;\nu-\mu) v(\mu) \tag{3.28}$$

with

$$K_N(z;\nu) = \sum_{\widetilde{\nu} \in \pi^{-1}(\nu)} K_\infty(z;\widetilde{\nu}) \tag{3.29}$$

where $\pi : \mathbb{Z} \to S_N$ is the natural projection.

A consequence of (3.26) is the following: Choose $\Omega = [-\frac{N}{2}, \frac{N}{2}[$ when N is even and $\Omega = [-\frac{N-1}{2}, \frac{N+1}{2}[$ when N is odd. Then,

$$K_N(z;\nu) = K_\infty(z;\nu) + \mathcal{O}\left(\mathrm{e}^{-\frac{N}{C}}\right), \quad \nu \in \Omega. \tag{3.30}$$

3.3. **The spectrum of P_{S_N}.** Using the finite Fourier transform $\ell^2(S_N) \to \ell^2(\widehat{S}_N)$, with $\widehat{S}_N = \{\mathrm{e}^{\frac{2\pi i k}{N}}; k = 0, \dots, N-1\}$, it is easy to prove that

$$\operatorname{Spec}(P_{S_N}) = p(\widehat{S}_N). \tag{3.31}$$

In this section we study the spectrum of the normal operator P_{S_N}, see (3.27) and (4.9) below, in

$$\gamma = p(S^1) \cap \Omega \tag{3.32}$$

with Ω as in Section 2.1.

3.3.1. *A Weyl law for P_{S_N}.* We present a Weyl law for the eigenvalues of P_{S_N}, which we shall use later on to count the eigenvalues of small perturbations of the operator P_N (1.4).

Let γ be as in (3.32). First notice that by (3.31)

$$\#\{\operatorname{Spec}(P_{S_N}) \cap \gamma\} = \#\{\widehat{S}_N \cap p^{-1}(\gamma)\}. \tag{3.33}$$

Since two consecutive points of $\widehat{S}_N$ differ by an angle of $2\pi/N$, we get that

$$\#\{\widehat{S}_N \cap p^{-1}(\gamma)\} = \frac{N}{2\pi} \int_{p^{-1}(\gamma)} L_{S^1}(d\theta) + \mathcal{O}(1), \tag{3.34}$$

where the measure $L_{S^1}(d\theta)$ in the integral denotes the Lebesgue measure on S^1. Combining (3.33), (3.34), we get

$$\#\{\operatorname{Spec}(P_{S_N}) \cap \gamma\} = \frac{N}{2\pi} \int_{p^{-1}(\gamma)} L_{S^1}(d\theta) + \mathcal{O}(1). \tag{3.35}$$

3.3.2. *Local eigenvalue spacing for P_{S_N}.* Let $z_0 \in p(S^1)$ be such that

$$dp \neq 0 \text{ on } p^{-1}(z_0). \tag{3.36}$$

Proposition 3.5. *Let p be as in (1.3) and let $z_0 \in p(S^1)$ be such that (3.36) holds. Then, there exist a constant $C > 0$ and an open neighborhood $U \subset \mathbb{C}$ of z_0, such that $p^{-1}(U)$ is the union of finitely many disjoint open sets $V_i \subset \mathbb{C}$, $i = 1, \dots, M$. Moreover, on each non-empty segment $\gamma_i = p(V_i \cap S^1)$ we have that*

$$\min_{\substack{z,w \in p(\widehat{S}_{\widetilde{N}}) \cap \gamma_i \\ w \neq z}} |z - w| \geq \frac{1}{CN}. \tag{3.37}$$

Proof of Proposition 3.5. For $i = 1, \dots, M$ let $\zeta_i \in p^{-1}(z_0)$ and notice that $M < +\infty$. By (3.36) and the implicit function theorem, there exist complex open neighborhoods U_i of z_0 and V_i of ζ_i such that $p : V_i \to U_i$ is a diffeomorphism. Setting $\gamma_i = p(S^1 \cap V_i) \subset U_i$ when $S^1 \cap V_i \neq \emptyset$, we have that

$$|\widehat{\zeta}^i_n - \widehat{\zeta}^i_m| \asymp |\widehat{z}^i_n - \widehat{z}^i_m|, \tag{3.38}$$

where $\widehat{\zeta}^i_n \in \widehat{S}_{\widetilde{N}} \cap V_i$ and $\widehat{z}^i_n \in p(\widehat{S}_{\widetilde{N}}) \cap \gamma_i$, for $n \in J_i \subset \mathbb{N}$, some index set which is non-empty for $N > 1$ sufficiently large. Since M is finite, the claim follows by (3.31) and by taking $U = \bigcap_{i=1}^M U_i$ and by potentially shrinking the segements γ_i. ☐

3.4. **Restrictions to intervals.** If $K \subset \mathbb{Z}$ is a finite set or an infinite interval, we identify

$$\ell^2(K) \simeq \ell^2_K \stackrel{\text{def}}{=} \{u \in \ell^2(\mathbb{Z});\ \operatorname{supp} u \subset K\}. \tag{3.39}$$

We define,

$$P_K \stackrel{\text{def}}{=} 1_K\, p(\tau) :\ \ell^2_K \longrightarrow \ell^2_K, \quad \text{and} \quad P_{\mathbb{Z}} = p(\tau). \tag{3.40}$$

In the following we assume (3.9). When K is an interval we define the length of K to be $\#K = |K|$.

Proposition 3.6. *Let K be an interval of length $\leq N_+ + N_-$. Any function $u :\ K \to \mathbb{C}$ can be extended to a solution $\widetilde{u} :\ \mathbb{Z} \to \mathbb{C}$ to $(p(\tau) - z)u = 0$. The space of such extensions is affine of dimension $N_+ + N_- - \#K$. In particular the extension is unique when $N_+ + N_- = \#K$.*

Proof. If $\#K < N_+ + N_-$, let $\widetilde{K} \supset K$ be an interval with $\#\widetilde{K} = N_+ + N_-$. The extensions $\hat{u} :\ \widetilde{K} \to C$ form an affine space of dimension $N_+ + N_- - K$, so it suffices to treat the case $\#K = N_+ + N_-$.

Let $K = [M, M + N_+ + N_-[$ and write $(p(\tau) - z)\widetilde{u} = 0$, i.e.

$$a_{N_+}\widetilde{u}(\nu - N_+) + \dots + (a_0 - z)\widetilde{u}(\nu) + \dots + a_{-N_-}\widetilde{u}(\nu + N_-) = 0. \tag{3.41}$$

For $\nu = N_+ + M$, $\nu + N_-$ is the first point in $\mathbb{Z}\backslash K$ to the right of K and $\nu + N_- - 1, \dots, \nu - N_+$ belong to K, so (3.41) defines $\widetilde{u}(\nu + N_-)$ uniquely. Replacing ν with $\nu + 1 = M + N_+ + 1$, we get $\widetilde{u}(M + N_+ + N_- + 1)$ and by repeating the procedure we get $\widetilde{u}(M + N_+ + N_- + \mu)$ for all $\mu \geq 0$.

For $\nu = M + N_+ - 1$, we have $\nu - N_+ \notin K$ while $\nu - N_+ + 1, \dots, \nu + N_- \in K$, and therefore (3.41) determines $\widetilde{u}(M - 1)$ uniquely. Iterating the procedure, we get all values of $\widetilde{u}(M - \mu)$, for $\mu > 0$. ☐

It follows that the space of solutions to $(p(\tau) - z)u = 0$ is of dimension $N_+ + N_- = m_+ + m_-$, cf. (3.8). Recall (3.11), (3.12), (3.8), (3.9), (3.3) and (3.10). The space of exponential solutions, spanned by the functions

$$\mathbb{Z} \ni \nu \mapsto \nu^k (\zeta^{\pm}_{j_\pm})^\nu, \quad \text{for } 1 \leq j_\pm \leq \widetilde{m}_\pm,\ 0 \leq k \leq \operatorname{mult}(\zeta^{\pm}_{j_\pm}) - 1, \tag{3.42}$$

is also of dimension $m_+ + m_-$, since these functions form a linearly independent system by Proposition 3.1. Hence, assuming (3.9), (3.3), they form a basis of the space of solutions $u : \mathbb{Z} \to \mathbb{C}$ to $(p(\tau) - z)u = 0$. We conclude the following

Proposition 3.7. *Suppose (3.9) and (3.3). Then, the general solution $u : \mathbb{Z} \to \mathbb{C}$ of $(p(\tau) - z)u = 0$ is of the form*

$$u(\nu) = \sum_{j=1}^{\widetilde{m}_+} \sum_{k=0}^{\operatorname{mult}(\zeta^+_j)-1} a^+_{j,k} \nu^k (\zeta^+_j)^\nu + \sum_{j=1}^{\widetilde{m}_-} \sum_{k=0}^{\operatorname{mult}(\zeta^-_j)-1} a^-_{j,k} \nu^k (\zeta^-_j)^\nu, \quad a^{\pm}_{j,k} \in \mathbb{C}. \tag{3.43}$$

The subspace of solutions decaying at $\nu \to \pm\infty$ is given by

$$a^{\mp}_{j,k} = 0, \quad \textit{for } j = 1, \dots, \widetilde{m}_\mp,\ k = 0, \dots, \operatorname{mult}(\zeta^+_j) - 1. \tag{3.44}$$

Remark 3.8. *Enumerate all the roots of* (3.5) *as*

$$w_j = \begin{cases} \zeta_j^+, \ \textit{for } j = 1, \dots, \widetilde{m}_+ \\ \zeta_{j-\widetilde{m}_+}^-, \ \textit{for } j = \widetilde{m}_+ + 1, \dots, \widetilde{m}_+ + \widetilde{m}_- \end{cases}$$

so that (3.43) *takes the form*

$$u(\nu) = \sum_{j=1}^{\widetilde{m}_+ + \widetilde{m}_-} \sum_{k=0}^{\mathrm{mult}(w_j)-1} b_{j,k} \nu^k (w_j)^\nu, \quad b_{j,k} \in \mathbb{C}. \tag{3.45}$$

We then recover the fact that the following Van der Monde type determinant

$$\det(A_1, \dots, A_{\widetilde{m}_+ + \widetilde{m}_-}) \tag{3.46}$$

is non-vanishing. Here, the block matrices A_j, $j = 1, \dots, \widetilde{m}_+ + \widetilde{m}_-$, are given by

$$A_j = (\nu^k w_j^\nu)_{\substack{\nu \in K \\ 0 \le k \le \mathrm{mult}(w_j)-1}} \in \mathbb{C}^{|K| \times \mathrm{mult}(w_j)} \tag{3.47}$$

where K is any interval of length $|K| = m_+ + m_-$.

We next look at P_K where K is the half-axis $[A, +\infty[$ or $]-\infty, A]$. The two cases are similar and we may assume by translation invariance that $K = [0, +\infty[$.

Let $u : K \to \mathbb{C}$ have its support in $[0, \infty[$ and satisfy

$$(p(\tau) - z)u = 0 \text{ in } [0, +\infty[. \tag{3.48}$$

More explicitly, by (1.1),

$$\left(\sum_{j=-N_-}^{N_+} a_j \tau^j - z \right) u(\nu) = 0, \quad \nu = 0, 1, \dots \tag{3.49}$$

The left most equation for $\nu = 0$ is

$$a_{N_+} u(-N_+) + a_{N_+-1} u(1 - N_+) + \dots + (a_0 - z)u(0) + \dots + a_{-N_-} u(N_-) = 0.$$

Here, $u(-N_+) = \dots = u(-1) = 0$, when $N_+ \le 1$. We know how to extend $u|_{[-N_+, +\infty[}$ to a function $\widetilde{u} : \mathbb{Z} \to \mathbb{C}$, by solving (3.49) with u replaced by $\widetilde{u}$ for $\nu = -1, -2, \dots$. The equation for $\nu = -1$ defines $\widetilde{u}(-N_+ - 1)$, the next one gives $\widetilde{u}(-N_+ - 2)$ and so on. In this way we get a solution $\widetilde{u}$ on $\mathbb{Z}$ of

$$(p(\tau) - z)\widetilde{u} = 0. \tag{3.50}$$

Consequently $\widetilde{u}$ has the form of the right hand side in (3.43). Now restrict the attention to solutions $u \in \ell^2_{[0,+\infty[}(\mathbb{Z})$ of (3.48). The corresponding extension $\widetilde{u}$ is of the form (3.43) with $a_{j,k}^- = 0$, since it must decay to the right. Hence,

$$\widetilde{u}(\nu) = \sum_{j=1}^{\widetilde{m}_+} \sum_{k=0}^{\mathrm{mult}(\zeta_j^+)-1} a_{j,k}^+ \nu^k (\zeta_j^+)^\nu \tag{3.51}$$

and by construction $\widetilde{u}(\nu) = u(\nu) = 0$ for $\nu \in [-N_+, -1]$. More explicitly, using (3.13), we have

$$0 = A \begin{pmatrix} a_{1,0}^+ \\ \vdots \\ a_{\widetilde{m}_+, \mathrm{mult}(\zeta_{\widetilde{m}_+})-1}^+ \end{pmatrix}, \quad A = (A_1^+, \dots, A_{\widetilde{m}_+}^+) \in \mathbb{C}^{N_+ \times m_+}, \tag{3.52}$$

$$A_j^+ = (\nu^0 (\zeta_j^+)^\nu, \dots, \nu^{\mathrm{mult}(\zeta_j^+)-1} (\zeta_j^+)^\nu)_{-N_+ \le \nu \le -1}, \quad \text{for } j = 1, \dots, \widetilde{m}_+.$$

Notice that A is a rectangular generalized matrix of Van der Monde type, of size $N_+ \times m_+$. Arguing as at the end of the proof of Proposition 3.1 and using (3.13), we see that A is of maximal rank $\min(N_+, m_+)$. Thus

- if $N_+ \ge m_+$, then
$$\ker\left(P_{[0,+\infty[} - z\right) = 0.$$

- If $N_+ < m_+$, then
$$\dim\ker\left(P_{[0,+\infty[} - z\right) = m_+ - N_+.$$

For $(P_{]-\infty,0]} - z)$ we have the corresponding statements with N_+, m_+ replaced by N_-, m_-.

Lemma 3.9. *Let $z \notin p(S^1)$, then the operators $(P_{[0,+\infty[} - z) : \ell^2([0,+\infty[) \to \ell^2([0,+\infty[)$ and $(P_{]-\infty,0]} - z) : \ell^2(]-\infty,0]) \to \ell^2(]-\infty,0])$ are Fredholm.*

Proof. We give the proof for $(P_{[0,+\infty[} - z)$, the one for $(P_{]-\infty,0]} - z)$ is similar.

Recall (3.19), and define for $z \notin p(S^1)$
$$E(z) = \mathbf{1}_{[0,+\infty[}(p(\tau) - z)^{-1}\mathbf{1}_{[0,+\infty[}.$$
Then,
$$(P_{[0,+\infty[} - z)E(z) = \mathbf{1}_{[0,+\infty[} + R(z)$$
and
$$E(z)(P_{[0,+\infty[} - z) = \mathbf{1}_{[0,+\infty[} + L(z).$$
where $R(z), L(z)$ are compact. Indeed, we have
$$R(z) = -\mathbf{1}_{[0,+\infty[}(p(\tau) - z)\mathbf{1}_{]-\infty,0[}(p(\tau) - z)^{-1}\mathbf{1}_{[0,+\infty[}.$$
By (1.1) we see that $R(z) = 1_{[0,N_+[}R(z)$, so $R(z)$ is of finite rank and thus compact. Similarly, we have
$$L(z) = -\mathbf{1}_{[0,+\infty[}(p(\tau) - z)^{-1}\mathbf{1}_{]-\infty,0[}(p(\tau) - z)\mathbf{1}_{[0,+\infty[}.$$
We notice that $L(z) = L(z)1_{[0,N_+[}$ is of finite rank, hence compact. ☐

Next, notice that by (3.2), (3.6), (3.7), $p(\tau)^*$ is similar to $p(\tau)$ just with the roles of N_+, m_+ and N_-, m_- exchanged. More explicitly, by (1.1),
$$p^*(\tau) = \overline{p}(\tau^{-1}) = \sum_{-N_-}^{N_+} \overline{a}_j \tau^{-j} = \sum_{-N_+}^{N_-} \overline{a}_{-j}\tau^j.$$
The analogue of (3.2) is $\overline{p}(\omega) - \overline{z} = 0$, or equivalently $p(\overline{\omega}) - z = 0$, since $\overline{p}(\omega) = \overline{p(\overline{\omega})}$. In view of (3.6), (3.7), we get the roots $\omega_j^{\pm} = 1/\overline{\zeta_j^{\pm}}$. Remembering (3.40), we have
$$P_K^* = 1_K\overline{p}(\tau^{-1})1_K$$
Therefore, the above statements remain valid with $(p(\tau) - z)$ replaced by $(p^*(\tau) - \overline{z})$ and N_+, m_+ exchanged with N_-, m_-.

