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Abstract

The 1-2-3 Conjecture states that every connected graph different from K_2 admits a proper 3-labelling, i.e., can have its edges labelled with $1, 2, 3$ so that no two adjacent vertices are incident to the same sum of labels. In connection with some recent optimisation variants of this conjecture, in this paper we investigate the role of label 3 in proper 3-labellings of graphs. An intuition from previous investigations is that, in general, it should always be possible to produce proper 3-labellings assigning label 3 to a only few edges.

We prove that, for every $p \geq 0$, there are various graphs needing at least p 3's in their proper 3-labellings. Actually, deciding whether a given graph can be properly 3-labelled with p 3's is NP-complete for every $p \geq 0$. We also focus on classes of 3-chromatic graphs. For various classes of such graphs (cacti, cubic graphs, triangle-free planar graphs, etc.), we prove that there is no $p \geq 1$ such that they all admit proper 3-labellings assigning label 3 to at most p edges. In such cases, we provide lower and upper bounds on the number of needed 3's.

Keywords: Proper labellings, 3-chromatic graphs, 1-2-3 Conjecture.

1 Introduction

This work is mainly motivated by the so-called **1-2-3 Conjecture**, which can be defined through the following terminology and notation. Let G be a graph and consider a k-labelling $\ell : E(G) \to \{1, \dots, k\}$, i.e., an assignment of labels $1, \dots, k$ to the edges of G. To every vertex $v \in V(G)$, we can associate, as its *colour* $c_\ell(v)$, the sum of labels assigned by ℓ to its incident edges. That is, $c_\ell(v) = \sum_{u \in N(v)} \ell(vu)$. We say that ℓ is *proper* if we have $c_\ell(u) \neq c_\ell(v)$ for every $uv \in E(G)$, that is, if no two adjacent vertices of G get incident to the same sum of labels by ℓ.

It turns out that K_2, the complete graph on two vertices, is the only connected graph admitting no proper labellings at all. Thus, when investigating the 1-2-3 Conjecture, we generally focus on *nice graphs*, which are those graphs with no connected component isomorphic to K_2, i.e., admitting proper labellings. If a graph G is nice, then we can investigate the smallest $k \geq 1$ such that proper k-labellings of G exist. This parameter is denoted by $\chi_\Sigma(G)$.

A natural question to ask, is whether this parameter $\chi_\Sigma(G)$ can be large for a given graph G. This question is precisely at the heart of the 1-2-3 Conjecture:

1-2-3 Conjecture (Karoński, Łuczak, Thomason [14])**.** *If G is a nice graph, then $\chi_\Sigma(G) \leq 3$.*

To date, most of the progress towards the 1-2-3 Conjecture can be found in [16]. Let us highlight that the conjecture was verified mainly for 3-colourable graphs [14] and complete graphs [7]. Regarding the tightness of the conjecture, it was proved that deciding if a given graph G verifies $\chi_\Sigma(G) \leq 2$ is NP-complete in general [11], and remains so even in the case of cubic graphs [9]. This means there is no nice characterisation of graphs admitting proper 2-labellings (or, the other way round, of graphs needing 3's in their proper 3-labellings), unless P=NP. Lastly, to date, the best result towards the 1-2-3 Conjecture, from [13], is that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G.

This work takes place in a recent line of research dedicated to studying optimisation problems related to the 1-2-3 Conjecture which arise when wondering about proper labellings fulfilling additional constraints. In a way, one of the main sources of motivation here is further understanding the very mechanisms that lie behind proper labellings. In particular, towards better understanding the connection between proper labellings and proper vertex-colourings, the authors of [1, 5] studied proper labellings ℓ for which the resulting vertex-colouring c_ℓ is required to be close to an optimal proper vertex-colouring (i.e., with the number of distinct resulting vertex colours being close to the chromatic number). Due to one of the core motivations behind the 1-2-3 Conjecture, the authors of [4] also investigated proper labellings minimising the sum of labels assigned to the edges.

Each of these previous investigations led to presumptions of independent interest. In particular, it is believed in [5], that every nice graph G should admit a proper labelling where the maximum vertex colour is at most $2\Delta(G)$ (recall that $\Delta(G)$ and $\delta(G)$ are used to denote the maximum and the minimum, respectively, degree of any vertex of G), while, from [4], it is believed that every G should admit a proper labelling where the sum of assigned labels is at most $2|E(G)|$. One of the main reasons why these presumptions are supposed to hold, is the fact that, in general, it seems that nice graphs admit 2-labellings that are almost proper, in the sense that they need only a few 3's to design proper 3-labellings. Note that if this was true, then indeed the presumptions from [5] and [4] above would be likely to hold. It is also worth mentioning that this belief on the number of 3's is actually a long-standing one of the field, as, in a way, it lies behind the 1-2 Conjecture raised by Przybyło and Woźniak [15], which states that we should be able to build a proper 2-labelling of every graph if we are additionally allowed to locally alter every vertex colour by a bit.

Our goal in this work is to study and formally establish the intuition that, in general, graphs should admit proper 3-labellings assigning only a few 3's. We study this through two questions.

- The very first question to consider is whether, given a (possibly infinite) class $\mathcal{F}$ of graphs, the members of $\mathcal{F}$ admit proper 3-labellings assigning only a constant number of 3's, i.e., whether there is a constant $c_\mathcal{F} \geq 0$ such that all graphs of $\mathcal{F}$ admit proper 3-labellings assigning label 3 to at most $c_\mathcal{F}$ edges. Note that this is something that is already known to hold for a few graph classes. For instance, all nice trees admit proper 2-labellings, thus proper 3-labellings assigning label 3 to no edge [7]. Similarly, from results in [4], it can be deduced that all nice bipartite graphs admit proper 3-labellings assigning label 3 to at most two edges.
- In case $\mathcal{F}$ admits no such constant $c_\mathcal{F}$, i.e., the number of 3's the members of $\mathcal{F}$ need in their proper 3-labellings is a function of their number of edges, the second question we consider is whether the number of 3's needed can be "large" for a given member of $\mathcal{F}$, with respect to the number of its edges.

Throughout this work, we investigate these two questions in general and for more restricted classes of graphs. We start off in Section 2 by formally introducing the terminology that we employ throughout this work to treat these concerns, and by raising preliminary observations and results. Then, in Section 3, we introduce proof techniques for establishing lower and upper bounds on the number of 3's needed to construct proper 3-labellings for some graph classes. In Section 4, we use these tools to establish that, for several classes of graphs, the number of needed 3's in their proper 3-labellings is not bounded by an absolute constant. In such cases, we exhibit bounds (functions depending on the size of said graphs) on this number.

2 Terminology, preliminary results, and a conjecture

2.1 Proper 3-labellings assigning few 3's

Let G be a graph and G' be a subgraph of G (i.e., created by deleting vertices and/or edges of G). For any vertex $v \in V(G)$, let $N(v) = \{u \in V(G) : uv \in E(G)\}$ denote the *neighbourhood* of v, and let $d(v) = |N(v)|$ denote the *degree* of v. Also, for any vertex $v \in V(G) \cap V(G')$, let $d_{G'} = |\{u \in V(G') : uv \in E(G')\}|$ denote the degree of v in G'. Finally, recall that G' is said to be *induced* if it can be created by only deleting vertices of G. That is, for each edge $uv \in E(G)$, if $u, v \in V(G')$, then $uv \in E(G')$. For any additional notation on graph theory not defined here, we refer the reader to [10].

Let G be a (nice) graph, and ℓ be a k-labelling of G. For any $i \in \{1, \dots, k\}$, we denote by $\mathrm{nb}_\ell(i)$ the number of edges assigned label i by ℓ. Focusing now on proper 3-labellings, we denote by $\mathrm{mT}(G)$ the minimum number of edges assigned label 3 by a proper 3-labelling of G. That is,

$$\mathrm{mT}(G) = \min\{\mathrm{nb}_\ell(3) : \ell \text{ is a proper 3-labelling of } G\}.$$

We extend this parameter mT to classes $\mathcal{F}$ of graphs by defining $\mathrm{mT}(\mathcal{F})$ as the maximum value of $\mathrm{mT}(G)$ over the members G of $\mathcal{F}$. Clearly, $\mathrm{mT}(\mathcal{F}) = 0$ for every class $\mathcal{F}$ of graphs admitting proper 2-labellings (i.e., $\chi_\Sigma(G) \leq 2$ for every $G \in \mathcal{F}$).

Given a graph class $\mathcal{F}$, we are interested in determining whether $\mathrm{mT}(\mathcal{F}) \leq p$ for some $p \geq 0$. From that perspective, for every $p \geq 0$, we denote by $\mathcal{G}_p$ the class of graphs G with $\mathrm{mT}(G) = p$. For convenience, we also define $\mathcal{G}_{\leq p} := \mathcal{G}_0 \cup \dots \cup \mathcal{G}_p$.

As it was proved, for instance in [7], that nice trees admit proper 2-labellings, if we denote by $\mathcal{T}$ the class of all nice trees, then the terminology above allows us to state that $\mathcal{T} \subset \mathcal{G}_0$. More generally speaking, bipartite graphs form perhaps the most investigated class of graphs in the context of the 1-2-3 Conjecture. A notable result, due to Thomassen, Wu, and Zhan [17], is that a bipartite graph G verifies $\chi_\Sigma(G) = 3$ if and only if G is an *odd multi-cactus*, where odd multi-cacti form a particular class of 2-edge-connected bipartite graphs obtained through pasting cycles with certain lengths onto each other in a particular way. This specific class of graphs was further studied in several works, such as [4], in which it was proved that odd multi-cacti admit proper 3-labellings assigning label 3 at most twice.

Theorem 2.1 (Bensmail, Fioravantes, Nisse [4])**.** *If G is a nice bipartite graph, then $G \in \mathcal{G}_{\leq 2}$. More precisely, $G \in \mathcal{G}_0$ if G is not an odd multi-cactus, $G \in \mathcal{G}_2$ if G is a cycle of length congruent to 2 modulo 4, and $G \in \mathcal{G}_1$ otherwise (i.e., if G is an odd multi-cactus different from a cycle C_{4k+2}).*

Theorem 2.1 is troublesome in the sense that, even without considering any additional constraint, we do not know much about how proper 3-labellings behave beyond the scope of bipartite graphs. Our take in this work is to focus on the next natural case to consider, that of 3-chromatic graphs, which fulfil the 1-2-3 Conjecture [14]. Unfortunately, as will be seen later on, a result equivalent to Theorem 2.1 for 3-chromatic graphs does not exist, even for very restricted classes of 3-chromatic graphs.

Regarding the classes $\mathcal{G}_0, \mathcal{G}_1, \dots$, it is worth mentioning right away that each $\mathcal{G}_p$ is well populated, in the sense that there exist infinitely many graphs with various substructures belonging to $\mathcal{G}_p$. Actually, it turns out that deciding whether a given graph G belongs to $\mathcal{G}_p$ is NP-complete for every $p \geq 0$. We postpone the proofs of these statements for Section 3 (Observation 3.3 and Theorem 3.4), as they require the tools and results introduced in that same section.

As mentioned earlier, we will see throughout this work that, for several graph classes $\mathcal{F}$, there is no $p \geq 0$ such that $\mathcal{F} \subset \mathcal{G}_{\leq p}$. For such a class, we want to know whether the proper 3-labellings of their members require assigning label 3 many times, with respect to their number of edges. We study this aspect through the following terminology. For a nice graph G, we define $\rho_3(G) := \mathrm{mT}(G)/|E(G)|$. We extend this ratio to a class $\mathcal{F}$ by setting $\rho_3(\mathcal{F}) = \max\{\rho_3(G) : G \in \mathcal{F}\}$.

In this work, we are thus interested in determining bounds on $\rho_3(\mathcal{F})$ for some graph classes $\mathcal{F}$ of 3-chromatic graphs, and, more generally speaking, in how large this ratio can be. Note that this is similar to considering how large $\rho_3(G)$ can be for a given graph G. Also, notice that small graphs G with $\chi_\Sigma(G) = 3$ are more likely to have $\rho_3(G)$ close to 1. Through a quick study, it is easy to see that, among the sample of small connected graphs (e.g., of order at most 6), the maximum ratio ρ_3 is exactly 1/3, which is attained by C_3 and C_6. As will be seen through the next sections, at the moment, these are the worst graphs we know of, which leads us to raising the following conjecture.

Conjecture 2.2. *If G is a nice connected graph, then $\rho_3(G) \leq 1/3$.*

It is worth adding that Conjecture 2.2 can be sort of seen as a much weaker version of an equitable version of the 1-2-3 Conjecture, investigated in [2, 3]. In that version, it is believed that, a few exceptions apart, every graph should admit a proper 3-labelling ℓ where all labels are assigned about the same number of times, i.e., the difference between $\mathrm{nb}_\ell(i)$ and $\mathrm{nb}_\ell(j)$ is at most 1 for any two assigned labels i, j. Such a labelling ℓ is called *equitable*.

