GEVREY ESTIMATES OF THE RESOLVENT AND SUB-EXPONENTIAL TIME-DECAY FOR THE HEAT AND SCHRÖDINGER SEMIGROUPS. II

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Abstract. In this paper, we improve and generalize the results of [12] on large-time expansions for the heat and Schrödinger semigroups with sub-exponential time-decay estimates on the remainder.

1. Introduction

In [12], one of the authors proved the Gevrey estimates of the resolvent for a class of non selfadjoint second order elliptic operators satisfying a weighted coercive condition and applied them to establish large time expansions of the heat and Schrödinger semigroups with sub-exponential time-decay estimates on the remainder. For the Schrödinger semigroup, a global virial condition on the model potential has been used to ensure the absence of quantum resonances near threshold zero. In the present work we want to extend the results of [12] to larger classes of potentials in which only a virial condition outside some compact is needed for the Schrödinger case. We also consider threshold eigenvalue with arbitrary geometric multiplicity instead of geometrically simple one, which requires an analysis of more general Jordan structure analogous to that carried out in [1] for quickly decreasing potentials.

Consider the non-selfadjoint Schrödinger operator

$$H = -\Delta + V(x) \tag{1.1}$$

which is regarded as perturbation of a model operator $H_0 = -\Delta + V_0(x)$ where $V(x)$ and $V_0(x)$ are complex-valued measurable functions on $\mathbb{R}^n$, $n \geq 1$. Assume that V and V_0 are $-\Delta$-compact satisfying for some constants $\mu \in]0,1[$ and $\rho > 2\mu$:

$$|V_0(x)| \leq C\langle x\rangle^{-2\mu}, \quad |W(x)| \leq C\langle x\rangle^{-\rho}, \tag{1.2}$$

for x outside some compact of $\mathbb{R}^n$. Here $W(x) = V(x) - V_0(x)$. An important condition for sub-exponential time-decay for the associated heat and Schödinger semigroups is the following weighted coercive condition on the model operator H_0: there exist $\mu \in]0,1[$ and $c_0 > 0$ such that

$$|\langle H_0 u, u\rangle| \geq c_0(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu} u\|^2), \quad \forall\, u \in H^2. \tag{1.3}$$

Throughout this work, conditions (1.2) and (1.3) are always assumed to be satisfied. Set $V_0(x) = V_1(x) - iV_2(x)$ with $V_1(x)$ and $V_2(x)$ real and denote

$$H_1 = -\Delta + V_1(x), \quad H = H_0 + W(x). \tag{1.4}$$

Under the condition 1.3, Gevrey estimates of the resolvent $R_0(z) = (H_0 - z)^{-1}$ are proved at threshold zero in appropriately weighted spaces ([12]). In order to apply them to the semigroups e^{-tH} and e^{-itH}, $t \geq 0$, we need additional conditions on the potentials. Recall the two classes of potentials introduced in [12] for the heat and Schrödinger semigroups, respectively.

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Definition 1.1. *Let $\mathcal{V}$ be the class of complex-valued potentials $V_0(x) = V_1(x) - iV_2(x)$ satisfying (1.2), (1.3) and the condition*

$$H_1 \geq -\alpha\Delta \tag{1.5}$$

for some constant $\alpha > 0$. Here $H_1 = -\Delta + V_1(x)$ is the selfadjoint part of H_0.

Definition 1.2. *Let $\mathcal{A}$ denote the class of complex-valued potentials $V_0(x)$ such that (1.2) and (1.3) are satisfied for some constant $\mu \in]0,1[$. Assume in addition that V_1 and V_2 are dilation analytic and extend holomorphically into a complex region of the form*

$$\Omega = \{x \in \mathbb{C}^n; |x| > c^{-1}, |\mathrm{Im}\, x| < c\,|\mathrm{Re}\, x|\} \tag{1.6}$$

for some $c > 0$ and satisfy for some constants $c_1, c_2 > 0$ and $R \in [0, +\infty]$

$$|V_j(x)| \leq c_1\langle \mathrm{Re}\, x\rangle^{-2\mu}, x \in \Omega, \quad j = 1, 2, \tag{1.7}$$

$$V_2(x) \geq 0, \quad \forall x \in \mathbb{R}^n, \tag{1.8}$$

$$x \cdot \nabla V_1(x) \leq -c_2 \frac{x^2}{\langle x\rangle^{2\mu+2}}, \quad x \in \mathbb{R}^n \text{ with } |x| \geq R, \text{ and} \tag{1.9}$$

$$V_2(x) \geq c_2\langle x\rangle^{-2\mu}, \quad x \in \mathbb{R}^n \text{ with } |x| < R. \tag{1.10}$$

If Condition (1.10) is satisfied with $R = 0$, we assume in addition

$$0 < \mu < \tfrac{3}{4}, \text{ if } n = 2 \text{ and } 0 < \mu < 1, \text{ if } n \geq 3. \tag{1.11}$$

The conditions (1.9) and (1.10) can be regarded as a (damped) global virial condition and is used to prove the absence of quantum resonances near threshold zero in [12], where sub-exponential time-decay estimates are obtained for compactly supported perturbations of model potentials in $\mathcal{V}$ (for the heat semigroup) or in $\mathcal{A}$ (for the Schrödinger semigroup) and for geometrically simple zero eigenvalue of H. Note that for $n = 3$, the repulsive Coulomb potential $V_0(x) = \frac{c_0}{|x|}$, $c_0 > 0$, belongs to $\mathcal{A}$.

In this paper, we consider more general potentials and need the following condition to compute the singularity of the resolvent $R(z) = (H - z)^{-1}$ at $z = 0$.

Assumption $(A1)$ Let zero be an eigenvalue of H of geometric multiplicity k. Assume that there exists a basis $\{\varphi_1, \cdots, \varphi_k\}$ of the eigenspace of H with eigenvalue zero verifying

$$\det\left(\langle \varphi_j, J\varphi_i\rangle\right)_{1\leq i,j\leq k} \neq 0. \tag{1.12}$$

The following result on the heat semigroup e^{-tH} will be proven by combining the ideas from [1] and [12].

Theorem 1.1. *Let $V_0 \in \mathcal{V}$ and $W = V - V_0$ satisfy the condition (1.2) with $\rho > 2\kappa\mu$, where κ is the integer given by Proposition 2.2. Assume that zero is an eigenvalue of H with geometric multiplicity k and assumption $(A1)$ holds. Then for any $a > 0$ there exist some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x\rangle^{1-\mu}}\left(e^{-tH} - \sum_{\lambda\in\sigma_d(H), \mathrm{Re}\,\lambda\leq 0} e^{-tH}\Pi_\lambda - \Pi_0\right)\| \leq C_a e^{-c_a t^{\beta'}} \quad t > 0, \tag{1.13}$$

where $\beta' = \frac{1-\mu}{1+\kappa\mu}$ and Π_0 is the spectral projection defined by

$$\Pi_0 f = \sum_{j=0}^{k} \langle f, J\Psi_j\rangle \Psi_j, \qquad \forall\, f \in L^2, \tag{1.14}$$

where $\{\Psi_1, \cdots, \Psi_k\} \subset L^2$ is a basis of the eigenspace of H associated with the eigenvalue zero satisfying the relations

$$\langle \Psi_i, J\Psi_j\rangle = \delta_{ij}, \tag{1.15}$$

with $\delta_{ij} = 1$ *if* $i = j$ *and* $\delta_{ij} = 0$*, otherwise.*

Theorem 1.1 is proven in [12] for W compactly supported and $k = 1$. The main step of the proof of Theorem 1.1 is the asymptotic expansion of the resolvent $R(z) = (H - z)^{-1}$ at threshold zero with Gevrey estimates on the remainder (Theorem 3.5). To prove this result, we decompose H as $H = h_0 + W_c$ with $h_0 = -\Delta + v_0$ verifying (1.3) and W_c of compact support and use the equation $R(z) = (1 + K(z))^{-1} r_0(z)$ for $z \notin \sigma(H)$, where $r_0(z) = (h_0 - z)^{-1}$ and $K(z) = r_0(z) W_c$. The condition (A1) is used to ensure the existence of an asymptotic expansion for $(1 + K(z))^{-1}$ at $z = 0$. In fact when 0 is an eigenvalue of geometric multiplicity k of H, -1 is an eigenvalue of $K(0)$ with the same geometric multiplicity k and with finite algebraic multiplicity $m = \dim E$, where $E = \{u \in L^2; \exists j, (1 + K)^j u = 0\}$. Since $\dim \mathrm{Ker}(1 + K(0)) = k$, there exists a Jordan basis $\mathcal{B}$ of E such that the matrix of $1 + K(0)$ restricted onto E is of the Jordan form:

$$A = \mathrm{Diag}(J_{m_1}, \cdots, J_{m_k})$$

where J_{m_l} is the Jordan bloc of order m_l associated with the eigenvalue 0 and $m_1 + \cdots m_k = m$. By Grushin method, the existence of an asymptotic expansion for $1 + K(z)$ is reduced to that of an $m \times m$ matric $E_{-+}(z)$ of the form

$$E_{-+}(z) = -A + zB + O(z^2).$$

Set

$$\begin{aligned} I &= \{1, \cdots, m\} \setminus \{m_1, m_1 + m_2, \cdots, m_1 + m_2 + \cdots + m_k\}, \\ J &= \{1, \cdots, m\} \setminus \{1, m_1 + 1, \cdots, m_1 + m_2 + \cdots + m_{k-1} + 1\}. \end{aligned}$$

If A is denoted by $A = (a_{ij})_{1 \le i,j \le m}$, then $a_{ij} = 1$ if $i \in I$ and $j \in J$ and $a_{ij} = 0$, otherwise. Then one can check that

$$\det E_{-+}(z) = z^k D_k + O(z^{k+1}) \tag{1.16}$$

for z near 0, where D_k is, up to a sign, equal to the minor of order k obtained from B by deleting all i-th lines and j-th colons with $i \in I$ and $j \in J$. One can prove by choosing appropriate bases of E that $D_k \neq 0$ if and only if the assumption (A1) is satisfied (cf. Lemma 3.3). See also [1] for non selfadjoint Schrödinger operators with quickly decreasing potentials and [12] in the case $k = 1$. Therefore the assumption (A1) is in some sense a necessary condition for $R(z)$ to admit an asymptotic expansion at $z = 0$ with the leading term of the order $\frac{1}{z}$. If one has for some $l \in \mathbb{N}$ $\det E_{-+}(z) = z^{k+l} D_{k+l} + O(z^{k+l+1})$ with $D_{k+l} \neq 0$, then one can show that $R(z)$ admits an asymptotic expansion at $z = 0$ with the leading term of the order $\frac{1}{z^{l+1}}$. But we do not have any example for which this condition is satisfied with $l \geq 1$.

The second main result of this work is about large-time expansion of the Schrödinger semi-group with sub-exponential time-decay estimates on the remainder. For this purpose, we introduce the following class of model potentials.

Definition 1.3. *Let* $\mathcal{A}_1$ *denote the subclass of potentials* $V_0 \in \mathcal{V}$ *satisfying that* $\mathrm{Im}\, V_0 \le 0$ *and there exists some potential* $\widetilde{V}_0 \in \mathcal{A}$ *with*

$$V_0(x) = \widetilde{V}_0(x), \quad |x| > R_1, \tag{1.17}$$

for some $R_1 > 0$*.*

For $V_0 \in \mathcal{A}_1$, denote $H_0 = -\Delta + V_0(x)$ and $\widetilde{H}_0 = -\Delta + \widetilde{V}_0(x)$ where $\widetilde{V}_0 \in \mathcal{A}$ coincides with $V_0(x)$ outside some compact. We use the technique of analytic distortion from quantum resonances to study e^{-itH} and need the following analyticity condition on $W(x) = V(x) - V_0(x)$:

$$\left\{\begin{array}{l} W \text{ is holomorphic in a complex region } \Omega \text{ of the form (1.6) for some } R_2>0 \text{ and} \\ \qquad\qquad |W(x)| \le C\langle \operatorname{Re} x\rangle^{-2\mu-\epsilon},\ \forall x\in\Omega. \end{array}\right. \tag{1.18}$$

for some constants $C>0$ and $\epsilon>0$. In addition, one assumes that there exists $R_3>0$ such that

$$\operatorname{Im} W(x) \le 0, \qquad x\in\mathbb{R}^n \text{ with } |x|>R_3. \tag{1.19}$$

Since $\operatorname{Im} W(x)$ is allowed to change sign in some bounded region, H may have outgoing positive resonances which are relevant to the asymptotic expansion of e^{-itH} as $t\to+\infty$.

If V is of short-range: $V(x)=O(\langle x\rangle^{-1-\epsilon})$ for some $\epsilon>0$, $\lambda>0$ is called real resonance of $H=-\Delta+V(x)$ if the equation $Hu=\lambda u$ admits a non-trivial solution $u\in H^2_{\rm loc}(\mathbb{R}^n)$ satisfying one of Sommerfeld radiation conditions:

$$u(x)=\frac{e^{\pm i\sqrt{\lambda}|x|}}{|x|^{\frac{n-1}{2}}}(a_\pm(\omega)+o(1)), \quad |x|\to\infty, \tag{1.20}$$

for some $a_\pm\in L^2(\mathbb{S}^{n-1}), a_\pm\neq 0$. λ is called an outgoing (resp., incoming) positive resonance of H if u verifies (1.20) with sign $+$ (resp. with sign $-$). In this paper, we use the same definition of real resonances as in [12] because potential $V(x)$ has a complex long-range tail.

Let $\chi_1\in C^\infty(\mathbb{R}^n)$ with $\chi_1(x)=0$ for $|x|\le 1$ and $\chi_1(x)=1$ for $|x|>2$. For $R>0$ sufficiently large : $R>2\max\{R_1,R_2,R_3\}$, set $\chi_R(x)=\chi_1(\frac{x}{R})$ and

$$h_0=-\Delta+v_0(x), \quad v_0=V_0+\chi_R W, \quad V_c=(1-\chi_R)V, \quad W_c=(1-\chi_R)W. \tag{1.21}$$

h_0 is a dissipative operator: $\operatorname{Im} h_0\le 0$ and the boundary value of the resolvent $r_0(\lambda+i0)=\lim_{\epsilon\to0^+}(h_0-(\lambda+i\epsilon))^{-1}$ exists in $\mathcal{B}(-1,s;1,-s)$ for any $s>\frac12$ and is continuous in $\lambda>0$ ([9]). The resolvent equation $R(z)=(1+r_0(z)W_c)^{-1}r_0(z)$ motivates the following definition

Definition 1.4. *Let $\lambda>0$ and $G_0^+(\lambda)=r_0(\lambda+i0)$. $\lambda>0$ is called outgoing positive resonance of $H=-\Delta+V(x)$ if -1 is an eigenvalue of the compact operator $G_0^+(\lambda)W_c$ in $L^{2,-s}$ for $s>\frac12$. Denote $r_+(H)$ the set of outgoing positive resonances of H. For $\lambda\in r_+(H)$, denote $m(\lambda)$ the algebraic multiplicity of eigenvalue -1 of $G_0^+(\lambda)W_c$ and $k(\lambda)$ its geometric multiplicity.*

The above definition is independent of the cut-off χ_R used. In fact if h_0 and $\widetilde{h}_0$ are two operators constructed as above with two different cut-offs χ_R and $\widetilde{\chi}_{\widetilde{R}}$ with R and $\widetilde{R}$ sufficiently large and $H=\widetilde{h}_0+\widetilde{W}_c$, one deduces from the formula

$$1+r_0(z)W_c=\left(1+r_0(z)(\chi_R-\widetilde{\chi}_{\widetilde{R}})W\right)\left(1+\widetilde{r}_0(z)\widetilde{W}_c\right)$$

and the similar one with χ_R and $\widetilde{\chi}_{\widetilde{R}}$ interchanged that -1 is an eigenvalue of $G_0^+(\lambda)W_c$ if and only if it is an eigenvalue of $\widetilde{G}_0^+(\lambda)\widetilde{W}_c$. If V_0 and V are short-range potentials, one can check that $\lambda>0$ is an outgoing resonance if and only if the equation $(H-\lambda)u=0$ admits a solution satisfying the outgoing Sommerfeld radiation condition (1.20).