By Lemma 3.9 we get that for $z \notin p(S^1)$
$$\dim\ker\left(P_{[0,+\infty[} - z\right)^* = \dim\operatorname{coker}\left(P_{[0,+\infty[} - z\right)$$
Hence, using (3.8) we conclude the following

Proposition 3.10. *Assume that $z \notin \{0,+\infty\} \cup p(S^1)$ and that (3.9) holds.*

- *If $N_+ \geq m_+$, then*
$$\ker\left(P_{[0,+\infty[} - z\right) = 0$$
and
$$\dim\operatorname{coker}\left(P_{[0,+\infty[} - z\right) = N_+ - m_+.$$
- *If $N_+ < m_+$, then*
$$\operatorname{coker}\left(P_{[0,+\infty[} - z\right) = 0$$
and
$$\dim\ker\left(P_{[0,+\infty[} - z\right) = m_+ - N_+.$$

For $(P_{]-\infty,0]} - z)$ we have the corresponding statements with N_+, m_+ replaced by N_-, m_-.

It will be convenient to replace $P_{[0,+\infty[}$ with the unitarily equivalent operator $P_{[N_+,+\infty[}$. Moreover, let us recall that the *index* of a Fredholm operator A is defined by

$$\mathrm{Ind}\, A \overset{\mathrm{def}}{=} \dim\ker A - \dim\operatorname{coker} A.$$

There is a very nice relation between the index of the Fredholm operator $(P_{[N_+,+\infty[}-z)$ and the winding number of the curve $p(S^1)$ around the point z.

Proposition 3.11. *Let $z \notin \{0,+\infty\}\cup p(S^1)$ and suppose that (3.9) holds. Then $(P_{[N_+,+\infty[}-z)$ is Fredholm of index*

$$\mathrm{Ind}(P_{[N_+,+\infty[}-z) = m_+ - N_+ = -\mathrm{ind}_{p(S^1)}(z). \tag{3.53}$$

Proof. The first equality follows from Proposition 3.10 (see also (4.21), (4.25)). To see the second equality, notice that

$$-\,\mathrm{ind}_{p(S^1)}(z) = \frac{1}{2\pi i}\int_{S^1}\frac{d}{d\eta}\log(p(1/\eta)-z)d\eta. \tag{3.54}$$

The integral on the right hand side is equal to the number of zeros minus the number of poles of $p(1/\eta)-z$ in $D(0,1)$, where both are counted including multiplicity. This is equal to $m_+ - N_+$ by (3.5), (3.6) and (3.7). □

Remark 3.12. *This result has been obtained by M.G. Krein via a different method. See* [BöSi99, Chapter 1.5] *for a detailed exposition.*

3.5. **Zone of zero winding number.** In this section we show that in regions in $\mathbb{C}$, for which the winding number of the curve $p(S^1)$ is zero, the norm of the resolvent of P_N is controlled by a constant. Hence, we can consider such regions to "spectrally stable" for P_N.

Proposition 3.13. *Let $\Omega \Subset \mathbb{C}\backslash(\{0\}\cup p(S^1))$ be a compact set and suppose that for every $z\in\Omega$ (3.9) holds and*

$$\mathrm{ind}_{p(S^1)}(z) = 0. \tag{3.55}$$

Then, there exists a constant $C>0$ such that for $N>0$ sufficiently large and for any $z\in\Omega$

$$\|(P_N - z)^{-1}\| \leq C.$$

Proof. By Propositions 3.11, 3.10 and by (3.55), we know that $(P_{[1,+\infty[}-z)$ and $(P_{]-\infty,N]}-z)$ are bijective on ℓ^2 with uniformly bounded inverses when $z\in\Omega$. By the Combes-Thomas argument the same holds after conjugation with a factor $\mathrm{e}^{\varepsilon\varphi}$ if φ is Lipschitz of modulus ≤ 1 and $|\varepsilon|$ is small enough.

Let

$$Q_N(z) = \mathbf{1}_{[1,N]}((P_{[1,+\infty[}-z)^{-1}\mathbf{1}_{[1,[N/2]]} + (P_{]-\infty,N]}-z)^{-1}\mathbf{1}_{[[N/2]+1,N]}).$$

Then, using the stability under exponential conjugation, it follows that

$$(P_{[1,N]}-z)Q_N(z) = 1 + R, \quad \|R\| \leq \mathcal{O}(1)\mathrm{e}^{-N/C}.$$

Hence, for $N>1$ large enough, $P_{[1,N]}:\ell^2([1,N])\to\ell^2([1,N])$ has a uniformly bounded right inverse which is also a left inverse since $P_{[1,N]}$ is a finite square matrix. □

4. A Grushin Problem

We begin by giving a short refresher on Grushin problems. See [SjZw07] for a review. The central idea is to set up an auxiliary problem of the form

$$\begin{pmatrix} P(z) & R_- \\ R_+ & R_{+-}\end{pmatrix} : \mathcal{H}_1\oplus\mathcal{H}_- \longrightarrow \mathcal{H}_2\oplus\mathcal{H}_+,$$

where $P(z)$ is the operator under investigation and $R_\pm, R_{+-}$ are suitably chosen. We say that the Grushin problem is well-posed if this matrix of operators is bijective. If $\dim\mathcal{H}_- = \dim\mathcal{H}_+ < \infty$, one typically writes

$$\begin{pmatrix} P(z) & R_- \\ R_+ & R_{+-}\end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z)\end{pmatrix}.$$

The key observation goes back to the Shur complement formula or, equivalently, the Lyapunov-Schmidt bifurcation method, i.e. the operator $P(z) : \mathcal{H}_1 \to \mathcal{H}_2$ is invertible if and only if the finite dimensional matrix $E_{-+}(z)$ is invertible and when $E_{-+}(z)$ is invertible, we have

$$P^{-1}(z) = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z).$$

$E_{-+}(z)$ is sometimes called effective Hamiltonian.

4.1. **A Grushin problem for the unperturbed operator.** Let $J \subset \mathbb{Z}$ be a fixed interval of length $\#J = N_+ + N_-$. More precisely, we choose

$$J = [-N_-, N_+[. \tag{4.1}$$

If $M > N_+ + N_-$ we view J as a segment of S_M, cf. the beginning of Section 3.2. More precisely we define a segment $[a, b] \subset S_M$, $a, b \in S_M$, to be the set of points in S_M that we get by picking first a, then $a+1$ and so on until we reach b (mod $M\mathbb{Z}$) with the last point b included. Similarly we define $[a, b[$, $]a, b[$, $]a, b]$. Recall that $S_\infty = \mathbb{Z}$.

Suppose that

$$N \geq N_+ + N_- + 1. \tag{4.2}$$

When N is finite we decompose

$$S_{N+N_++N_-} = J \cup I_N, \tag{4.3}$$

$$I_N = [N_+, -N_- - 1] \tag{4.4}$$

where $I_N \simeq [N_+, -N_- - 1 + N_+ + N_- + N] = [N_+, N_+ + N - 1]$ in $\mathbb{Z}$. When $N = \infty$, we decompose

$$\mathbb{Z} = S_\infty = J \cup I_\infty, \tag{4.5}$$

$$I_\infty =]-\infty, -N_- - 1] \cup [N_+, \infty[. \tag{4.6}$$

Since $\#I_N = N$, we can identify

$$P_N \simeq P_{I_N}, \tag{4.7}$$

in view of (1.4), when N is finite, while P_{I_∞} is the direct sum

$$P_{]-\infty,-N_--1]} \oplus P_{[N_+,\infty[} \simeq P_{]-\infty,0]} \oplus P_{[0,\infty[}. \tag{4.8}$$

In both cases we identify

$$\ell^2(S_{N+N_-+N_-}) \simeq \ell^2(I_N) \oplus \ell^2(J)$$

so that

$$(P_{S_{N+N_-+N_-}} - z) \stackrel{\text{def}}{=} \mathcal{P}_N(z) = \begin{pmatrix} P_{I_N} - z & R_-^N \\ R_+^N & R_{+-}^N(z) \end{pmatrix} : \ell^2(I_N) \oplus \ell^2(J) \to \ell^2(I_N) \oplus \ell^2(J) \tag{4.9}$$

where

$$\begin{aligned} &P_{I_N} - z = 1_{I_N}(p(\tau) - z)1_{I_N}, \quad R_-^N = 1_{I_N}p(\tau)1_J, \\ &R_+^N = 1_J p(\tau)1_{I_N}, \quad R_{+-}^N(z) = 1_J(p(\tau) - z)1_J. \end{aligned} \tag{4.10}$$

Lemma 4.1. *R_+^N is surjective and R_-^N is injective.*

Proof. Suppose that $\operatorname{supp} u \subset [-N_- - N_+, -N_-[\subset I_N$. Then, $\operatorname{supp} R_+^N u \subset [-N_-, -N_- + N_+[$. By fixing the values of $u(-N_- - 1), \dots, u(-N_+ - N_-)$ we can arrange so that $R_+^N u$ is equal to any given function with support in $[-N_-, -N_- + N_+[$.

Similarly, if $\operatorname{supp} u \subset [N_+, N_+ + N_-[$ then $\operatorname{supp} R_+^N u \subset [-N_+ - N_-, N_+[$ and a convenient choice of such a u will produce any given function with support in $[N_+ - N_-, N_+[$. Since $J = [-N_-, -N_- + N_+[\cup [N_+ - N_-, N_+[$ and $[-N_- - N_+, -N_-[$, $[N_+, N_+ + N_-[$ are by (4.2) disjoint subsets of I_N, we see that R_+^N is surjective.

For the same reason ${}^t(R_-^N) = 1_J {}^t p(\tau) 1_{I_N}$ is surjective and therefore R_-^N is injective. $\square$

Recall (3.31). If $z \notin \mathrm{Spec}(P_{S_{\widetilde{N}}})$, where $\widetilde{N} = N + N_- + N_-$, then $\mathcal{P}_N(z)$ in (4.9) is bijective and invertible with bounded inverse

$$\mathcal{E}_N(z) = \begin{pmatrix} E^N(z) & E_+^N(z) \\ E_-^N(z) & E_{-+}^N(z) \end{pmatrix} : \ell^2(I_N) \oplus \ell^2(J) \to \ell^2(I_N) \oplus \ell^2(J). \tag{4.11}$$

We have

$$\begin{aligned} E^N(z) &= 1_{I_N}(P_{S_{\widetilde{N}}} - z)^{-1} 1_{I_N}, \quad E_+^N(z) = 1_{I_N}(P_{S_{\widetilde{N}}} - z)^{-1} 1_J, \\ E_-^N(z) &= 1_J(P_{S_{\widetilde{N}}} - z)^{-1} 1_{I_N}, \quad E_{-+}^N(z) = 1_J(P_{S_{\widetilde{N}}} - z)^{-1} 1_J. \end{aligned} \tag{4.12}$$

If $z \notin p(S^1)$, then this also holds for $N = \infty$. We now recall Proposition 3.4 and (3.26) with N replaced by $\widetilde{N}$. On the level of matrices we get with $\pi = \pi_{\widetilde{N}} : \mathbb{Z} \to S_{\widetilde{N}}$

$$\begin{aligned} E^N(z;\nu,\mu) &= \sum_{\widetilde{\nu} \in \pi^{-1}(\nu)} E^\infty(z;\widetilde{\nu},\widetilde{\mu}), \quad \widetilde{\mu} \in \pi^{-1}(\mu),\ \mu,\nu \in I_N, \\ E_+^N(z;\nu,\mu) &= \sum_{\widetilde{\nu} \in \pi^{-1}(\nu)} E_+^\infty(z;\widetilde{\nu},\mu), \quad \mu \in J,\ \nu \in I_N, \\ E_-^N(z;\nu,\mu) &= \sum_{\widetilde{\mu} \in \pi^{-1}(\mu)} E_-^\infty(z;\nu,\widetilde{\mu}), \quad \nu \in J,\ \mu \in I_N, \end{aligned} \tag{4.13}$$

and

$$E_{-+}^N(z;\nu,\mu) = \sum_{j \in \mathbb{Z}} (p(\tau) - z)^{-1}(\nu + j\widetilde{N}, \mu) \quad \mu,\nu \in J. \tag{4.14}$$

In these formulas we used that J is naturally defined both as a subset of $S_{\widetilde{N}}$ and of $\mathbb{Z}$. We can consider a similar non-canonical identification of I_N with $\widetilde{I}_N \subset \mathbb{Z}$ given by $[-M, -N_-[\cup[N_+, -M + \widetilde{N}[$, $\widetilde{N} = N + N_- + N_+$, where we choose M so that $\Theta N \leq M \leq (1-\Theta)N$ for some $\Theta \in]0,1[$, with $N \gg 1$. Then, (4.13) has a more explicit form:

$$\begin{aligned} E^N(z;\nu,\mu) &= \sum_{j \in \mathbb{Z}} E^\infty(z;\nu + j\widetilde{N},\mu), \quad \mu,\nu \in \widetilde{I}_N, \\ E_+^N(z;\nu,\mu) &= \sum_{j \in \mathbb{Z}} E_+^\infty(z;\nu + j\widetilde{N},\mu), \quad \nu \in \widetilde{I}_N,\ \mu \in J, \\ E_-^N(z;\nu,\mu) &= \sum_{j \in \mathbb{Z}} E_-^\infty(z;\nu,\mu + j\widetilde{N}), \quad \nu \in J,\ \mu \in \widetilde{I}_N, \\ E_{-+}^N(z;\nu,\mu) &= \sum_{j \in \mathbb{Z}} E_{-+}^\infty(z;\nu + j\widetilde{N},\mu), \quad \nu,\mu \in J. \end{aligned} \tag{4.15}$$

In particular, due to the exponential decay,

$$\begin{aligned} E_+^N(z;\nu,\mu) &= E_+^\infty(z;\nu,\mu) + \mathcal{O}\left(\mathrm{e}^{-\frac{N}{C}}\right), \quad \nu \in \widetilde{I}_N,\ \mu \in J, \\ E_-^N(z;\nu,\mu) &= E_-^\infty(z;\nu,\mu) + \mathcal{O}\left(\mathrm{e}^{-\frac{N}{C}}\right), \quad \nu \in J,\ \mu \in \widetilde{I}_N, \\ E_{-+}^N(z;\nu,\mu) &= E_{-+}^\infty(z;\nu,\mu) + \mathcal{O}\left(\mathrm{e}^{-\frac{N}{C}}\right), \quad \nu,\mu \in J. \end{aligned} \tag{4.16}$$

We next look at some general properties of E_{-+}^N. We are mainly interested in the case $N = +\infty$, but the discussion holds for all N, so we drop the superscript N. From $(P-z)E_+ + R_-E_{-+} = 0$, conclude that

$$\ker(E_{-+}) \xrightarrow{E_+} \ker(P - z). \tag{4.17}$$

From $E_{-+}R_+ + E_-(P-z) = 0$ we see that

$$\ker(P - z) \xrightarrow{R_+} \ker(E_{-+}). \tag{4.18}$$

Also notice that since $R_+E_+ + R_{+-}E_{-+} = 1$, we have

$$R_+E_+ = 1 \text{ on } \ker(E_{-+}). \tag{4.19}$$

Similarly for $E(P-z)+E_+R_+=1$, we have

$$E_+R_+ = 1 \text{ on } \ker(P-z), \tag{4.20}$$

so (4.17), (4.18) are bijective and inverse to each other.

Let $N=+\infty$. From Proposition 3.10 we know that

(1) if $N_+ = m_+$, then $N_- = m_-$, by (3.8), and
$$\ker(P_{I_\infty} - z) = 0.$$
Since $\ker(E^\infty_{-+}) = R_+ \ker(P_{I_\infty} - z)$ we conclude that E^∞_{-+} is injective and hence bijective.