Unfortunately, this equitable version is obviously much stronger than our concerns in this paper, and, as a consequence, we actually do not get much from the results in [2, 3]. Indeed, most of these results are actually about equitable proper 3-labellings of classes of bipartite graphs, while bipartite graphs form a pretty understandable case in our context (recall Theorem 2.1). One result we actually get from [2] is an upper bound on ρ_3 for complete graphs, which is actually improved by another result (see Section 5).

2.2 General results on proper labellings

In this subsection, we prove results on proper labellings, which will be useful in the next sections.

Observation 2.3. *Let G be a graph with a path (v_1, v_2, v_3, v_4) such that $d(v_2) = d(v_3) = 2$. Then, by any proper labelling ℓ of G, we have $\ell(v_1v_2) \neq \ell(v_3v_4)$.*

Proof. Since, by any proper labelling ℓ of G, we have that $c_\ell(v_2) = \ell(v_1v_2) + \ell(v_2v_3)$, $c_\ell(v_3) = \ell(v_2v_3) + \ell(v_3v_4)$, and $c_\ell(v_2) \neq c_\ell(v_3)$, then $\ell(v_1v_2) \neq \ell(v_3v_4)$. □

Let ℓ be a k-labelling of some graph, and let $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$ be a permutation of $\{1, \dots, k\}$. We denote by $\mathrm{sw}(\ell, \sigma)$ the k-labelling obtained from ℓ by switching labels as indicated by σ. That is, if $\ell(e) = i$ for some edge e and label i, then $\mathrm{sw}(\ell, \sigma)(e) = \sigma(i)$. Assuming the set of labels $\{1, \dots, k\}$ is clear from the context, for any two $i, j \in \{1, \dots, k\}$, we denote by $\sigma_{i\leftrightarrow j}$ the permutation only swapping labels i and j. That is, $\sigma_{i\leftrightarrow j}(i) = j$, $\sigma_{i\leftrightarrow j}(j) = i$, and $\sigma_{i\leftrightarrow j}(l) = l$ for every $l \in \{1, \dots, k\} \setminus \{i, j\}$.

Let G be a connected graph. If, for every vertex $v \in V(G)$, we have that $d(v) = d$, then G is said to be *d-regular*. If, in addition, G also has vertices of degree 1, then we say that G is *d-quasi regular*. It is clear that every graph that is d-regular is also d-quasi regular.

Lemma 2.4. *If ℓ is a proper 3-labelling of a quasi regular graph G, then $\mathrm{sw}(\ell, \sigma_{1\leftrightarrow 3})$ is also proper.*

Proof. Assume G is d-quasi regular for some $d \geq 2$, and set $\ell' = \mathrm{sw}(\ell, \sigma_{1\leftrightarrow 3})$. Observe that, by a labelling, a vertex of degree 1 can never be involved in a colour conflict with its neighbour. It is thus sufficient to show that, for every vertex $v \in V(G)$ with $d(v) = d$, the pair $(c_\ell(v), c_{\ell'}(v))$ is unique. Consider any $v \in V(G)$ and let x be the number of its incident edges labelled 1, y be the number of its incident edges labelled 2, and z be the number of its incident edges labelled 3 by ℓ. We have that $c_\ell(v) = x + 2y + 3z$ and $c_{\ell'}(v) = 3x + 2y + z$, while $x + y + z = d$ since G is d-quasi regular. It follows that $c_\ell(v) + c_{\ell'}(v) = 4(x + y + z) = 4d$. Moreover, we have that $c_\ell(v), c_{\ell'}(v) \in \{d, \dots, 3d\}$. Let $c_\ell(v) = d + \lambda$, with $\lambda \in \{0, \dots, 2d\}$. Then, $c_{\ell'}(v) = 3d - \lambda$ and this is a unique number in $\{d, \dots, 3d\}$. □

A particular case of Lemma 2.4 is the following:

Observation 2.5. *If ℓ is a proper 2-labelling of a 3-quasi regular graph, then $\mathrm{sw}(\ell, \sigma_{1\leftrightarrow 2})$ is also proper.*

3 Tools for establishing bounds on mT and ρ_3

3.1 Weakly induced subgraphs – A tool for lower bounds

Most of the lower bounds on mT and ρ_3 that we exhibit in Section 4 are through a particular graph construction. The general idea is that, if we have a collection of graphs $H_1, \dots, H_n$ with certain structural and labelling properties, then, under particular circumstances, it is possible to combine these H_i's in some fashion to form a bigger graph G in which the H_i's retain their respective labelling properties, from which we can deduce that G itself has certain labelling properties.

In order to state that construction formally, we need to introduce some terminology first (see Figure 1 for an illustration). Let G and H be two graphs. We say that *G contains H as a weakly induced subgraph X* if there exists an induced subgraph X of G such that H is a spanning subgraph of X, and, for every vertex $v \in V(H)$, either $d_H(v) = 1$ or $d_H(v) = d_G(v)$. In other words, if we add to H the edges of G that connect the vertices of degree 1 in H, we obtain X. That is, for

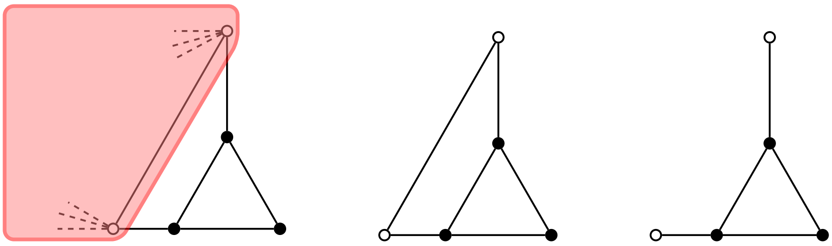


Figure 1: A graph G containing another graph H as a weakly induced subgraph X. In G, the white vertices can have arbitrarily many neighbours in the red part, while the full neighbourhood of the black vertices is as displayed. In H, the white vertices are the border vertices, while the black vertices are the core vertices.

every edge $uv \in E(G)$, if $u \in V(X)$ and $v \in V(G) \setminus V(X)$, then $d_H(u) = 1$; we call these the *border vertices* of H. Also, we call the other vertices of H (i.e., those that are not border vertices) its *core vertices*. By the definitions, note that if G contains H as a weakly induced subgraph and $\delta(H) \geq 2$, then G is isomorphic to H. For this reason, this notion makes more sense when $\delta(H) = 1$.

Let X_1, X_2 be two weakly induced subgraphs of a graph G. We say that X_1 and X_2 are *disjoint* (in G) if they share no core vertices. It follows directly from the definition that, for every $v \in V(G)$, if $v \in V(X_1) \cap V(X_2)$, then v is a border vertex of both X_1 and X_2.

Let ℓ be a labelling of G. For a subgraph H of G, we denote by $\ell|_H$ the *restriction* of ℓ to the edges of H, i.e., we have $\ell|_H(e) = \ell(e)$ for every edge $e \in E(H)$. Assume now that G contains H as a weakly induced subgraph X. Abusing the notations, we will sometimes write $\ell|_H$, which refers to the labelling of H inferred from $\ell|_X$, i.e., where $\ell|_H(e) = \ell|_X(e)$ for every $e \in E(H)$.

The key result is that, if a graph G contains other graphs $H_1, \dots, H_n$ as pairwise disjoint weakly induced subgraphs, then the labelling properties of the H_i's can be inferred to those of G:

Lemma 3.1. *Let G be a graph containing nice graphs $H_1, \dots, H_n$ as pairwise disjoint weakly induced subgraphs $X_1, \dots, X_n$. If ℓ is a proper 3-labelling of G, then $\ell|_{H_i}$ is a proper 3-labelling of H_i for every $i \in \{1, \dots, n\}$. Consequently, $\mathrm{mT}(G) \geq \sum_{i=1}^{n} \mathrm{mT}(H_i)$.*

Proof. Consider H_j for some $1 \leq j \leq n$. Since, by any labelling of a nice graph, a vertex of degree 1 cannot get the same colour as its unique neighbour, then it cannot be involved in a conflict. This implies that $\ell|_{H_j}$ is proper if and only if any two adjacent core vertices of H_j get distinct colours by $\ell|_{H_j}$. By the definition of a weakly induced subgraph, recall that we have $d_{H_j}(v) = d_{X_j}(v) = d_G(v)$ for every core vertex v of H_j, which implies that $c_{\ell|_{H_j}}(v) = c_{\ell|_{X_j}}(v) = c_\ell(v)$. Thus, for every edge $uv \in E(H_j)$ joining core vertices, we have $c_\ell(u) = c_{\ell|_{H_j}}(u) = c_{\ell|_{X_j}}(u) \neq c_{\ell|_{X_j}}(v) = c_{\ell|_{H_j}}(v) = c_\ell(v)$ since ℓ is proper, meaning that $\ell|_{H_j}$ is also proper. Now, since G contains nice graphs $H_1, \dots, H_n$ as pairwise disjoint weakly induced subgraphs $X_1, \dots, X_n$, then $\mathrm{mT}(G) \geq \sum_{i=1}^{n} \mathrm{mT}(H_i)$. □

In the next lemma, we point out that, in some contexts, we can add some structure to a given graph without altering its value of mT. In some of the later proofs, this will be particularly convenient for applying inductive arguments or simplifying the structure of a considered graph.

Lemma 3.2. *Let G be a nice graph with minimum degree 1 and $v \in V(G)$ be such that $d(v) = 1$. If G' is the graph obtained from G by adding $x > 0$ vertices of degree 1 adjacent to v, then $\mathrm{mT}(G') = \mathrm{mT}(G)$.*

Proof. Since G' contains G as a weakly induced subgraph, then by Lemma 3.1, we have that $\mathrm{mT}(G') \geq \mathrm{mT}(G)$. To show that $\mathrm{mT}(G') \leq \mathrm{mT}(G)$, it suffices to extend a proper 3-labelling of G to one of G' that uses the same number of edges labelled 3. To do this, simply note that since each one of the leaves adjacent to v has degree 1, its colour cannot be in conflict with that of v. Thus, the only colour conflict that can occur when extending the labelling, is between v and its unique neighbour in G. If, by labelling all of the edges incident to the leaves adjacent to v with 1's, there is a colour conflict between v and its neighbour in G, then it suffices to change exactly one of those labels to 2. □

Through an easy use of Lemma 3.1, we can already establish results of interest regarding the parameter mT. For instance, we can prove that each graph class $\mathcal{G}_p$ ($p \geq 1$) contains infinitely many graphs with various substructures.

Observation 3.3. *$\mathcal{G}_p$ contains infinitely many graphs with various substructures for every $p \geq 1$.*

Proof. Let H be a graph with $\delta(H) = 1$ and $\mathrm{mT}(H) = 1$ (such graphs exist, see, e.g., our results from Section 4). Let uv be an edge of H such that $d(u) = 1$ and $d(v) \geq 2$. Also, let T be any locally irregular graph[1] with an edge $u'v'$ such that $d(u') = 1$ and $d(v') \geq 3p + 3$.

Now, let G be the graph that is the disjoint union of T and of p copies $X_1, \dots, X_p$ of H, and identify u' and the p copies of u to a single vertex w. Clearly, G contains T and the disjoint union of p copies of H as pairwise disjoint weakly induced subgraphs $T, X_1, \dots, X_p$ (abusing the notation, for simplicity we refer to both the original T and its copy in G as T). By Lemma 3.1, we have $\mathrm{mT}(G) \geq \mathrm{mT}(T) + p \cdot \mathrm{mT}(H) = p$ since T is locally irregular (thus, $\mathrm{mT}(T) = 0$) and $\mathrm{mT}(H) = 1$.

To prove that the equality actually holds, it suffices to construct a proper 3-labelling ℓ of G with $\mathrm{nb}_\ell(3) = p$. Let ℓ' be a proper 3-labelling of H such that $\mathrm{nb}_{\ell'}(3) = 1$, which exists since $\mathrm{mT}(H) = 1$. To obtain ℓ, for each X_i, we set $\ell(e) = \ell'(e)$ for every edge e of that X_i, while we set $\ell(e) = j$ for every edge e of T, where $j \in \{1, 2\}$ is chosen so that $c_\ell(w) \neq c_{\ell'}(v)$ for v in each copy of X_i (recall that $c_{\ell'}(v)$ is the same for each copy of X_i). As a result, for any X_i, for every vertex $x \neq w$ from that X_i, we get $c_\ell(x) = c_{\ell'}(x)$. Hence, for any X_i, for every edge xy not containing w from that X_i, we have $c_\ell(x) \neq c_\ell(y)$. Furthermore, for every vertex x of T different from w, we have either $c_\ell(x) = d(x)$ or $c_\ell(x) = 2d(x)$, meaning that, for every edge xy of T not containing w, we have $c_\ell(x) \neq c_\ell(y)$ since T is locally irregular. Now, by the construction of ℓ, note that w cannot be in conflict with its neighbours in the X_i's (due to the choice of j), and $c_\ell(w) < 3p + 3 \leq d(v') \leq c_\ell(v')$, meaning that w and v' cannot be in conflict. Thus, ℓ is proper.