For $\lambda\in r_+(H)$, we define the symmetric bilinear form B_λ^+ associated with λ

$$B_\lambda^+(\varphi,\psi)=\langle G_1^+(\lambda)W_c\psi, JW_c\varphi\rangle \ \text{ for all } \ \varphi,\psi\in L^{2,-s}(\mathbb{R}^n), \tag{1.22}$$

where $G_1^+(\lambda)=\lim_{z\to\lambda,\operatorname{Im} z>0}\frac{d}{dz}r_0(z)$ in the norm sense of $\mathcal{B}(-1,s,1,-s)$, $s>3/2$, and $J: f(x)\mapsto \overline{f(x)}$ is the complex conjugation.

Assumption $(A2)$**:** For all $\lambda \in r_+(H)$, one assumes that there exists $\{\psi_1^+, \cdots, \psi_{k(\lambda)}^+\} \subset L^{2,-s}$, $\forall s > 1/2$, a basis of $\mathrm{Ker}(I + G_0^+(\lambda)W_c)$ such that

$$\det \left(B_\lambda^+(\psi_i^+, \psi_j^+) \right)_{1 \le i,j \le k(\lambda)} \neq 0. \tag{1.23}$$

The condition (1.23) is similar to that used in [1, 11] for quickly decreasing potentials.

Theorem 1.2. *Let $V_0 \in \mathcal{A}_1$ and $W(x) = V(x) - V_0(x)$ satisfy (1.2), (1.18) and (1.19). Suppose that the conditions $(A1)$ and $(A2)$ hold for zero eigenvalue and positive outgoing resonances, respectively. Then the set of outgoing resonances $r_+(H)$ of H is at most finite and there exists $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, one has*

$$\|\chi \left(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \overline{\mathbb{C}}_+} e^{-itH}\Pi_\lambda - \Pi_0 - \sum_{\nu \in r_+(H)} e^{-it\nu}\Pi_0^+(\nu) \right) \chi\| \le C_\chi e^{-c\, t^\beta} \quad t > 0. \tag{1.24}$$

Here $\beta = \frac{1-\mu}{1+\mu}$, Π_λ denotes the Riesz projection associated with the discrete eigenvalue λ of H, Π_0 is the projector given in (1.14) and $\Pi_0^+(\nu)$ is an operator of rank $k(\nu)$ given by

$$\Pi_0^+(\nu)f = \sum_{j=0}^{k(\nu)} \langle f, J\Phi_j^+(\nu)\rangle \Phi_j^+(\nu) \qquad \text{for all} \quad f \in L^{2,s}, s > 1/2, \tag{1.25}$$

with

$$B_\nu^+(\Phi_i^+(\nu), \Phi_j^+(\nu)) = \delta_{ij}, \tag{1.26}$$

where B_ν^+ is the bilinear form defined in (1.22) and $\{\Phi_1^+(\nu), \cdots, \Phi_{k(\nu)}^+(\nu)\} \subset L^{2,-s}$, $\forall s > 1/2$, is a basis of the eigenspace associated with the eigenvalue -1 of $r_0(\lambda + i0)W_c$.

Of course, if zero eigenvalue and/or positive outgoing resonances are absent, the asymptotic expansion (1.24) still holds with the associated terms disappeared.

Remark 1.5. *1. Theorem 1.2 can be compared with* [12, Theorem 2.4 (b)], *where the perturbation $W = V - V_0$ is compactly-supported with $V_0 \in \mathcal{A}$ (satisfying a global virial condition), the zero eigenvalue is supposed to be geometrically simple, the condition $(A2)$ is not assumed and the contribution from positive outgoing resonances has not been explicitly calculated. Here V_0 satisfies only a virial condition outside some compact, the geometrical multiplicity of zero eigenvalue is arbitrary and the contribution of positive outgoing resonances is explicitly calculated in terms of an orthonormal basis relatively to $B_\nu^+(\cdot,\cdot)$.*

2. The proofs of Theorem 1.2 and [12, Theorem 2.4 (b)] *are both based on the resolvent expansions near the threshold eigenvalue and outgoing resonances. But the methods to establish the resolvent expansions are different. To study the asymptotic expansion of $(1 + R_0(z)W)^{-1}$ in a sector below the positive half-axis, in* [12] *we constructed an approximate Grushin problem with θ-independent bases (where θ is the complex parameter used in analytic deformation of operators), while in this work we study θ-dependent Grushin problems by constructing Jordan bases depending holomorphically on θ.*

Example 1.6. *Let $\varphi \in C^2(\mathbb{R}^n; \mathbb{C})$ such that there exist some constants $\mu \in]0,1[$ and $R > 0$ such that $\varphi(x) = \langle x\rangle^{1-\mu}$ for $|x| > R$. The associated Witten Laplacian acting on functions is given by*

$$-\Delta_\varphi = -\Delta + V(x) \quad \text{with } V(x) = (\nabla\varphi \cdot \nabla\varphi)(x) - \Delta\varphi(x).$$

Take $V_0(x) = \chi(\frac{x}{R}) + (1 - \chi(\frac{x}{R}))\frac{(1-\mu)^2}{\langle x\rangle^{2\mu}}$ where χ is a cut-off with $0 \le \chi \le 1$ and $\chi(x) = 1$ if $|x| \le 2$; 0 if $|x| \ge 3$. Then $V(x)$ can be decomposed as $V(x) = V_0(x) + W(x)$ with $V_0 \in \mathcal{A}_1$ and W satisfying the conditions of Theorem 1.2. But $V(x)$ can not be decomposed *as* $V(x) = \widetilde{V}_0(x) + \widetilde{W}(x)$ *with* $\widetilde{V}_0 \in \mathcal{A}$ *(which in particular requires* $\widetilde{V}_0(x)$ *to be dilation analytic) and* $\widetilde{W}$ *compactly supported. Therefore Theorem 1.2 can be applied to* $e^{it\Delta_\varphi}$*, while* [12, Theorem 2.4 (b)] *cannot.*

The remaining part of this work is organized as follows. In Section 2, we recall from [12] the Gevrey estimates for the resolvent of the model operator needed in this paper. The asymptotic expansion of the resolvent $R(z)$ (or the cut-off resolvent $\chi R(z)\chi$ with $\chi \in C_0^\infty(\mathbb{R}^n)$) near $z = 0$ is calculated in Section 3. In Section 4, we study resolvent expansion near positive resonances. Finally Theorems 1.1 and 1.2 are proved in Section 5.

Notation. We denote $H^{r,s}$, $r \geq 0, s \in \mathbb{R}$ the weighted Sobolev space of order r with the weight $\langle x\rangle^s$ on $\mathbb{R}^n$: $H^{r,s} = \{u \in \mathcal{S}'(\mathbb{R}^n); \|u\|_{r,s} = \|\langle x\rangle^s(1-\Delta)^{\frac{r}{2}}u\|_{L^2} < \infty\}$. For $r < 0$, $H^{r,s}$ is defined as the dual space of $H^{-r,-s}$ with dual product identified with the scalar product $\langle\cdot,\cdot\rangle$ of $L^2(\mathbb{R}^n)$. Set $H^{0,s} = L^{2,s}$. $\mathcal{B}(r, s; r', s')$ stands for the space of continuous linear operators from $H^{r,s}$ to $H^{r',s'}$. If $(r, s) = (r', s')$, we denote $\mathcal{B}(r, s) = \mathcal{B}(r, s; r', s')$. Unless otherwise mentioned explicitly, $\|\cdot\|$ denotes norm in $L^2(\mathbb{R}^n)$ or in $\mathcal{B}(L^2)$ when no confusion is possible. $\mathbb{C}_\pm$ denote respectively the upper and the lower open half-plane and $\overline{\mathbb{C}}_\pm$ their closure. Set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $\theta_1 < \theta_2$ and $r > 0$, $S(\theta_1, \theta_2)$ denotes the sector

$$S(\theta_1, \theta_2) = \{z \in \mathbb{C}^*; \theta_1 < \arg z < \theta_2\}$$

and $\Omega(r, \theta_1, \theta_2)$ is a part of $S(\theta_1, \theta_2)$ near zero :

$$\Omega(r, \theta_1, \theta_2) = \{z \in S(\theta_1, \theta_2); |z| < r\}.$$

In this work, the scalar product denoted as $\langle\cdot,\cdot\rangle$ is assumed to be linear with respect to the left variable.

2. Gevrey estimates for the resolvent of the model operators

For $V_0 \in \mathcal{V}$ and $W = V - V_0$ verifying (1.2), set $v_0(x) = V_0(x) + \chi_1(\frac{x}{R})W(x)$ where $\chi_1 \in C^\infty(\mathbb{R}^n)$ is a cut-off function with $\chi_1(x) = 0$ for $|x| \leq 1$ and $\chi_1(x) = 1$ for $|x| \geq 2$. Since $\rho > 2\mu$, for $R > 1$ large enough, v_0 satisfies condition (1.3) and the results on Gevrey estimates of the resolvent of [12] can be applied to $h_0 = -\Delta + v_0(x)$. Let $r_0(z) = (h_0 - z)^{-1}$. Define operator $G_0 : \mathrm{Im}\, h_0 \to L^2$ by

$$G_0 f = \lim_{\mathrm{Re}\, z<0, z\to 0} r_0(z)f, \quad f \in \mathrm{Im}\, h_0.$$

G_0 is an unbounded closed operator with $D(G_0) = \mathrm{Im}\, h_0$. It is proven that $G_0 : L^{2,s} \to L^{2,s-2\mu}$ is continuous (cf. [12, Lemma 3.3]). The following result is consequence of [12, Theorem 2.1].

Theorem 2.1. *Let* $V_0 \in \mathcal{V}$*. Then for any* $a > 0$*, there exist some constants* $C_a, c_a > 0$ *such that*

$$\|\langle x\rangle^{-\tau}e^{-a\langle x\rangle^{1-\mu}}G_0^N\langle x\rangle^\tau\| + \|\langle x\rangle^\tau G_0^N e^{-a\langle x\rangle^{1-\mu}}\langle x\rangle^{-\tau}\| \leq C_a c_a^{N+\tau}(N+\tau)^{\frac{\tau}{1-\mu}+\gamma N} \tag{2.1}$$

for all $N \in \mathbb{N}^*$ *and* $\tau \geq 0$*. Here* $\gamma = \frac{2\mu}{1-\mu}$*.*

Theorem 2.1 is Gevrey estimates for the resolvent $r_0(z)$ at $z = 0$. To obtain sub-exponential time-decay estimates for the heat semi-group, we need Gevrey estimates of the resolvent in some region of the right half-plane. Recall first the following result on $R_0(z) = (H_0 - z)^{-1}$ ([12, Proposition 4.1]).

Proposition 2.2. *Let* $V_0 \in \mathcal{V}$ *for some* $\mu \in]0, 1[$*. Then*

(1) *There exist $\mu' \in]0,1]$ and $C_0 > 0$ such that the numerical range $N(H_0)$ of H_0 is contained in a region of the form $\{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| \leq C_0 (\operatorname{Re} z)^{\mu'}\}$. Consequently, for $\delta > 0$ small enough there exists some constant M_0 such that*

$$\|R_0(z)\| \leq \frac{M_0}{|z|^{\frac{1}{\mu'}}} \tag{2.2}$$

for $z \in O(\delta)$ where

$$O(\delta) = \{z \in \mathbb{C}^*; |z| < \delta, \operatorname{Re} z < \delta |\operatorname{Im} z|^{\frac{1}{\mu'}}\}. \tag{2.3}$$

(2) *If κ is the smallest integer such that $\kappa \geq \frac{1}{\mu'}$, one has*

$$\|\langle x\rangle^{-2\kappa\mu} R_0(z)\| \leq C \tag{2.4}$$

uniformly in $z \in O(\delta)$.

Note that in [12], the constant μ' used to describe the numerical range of H_0 is obtained from the generalized Hardy inequality. If one knows $\mu' = 1$ (for example if H_0 is selfadjoint), then κ can be taken to be 1 in the above proposition.

Proposition 2.3. *Let $V_0 \in \mathcal{V}$ for some $\mu \in]0,1[$ and κ be given by Proposition 2.2. Assume the condition (1.2) is satisfied for some $\rho > 2\kappa\mu$. Let h_0 be defined as above with $R > 1$ appropriately large. Then one has*

(*a*) *h_0 has no eigenvalues in $O(\delta)$ and*

$$\|\langle x\rangle^{-2\kappa\mu} r_0(z)\| \leq C \tag{2.5}$$

uniformly in $z \in O(\delta)$. In addition,

$$\lim_{z \in O(\delta), z\to 0} r_0(z) = G_0 \tag{2.6}$$

as bounded operators from $L^{2,s}$ to $L^{2,s-r}$ for all $s \in \mathbb{R}$ and $r > 2\kappa\mu$.

(*b*) *One has the following Gevrey estimates of the resolvent $r_0(z)$: for any $a > 0$, there exist $c, C > 0$ such that*

$$\|e^{-a\langle x\rangle^{1-\mu}} \frac{d^{N-1}}{dz^{N-1}} r_0(z)\| \leq C c^N N^{(1+\kappa\gamma)N}, \quad \forall N \geq 1, \tag{2.7}$$

uniformly in $z \in O_0(\delta) \equiv O(\delta) \cup \{0\}$, where $\gamma = \frac{2\mu}{1-\mu}$.

Proof. (*a*). One has

$$r_0(z) = R_0(z) - r_0(z)\chi_R W R_0(z). \tag{2.8}$$

Since $\rho > 2\kappa\mu$, it follows from (2.4) that if $R > 1$ is appropriately large,

$$\|\chi_R W R_0(z)\| \leq C R^{-(\rho - 2\kappa\mu)} \|\langle x\rangle^{-2\kappa\mu} R_0(z)\| < 1$$

uniformly in $z \in O(\delta)$. This proves $1 + \chi_R W R_0(z)$ is invertible with uniformly bounded inverse. Therefore h_0 has no eigenvalues in $O(\delta)$ and

$$r_0(z) = R_0(z)(1 + \chi_R W R_0(z))^{-1}, \quad z \in O(\delta),$$

which together with (2.4) gives the uniform estimate (2.5). (2.6) is deduced from (2.5) and the resolvent equation

$$r_0(z) - G_0 = z G_0 r_0(z), \tag{2.9}$$

using the compactness of $r_0(z) - G_0$ as operators from $L^{2,s}$ to $L^{2,s-r}$ for $r > 2\kappa\mu$.

(*b*) is derived from (*a*), Theorem 2.1 and the formula

$$r_0(z) = \sum_{j=0}^{q-1} z^j G_0^{j+1} + z^q G_0^q r_0(z), \quad z \in O(\delta) \tag{2.10}$$

with $q = \kappa$. See [12, Corollary 4.2] for details when $r_0(z)$ is replaced by $R_0(z)$. ∎

3. Resolvent expansion at threshold zero

In this section, we look at the case where H has an embedded eigenvalue zero with arbitrary geometric multiplicity. To obtain the asymptotic expansion of $R(z)$ with Gevrey estimates on the remainder, we use the resolvent equation $R(z) = (I + r_0(z)W_c)^{-1}r_0(z)$ for $z \notin \sigma(H)$.

3.1. Asymptotic expansion of $R(z)$.

In this subsection, we assume $V_0 \in \mathcal{V}$. Set $K_0(z) = r_0(z)W_c : L^2 \to L^2$. The compact operator-valued function $z \mapsto K_0(z)$ is meromorphic on $\mathbb{C}\setminus\mathbb{R}_+$ with poles at discrete eigenvalues of h_0. The condition (1.3) is satisfied for h_0 and zero is not an eigenvalue of h_0. According to Proposition 2.3, if $V_0 \in \mathcal{V}$ and W satisfies (1.2) for some $\rho > 2\kappa\mu$, then G_0 is continuous as operator from $L^{2,s}$ to $L^{2,s-r}$ for all $s \in \mathbb{R}$ and $r > 2\kappa\mu$. In addition, $K_0 := K_0(0) = G_0W_c$ is a compact operator on L^2. One can check that zero is an embedded eigenvalue of H if and only if -1 is a discrete eigenvalue of K_0 and their eigenspaces coincide in L^2. Although the Riesz projection for the eigenvalue zero of H can not be defined, that for eigenvalue -1 of K_0 is well defined and is given by

$$\Pi_1 = \frac{1}{2i\pi}\int_{|z+1|=\epsilon_0} (z - K_0)^{-1}\, dz,$$

for some $\epsilon_0 > 0$ small enough such that -1 is the only eigenvalue of K_0 inside the cercle $\{z; |z+1| = \epsilon_0\}$.