(2) if $N_+ < m_+$, then $N_- > m_-$ and
$$u \in \ker(P_{I_\infty} - z) \iff \begin{cases} u|_{]-\infty,-N_- -1]} = 0, \\ u \in \ker(P_{[N_+,+\infty[} - z). \end{cases}$$
Moreover,
$$\begin{aligned} \dim\ker(E^\infty_{-+}(z)) &= \dim\ker(P_{I_\infty} - z) \\ &= \dim\ker(P_{[N_+,+\infty[} - z) = m_+ - N_+. \end{aligned} \tag{4.21}$$

(3) if $N_+ > m_+$, then $N_- < n_-$ and
$$u \in \ker(P_{I_\infty} - z) \iff \begin{cases} u|_{[N_+,+\infty[} = 0, \\ u \in \ker(P_{]-\infty,-N_- -1]} - z). \end{cases}$$
Moreover,
$$\begin{aligned} \dim\ker(E^\infty_{-+}(z)) &= \dim\ker(P_{I_\infty} - z) \\ &= \dim\ker(P_{]-\infty,-N_- -1]} - z) = m_- - N_-. \end{aligned}$$

In all cases $\ker(E^\infty_{-+}(z)) = R_+ \ker(P_{I_\infty} - z)$.

Suppressing again the superscripts, we can describe by duality $\mathcal{R}(E_{-+})^\perp = \ker(E^*_{-+})$. In fact by (4.10)

$$\begin{aligned} R_-^* &= 1_J\, p(\tau)^*\, 1_{I_N}, \quad R_+^* = 1_{I_N}\, p(\tau)^*\, 1_J \\ R_{+-}^* &= 1_J\, (p(\tau) - z)^*\, 1_J. \end{aligned} \tag{4.22}$$

So

$$\mathcal{P}^*(z) = \begin{pmatrix} (P_I - z)^* & R_+^* \\ R_-^* & R_{+-}^* \end{pmatrix} \tag{4.23}$$

is obtained from $(P_{S_{N+N_-+N_+}} - z)^*$ in exactly the same way as $\mathcal{P}(z)$ from $(P_{S_{N+N_-+N_+}} - z)$, cf. (4.9). The inverse is

$$\mathcal{E}^*(z) = \begin{pmatrix} E(z)^* & E_-(z)^* \\ E_+(z)^* & E_{-+}(z)^* \end{pmatrix}, \tag{4.24}$$

and we get from $N=+\infty$ that $\ker(E_{+-}(z)^*) = (R_-)^* \ker((P_{I_\infty} - z)^*)$.

For $u \in \ell^2(\mathbb{Z})$ let $\Gamma u = \overline{u}$. In view of (1.1) we see that
$$(p(\tau) - z)^* = \Gamma(p(\tau^{-1}) - z)\Gamma$$
as operators acting on $\ell^2(\mathbb{Z})$. By Proposition 3.10 we get

(1) if $N_+ = m_+$, then $N_- = m_-$, by (3.8), and
$$\ker\left((P_{I_\infty} - z)^*\right) = 0.$$
Since $\ker((E^\infty_{-+})^*) = R_-^* \ker((P_{I_\infty} - z)^*)$, we conclude that $(E^\infty_{-+})^*$ is injective and hence bijective.

(2) if $N_+ < m_+$, then $N_- > m_-$ and

$$u \in \ker\big((P_{I_\infty} - z)^*\big) \iff \begin{cases} u|_{[N_+,\infty[} = 0, \\ u \in \ker\big((P_{]-\infty,-N_--1]} - z)^*\big). \end{cases}$$

Moreover,

$$\begin{aligned}\dim\ker\big(E_{-+}^\infty(z)^*\big) &= \dim\ker\big((P_{I_\infty} - z)^*\big) \\ &= \dim\ker\big((P_{]-\infty,-N_--1]} - z)^*\big) = m_+ - N_+.\end{aligned}$$

(3) if $N_+ > m_+$, then $N_- < m_-$ and

$$u \in \ker\big((P_{I_\infty} - z)^*\big) \iff \begin{cases} u|_{]-\infty,-N_--1]} = 0, \\ u \in \ker\big((P_{[N_+,+\infty[} - z)^*\big). \end{cases} \tag{4.25}$$

Moreover,

$$\begin{aligned}\dim\ker\big(E_{-+}^\infty(z)^*\big) &= \dim\ker\big((P_{I_\infty} - z)^*\big) \\ &= \dim\ker\big((P_{[N_+,+\infty[} - z)^*\big) = m_- - N_-.\end{aligned}$$

4.2. **Estimates on the singular values of $E_\pm$.** In this section we will give bounds on the singular values of $E_\pm$, see (4.12). We will treat both the case when $N \geq N_+ + N_- + 1$. and the limiting case when $N = +\infty$. First, notice that

$$\operatorname{rank}(E_\pm^N) \leq |J| = N_+ + N_-. \tag{4.26}$$

When $N \geq N_+ + N_- + 1$. and $z \notin \operatorname{Spec}(P_{S_{\tilde{N}}})$, let

$$0 \leq s_{|J|}^{N,\pm} \leq \dots \leq s_1^{N,\pm} = \|E_\pm^N\| \tag{4.27}$$

denote the singular values of $E_\pm^N$. When $N = +\infty$ and $z \notin p(S^1)$, let

$$0 \leq s_{|J|}^{\infty,\pm} \leq \dots \leq s_1^{\infty,\pm} = \|E_\pm^\infty\| \tag{4.28}$$

denote the singular values of $E_\pm^\infty$. Although we have not denoted it explicitly here, the singular values (4.28), (4.28), depend on z. Recall (4.9) and notice that since the operator $p(\tau)$ acting on $\ell^2(S_{\tilde{N}})$ and on $\ell^2(\mathbb{Z})$ is normal, we have the trivial upper bounds

$$s_1^{N,\pm} \leq \frac{1}{\operatorname{dist}(z, \operatorname{Spec}(P_{S_{\tilde{N}}}))}, \quad s_1^{\infty,\pm} \leq \frac{1}{\operatorname{dist}(z, p(S^1))} \tag{4.29}$$

Lemma 4.2. *Let $N \geq 2(N_+ + N_-) + 1$ and let $\Omega \Subset \mathbb{C}$ be a compact set. Then,*

(1) there exists a constant $C > 0$, such that for all $z \in \Omega \backslash \operatorname{Spec}(P_{S_{\tilde{N}}})$

$$\frac{1}{C} \leq s_j^{N,\pm} \leq \frac{1}{\operatorname{dist}(z, \operatorname{Spec}(P_{S_{\tilde{N}}}))}, \quad j = 1, \dots, N_+ + N_-.$$

In particular E_+^N is injective and E_-^N is surjective.

(2) there exists a constant $C > 0$, such that for all $z \in \Omega \backslash p(S^1)$

$$\frac{1}{C} \leq s_j^{\infty,\pm} \leq \frac{1}{\operatorname{dist}(z, p(S^1))}, \quad j = 1, \dots, N_+ + N_-.$$

In particular E_+^∞ is injective and E_-^∞ is surjective.

Remark 4.3. *Notice that in both cases the lower bound on the singular values only depends on the compact set Ω and is independent of N. This is due to the fact that the only moment in the proof of Lemma 4.2 where we need that $z \notin \operatorname{Spec}(P_{S_{\tilde{N}}})$ (respectively $z \notin p(S^1)$ when $N = +\infty$) is when we use that $\mathcal{E}$ (4.11)- the inverse of the Grushin problem $\mathcal{P}$ (4.9) - exists, see (4.44) below.*

Proof of Lemma 4.2. We begin with the case (1): The upper bounds follow from (4.29).

Let us now turn to the lower bounds. We begin by recalling the Grushin problem (4.9): for $z \in \Omega\backslash\mathrm{Spec}(P_{S_{\widetilde{N}}})$, the operator

$$(p(\tau)-z) : \ell^2(S_{\widetilde{N}}) \longrightarrow \ell^2(S_{\widetilde{N}})$$

is bijective with bounded inverse $\mathcal{E}_N(z)$, see (4.11). Here, $S_{\widetilde{N}} \simeq \mathbb{Z}/\widetilde{N}\mathbb{Z}$, $\widetilde{N} = N + N_+ + N_-$. Recall the notation introduced in the discussion after (4.1) where we write segments of $S_{\widetilde{N}}$ as intervals modulo $\widetilde{N}\mathbb{Z}$. We write

$$S_{\widetilde{N}} = J \cup I_N$$

where $J = [-N_-, N_+[$ is naturally defined both as a subset of $S_{\widetilde{N}}$ and of $\mathbb{Z}$. For I_N we write

$$I_N = S_{\widetilde{N}}\backslash J \equiv [N_+, -N_- - 1] \subset S_{\widetilde{N}}.$$

Moreover, we will use the notation $a + J = [a - N_-, a + N_+[$, $a \in S_{\widetilde{N}}$.

Next, suppose that $z \in \Omega$ and let

$$(p(\tau)-z)u = v \quad \text{on } S_{\widetilde{N}}, \quad \text{with } \operatorname{supp} v \subset J. \tag{4.30}$$

Fix $a_+, a_- \in S_{\widetilde{N}}\backslash J$, so that

$$N_+ + N_- + 1 \leq \operatorname{dist}_{S_{\widetilde{N}}}(a_+, N_+ - 1) = \operatorname{dist}_{S_{\widetilde{N}}}(a_+, J) = \mathcal{O}(1) \tag{4.31}$$

and

$$N_+ + N_- + 1 \leq \operatorname{dist}_{S_{\widetilde{N}}}(a_-, -N_-) = \operatorname{dist}_{S_{\widetilde{N}}}(a_-, J) = \mathcal{O}(1). \tag{4.32}$$

Notice that

$$(a_\pm + [-N_- - N_+, N_+ + N_-]) \cap J = \emptyset. \tag{4.33}$$

By (4.30) we see that

$$(p(\tau)-z)\mathbf{1}_{[a_-,a_+[}\, u = \begin{cases} 0, \text{ on } S_{\widetilde{N}}\backslash[a_- - N_-, a_+ + N_+[\,, \\ v, \text{ on } [a_- + N_+, a_+ - N_-[\,, \\ w_-, \text{ on } a_- + J\,, \\ w_+, \text{ on } a_+ + J\,, \end{cases} \tag{4.34}$$

where $w_\pm \in \ell^2(S_{\widetilde{N}})$ and $\operatorname{supp} w_\pm \subset a_\pm + J$. Since $\operatorname{supp} v \subset J$, we see by (4.31), (4.32) and (4.34), that

$$(p(\tau)-z)\mathbf{1}_{[a_-,a_+[}\, u = v + w_+ + w_-\,, \tag{4.35}$$

and

$$\|w_\pm\| \leq \mathcal{O}(1)\|\mathbf{1}_{a_\pm+[-N_+-N_-,N_++N_-[}\, u\|. \tag{4.36}$$

Next, write

$$\tau^{-N_+}(p(\tau)-z)\mathbf{1}_{[a_-,a_+[}\, u = \tau^{-N_+}(v + w_+ + w_-) \tag{4.37}$$

and

$$\tau^{N_-}(p(\tau)-z)\mathbf{1}_{[a_-,a_+[}\, u = \tau^{N_-}(v + w_+ + w_-). \tag{4.38}$$

We will use these two equations to estimate $\|\mathbf{1}_{[0,N_+[}\, u\|$, when $N_+ \geq 1$, and $\|\mathbf{1}_{[-N_-,0[}\, u\|$, when $N_- \geq 1$.

In view of (1.1), (3.9), we see that $\tau^{-N_+}(p(\tau)-z)$ is upper triangular with a non-vanishing constant entry at the diagonal. Since $\operatorname{supp} \tau^{-N_+}(v + w_+ + w_-) \subset [a_- - N_+ - N_-, a_+[$ and $\operatorname{supp} \mathbf{1}_{[a_-,a_+[}\, u \subset [a_-, a_+[$, we see that

$$\|\mathbf{1}_{[0,a_+[}\, u\| \leq \mathcal{O}(1)\|\mathbf{1}_{[0,a_+[}\tau^{-N_+}(v + w_+ + w_-)\|, \tag{4.39}$$

where the constant is uniform in $z \in \Omega$ and independent of N. Here,

$$\mathbf{1}_{[0,a_+[}\tau^{-N_+}(v + w_+ + w_-) = \mathbf{1}_{[0,a_+[}\tau^{-N_+} w_+\,,$$

so, by (4.39), (4.36),

$$\|\mathbf{1}_{[0,a_+[}u\| \le \mathcal{O}(1)\|\mathbf{1}_{a_++[-N_+-N_-,N_++N_-[}u\|$$

which, using (4.31), implies

$$\|\mathbf{1}_{[0,a_++(N_++N_-)[}u\| \le \mathcal{O}(1)\|\mathbf{1}_{[N_+,a_++N_++N_-[}u\|. \tag{4.40}$$

Notice that when $N_+=0$ this holds trivially.

When $N_-\ge 1$, we use that $\tau^{N_-}(p(\tau)-z)$ is lower triangular with a non-vanishing constant entry at the diagonal. In (4.38) we have that $\operatorname{supp}\tau^{N_-}(v+w_++w_-)\subset[a_-,a_++N_++N_-[$ and $\operatorname{supp}\mathbf{1}_{[a_-,a_+[}u\subset[a_-,a_+[$. We therefore deduce that

$$\|\mathbf{1}_{[a_-,0[}u\| \le \mathcal{O}(1)\|\mathbf{1}_{[a_-,0[}\tau^{N_-}(v+w_++w_-)\|, \tag{4.41}$$

where the constant is uniform in $z\in\Omega$ and independent of N. Since

$$\mathbf{1}_{[a_-,0[}\tau^{N_-}(v+w_++w_-)=\mathbf{1}_{[a_-,0[}\tau^{N_-}w_-,$$

we obtain by (4.41), (4.36), (4.32) that

$$\|\mathbf{1}_{[a_--(N_++N_-),0[}u\| \le \mathcal{O}(1)\|\mathbf{1}_{[a_--(N_++N_-),-N_-[}u\|, \tag{4.42}$$

which holds trivially when $N_-=0$.

Combining (4.40), (4.42) gives

$$\|\mathbf{1}_{[a_--(N_++N_-),a_++(N_++N_-)[}u\| \le \mathcal{O}(1)\|\mathbf{1}_{[a_--(N_++N_-),a_++(N_++N_-)[\backslash J}u\|. \tag{4.43}$$

Since $v=(p(\tau)-z)u$ is supported in J, we have that

$$\begin{aligned}\|v\| &\le \mathcal{O}(1)\|\mathbf{1}_{[-(N_++N_-),(N_++N_-)[}u\|\\ &\le \mathcal{O}(1)\|\mathbf{1}_{[a_--(N_++N_-),a_++(N_++N_-)[}u\|,\end{aligned}$$

where the constant in the estimate is uniform in $z\in\Omega$ and independent of N. Combining this with (4.43) shows that

$$\|v\| \le \mathcal{O}(1)\|\mathbf{1}_{[a_--(N_++N_-),a_++(N_++N_-)[\backslash J}u\|.$$

Now suppose that $z\in\Omega\backslash\mathrm{Spec}(P_{S_{\widetilde{N}}})$ and recall from (4.9), (4.11), that when $u\in\ell^2(S_{\widetilde{N}})$, we have that

$$u=\mathcal{E}_N(z)v,\quad \text{with } v=\mathbf{1}_Jv_+,\ v_+\in\ell^2(J). \tag{4.44}$$

Hence, by (4.11), $u=E_+^Nv_+$ on $I_N=S_{\widetilde{N}}\backslash J$. Thus,

$$\begin{aligned}\|v_+\| &\le \mathcal{O}(1)\|\mathbf{1}_{[a_--(N_++N_-),a_++(N_++N_-)[\backslash J}E_+^Nv_+\|\\ &\le \mathcal{O}(1)\|\mathbf{1}_{I_N}E_+^Nv_+\|,\end{aligned}$$

where the constant in the estimate is uniform in $z\in\Omega$ and independent of N. This concludes the proof for the singular values of E_+^N. The proof of the statement for E_-^N follows exactly the same lines using $(E_-^N)^*$ instead of E_+^N.