Note that the "various substructures" part of the statement is implied by the fact that the structure of T does not matter, and can be anything as long as T is locally irregular and has the particular edge $u'v'$. In particular, T can potentially contain any graph as an induced subgraph. $\square$

Through the same ideas, we can actually prove that, more generally speaking, a nice characterisation of any $\mathcal{G}_p$ should not exist, unless P=NP.

Theorem 3.4. *Let $p \geq 1$ and G be a graph. Deciding if $G \in \mathcal{G}_p$ is NP-complete.*

Proof. The problem is obviously in NP. Let us focus on proving it is also NP-hard. This is done by a reduction from the 2-Labelling problem, which was proved to be NP-hard, e.g., by Dudek and Wajc in [11]. In that problem, a graph H is given, and the goal is to decide whether H admits proper 2-labellings. Given an instance H of 2-Labelling, we construct, in polynomial time, a graph G such that $\mathrm{mT}(G) = p$ if and only if H admits proper 2-labellings.

Looking closely at the proof from [11], it can be noted that 2-Labelling remains NP-hard when restricted to graphs with minimum degree 1. Thus, we can assume H has this property.

The construction of G is achieved as follows. Let H' be a graph with $\delta(H') = 1$ and $\mathrm{mT}(H') = 1$ (as mentioned in the proof of Observation 3.3, such graphs exist). Let uv be an edge of H' such that $d(u) = 1$ and $d(v) \geq 2$. Now, start from G being the disjoint union of H and of p copies $X_1, \dots, X_p$ of H', and then identify a vertex of degree 1 of H and of the p copies of u to a single vertex w. Finally, attach new vertices of degree 1 to w so that the degree of w in G gets at least four times bigger than the degree of any of its neighbours. Clearly, the construction of G is achieved in polynomial time.

We now prove the equivalence between the two problems.

- Assume ℓ is a proper 3-labelling of G such that $\mathrm{nb}_\ell(3) = p$. Note that G contains H and p copies of H' as pairwise disjoint weakly induced subgraphs $H, X_1, \dots, X_p$. Due to Lemma 3.1, and because $\mathrm{mT}(H') = 1$, this means that we must have $\mathrm{nb}_{\ell|_{X_i}}(3) = 1$ for every $i \in \{1, \dots, p\}$, and, thus, $\mathrm{nb}_{\ell|_H}(3) = 0$. Then, $\ell|_H$ must be a proper 2-labelling of H.
- Assume ℓ is a proper 2-labelling of H. Since $\mathrm{mT}(H') = 1$, there exists a proper 3-labelling ℓ' of H' where $\mathrm{nb}_{\ell'}(3) = 1$. Now, let ℓ'' be the 3-labelling of G obtained by setting $\ell''(e) = \ell(e)$

[1] A graph is *locally irregular* if no two of its adjacent vertices have the same degree.

for every $e \in E(H)$, setting $\ell''(e) = \ell'(e)$ for every $e \in E(X_i)$ for each $i \in \{1, \dots, p\}$, and setting $\ell''(e) = 1$ for every remaining pending edge attached at w. By the properties of ℓ and ℓ', and by arguments similar to those used in the proof of Observation 3.3, no conflict can occur along an edge not containing w. Now, regarding w, due to the choice of its degree, it can be noted that $c_{\ell''}(w)$ must be strictly bigger than the colour of each of the neighbours of w. Thus, ℓ'' is a proper 3-labelling of G, and $\mathrm{nb}_{\ell''}(3) = p$. □

3.2 Switching closed walks – A tool for upper bounds

Due to Theorem 2.1, investigating the parameters mT and ρ_3 only makes sense for graphs with chromatic number at least 3, i.e., that are not bipartite. These graphs have odd-length cycles. We take advantage of these cycles to prove the following upper bound on ρ_3 for 3-chromatic graphs.

Theorem 3.5. *If G is a connected 3-chromatic graph, then $\rho_3(G) \leq |V(G)|/|E(G)|$.*

Proof. Since G is not bipartite, there exists an odd-length cycle C in G. Let H be a subgraph of G constructed as follows. Start from $C = H$. Then, until $V(H) = V(G)$, repeatedly choose a vertex $v \in V(G) \setminus V(H)$ such that there exists a vertex $u \in V(H)$ with $uv \in E(G)$, and add the edge uv to H. In the end, H is a connected spanning subgraph of G containing only one cycle, C, which is of odd length. Then, we have $|E(H)| = |V(G)|$.

Let $\phi : V(G) \to \{0, 1, 2\}$ be a proper 3-vertex-colouring of G. In what follows, our goal is to construct a 3-labelling ℓ of G such that $c_\ell(v) \equiv \phi(v) \bmod 3$ for every vertex $v \in V(G)$, thus making ℓ proper. Additionally, to prove the full statement, we want ℓ to verify $\mathrm{nb}_\ell(3) \leq |V(G)|/|E(G)|$. Note that, aiming at vertex colours modulo 3, we can instead assume that ℓ assigns labels $0, 1, 2$, and require $\mathrm{nb}_\ell(0) \leq |V(G)|/|E(G)|$. To obtain such a labelling, we start from ℓ assigning label 2 to all edges of G. We then modify ℓ iteratively until all vertex colours are as desired modulo 3.

As long as G has a vertex v with $c_\ell(v) \not\equiv \phi(v) \bmod 3$, we apply the following procedure. Choose $W = (v, v_1, \dots, v_n, v)$, a closed walk[2] of odd length in G starting and ending at v, and going through edges of H only. This walk is sure to exist. Indeed, consider, in H, a (possibly empty) path P from v to the closest vertex u of C (if v lies on C, then note that $u = v$ and P has no edge). Then, the closed walk $vPuCuPv$ is a possible W. We then follow the consecutive edges of W, starting from v and ending at v, and, going along, we apply $+2, -2, +2, -2, \dots, +2$ (modulo 3) to the labels assigned by ℓ to the traversed edges. As a result, note that $c_\ell(x)$ is not altered modulo 3 for every vertex $x \neq v$, while $c_\ell(v)$ is incremented by 1 modulo 3. If $c_\ell(v) \equiv \phi(v) \bmod 3$, then we are done with v. Otherwise, we repeat this switching procedure once again, so that v fulfils that property.

Eventually, we get $c_\ell(v) \equiv \phi(v) \bmod 3$ for every $v \in V(G)$, meaning that ℓ is proper. Recall that we have $\ell(e) = 2$ for every $e \in E(G) \setminus E(H)$. Thus, only the edges of H can be assigned label 0 by ℓ. Since there are exactly $|V(G)|$ such edges, and we can replace all assigned 0's with 3's without breaking the modulo 3 property, we have $\mathrm{mT}(G) \leq |V(G)|$, which implies that $\rho_3(G) \leq |V(G)|/|E(G)|$. □

Theorem 3.5, by itself, has implications on Conjecture 2.2. In particular, every sufficiently dense connected 3-chromatic graph verifies the conjecture. This remark applies to, e.g., every connected 3-chromatic graph G with $\delta(G) \geq 6$, since it obviously verifies $|E(G)| \geq 3|V(G)|$. Note that, in that case, the connectivity condition can actually be dropped, as every connected component of a 3-chromatic graph is 3-colourable (so, for each component, one of Theorems 2.1 and 3.5 applies).

Corollary 3.6. *If G is a 3-chromatic graph with $\delta(G) \geq 6$, then $\rho_3(G) \leq 1/3$.*

In general, and more particularly for less dense graphs, it would be interesting to find ways to improve the arguments in the proof of Theorem 3.5 to further reduce the number of assigned 3's. Note that several of our arguments could actually be subject to improvement. For instance, in the current proof, we always set $\ell(e) = 2$ for an edge $e \in E(G) \setminus E(H)$, which might be one of the reasons why many 3's might appear through the eventual walk-switching procedure. It seems, however, that in general, this is tough to improve upon significantly without further assumptions on G. Similarly, in some contexts, it might be possible to choose the unicyclic subgraph H in a

[2] Recall that a *walk* in a graph is a path in which vertices and edges can be repeated.

clever way, but this seems hard to do in general. A more interesting direction is about choosing the proper 3-vertex-colouring ϕ in a more clever way. In the next lemma, we show a way to play with ϕ in order to reduce the number of 3's assigned by ℓ to certain sets of edges.

Lemma 3.7. *Let G be a graph and ℓ be a proper $\{0,1,2\}$-labelling of G such that $c_\ell(u) \not\equiv c_\ell(v) \bmod 3$ for every edge $uv \in E(G)$. If H is a (not necessarily connected) spanning d-regular subgraph of G for some $d \geq 1$, then there exists a proper $\{0,1,2\}$-labelling ℓ' of G such that $c_{\ell'}(u) \not\equiv c_{\ell'}(v) \bmod 3$ for every edge $uv \in E(G)$ and that assigns label 0 to at most a third of the edges of $E(H)$. Moreover, for every edge $e \in E(G) \setminus E(H)$, $\ell'(e) = \ell(e)$.*

Proof. We construct the following new labelling: starting from ℓ, add 1 (modulo 3) to all the labels assigned by ℓ to the edges of H. The resulting labelling ℓ_1 is a proper $\{0,1,2\}$-labelling of G such that $c_{\ell_1}(u) \not\equiv c_{\ell_1}(v) \bmod 3$ for every edge $uv \in E(G)$. Indeed, for every $v \in V(G)$, we have $c_{\ell_1}(v) \equiv c_\ell(v) + d \bmod 3$. Thus, if there exist two vertices $u, v \in V(G)$ such that $c_{\ell_1}(u) \equiv c_{\ell_1}(v) \bmod 3$, then $c_\ell(u) \equiv c_\ell(v) \bmod 3$, a contradiction. We define ℓ_2 in a similar fashion, by adding 1 (modulo 3) to all the labels assigned by ℓ_1 to the edges of H. Similarly, ℓ_2 is proper. Note that, for every edge $e \in E(H)$, we have $\{\ell(e), \ell_1(e), \ell_2(e)\} = \{0,1,2\}$. This implies that at least one of ℓ, ℓ_1, ℓ_2 assigns label 0 to at most a third of the edges of $E(H)$. Finally, since none of the labels of the edges of $E(G) \setminus E(H)$ were changed to obtain ℓ_1 from ℓ and to obtain ℓ_2 from ℓ_1, the last statement of the lemma holds. □

In Lemma 3.7, if $d = 2$, then H forms a cycle cover of G. Thus, when H is also a unicyclic spanning connected subgraph of G, a particular application of Lemma 3.7 in conjunction with the proof of Theorem 3.5 gives the following corollary:

Corollary 3.8. *If G is a 3-chromatic Hamiltonian graph of odd order, then $\rho_3(G) \leq 1/3$.*

Another application of Lemma 3.7 is for $d = 1$, i.e., H forms an independent edge cover. That is, Lemma 3.7 in conjunction with the proof of Theorem 3.5 can be used, for instance, to prove that class-1 cubic graphs (i.e., admitting three disjoint perfect matchings) verify Conjecture 2.2. Indeed, let G be a class-1 cubic graph, and let M_1, M_2, M_3 be three disjoint perfect matchings of G. We can assume that G is not bipartite, as otherwise Theorem 2.1 would apply, and also that G is not K_4 (as it can be checked by hand that $\mathrm{mT}(K_4) = 1$). Thus, G is 3-chromatic. Mimicking the proof of Theorem 3.5, we can use an odd-length cycle of G to deduce a $\{0,1,2\}$-labelling ℓ of G where $c_\ell(u) \not\equiv c_\ell(v) \bmod 3$ for every $uv \in E(G)$, and, by Lemma 3.7, we can assume that, for every M_i, at most a third of its edges are assigned label 0 by ℓ. Since the M_i's partition $E(G)$, turning all 0's by ℓ into 3's, we end up with a proper 3-labelling of G where at most a third of the edges are assigned label 3. In Section 4, via a different approach we will actually prove Conjecture 2.2 for all cubic graphs.

Regarding the proof of Theorem 3.5 and the previous arguments, it would be interesting if we could always choose the unicyclic subgraph H in such a way that it admits several disjoint perfect matchings, so that Lemma 3.7 can be employed to reduce the number of assigned 3's. In the proof of Theorem 4.19, we will point out one graph class in which this strategy can be employed.