Let $E = \text{Range}\, \Pi_1$, $m = \text{Rank}\, \Pi_1$ and $k = \dim \text{Ker}(I + K_0)$. E is a subspace of L^2. Consider the non degenerate bilinear form B on $E \times E$:

$$\forall u, v \in E, \qquad B(u, v) = \langle u, v^*\rangle, \tag{3.1}$$

where $v^* := JW_cv$. Note that the relations $h_0^* = Jh_0J$ and $H^* = JHJ$ imply that $JW_cK_0 = K_0^*JW_c$ and $JW_c\Pi_1 = \Pi_1^*JW_c$. The mapping $S : \phi \to \phi^* = JW_c\phi$ sends Range Π_1 to Range Π_1^* and $\text{Ker}(1 + K)$ to $\text{Ker}(1 + K^*)$. It is injective on $\text{Ker}(1 + K)$, because if $\phi \in \text{Ker}(1 + K)$ with $\phi \neq 0$ and $\phi^* = 0$, then $W_c\phi = 0$ and hence $h_0\phi = 0$. This is impossible because 0 is not an eigenvalue of h_0. We infer that if $\mathcal{B}$ is a Jordan basis of Range Π_1, then $\mathcal{B}^* = S(\mathcal{B})$ is free in Range Π_1^*, hence S is injective from Range Π_1 into Range Π_1^*. Since Rank Π_1 = Rank Π_1^*, S is bijective from Range Π_1 onto Range Π_1^*. The argument of Lemma 5.13 in [12] based on the bijectivity of S shows that the bilinear form $B(\cdot,\cdot)$ defined above is non-degenerate on $E \times E$.

In order to study the case where the eigenvalue -1 of K_0 has arbitrary geometric multiplicity $k \in \mathbb{N}^*$, we decompose E into k invariant subspaces of K_0, such that B restricted to each one is non-degenerate. More precisely, we have

Lemma 3.1. *Assume that -1 is an eigenvalue of K_0 of geometric multiplicity $k \in \mathbb{N}^*$ and algebraic multiplicity m. Then there exist k invariant subspaces of K_0, denoted by $E_1, \cdots, E_k$, such that*

(1) *$E = E_1 \oplus \cdots \oplus E_k$, where $\forall i \neq j$: $E_i \perp E_j$ with respect to the bilinear form B.*

(2) *$\forall 1 \le i \le k$, there exists a basis $\mathcal{U}_i := \{u_r^{(i)} = (I + K_0)^{m_i - r}u_{m_i}^{(i)}, 1 \le r \le m_i\} \subset L^2$ of E_i such that*

$$(I + K_0)^{m_i}u_{m_i}^{(i)} = 0 \text{ and } B(u_1^{(i)}, u_{m_i}^{(i)}) = c_i \neq 0. \tag{3.2}$$

(3) *$\forall 1 \le j \le k$, there exists a dual basis $\mathcal{W}_j := \{w_1^{(j)}, \cdots, w_{m_j}^{(j)}\} \subset L^2$ of E_j such that*

$$w_r^{(j)} \in Ker(I+K_0)_{|E_j}^{m_j+1-r}, \ \ B(u_l^{(i)}, w_r^{(j)}) = \delta_{lr}\delta_{ij}.$$

(4) *$dim\ ker(1+K_0)_{|E_j} = 1$, $\forall j = 1, \cdots, k$.*

Moreover, the matrix of $\Pi_1(I+K_0)\Pi_1$ in the basis $\mathcal{U} := \bigcup_{i=1}^k \mathcal{U}_i$ of E is a $k\times k$ block diagonal matrix given by

$$J = diag(J_{m_1}, J_{m_2}, \cdots, J_{m_k}), \tag{3.3}$$

where

$$J_{m_j} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m_j\times m_j} \tag{3.4}$$

is a Jordan block. We have also denoted $m_j = dim\ E_j$ for $j = 1, \cdots, k$, such that $m = m_1 + \cdots + m_k$.

For the proof of the above lemma, we refer to that of Lemma 3.1 in [1] in the case where 0 is an eigenvalue of $-\Delta + V(x)$ with a quickly decreasing complex-valued potential $V(x)$. The proof is algebraic in nature and is still valid in the present case.

As consequence of Lemma 3.1, one has the following

Corollary 3.2. *Let a basis $\mathcal{B} \equiv \bigcup_{i=1}^{k} \mathcal{U}_i$ of E and its B-dual basis $\mathcal{B}^* \equiv \bigcup_{i=1}^{k} \mathcal{W}_i$ be constructed as in Lemma 3.1. Then the Riesz projection Π_1 can be represented as*

$$\Pi_1 f = \sum_{i=1}^{k}\sum_{j=1}^{m_i} B(f, w_j^{(i)})u_j^{(i)}, \qquad \forall f \in L^2. \tag{3.5}$$

3.1.1. ***Grushin problem for** $I + K_0(z)$.* Denote $\Pi_1' = I - \Pi_1$. Since $I + K_0$ is injective on Range Π_1', it is invertible on Range Π_1' and by an argument of perturbation $\Pi_1'(I+K_0(z))\Pi_1'$ is invertible on Range Π_1' for $z \in O(\delta)$, $\delta > 0$ small, with the inverse denoted by

$$E(z) = \left(\Pi_1'(I+K_0(z))\Pi_1'\right)^{-1}\Pi_1' \in \mathcal{B}(L^2). \tag{3.6}$$

According to Proposition 2.3, $K_0(z)$ is continuous and uniformly bounded in $z \in O(\delta)$. Since W_c is of compact support, (2.7) leads to the following Gevrey estimates on $K_0(z)$: there exist some constants $C, c > 0$ such that

$$\|K_0(z)^{(N)}\| \le Cc^N N^{(1+\kappa\gamma)N}, \quad N \in \mathbb{N}^*,$$

for $z \in O_0(\delta)$. Here $\|\cdot\|$ is the norm of bounded operators in L^2 and $K_0(z)^{(N)}$ denotes the N-th derivative of $K_0(z)$. By operations in Gevrey classes, one deduces that there exist constants C' and $c_1 > 0$ such that $E(z)$ satisfies the following Gevrey estimates:

$$\|E(z)^{(N)}\| \le C' c_1^N N^{(1+\kappa\gamma)N}, \tag{3.7}$$

for all $N \in \mathbb{N}^*$ and $z \in O_0(\delta)$. In addition, for all $q \in \mathbb{N}$, one has

$$E(z) = \sum_{j=0}^{q} z^j E_j + z^q \epsilon_q(z), \qquad z \in O(\delta), \tag{3.8}$$

in $\mathcal{B}(L^2)$, where $E_0 = \left(\Pi_1^{'}(I+K_0)\Pi_1^{'}\right)^{-1}\Pi_1^{'}$ and other terms $E_j, j = 1, \cdots, q$, can be computed explicitly. Moreover, the remainder $\epsilon_q(z)$ satisfies the following Gevrey estimates: there exist $C_q, c_q > 0$ such that

$$\|\epsilon_q(z)^{(N)}\| \le C_q c_q^N N^{(1+\kappa\gamma)N}, \tag{3.9}$$

for all $N \in \mathbb{N}^*$ and $z \in O_0(\delta)$.

Define $S : \mathbb{C}^m \to \text{Range}\ \Pi_1$ and $T : L^2 \to \mathbb{C}^m$ by

$$Sc = \sum_{i=1}^{k}\sum_{j=1}^{m_j} c_j^{(i)} u_j^{(i)}, \qquad \forall\, c = \bigoplus_{i=1}^{k}\left(c_1^{(i)}, \cdots, c_{m_i}^{(i)}\right) \in \bigoplus_{i=1}^{k}\mathbb{C}^{m_i},$$

$$Tu = \bigoplus_{i=1}^{k}\left(\langle u, (w_1^{(i)})^*\rangle, \cdots, \langle u, (w_{m_i}^{(i)})^*\rangle\right), \qquad \forall\, u \in L^2.$$

Then one has $TS = I_m$ and $ST = \Pi_1$. Set $W(z) = 1 + K_0(z)$ and

$$E_+(z) = S - E(z)W(z)S, \tag{3.10}$$

$$E_-(z) = T - TW(z)E(z), \tag{3.11}$$

$$E_{-+}(z) = -TW(z)S + TW(z)E(z)W(z)S. \tag{3.12}$$

Since $E(z)$ and $W(z)$ satisfy Gevrey estimates of the form (3.9) on $O_0(\delta)$, $E_\pm(z)$ and $E_{-+}(z)$ satisfy similar estimates on $O_0(\delta)$. In addition, $E_{-+}(z)$ is invertible if and only if $(I + K_0(z))$ is it and one has the formula

$$(I + r_0(z)W_c)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \text{ on } L^2(\mathbb{R}^n). \tag{3.13}$$

In order to study the singularity of $(I+r_0(z)W_c)^{-1}$ we calculate the expansion of $\det E_{-+}(z)$ in power of z, as well as that of $E_{-+}(z)^{-1}$ by using the method developed in [1].

Lemma 3.3. *Let $V_0 \in \mathcal{V}$ and W satisfy condition (1.2) for some $\rho > 2\kappa\mu$ with $\kappa \ge 1$ given by Proposition 2.2. Assume that -1 is an eigenvalue of $K_0 = G_0W_c$ of geometric multiplicity $k \ge 1$. Assume in addition that (A1) holds. Then, for all $q \in \mathbb{N}^*$, one has*

$$det\, E_{-+}(z) = \sum_{j=0}^{q}\sigma_j z^{k+j} + o(|z|^{k+q}), \tag{3.14}$$

for $z \in O(\delta)$ with $\delta > 0$ small, where $\sigma_0 = \sigma^{'} \times det(\langle\varphi_j, J\varphi_i\rangle)_{1\le i,j\le k} \neq 0$. Moreover, $det\, E_{-+}(z)$ satisfies Gevrey estimates of order $1+\kappa\gamma$.

Proof. Using Lemma 3.1, we partition the matrix $E_{-+}(z)$ as follows:

$$E_{-+}(z) = \left(E_{-+}^{(ij)}(z)\right)_{1\le i,j\le k} \text{ with}$$

$$E_{-+}^{(ij)}(z) = -\left(\left\langle\left(I - (I + r_0(z)W_c)E(z)\right)(I + r_0(z)W_c)u_r^{(j)}, JW_c w_\ell^{(i)}\right\rangle\right)_{\ell,r},$$

is a $m_i \times m_j$ block. Introducing the expansion (2.10) for $q \in \mathbb{N}^*$ into the previous formula, we obtain

$$E_{-+}(z) = \sum_{\ell=0}^{q-1} z^\ell E_{-+,\ell} + z^q\epsilon_{-+,q}(z), \tag{3.15}$$

where

$$
\begin{aligned}
E^{(ij)}_{-+,0} &= -\left(\left\langle (I+G_0W_c)u^{(j)}_r, JW_cw^{(i)}_\ell \right\rangle\right)_{\ell,r},\\
E^{(ij)}_{-+,1} &= -\left(\left\langle G_0W_cu^{(j)}_r, JG_0W_cw^{(i)}_\ell \right\rangle\right)_{\ell,r},\\
E^{(ij)}_{-+,2} &= -\left(\left\langle (I-G_0W_cE(0))G_0^2W_cu^{(j)}_r, JG_0W_cw^{(i)}_\ell \right\rangle\right)_{\ell,r} \text{ with}\\
E(0) &= \left(\Pi_1'(I+G_0W_c)\Pi_1'\right)^{-1}\Pi_1'.
\end{aligned}
$$

Also other terms $E_{-+,\ell}, \ell = 3, \cdots, q-1$ can be calculated explicitly. In addition, it can be seen from (3.9) that the remainder $\epsilon_{-+,q}(z)$ satisfies the following Gevrey estimates: $\exists\, c, C > 0$ such that

$$\left\| \frac{d^{N-1}}{dz^{N-1}} \epsilon_{-+,q}(z) \right\| \le C c^N N^{(1+\kappa\gamma)N}, \tag{3.16}$$

for all $N \ge 1$ and $z \in O_0(\delta)$, where the above norm is the matrix norm.

We can simplify the form of the above matrices as follows: Since $u^{(j)}_1 \in \mathrm{Ker}(I + G_0W_c)$, $(I + G_0W_c)u^{(j)}_r = u^{(j)}_{r-1}$, $r = 2, \cdots, m_j$, $1 \le j \le k$, and $w^{(i)}_{m_i} = c_i^{-1}u^{(i)}_1$ (c_i, $1 \le i \le k$, are given in (3.2)), then for all $1 \le i, j \le k$ we have

$$E^{(ij)}_{-+,0} = \begin{pmatrix} 0 & -\delta_{ij} & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & -\delta_{ij}\\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m_i\times m_j} \quad \text{and} \tag{3.17}$$

$$E^{(ij)}_{-+,1} = \begin{pmatrix} * & * & \cdots & \cdots & *\\ \vdots & \vdots & \ddots & & \vdots\\ \vdots & \vdots & & \ddots & \vdots\\ * & * & \cdots & \cdots & *\\ a_{ij} & * & \cdots & \cdots & * \end{pmatrix}_{m_i\times m_j} \quad \text{with } a_{ij} = -c_i^{-1}\langle u^{(j)}_1, Ju^{(i)}_1\rangle. \tag{3.18}$$

Thus, to obtain (3.14) we shall calculate $\det(E_{-+,0} + zE_{-+,1})$ by following the method used in [1] which is based on Lidskii's idea ([6]).

First, for $z \ne 0$ we introduce the $k \times k$ block diagonal matrix $L(z)$ partitioned conformally with the block structure of $E_{-+}(z)$, given by

$$L(z) = \mathrm{diag}(L_1(z), \cdots, L_k(z)), \tag{3.19}$$

where

$$L_i(z) = \mathrm{diag}(1, \cdots, 1, z^{-1}),\ \forall 1 \le i \le k.$$

Let $q \ge 2$. Set $E_{0,1}(z) = E_{-+,0} + zE_{-+,1}$. Multiplying $E_{0,1}(z)$ on the left by $L(z)$, we obtain

$$\widetilde{E}_{0,1}(z) := L(z)E_{0,1}(z) = \widetilde{E}_0 + z\,\widetilde{E}_1, \tag{3.20}$$

where the block entries of the obtained matrices $\widetilde{E}_0$ and $\widetilde{E}_1$ have the following forms:

$$\widetilde{E}_0^{(ij)} = \begin{pmatrix} 0 & -\delta_{ij} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -\delta_{ij} \\ a_{ij} & * & \cdots & * & * \end{pmatrix}_{m_i\times m_j}, \qquad \forall 1\leq i,j\leq k,$$

$$\widetilde{E}_1^{(ij)} = \begin{pmatrix} * & * & \cdots & \cdots & * \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ * & * & \cdots & \cdots & * \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}_{m_i\times m_j}, \qquad \forall 1\leq i,j\leq k. \tag{3.21}$$

The above terms a_{ij} are given in (3.18).

Next, we calculate $\det \widetilde{E}_{0,1}(z)$ where it is clear from (3.20) that it is a polynomial of z. We have

$$\det \widetilde{E}_{0,1}(z) = \det \widetilde{E}_0 + O(|z|), \qquad \forall z\in O(\delta). \tag{3.22}$$

Then, let us calculate $\det \widetilde{E}_0(0)$. By expanding the determinant of $\widetilde{E}_{0,1}(0)$ along the rows of $\widetilde{E}_{0,1}(0)$ that are containing only -1, we obtain

$$\det \widetilde{E}_0 = \det\,(a_{ij})_{1\leq i,j\leq k} = \sigma'' \times \det\,\left(\langle u_1^{(j)}, Ju_1^{(i)}\rangle\right)_{1\leq i,j\leq k} \neq 0, \tag{3.23}$$

where $\sigma'' = (-1)^k c_1^{-1}\cdots c_k^{-1}$ by (3.18). (See also the proof of Theorem 2.1 in [7] for a specific example with 12×12 matrix that illustrates the method to calculate the determinant).