The proof of the statement in the case (2), when $N=\infty$, is similar, using that $S_\infty\simeq\mathbb{Z}=]-\infty,N_-[\cup[N_-,N_+[\cup[N_+,+\infty[$. $\square$

5. A Grushin Problem for the perturbed operator

Our aim is to study the following random perturbation of $P_0=P_{\mathrm{I_N}}$:

$$P_N^\delta \overset{\mathrm{def}}{=} P_N^0+\delta Q_\omega,\quad Q_\omega=(q_{j,k}(\omega))_{1\le j,k\le N}, \tag{5.1}$$

where $0\le\delta\ll1$ and $q_{j,k}(\omega)$ are independent and identically distributed complex Gaussian random variables, following the complex Gaussian law $\mathcal{N}_{\mathbb{C}}(0,1)$. Here, $1\ll N<\infty$. Consider the space $\mathcal{H}_N\overset{\mathrm{def}}{=}(\mathbb{C}^{N\times N},\|\cdot\|_{\mathrm{HS}})$ of $N\times N$ complex valued matrices equipped with the Hilbert-Schmidt norm. We equip $\mathcal{H}_N$ with the probability measure

$$\mu_N(dQ)\overset{\mathrm{def}}{=}\pi^{-N^2}\mathrm{e}^{-\|Q\|^2_{\mathrm{HS}}}L(dQ), \tag{5.2}$$

where $L(dQ)$ denotes the Lebesgue measure on $\mathcal{H}_N$. For $C_1 > 0$, let $\mathcal{Q}_{C_1N} \subset \mathcal{H}_N$ be the subset where

$$\|Q\|_{\mathrm{HS}} \leq C_1 N. \tag{5.3}$$

Markov's inequality [Ka97, Lemma 3.1] implies that if $C_1 > 0$ is large enough,

$$\mathbb{P}\left[\|Q_\omega\|_{\mathrm{HS}} \leq C_1 N\right] = \mu_N(\mathcal{Q}_{C_1N}) \geq 1 - e^{-N^2}. \tag{5.4}$$

5.1. **A general discussion.** We begin with a formal discussion of a Grushin problem for the perturbed operator P_δ. Recall from Section 4 that the Grushin problem for the unperturbed operator is of the form

$$\mathcal{P}_0 = \begin{pmatrix} P_0 - z & R_- \\ R_+ & R_{+-} \end{pmatrix} : \ \ell^2(I_N) \times \ell^2(J) \longrightarrow \ell^2(I_N) \times \ell^2(J),$$

We added a subscript 0 to indicate that we deal with the unperturbed operator. Suppose that $\mathcal{P}_0$ is bijective with inverse

$$\mathcal{E}_0 = \begin{pmatrix} E^0 & E^0_+ \\ E^0_- & E^0_{-+} \end{pmatrix} : \ \ell^2(I_N) \times \ell^2(J) \longrightarrow \ell^2(I_N) \times \ell^2(J),$$

where we added a superscript 0 for the same reason. Supposing that

$$\|\delta Q_\omega\| \|E^0\| < 1, \tag{5.5}$$

we see by a Neumann series argument that

$$\mathcal{P}_\delta \stackrel{\mathrm{def}}{=} \begin{pmatrix} P_\delta - z & R_- \\ R_+ & R_{+-} \end{pmatrix} : \ \ell^2(I_N) \times \ell^2(J) \longrightarrow \ell^2(I_N) \times \ell^2(J),$$

is bijective and admits the inverse

$$\mathcal{E}_\delta = \begin{pmatrix} E^\delta & E^\delta_+ \\ E^\delta_- & E^\delta_{-+} \end{pmatrix} : \ \ell^2(I_N) \times \ell^2(J) \longrightarrow \ell^2(I_N) \times \ell^2(J),$$

where

$$\begin{aligned} E^\delta_+ &= (1 + E^0(\delta Q_\omega))^{-1} E^0_+, \\ E^\delta_- &= E^0_-(1 + \delta Q_\omega E^0)^{-1}, \\ E^\delta &= E^0(1 + \delta Q_\omega E^0)^{-1}, \\ E^\delta_{-+} &= E^0_{-+} - E^0_- \delta Q_\omega (1 + E^0(\delta Q_\omega))^{-1} E^0_+. \end{aligned} \tag{5.6}$$

One obtains the following estimates

$$\begin{aligned} &\|E^\delta\| \leq \frac{\|E^0\|}{1 - \|\delta Q_\omega\|\|E^0\|}, \ \|E^\delta_\pm\| \leq \frac{\|E^0_\pm\|}{1 - \|\delta Q_\omega\|\|E^0\|}, \\ &\|E^\delta_{-+} - E^0_{-+}\| \leq \frac{\|E^0_+\|\|E^0_-\|\|\delta Q_\omega\|}{1 - \|\delta Q_\omega\|\|E^0\|}. \end{aligned} \tag{5.7}$$

Differentiating the equation $\mathcal{E}^\delta \mathcal{P}^\delta = 1$ with respect to δ yields

$$\partial_\delta \mathcal{E}^\delta = -\mathcal{E}^\delta(\partial_\delta \mathcal{P}^\delta)\mathcal{E}^\delta = -\begin{pmatrix} E^\delta Q_\omega E^\delta & E^\delta Q_\omega E^\delta_+ \\ E^\delta_- Q_\omega E^\delta & E^\delta_- Q_\omega E^\delta_+ \end{pmatrix}. \tag{5.8}$$

Integrating this relation from 0 to δ yields

$$\|E^\delta - E^0\| \leq \frac{\|\delta Q_\omega\|\|E^0\|^2}{(1 - \|\delta Q_\omega\|\|E^0\|)^2}, \ \|E^\delta_\pm - E^0_\pm\| \leq \frac{\|\delta Q_\omega\|\|E^0_\pm\|\|E^0\|}{(1 - \|\delta Q_\omega\|\|E^0\|)^2}. \tag{5.9}$$

Since $\mathcal{P}^\delta$ is invertible and of finite rank, we know that

$$|\partial_\delta \ln \det \mathcal{P}^\delta| = |\mathrm{tr}(\mathcal{E}^\delta \partial_\delta \mathcal{P}^\delta)|.$$

Letting $\|\cdot\|_{\mathrm{tr}}$ denote the trace class norm, we get

$$|\partial_\delta \ln \det \mathcal{P}^\delta| = |\mathrm{tr}(Q_\omega E^\delta)| \leq \|Q_\omega\|_{\mathrm{tr}} \|E^\delta\| \leq \frac{\|E^0\|\|Q_\omega\|_{\mathrm{tr}}}{1 - \|\delta Q_\omega\|\|E^0\|}, \tag{5.10}$$

where $\|Q_\omega\|_{\mathrm{tr}} \le N^{1/2}\|Q_\omega\|_{\mathrm{HS}}$. Integration from 0 to δ yields

$$\left|\ln|\det \mathcal{E}^\delta| - \ln|\det \mathcal{E}^0|\right| = \left|\ln|\det \mathcal{P}^\delta| - \ln|\det \mathcal{P}^0|\right| \le \frac{\|E^0\|\|\delta Q_\omega\|_{\mathrm{tr}}}{1-\|\delta Q_\omega\|\|E^0\|}. \tag{5.11}$$

Sharpening the assumption (5.5) to

$$\|\delta Q_\omega\|\|E^0\| < \frac{1}{2}, \tag{5.12}$$

we get

$$\|E^\delta\| \le 2\|E^0\|,\ \|E^\delta_\pm\| \le 2\|E^0_\pm\|,\ \|E^\delta_{-+} - E^0_{-+}\| \le 2\|E^0_+\|\|E^0_-\|\|\delta Q_\omega\|. \tag{5.13}$$

By (5.8) we know that $\partial_\delta E^\delta_{-+} = -E^\delta_- Q_\omega E^\delta_+$. Therefore, using (5.7), (5.9) and (5.13) we get

$$\begin{aligned}\|\partial_\delta E^\delta_{-+} + E^0_- Q_\omega E^0_+\| &\le \|E^0_- Q_\omega\|\|E^\delta_+ - E^0_+\| + \|Q_\omega E^\delta_+\|\|E^\delta_- - E^0_-\| \\ &\le 12\delta\|Q_\omega\|^2\|E^0_-\|\|E^0_+\|\|E^0\|.\end{aligned} \tag{5.14}$$

By integration from 0 to δ, we conclude

$$E^\delta_{-+} = E^0_{-+} - E^0_-(\delta Q_\omega)E^0_+ + \mathcal{O}(\|\delta Q_\omega\|^2\|E^0_-\|\|E^0_+\|\|E^0\|). \tag{5.15}$$

5.2. **A Grushin problem for the perturbed operator.** Recall from (4.9) that

$$\mathcal{P}_N(z) = (P_{S_{\widetilde{N}}} - z), \quad \widetilde{N} = N + N_- + N_+$$

and from (3.31) that its spectrum is equal to $p(\widehat{S}_{\widetilde{N}})$. Suppose that $z \notin \mathrm{Spec}(P_{S_{\widetilde{N}}})$. As in (4.11), $\mathcal{P}_N(z)$ is invertible with bounded inverse $\mathcal{E}_N(z)$.

Suppose that

$$\mathrm{dist}(z, \mathrm{Spec}(P_{S_{\widetilde{N}}})) \ge \frac{1}{CN} \tag{5.16}$$

for some fixed sufficiently large constant $C > 1$ to be determined later on. Since the operator $\mathcal{P}_N(z)$ is normal, it follows that

$$\|\mathcal{E}_N(z)\| = \frac{1}{\mathrm{dist}(z, \mathrm{Spec}(P_{S_{\widetilde{N}}}))}. \tag{5.17}$$

In particular

$$\|E^N(z)\|, \|E^N_-(z)\|, \|E^N_+(z)\|, \|E^N_{-+}(z)\| \le \frac{1}{\mathrm{dist}(z, \mathrm{Spec}(P_{S_{\widetilde{N}}}))}. \tag{5.18}$$

Suppose that

$$0 < \delta \ll N^{-2}. \tag{5.19}$$

Then, by (5.4), (5.18), (5.16), with probability $\ge 1 - \mathrm{e}^{-N^2}$, the assumption (5.12) is satisfied. Therefore, by the discussion in Section 5.1 we conclude

Proposition 5.1. *With probability $\ge 1 - \exp(-N^2)$ we have: Suppose (5.16), (5.19). Let $\mathcal{P}^0_N(z) = \mathcal{P}_N(z)$ be as in (4.9) and let $\mathcal{E}^0_N(z) = \mathcal{E}_N(z)$ be as in (4.11). Then,*

$$\mathcal{P}^\delta_N(z) \stackrel{\mathrm{def}}{=} \begin{pmatrix} P^\delta_{I_N} - z & R^N_- \\ R^N_+ & R^N_{+-}(z)\end{pmatrix} : \ell^2(I_N)\oplus\ell^2(J) \to \ell^2(I_N)\oplus\ell^2(J)$$

is bijective with bounded inverse

$$\mathcal{E}^\delta_N(z) = \begin{pmatrix} E^{N,\delta}(z) & E^{N,\delta}_+(z) \\ E^{N,\delta}_-(z) & E^{N,\delta}_{-+}(z)\end{pmatrix} : \ell^2(I_N)\oplus\ell^2(J) \to \ell^2(I_N)\oplus\ell^2(J).$$

Moreover,

$$\|E^{N,\delta}(z)\|, \|E^{N,\delta}_-(z)\|, \|E^{N,\delta}_+(z)\|, \|E^{N,\delta}_{-+}(z)\| \le \frac{2}{\mathrm{dist}(z, \mathrm{Spec}(P_{S_{\widetilde{N}}}))}.$$

5.3. **A lower bound on the determinant of the effective Hamiltonian.** Suppose that $\Omega \Subset \mathbb{C}$ is a compact set. Let $E_{-+}^{N,\delta}$ be as in Proposition 5.1. In this section we are interested in estimating the probability that $\log|\det E_{-+}^{N,\delta}(z)| \leq a$ for $a \in \mathbb{R}$ and for some $z \in \Omega \backslash p(\widehat{S}_{\widetilde{N}})$ which may depend on N. To obtain this bound we will adapt the approach developed in [HaSj08, Section 9].

Set

$$\alpha = \alpha(z;N) \stackrel{\text{def}}{=} \operatorname{dist}(z, p(\widehat{S}_{\widetilde{N}})). \tag{5.20}$$

Until further notice we suppose that

$$\alpha \geq \frac{1}{CN^{\kappa}}, \quad \text{for some } C > 1, \tag{5.21}$$

where $\kappa \geq 1$ is fixed, and we strengthen assumption (5.19) to

$$0 < \delta \ll N^{-1}\min(\alpha, N^{-1}). \tag{5.22}$$

Recall Proposition 5.1 and (5.6). We want to study the map

$$\begin{aligned} \mathcal{Q}_{C_1 N} \ni Q \mapsto E_{-+}^{\delta}(z,Q) &= E_{-+}^{0}(z) - \delta E_{-}^{0}(z)\left(Q + \sum_{1}^{\infty}(-\delta)^n Q(E^0(z)Q)^n\right)E_{+}^{0}(z) \\ &\stackrel{\text{def}}{=} E_{-+}^{0}(z) - \delta E_{-}^{0}(z)(Q + T(z,Q,\delta,N))E_{+}^{0}(z) \end{aligned} \tag{5.23}$$

where by (5.18), (5.3),

$$\|T\|_{\text{HS}} \leq \mathcal{O}\left(\frac{\delta(C_1 N)^2}{\alpha}\right). \tag{5.24}$$

Next, recall (5.2), and notice that the measure μ_N is invariant under the left and right action of the group of unitary matrices $\mathcal{U}(N,\mathbb{C})$ on $\mathcal{H}_N$, i.e. for any $U, V \in \mathcal{U}(N,\mathbb{C})$, we have that

$$\mu_N(d(UQV)) = \mu_N(dQ). \tag{5.25}$$

Furthermore, the left and right action of the group of unitary matrices $\mathcal{U}(N,\mathbb{C})$ leaves $\mathcal{Q}_{C_1 N}$ invariant, see (5.3), and therefore also the probability (5.4). Thus, we may choose any orthonormal bases (ONB) to represent the matrix $Q \in \mathcal{H}_N$. Let $\widetilde{e}_1, \dots, \widetilde{e}_N$ and $\widehat{e}_1, \dots, \widehat{e}_N$ be two orthonormal bases of $\mathbb{C}^N$ and write

$$Q = \sum_{i,j=1}^{N} q_{i,j}\, \widetilde{e}_i \circ \widehat{e}_j^*, \quad \text{where } q_{i,j} \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ (iid)}. \tag{5.26}$$

By Lemma 4.2 and (5.21), we have for a compact set $\Omega \Subset \mathbb{C}$ and for $z \in \Omega \backslash p(\widehat{S}_{\widetilde{N}})$, the following bound on the singular values of $E_{\pm}^{N}$

$$\frac{1}{C} \leq s_j^{N,\pm} \leq \frac{1}{\alpha}, \quad j = 1, \dots, |J| = N_+ + N_-, \tag{5.27}$$

where the constant $C > 0$ is uniform in $z \in \Omega$ and independent of N.

By the polar decomposition we write $E_{+}^{0} = S_+ D_+$ where $S_+ : \mathbb{C}^{|J|} \to \mathbb{C}^N$ is an isometry, with $S_+^* S_+ = 1$ and $S_+ S_+^*$ is the orthogonal projection $\mathbb{C}^N \to \mathcal{R}(E_+^0)$, and $D_+ : \mathbb{C}^{|J|} \to \mathbb{C}^{|J|}$ is selfadjoint with eigenvalues $s_1^+, \dots, s_{|J|}^+$. Similarly,

$$(E_{-}^{0})^* = S_- D_-, \quad E_{-}^{0} = D_- S_-^*, \tag{5.28}$$

where $S_- : \mathbb{C}^{|J|} \to \mathbb{C}^N$ is an isometry, with $S_-^* S_- = 1$ and $S_- S_-^*$ is the orthogonal projection $\mathbb{C}^N \to \mathcal{R}((E_-^0)^*)$, and $D_- : \mathbb{C}^{|J|} \to \mathbb{C}^{|J|}$ is selfadjoint with eigenvalues $s_1^-, \dots, s_{|J|}^-$.