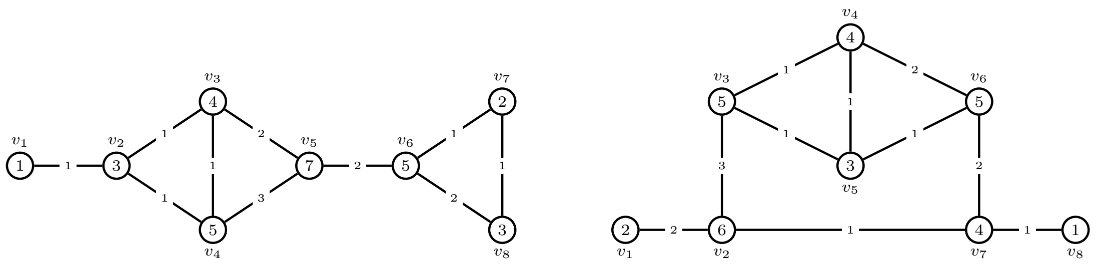
4 Results on the parameters mT and ρ_3 for some graph classes

We now use the tools introduced in Section 3 to exhibit results on the parameters mT and ρ_3 for some particular classes of 3-chromatic graphs (and beyond sometimes). In particular, we prove that, for many classes $\mathcal{F}$ of 3-chromatic graphs, there is no $p \geq 1$ such that $\mathcal{F} \subset \mathcal{G}_{\leq p}$ (i.e., a constant number p of 3's is not sufficient to construct a proper 3-labelling of at least one of the graphs in $\mathcal{F}$). In such cases, we provide upper bounds for $\rho_3(\mathcal{F})$.

4.1 Connected graphs needing lots of 3's

As mentioned earlier, we are aware of only two connected graphs for which the parameter ρ_3 is exactly 1/3, and these are C_3 and C_6[3]. A legitimate question to ask, is whether the bound in Conjecture 2.2 is accurate in general, i.e., whether it can be attained by arbitrarily large graphs.

[3] Any disjoint union of C_3's and C_6's reaches that value. This is why Conjecture 2.2 focuses on connected graphs.



(a) A_1 (b) A_2

Figure 2: Proper 3-labellings ℓ of A_1 and A_2 with $\mathrm{nb}_\ell(3) = 1$. The colours by c_ℓ are indicated by integers within the vertices.

In light of these thoughts, our goal in this subsection is to provide a class of arbitrarily large connected graphs achieving the largest possible ratio ρ_3. Our arguments are based on our notion of weakly induced subgraphs, introduced in Section 3. Basically, the idea is to have a connected graph H with $\mathrm{mT}(H) \geq 1$, and to combine p copies of H to a single connected graph G so that $\mathrm{mT}(G) \geq p$. To guarantee that $\rho_3(G)$ is large, the main ideas are 1) to choose H so that $|E(H)|$ is as small as possible, and 2) to construct G so that only a few edges join the p copies of H. These two conditions are to ensure that $|E(G)|$ itself is as small as possible.

We ran computer programs to find graphs H with $\delta(H) = 1$, $\mathrm{mT}(H) \geq 1$, and with the fewest edges possible. It turns out that the smallest such graphs have 10 edges. Two such graphs, which we call A_1 and A_2 throughout this section, are depicted in Figure 2. Since these two graphs will allow us to prove several lower bounds on ρ_3 for various graph classes, let us formally establish that they do have the desired property.

Observation 4.1. $\mathrm{mT}(A_1) = 1$.

Proof. A proper 3-labelling ℓ of A_1 with $\mathrm{nb}_\ell(3) = 1$ is depicted in Figure 2(a), which shows that $\mathrm{mT}(A_1) \leq 1$. We now prove that $\mathrm{mT}(A_1) > 0$, i.e., that there is no proper 2-labelling of A_1. Towards a contradiction, assume a proper 2-labelling ℓ of A_1 exists.

By Observation 2.3, we have $\ell(v_6v_7) \neq \ell(v_6v_8)$. Also, since ℓ is a 2-labelling, we have $c_\ell(v_5) \in \{3, 4, 5, 6\}$. We distinguish the following cases:

- **Case 1:** $c_\ell(v_5) = 3$. Then, $\ell(v_3v_5) = \ell(v_4v_5) = \ell(v_5v_6) = 1$, and so, $\{c_\ell(v_3), c_\ell(v_4)\} = \{4, 5\}$. Assume w.l.o.g., that $c_\ell(v_3) = 4$ and $c_\ell(v_4) = 5$. It follows that $\ell(v_2v_3) = 1$ and $\ell(v_2v_4) = \ell(v_3v_4) = 2$, and thus, $c_\ell(v_2) \in \{4, 5\} = \{c_\ell(v_3), c_\ell(v_4)\}$, which contradicts that ℓ is proper.
- **Case 2:** $c_\ell(v_5) = 4$. Then, v_5 has exactly one incident edge labelled 2. First, assume that $\ell(v_5v_6) = 2$. It follows that $\ell(v_3v_5) = \ell(v_4v_5) = 1$, and thus, $\{c_\ell(v_3), c_\ell(v_4)\} = \{3, 5\}$. Assume w.l.o.g, that $c_\ell(v_3) = 3$ and $c_\ell(v_4) = 5$. Since $c_\ell(v_3) = 3$, we have that $\ell(v_2v_3) = \ell(v_3v_4) = 1$, and thus, $c_\ell(v_4) \leq 4$, a contradiction. Second, assume that $\ell(v_5v_6) = 1$. Since $\{\ell(v_6v_7), \ell(v_6v_8)\} = \{1, 2\}$, we have $c_\ell(v_6) = 4 = c_\ell(v_5)$, a contradiction.

For the next two cases, let $A_1' = A_1 - \{v_7v_8\}$ and observe that A_1' is 3-quasi regular.

- **Case 3:** $c_\ell(v_5) = 5$. Then, by Observation 2.5, the 2-labelling $\ell' = \mathrm{sw}(\ell|_{A_1'}, \sigma_{1\leftrightarrow 2})$ is also proper for A_1'. Moreover, recall that $\{\ell'(v_6v_7), \ell'(v_6v_8)\} = \{1, 2\}$. It follows that ℓ' can be extended to a proper 2-labelling ℓ'' of A_1 by setting $\ell''(v_7v_8) = 1$. But then, $c_{\ell''}(v_5) = 4$, and we get a contradiction to **Case 2** above.
- **Case 4:** $c_\ell(v_5) = 6$. Similarly to the previous case, the 2-labelling $\ell' = \mathrm{sw}(\ell|_{A'}, \sigma_{1\leftrightarrow 2})$ is proper for A_1' and it can be extended to a proper 2-labelling ℓ'' of A_1 by setting $\ell''(v_7v_8) = 1$. But then, $c_{\ell''}(v_5) = 3$, and we get a contradiction to **Case 1** above. □

Observation 4.2. $\mathrm{mT}(A_2) = 1$.

Proof. A proper 3-labelling ℓ of A_2 with $\mathrm{nb}_\ell(3) = 1$ is depicted in Figure 2(b). Thus, $\mathrm{mT}(A_2) \leq 1$. Let us prove now that $\mathrm{mT}(A_2) > 0$, i.e., that there is no proper 2-labelling of A_2. Towards a contradiction, assume a proper 2-labelling ℓ of A_2 exists.

Since ℓ is a 2-labelling, we have $c_\ell(v_3) \in \{3,4,5,6\}$. We distinguish the following cases:

- **Case 1:** $c_\ell(v_3) = 3$. Then, $\ell(v_2v_3) = \ell(v_3v_4) = \ell(v_3v_5) = 1$, and so, $\{c_\ell(v_4), c_\ell(v_5)\} = \{4,5\}$. Assume w.l.o.g., that $c_\ell(v_4) = 4$ and $c_\ell(v_5) = 5$. It follows that $\ell(v_5v_4) = \ell(v_5v_6) = 2$ and $\ell(v_4v_6) = 1$, and thus, $c_\ell(v_6) \in \{4,5\} = \{c_\ell(v_4), c_\ell(v_5)\}$, which contradicts that ℓ is proper.
- **Case 2:** $c_\ell(v_3) = 4$. Then, v_3 has exactly one incident edge labelled 2. First, assume that $\ell(v_3v_2) = 2$. It follows that $\ell(v_3v_4) = \ell(v_3v_5) = 1$, and thus, $\{c_\ell(v_4), c_\ell(v_5)\} = \{3,5\}$. Assume w.l.o.g., that $c_\ell(v_4) = 3$ and $c_\ell(v_5) = 5$. Since $c_\ell(v_4) = 3$, we have that $\ell(v_4v_5) = 1$, and thus, $c_\ell(v_5) \leq 4$, a contradiction. Then, assume w.l.o.g., that $\ell(v_3v_5) = 2$ (and $\ell(v_3v_2) = \ell(v_3v_4) = 1$). It follows that $c_\ell(v_5) \in \{5,6\}$ and $c_\ell(v_4) \in \{3,5\}$. If $c_\ell(v_4) = 5$, then $c_\ell(v_5) = 6$. This implies that $\ell(v_4v_6) = \ell(v_5v_6) = 2$, and thus, $c_\ell(v_6) \in \{5,6\} = \{c_\ell(v_4), c_\ell(v_5)\}$, a contradiction. Otherwise, $c_\ell(v_4) = 3$, and so, $c_\ell(v_5) = 5$ and $c_\ell(v_6) = 4$. Hence, $\ell(v_6v_7) = \ell(v_3v_2) = 1$, $c_\ell(v_2), c_\ell(v_7) \in \{3,5\}$ (because $c_\ell(v_3) = c_\ell(v_6) = 4$). We now get a contradiction no matter how v_1v_2, v_2v_7, and v_7v_8 are labelled, as either $c_\ell(v_2) = c_\ell(v_7)$ or $4 \in \{c_\ell(v_2), c_\ell(v_7)\}$.
- **Case 3:** $c_\ell(v_3) = 5$. Then, by Observation 2.5, the 2-labelling $\ell' = \mathrm{sw}(\ell, \sigma_{1\leftrightarrow 2})$ is also proper (note that A_2 is 3-quasi regular). Since $c_{\ell'}(v_3) = 4$, we get a contradiction to **Case 2** above.
- **Case 4:** $c_\ell(v_3) = 6$. Then, by Observation 2.5, the 2-labelling $\ell' = \mathrm{sw}(\ell, \sigma_{1\leftrightarrow 2})$ is also proper. Since $c_{\ell'}(v_3) = 3$, we get a contradiction to **Case 1** above. □

We can now use A_1 or A_2 to build arbitrarily large connected graphs with large ρ_3.

Theorem 4.3. *There exist arbitrarily large connected graphs G with $\rho_3(G) \geq 1/10$.*

Proof. Let $p \geq 1$ be fixed. We construct a connected graph G with $10p$ edges, such that $\mathrm{nb}_\ell(3) \geq p$ for any proper 3-labelling ℓ of G, which implies that $\rho_3(G) \geq 1/10$. Start, as G, with p disjoint copies of A_1 (or A_2), and identify a vertex of degree 1 from each of these p copies to a single vertex. Clearly, G has the desired connectivity and size properties. The labelling property follows from Lemma 3.1 and Observation 4.1 or 4.2, since G contains p copies of A_1 or A_2 as pairwise disjoint weakly induced subgraphs. Moreover, these arguments apply for any value of p, and so, G can be as large as desired. □

It is worth mentioning that it can be checked that we actually have $\rho_3(G) = 1/10$ for the graphs G constructed in the proof of Theorem 4.3.

4.2 Bounds for connected cubic graphs

Recall that, given a cubic graph G, it is NP-complete to decide whether $\chi_\Sigma(G) \leq 2$ (see [9]). In other words, unless P=NP, there is no nice characterisation of cubic graphs admitting proper 2-labellings. Then, a legitimate question to ask is whether they always admit proper 3-labellings assigning only a limited number of 3's. We prove that there is actually no $p \geq 1$ such that the class of all cubic graphs lies in $\mathcal{G}_{\leq p}$. In contrast, we verify Conjecture 2.2 for this class of graphs.

First off, we note that the construction in the proof of Theorem 4.3 can be modified slightly to reach the same conclusion for cubic graphs.

Theorem 4.4. *There exist arbitrarily large connected cubic graphs G with $\rho_3(G) \geq 1/10$.*

Proof. The proof is essentially the same as that of Theorem 4.3, except that we must combine copies of A_1 and of A_2 in such a way that the resulting graph G is cubic. One way to proceed is as follows. Let $p \geq 2$ be fixed. The construction of G is as follows. Add p copies $X_0, \dots, X_{p-1}$ of A_2, and identify their vertices of degree 1 sequentially in a "cyclic" way. That is, for every $i \in \{0, \dots, p-1\}$, identify the vertex v_8 of X_i and the vertex v_1 of X_j, where $j = i + 1 \bmod p$, to a single vertex (where we refer to the vertices of A_2 following the terminology in Figure 2(b)). Note that, at this point, G is not cubic because of p vertices of degree 2 (those we have identified), which we denote by $x_0, \dots, x_{p-1}$.

Let us now further modify G as follows. For every $i \in \{0, \dots, p-1\}$, add a copy Y_i of A_1 to the graph, and identify its vertex v_1 (following Figure 2(a)) and x_i. As a result, the x_i's become of degree 3, but in each of the Y_i's there remain two vertices of degree 2 (vertices v_7 and v_8 in Figure 2(a)). We denote by $y_{1,i}, y_{2,i}$ those two vertices in a given Y_i.

Finally, for every $i \in \{0, \dots, p-1\}$, add a new copy Z_i of A_2 to G, identify one of its vertices of degree 1 and $y_{1,i}$, and identify its other vertex of degree 1 and $y_{2,i}$. Note that G is now cubic.