Finally, for $z\in O(\delta)$ with $\delta>0$ small, it follows from (3.20), (3.22) and (3.23) that

$$\begin{aligned}\det E_{0,1}(z) &= (\det L(z))^{-1}\left(\det \widetilde{E}_0 + O(|z|)\right)\\ &= z^k \det\,(a_{ij})_{1\leq i,j\leq k} + O(|z|^{k+1}).\end{aligned}$$

Thus (3.14) can be deduced from (3.15) and the previous equation. Moreover, Gevrey estimates of order $1+\kappa\gamma$ for $\det E_{-+}(z)$ on $O_0(\delta)$ can be derived from that of $E_{-+}(z)$. ■

In the previous lemma we have given an sufficient condition for the invertibility of $E_{-+}(z)$ for $z\in O(\delta)$. Now we give the asymptotic expansion of its inverse.

Lemma 3.4. *Suppose that assumptions in Lemma 3.3 are satisfied. Then*

$$E_{-+}(z)^{-1} = \frac{\mathcal{F}_{-1}}{z} + \mathcal{F}(z) \tag{3.24}$$

for $z\in O(\delta)$, $\delta>0$ small, where $\mathcal{F}_{-1}$ is a matrix of order m and of rank k, whose block entries are of the form

$$\mathcal{F}_{-1}^{(ij)} = \begin{pmatrix} 0 & \cdots & 0 & \alpha_{ij} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad \forall 1\leq i,j\leq k, \tag{3.25}$$

with

$$(\alpha_{ij})_{1\leq i,j\leq k} = A_k^{-1}, \tag{3.26}$$

and the remainder $\mathcal{F}(z)$ is continuous up to $z=0$ and satisfies Gevrey estimates of the form (3.16) on $O_0(\delta)$.

Proof. To obtain the expansion of $E_{-+}(z)^{-1}$ we compute the inverse of the leading term $E_{0,1}(z) = E_{-+,0} + zE_{-+,1}$ in (3.12). By regularity of the matrix $L(z)$ for $z \neq 0$ given in (3.19), we observe that $E_{0,1}(z)$ is invertible if and only if $\widetilde{E}_{0,1}(z)$ in (3.20) is invertible. Thus our problem is reduced to calculate $\widetilde{E}_{0,1}(z)^{-1}$ if it does exist. Indeed, if the condition (1.12) holds, then by (3.23) the matrix $\widetilde{E}_0$ is invertible with

$$\widetilde{E}_0^{-1} = \frac{{}^t Com \widetilde{E}_0}{\det \widetilde{E}_0}.$$

By using the same technique to calculate $\det \widetilde{E}_0$, we can calculate the minors of the matrix $\widetilde{E}_0$. We obtain

$$\widetilde{E}_0^{-1} = \mathcal{F}_{-1} + \widetilde{\mathcal{F}}_0,$$

where $\mathcal{F}_{-1}$ is given in (3.25) and the block entries of $\widetilde{\mathcal{F}}_0$ are of the form

$$\widetilde{\mathcal{F}}_0^{(ij)} = \begin{pmatrix} * & * & \cdots & * & 0 \\ * & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 \end{pmatrix}_{m_i \times m_j} , \quad \forall 1 \leq i, j \leq k.$$

Moreover, to prove (3.26) we check by using the definition of coefficients $\alpha_{ij}, 1 \leq i, j \leq k$, that

$$(\det A_k)\alpha_{ij} = (-1)^{i+j} |[A_k]_j^i|,$$

where we denoted by $|[A_k]_j^i|$ the (j,i)-th minor of the matrix A_k. Thus, for $z \in O(\delta), \delta > 0$ small enough, $\widetilde{E}_{0,1}(z)$ is invertible, with

$$\widetilde{E}_{0,1}(z)^{-1} = \mathcal{F}_{-1} + \widetilde{\mathcal{F}}_0 - z\widetilde{E}_0^{-1}\widetilde{E}_1\widetilde{E}_0^{-1} + \widetilde{\mathcal{F}}(z), \tag{3.27}$$

where the remainder $\widetilde{\mathcal{F}}(z)$ is analytic in $z \in O(\delta)$ and continuous for $z \in O_0(\delta)$. By regularity of the matrix $L(z)$ for $z \neq 0$, it follows from (3.20) that

$$E_{0,1}(z)^{-1} = \widetilde{E}_{0,1}(z)^{-1} L(z).$$

In addition, we can easily check that

$$\mathcal{F}_{-1} L(z) = \frac{\mathcal{F}_{-1}}{z} \ \text{ and } \widetilde{\mathcal{F}}_0 L(z) = \widetilde{\mathcal{F}}_0.$$

This yields

$$E_{0,1}(z)^{-1} = \frac{\mathcal{F}_{-1}}{z} + \widetilde{\mathcal{F}}_0(I - \widetilde{E}_1 \mathcal{F}_{-1}) + \mathcal{F}_1(z),$$

for $z \in O(\delta)$, where $\mathcal{F}_1(z)$ is analytic in $z \in O(\delta)$ and continuous for $z \in O_0(\delta)$.

Finally, for $z \in O(\delta)$, the expansion (3.24) of $E_{-+}(z)^{-1}$ follows directly from (3.15) and the above expansion of $E_{0,1}(z)^{-1}$. In addition, Gevrey estimates of the remainder $\mathcal{F}(z)$ are derived from (3.16). ■

We can now prove the asymptotic expansion of the resolvent $R(z)$ at threshold zero.

Theorem 3.5. *Let $\kappa \in \mathbb{N}^*$ be given by Proposition 2.2. Assume that zero is an eigenvalue of H of geometric multiplicity k, $k \in \mathbb{N}^*$, and that assumption (A1) holds. Suppose that $V_0 \in \mathcal{V}$*

and $W = V - V_0$ satisfies condition (1.2) for some $\rho > 2\kappa\mu$. Then, for $z \in O(\delta)$ and $\delta > 0$ small enough

$$R(z) = -\frac{\Pi_0}{z} + R_1(z), \tag{3.28}$$

where Π_0 is a spectral projection given by

$$\Pi_0 f = \sum_{i=1}^{k} \langle f, J\Psi_i \rangle \Psi_i, \qquad \forall f \in L^2, \tag{3.29}$$

with $\{\Psi_1, \cdots, \Psi_k\}$ is a basis of the eigenspace of H associated with eigenvalue 0 such that

$$\langle \Psi_i, J\Psi_j \rangle = \delta_{ij}, \qquad \forall 1 \le i, j \le k.$$

The remainder $R_1(z)$ satisfies the estimate: $\exists C, \delta > 0$ such that

$$\|\langle x \rangle^{-2\kappa\mu} R_1(z)\| + \|R_1(z)\langle x \rangle^{-2\kappa\mu}\| \le C \tag{3.30}$$

for $z \in O(\delta)$. In particular,

$$\sigma_d(H) \cap O(\delta) = \emptyset. \tag{3.31}$$

In addition $R_1(z)$ is continuous up to $z = 0$ and for any $a > 0$, there exist $C_a, c_a > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} R_1^{(N)}(z)\| + \|R_1^{(N)}(z) e^{-a\langle x \rangle^{1-\mu}}\| \le C_a c_a^N N^{(1+\kappa\gamma)N}, \tag{3.32}$$

for any $N \in \mathbb{N}^*$ and $z \in O(\delta) \cup \{0\}$.

Proof. Since condition (1.12) holds, $E_{-+}(z)$ is invertible by Lemma 3.3. It follows from formulae (3.13) and (3.24) that $(I + r_0(z)W_c)^{-1}$ exists for $z \in O(\delta)$, $\delta > 0$ small, with

$$\begin{aligned}(I + r_0(z)W_c)^{-1} g &= -\frac{S\mathcal{F}_{-1}Tg}{z} + B_0(z)g \\ &= -\frac{1}{z}\sum_{i,j=1}^{k} \alpha_{ij} \langle g, JW_c w_{m_j}^{(j)} \rangle u_1^{(i)} + B_0(z)g \\ &= -\frac{1}{z}\sum_{i,j=1}^{k} \alpha_{ij} c_j^{-1} \langle g, JW_c u_1^{(j)} \rangle u_1^{(i)} + B_0(z)g, \end{aligned} \tag{3.33}$$

for $g \in L^2$, where $B_0(z)$ is uniformly bounded for $z \in O(\delta)$ as operator in $\mathcal{B}(L^2)$ and there $C_0, c_0 > 0$ such that

$$\|B_0^{(N)}(z)\| \le C_0 c_0^N N^{(1+\kappa\gamma)N}, \tag{3.34}$$

for all $N \in \mathbb{N}^*$ and $z \in O_0(\delta)$.

Set

$$A_k := (a_{ij})_{1 \le i,j \le k}, \qquad Q_k := (\langle u_1^{(j)}, Ju_1^{(i)} \rangle)_{1 \le i,j \le k}, \tag{3.35}$$

where a_{ij} are given in (3.18). We have

$$A_k = -C_k \times Q_k, \qquad C_k = \mathrm{diag}(c_1^{-1}, \cdots, c_k^{-1}). \tag{3.36}$$

For $f \in L^{2,2\kappa\mu}$, putting $g = r_0(z)f$ in the first term in the right hand side of (3.33), we obtain

$$\begin{aligned}\sum_{i,j=1}^{k} \alpha_{ij} c_j^{-1} \langle r_0(z)f, JW_c u_1^{(j)}\rangle u_1^{(i)} &= \sum_{i,j=1}^{k} \alpha_{ij} c_j^{-1} \left[\langle f, JG_0 W_c u_1^{(j)}\rangle + z\langle r_0(z)f, JG_0W_c u_1^{(j)}\rangle\right] u_1^{(i)} \\ &= -\sum_{i,j=1}^{k} \alpha_{ij} c_j^{-1}\langle f, Ju_1^{(j)}\rangle u_1^{(i)} + B_1(z)f \\ &= -\sum_{i}^{k} \langle f, Jv_i\rangle u_1^{(i)} + B_1(z)f,\end{aligned}$$

where by the identity (3.26), one has

$$\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = A_k^{-1} \cdot C_k \begin{pmatrix} u_1^{(1)} \\ \vdots \\ u_1^{(k)} \end{pmatrix} = -Q_k^{-1} \begin{pmatrix} u_1^{(1)} \\ \vdots \\ u_1^{(k)} \end{pmatrix}.$$

In addition, the remainder $B_1(z)$ has the same regularity properties as $B_0(z)$ in (3.33). Finally, let

$$Q_k^{-1} = {}^tP_k P_k$$

be the Cholesky decomposition of the complex symmetric matrix Q_k^{-1}, where $P_k = (p_{ij})_{1\le i,j\le k}$ is an upper triangular matrix (cf. [8, Proposition 25]). Then,

$$\begin{aligned} R(z)f &= (I + r_0(z)W_c)^{-1} r_0(z)f \\ &= -\frac{1}{z}\sum_{\ell=1}^{k}\sum_{i,j=1}^{k} p_{\ell i}\, p_{\ell j}\langle f, Ju_1^{(j)}\rangle u_1^{(i)} + R_1(z)f.\end{aligned}$$

This establishes (3.29) with $\Psi_i = \sum_{\ell=1}^{k} p_{i\ell} u_1^{(\ell)}$, $\forall 1 \le i \le k$. In addition, one checks that

$$\langle \Psi_j, J\Psi_i\rangle = \sum_{\ell,r=1}^{k} p_{jr} p_{i\ell}\langle u_1^{(r)}, Ju_1^{(\ell)}\rangle = \sum_{r=1}^{k} p_{jr}(P_k Q_k)_{ir} = (P_k Q_k^t P_k)_{ij} = \delta_{ij}.$$

The estimate (3.30) follows from uniformly boundedness of $r_0(z)$ in $B(0, s, 0, s - 2\kappa\mu)$, $\forall s \in \mathbb{R}$ (see (2.5)) and the resolvent equation $R(z) = (I + r_0(z)W_c)^{-1} r_0(z)$. Moreover, (3.32) can be obtained from (2.7) and (3.34). See also [12, Proposition 5.12]. ■

3.2. **Expansion of the cut-off resolvent.**

In this subsection we suppose $V_0 \in \mathcal{A}_1$ and W satisfies conditions (1.2), (1.18) and (1.19). We want to use technics from quantum resonances to establish an asymptotic expansion for the cut-off resolvent $\chi R(z)\chi$ valid in a sector below the positive real axis. As before, denote $h_0 = -\Delta + v_0(x)$, $v_0 = V_0 + \chi_R W$ and $W_c = (1 - \chi_R)W$ where $\chi_R(x) = \chi_1(\frac{x}{R})$ for all $x \in \mathbb{R}^n$, $\chi_1 \in C^\infty(\mathbb{R}^n)$ is a cut-off function with $\chi_1(x) = 0$ if $|x| \le 1$ and $\chi_1(x) = 1$ if $|x| \ge 2$, where R is chosen such that $R > 2\max\{R_1, R_2, R_3\}$, where R_1, R_2 and R_3 are given in (1.17), (1.18) and (1.19), respectively. Remark that v_0 is still distorsion analytic outside some compact and $\operatorname{Im} v_0 \le 0$, but v_0 may no longer belong to $\mathcal{A}_1$.

In order to obtain the asymptotic expansion at low energy of the cut-off resolvent $\chi R(z)\chi$, $\chi \in C_0^\infty(\mathbb{R})$, we use analytic distortion of H outside the support of χ. Let $R_0 > 2R$ such that supp $\chi \subset B(0, R_0)$. Let $\rho \in C^\infty(\mathbb{R})$ with $0 \le \rho \le 1$, $\rho(r) = 0$ if $r \le 1$ and $\rho(r) = 1$ if $r \ge 2$. Set

$$F_\theta(x) = x\left(1 + \theta\rho\Big(\frac{|x|}{R_0}\Big)\right), \quad x \in \mathbb{R}^n. \tag{3.37}$$

When $\theta \in \mathbb{R}$ with $|\theta|$ sufficiently small, $x \mapsto F_\theta(x)$ is a global diffeomorphism on $\mathbb{R}^n$. Set

$$U_\theta f(x) = |DF_\theta(x)|^{\frac{1}{2}} f(F_\theta(x)), \quad f \in L^2(\mathbb{R}^n),$$

where $DF_\theta(x)$ is the Jacobi matrix and $|DF_\theta(x)|$ is the Jacobian of the change of variables: $x \mapsto F_\theta(x)$. One has

$$|DF_\theta(x)| = \begin{cases} 1, & |x| < R_0, \\ (1+\theta)^n, & |x| > 2R_0. \end{cases} \tag{3.38}$$

U_θ is unitary in $L^2(\mathbb{R}^n)$ for θ real with $|\theta|$ sufficiently small. Define the distorted operator $H(\theta)$ by

$$H(\theta) = U_\theta H U_\theta^{-1}, \qquad \theta \in \mathbb{R}. \tag{3.39}$$

One can calculate that

$$H(\theta) = h_0(\theta) + W_c(x), \tag{3.40}$$

where, for $\theta \in \mathbb{R}$,

$$h_0(\theta) = U_\theta h_0 U_\theta^{-1} = -\Delta_\theta + v_0(x, \theta) \tag{3.41}$$

with

$$\Delta_\theta = {}^t\nabla_\theta \cdot \nabla_\theta \ \text{ and } v_0(x,\theta) = v_0(F_\theta(x)),$$

such that

$$\nabla_\theta = ({}^tDF_\theta)^{-1} \cdot \nabla - \frac{1}{|DF_\theta|^2}({}^tDF_\theta)^{-1} \cdot (\nabla|DF_\theta|). \tag{3.42}$$

In particular, $\nabla_\theta f = (1+\theta)^{-1}\nabla f$ if f is supported outside the ball $B(0, 2R_0)$.