From (5.42), we get

$$\begin{aligned} E_{-+}^{\delta} &= E_{-+}^{0} - \delta D_- S_-^*(Q + T)S_+ D_+ \\ &= D_-\big(\widehat{E}_{-+}^{0} - \delta(S_-^* Q S_+ + S_-^* T S_+)\big)D_+, \end{aligned} \tag{5.29}$$

where $\widehat{E}^0_{-+} = D_-^{-1}E^0_{-+}D_+^{-1}$. Moreover, set

$$\widehat{T} = S_-^* T S_+. \tag{5.30}$$

View $\mathbb{C}^{|J|}$ as a subspace of $\mathbb{C}^N$ by considering that $J \subset \{1,\dots,N\}$. Let $\Pi_0 : \mathbb{C}^N \to \mathbb{C}^{|J|}$ be the orthogonal projection and, whenever convenient, view Π_0 as the inclusion map $\Pi_0 : \mathbb{C}^{|J|} \hookrightarrow \mathbb{C}^N$. Let $\mathcal{S}_+ : \mathbb{C}^N \to \mathbb{C}^N$ be unitary with $\mathcal{S}_+|_{\mathbb{C}^{|J|}} = S_+$ and similarly for $\mathcal{S}_-$. Then,

$$S_+ = \mathcal{S}_+\Pi_0, \tag{5.31}$$

where Π_0 is viewed as a map $\mathbb{C}^{|J|} \to \mathbb{C}^N$. Similarly,

$$S_- = \mathcal{S}_-\Pi_0, \quad S_-^* = \Pi_0\mathcal{S}_-^* = \Pi_0 S_-^{-1}. \tag{5.32}$$

Then,

$$E^\delta_{-+} = D_-\big(\widehat{E}^0_{-+} - \delta(\Pi_0\widehat{Q}\Pi_0 + \widehat{T})\big)D_+, \quad \widehat{Q} = \mathcal{S}_-^* Q\mathcal{S}_+. \tag{5.33}$$

Let $\delta_j \in \mathbb{C}^N$, with $\delta_j(i) = 1$ if $i = j$ and $= 0$ else, denote the standard ONB of $\mathbb{C}^N$. For $k = 1,\dots,N$ set

$$\widehat{e}_k \stackrel{\mathrm{def}}{=} \mathcal{S}_+\delta_k, \quad \widetilde{e}_k \stackrel{\mathrm{def}}{=} \mathcal{S}_-^*\delta_k$$

in (5.26). Hence,

$$\widehat{Q} = \mathcal{S}_-^* Q\mathcal{S}_+ = (q_{j,k})_{1\le j,k\le N}$$

where $q_{j,k} \sim \mathcal{N}_{\mathbb{C}}(0,1)$ are independent and identically distributed complex Gaussian random variables.

By (5.30), (5.31) and (5.32), we see that $\widehat{T}(Q) = \Pi_0\widehat{T}(Q)\Pi_0$ and that the map $\mathcal{H}_N \ni Q \mapsto \widehat{T}(Q) \in \mathcal{H}_{|J|}$ satisfies

$$\|\widehat{T}(Q)\|_{\mathrm{HS}} \le \mathcal{O}\left(\frac{\delta(C_1N)^2}{\alpha}\right). \tag{5.34}$$

where the estimate is uniform in $Q \in \mathcal{Q}_{C_1N}$.

By (5.33)

$$\mathcal{Q}_{C_1N} \ni Q \mapsto \det E^\delta_{-+}(z,Q) = \prod_{k=1}^{|J|}(s_k^+ s_k^-)\det\left(\widehat{E}^0_{-+}(z) - \delta(\Pi_0\widehat{Q}\Pi_0 + \widehat{T}(Q))\right) \tag{5.35}$$

Recall from (5.28) and from the discussion after (5.27) that s_k^+ (resp. s_k^-) denote the singular values of E^0_+ (resp. $(E^0_-)^*$).

The Cauchy inequalities and (5.34) imply that

$$\|d_Q\widehat{T}\|_{\mathcal{H}_N\to\mathcal{H}_{|J|}} \le \mathcal{O}\left(\frac{\delta C_1N}{\alpha}\right), \tag{5.36}$$

uniformly for $Q \in \mathcal{Q}_{C_1N}$. Technically, we can only apply the Cauchy inequalities in $\|Q\|_{\mathrm{HS}} \le \eta C_1N$ for some $\eta \in]0,1[$. However, we have room for that if we start with a slightly large parameter $C_1 > 0$ to begin with and then restrict to a $C_1 > 0$ such that (5.36) and (5.4) hold.

Next, we define the maps

$$\begin{aligned} \kappa : \mathcal{H}_N \supset \mathcal{Q}_{C_1N} &\longrightarrow \kappa(\mathcal{Q}_{C_1N}) \subset \mathcal{H}_N \\ Q &\longmapsto \kappa(Q) \stackrel{\mathrm{def}}{=} \widehat{Q} + \widehat{T}(Q), \end{aligned} \tag{5.37}$$

where we identify $\widehat{T}(Q)$ with its image in $\mathcal{H}_N$ under the natural inclusion map $\mathcal{H}_{|J|} \hookrightarrow \mathcal{H}_N$, which has the left inverse

$$\widetilde{\Pi}_0 : \ \mathcal{H}_N \to \mathcal{H}_{|J|} : \ Q \mapsto \widetilde{\Pi}_0(Q) \stackrel{\mathrm{def}}{=} \Pi_0 Q\Pi_0 \tag{5.38}$$

Moreover, we define the map $\Pi : \mathcal{H}_N \supset \mathcal{Q}_{C_1N} \to \mathcal{H}_{|J|}$ by

$$\Pi \stackrel{\text{def}}{=} \widetilde{\Pi}_0 \circ \kappa. \tag{5.39}$$

In analogy with (5.2) we define the probability measure μ_J on $\mathcal{H}_{|J|}$ by

$$\mu_J(dQ) \stackrel{\text{def}}{=} \pi^{-|J|^2} \mathrm{e}^{-\|Q\|^2_{\mathrm{HS}}} L(dQ). \tag{5.40}$$

We will estimate the probability

$$\mu_N \left(\log|\det E^{\delta}_{-+}(z,Q)|^2 \leq a \text{ and } Q \in \mathcal{Q}_{C_1N} \right). \tag{5.41}$$

To begin, we strengthen (5.22) to

$$0 < \delta \ll \frac{\alpha}{(C_1N)^3}. \tag{5.42}$$

By (5.36), (5.37), we see that κ is injective, since for $Q_1, Q_2 \in \mathcal{Q}_{C_1N}$

$$\begin{aligned}\|\kappa(Q_1) - \kappa(Q_2)\| &\geq \|Q_1 - Q_2\| - \int_0^1 \|d_Q\widehat{T}(tQ_1 + (1-t)Q_2)\| \cdot \|Q_1 - Q_2\| dt \\ &\geq \left(1 - \mathcal{O}\left(\frac{\delta C_1 N}{\alpha}\right)\right) \|Q_1 - Q_2\|.\end{aligned}$$

Define the restricted measure

$$(\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N)(A) \stackrel{\text{def}}{=} \mu_N(A \cap \mathcal{Q}_{C_1N}), \quad \forall A \in \mathcal{B}(\mathcal{H}_N), \tag{5.43}$$

where $\mathcal{B}(\mathcal{H}_N)$ denotes the Borel σ-algebra of $\mathcal{H}_N$. In view of the discussion after (5.24), the measure $\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N$ is invariant under the change of orthonormal basis of $\mathcal{Q}_{C_1N}$. Thus, by (5.39), (5.35), the probability in (5.41) is equal to

$$\begin{aligned}&(\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N) \left[\log \left| \prod_{k=1}^{|J|} (s_k^+ s_k^-) \det\left(\widehat{E}^0_{-+}(z) - \delta(\Pi_0 \widehat{Q} \Pi_0 + \widehat{T}(Q))\right)\right|^2 \leq a \right] \\ &= (\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N) \left[\log \left| \det(\delta^{-1}\widehat{E}^0_{-+}(z) - \Pi(Q))\right|^2 \leq b \right] \\ &= \Pi_*(\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N) \left[\log|\det(\delta^{-1}\widehat{E}^0_{-+}(z) - Q')|^2 \leq b \right],\end{aligned} \tag{5.44}$$

where by (5.35), (5.27),

$$\begin{aligned} b &= a - 2|J|\log\delta - 2\sum_{j=1}^{|J|} \log(s_j^+ s_j^-) \\ &\leq a - 2|J|\log\delta + 4|J|\log C. \end{aligned} \tag{5.45}$$

Continuing, we will estimate the measure $\Pi_*(\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N)$. We begin by studying the Jacobian of κ, (5.37). By (5.36) and (5.42), we see that the differential of $\widehat{T}$ is bounded with norm $\ll 1$. Moreover, since the rank of $d_Q\widehat{T}$ is bounded by $|J|^2$, it follows that $\|d_Q\widehat{T}\|_{\mathrm{tr}} \leq |J|^2 \|d_Q\widehat{T}\|$. Thus, by (5.36)

$$\begin{aligned}\det\frac{\partial\kappa}{\partial Q} &= \det\left(1 + d_Q\widehat{T}\right) \\ &= 1 + \mathcal{O}(\|d_Q\widehat{T}\|_{\mathrm{tr}}) \\ &= 1 + \mathcal{O}\left(\frac{\delta C_1 N}{\alpha}\right),\end{aligned} \tag{5.46}$$

where in the last line we used as well that $|J|$ is a constant independent of N.

Since κ is a holomorphic map, it follows that

$$\begin{aligned} L(d\kappa(Q)) &= \left|\det \frac{\partial \kappa}{\partial Q}\right|^2 L(dQ) \\ &= \left(1+\mathcal{O}\left(\frac{\delta C_1 N}{\alpha}\right)\right) L(dQ). \end{aligned} \tag{5.47}$$

Next, we see by (5.37), (5.34), that for $Q \in \mathcal{Q}_{C_1N}$

$$\begin{aligned} \left|\|\kappa(Q)\|_{\mathrm{HS}}^2 - \|Q\|_{\mathrm{HS}}^2\right| &= |\|\kappa(Q)\|_{\mathrm{HS}} - \|Q\|_{\mathrm{HS}}| \left(\|\kappa(Q)\|_{\mathrm{HS}} + \|Q\|_{\mathrm{HS}}\right) \\ &\leq \|\kappa(Q) - Q\|_{\mathrm{HS}}(\|\kappa(Q)\|_{\mathrm{HS}} + \|Q\|_{\mathrm{HS}}) \\ &= \mathcal{O}\left(\frac{\delta (C_1N)^2}{\alpha}\right)\left(C_1 N + \mathcal{O}\left(\frac{\delta (C_1N)^2}{\alpha}\right)\right) \\ &= \mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right) \ll 1, \end{aligned}$$

which implies that on $\mathcal{Q}_{C_1N}$

$$\mathrm{e}^{-\|Q\|_{\mathrm{HS}}^2} = \left(1+\mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right)\right) \mathrm{e}^{-\|\kappa(Q)\|_{\mathrm{HS}}^2}. \tag{5.48}$$

(5.47), (5.48) imply that for any bounded continuous function $\varphi \in \mathcal{C}_b(\mathcal{H}_N;\mathbb{R}_+)$ with values in $\mathbb{R}_+$,

$$\begin{aligned} \int \varphi\, \kappa_*(\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N) &= \int_{\mathcal{Q}_{C_1N}} \varphi(\kappa(Q))\mu_N(dQ) \\ &= \left(1+\mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right)\right) \int_{\mathcal{Q}_{C_1N}} \varphi(\kappa(Q))\mathrm{e}^{-\|\kappa(Q)\|_{\mathrm{HS}}^2} \frac{L(d\kappa(Q))}{\pi^{N^2}} \\ &= \left(1+\mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right)\right) \int_{\kappa(\mathcal{Q}_{C_1N})} \varphi(\widetilde{Q})\mathrm{e}^{-\|\widetilde{Q}\|_{\mathrm{HS}}^2} \frac{L(d\widetilde{Q})}{\pi^{N^2}}. \end{aligned}$$

Thus,

$$\kappa_*(\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N) = \left(1+\mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right)\right) \mathbf{1}_{\kappa(\mathcal{Q}_{C_1N})}\mu_N. \tag{5.49}$$

This, together with (5.39), implies that for any $\varphi \in \mathcal{C}_b(\mathcal{H}_{|J|};\mathbb{R}_+)$

$$\begin{aligned} \Pi_*(\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N)(\varphi) &= \int (\varphi\circ\widetilde{\Pi}_0)\, \kappa_*(\mathbf{1}_{\mathcal{Q}_{C_1N}}\mu_N) \\ &= \left(1+\mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right)\right) \int \varphi\circ\widetilde{\Pi}_0\, \mathbf{1}_{\kappa(\mathcal{Q}_{C_1N})}\mu_N \\ &\leq \left(1+\mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right)\right) \int \varphi\circ\widetilde{\Pi}_0\, \mu_N \\ &\leq \left(1+\mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right)\right) \int \varphi(Q')\, \mu_J(dQ'), \end{aligned}$$

where in the last line we used that $(\widetilde{\Pi}_0)_*\mu_N = \mu_J$. Hence, by (5.44) and a density argument, we deduce that the probability in (5.41) is

$$\leq \left(1+\mathcal{O}\left(\frac{\delta (C_1N)^3}{\alpha}\right)\right) \mu_J\left[\log|\det(\delta^{-1}\widehat{E}^0_{-+}(z) - Q')|^2 \leq b\right]. \tag{5.50}$$

The right hand side can be estimated by [HaSj08, Proposition 7.3].

Proposition 5.2. *Let $\mathbb{N} \ni M \geq 1$, let μ_M be the Gaussian measure on $\mathcal{H}_M$ defined in (5.2). Then, there exist constants $\widetilde{C}, C' > 0$ such that for any fixed (deterministic) matrix $D \in \mathcal{H}_M$*

$$\begin{aligned} \mu_M(\log|\det(D+Q)|^2 \leq b) &\leq \mu_M(\log|\det Q)|^2 \leq b) \\ &\leq \widetilde{C} \exp\left[-\frac{1}{2}\left(C' + \left(M - \frac{1}{2}\right)\ln M - 2M - b\right)\right], \end{aligned}$$

when $b \leq C' + \left(M + \frac{1}{2}\right)\ln M - 2M$.

Combining, (5.50), (5.41), (5.44), (5.45) and (5.27) with Proposition 5.2, we deduce that there exist constants $\widetilde{C}, C' > 0$ such that

$$\begin{aligned} &\mu_N(\{\log|\det E^{\delta}_{-+}(z,Q)|^2 \leq a\} \cap \mathcal{Q}_{C_1 N}) \\ &\leq \widetilde{C} \exp\left[-\frac{1}{2}\left(C' + \left(|J| - \frac{1}{2}\right)\ln|J| - 2|J| - b\right)\right] \\ &\leq \widetilde{C} \exp\left[-\frac{1}{2}\left(C' + \left(|J| - \frac{1}{2}\right)\ln|J| - 2|J| - a + 2|J|\log\delta - 4|J|\log C\right)\right] \end{aligned}$$

when $b \leq C' + \left(|J| + \frac{1}{2}\right)\ln|J| - 2|J|$ and thus, by (5.45), when

$$a \leq C' + \left(|J| + \frac{1}{2}\right)\ln|J| - 2|J| + 2|J|\log\delta - 4|J|\log C.$$

Here, the constants $\widetilde{C}, C'$ only depend on J and the constant C is given by the lower bounds in (5.27) which are uniform in $z \in \Omega$. Setting

$$\begin{aligned} C_0 &= C' + \left(|J| + \frac{1}{2}\right)\ln|J| - 2|J| - 4|J|\log C, \\ a &= -t, \end{aligned}$$

we conclude, by absorbing the factor $\mathrm{e}^{-\frac{1}{2}(C_0 - \log|J|)}$ into the constant $\widetilde{C}$, that

$$\mu_N(\{\log|\det E^{\delta}_{-+}(z,Q)|^2 \leq -t\} \cap \mathcal{Q}_{C_1 N}) \leq \widetilde{C} \exp\left[-\frac{1}{2}t - |J|\log\delta\right] \tag{5.51}$$

when $t \geq C_0 - 2|J|\log\delta$. Finally, since

$$\mathbb{P}[A^c \cap B] = \mathbb{P}[B] - \mathbb{P}[A \cap B],$$

where A^c denotes the complement of the measurable set A, we obtain, by combining (5.51)and (5.4),

Proposition 5.3. *Let $\kappa \geq 1$, let $\Omega \Subset \mathbb{C}$ be a compact set, let $C > 0$ and let $C_1 > 0$ be such that (5.4) holds. Then, there exist constants $C_0 \in \mathbb{R}$ and $C_2 > 0$, such that for any $z \in \Omega$, with*

$$\alpha(z; N) = \mathrm{dist}(z, p(\widehat{S}_{\widetilde{N}})) \geq \frac{1}{CN^{\kappa}},$$

we have that

$$\mathbb{P}\left[\log|\det E^{\delta}_{-+}(z,Q)|^2 \geq -t \text{ and } \|Q\|_{\mathrm{HS}} \leq C_1 N\right] \geq 1 - \mathrm{e}^{-N^2} - C_2\,\delta^{-|J|}\mathrm{e}^{-t/2},$$

when

$$t \geq C_0 - 2|J|\log\delta$$

and

$$0 < \delta \ll \frac{\alpha}{(C_1 N)^3}.$$

6. Counting eigenvalues

In this section we count the eigenvalues of the perturbed operator

$$P_N^\delta = P_N^0 + \delta Q_\omega, \tag{6.1}$$

near the curve $p(S^1)$, see also (5.1). Recall from (4.7) that $P_N^0 = P_{I_N}$, see also(4.9). Similarly, we have $P_N^\delta = P_{I_N}^\delta$ as in Proposition 5.1.