To summarise, G was constructed using $2p$ copies (the X_i's and Z_i's) of A_2 and p copies (the Y_i's) of A_1, that were combined in an edge-disjoint way. Then $|E(G)| = 30p$. Also, these $3p$ copies of A_1 and A_2 appear as pairwise disjoint weakly induced subgraphs of G, and thus, from Lemma 3.1 we get that $\mathrm{mT}(G) \geq 3p$, thereby our conclusion. □

Again, it is not too complicated to check that, for any cubic graph G constructed in the proof of Theorem 4.4, we actually have $\mathrm{mT}(G) = |E(G)|/10$. Thus, our construction cannot be used to further improve the $1/10$ lower bound in the statement.

Regarding upper bounds, we prove that the parameter ρ_3 cannot exceed the $1/3$ barrier in cubic graphs. In other words, we prove Conjecture 2.2 for these graphs.

Theorem 4.5. *If G is a cubic graph, then $\rho_3(G) \leq 1/3$.*

Proof. We can assume that G is connected. Also, we can assume that G is neither K_4 (in which case the claim can be verified by hand) nor bipartite (due to Theorem 2.1). Thus, by Brooks' Theorem [6], we know that G is 3-chromatic. Recall that $|E(G)| = \frac{3}{2}|V(G)|$.

Let us now mimic the proof of Theorem 3.5 to get a proper 3-labelling ℓ of G such that, for every edge $e \in E(G) \setminus E(H)$ (where, recall, H is a particular unicyclic spanning connected subgraph of G), we have $\ell(e) = 2$. This means that only the edges of H can be labelled 1 or 3 by ℓ. If $\mathrm{nb}_\ell(3) \leq \frac{1}{2}|E(H)|$, then the result follows since $|E(H)| = \frac{2}{3}|E(G)|$. So, assume now that $\mathrm{nb}_\ell(3) > \frac{1}{2}|E(H)|$, and hence, $\mathrm{nb}_\ell(1) < \frac{1}{2}|E(H)|$. Since G is regular, by Lemma 2.4, the 3-labelling $\ell' = \mathrm{sw}(\ell, \sigma_{1\leftrightarrow 3})$ of G is also proper. Since only the edges of H are labelled 1 or 3 by ℓ, we deduce that $\mathrm{nb}_{\ell'}(3) = \mathrm{nb}_\ell(1) < \frac{1}{2}|E(H)| = \frac{1}{3}|E(G)|$, and the result follows. □

4.3 Bounds for connected planar graphs with large girth

Recall that the *girth* $g(G)$ of a graph G is the length of its shortest cycle. For any $g \geq 3$, we denote by $\mathcal{P}_g$ the class of planar graphs with girth at least g. Note, for instance, that $\mathcal{P}_3$ is the class of all planar graphs, and that $\mathcal{P}_4$ is the class of all triangle-tree planar graphs. Recall that the girth of a tree is set to ∞, since it has no cycle.

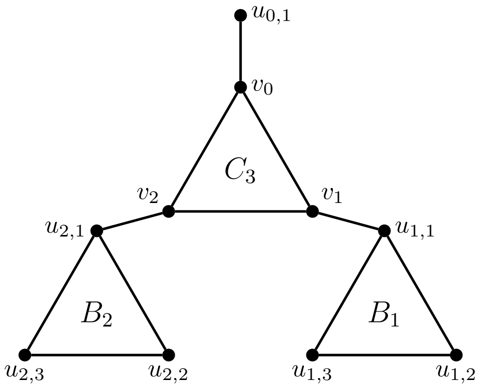
To date, it is still unknown whether planar graphs verify the 1-2-3 Conjecture, which makes the study of the parameters mT and ρ_3 adventurous for this class of graphs. Something we can state, however, is that there is no $p \geq 1$ such that planar graphs lie in $\mathcal{G}_{\leq p}$. This can be established from the construction in the proof of Theorem 4.3 (or from that of Theorem 4.4 to additionally get a cubic graph assumption), since the graphs A_1 and A_2 are planar.

Theorem 4.6. *There exist arbitrarily large connected planar graphs G with $\rho_3(G) \geq 1/10$.*

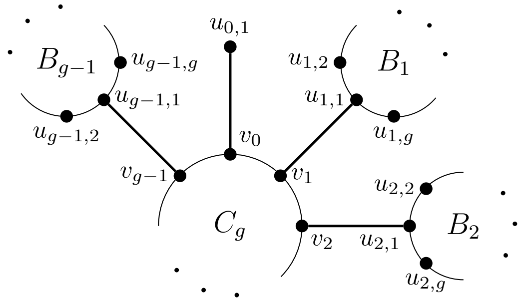
To go further, we can consider planar graphs with large girth. Indeed, as established by Grötzsch's Theorem [12], triangle-tree planar graphs are 3-colourable, which means that they verify the 1-2-3 Conjecture (see [14]). In what follows, we prove two main results. First, we prove that, for every $g \geq 3$, there is no $p \geq 1$ such that $\mathcal{P}_g \subset \mathcal{G}_{\leq p}$. Second, we prove that, as the girth $g(G)$ of a planar graph G grows, the ratio $\rho_3(G)$ decreases. As a side result, we prove Conjecture 2.2 for planar graphs with girth at least 36.

In order to prove the first result above, note that we cannot use the graphs A_1 and A_2 introduced previously, as they contain triangles. Instead, we provide another construction, yielding, for any $g \geq 3$, a planar graph S_g with girth g. Start from S_g being the cycle $C_g = (v_0, \dots, v_{g-1}, v_0)$ on g vertices. Then, for each $i \in \{0, \dots, g-1\}$, add a new vertex $u_{i,1}$ and the edge $v_i u_{i,1}$ to S_g. Then, for every $i \in \{1, \dots, g-1\}$, add a cycle $B_i = (u_{i,1}, u_{i,2}, \dots, u_{i,g}, u_{i,1})$ to S_g, where $u_{i,2}, \dots, u_{i,g}$ are new vertices. Finally, let $u_{0,1}$ be the *root* of S_g. See Figure 3 for an illustration of S_3 and S_g. It is clear that all the cycles of S_g have length g, and thus, $g(S_g) = g$. Moreover, S_g is clearly planar, and $|E(S_g)| = g^2 + g$.

Note that S_g is bipartite whenever g is even. Since $\delta(S_g) = 1$, in such cases we have $\mathrm{mT}(S_g) = 0$ by Theorem 2.1. When $g \equiv 1 \bmod 4$, it can be checked (for instance, by using some of the



(a) S_3



(b) S_g

Figure 3: The planar graphs S_3 (left) and S_g (right) of girths 3 and g, respectively.

arguments in the proof of upcoming Lemma 4.7) that S_g admits proper 2-labellings, and thus we have $\mathrm{mT}(S_g) = 0$ in those cases as well. The main point for considering this construction is for the last possible values of g, the values where $g \equiv 3 \bmod 4$, for which the following is verified:

Lemma 4.7. *For every $g \geq 3$ with $g \equiv 3 \bmod 4$, we have $\mathrm{mT}(S_g) = 1$.*

Proof. We begin by showing that a proper 2-labelling of S_g must have specific properties. In what follows, for every $i \in \{1, \dots, g-1\}$, we denote by H_i the subgraph of S_g induced by $V(B_i) \cup \{v_i\}$.

Claim 4.8. *Let $i \in \{1, \dots, g-1\}$. By any proper 2-labelling ℓ of H_i, we have $\ell(u_{i,1}u_{i,2}) \neq \ell(u_{i,g}u_{i,1})$, and thus, $c_\ell(u_{i,1}) = \ell(u_{i,1}v_i) + 3$. Furthermore, such a proper 2-labelling exists.*

Proof of the claim. The first part of the claim follows from Observation 2.3. Indeed, since $g \equiv 3 \bmod 4$, it follows that we must have $\ell(u_{i,1}u_{i,2}) \neq \ell(u_{i,3}u_{i,4}) \neq \cdots \neq \ell(u_{i,g}u_{i,1})$. Now, it is easy to check that the following is a proper 2-labelling ℓ of H_i. Start by setting $\ell(u_{i,1}u_{i,2}) = 2$. Then, continue from $u_{i,2}u_{i,3}$ and, following the edges of B_i until reaching $u_{i,g}u_{i,1}$, assign labels $1, 1, 2, 2, 1, 1, \dots, 2, 2, 1, 1, 2, 2, 1, 1$. The edge $u_{i,1}v_i$ can then be assigned any label in $\{1, 2\}$. ◇

Assume now that there exists a proper 2-labelling ℓ of S_g, and let $\{\alpha, \beta\}$ be a permutation of $\{1, 2\}$. Then, there exists at least one $i \in \{0, \dots, g-1\}$ such that $\ell(v_{i-1}v_i) = \alpha$ and $\ell(v_iv_{i+1}) = \beta$ (where such operations over the indexes, here and further, are understood modulo g). Indeed, assume that this is not the case. Then, we would have that all the edges of C_g receive the same label α or β. For ℓ to be proper, the labels of the consecutive edges in $E^* = \{e_i = v_iu_{i,1} : 0 \leq i \leq g-1\}$ have to alternate between α and β. That is, we must have $\ell(e_0) = \alpha, \ell(e_1) = \beta, \ell(e_2) = \alpha, \dots, \ell(e_{g-1}) = \beta$. But, since g is odd, this is impossible. So let $v' = v_x$ (for $0 \leq x \leq g-1$) be one vertex of C_g whose incident edges in C_g are labelled α and β respectively, and assume w.l.o.g. that $\ell(v_xu_{x,1}) = \alpha$. It follows that $c_\ell(v_x) = \alpha + 3$. Note that the existence of at least one such vertex v' guarantees the existence of at least one additional vertex $v'' = v_y$, where $0 \leq y \leq g-1$ and $x \neq y$, such that $\ell(v_yv_{y-1}) = \beta$ and $\ell(v_yv_{y+1}) = \alpha$. Indeed, if this v'' did not exist, then, w.l.o.g., all the edges of C_g that are "after" v' (following the ordering of the vertices of C_g given above) would be labelled β. Cycling around, this would imply that $\ell(v_{x-1}v_x) = \beta$ as well, a contradiction. Thus, we can assume that $v' = v_x \neq v_0$, and so, $v_x \in V(H_x)$ (i.e., H_x exists). Then, by Claim 4.8, we have that $c_\ell(u_{x,1}) = \alpha + 3 = c_\ell(v_x)$, a contradiction.

So far, we have proven that $\mathrm{mT}(S_g) \geq 1$. In order to show that $\mathrm{mT}(S_g) = 1$, it suffices to provide a proper 3-labelling ℓ of S_g such that $\mathrm{nb}_\ell(3) = 1$. We construct one such labelling as follows. For every $i \in \{1, \dots, g-1\}$, we label the subgraph B_i following the 2-labelling scheme provided in Claim 4.8. Then, we set $\ell(v_0v_1) = 3$ and, for every edge $e \in E(C_g) \setminus \{v_0v_1\}$, we set $\ell(e) = 1$. Finally, for the edges $e_i \in E^*$, we set $\ell(e_0) = 1, \ell(e_1) = 2, \ell(e_2) = 1, \dots, \ell(e_{g-2}) = 2, \ell(e_{g-1}) = 1$. It is clear that $c_\ell(v_0) = 5$, and $c_\ell(v_1) = 6$, while the colours of the vertices of the rest of the cycle C_g alternate between 3 and 4. Moreover, for all $2 \leq i \leq g-1$, if $c_\ell(v_i) = 3$, then $c_\ell(u_{i,1}) = 4$, and if $c_\ell(v_i) = 4$, then $c_\ell(u_{i,1}) = 5$ (by Claim 4.8). Thus, ℓ is a proper 3-labelling that assigns label 3 to only one edge of S_g. □

We are now ready to prove our lower bound.

Theorem 4.9. *For every $g' \geq 3$, there exist arbitrarily large connected planar graphs G with $g(G) \geq g'$ and $\rho_3(G) \geq \frac{1}{g^2+g}$, where g is the smallest natural number such that $g \geq g'$ and $g \equiv 3 \bmod 4$.*

Proof. For any integer $q \geq 1$, denote by G the graph obtained from q disjoint copies $X_1, \dots, X_q$ of S_g by identifying their roots to a single vertex. Clearly, G is planar and has girth $g \geq g'$. Furthermore, G clearly contains q copies of S_g as pairwise disjoint weakly induced subgraphs $X_1, \dots, X_q$. Then, Lemma 3.1 implies that $\mathrm{mT}(G) \geq q \cdot \mathrm{mT}(S_g)$, which is at least q by Lemma 4.7. Since G has $q|E(S_g)| = q(g^2 + g)$ edges, the result follows. Moreover, these arguments apply for any value of q, and so, G can be as large as desired. □

Again, it is not too complicated to check that our construction in the proof of Theorem 4.9 yields planar graphs G of girth g verifying $\mathrm{mT}(G) = |E(G)|/(g^2 + g)$ (when $g \equiv 3 \bmod 4$), which implies that the lower bound in the statement is somewhat tight here.