For $z_0 \in \mathbb{C}$ and $\epsilon > 0$ small, let

$$D(z_0, \epsilon) = \{z \in \mathbb{C};\ |z - z_0| < \epsilon\}, \tag{3.43}$$

be small complex neighborhood of z_0 and $D_+(z_0, \epsilon) = D(z_0, \epsilon) \cap \mathbb{C}_+$. If $V_0 \in \mathcal{A}_1$ and W satisfies (1.18), then for $R_0 > 2R$ with $R > \max\{R_1, R_2\}$, $H(\theta)$ and $h_0(\theta)$ can be extended to holomorphic families of type A for $\theta \in D_+(0, \epsilon)$, $\epsilon > 0$ small. We observe that the distorted operator $h_0(\theta)$ is a perturbation of $H_0(\theta) = -\Delta_\theta + V_0(x, \theta)$ by $(\chi_R W)(x, \theta)$ which is relatively $H_0(\theta)$-compact by condition (1.18). From this we deduce that the essential spectrum of $h_0(\theta)$ coincides with that of $H_0(\theta)$ given by

$$\sigma_{ess}(H_0(\theta)) = \{\frac{r}{(1+\theta)^2} : r \ge 0\} \tag{3.44}$$

(see [12, Section 4.2]). Set $r_0(z, \theta) = (h_0(\theta) - z)^{-1}$. For $\theta \in \mathbb{R}$, $r_0(z, \theta)$ is holomorphic in $z \in \mathbb{C}_+$ and meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$. For $\operatorname{Im}\theta > 0$, it follows from (3.44) that the resolvent $r_0(z, \theta)$ defined for $z \in \mathbb{C}$ and $\operatorname{Im} z >> 1$ can be meromorphically extended across the positive real axis $\mathbb{R}_+$ into the sector $\{z;\ \arg z > -\operatorname{Im}\theta\}$.

The aim of this subsection is to establish an expansion for the cut-off resolvent $\chi R(z)\chi$ for z in some complex domain defined for $\operatorname{Im}\theta > 0$ and $\delta > 0$ small by

$$\Omega(\delta, \theta) = \{z \in \mathbb{C}^*; |z| < \delta, -\delta \operatorname{Im}\theta < \arg z < \frac{3\pi}{2} - \delta\}. \tag{3.45}$$

Let us begin by analyzing properties of the distorted model operator $h_0(\theta)$. Let $V_0 \in \mathcal{A}_1$ and W satisfy (1.2), (1.18) and (1.19). Since h_0 satisfies condition (1.3), for $\theta \in \mathbb{C}$ with $|\theta|$

small the operator $h_0(\theta)$ still satisfies (1.3) with some constant $c_0 > 0$ independent of $R_0 > 0$ and θ. Therefore, Lemma 3.1 in [12] can be applied to $h_0(\theta)$, so we can define $G_0(\theta)$ by

$$G_0(\theta) = s - \lim_{\operatorname{Re} z<0, z\to 0} r_0(z,\theta) \tag{3.46}$$

as operators from $L^{2,s}$ to $L^{2,s-2\mu}$ for all $s \in \mathbb{R}$. Moreover, an argument of perturbation shows that Theorem 3.4 in [12] holds for $G_0(\theta)$ uniformly in θ when $|\theta|$ is small.

The following proposition gives a uniform estimate on $r_0(z,\theta)$ for z in the sector

$$S(-c_0\theta, \gamma_0) := \{z \in \mathbb{C}^*; -c_0 \operatorname{Im}\theta < \arg z < \gamma_0\} \tag{3.47}$$

for some $c_0 > 0$, $\gamma_0 \in]\pi, \frac{3\pi}{2}[$ and $\theta \in \mathbb{C}$ with $|\theta|$ small. We want to show that the resolvent estimate given in [12, Proposition 4.7] still holds if the global virial condition is replaced by a virial condition at the infinity.

Proposition 3.6. *Let $V_0 \in \mathcal{A}_1$ and W satisfy conditions (1.2), (1.18) and (1.19). Then there exist some constants $c_0 > 0$ and $\gamma_0 \in]\pi, \frac{3\pi}{2}[$ such that for $\theta \in \mathbb{C}$ with $|\theta|$ sufficiently small and $\operatorname{Im}\theta > 0$, one has*

$$\sigma(h_0(\theta)) \cap S(-c_0\theta, \gamma_0) = \emptyset \tag{3.48}$$

and there exists $C > 0$ such that

$$\|\langle x\rangle^{-2\mu} r_0(z,\theta)\| \le \frac{C}{\langle z\rangle}, \qquad \forall z \in S(-c_0\theta, \gamma_0). \tag{3.49}$$

Proof. First, we prove that

$$\|\langle x\rangle^{-2\mu}(H_0(\theta) - z)^{-1}\| \le \frac{C}{\langle z\rangle}, \quad z \in S(-c_0\theta, \gamma_0). \tag{3.50}$$

Let $R_0 > 2R$ where $\operatorname{supp} W_c \subset B(0, 2R)$ and R is chosen such that $R > \max\{R_1, R_2, R_3\}$, where $R_j, j = 1, 2, 3$, are given in (1.17)-(1.19). Let $\widetilde{H}_0 = -\Delta + \widetilde{V}_0(x)$. Set $H_0(\theta) = U(\theta)^{-1} H_0 U(\theta)$ and $\widetilde{H}_0(\theta) = U(\theta)^{-1}\widetilde{H}_0 U(\theta)$ for $\theta \in \mathbb{R}$, $|\theta|$ small, where the distortion is made outside the ball $B(0, R_0)$. Then $H_0(\theta)$ and $\widetilde{H}_0(\theta)$ can be extended holomorphically in $\theta \in D_+(0, \delta)$ with $\delta > 0$ small and their domains are constant. Fix $\theta \in \mathbb{C}_+$, $|\theta|$ small. Denote $\widetilde{R}_0(z, \theta) = (\widetilde{H}_0(\theta) - z)^{-1}$ for $z \notin \sigma(\widetilde{H}_0(\theta))$. By Proposition 4.7 in [12], there exists $c_0, C > 0$ and $\gamma_0 \in]\pi, \frac{3\pi}{2}[$ such that

$$\|\langle x\rangle^{-2\mu}\widetilde{R}_0(z,\theta)\| \le \frac{C}{\langle z\rangle}, \quad z \in S(-c_0\theta, \gamma_0). \tag{3.51}$$

Using an argument of perturbation, the estimate (3.50) for $|z|$ large follows from (3.51). To prove (3.50) for $|z|$ bounded we use the same argument as in the proof of [12, Proposition 4.7]. We compare $R_0(z, \theta)$ with $\widetilde{R}_0(z, \theta)$ for $|x|$ large and with $R_0(0, \theta)$ for $|z|$ small.
Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ if $|x| \le 2R_0$. Take $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ such that $\widetilde{\chi}\chi = \chi$. On the support of $1 - \chi$, $H_0(\theta) = \widetilde{H}_0(\theta)$. For $z \in S(-c_0\theta, \gamma_0)$ and $|z|$ small, one has

$$\begin{aligned} R_0(z,\theta) &= R_0(0,\theta) + zR_0(0,\theta)R_0(z,\theta) \\ &= R_0(0,\theta) + zR_0(0,\theta)(\widetilde{\chi} + (1-\widetilde{\chi}))R_0(z,\theta) \\ &= R_0(0,\theta) + zR_0(0,\theta)\widetilde{\chi}R_0(z,\theta) \\ &\quad + zR_0(0,\theta)(1-\widetilde{\chi})\widetilde{R}_0(z,\theta)(1-\chi) \\ &\quad + zR_0(0,\theta)(1-\widetilde{\chi})\widetilde{R}_0(z,\theta)[\Delta_\theta, \chi]R_0(z,\theta). \end{aligned} \tag{3.52}$$

Then, it follows from formula (3.52) that for $z \in S(-c_0\theta, \gamma_0)$ and $|z|$ small

$$\begin{aligned} (I + \mathcal{K}(z,\theta))\langle x\rangle^{-2\mu}R_0(z,\theta)\langle x\rangle^{-2\mu} &= \langle x\rangle^{-2\mu}R_0(0,\theta)\langle x\rangle^{-2\mu} \\ &\quad + z\langle x\rangle^{-2\mu}R_0(0,\theta)(1-\widetilde{\chi})\widetilde{R}_0(z,\theta)(1-\chi)\langle x\rangle^{-2\mu} \end{aligned} \tag{3.53}$$

where

$$\mathcal{K}(z,\theta) = -z\langle x\rangle^{-2\mu}R_0(0,\theta)\widetilde{\chi}\langle x\rangle^{2\mu} - z\langle x\rangle^{-2\mu}R_0(0,\theta)(1-\widetilde{\chi})\widetilde{R}_0(z,\theta)[\Delta_\theta,\chi]\langle x\rangle^{2\mu} \tag{3.54}$$

One has that $\|\langle x\rangle^{-2\mu}R_0(0,\theta)\|$ is uniformly bounded in θ for $\operatorname{Im}\theta > 0$ and $|\theta|$ small. In addition, one can check that $\|\widetilde{R}_0(z,\theta)[\Delta_\theta,\chi]\langle x\rangle^{2\mu}\|$ is uniformly bounded for $z \in S(-c_0\theta,\gamma_0)$ and $|z| \le 1$ (see proof of Proposition 4.7 in [12]). This yields

$$\|\mathcal{K}(z,\theta)\| = O(|z|)$$

for $z \in S(-c_0\theta,\gamma_0)$ uniformly in θ. Thus, for $z \in S(-c_0\theta,\gamma_0)$ and $|z|$ small enough, $I+\mathcal{K}(z,\theta)$ is invertible in L^2 and there exists $C_2 > 0$ such that

$$\|(I+\mathcal{K}(z,\theta))^{-1}\| \le C_2,$$

uniformly in z. In addition one observes from (3.51) that

$$\|\langle x\rangle^{-2\mu}\widetilde{R}_0(z,\theta)\| \le C,$$

uniformly in $z \in S(-c_0\theta,\gamma_0)$ and $\theta \in \mathbb{C}$ with $\operatorname{Im}\theta > 0$ and $|\theta|$ small. Consequently, for $z \in S(-c_0\theta,\gamma_0)$ and $|z|$ small enough

$$\|\langle x\rangle^{-2\mu}R_0(z,\theta)\langle x\rangle^{-2\mu}\| \le C_2\|\langle x\rangle^{-2\mu}R_0(0,\theta)\|\Big(1+C_3|z|\Big) \le C_4, \tag{3.55}$$

for some constants $C_3, C_4 > 0$ independent of z and θ. Finally, (3.50) can be derived from (3.51), (3.55) and the equation

$$R_0(z,\theta) = \widetilde{R}_0(z,\theta) + R_0(z,\theta)(\widetilde{V}_0(\theta) - V_0(\theta))\widetilde{R}_0(z,\theta),$$

where $\widetilde{V}_0(\theta) - V_0(\theta)$ is compactly supported and relatively bounded with respect to $-\Delta$.

Next, one derives (3.49) from (3.50) by using an argument of perturbation. Indeed, for $|x|$ large enough, $(\chi_R W)(F_\theta(x)) = W((1+\theta)x)$ which tends to 0 as $|x| \to +\infty$ by condition (1.18). Then, using (3.50) we can choose $R > 0$ large enough so that

$$\|(\chi_R W)(\theta)R_0(z,\theta)\| \le \frac{C_1C_w}{\langle R\rangle^\epsilon} < 1$$

uniformly in θ for $\theta \in D_+(0,\theta_0)$ and $z \in S(-c_0\theta,\gamma_0)$. This yields $(I+\chi_R W(\theta)R_0(z,\theta))^{-1}$ exists for z in $S(-c_0\theta,\gamma_0)$ and $\theta \in D_+(0,\theta_0)$ with

$$\|(I+\chi_R W(\theta)R_0(z,\theta))^{-1}\| \le C_2 \tag{3.56}$$

uniformly in z and θ. Hence, (3.49) derives from the resolvent equation

$$r_0(z,\theta) = R_0(z,\theta)(I+\chi_R W(\theta)R_0(z,\theta))^{-1} \tag{3.57}$$

and uniform estimates (3.50) and (3.56). It follows from (3.50) that $\sigma_d(H_0(\theta)) \cap S(-c_0\theta,\gamma_0) = \emptyset$, so does $r_0(z)$ using the resolvent equation (3.57) and (3.56). (3.48) is proved. (3.49) follows from (3.50) and (3.57). ■

As consequence of the above proposition, we obtain the following Gevrey estimates (see [12, Corollary 4.8]):

Corollary 3.7. *For any $a > 0$ there exist some constants $c, C > 0$ such that*

$$\|e^{-a\langle x\rangle^{1-\mu}}r_0(z,\theta)^N\| \le Cc^N N^{\gamma N}, \qquad N \in \mathbb{N}^*, \tag{3.58}$$

for all $z \in \Omega_0(\delta,\theta) \equiv \Omega(\delta,\theta) \cup \{0\}$, where $\Omega(\delta,\theta)$ is given in (3.45).

Set $K_0(\theta) = G_0(\theta)W_c$, for $\theta \in \mathbb{C}$ with $|\theta|$ small, where the distortion is made outside the ball $B(0,R_0)$ for $R_0 > 0$ large enough chosen as in the proof of Proposition 3.6. $K_0(\theta)$ is a family of compact operator on $L^2(\mathbb{R}^n)$ for $\theta \in \mathbb{C}$ with $|\theta|$ small.

Lemma 3.8. *Assume that assumptions in Proposition 3.6 hold.*
Then the mapping $\theta \mapsto K_0(\theta) \in \mathcal{B}(L^2)$ is holomorphic in operator norm for $\theta \in D_+(0,\delta)$ with $\delta > 0$ small and continuous for $\theta \in \overline{D_+(0,\delta)}$. Here $\overline{D_+(0,\delta)} = D(0,\delta) \cap \overline{\mathbb{C}}_+$ (see (3.43)).