Until further notice, we will work in the restricted probability space where (5.3) holds (see also (5.4)) and work under the assumptions that

$$0 < \delta \ll \frac{\alpha}{N^3}, \quad \frac{1}{CN} \leq \alpha \leq \mathcal{O}(1), \tag{6.2}$$

for some sufficiently large constant $C > 0$ to be determined later on, see also (5.16), (5.42). Here α is as in (5.20).

Counting the number of eigenvalues of $P_{I_N}^\delta$ in some domain $\Omega \Subset \mathbb{C}$ is equivalent to counting the number of zeros of the holomorphic function $u(z;N) = \det(P_{I_N}^\delta - z)$ in Ω. The Shur complement formula and Proposition 5.1 imply that, away from $\mathrm{Spec}(P_{S_{\widetilde{N}}})$, $P_{I_N}^\delta - z$ is invertible if and only if $E_{-+}^{N,\delta}(z)$ is invertible, and that

$$\log|\det(P_{I_N}^\delta - z)| = \log|\det \mathcal{P}_N^\delta(z)| + \log|\det E_{-+}^{N,\delta}(z)|. \tag{6.3}$$

6.1. **Counting zeros of holomorphic functions of exponential growth.** We recall Theorem 1.1 in [Sj10], in a form somewhat adapted to our formalism:

1) Domains with associated Lipschitz weight. Let $N \geq 1$ be a large parameter, and let $\Omega \Subset \mathbb{C}$ be an open simply connected set with Lipschitz boundary $\omega = \partial\Omega$ which may depend on N. More precisely, we assume that $\partial\Omega$ is Lipschitz with an associated Lipschitz weight $r : \omega \to]0, +\infty[$, which is a Lipschitz function of modulus $\leq 1/2$, in the following way :

There exists a constant $C_0 > 0$ such that for every $x \in \omega$ there exist new affine coordinates $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2)$ of the form $\widetilde{y} = U(y - x)$, $y \in \mathbb{C} \simeq \mathbb{R}^2$ being the old coordinates, where $U = U_x$ is orthogonal, such that the intersection of Ω and the rectangle $R_x := \{y \in \mathbb{C}; |\widetilde{y}_1| < r(x), |\widetilde{y}_2| < C_0 r(x)\}$ takes the form

$$\{y \in R_x;\ \widetilde{y}_2 > f_x(\widetilde{y}_1), |\widetilde{y}_1| < r(x)\}, \tag{6.4}$$

where $f_x(\widetilde{y}_1)$ is Lipschitz on $[-r(x), r(x)]$, with Lipschitz modulus $\leq C_0$.

Remark 6.1. *Notice that (6.4) remains valid if we shrink the weight function r.*

2) Thickening of the boundary and choice of points. Define

$$\widetilde{\omega}_r = \bigcup_{x\in\omega} D(x, r(x))$$

and let $z_j^0 \in \omega$, $j \in \mathbb{Z}/M\mathbb{Z}$, with $M \in \mathbb{N}$ which may depend on N, be distributed along the boundary in the positively oriented sense such that

$$r(z_j^0)/4 \leq |z_{j+1}^0 - z_j^0| \leq r(z_j^0)/2.$$

Theorem 6.2 (Theorem 1.1 in [Sj10])**.** *Let $C_0 > 0$ be as in 1) above. There exists a constant $C_1 > 0$, depending only on C_0, such that if $z_j \in D(z_j^0, r(z_j^0)/(2C_1))$ we have the following :*

Let $N \geq 1$ and let ϕ be a continuous subharmonic function on $\widetilde{\omega}_r$ with a distributional extension to $\Omega \cup \widetilde{\omega}_r$, denoted by the same symbol. Then, there exists a constant $C_2 > 0$ such that if u is a holomorphic function on $\Omega \cup \widetilde{\omega}_r$ satisfying

$$\log|u| \ \leq N\phi \ \text{on } \widetilde{\omega}_r, \tag{6.5}$$

$$\log|u(z_j)| \ \geq N(\phi(z_j) - \varepsilon_j),\ \text{for } j = 1, \dots, M, \tag{6.6}$$

where $\varepsilon_j \geq 0$, then the number of zeros of u in Ω satisfies

$$\left|\#(u^{-1}(0)\cap\Omega) - \frac{N}{2\pi}\mu(\Omega)\right| \leq C_2 N\left(\mu(\widetilde{\omega}_r) + \sum_{j=1}^{M}\left(\varepsilon_j + \int_{D\left(z_j,\frac{r(z_j)}{4C_1}\right)}\left|\log\frac{|w-z_j|}{r(z_j)}\right|\mu(dw)\right)\right).$$

Here $\mu \stackrel{\text{def}}{=} \Delta\phi \in \mathcal{D}'(\Omega\cup\widetilde{\omega}_r)$ is a positive measure on $\widetilde{\omega}_r$ so that $\mu(\Omega)$ and $\mu(\widetilde{\omega}_r)$ are well-defined. Moreover, the constant $C_2 > 0$ only depends on C_0.

6.2. **Upper bound on** $\log|\det(P_{I_N}^\delta - z)|$**.** Recall from (3.31), that $\#(p(\widehat{S}_{\widetilde{N}})) = \widetilde{N}$ where $\widetilde{N} = N + N_- + N_+$. Then, define the subharmonic function

$$\phi(z) \stackrel{\text{def}}{=} \phi(z;N) \stackrel{\text{def}}{=} \frac{1}{N}\sum_{\lambda\in p(\widehat{S}_{\widetilde{N}})}\log|\lambda - z|. \tag{6.7}$$

Applying (5.11), (5.18), (6.2) to (6.3) we can express the contribution from the perturbed Grushin problem in (6.3) by the function ϕ and a small error term, i.e.

$$\begin{aligned}\log|\det(P_{I_N}^\delta - z)| &= \log|\det\mathcal{P}_N^0(z)| + \mathcal{O}(\delta\|Q_\omega\|_{\mathrm{tr}}\|E^{N,0}\|) + \log|\det E_{-+}^{N,\delta}(z)| \\ &= N\left(\phi(z) + \frac{\log|\det E_{-+}^{N,\delta}(z)|}{N} + \mathcal{O}\left(\frac{\delta\|Q_\omega\|_{\mathrm{HS}}}{N^{1/2}\alpha}\right)\right).\end{aligned} \tag{6.8}$$

In the last line we used that $\|Q_\omega\|_{\mathrm{tr}} \leq N^{1/2}\|Q_\omega\|_{\mathrm{HS}}$.

By (6.2), (5.4) we have that $\alpha^{-1}\delta\|Q_\omega\|_{HS} \ll N^{-2}$. Recall that the dimension of the matrix E_{-+}^δ is $|J| = N_+ + N_-$. Therefore, using (6.2), (5.18) and Proposition 5.1, we can bound (6.8) from above and get

$$\log|\det(P_{I_N}^\delta - z)| \leq N\left(\phi(z) + \mathcal{O}(N^{-1}|\log\alpha|) + \mathcal{O}(N^{-5/2})\right). \tag{6.9}$$

In conclusion, assuming (6.2), we have that

$$\log|\det(P_{I_N}^\delta - z)| \leq N\psi(z;N) \tag{6.10}$$

with probability $\geq 1 - \mathrm{e}^{-N^2}$. Here,

$$\psi(z;N) \stackrel{\text{def}}{=} \phi(z) + \frac{C\log N}{N}, \tag{6.11}$$

for some sufficiently large constant $C > 0$.

6.3. **Lower bound on** $\log|\det(P_{I_N}^\delta - z)|$**.** Fix a $\varepsilon_0 \in]0,1[$. By (6.2) and Proposition 5.3 we have for any z_0, satisfying

$$\alpha(z_0;N) \geq \frac{1}{CN},$$

that

$$\mathbb{P}\left[\log|\det E_{-+}^\delta(z_0,Q)|^2 \geq -N^{\varepsilon_0} \text{ and } \|Q\|_{\mathrm{HS}} \leq C_1 N\right] \geq 1 - \mathrm{e}^{-N^2} - C_2\delta^{-|J|}\mathrm{e}^{-\frac{1}{2}N^{\varepsilon_0}}, \tag{6.12}$$

for

$$\exp\left[\frac{C_0}{2|J|} - \frac{N^{\varepsilon_0}}{2|J|}\right] \leq \delta \ll \frac{\alpha(z_0;N)}{N^3}. \tag{6.13}$$

Thus, assuming (6.13) and combining (6.12), (6.8), (6.2) and (6.11), we get that $\|Q\|_{\mathrm{HS}} \leq C_1 N$ and

$$\log|\det(P_{I_N}^\delta - z_0)| \geq N\left(\psi(z_0;N) - CN^{\varepsilon_0 - 1}\right) \tag{6.14}$$

hold with probability

$$\geq 1 - \mathrm{e}^{-N^2} - C_2\delta^{-|J|}\mathrm{e}^{-\frac{1}{2}N^{\varepsilon_0}}. \tag{6.15}$$

6.4. **Counting eigenvalues in a fixed smooth domain.** Let $\Omega \Subset \mathbb{C}$ be an open simply connected set with smooth boundary $\partial\Omega$ which is independent of N. Moreover, suppose that (Ω1)–(Ω3) hold.

To estimate the number of zeros of $\det(P_{I_N} - z)$, see (6.3), in Ω, we will apply Theorem 6.2. The boundary $\partial\Omega$ is uniformly Lipschitz at scale

$$r(x) \stackrel{\text{def}}{=} \frac{1}{C}\left(\mathrm{dist}(x, p(S^1)) + \frac{1}{N}\right), \quad x \in \partial\Omega \tag{6.16}$$

which is Lipschitz of modulus $\leq 1/2$. Here, $C > 0$ is chosen sufficiently large, and we will potentially increase it later on.

Due to the singularities of ψ at $p(\widehat{S}_{\widetilde{N}})$, see (6.11), (6.7), we cannot in general assure that the weight function ψ (6.11) be continuous in

$$\bigcup_{x\in\partial\Omega} D(x, r(x)).$$

To remedy this problem we will consider two N-dependent perturbations of the boundary $\partial\Omega$: let $z_0 \in p(S^1)\cap\partial\Omega$ and pass to new affine coordinates $\widetilde{y} \in \mathbb{R}^2 \simeq \mathbb{C}$ (as in Section 6.1) so that the boundary $\partial\Omega$ is given by the graph of the smooth function f_{z_0} near 0, with derivative bounded by $C_0 > 0$. For $C' > 1$ and $N > 0$ sufficiently large, the intersection of $\partial\Omega$ with the rectangle

$$R_{z_0}(N) \stackrel{\text{def}}{=} \{y \in \mathbb{C} \simeq \mathbb{R}^2; |\widetilde{y}_1| \leq 1/(C'N),\ |\widetilde{y}_2| \leq 2C_0/(C'N)\} \tag{6.17}$$

takes the form

$$\{y \in \mathbb{C} \simeq \mathbb{R}^2; |\widetilde{y}_1| \leq 1/(C'N),\ \widetilde{y}_2 > f_{z_0}(\widetilde{y}_1)\}.$$

Here, $y \in \mathbb{C} \simeq \mathbb{R}^2$ denote the old coordinates and $\widetilde{y} \in \mathbb{C} \simeq \mathbb{R}^2$ denote the new ones.

Next, define the continuous function $\widetilde{\chi}$, supported in $[-1, 1]$ and of Lipschitz modulus 2, by

$$\widetilde{\chi}(x) = \begin{cases} 2(x+1),\ -1 \leq x < -1/2, \\ 1,\ |x| \leq 1/2, \\ 1 - 2(x - 1/2),\ 1/2 < x \leq 1, \end{cases}$$

and set

$$\chi(\widetilde{y}_1) \stackrel{\text{def}}{=} \chi(\widetilde{y}_1; N) \stackrel{\text{def}}{=} \frac{C_0}{4C'N}\widetilde{\chi}(C'N\widetilde{y}_1).$$

Moreover, we define for $\eta_\pm \in [0, 1]$

$$f_{z_0}^{\eta_\pm}(\widetilde{y}_1) \stackrel{\text{def}}{=} f_{z_0}(\widetilde{y}_1) \pm \eta_\pm\chi(\widetilde{y}_1).$$

Since f_{z_0} has Lipschitz modulus $\leq C_0$, if follows that $f_{z_0}^{\eta_\pm}$ has Lipschitz modulus $\leq 3C_0/2$, for $N > 0$ sufficiently large.

By Proposition 3.5, it follows that the number of eigenvalues of $P_{S_{\widetilde{N}}}$ contained in $R_{z_0}(N)$ is bounded by a constant depending only p, C' and C_0. Since the are only finitely many points to avoid, there exist $\eta_\pm \in [0, 1]$ such that

$$\{y \in \mathbb{R}^2 \simeq \mathbb{C}; |\widetilde{y}_1| \leq 1/C'N,\ \widetilde{y}_2 = f_{z_0}^{\eta_\pm}(\widetilde{y}_1)\} \cap \big(\mathrm{Spec}(P_{S_{\widetilde{N}}}) \cap R_{z_0}(N)\big) = \emptyset. \tag{6.18}$$

For $C, C', \widetilde{C} > 0$ large enough we can arrange that

$$\Bigg(\bigcup_{\substack{y\in\mathbb{R}^2 \text{ s.t. } (\widetilde{y}_1, f_{z_0}^{\eta,\pm}(\widetilde{y}_1)) \\ |\widetilde{y}_1|\leq 1/C'N}} \overline{D(y, r(y))}\Bigg) \cap \bigcup_{\lambda\in\mathrm{Spec}(P_{S_{\widetilde{N}}})} \overline{D(\lambda, 1/(\widetilde{C}N))} \cap R_{z_0}(N) = \emptyset. \tag{6.19}$$

We perform these two deformations of $\partial\Omega$ near every point $z_0 \in p(S^1)\cap\partial\Omega$, pick $C > 0$ in (6.16) at least as large as the maximum over all constants C so that (6.19) holds, and call the resulting deformed sets

$$\Omega_\pm \text{ with boundary } \partial\Omega_\pm. \tag{6.20}$$

Here, we always take the local deformation $f_{z_0}^{\eta+}$ for Ω_+, and $f_{z_0}^{\eta-}$ for Ω_-. Notice that since

$$f_{z_0}^{\eta-}(\widetilde{y}_1) \leq f_{z_0}(\widetilde{y}_1) \leq f_{z_0}^{\eta+}(\widetilde{y}_1), \quad |\widetilde{y}_1| \leq 1/(C'N),$$

we have

$$\Omega_+ \subset \Omega \subset \Omega_-, \tag{6.21}$$

where we do not denote the N dependence explicitly.

By (6.19) , (Ω1) and (Ω3), there exists a $C>0$ such that

$$\operatorname{dist}\left(\bigcup_{x\in\partial\Omega_\pm} \overline{D(x,r(x))}, \operatorname{Spec}(P_{S_{\widetilde{N}}})\right) \geq \frac{1}{CN}, \tag{6.22}$$

which also determines the constant $C>0$ in (6.2). Next, choose points $z_j^{0,\pm} \in \partial\Omega_\pm$, $j\in\mathbb{Z}/M\mathbb{Z}$, such that

$$\partial\Omega_\pm \subset \bigcup_{j\in\mathbb{Z}/M\mathbb{Z}} D(z_j^0, r_j^\pm/2), \text{ and } r_j^\pm/4 \leq |z_{j+1}^{0,\pm} - z_j^{0,\pm}| \leq r_j^\pm/2, \tag{6.23}$$

where $r_j^\pm = r(z_j^{0,\pm})$.