We now proceed to prove that $\rho_3(G) \leq \frac{2}{k-1}$ for any nice planar graph G of girth $g \geq 5k + 1$, when $k \geq 7$. In other words, the bigger the girth of a planar graph G, the smaller $\rho_3(G)$ gets.

The following theorem from [8] is one of the main tools we use to prove this result. Recall that, for any $k \geq 1$, a *k-thread* in a graph G is a path $(u_1, \dots, u_{k+2})$, where the k inner vertices $u_2, \dots, u_{k+1}$ all have degree 2 in G.

Theorem 4.10 (Chang, Duh [8])**.** *For any integer $k \geq 1$, every planar graph with minimum degree at least 2 and girth at least $5k + 1$ contains a k-thread.*

We can now proceed with the main theorem.

Theorem 4.11. *Let $k \geq 7$. If G is a nice planar graph with $g(G) \geq 5k + 1$, then $\rho_3(G) \leq \frac{2}{k-1}$.*

Proof. Throughout this proof, we set $g = g(G)$. The proof is by induction on the order of G. The base case is when $|V(G)| = 3$. In that case, G must be a path of length 2 (due to the girth assumption), and the claim is clearly true. So let us focus on proving the general case.

We can assume that G is connected. If G is a tree, then $\chi_\Sigma(G) \leq 2$ and we have $\rho_3(G) = 0$. So, from now on, we may assume that G is not a tree. We can also assume that G has no vertex v to which is attached a pending tree T_v that is not a star with center v. Indeed, if such a T_v exists, then we can find a vertex $u \in V(T_v) \setminus \{v\}$ whose all neighbours $u_1, \dots, u_x$ but one are degree-1 vertices. Since G is not a tree, the graph $G' = G - \{u_1, \dots, u_x\}$ is clearly a nice planar graph with girth g, admitting, by the induction hypothesis, a proper 3-labelling attesting that $\rho_3(G') \leq \frac{2}{k-1}$. Lemma 3.2 tells us that such a labelling can be extended to one of G.

Let G^- be the graph obtained from G by removing all vertices of degree 1. Note that removing vertices of degree 1 from G can neither decrease its girth nor result in a tree. Since G has girth $g \geq 5k + 1$ and does not contain any cut vertex $v \in V(G)$ as described above, the graph G^- has minimum degree 2. By Theorem 4.10, G^- contains a k-thread P. Let $u_1, \dots, u_{k+2}$ be the vertices of P, where $d_H(u_i) = 2$ for all $2 \leq i \leq k + 1$. Thus, the vertices of P exist in G except that each of the vertices u_i (for $2 \leq i \leq k + 1$) may be adjacent to some vertices of degree 1 in addition to their adjacencies in G^-. Let G' be the graph obtained from G by removing the vertices $u_3, \dots, u_k$ and all of their neighbours that have degree 1 in G. Note that G' might contain up to two connected components. In case G' has exactly two connected components, then, due to a previous assumption, none of these can be a tree, which implies that G' is nice. If G' is connected, then, because it has at least two edges (u_1u_2 and $u_{k+1}u_{k+2}$), it must be nice. Furthermore, in both cases, the girth of G' is at least that of G. Then, by combining the inductive hypothesis and the fact that every nice tree T verifies $\rho_3(T) = 0$, we deduce that $\rho_3(G') \leq \frac{2}{k-1}$.

To obtain a proper 3-labelling ℓ of G such that $\rho_3(G) \leq \frac{2}{k-1}$, we extend a proper 3-labelling ℓ' of G' corresponding to $\rho_3(G') \leq \frac{2}{k-1}$, as follows. First, label with 1 all of the edges incident to the vertices of degree 1 we have removed. Note that none of these vertices of degree 1 can, later on, be in conflict with their neighbour since they have degree 1. Now, for each $2 \leq j \leq k - 2$, in increasing order of j, label the edge u_ju_{j+1} with 1 or 2, so that the resulting colour of u_j does not conflict with the colour of u_{j-1}. Finally, label the edges $u_{k-1}u_k$ and u_ku_{k+1} with 1, 2 or 3, so that the resulting colour of u_{k-1} does not conflict with that of u_{k-2}, the resulting colour of u_k does not conflict with that of u_{k-1} nor with that of u_{k+1}, and the resulting colour of u_{k+1} does not conflict with that of u_{k+2}. Indeed, this is possible since there exist at least two distinct labels $\{\alpha, \beta\}$ ($\{\alpha', \beta'\}$, respectively) in $\{1, 2, 3\}$ for $u_{k-1}u_k$ (u_ku_{k+1}, respectively) such that the colour of u_{k-1} (u_{k+1}, respectively) is not in conflict with that of u_{k-2} (u_{k+2}, respectively). Thus, w.l.o.g., choose α and α' for the labels of $u_{k-1}u_k$ and u_ku_{k+1}, respectively. If the colour of u_k does not conflict with that of u_{k-1} nor with that of u_{k+1}, then we are done. If the colour of u_k conflicts with both that of u_{k-1} and that of u_{k+1}, then it suffices to change both the labels of $u_{k-1}u_k$ and u_ku_{k+1} to β and β', respectively. Lastly, w.l.o.g., if the colour of u_k only conflicts with that of u_{k-1}, then it suffices to change the label of u_ku_{k+1} to β'. The resulting labelling ℓ of G is thus proper. Moreover, $|E(G) \setminus E(G')| \geq k - 1$ and ℓ uses label 3 at most twice more than ℓ', and so, the result follows. □

4.4 Bounds for connected cacti

Recall that a *cactus* is a graph in which any two cycles have at most one vertex in common.

First off, note that the graphs S_g introduced in Section 4.3, and those we have constructed from them in the proof of Theorem 4.9, are all cacti (all of their cycles are actually disjoint). Since the smallest graph S_g is S_3, which has 12 edges, the proof of that theorem implies the following.

Theorem 4.12. *There exist arbitrarily large connected cacti G with $\rho_3(G) \geq 1/12$.*

We now focus on the upper bound. We actually end up proving Conjecture 2.2 for cacti.

Theorem 4.13. *If G is a nice cactus, then $\rho_3(G) \leq 1/3$.*

Proof. The proof is done by induction on $|V(G)|$. Since the claim is clearly true when G has only three vertices, let us consider the general case. Clearly, we can assume that G is connected (as otherwise we could use the induction hypothesis on each connected component), is not a tree (since $\mathrm{mT}(T) = 0$ for every nice tree T), is not bipartite (by Theorem 2.1), and is not a cycle (see [4]).

Throughout this proof, for readability reasons, we say that a proper 3-labelling is *good* if it assigns label 3 to at most a third of the edges of the labelled graph. We first prove that if G has some specific properties, then we can remove some vertices from G, resulting in a nice cactus G' that is smaller than G, and extend a good labelling ℓ' of G', obtained by induction, into a good labelling ℓ of G, thus proving the statement for G. It can then be assumed that G does not have these properties, which will simplify its structure and allow us to prove the final inductive step.

Let us state a few more remarks. Let ℓ be an extension of ℓ' that assigns labels from $\{1, 2\}$ to the edges of G that are not in G'. If this ℓ is proper, then note that it is also good. Similarly, for $m = |E(G)| - |E(G')|$ and $m \geq 3$, if ℓ assigns label 3 to at most a third of the edges of G that are not in G' and ℓ is proper, then it is also good.

We start by analysing certain cycles of G. To define those cycles, let us consider the following terminology. We denote by G^- the cactus obtained from G by repeatedly deleting vertices of degree 1, until the remaining graph has minimum degree 2. Since G contains cycles, note that G^- is not empty. In what follows, we study structures around *end-cycles*, where an end-cycle C of G refers to a cycle of G^- which is connected to the rest of the graph via a single vertex r. That is, in G^-, all vertices of C but r, have degree 2. We call r the *root* of C, while its other vertices are the *inner vertices*. Note that end-cycles are better defined as soon as G has at least two cycles. In case G has only one cycle C, then we consider C as an end-cycle, its root being any of its vertices of degree more than 2 (at least one exists since G is not a cycle).

In what follows, we consider any end-cycle C of G. We first investigate properties of pending trees attached to the vertices of C. For every vertex v of C, we define T_v as the pending tree rooted at v in G. Note that there might be no edges in such a T_v, i.e., we can have $V(T_v) = \{v\}$. We implicitly assume that every T_v comes with the natural (virtual) orientation of its edges from the root (v) to the leaves. Also, we say that T_v is *inner* if v is indeed an inner vertex of C.

Claim 4.14. *If some T_v has edges and is not a star, then there is a good labelling of G.*

Proof of the claim. Let us consider a deepest (i.e., farthest from v) vertex u of T_v, where all of its $x \geq 1$ children are leaves. Since T_v is not a star, we have $u \neq v$. Then, the graph G' obtained from G by removing all these x leaves is a nice cactus (due to the presence of the cycle C) in which u has degree 1. Thus, G' admits a good labelling by the induction hypothesis. Lemma 3.2 tells us that this good labelling of G' can be extended to one of G. ◇

Claim 4.15. *If some inner T_v is a star with at least two edges, then there is a good labelling of G.*

Proof of the claim. Let G' be the graph obtained from G by removing two leaves u, u' of T_v. Clearly, G' is a cactus, and G' is nice due to the presence of C. By the induction hypothesis, there is a good labelling of G'. To obtain one of G, it suffices to extend this labelling to vu and vu' by assigning labels 1 and 2, in such a way that no colour conflict arises. Recall that, by a labelling of a nice graph, a vertex of degree 1 cannot be involved in a conflict with its neighbour. Then, it suffices to label vu and vu' so that no colour conflict arises between v and its two neighbours in C. Since there are three ways to alter the colour of v by labelling vu and vu' this way (assigning label 1 twice, assigning 2 twice, or assigning both 1 and 2 once), there is a way to extend the good labelling of G' to one of G. ◇

Thus, in C, any inner T_v can be assumed to have at most one edge.

Claim 4.16. *If C has length at least 4 and some inner T_v has an edge, then there is a good labelling of G.*

Proof of the claim. Assume $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$, where $v_0 = r$ is the root of C and $n \geq 4$. By Claims 4.14 and 4.15, each T_{v_i} (where $i \in \{1, \ldots, n-1\}$) has at most one edge.

Assume first that there is an $i \in \{2, \ldots, n-2\}$ such that T_{v_i} has an edge v_iu. Let G' be the graph obtained from G by removing u and v_i. Clearly, G' is a cactus with at least two edges (v_0v_1 and $v_{n-1}v_0$), so it is nice. By the induction hypothesis, there is a good labelling of G', which we want to extend to one of G. To that aim, we have to label the three edges $v_iu, v_iv_{i-1}, v_iv_{i+1}$ (where, here and further, the operations are understood modulo n) so that no colour conflict arises, and label 3 is assigned at most once. First, we assign 1 or 2 to v_iv_{i-1} so that v_{i-1} does not get in conflict with v_{i-2}. Second, we assign 1 or 2 to v_iv_{i+1} so that v_{i+1} does not get in conflict with v_{i+2}. Third, we assign 1, 2 or 3 to v_iu so that v_i gets in conflict with neither v_{i-1} nor v_{i+1}. As mentioned earlier, u cannot get in conflict with v_i due to its degree, so the resulting labelling of G is good.

Assume now that T_{v_i} has no edge for every $i \in \{2, \ldots, n-2\}$, but T_{v_1} has an edge v_1u (the case where $T_{v_{n-1}}$ has an edge is symmetrical). This means that each of $v_2, \ldots, v_{n-2}$ has degree 2. In this case, we consider G' the cactus obtained from G by removing u and v_2. Note that G' has more than one edge since r has degree at least 3 in G. Then, G' is nice. By the induction hypothesis, there is a good labelling of G'. To extend it to one of G, we must label the edges v_1u, v_1v_2, v_2v_3 so that no colour conflicts arise, and label 3 is assigned at most once. Similarly as in the previous case, this can be achieved by first labelling v_2v_3 with 1 or 2 so that no conflict between v_3 and v_4 arises, then labelling v_1v_2 with 1 or 2 so that no conflict between v_2 and v_3 arises, and lastly labelling v_1u with 1, 2 or 3 so that v_1 is not in conflict with v_0 nor v_2. ◇

Due to the previous claims, in G we can assume that C is either a cycle of any length at least 3 (i.e., all inner vertices have degree 2), or a triangle where one or two of its inner vertices have a pending edge attached (i.e., one or two of the T_v's have size 1). We call the first of these two triangle configurations a 1-*triangle*, while we call the second configuration a 2-*triangle*. For convenience, we also regard these configurations as end-cycles, though they are technically not cycles in G.

We are now ready to conclude the proof. If G has only one cycle, then, by the previous claims and our original assumption that G is not just a cycle, it must be that G is a triangle (u, v, w, u) with a pending vertex attached to u and possibly one attached to v, in which case the claim can be verified easily. So G has at least two cycles. Let us consider a set of end-cycles that are the "deepest" ones, in the following sense. Choose a cycle C_x of G. For every other cycle C of G, we define the *distance* from C to C_x as the length of a shortest path from a vertex of C to a vertex of C_x. In case C and C_x share a vertex, note that the distance from C to C_x is 0.