Proof. We shall show the derivability of $\theta \mapsto K_0(\theta)$ in operator norm of $\mathcal{B}(L^2)$ for $\theta \in D_+(0,\delta)$. For this we prove that

$$\frac{K_0(\theta+h) - K_0(\theta)}{h}$$

has a limit in operator norm as $|h| \to 0$. Notice that

$$\frac{dF_\theta}{d\theta}(x) = \rho\Big(\frac{|x|}{R_0}\Big)x, \qquad \forall x \in \mathbb{R}^n$$

(see (3.37)) and $\chi_1(\frac{x}{R})\rho(\frac{|x|}{R_0}) = \rho(\frac{|x|}{R_0})$ if $R_0 > 2R$. Let $\theta \in D_+(0,\delta)$ and $h \in \mathbb{C}^*$ with $|h|$ small. We have

$$\begin{aligned} K_0(\theta+h) - K_0(\theta) &= \Big(G_0(\theta+h) - G_0(\theta)\Big)W_c(x) \\ &= G_0(\theta+h)\Big(g_0(\theta) - g_0(\theta+h)\Big)G_0(\theta)W_c(x), \end{aligned}$$

where

$$\begin{aligned} g_0(\theta) - g_0(\theta+h) &= (h + o(|h|))\Big[(-\Delta_\theta + \Delta_{\theta+h}) + v_0(F_\theta(x)) - v_0(F_{\theta+h}(x))\Big] \\ &= (h + o(|h|))\Big[\,{}^t(A_\theta(x)\nabla)\cdot\nabla + b_\theta(x,\nabla) - \rho\Big(\frac{|x|}{R_0}\Big)x\cdot\nabla V_0(F_\theta(x)) \\ &\quad - \rho\Big(\frac{|x|}{R_0}\Big)x\cdot\nabla W(F_\theta(x))\Big], \end{aligned}$$

with $A_\theta(x)$ and $b_\theta(x,\nabla)$ are independent of h and it can be calculated explicitly from (3.42), where $b_\theta(x,\nabla)$ is a first order differential operator. Using (3.38), we can check that

$$G_0(\theta)^t(A_\theta(x)\nabla)\cdot\nabla \text{ and } G_0(\theta)b_\theta(x,\nabla)$$

are uniformly bounded in $\mathcal{B}(L^{2,\mu}, L^2)$ for $\theta \in \overline{D_+(0,\delta)}$ with $\delta > 0$ small enough, since $G_0(\theta)$ is uniformly bounded in θ as operator from $L^{2,2\mu}$ to L^2 (see 3.46). In addition, $V_0(x) = \widetilde{V}_0(x)$ for $|x| > R_0$, $\widetilde{V}_0 \in \mathcal{A}$, and W satisfying condition (1.18) can be extended holomorphically into the region

$$\{y = F_\theta(x),\, x \in \mathbb{R}^n, |x| > R_0, \theta \in D_+(0,\delta)\} \subset \mathbb{C}^n.$$

It follows that

$$G_0(\theta+h)x\cdot\Big(\nabla V_0(F_\theta(x)) + \nabla W(F_\theta(x))\Big)\rho\Big(\frac{|x|}{R_0}\Big) \,:\; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

is uniformly bounded with respect to θ and h for $|\theta|$ and $|h|$ small. Indeed,

$$G_0(\theta+h)\rho\Big(\frac{|x|}{R_0}\Big)x\cdot\nabla V_0(F_\theta(x)) = G_0(\theta+h)(-\Delta+1)\frac{d}{d\theta}\left[(-\Delta+1)^{-1}V_0(F_\theta(x))\right],$$

and the same argument done for $\rho\Big(\frac{|x|}{R_0}\Big)x\cdot\nabla W(F_\theta(x))$. Consequently

$$\Big|\Big\langle\, \Big(K_0(\theta+h) - K_0(\theta) - h\mathcal{K}^{(1)}(\theta)\Big)\,u, v\Big\rangle\Big| = o(h)\|u\|\|v\| \tag{3.59}$$

uniformly for $u, v \in L^2$ and $|\theta|, |h|$ small, where

$$\begin{aligned} \mathcal{K}^{(1)}(\theta) \;=\; & G_0(\theta)\Big[\,{}^t(A_\theta(x)\nabla)\cdot\nabla + b_\theta(x,\nabla) - \rho\Big(\frac{|x|}{R_0}\Big)x\cdot\nabla V_0(F_\theta(x)) \\ & -\rho\Big(\frac{|x|}{R_0}\Big)x\cdot\nabla W(F_\theta(x))\Big]G_0(\theta)W_c(x). \end{aligned}$$

Thus, (3.59) shows that $\theta \mapsto K_0(\theta)$ is derivable in operator norm for $\theta \in D_+(0,\delta)$. Moreover, the same arguments used to prove (3.59) show that $\|\mathcal{K}^{(1)}(\theta)\|$ is uniformly bounded in $\theta \in \overline{D_+(0,\delta)}$. This proves the continuity of $K_0(\theta)$ in $\theta \in \overline{D_+(0,\delta)}$. ∎

Consider the resolvent equation $R(z,\theta) = (Id + r_0(z,\theta)W_c)^{-1} r_0(z,\theta)$ for $z \in S(-c_0\theta, \gamma_0)$ and $\theta \in \mathbb{C}_+$ with $|\theta|$ small. It is not difficult to check that -1 is a discrete eigenvalue of $K_0(\theta)$ if and only if 0 is an embedded eigenvalue of $H(\theta)$ and they have the same geometric multiplicity.

3.2.1. *Analyticity of the Riesz projection.*

In order to express the Riesz projection associated with the eigenvalue -1 of $K_0(\theta)$ in term of the resolvent of $K_0(\theta)$ for $\theta \in D_+(0,\delta_0)$, we show that there exists some $\epsilon_0 > 0$ small such that the disk $D(-1,\epsilon_0)$ does contain no eigenvalue of $K_0(\theta)$ other than -1.

Let $\epsilon_1 > 0$ be such that $\gamma_1 \equiv \{|z+1| = \epsilon_1\} \subset \rho(K_0(0))$ and $Int\gamma_1 \cap \sigma(K_0(0)) = \{-1\}$. For $z \in \gamma_1$ and $\theta \in \overline{D_+(0,\delta)}$:

$$(z - K_0(\theta))(z - K_0(0))^{-1} = I + (K_0(0) - K_0(\theta))\,(z - K_0(0))^{-1}.$$

Since the mapping $\theta \mapsto K_0(\theta) \in \mathcal{B}(L^2)$ is continuous in operator norm for $\theta \in \overline{D_+(0,\delta)}$, $\delta > 0$ small, then there exists $\delta_1 > 0$ such that for $\theta \in \overline{D_+(0,\delta_1)}$

$$\|K_0(\theta) - K_0(0)\| < \frac{1}{\sup_{z\in\gamma_1} \|(z - K_0(0))^{-1}\|}.$$

It follows that $(z - K_0(\theta))^{-1}$ exists for $z \in \gamma_1$ and $\theta \in \overline{D_+(0,\delta_1)}$. Hence, the operator

$$\Pi_1(\theta) := \frac{1}{2i\pi} \int_{\gamma_1} (z - K_0(\theta))^{-1}\, dz \tag{3.60}$$

is well defined and is a projection. In particular $\Pi_1(0) = \Pi_1$, the Riesz projection associated with the eigenvalue -1 of $K_0(0) = G_0 W_c$.

Lemma 3.9. *1). The mapping $\theta \mapsto \Pi_1(\theta) \in \mathcal{B}(L^2)$ is holomorphic in operator norm for $\theta \in D_+(0,\delta)$ with some $\delta > 0$ and*

$$\lim_{\theta \in \overline{D_+(0,\delta)}, \theta \to 0} \|\Pi_1(\theta) - \Pi_1\| = 0. \tag{3.61}$$

2). For $\theta \in \overline{D_+(0,\delta)}$, -1 is an eigenvalue of $K_0(\theta)$ with the same algebraic multiplicity as $K_0(0)$ and $\Pi_1(\theta)$ is the Riesz projection associated with the eigenvalue -1 of $K_0(\theta)$.

Proof. 1) follows from Lemma 3.8. To show 2), remark first that since $\Pi_1(\theta)$ is a projection, the norm-continuity of the mapping $\theta \mapsto \Pi_1(\theta)$ implies that rank $\Pi_1(\theta)$ = rank $\Pi_1 = m$ (cf. [5, Lemma I-4.10]) where m is the algebraic multiplicity of the eigenvalue -1 of $G_0 W_c$. It remains to show that for $\epsilon_1 > 0$ sufficiently small, -1 is the only eigenvalue of $K_0(\theta)$ inside the disc $D(-1,\epsilon_1)$ for $\theta \in \overline{D_+(0,\delta)}$ with $\delta > 0$ small enough. Indeed, it suffices to restrict ourselves to the holomorphic family of finite rank operators $K_0(\theta)\Pi_1(\theta)$. For $\theta \in \mathbb{R}$ and $|\theta| < \delta$ with $\delta > 0$ small, $K_0(\theta)\Pi_1(\theta)$ is unitary equivalent to $K_0(0)\Pi_1$. So there exists $\epsilon_1 > 0$ such that $K_0(\theta)\Pi_1(\theta)$ has no eigenvalues in $D(-1,\epsilon_1)$ other than -1. By the theorem on finite dimensional analytic perturbation of eigenvalues ([5, Theorem 1.8 in Ch.2]), the eigenvalues of $K_0(\theta)\Pi_1(\theta)$ are branches of analytic functions in $\theta \in D_+(0,\delta)$ with at worst algebraic singularity. Since theses functions are equal to -1 when θ is real, we deduce that -1 is the only eigenvalue of $K_0(\theta)P(\theta)$ in $D(-1,\epsilon_1)$ for $\theta \in \overline{D_+(0,\delta)}$ if δ is small enough. This proves that $\Pi_1(\theta)$ defined by (3.60) is the Riesz projection of $K_0(\theta)$ associated with eigenvalue -1. ∎

Denote $E(\theta) = \text{Range}\,\Pi_1(\theta)$ and $E = \text{Range}\,\Pi_1$. We construct a basis of $E(\theta)$ as follows. Let $\mathcal{U}$ and $\mathcal{W}$ (respectively, $\mathcal{U}_i$) be bases of E (respectively, E_i) given by Lemma 3.1. Set

$$\mathcal{U}(\theta) \equiv \Pi_1(\theta)\mathcal{U} = \bigcup_{i=1}^{k}\{\Pi_1(\theta)u_\ell^{(i)}, 1 \le \ell \le m_i,\, u_\ell^{(i)} \in \mathcal{U}_i\}.$$

For $1 \le i \le k$, the m_i functions $\Pi_1(\theta)u_\ell^{(i)} \in E(\theta)$ are linearly independent for $\text{Im}\,\theta > 0$ with $|\theta|$ small since $\theta \mapsto \Pi_1(\theta)$ is holomorphic for $\theta \in D_+(0,\delta)$ and norm continuous at $\theta = 0$. This shows that the above family $\mathcal{U}(\theta)$ is a basis of $E(\theta)$ with the same properties as in Lemma 3.1 (2). Similarly, we observe that $\mathcal{W}(\theta) \equiv \Pi_1(\theta)\mathcal{W}$ is the dual basis of $E(\theta)$ with respect to the bilinear form B defined in (3.1).

Lemma 3.10. *Assume that $V_0 \in \mathcal{A}$ and W satisfies conditions (1.18) and (1.19). Then we have*

$$\Pi_1(\theta)f = \sum_{j=1}^{k}\sum_{i=1}^{m_j}\langle f, JW_c w_i^{(j)}(\theta)\rangle u_i^{(j)}(\theta), \qquad \forall f \in L^2, \tag{3.62}$$

for $\theta \in D(0,\delta_0)$ with $\delta_0 > 0$ small.

Proof. For $\theta \in \mathbb{R}$ and $|\theta| < \theta_0$, the equality holds by the unitary equivalence. See Corollary 3.2. Since the both sides of (3.62) are continuous in θ in the closed half-disc $\overline{D_+(0,\delta_0)}$ and holomorphic in its interior, the equality still holds true for $\theta \in D(0,\delta_0)$. ■

Theorem 3.11. *Let $V_0 \in \mathcal{A}_1$ and $W = V - V_0$ satisfy (1.2), (1.18) and (1.19). Assume that zero is an embedded eigenvalue of $H = -\Delta + V(x)$ of geometric multiplicity k and assumption (A1) holds. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\Omega(\delta,\theta)$ be defined in (3.45) for some $\theta \in \mathbb{C}$ with $\text{Im}\,\theta > 0$. Then*

$$\chi R(z)\chi = -\frac{\chi\Pi_0\chi}{z} + R_2(z) \tag{3.63}$$

for $z \in \Omega(\delta,\theta)$, where Π_0 is the same projector as in Theorem 3.5 and the remainder $R_2(z)$ is continuous up to $z = 0$ and satisfies Gevrey estimates: $\exists C_\chi, C > 0$ such that

$$\|R_2(z)^{(N)}\| \le C_\chi C^N N^{(1+\gamma)N} \tag{3.64}$$

for $z \in \Omega(\delta,\theta) \cup \{0\}$ and for all $N \in \mathbb{N}^*$.

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^n)$. Let $h_0 = -\Delta + v_0(x)$ with $v_0 = V_0 + \chi_R W$ constructed as before with $R > 1$ large enough so that $\chi\chi_R = 0$. Then $\chi W_c = W_c$ (see (1.21)). Let U_θ be the analytic distorsion made outside the ball $B(0,R_0)$ with R_0 sufficiently large, one has for $\theta \in \mathbb{R}$ with $|\theta| < \delta_0$

$$\chi R(z)\chi = \chi R(z,\theta)\chi = \chi(I + r_0(z,\theta)W_c)^{-1}r_0(z,\theta)\chi. \tag{3.65}$$

The above equality initially valid for $\theta \in \mathbb{R}$ and $z \in \mathbb{C}_+$ with $\text{Im}\,z > 0$ sufficiently large allows to extend $\chi R(z)\chi$ meromorphically into a sector below positive real axis when $\text{Im}\,\theta > 0$ and appropriately small. Fix $\theta \in \mathbb{C}_+$ with $|\theta|$ small. In order to inverse the holomorphic family $W(z,\theta) \equiv I + r_0(z,\theta)W_c$ for z in $\Omega(\delta,\theta)$ with $\delta > 0$ small, we shall construct θ-dependent Grushin problem similar to the θ-independent one considered in the proof of Theorem 3.5. For this purpose we use holomorphic families of vectors $\mathcal{U}(\theta)$ and $\mathcal{W}(\theta)$ constructed above.

Let $\Pi_1'(\theta) = I - \Pi_1(\theta)$. $I + G_0(\theta)W_c$ is injective on Range $\Pi_1'(\theta)$. Since the operator $I + G_0(\theta)W_c$ is compact, the Fredholm theorem implies that $\Pi_1'(\theta)(I + G_0(\theta)W_c)\Pi_1'(\theta)$ is invertible on Range $\Pi_1'(\theta)$. So is $\Pi_1'(\theta)(I + r_0(z,\theta)W_c)\Pi_1'(\theta)$ for $z \in \Omega(\delta,\theta)$ when $\delta > 0$ is small. The inverse

$$E(z,\theta) := (\Pi_1'(\theta)(1 + r_0(z,\theta)W_c)\Pi_1'(\theta))^{-1}\Pi_1'(\theta) \tag{3.66}$$

is uniformly bounded in $z \in \Omega(\delta, \theta)$ (see Proposition 3.6) and satisfies by Corollary 3.7 Gevrey estimates of the form (3.64) on $\Omega_0(\delta, \theta)$.