Lemma 6.3. *Let M be as in (6.23). Then,*

$$M = \mathcal{O}(\log N).$$

We will postpone the proof of Lemma 6.3 to the end of this section and carry on with the proof of our main result.

First, notice that (6.10) holds in $\bigcup_{j=1}^M D(z_j^0, r_j)$ with probability $\geq 1 - \mathrm{e}^{-N^2}$. By (6.22), it follows that the weight function $\psi(z;N)$ (6.11) is continuous on $\bigcup_{x\in\partial\Omega_\pm}\overline{D(x,r(x))}$. Moreover, by (6.22), we have that for any $z_j \in D(z_j^0, r_j/2)$

$$\alpha(z_j;N) \geq \frac{1}{CN}, \tag{6.24}$$

and so it follows that (6.14) holds with probability (6.15), assuming (6.13). Hence, using Lemma 6.3, we have that (6.14) holds for $z_1^0,\dots,z_M^0$ with probability

$$\geq 1 - \mathcal{O}(\log N)\left(\mathrm{e}^{-N^2} + C_2\delta^{-|J|}\mathrm{e}^{-\frac{1}{2}N^{\varepsilon_0}}\right). \tag{6.25}$$

In view of (6.14), we can pick $\varepsilon_j = CN^{\varepsilon_0-1}$ in Theorem 6.2, so using Lemma 6.3, we get

(6.26)

$$\begin{aligned}&\left|\#(\operatorname{Spec}(P_N^\delta)\cap\Omega_\pm) - \frac{N}{2\pi}\int_{\Omega_\pm}\Delta\phi(z)L(dz)\right| \\ &\leq \mathcal{O}(N)\left(N^{\varepsilon_0-1}\log N + \mu\left(\bigcup_{x\in\partial\Omega_\pm} D(x,r(x))\right) + \sum_{j=1}^M \int_{D\left(z_j^0,\frac{r(z_j^0)}{4C_1}\right)}\left|\log\frac{|w-z_j^0|}{r(z_j^0)}\right|\mu(dw)\right),\end{aligned}$$

with probability (6.25), where we used as well that $\Delta\psi(z;N) = \Delta\phi(z)$, see (6.11). Moreover, since $\Delta_z \log|z-w| = 2\pi\delta_w$, we have

$$\Delta\phi = \mu = \frac{2\pi}{N}\sum_{\lambda\in p(\widehat{S}_{\widetilde{N}})}\delta_\lambda \text{ in } \mathcal{D}'(\mathbb{C}). \tag{6.27}$$

The integral in the first line is up to an error of order $\mathcal{O}(1)$ the number of eigenvalues of $P_{S_{\widetilde{N}}}$ contained in $\Omega\cap p(S^1)$. Hence, by (6.7) and (3.35),

$$\frac{N}{2\pi}\int_{\Omega_\pm}\Delta\phi(z)L(dz) = \frac{N}{2\pi}\int_{p^{-1}(\Omega\cap p(S^1))} L_{S^1}(d\theta) + \mathcal{O}(1). \tag{6.28}$$

By (6.22)

$$\mu\left(\bigcup_{x\in\partial\Omega} D(x,r(x))\right) = 0 \tag{6.29}$$

Similarly, the discs $\overline{D(z_j^0, r(z_j^0)/2)}$ do not contain any eigenvalues of $\mathcal{P}_N^0$. Thus,

$$\sum_{j=1}^{M} \int_{D\left(z_j^0, \frac{r(z_j^0)}{4C_1}\right)} \left|\log \frac{|w-z_j^0|}{\widetilde{r}(z_j^0)}\right| \mu(dw) = 0 \tag{6.30}$$

Finally, from (6.21), it follows that

$$\#(\mathrm{Spec}(P_N^\delta)\cap\Omega_+) \leq \#(\mathrm{Spec}(P_N^\delta)\cap\Omega) \leq \#(\mathrm{Spec}(P_N^\delta)\cap\Omega_-). \tag{6.31}$$

Combining (6.26), (6.28), (6.29), (6.30) and (6.31) we get that

$$\left|\#(\mathrm{Spec}(P_N^\delta)\cap\Omega) - \frac{N}{2\pi}\int_{p^{-1}(\Omega\cap p(S^1))} L_{S^1}(d\theta)\right| \leq \mathcal{O}(N^{\varepsilon_0}\log N). \tag{6.32}$$

with probability (6.25), provided (6.13) holds. This completes the proof of Theorem 2.1.

Proof of Lemma 6.3. 1. The perturbed boundaries $\partial\Omega_\pm$ (6.20) coincide with $\partial\Omega$ outside the rectangles (6.17). Recall from (Ω1) that there are only finitely many such rectangles. The number of discs of radius $r_j^\pm$ (6.23) needed to cover $\partial\Omega_\pm$, as in (6.23), inside these rectangles is by (6.16) of order

$$\mathcal{O}(1). \tag{6.33}$$

It remains to estimate the number of discs needed to cover $\partial\Omega$ outside these rectangles, which differs from order of the number of discs needed to cover the unperturbed $\partial\Omega$ by $\mathcal{O}(1)$. Hence, it is sufficient to estimate the number of discs needed to cover $\partial\Omega$.

2. Since Ω is relatively compact and intersects with $p(S^1)$ at most finitely many points, we see that for any fixed constant $C > 1$ the number of discs needed to cover $\partial\Omega\cap\{z\in\mathbb{C}; \mathrm{dist}(z,p(S^1)) \geq 1/C\}$, is of order

$$\mathcal{O}(1). \tag{6.34}$$

3. It remains to estimate the number of discs needed to cover $\partial\Omega$ inside $\{z\in\mathbb{C}; \mathrm{dist}(z,p(S^1)) \leq 1/C\}$. By assumption ($\Omega$1) and the fact that Ω is relatively compact we see that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x\in\partial\Omega$

$$\mathrm{dist}(x,p(S^1)) < \delta \quad\Longrightarrow \min_{z_0\in p(S^1)\cap\partial\Omega} \mathrm{dist}(x,z_0) < \varepsilon. \tag{6.35}$$

Hence, for any fixed $C' > 0$, we have for $C > 0$ sufficiently large

$$\partial\Omega\cap\{z\in\mathbb{C}; \mathrm{dist}(z,p(S^1)) \leq 1/C\} \subset \bigcup_{z_0\in p(S^1)\cap\partial\Omega} D(z_0, 1/C').$$

By (Ω1), may restrict our attention to one $z_0\in\partial\Omega\cap p(S^1)$ and

$$\beta = \partial\Omega\cap\{z\in\mathbb{C}; \mathrm{dist}(z,p(S^1)) \leq 1/C\}\cap D(z_0,1/C'). \tag{6.36}$$

For $x,y\in\beta$ let $\mathrm{dist}_\beta(x,y)$ denote the length of the curve in β with endpoints x and y. By the transversality assumption (Ω3), we see that for $C > 0$ sufficiently large

$$\mathrm{dist}_\beta(x,z_0) \asymp \mathrm{dist}(x,p(S^1)), \quad x\in\beta, \tag{6.37}$$

and

$$\mathrm{dist}_\beta(x,y) \asymp |x-y|, \quad x,y\in\beta. \tag{6.38}$$

4. Notice that M_β, the number of discs $D(z_i^0, r_i/2)$ needed to cover β, as in (6.23), increases when decreasing the scale r (6.16). Using (6.37) and by possibly increasing $C > 0$ in (6.16), we shrink r to the new scale

$$r(x) = \frac{1}{C}\left(\mathrm{dist}_\beta(x, z_0) + \frac{1}{N}\right), \quad x \in \beta, \tag{6.39}$$

denoted by the same letter. Set

$$d_j \stackrel{\text{def}}{=} \mathrm{dist}_\beta(z_j^0, z_0), \quad 1 \leq j \leq M_\beta, \tag{6.40}$$

and let j_1 be the smallest index so that $d_{j_1} \geq N^{-1}$. Notice that $j_1 = \mathcal{O}(1)$ and that $d_{j_1} \asymp N^{-1}$. By (6.40), (6.38), (6.39) we have for $j > j_1$

$$\begin{aligned} d_j &= \mathrm{dist}_\beta(z_j^0, z_{j-1}^0) + \mathrm{dist}_\beta(z_{j-1}^0, z_0) \\ &\geq \frac{1}{C}|z_j^0 - z_{j-1}^0| + d_{j-1} \\ &\geq (1 + C^{-1})d_{j-1} \\ &\geq (1 + C^{-1})^{j-j_1} d_{j_1}, \end{aligned} \tag{6.41}$$

where the constant $C > 0$ changes from the second to the third line. Similarly

$$d_j \leq (1 + C)^{j-j_1} d_{j_1}. \tag{6.42}$$

Thus,

$$(1 + \widehat{C}^{-1})^{M_\beta - j_1} d_{j_1} \leq d_{M_\beta} \leq (1 + C)^{M_\beta - j_1} d_{j_1}. \tag{6.43}$$

Using that the length of β is $\asymp 1$, we get that $M_\beta \asymp \log N$ and therefore, by (6.33), (6.34), that

$$M = \mathcal{O}(\log N).$$

□

6.5. **Counting eigenvalues in thin N-dependent domains.** In Section 6.4 we saw that most eigenvalues of P_N^δ lie "near" the curve $p(S^1)$. Now we want to give a quantitative estimate on *how close* these eigenvalues are to the $p(S^1)$. For this purpose let $\Omega \Subset \mathbb{C}$ be an open simply connected set with smooth boundary $\partial\Omega$ which is independent of N and satisfies (Ω1)–(Ω3), as in Section 2.1.

We consider an open simply connected N-dependent set Ω_N, with a unifromly Lipschitz boundary $\partial\Omega_N$, which coincides with Ω in small tube around $p(S^1)$. More precisely, let

$$\frac{C}{N} \leq \tau \leq \mathcal{O}(1), \quad C > 1, \tag{6.44}$$

and suppose that

$$\Omega_N \cap \{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) < \tau\} = \Omega \cap \{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) < \tau\}, \tag{6.45}$$

and that $\partial\Omega_N$ is uniformly Lipschitz, as in Section 6.1, with weight function

$$r(x) \stackrel{\text{def}}{=} \frac{1}{C}\left(\mathrm{dist}(x, p(S^1)) + \frac{1}{N}\right), \quad x \in \partial\Omega_N \cap \{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) < \tau\}, \tag{6.46}$$

inside $\{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) < \tau\}$ and with constant weight function

$$r(x) \stackrel{\text{def}}{=} \tau, \quad x \in \partial\Omega_N \cap \{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) \geq \tau\} \tag{6.47}$$

outside. Let

$$\ell(N) > 0 \tag{6.48}$$

be the length of $\partial\Omega_N \cap \{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) \geq \tau\}$. To prove Theorem 6.5, we can follow the proof of Theorem 2.1 in Section 6.4 with some modifications:

By (6.44) and (6.45), we may perform the same perturbations of $\partial\Omega_N$ as for $\partial\Omega$ in (6.17)–(6.18) so that (6.21) and (6.22) hold for the perturbed sets

$$\Omega_N^\pm \text{ with boundary } \partial\Omega_N^\pm. \tag{6.49}$$

Next, choose points $z_j^{0,\pm} \in \partial\Omega_N^\pm$, $j \in \mathbb{Z}/M\mathbb{Z}$, such that

$$\partial\Omega_N^\pm \subset \bigcup_{j\in\mathbb{Z}/M\mathbb{Z}} D(z_j^{0,\pm}, r_j^\pm/2), \text{ and } r_j^\pm/4 \leq |z_{j+1}^{0,\pm} - z_j^{0,\pm}| \leq r_j^\pm/2, \tag{6.50}$$

where $r_j^\pm = r(z_j^{0,\pm})$.

Lemma 6.4. *Let M be as in (6.50). Then,*

$$M = \mathcal{O}(\ell(N)\tau^{-1}) + \mathcal{O}(\log(\tau N)).$$

Proof. Following the exact same lines of *Step 1, 3* and *4* of the proof of Lemma 6.3, while keeping in mind (6.46) and that by (6.44), (6.45) the length of $\partial\Omega_N \cap \{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) \leq \tau\}$ is of order $\asymp \tau$, we see that the number of discs needed to cover $\partial\Omega_N \cap \{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) \leq \tau\}$ is of order

$$\mathcal{O}(\log(\tau N)). \tag{6.51}$$

By (6.48), (6.47) we have that we have that the number of discs needed to cover $\partial\Omega_N \cap \{z \in \mathbb{C}; \mathrm{dist}(z, p(S^1)) \geq \tau\}$ is of order

$$\mathcal{O}(\ell(N)\tau^{-1}). \tag{6.52}$$

□

Since (6.22) holds for $\partial\Omega_N^\pm$, the weight function $\psi(z;N)$ (6.11) is continuous on

$$\bigcup_{x\in\partial\Omega_N^\pm} \overline{D(x, r(x))},$$

and that (6.10) holds in $\bigcup_{j=1}^M D(z_j^0, r_j)$ (6.50) with probability $\geq 1 - \mathrm{e}^{-N^2}$. Moreover, since (6.22) holds for $\partial\Omega_N^\pm$, we have that for any $z_j \in D(z_j^0, r_j/2)$

$$\alpha(z_j;N) \geq \frac{1}{CN},$$

and it follows that (6.14) holds with probability (6.15), assuming (6.13). Hence, using Lemma 6.4, we have that (6.14) holds for $z_1^0, \ldots, z_M^0$ with probability

$$\geq 1 - \mathcal{O}(M)\left(\mathrm{e}^{-N^2} + C_2\delta^{-|J|}\mathrm{e}^{-\frac{1}{2}N^{\varepsilon_0}}\right). \tag{6.53}$$

In view of (6.14), we may set $\varepsilon_j = CN^{\varepsilon_0 - 1}$ in Theorem 6.2 and, by following the exact same arguments as above, from (6.26) to (6.31), while keeping in mind Lemma 6.4, we obtain

Theorem 6.5. *Let p be as in (1.1), set $M = N_+ + N_-$ and let P_N^δ be as in (1.9). Let τ be as in (6.44) and let $\Omega_N \Subset \mathbb{C}$ be a relatively compact open simply connected set satisfying (6.45)–(6.48). Pick a $\varepsilon_0 \in]0,1[$.*

There exists a constant $C > 0$ such that for $N > 1$ sufficiently large, if (2.2) holds,

$$C\mathrm{e}^{-N^{\varepsilon_0}/(2M)} \leq \delta \leq \frac{N^{-4}}{C},$$

then,

$$\left| \#(\mathrm{Spec}(P_N^\delta) \cap \Omega) - \frac{N}{2\pi}\int_{p^{-1}(\Omega\cap p(S^1))} L_{S^1}(d\theta) \right| \leq \mathcal{O}(N^{\varepsilon_0}\ell(N)\tau^{-1} + N^{\varepsilon_0}\log(\tau N)). \tag{6.54}$$

with probability

$$\geq 1 - \mathcal{O}(\ell(N)\tau^{-1} + \log(\tau N))\left(\mathrm{e}^{-N^2} + C_2\delta^{-|J|}\mathrm{e}^{-\frac{1}{2}N^{\varepsilon_0}}\right). \tag{6.55}$$

Remark 6.6. *In the assumption 6.45 on Ω_N we assumed that it coincides with an Ω with smooth boundary, which is independent of N, inside a tube of radius τ around $p(S^1)$. Therefore, Assumption 6.45 implies that $\ell(N) \geq 1/C > 0$. However, the proof of Theorems 2.1 and 6.5 shows that we can allow Ω to be N dependent as long as its boundary $\partial\Omega$ remains uniformly*

Lipschitz in the sense discussed at the beginning of Section 6.1 and satisfies (Ω1)–(Ω3). Hence, Theorem 6.5 holds as well for sets Ω_N, satisfying (6.44)-(6.47) with

$$\frac{C}{N} \leq \ell(N). \tag{6.56}$$

7. Convergence of the empirical measure

In this section we present the two proofs of Corollary 2.2. The first one, in Section 7.1, shows that it is a consequence of Theorem 2.1. The second (alternative) proof in Sections Section 7.2, Section 7.3, shows how one can obtain the result from our methods via analysing the convergence of the associated logarithmic potentials, in perhaps a more direct way.