Now, let $d \geq 0$ be the maximum distance from C_x to a cycle in G. Let C_1 be a cycle at distance d from C_x. Note that C_1 is an end-cycle in G, and let r denote its root. Observe that there might be other (end-)cycles of G at distance d from C_x, with root r. In case they exist, we denote them by $C_2, \dots, C_q$. Then $C_1, \dots, C_q$ are end-cycles in G with the same root r, and, by how these C_i's were chosen, r either has only one neighbour u or only two neighbours u, u' of degree at least 2 that do not belong to the C_i's. More precisely, r is connected to the rest of the graph either via a path (through an edge ru), or via a unique cycle (containing both u and u'). Furthermore, there might be vertices of degree 1 adjacent to r. Indeed, by Claim 4.14, if there is a pending tree T_r attached at r, then T_r must be a star with center r. Recall that each of the C_i's is a cycle, a 1-triangle, or a 2-triangle, due to previous claims.

Now, let G' be the cactus obtained from G by removing all non-root vertices of the C_i's (i.e., all their inner vertices, plus the at most two pending vertices of the 1-triangles and 2-triangles). Since G' contains at least one cycle, it is nice, and thus, admits a good labelling by the induction hypothesis. Our goal is to extend it to one of G, by labelling the removed edges so that no conflict arises and at most a third of these edges are assigned label 3.

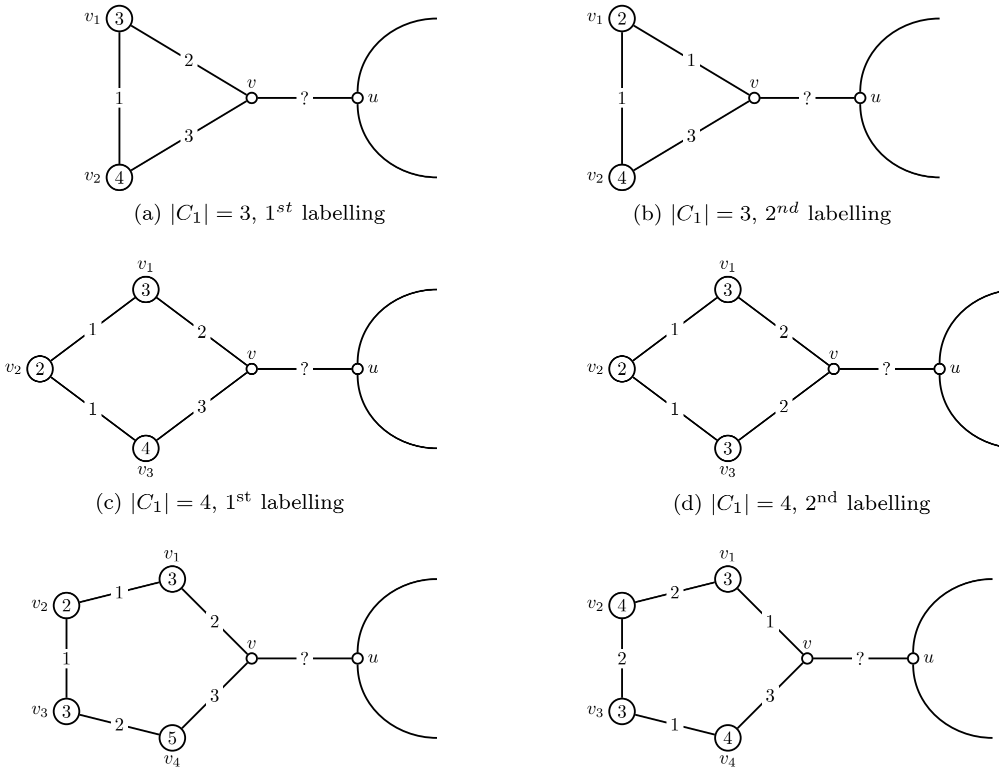
- Assume $q \geq 2$. We first label the edges of every C_i that is a cycle, assigning consecutive labels $2, 1, 1, 2, 2, 1, 1, \dots$ while going around, starting and ending with an edge incident to r. Note that this avoids any conflict between the inner vertices of C_i, that their colours are at most 4, and that this alters the colour of r by at least 3. For every C_i that is a 1-triangle, we assign label 2 to its two edges incident to r, and label 1 to its two other edges. Note that this raises no conflict between the inner vertices of C_i, that their colours are at most 4, and that the colour of r is altered by 4. Finally, for every C_i that is a 2-triangle, we assign label 2 to its two edges incident to r and to one pending edge, and label 1 to the two other edges. As a result, no conflict arises between inner vertices, their colours are at most 5, and the colour of r is altered by 4.

  Since $q \geq 2$ and r has at least one neighbour not in the C_i's, the colour of r is at least 7, and thus, r cannot be in conflict with its neighbours in the C_i's. However, we still have to make sure that the colour of r is different from that of u and u' (if it exists). Note that, in each C_i, there is an edge labelled 2 incident to r that can be relabelled 3 without causing conflicts between the inner vertices. Indeed, if C_i is a cycle, then the very first labelled edge is such an edge. If C_i is a 1-triangle or 2-triangle, then the one of its two edges labelled 2 incident to r going to the inner vertex with the largest colour, is such an edge. Thus, by changing the label from 2 to 3, of one or two of these edges, we can increment the colour of r by 1 or 2 to avoid the colours of u and u' (if it exists). This means that, by introducing at most two 3's, we can get a proper 3-labelling of G, which is good since $q \geq 2$.

- Assume $q = 1$. Assume first that C_1 is a 1-triangle or a 2-triangle. Let (r, v_1, v_2, r) denote the vertices of the cycle of C_1, and u_1 and u_2 (if it exists) denote the pending vertices attached to v_1 and v_2, respectively. We first label rv_1 and rv_2 with 1 or 2 so that no conflict arises between r and u and u' (if it exists). This is possible, since there are three possible combinations. In case C_1 is a 1-triangle, then we label v_1v_2 with 1 or 2 so that no conflict arises between v_2 and r. In case C_1 is a 2-triangle, then we label v_1v_2 with any of 1 and 2. Now, if C_1 is a 1-triangle, then we label v_1u_1 with 1, 2 or 3 so that no conflict arises between v_1 and r, and v_1 and v_2. If C_1 is a 2-triangle, then we label v_1u_1 with 1 or 2 so that no conflict arises between v_1 and r, and then we label v_2u_2 with 1, 2 or 3 so that no conflict arises between v_2 and r, and v_1 and v_2. In all cases, we assign label 3 to only one edge, so the resulting proper 3-labelling of G is good since no conflict arises.

  Assume now that C_1 is a cycle. First assume that u' exists. We consider the edges of C_1, and assign to them labels 1 and 2 as previously, i.e., by applying the labelling pattern $2, 1, 1, 2, 2, 1, 1, \dots$ from one edge incident to r to the other. We consider two cases:

  - Assume first that, in the labelling of C_1, the two edges incident to r get assigned distinct labels (1 and 2). As earlier, no two inner vertices of C_1 are in conflict, their colours are at most 4, and, since u' exists, the colour of r is at least 5. If this raises no conflict between r and u or u', then we are done. Otherwise, note that turning into a 3 the label assigned to any of the two edges of C_1 incident to r, raises no conflict between two vertices of C_1. Since these two edges are labelled differently, 1 and 2, this means that by introducing



(a) $|C_1| = 3$, 1^{st} labelling

(b) $|C_1| = 3$, 2^{nd} labelling

(c) $|C_1| = 4$, 1^{st} labelling

(d) $|C_1| = 4$, 2^{nd} labelling

(e) $|C_1| = 5$, 1^{st} labelling

(f) $|C_1| = 5$, 2^{nd} labelling

Figure 4: Labelling a pending cycle in the proof of Theorem 4.13. Some colours by the labelling are indicated by integers within the vertices.

label 3 once in C_1, we can increment the colour of r by 1 or 2 so that we avoid any conflict between r, and u and u'. Then we can deduce a good labelling of G.

– Assume now that both edges incident to r in C_1 get assigned label 2. Then, this time, the colour of r is at least 6. If there is no conflict between r and one of u and u', then we are done. So we can assume there is a conflict, and also that changing to 3 the label of one of the two edges of C_1 incident to r, makes r being in conflict with the second one of these two vertices. Then, note that we get a good labelling when labelling C_1 following the pattern $1, 2, 2, 1, 1, 2, 2, \dots$ instead, since r gets its two incident edges in C_1 being assigned label 1, the colour of r is at least 4 and smaller than the previous colours we have produced for r, and the colours of the two neighbours of r in C_1 are at most 3.

Now assume u' does not exist. We start by considering the cases where C_1 has length at least 6. Start by applying the labelling pattern $2, 1, 1, 2, 2, 1, 1, 2, 2, \dots$ to the edges of C_1 as before. Assume first that the two edges of C_1 incident to r get assigned distinct labels. Then change the 1 assigned as a label to one of these two edges into a 3. As a result, no conflicts arise between inner vertices of C_1, their colours are at most 5, while the colour of r is at least 6 due to the edge ru. So the only possible conflict is between r and u. Suppose it occurs. Then no conflict remains when assigning label 3 to the second edge of C_1 incident to r and we get a good labelling (in particular, only two edges of C_1 get assigned label 3 while its length is at least 6, and this assumption also guarantees that no two inner vertices of C_1 get in conflict). Lastly, assume that both edges of C_1 incident to r get assigned label 2 by the initial labelling scheme. Then the colour of r is at least 5, which thus cannot be in conflict with its neighbours in C_1. If r is not in conflict with u, then we get a good labelling of G. Otherwise, we get one by changing to 3 the label of one of the two edges of C_1 incident to r.

It remains to check three length values for C_1. The labelling schemes described below are illustrated in Figure 4.

– If C_1 has length 3, then assigning either labels $2, 1, 3$ or $1, 1, 3$ to the edges while going around, starting and ending with r, yields a good labelling, since r gets colour at least 6 or 5, respectively, while the inner vertices of C_1 get colours at most 4, and the colour of u is the only other colour to avoid. In particular, note that these two labelling schemes alter the colour of r in two different ways ($+5$ and $+4$, respectively).
– If C_1 has length 4, then we get the same conclusion from applying the labelling scheme $2, 1, 1, 3$ or $2, 1, 1, 2$. Note indeed that the inner vertices get colours at most 4 and 3, respectively, while r gets colour at least 6 and 5, respectively. Also, these two schemes alter the colour of r differently, by $+5$ and $+4$, respectively.
– If C_1 has length 5, then the sequence $2, 1, 1, 2, 3$ or $1, 2, 2, 1, 3$ yields the same conclusion. Indeed, the inner vertices get colours at most 5 and 4, respectively, while r gets colour at least 6 and 5, respectively. Also, these two schemes alter the colour of r differently, by $+5$ and $+4$, respectively.

In all cases, we can deduce a good labelling of G, which concludes the proof. □

4.5 Bounds for other graph classes

In this section, we state, in the similar spirit as in the previous subsections, some lower or upper bounds on ρ_3 that can be obtained for other classes of graphs that are mostly 3-chromatic. Indeed, we mostly focus on outerplanar graphs and Halin graphs. The difference in this section is that for the considered graph classes, one of the two bounds is partially missing.

4.5.1 Outerplanar graphs

First off, we note that the construction described in the proof of Theorem 4.3, when performed with copies of A_1 only, provides graphs that are outerplanar[4], since A_1 is itself outerplanar.

Theorem 4.17. *There exist arbitrarily large connected outerplanar graphs G with $\rho_3(G) \geq 1/10$.*

Recall as well that outerplanar graphs form a subclass of series-parallel graphs. Thus, Theorem 4.17 also holds for arbitrarily large connected series-parallel graphs.

Note however that the outerplanar graphs constructed above have cut vertices. So the question remains, whether or not this lower bound still holds when considering 2-connected outerplanar graphs (recall that outerplanar graphs are 2-degenerate, and thus, each of them is either separable or 2-connected). As for an upper bound, we can provide the following:

Theorem 4.18. *If G is a 2-connected outerplanar graph such that $|E(G)| \geq |V(G)| + 3$, then $\rho_3(G) \leq 1/3$.*

Proof. We can assume that G is not bipartite, as otherwise the claim follows from Theorem 2.1. Then $\chi(G) = 3$ since outerplanar graphs are 2-degenerate. Now, if $|V(G)|$ is odd, then the result follows from Corollary 3.8. So, in what follows, we assume that $|V(G)|$ is even.