Define $S(\theta) : \mathbb{C}^m \to L^2$ and $T(\theta) : L^2 \to \mathbb{C}^m$ by

$$S(\theta)c = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_j^{(i)} u_j^{(i)}(\theta), \qquad c = \oplus_{i=1}^{k} (c_1^{(i)}, \cdots, c_{m_i}^{(i)}) \in \oplus_{i=1}^{k} \mathbb{C}^{m_i},$$

$$T(\theta)u = \oplus_{i=1}^{k} \left(B(u, w_1^{(i)}(\theta)), \cdots, B(u, w_{m_i}^{(i)}(\theta)) \right), \qquad u \in L^2.$$

Then one has $S(\theta)T(\theta) = \Pi_1(\theta)$ in view of Lemma 3.10 and $T(\theta)S(\theta) = I_m$. Indeed, for $\theta \in]-\delta, \delta[$, $T(\theta)S(\theta) = I_m$ since $\Pi_1(\theta)$ and Π_1 are unitary equivalent. Thus the equality still holds true for $\theta \in \overline{D_+(0, \delta)}$ with $\delta > 0$ small since $\theta \mapsto \Pi_1(\theta)$ is holomorphic in $D_+(0, \delta)$ and norm-continuous for $\theta \in \overline{D_+(0, \delta)}$ by Lemma 3.9. One can follow the same way as in the proof of Theorem 3.5 by using the following asymptotic expansion at threshold zero: for all $N \in \mathbb{N}^*$

$$r_0(z, \theta) = \sum_{j=0}^{N-1} z^j G_0(\theta)^{j+1} + z^N G_0(\theta)^N r_0(z, \theta), \qquad z \in \Omega(\delta, \theta),$$

in $\mathcal{B}(0, s, 0, s - 2N\mu), \forall s \in \mathbb{R}$. For $z \in \Omega(\delta, \theta)$ with $\delta > 0$ small and $L \in \mathbb{N}$, we obtain

$$\det E_{-+}(z, \theta) = \sigma_0(\theta) z^k + \sigma_1(\theta) z^{k+1} + \cdots + \sigma_{L-1}(\theta) z^{k+L-1} + O(|z|^{k+L}), \tag{3.67}$$

where

$$\sigma_0(\theta) = \mu(\theta) \times \det \left(\langle u_1^{(j)}(\theta), J u_1^{(i)}(\theta) \rangle \right)_{1 \le i,j \le k} \text{with} \qquad \mu(\theta) = (-1)^k c_1(\theta)^{-1} \cdots c_k(\theta)^{-1}.$$

Indeed, if condition (1.12) holds, then $\det \left(\langle u_1^{(j)}, J u_1^{(i)} \rangle \right)_{1 \le i,j \le k} \neq 0$. Thus, for $\operatorname{Im} \theta \ge 0$ and $|\theta|$ small enough

$$\det \left(\langle u_1^{(j)}(\theta), J u_1^{(i)}(\theta) \rangle \right)_{1 \le i,j \le k} \neq 0,$$

since $\theta \mapsto \Pi_1(\theta)$ is continuous at $\theta = 0$. In the same way one can see that

$$c_i(\theta) := B(u_1^{(i)}(\theta), u_{m_i}^{(i)}(\theta)) \neq 0, \, 1 \le i \le k,$$

for $\operatorname{Im} \theta \ge 0$ and $|\theta|$ small. From (3.65), one can deduce that

$$\begin{aligned} R(z, \theta) f &= -\frac{1}{z} \sum_{\ell=1}^{k} \sum_{i,j=1}^{k} p_{\ell i}(\theta) \, p_{\ell j}(\theta) \langle f, J u_1^{(j)}(\theta) \rangle u_1^{(i)}(\theta) + R_2(z, \theta) f \\ &:= -\frac{1}{z} \sum_{i=1}^{k} \langle f, J \Psi_i(\theta) \rangle \Psi_i(\theta) + R_2(z, \theta) f \end{aligned}$$

for $z \in \Omega(\delta, \theta)$, where $\Psi_i(\theta) = \sum_{\ell=1}^{k} p_{\ell i}(\theta) u_1^{(i)}(\theta)$ and $P_k(\theta) = (p_{ij}(\theta))_{1 \le i,j \le k}$ is such that ${}^t P_k(\theta) P_k(\theta) = Q_k(\theta)^{-1}$ with

$$Q_k(\theta) = (\langle u_1^{(j)}(\theta), J u_1^{(i)}(\theta) \rangle)_{1 \le i,j \le k}$$

is a holomorphic family of invertible matrices for $\theta \in \mathbb{C}$ with $|\theta|$ small enough, by the above argument. Moreover, the remainder $R_2(z, \theta)$ is continuous up to $z = 0$ and satisfies uniform Gevrey estimates as in (3.58). The detail of calculation is similar to the proof of Theorem 3.5 and is not repeated here. This shows that

$$\chi R(z) \chi = -\frac{\mathcal{P}_0}{z} + R_2(z), \qquad \forall z \in \Omega(\delta, \theta), \tag{3.68}$$

where

$$\mathcal{P}_0 = \sum_{i=1}^{k} \langle \cdot, J\chi\Psi_i(\theta)\rangle \chi\Psi_i(\theta),$$

and the remainder $R_2(z)$ is continuous up to $z = 0$ and satisfies the estimates (3.64). See also the proof of Theorem 5.20 in [12] for Gevrey estimates of the remainder $R_2(z)$.

Finally, we affirm that $\mathcal{P}_0$ is independent of θ and is equal to $\chi\Pi_0\chi$. In fact, since $V_0 \in \mathcal{V}$, we can apply Theorem 3.5 to $R(z)$ for $\operatorname{Re} z < 0$ without utilizing analytic deformation. It follows that

$$\chi R(z)\chi = -\frac{\chi\Pi_0\chi}{z} + \chi R_1(z)\chi. \tag{3.69}$$

Thus the comparison between (3.68) and (3.69) yields $\mathcal{P}_0 = \chi\Pi_0\chi$ and $R_2(z) = \chi R_1(z)\chi$ for $\operatorname{Re} z < 0$ and $|z|$ small. ■

4. Resolvent expansion near positive resonances

In this section we compute the expansions of $R(z)$ and $\chi R(z)\chi$ near outgoing positive resonances of H by the method similar to that used in [1] for quickly decreasing potentials. When there is no analyticity condition on the potential, the expansion of $R(z)$ is calculated for z in a half-disk in $\mathbb{C}_+$ around an outgoing positive resonance, while if the potential is analytic, we use the analytic distortion method to calculate the expansion of $\chi R(z)\chi$ for z in a pointed disk in $\mathbb{C}$ centered at an outgoing positive resonance.

4.1. **The case of non-analytic potentials.**

Consider the non-selfadjoint Schrödinger operator $H = -\Delta + V$ which is a compactly supported perturbation of the model operator $h_0 = -\Delta + v_0$, where $V - v_0 = W_c$.

Assumption *(A3). 1. Assume that* $\operatorname{Im} v_0 \leq 0$ *and that* v_0 *satisfies for some* $C > 0$ *and* $\rho_1 > 0$

$$|v_0(x)| \leq C\langle x\rangle^{-\rho_1}, \qquad \forall x \in \mathbb{R}^n \text{ and } |x| > R \tag{4.70}$$

for some $R > 0$*;*

2. $(x \cdot \nabla_x)^j v_0$ *is* $-\Delta-$*compact for all integers* $j \in \mathbb{N}$*;*

3. W_c *is of compact support and is* $-\Delta$*-compact.*

Let $r_+(H)$ be the set of outgoing positive resonances of H defined by Definition 1.4. One has $r_+(h_0) = \emptyset$ because h_0 is dissipative, and the boundary value of the resolvent

$$r_0(\lambda + i0) = \lim_{z\to\lambda, \operatorname{Im} z>0} (h_0 - z)^{-1}$$

exists in $\mathcal{B}(-1, s, 1, -s)$, $\forall s > 1/2$, for all $\lambda > 0$. Moreover, for $\ell \in \mathbb{N}^*$ and $s > \ell + 1/2$ the above limit $r_0(\lambda + i0)$ defines a function of class C^ℓ on $]0, +\infty[$ with values in $\mathcal{B}(-1, s, 1, -s)$ (cf. [9]), where

$$\frac{1}{j!}\frac{d^j}{d\lambda^j} r_0(\lambda + i0) = \lim_{z\to\lambda, \operatorname{Im} z>0} r_0(z)^{j+1}, \qquad j = 1, 2, \cdots, \ell. \tag{4.71}$$

For simplicity, we denote for $\lambda > 0$

$$G_j^+(\lambda) \;=\; \frac{1}{j!}\frac{d^j}{d\lambda^j} r_0(\lambda + i0), \qquad j = 0, 1, \cdots, \ell. \tag{4.72}$$

By Definition 1.4, $\lambda_0 \in r_+(H)$ if and only if -1 is an eigenvalue of the compact operator $K_0^+(\lambda_0) := G_0^+(\lambda_0)W_c$ in $L^{2,-s}$ for all $s > 1/2$. We denote by Π_1^+ the Riesz projection associated with the eigenvalue -1 of $K_0^+(\lambda_0)$ given by

$$\Pi_1^+ = \frac{1}{2i\pi}\int_{|z+1|=\epsilon_0} (z - K_0^+(\lambda_0))^{-1}\, dz : L^{2,-s} \to L^{2,-s}$$

for some $\epsilon_0 > 0$ small. Set $E^+(\lambda_0) = \text{Range}\,\Pi_1^+(\lambda_0)$. Denote $m(\lambda_0) = \dim E^+(\lambda_0)$ and $k_0 = \dim \text{Ker}(I + K_0^+(\lambda_0))$. Notice that Lemma 3.1 is still true for $E^+(\lambda_0) \subset L^{2,-s}$ in the present case. In particular, we have the following basis of $E^+(\lambda_0)$:

$$\mathcal{U}^+ = \bigcup_{i=1}^{k_0} \mathcal{U}_i^+ \text{ with } \mathcal{U}_i^+ = \{\varphi_1^{(i)}, \cdots, \varphi_{m_i(\lambda_0)}^{(i)}\},\ i = 1, \cdots, k_0,$$

and its dual basis with respect to the bilinear form $B(\cdot,\cdot)$ in (3.1) well defined on $E^+(\lambda_0) \times E^+(\lambda_0)$:

$$\mathcal{W}^+ = \bigcup_{i=1}^{k_0} \mathcal{W}_i^+ \text{ with } \mathcal{W}_i^+ = \{\psi_1^{(i)}, \cdots, \psi_{m_i(\lambda_0)}^{(i)}\},\ i = 1, \cdots, k_0.$$

In order to compute the asymptotic expansion of $R(z)$ for z in a small domain of the form

$$\Omega^+(\delta) = \{z \in \mathbb{C}_+, |z - \lambda_0| < \delta\} \tag{4.73}$$

for $\delta > 0$ small, we use the formula

$$R(z) = (1 + r_0(z)W_c)^{-1} r_0(z) \tag{4.74}$$

and construct for $1 + r_0(z)W_c$ a Grushin problem similarly as in Section 3 (with $z \in \Omega^+(\delta)$ and $\mathcal{U}$ and $\mathcal{W}$ replaced by $\mathcal{U}^+$ and $\mathcal{W}^+$, respectively). Let

$$(1 + r_0(z)W_c)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \tag{4.75}$$

be the resulting representation formula. Recall that

$$r_0(z) = \sum_{j=0}^{\ell} (z - \lambda_0)^j G_j^+(\lambda_0) + r_\ell^+(z, \lambda_0), \qquad z \in \Omega^+(\delta), \tag{4.76}$$

which is valid in $\mathcal{B}(0, s, 0, -s)$ for $s > \ell + 1/2$ and $\ell \in \mathbb{N}$, where the remainder $r_\ell^+(z, \lambda_0)$ is holomorphic in $\Omega^+(\delta)$ and continuous up to $\lambda \in]\lambda_0 - \delta, \lambda_0 + \delta[$. In addition, the limit

$$r_\ell^+(\lambda + i0, \lambda_0) = \lim_{z \to \lambda, z \in \Omega^+(\delta)} r_\ell^+(z, \lambda_0)$$

defines a function of λ of class C^ℓ on $(]\lambda_0 - \delta, \lambda_0 + \delta[)$. Thus for $\ell \in \mathbb{N}$ and $z \in \Omega^+(\delta)$, the expansion of $E_{-+}(z)$ in (3.15) has the following form:

$$E_{-+}(z) = \sum_{j=0}^{\ell} (z - \lambda_0)^j E_j^+(\lambda_0) + E_{-+,\ell}(z, \lambda_0), \tag{4.77}$$

where

$$(E_0^+(\lambda_0))^{(ij)} = -\begin{pmatrix} 0 & \delta_{ij} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \delta_{ij} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m_i(\lambda_0)\times m_j(\lambda_0)} \quad \text{and} \tag{4.78}$$

$$(E_1^+(\lambda_0))^{(ij)} = \begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ a_{ij}(\lambda_0) & * & \cdots & * \end{pmatrix}_{m_i(\lambda_0)\times m_j(\lambda_0)}, \quad \forall 1 \le i,j \le k, \tag{4.79}$$

with

$$a_{ij}(\lambda_0) = -c_i(\lambda_0)^{-1}\langle G_1^+(\lambda_0)W_c\varphi_1^{(j)}, JW_c\varphi_1^{(i)}\rangle, \ \forall 1 \le i,j \le k_0. \tag{4.80}$$

In addition, the remainder $E_{-+,\ell}(z,\lambda_0)$ is analytic in $z \in \Omega^+(\delta)$ as matrix-valued function and for $\lambda \in]\lambda_0-\delta, \lambda_0+\delta[$ the limit $E_{-+,\ell}(\lambda+i0,\lambda_0)$ defines a function of class C^ℓ on $]\lambda_0-\delta,\lambda_0+\delta[$.

Lemma 4.1. *Let the assumptions (A2) and (A3) be satisfied and $\lambda_0 \in r_+(H)$. Then for any $N \in \mathbb{N}$*

$$det\, E_{-+}(z) = \sigma_{k_0}(\lambda_0)(z-\lambda_0)^{k_0} + \cdots + \sigma_{k_0+N}(\lambda_0)(z-\lambda_0)^{k_0+N} + o(|z-\lambda_0|^{k_0+N}), \tag{4.81}$$

for $z \in \Omega^+(\delta)$, where $\sigma_{k_0}(\lambda_0) = det(a_{ij}(\lambda_0))_{1\le i,j\le k_0} \neq 0$.

The proof of Lemma 3.3 can be repeated here to prove Lemma 4.1, using the expansion (4.77). Lemma 4.1 implies in particular the following

Corollary 4.2. *Under the conditions of Lemma 4.1, λ_0 is an isolated point in $r_+(H)$.*

Proof. Lemma 4.1 shows that there exists some $\delta > 0$ such that $E_{-+}(z)^{-1}$ exists for z such that $0 < |z-\lambda_0| < \delta$ and $\operatorname{Im} z \ge 0$. From (4.75), it follows that for $\lambda \in]\lambda_0-\delta, 0[\cup]0, \lambda_0+\delta[$, $1 + r_0(\lambda+i0)W_c$ is invertible as operator in $L^{2,-s}$, $s > \frac{1}{2}$. Therefore λ is not an outgoing positive resonance of H. This proves $r_+(H)\cap]\lambda_0-\delta,\lambda_0+\delta[= \{\lambda_0\}$. ■

Theorem 4.3. *Suppose that Assumptions (A2) and (A3) are satisfied. Let $\ell \in \mathbb{N}$, $s > \ell + 3/2$ and $\lambda_0 \in r_+(H)$. Then the resolvent expansion has the following form*

$$R(z) = -\frac{\Pi_0^+(\lambda_0)}{z-\lambda_0} + R_1^+(z,\lambda_0) \tag{4.82}$$

in $\mathcal{B}(-1,s,1,-s)$ for $z \in \Omega^+(\delta)$. Here

$$\Pi_0^+(\lambda_0) = \sum_{i=1}^{k_0}\langle \cdot, J\Psi_i^+\rangle\Psi_i^+ \qquad \textit{with} \tag{4.83}$$

$$B_{\lambda_0}^+(\Psi_i^+, \Psi_j^+) = \delta_{ij}, \qquad \forall 1 \le i,j \le k_0, \tag{4.84}$$

where $\{\Psi_1^+, \cdots, \Psi_{k_0}^+\}$ is a basis of $Ker(I + K_0^+(\lambda_0))$ in $L^{2,-s}$ and $B_{\lambda_0}^+$ is the bilinear form defined in (1.22). Moreover, the remainder term $R_1^+(z,\lambda_0)$ is analytic in $\Omega^+(\delta)$ and for $\lambda > 0$ with $|\lambda-\lambda_0| < \delta$, the limit

$$R_1^+(\lambda+i0,\lambda_0) = \lim_{z\in\Omega^+(\delta), z\to\lambda} R_1^+(z,\lambda_0) \tag{4.85}$$

exists in $\mathcal{B}(-1,s,1,-s)$ and defines a function of λ of class C^ℓ on $(]\lambda_0-\delta,\lambda_0+\delta[)$ with values in $\mathcal{B}(-1,s,1,-s)$.