7.1. Proof of Corollary 2.2.

Let Ω be a fixed domain as in Theorem 2.1 and choose a sequence $\delta = \delta_N$ satisfying (2.2). By the Borel-Cantelli lemma, we know that a.s. (almost surely)

$$\frac{1}{N}\#(\sigma(P_\delta^N) \cap \Omega) \to \mathrm{vol}\,(p^{-1}(\Omega) \cap S^1), \ N \to \infty. \tag{7.1}$$

Let now Ω be a square of the form $a_1 \leq \mathrm{Re}\, z < a_2$, $b_1 \leq \mathrm{Im}\, z < b_2$, $a_2 - a_1 = b_2 - b_1 > 0$. Assume that the corners $a_j + ib_k$ do not belong to $p(S^1)$. Then the conditions (Ω1)–(Ω3) make sense. If they are fulfilled, then (7.1) holds a.s.. Indeed, let Ω_{int}, Ω_{ext} be sets with smooth boundary such that $\Omega_{\mathrm{int}} \subset \Omega \subset \Omega_{\mathrm{ext}}$ and coinciding with Ω away from a small neighborhood of the union of the corners of Ω. Then (7.1) holds a.s. for Ω_{int} and Ω_{ext}, and the common limit in the right hand side is $(2\pi)^{-1}\mathrm{vol}\,(p^{-1}(\Omega) \cap S^1)$. Since

$$\frac{1}{N}\#(\sigma(P_\delta^N) \cap \Omega_{\mathrm{int}}) \leq \frac{1}{N}\#(\sigma(P_\delta^N) \cap \Omega) \leq \frac{1}{N}\#(\sigma(P_\delta^N) \cap \Omega_{\mathrm{ext}}),$$

we conclude that (7.1) holds a.s. for Ω.

Write $p(\zeta) = p_1(\zeta) + ip_2(\zeta)$ so that $p_{j|_{S^1}}$ are real analytic. Then for $j = 1, 2$:

1) The set C_j of critical values of $p_{j|_{S^1}}$ is finite.
2) For $j = 1, 2$ and for every $a \in \mathbb{R}$ the equation $p_j(\zeta) = a$ has at most finitely many solutions in S^1.

Let $\epsilon > 0$. Then we can choose $a, b \in \mathbb{R}$ (depending on ϵ) such that $a + \mathbb{Z}\epsilon \cap C_1 = \emptyset$, $b + \mathbb{Z}\epsilon \cap C_2 = \emptyset$. After a slight shift of b we can arrange so that we also have

$$(a + \mathbb{Z}\epsilon) + i(b + \mathbb{Z}\epsilon) \cap p(S^1) = \emptyset.$$

Then for each $\epsilon > 0$ we have a.s. that (7.1) holds for $\Omega = \Omega_{\epsilon,j,k}$ for all $j, k \in \mathbb{Z}$. Here, we put $\Omega_{\epsilon,j,k} = (a + [j, j+1[\epsilon) + i(b + [k, k+1[\epsilon[)$. Let $\epsilon_\nu > 0$, $\nu \in \mathbb{N}$ be a decreasing sequence tending to zero. Then a.s., (7.1) holds for all the $\Omega_{\epsilon_\nu,j,k}$.

Let G be the set of all step functions of the form,

$$\psi = \sum_{j,k} g_{j,k} 1_{\Omega_{\epsilon_\nu,j,k}}, \ g_{j,k} \in \mathbb{Q}, \tag{7.2}$$

Then a.s. we have for every $\psi \in G$, that

$$\int \psi\, \xi_N(dz) \to \int \psi\, p_* \left(\frac{1}{2\pi} L_{S^1}\right)(dz), \ N \to \infty. \tag{7.3}$$

Let $\phi \in C_c(\mathbb{C}; \mathbb{R})$. For every $\epsilon > 0$, we can find $\psi = \psi_\epsilon \in G$, such that $|\phi - \psi| \leq \epsilon$. ξ_N and $p_*((2\pi)^{-1}L_{S^1})$ are probability measures, so

$$\left|\int \phi\, \xi_N(dz) - \int \psi\, \xi_N(dz)\right| \leq \epsilon,$$

$$\left|\int \phi\, p_* \left(\frac{1}{2\pi} L_{S^1}\right)(dz) - \int \psi\, p_* \left(\frac{1}{2\pi} L_{S^1}\right)(dz)\right| \leq \epsilon.$$

It follows that a.s., we have for all $\phi \in C_0(\mathbb{C})$,

$$\limsup_{N\to\infty}\left|\int \phi\,\xi_N(dz) - \int \phi\, p_*\left(\frac{1}{2\pi}L_{S^1}\right)(dz)\right| \leq$$
$$2\epsilon + \limsup_{N\to\infty}\left|\int \psi\,\xi_N(dz) - \int \psi\, p_*\left(\frac{1}{2\pi}L_{S^1}\right)(dz)\right|.$$

A.s. the last limit is 0 for all $\psi \in G$, hence a.s. we have that for all $\epsilon > 0$ and all $\phi \in C_c(\mathbb{C})$,

$$\limsup_{N\to\infty}\left|\int \phi\,\xi_N(dz) - \int \phi\, p_*\left(\frac{1}{2\pi}L_{S^1}\right)(dz)\right| \leq 2\epsilon.$$

In other words, a.s. we have

$$\lim_{N\to\infty}\int \phi\,\xi_N(dz) = \int \phi\, p_*\left(\frac{1}{2\pi}L_{S^1}\right)(dz),$$

for all $\phi \in C_c(\mathbb{C})$, so a.s.:

$$\xi_N(dz) \rightharpoonup p_*\left(\frac{1}{2\pi}L_{S^1}\right),\ N \to \infty.$$

Notice that almost surely, $\operatorname{supp}\xi_N$ is contained in a fixed compact set.

7.2. **Logarithmic potential and weak convergence of measure.** We begin by recalling some basic facts concerning the weak convergence of measures. Let $\mathcal{P}(\mathbb{C})$ denote the space of probability measures μ on $\mathbb{C}$, integrating the logarithm at infinity

$$\int \log(1+|x|)\mu(dx) < +\infty. \tag{7.4}$$

We define the *logarithmic potential* of μ by

$$U_\mu(z) \stackrel{\mathrm{def}}{=} -\int \log|z-x|\mu(dx). \tag{7.5}$$

Since $U_\mu \in L^1_{\mathrm{loc}}(\mathbb{C}, L(dz))$, it follows that $U_\mu(z) < +\infty$ for Lebesgue almost every (a.e.) $z \in \mathbb{C}$.

One property of the logarithmic potential is that for a given sequence of probability measures $\{\mu_n\}_n \in \mathcal{P}(\mathbb{C})$, satisfying some suitable uniform integrability assumption, one has that almost sure convergence of the associated logarithmic potentials $U_{\mu_n}(z) \to U_\mu(z)$, for some $\mu \in \mathcal{P}(\mathbb{C})$, implies the weak convergence $\mu_n \rightharpoonup \mu$.

There are various versions of the above observation known in the case of random measures, see for instance [Ta02, Theorem 2.8.3] or [BoCa13]. In the following we describe a slightly modified version of [Ta02, Theorem 2.8.3] for the reader's convenience.

Theorem 7.1. *Let $K, K' \Subset \mathbb{C}$ be open relatively compact sets with $\overline{K} \subset K'$, and let $\{\mu_n\}_{n\in\mathbb{N}} \in \mathcal{P}(\mathbb{C})$ be as sequence of random measures so that almost surely*

$$\operatorname{supp}\mu_n \subset K \text{ for } n \text{ sufficiently large.} \tag{7.6}$$

Suppose that for a.e. $z \in K'$ almost surely

$$U_{\mu_n}(z) \to U_\mu(z), \quad n \to \infty, \tag{7.7}$$

where $\mu \in \mathcal{P}(\mathbb{C})$ is some probability measure with $\operatorname{supp}\mu \subset K$. Then, almost surely,

$$\mu_n \rightharpoonup \mu, \quad n \to \infty, \quad \text{weakly.} \tag{7.8}$$

Proof. 1. Notice that the assumption that for a.e. $z \in K'$ (7.7) holds almost surely is equivalent to the statement that almost surely (7.7) holds for a.e. $z \in K'$. To see this, consider the set $E = \{(z,\omega) \in K' \times \Omega; U_{\mu_n}(z) \to U_\mu(z),\ \text{as } n \to \infty\} \subset K' \times \Omega$, where Ω denotes the underlying probability space. Applying the Tonelli theorem to $\mathbf{1}_{E^c}$ lets us conclude the claim.

2. Since $\log|\cdot - w| \in L^2(K')$ uniformly for $w \in K'$, it follows by the Minkowski integral inequalities that, almost surely, $U_{\mu_n}, U_\mu \in L^2(K')$ uniformly. Let us remark here that although μ_n depends on the random parameter ω, we do not denote that explicitly.

Combining this with (7.6) and *step 1.* above, we see that there exists an $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$, so that for each $\omega \in \Omega'$ we have that

- (7.7) holds for a.e. $z \in K'$,
- there exists an $n_0 \geq 1$ such that $\operatorname{supp} \mu_n \subset K$ for all $n \geq n_0$,
- there exists a $C_{K',\Omega'} > 0$, depending only on K' and Ω', such that $\|U_{\mu_n}\|_{L^2(K')}, \|U_\mu\|_{L^2(K')} \leq C_{K',\Omega'}$ for any $n \geq 1$.

To show (7.8) for any $\omega \in \Omega'$, it is enough to show that for any real-valued smooth function $\phi \in \mathcal{C}_c^\infty(K';\mathbb{R})$ with support contained in K',

$$\mu_n(\phi) \to \mu(\phi), \quad n \to \infty. \tag{7.9}$$

3. Let $\omega \in \Omega'$, and set $g_n^M(z) = \min(|U_{\mu_n}(z) - U_\mu(z)|, M)$, $z \in K'$, for $M > 0$. The dominated convergence theorem shows that $g_n^M \to 0$, as $n \to \infty$, in $L^1(K')$ for any $M > 0$. Using the $L^2(K)$ bound of U_{μ_n} and U_μ, we see that

$$\begin{aligned}
\|g_n^M - |U_{\mu_n} - U_\mu|\|_{L^1(K')} &\leq \int_{\substack{|U_{\mu_n} - U_\mu| \geq M \\ z \in K}} |U_{\mu_n}(z) - U_\mu(z)| L(dz) \\
&\leq \sqrt{2} C_{K'} \left(\int_{\substack{|U_{\mu_n} - U_\mu| \geq M \\ z \in K}} L(dz) \right)^{1/2} \\
&\leq \frac{\sqrt{2} C_{K'} L(K')^{1/2}}{M}.
\end{aligned}$$

Hence, for any $w \in \Omega'$ we have that $U_{\mu_n} \to U_\mu$ in $L^1(K')$ as $n \to \infty$. Thus, almost surely $U_{\mu_n} \rightharpoonup U_\mu$ in $\mathcal{D}'(K')$, and so (7.9) holds almost surely, since $\Delta_z U_{\mu_n} = -2\pi\mu_n$, $\Delta_z U_\mu = -2\pi\mu$ in $\mathcal{D}'(\mathbb{C})$. □

7.3. **Proof of Corollary 2.2.** Recall the definition of the empirical measure ξ_N (2.5) and (1.9). By (1.3), (1.4) and the Fourier transform $\mathcal{F}$ (3.16) we see that the operator norm of the unperturbed operator P_N^0 is satisfies

$$\|P_N^0\| \leq \|p\|_{L^\infty(S^1)}. \tag{7.10}$$

Suppose (5.19), then by (5.4), (7.10) it follows that

$$\|P_N^\delta\| \leq \|p\|_{L^\infty(S^1)} + 1 \tag{7.11}$$

for $N > 1$ sufficiently large, with probability $\geq 1 - \mathrm{e}^{-N^2}$. We deduce by a Borel-Cantelli argument that almost surely

$$\operatorname{supp} \xi_N \subset \overline{D(0, \|p\|_{L^\infty(S^1)} + 1)} \stackrel{\text{def}}{=} K \subset D(0, \|p\|_{L^\infty(S^1)} + 2) \stackrel{\text{def}}{=} K' \tag{7.12}$$

for N sufficiently large. For p as in (1.3), define the probability measure

$$\xi = p_* \left(\frac{1}{2\pi} L_{S^1} \right) \tag{7.13}$$

which has compact support,

$$\operatorname{supp} \xi = p(S^1) \subset K. \tag{7.14}$$

Here, $\frac{1}{2\pi} L_{S^1}$ denotes the normalized Lebesgue measure on S^1.

To conclude Corollary 2.2 from Theorem 7.1 it remains to show that for almost every $z \in K'$ we have that $U_{\xi_N}(z) \to U_\xi(z)$ almost surely.

By (7.5) we see that for $z \notin \operatorname{Spec}(P_N^\delta)$

$$U_{\xi_N}(z) = -\frac{1}{N} \log |\det(P_N^\delta - z)|. \tag{7.15}$$

For any $z \in \mathbb{C}$ the set $\Sigma_z = \{Q \in \mathbb{C}^{N\times N}; \det(P_0 + \delta Q - z) = 0\}$ has Lebesgue measure 0, since $\mathbb{C}^{N\times N} \ni Q \mapsto \det(P_N^\delta - z)$ is analytic and not constantly 0. Thus $\mu_N(\Sigma_z) = 0$, where μ_N is the Gaussian measure given in (5.2), and for every $z \in \mathbb{C}$ (7.15) holds almost surely (a.s.).

Next, define the set

$$E_N \stackrel{\text{def}}{=} \{z \in \mathbb{C}; \operatorname{dist}(z, p(S^1)) \leq 1/(CN)\} \tag{7.16}$$

which has Lebesgue measure $L(E_N) = \mathcal{O}(N^{-1})$. By (6.7), (6.8), (5.4) as well as Proposition 5.1 and (5.19) we have that for every $z \in K' \backslash E_N$

$$\left| \frac{1}{N} \log |\det(P_N^\delta - z)| - \phi(z) \right| \leq \mathcal{O}(\delta N^{3/2}) + N^{-1} \big| \log |\det E_{-+}^\delta(z)| \big|. \tag{7.17}$$

with probability $\geq 1 - \mathrm{e}^{-N^2}$. Using Proposition 5.1, we see that for every $z \in K' \backslash E_N$

$$\log |\det E_{-+}^\delta(z)| \leq \mathcal{O}(\log N). \tag{7.18}$$

with probability $\geq 1 - \mathrm{e}^{-N^2}$. Let $\varepsilon_0 \in]0, 1[$ be as in Corollary 2.2 and let $\varepsilon_1 \in]0, 1[$ with $\varepsilon_0 < \varepsilon_1$. Then, by replacing ε_0 in (6.12) with ε_1, we have that

$$\log |\det E_{-+}^\delta(z)| \geq -N^{\varepsilon_1} \tag{7.19}$$

with probability $\geq 1 - \mathrm{e}^{-N^2} - C_2 \delta^{-|J|} \mathrm{e}^{-\frac{1}{2} N^{\varepsilon_1}}$, when

$$\exp\left[\frac{C_0}{2|J|} - \frac{N^{\varepsilon_1}}{2|J|}\right] \leq \delta. \ll N^{-4}$$

For $z \notin p(S^1)$ the function $S^1 \ni \zeta \mapsto \log |z - p(\zeta)|$ is continuous. Hence, by (6.7), (7.13), (7.5), and a Riemann sum argument, we see that for

$$|\phi(z) + U_\xi(z)| \longrightarrow 0, \quad \text{as } N \to \infty. \tag{7.20}$$

For any $z \in K' \backslash p(S^1)$ we have that $z \in K' \backslash E_N$ for $N > 1$ suffciently large. Thus, by (7.15), (7.17), (7.18), (7.19), and (7.20) we have for any $z \in K' \backslash p(S^1)$ and $N > 1$ sufficiently large that

$$|U_{\xi_N}(z) - U_\xi(z)| = o(1) \tag{7.21}$$

with probability $\geq 1 - \mathcal{O}(1) \mathrm{e}^{-\frac{1}{2} N^{\varepsilon_1}(1 - |J| N^{\varepsilon_0 - \varepsilon_1})}$. Here we also used (2.2). Since $\varepsilon_0 < \varepsilon_1$, we conclude by the Borel-Cantelli theorem that for almost every $z \in K'$

$$U_{\xi_N}(z) \longrightarrow U_\xi(z), \quad \text{as } N \to \infty, \text{ almost surely}, \tag{7.22}$$

which by Theorem 7.1 concludes the proof of Corollary 2.2.

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