In 2-connected outerplanar graphs, the outer face forms a Hamiltonian cycle $(v_0, \dots, v_{n-1}, v_0)$. The other edges, which do not lie on the outer face, are called *chords*. Since G is not bipartite, it has an odd-length cycle C_x. Since $|V(G)|$ is even, this C_x is not the whole outer cycle of G. Furthermore, we can assume that C_x consists of consecutive vertices of the outer face, i.e., that $C_x = (v_a, v_{a+1}, \dots, v_{a+x-1}, v_a)$ for some $a \in \{0, \dots, n-1\}$ (where the operations over the indices, here and further, are understood modulo n), or, in other words, that $v_a v_{a+x-1}$ is the only chord of G in C_x. Indeed, assume C_x has at least two chords, one of which is $v_i v_j$, where $i < j$. Note that $\{v_i, v_j\}$ is a cut set of G. This means that $V(C_x)$ is fully included in either $\{v_j, v_{j+1}, \dots, v_i\}$ or $\{v_i, v_{i+1}, \dots, v_j\}$. Assume that $V(C_x) \subseteq \{v_j, v_{j+1}, \dots, v_i\}$ (the other case being symmetrical). Then note that $|\{v_i, v_{i+1}, \dots, v_j\}|$ must be even, as otherwise $(v_i, v_{i+1}, \dots, v_j, v_i)$ would be an odd-length cycle as desired. Now we note that replacing $v_i v_j$ in C_x by the path $(v_i, v_{i+1}, \dots, v_j)$ results in another odd-length cycle of G with one less chord. Repeating this process as long as the resulting odd-length cycle has more than one chord, eventually we end up with an odd-length cycle of G with only one chord, which is as desired.

[4]Recall that a graph is *outerplanar* if it admits a planar embedding where all vertices lie on the outer face.

Up to relabelling the vertices, we can assume, w.l.o.g., that $C_x = (v_1, \dots, v_x, v_1)$. Let us consider H, the subgraph of G containing the x edges of C_x, and all the (other) edges of the Hamiltonian cycle $(v_1, \dots, v_n, v_1)$ on the outer face of G except for the edge v_nv_1. Note that H is a unicyclic spanning connected subgraph of G, in which the only cycle (being of odd length) is C_x, to which is attached a hanging path $(v_x, v_{x+1}, \dots, v_n)$ containing all other vertices of G. Since H is spanning, connected, and unicyclic, $|E(H)| = |V(G)|$, which is at most $|E(G)| - 3$, since $|E(G)| \geq |V(G)| + 3$.

All conditions are now met to invoke the arguments in the proof of Theorem 3.5, from which we can deduce a proper $\{0, 1, 2\}$-labelling ℓ of G where adjacent vertices get distinct colours modulo 3 and in which only the edges of (our) H are possibly assigned label 0. Let us now consider the subgraph H' of G obtained from H by adding the edge v_nv_1, which is present in G. Recall that $\ell(v_nv_1) = 2$ by default. Note that H' contains at least two disjoint perfect matchings M_1, M_2. Indeed, since $|V(G)|$ is even, a first perfect matching M_1 of H' contains $v_1v_2, v_3v_4, \dots, v_{n-1}v_n$. A second perfect matching M_2 of H' contains $v_2v_3, v_4v_5, \dots, v_nv_1$. By Lemma 3.7, we can assume that at most a third of the edges in $M_1 \cup M_2$ are assigned label 0 by ℓ. Since $|M_1| + |M_2| = |E(H')| - 1 = |E(H)|$ but the edge $v_1v_x \in E(H)$ is not included in M_1 nor M_2 (and so may have label 0 too), this gives $\mathrm{nb}_\ell(0) \leq \frac{E(H)}{3} + 1$, which is less than $|E(G)|/3$ since $|E(G)| \geq |V(G)| + 3$. By turning 0's by ℓ into 3's, we get a proper 3-labelling of G with the same upper bound on the number of assigned 3's. □

Theorem 4.18 does not cover all 2-connected outerplanar graphs. However, it covers all such graphs with at least three chords. Thus, to get a generalisation of Theorem 4.18 for all 2-connected outerplanar graphs, it remains to prove a similar result for the 3-chromatic ones with at most two chords. Those with no chords are exactly odd-length cycles, for which the claim holds (see, e.g., [4]). For those with one or two chords, the claim can also be verified, for instance through considering the possible ways for the at most two chords to interact in 2-connected outerplanar graphs, and, for each possible configuration, extending a proper 3-labelling from face to face. Let us mention that the number of cases to consider can be reduced drastically by applying some of the arguments used in the proof of Theorem 4.11 to deal with long threads. We voluntarily omit a tedious proof, which would be less interesting than that of Theorem 4.18 (whose main purpose is to illustrate how some of the tools from Section 3 can be used).

4.5.2 Halin graphs

We now proceed by proving Conjecture 2.2 for a 4-chromatic family of graphs. A *Halin graph* is a planar graph with minimum degree 3 obtained as follows. Start from a tree T with no vertex of degree 2, and consider a planar embedding of T. Finally, add edges to form a cycle going through all the leaves of T in the clockwise ordering in this embedding. A Halin graph is called a *wheel* if it is constructed from a tree T with diameter 2 (i.e., being a star).

Halin graphs are known to have many properties of interest, such as having triangles, being Hamiltonian, and having Hamiltonian cycles going through any given edge (see, e.g., [18]). Also, Halin graphs are 3-degenerate, so, due to the presence of triangles, each of them has chromatic number 3 or 4. The dichotomy is well understood, as a Halin graph has chromatic number 4 if and only if it is a wheel of even order [19]. This allows us to use our tools from Section 3 to establish an upper bound on ρ_3 for most Halin graphs (the 3-chromatic ones), while we can treat the remaining ones separately.

Theorem 4.19. *If G is a Halin graph, then $\rho_3(G) \leq 1/3$.*

Proof. First, consider the case where G is a wheel of even order n. If $n = 4$, then $G = K_4$, and the statement holds (since it can be checked by hand that $\rho_3(K_4) = 1/6$). For $n \geq 6$, we have that $\mathrm{mT}(G) = 0$. Indeed, let v be the center of the star T, and let $v_2, \dots, v_n$ be the leaves of T. We can construct a proper 2-labelling ℓ of G as follows: start from v_2v_3 and, following the edges of the cycle joining the leaves of T in increasing order of their indices, assign labels $1, 1, 2, 2, 1, 1, \dots$, until v_nv_2 is labelled. If $\ell(v_nv_2) = 1$, then set $\ell(vv_2) = 1$ and $\ell(vv_i) = 2$ for every $3 \leq i \leq n$. Otherwise, if $\ell(v_nv_2) = 2$ (and so, $\ell(v_{n-1}v_n) = 1$), set $\ell(vv_2) = 2$ and $\ell(vv_i) = 1$ for every $3 \leq i \leq n$. It is easy to check that in both cases ℓ is a proper 2-labelling of G. Thus, $\rho_3(G) = 0$ and the statement holds.

Next, consider the case where G is not a wheel of even order. Then $\chi(G) = 3$. If $|V(G)|$ is odd, then the result follows from Corollary 3.8. Thus, we can assume that $|V(G)|$ is even.

By considering any non-leaf vertex r of T in G, and defining a usual root-to-leaf (virtual) orientation, since no vertex has degree 2 in T, it can be seen that G has a triangle (u, v, w, u), where v, w are leaves in T with parent u. Furthermore, $d_G(v) = d_G(w) = 3$, while $d_G(u) \geq 3$. Due to these degree properties, note that if we consider C a Hamiltonian cycle traversing uv, then C must also include either wu or vw. More precisely, if we orient the edges of C, resulting in a spanning oriented cycle $\vec{C}$, then, at some point, $\vec{C}$ enters (u, v, w, u) through one of its vertices, goes through another vertex of the triangle and then through the third of its vertices, before leaving the triangle. In other words, C traverses all vertices of (u, v, w, u) at once.

Up to relabelling the vertices of (u, v, w, u), we can assume that $\vec{C}$ enters the triangle through u, then goes to v, before going to w and leaving the triangle. Let us consider H, the subgraph of G containing the three edges of (u, v, w, u), and all successive edges traversed by C after leaving the triangle except for the edge going back to u. Note that H is a unicyclic spanning connected subgraph of G, in which the only cycle is the triangle (u, v, w, u) to which is attached a hanging path $(w, x_1, \dots, x_{n-3})$ containing all other vertices of G (i.e., $n = |V(G)|$). Furthermore, in $E(G) \setminus E(H)$, if we set $x = x_{n-3}$, then the edge xu exists. Since H is spanning, connected, and unicyclic, $|E(H)| = |V(G)|$, which is at most $2|E(G)|/3$, since $\delta(G) \geq 3$.

All conditions are now met to invoke the arguments in the proof of Theorem 3.5, from which we can deduce a proper $\{0, 1, 2\}$-labelling of G where adjacent vertices get distinct colours modulo 3 and in which only the edges of the chosen H are possibly assigned label 0. Let us now consider the subgraph H' of G obtained from H by adding the edge xu, which is present in G. Recall that $\ell(xu) = 2$ by default. Note that H' contains at least two disjoint perfect matchings M_1, M_2. Indeed, since $|V(G)|$ is even, then, in H, the hanging path attached at w has odd length. A first perfect matching M_1 of H' contains $x_{n-3}x_{n-4}, x_{n-5}x_{n-6}, \dots, wx_1$ and uv. A second perfect matching M_2 of H' contains $x_{n-4}x_{n-5}, x_{n-6}x_{n-7}, \dots, x_2x_1$, and wv and xu. Now, by Lemma 3.7, we can assume that at most a third of the edges in $M_1 \cup M_2$ are assigned label 0 by ℓ. Since $|M_1| + |M_2| = |E(H')| - 1 = |E(H)|$, this gives $\mathrm{nb}_\ell(0) \leq \frac{E(H)}{3} + 1$, which is less than $|E(G)|/3$ since $|E(G)| \geq 3|V(G)|/2$. By turning 0's by ℓ into 3's, we get a proper 3-labelling of G with the same upper bound on the number of assigned 3's. □

5 Conclusion

This work was dedicated to studying the importance of 3's in designing proper 3-labellings, this aspect being motivated by a presumption from previous works that proper 3-labellings of graphs, in general, should require only a few 3's. This led us to introducing the two quantifying parameters mT and ρ_3. As a main contribution, we have introduced, in Section 3, some tools for deducing bounds on these parameters. Applications of these, in Section 4, led us to results for specific classes of 3-chromatic graphs. In particular, we have established that, for several simple classes $\mathcal{F}$ of graphs, there is no $p \geq 0$ such that $\mathcal{F} \subset \mathcal{G}_{\leq p}$. In such cases, we have provided bounds on $\rho_3(\mathcal{F})$.

Several directions for further research sound particularly appealing. A first one is to prove Conjecture 2.2 for more classes of graphs, or to exhibit weaker upper bounds towards it. Another one is to investigate whether the bound of 1/3 in that conjecture is close to being tight or not, in general. Indeed, at the moment we only know of two small connected graphs, namely C_3 and C_6, which attain the bound, while the class of arbitrarily large graphs with the biggest value ρ_3 we could construct, achieves a ratio of 1/10 (Theorem 4.3).

Other directions of interest include bounds that are missing in Section 4. For instance, we are missing an upper bound on ρ_3 for a few classes of 3-chromatic graphs, such as separable outerplanar graphs and, more generally, series-parallel graphs. Regarding our upper bound for Halin graphs (Theorem 4.19), the main point of interest in the proof lies in that it shows an application of Lemma 3.7. However, we were not able to come up with examples of arbitrarily large Halin graphs needing more and more 3's in their proper 3-labellings. Actually, we are aware of only three Halin graphs that do not admit proper 2-labellings. Two of them are K_4 and the prism graph (Cartesian product of K_3 and K_2). The third one is constructed as follows: start with two perfect binary trees on 7 vertices each and add an edge between the roots (degree-2 vertices) of these trees; from the resulting tree T, construct G as explained in Section 4.5.2. All three of these graphs turn out to lie in $\mathcal{G}_1$. Thus, though we were not able to prove it, it is possible that there exists a $p \geq 1$ such that Halin graphs are in $\mathcal{G}_{\leq p}$, and even that $p = 1$.

Let us mention a last intriguing open question regarding complete graphs. It is known from [2] that complete graphs K_n with $n \geq 5$ admit equitable proper 3-labellings, which implies that they verify Conjecture 2.2, i.e., $\mathrm{mT}(K_n) \leq |E(K_n)|/3$ which is roughly of order $n^2/6$. In [4], the authors exhibited proper 3-labellings of complete graphs where the sum of assigned labels is as small as possible. Looking closely at the proof, it turns out that the designed proper 3-labellings assign label 3 to roughly $n/4$ edges, which yields a better upper bound on $\rho_3(K_n)$. Determining the precise ratio in general sounds like an interesting challenge. Through computer experimentation, we were able to verify that $K_n \in \mathcal{G}_1$ for $3 \leq n \leq 5$, while $K_n \in \mathcal{G}_2$ for $6 \leq n \leq 9$, and $K_n \in \mathcal{G}_3$ for $10 \leq n \leq 12$. However, we did not manage to prove a general result. We are not even sure if there exists a $p \geq 3$ such that all complete graphs are in $\mathcal{G}_{\leq p}$.

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