Remark 4.1. *(a) In the above theorem, the outgoing resonance λ_0 is shown to be a simple pole of the the resolvent under Assumption (A2) and the leading term of the resolvent expansion near λ_0 is computed explicitly.*

(b) The result of Theorem 4.3 was obtained in [1, Theorem 2.5] *for rapidly decreasing potential V under the condition that $V(x) = O(\langle x\rangle^{-\rho}), \forall x \in \mathbb{R}^3$, for some $\rho > 2\ell + 3$.*

Proof. Under the condition (1.23), the matrix

$$\left(B^+_{\lambda_0}(\varphi_1^{(i)}, \varphi_1^{(j)})\right)_{1\le i,j\le k_0} \tag{4.86}$$

is invertible. Then by Lemma 4.1, the matrix $E_{-+}(z)$ is invertible for $z \in \Omega^+(\delta)$. Thus, the same Grushin problem constructed in the proof of Theorem 3.5 for $I + G_0(z)W_c$ is invertible for $z \in \Omega^+(\delta)$. For $\ell \in \mathbb{N}$, $s > \ell + 1/2$ and $z \in \Omega^+(\delta)$, introducing the expansion (4.76) in (3.6) yields the following expansion of $E(z)$

$$E(z) = \sum_{j=0}^{\ell} (z - \lambda_0)^j E_j(\lambda_0) + \epsilon_\ell(z), \tag{4.87}$$

in the norm sense of $\mathcal{B}(L^{2,-s})$, where the terms E_j, $j = 0, \cdots, \ell$, can be obtained explicitly. By (4.85), the remainder term $\epsilon_\ell(z)$ is analytic in $\Omega^+(\delta)$ and for $|\lambda-\lambda_0| < \delta$ the limit $\epsilon_\ell(\lambda+i0)$ belongs to $C^\ell(]\lambda_0 - \delta, \lambda_0 + \delta[, \mathcal{B}(L^{2,-s}))$.
Following the same method used to calculate the expansion (3.24), we obtain

$$E_{-+}(z)^{-1} = \frac{\mathcal{F}_{-1}(\lambda_0)}{z - \lambda_0} + \mathcal{F}(z, \lambda_0), \qquad \forall z \in \Omega^+(\delta), \tag{4.88}$$

where $\mathcal{F}_{-1}(\lambda_0)$ is $k_0 \times k_0$ block matrix, with block entries $\mathcal{F}^{(ij)}_{-1}(\lambda_0)$ are of the same form as $\mathcal{F}^{(ij)}_{-1}$ in (3.25) with entries $\gamma_{ij}(\lambda_0)$ instead of α_{ij}, $1 \le i, j \le k_0$, such that

$$(\gamma_{ij}(\lambda_0))_{1\le i,j\le k_0} = \left((a_{ij}(\lambda_0))_{1\le i,j\le k_0}\right)^{-1},$$

where $a_{ij}(\lambda_0)$ are given in (4.80). Moreover, the remainder term $\mathcal{F}(z, \lambda_0)$ is continuous up to $]\lambda_0-\delta, \lambda_0+\delta[$ and the limit $\mathcal{F}(\lambda+i0, \lambda_0)$ defines a C^∞ matrix-valued function on $]\lambda_0-\delta, \lambda_0+\delta[$.

Since the techniques of the rest of the proof are close to those used in the proof of Theorem 3.5, we omit details. We only note that in the present case the matrix in (4.86) will play the role of Q_k in (3.35) which yields the leading term in (4.82).

Note that because of the singularity $(z-\lambda_0)^{-1}$ in (4.88), the expansion (4.87) of $E(z)$ up to order $\ell + 1$ is required, so $s > \ell + 3/2$. Moreover, for the remainder $R_1^+(z, \lambda_0)$, the regularity $C^\ell\Big(]\lambda_0 - \delta, \lambda_0 + \delta[, B(-1, s, 1, -s)\Big)$ of $R_1^+(\lambda + i0, \lambda_0)$ derives from the regularity properties of the remainders of $E(z)$ and $E_{-+}(z)^{-1}$ in (4.87) and (4.88), respectively, and from formulae (4.74) and (4.75). ■

4.2. **The case of analytic potentials.**

In this subsection, we assume that all conditions of Theorem 1.2 are satisfied. In particular, we can decompose H as $H = h_0 + W_c$ as in Subsection 3.2 with $h_0 = -\Delta + v_0(x)$ and $v_0 = V_0 + \chi_R W$. Let

$$H(\theta) = U_\theta H U_\theta^{-1} = h_0(\theta) + W_c(x), \qquad \theta \in \mathbb{R},$$

where

$$h_0(\theta) = U_\theta h_0 U_\theta^{-1} = -\Delta_\theta + v_0(x, \theta),$$

be the distorted operator defined in Section 3. Here the distortion is made outside a sufficiently large ball. Then $h_0(\theta)$ and $H(\theta)$ define holomporphic families of type A for θ in a complex neighborhood of zero. For $\mathrm{Im}\,\theta > 0$ and $|\theta| < \theta_0$ with $\theta_0 > 0$ small, it has been shown in [12, Theorem 5.1] that $\sigma_d(H(\theta))$ (the discrete spectrum of $H(\theta)$) verifies

$$\sigma_d(H(\theta)) \cap \mathbb{R}_+ = r_+(H), \tag{4.89}$$

$$\sigma_d(H(\theta)) \cap \mathbb{C}_+ = \sigma_d(H) \cap \mathbb{C}_+. \tag{4.90}$$

In particular, H has at most a countable set of outgoing positive resonances with zero as the only possible accumulation point. Theorem 3.11 says that zero is not an accumulation point of $\sigma_d(H(\theta))$, so the set $r_+(H)$ is finite under the condition of Theorem 1.2.

We observe that $r_0(z, \theta) = (h_0(\theta) - z)^{-1}$ defined initially for $\mathrm{Im}\, z > 0$ and $\theta \in \mathbb{R}$ can be holomorphically extended in θ for $\theta \in \mathbb{C}$ with $|\theta|$ small. Then, for $\theta \in D_+(0, \theta_0)$ with $\theta_0 > 0$ small, $r_0(z, \theta)$ is holomorphic in $z \in \{\mathrm{Im}\, z > -c\mathrm{Im}\,\theta \mathrm{Re}\, z\}$. In particular, for $\mathrm{Im}\,\theta > 0$ and $\lambda_0 > 0$ we have that $r_0^+(\lambda_0, \theta) := (h_0(\theta) - \lambda_0 - i0)^{-1}$ is a bounded operator on $L^2(\mathbb{R}^n)$. In addition, for any $L \in \mathbb{N}$, the expansion of $r_0^+(\lambda_0, \theta)$ in $\mathcal{B}(L^2)$ is written

$$r_0(z, \theta) = \sum_{j=0}^{L} (z - \lambda_0)^j G_0^{+,j}(\lambda_0, \theta) + r_1^+(z, \lambda_0, \theta), \qquad \forall z \in \Omega_{\lambda_0}(\delta), \tag{4.91}$$

where $G_0^{+,j}(\lambda_0, \theta), j = 0, 1, \cdots, L$, are bounded operators on $L^2(\mathbb{R}^n)$ defined as in (4.72) with $r_0(z)$ replaced by $r_0(z, \theta)$ and $\Omega_{\lambda_0}(\delta)$ is given by

$$\Omega_{\lambda_0}(\delta) = \Omega(\delta) \cup \{\lambda_0\} \text{ with } \Omega(\delta) := \{z \in \mathbb{C}; 0 < |z - \lambda_0| < \delta\}, \tag{4.92}$$

for some $0 < \delta < \lambda_0 \mathrm{Im}\,\theta$. Moreover, the remainder $r_1^+(z, \lambda_0, \theta)$ is holomorphic in $z \in \Omega_{\lambda_0}(\delta)$.

Set $K_0^+(\theta) = r_0^+(\lambda_0, \theta) W_c$. The same proof as that of Lemma 3.8 shows that $K_0^+(\theta)$ is a holomorphic family of compact operators on $L^{2,-s}(\mathbb{R}^n)$, $\forall s > 1/2$, for $\theta \in D_+(0, \epsilon)$ with $\epsilon > 0$ small, and norm-continuous for $\theta \in \overline{D_+(0, \epsilon)}$. We recall that by definition of outgoing positive resonances, $\lambda \in r_+(H)$ if -1 is a discrete eigenvalue of the compact operator $K_0^+ = K_0^+(0)$ on $L^{2,-s}$, $\forall s > 1/2$. Thus Lemma 3.9 can be applied here to $\langle x\rangle^{-s}\Pi_1^+(\theta)\langle x\rangle^s$, $\forall s > 1/2$, where $\Pi_1^+(\theta)$ is defined similarly to $\Pi_1(\theta)$ with $K_0(\theta)$ replaced by $K_0^+(\theta)$, because $\theta \mapsto K_0^+(\theta) \in \mathcal{B}(L^{2,-s})$ is norm-continuous for $\theta \in \overline{D_+(0, \epsilon)}$. We obtain that the family $\Pi_1^+(\theta) \in \mathcal{B}(L^{2,-s})$ of Riesz projections associated with the eigenvalue -1 of $K_0^+(\theta)$ is holomorphic in $\theta \in D_+(0, \epsilon)$ and $\forall s > 1/2$

$$\lim_{\theta \to 0,\, \theta \in \overline{D_+(0,\epsilon)}} \|\langle x\rangle^{-s}(\Pi_1^+(\theta) - \Pi_1^+(0))\langle x\rangle^s\| = 0.$$

Theorem 4.4. *Assume that $V_0 \in \mathcal{A}_1$ and that conditions (1.2), (1.18) and (1.19) are satisfied. Let $\lambda_0 \in r_+(H)$ verifying the assumption (A2) and $\chi \in C_0^\infty(\mathbb{R}^n)$. Then for $z \in \Omega(\delta)$, $0 < \delta < \lambda_0 \mathrm{Im}\,\theta$ with $\theta \in D_+(0, \epsilon)$ and $\epsilon > 0$ small, one has*

$$\chi R(z)\chi = -\frac{\chi \Pi_0^+(\lambda_0)\chi}{z - \lambda_0} + R_2^+(z, \lambda_0) \tag{4.93}$$

in $\mathcal{B}(L^2(\mathbb{R}^n))$, where $\Pi_0^+(\lambda_0)$ is given in Theorem 4.3 and the remainder $R_2^+(z, \lambda_0)$ is analytic in $z \in \Omega_{\lambda_0}(\delta)$.

Theorem 4.4 can be proven by combining the methods used in the proofs of Theorem 3.11 and Theorem 4.3 and by using formulae (3.65) for $z \in \Omega(\delta, \theta)$ and (4.91) for $z \in \Omega_{\lambda_0}(\delta)$ and by studying the Grushin problem constructed with bases $\Pi_1^+(\theta)\mathcal{U}^+$ and $\Pi_1^+(\theta)\mathcal{W}^+$. The details are omitted here.

5. Proofs of the main theorems

In this section we prove the main theorems. In theorems 1.1 and 1.2 when zero is an eigenvalue of H we are interested in the contribution of the eigenvalue zero to the leading term of the asymptotic expansion in time for the heat semigroup e^{-tH} and Shrödinger semigroup e^{-itH}. With the Gevrey estimates on the remainders in resolvent expansions, the sub-exponential time-decay estimates in (1.13) and (1.24) can be proved in the same way as in [12, Section 5], therefore the details are omitted here.

Proof of Theorem 1.1. The same proof as that of Theorem 2.4 in [12] can be done here. The large-time expansion of e^{-tH} follows directly from Theorem 3.5 and the formula

$$e^{-tH} - \sum_{\lambda\in\sigma_d(H),\operatorname{Re}\lambda\le 0} e^{-tH}\Pi_\lambda \quad = \quad \frac{i}{2\pi}\lim_{\epsilon\to 0_+}\int_{\Gamma(\epsilon)} e^{-tz}R(z)dz + O(e^{-ct}) \tag{5.94}$$

where $c>0$ and

$$\Gamma(\epsilon) \quad = \quad \{z; |z|\ge\epsilon, \operatorname{Re} z\ge 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^{\mu'}\}\cup\{z; |z|=\epsilon, |\arg z|\ge\omega_0\}$$

for some appropriate constants $C,\mu'>0$. Here ω_0 is the argument of the point z_0 with $|z_0|=\epsilon$, $\operatorname{Re} z_0>0$ and $\operatorname{Im} z_0 = C(\operatorname{Re} z_0)^{\mu'}$. According to Theorem 3.5, if the assumption $(A1)$ holds, then H has no discrete eigenvalues in some domain $O(\delta)$ given in (2.3) for $\delta>0$ small. Since discrete eigenvalues located on the left of the curve $\Gamma(0)=\{z;\operatorname{Re} z\ge 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^{\mu'}\}$ can accumulate only at zero, it follows that H has at most a finite number of eigenvalues there. These discrete eigenvalues contribute to e^{-tH} the term $\sum_{\lambda\in\sigma_d(H),\operatorname{Re}\lambda\le 0} e^{-tH}\Pi_\lambda + O(e^{-ct})$. By Theorem 3.5, one can evaluate

$$\|e^{-a\langle x\rangle^{1-\mu}}\left(\frac{i}{2\pi}\lim_{\epsilon\to 0_+}\int_{\Gamma(\epsilon)} e^{-tz}R(z)dz - \Pi_0\right)\| \le Ce^{-ct^{\beta'}}$$

with $C,c>0$ and $\beta' = \frac{1-\mu}{1+\kappa\mu}$. Theorem 1.1 is proved. ■

The large-time expansion for the Shrödinger semigroup e^{-itH} can be derived from Theorem 3.11 and Theorem 4.4 by taking into account the contribution from discrete eigenvalues in the upper-half plane and outgoing positive resonances.

Proof of Theorem 1.2. For $\chi\in C_0^\infty(\mathbb{R}^n)$, we use analytic distorsion outside some sufficiently large ball (including the support of χ). Let $R(z,\theta)=(H(\theta)-z)^{-1}$. The formula

$$\chi R(z)\chi = \chi R(z,\theta)\chi$$

with $\operatorname{Im}\theta>0$ gives a meromorphic extension of the cut-off resolvent into a sector below the positive half-axis. As seen before, the outgoing positive resonances are discrete eigenvalues of the distorted operator $H(\theta)$ in $]0,+\infty[$ and their number is finite. Under the conditions of Theorem 1.2, outgoing positive resonances are simple poles of the cut-off resolvent $\chi R(z)\chi$ according to Theorem 4.4.

Let $\Gamma_\eta(\epsilon)$ be the contour defined by

$$\Gamma_\eta(\epsilon) = \{z = re^{-i\eta}, r\ge\epsilon\}\cup\{z=-re^{i\eta}, r\ge\epsilon\}\cup\{z;|z|=\epsilon, -\eta\le\arg z\le\pi+\eta\}$$

for some $\eta>0$, where $\eta>0$ is chosen such that $\chi R(z)\chi$ has no poles with negative imaginary part between $\Gamma_\eta(\epsilon)$ and the real axis. Since $\chi R(z)\chi$ has only a finite number of poles located above $\Gamma_\eta(\epsilon)$ which are discrete eigenvalues of H in $\overline{\mathbb{C}_+}$ and the outgoing positive resonances, we obtain from Theorem 4.4 the formula

$$\chi\Big(e^{-itH} - \sum_{\lambda\in\sigma_d(H)\cap\overline{\mathbb{C}}_+} e^{-itH}\Pi_\lambda - \sum_{\nu\in r_+(H)} e^{-it\nu}\Pi_0^+(\nu)\Big)\chi$$
$$= \frac{i}{2\pi}\lim_{\epsilon\to 0_+}\int_{\Gamma_\eta(\epsilon)} e^{-itz}\chi R(z)\chi dz + O(e^{-ct}).$$

Since $\operatorname{Im} z < 0$ for $z \in \Gamma_\eta(0)$ and $z \neq 0$, one can derive from Theorem 3.11 that

$$\|e^{-a\langle x\rangle^{1-\mu}}\left(\frac{i}{2\pi}\lim_{\epsilon\to 0_+}\int_{\Gamma_\eta(\epsilon)} e^{-tz}\chi R(z)\chi dz - \chi\Pi_0\chi\right)\| \le Ce^{-ct^\beta}$$

with $C, c > 0$ and $\beta = \frac{1-\mu}{1+\mu}$. Theorem 1.2 is proved. ∎

References

[1] M. Aafarani, Large-time behavior of solutions to Schrödinger equation with complex-valued potential, *arXiv preprint arXiv:1910.03681*, 2019.

[2] J. Aguilar, J.M. Combes, A class of analytic perturbations for one-body Schrödinger Hamiltonians. Comm. Math. Phys. 22 (1971), 269-279.

[3] B. Helffer, A. Martinez, Comparaison entre les diverses notions de résonances. *Helvetica Physica Acta*, 60(8): 992-1003, 1987.

[4] W. Hunziker, Distortion analyticity and molecular resonance curves. *Annales de l'IHP Physique théorique*, 45(4): 339–358, 1986.

[5] T. Kato, Perturbation Theory for Linear Operators. Second Edition. Springer, Berlin, 1984.

[6] V.B. Lidskii, Perturbation theory of non-conjugate operators. *USSR Computational Mathematics and Mathematical Physics*, 6(1):73-85, 1966.

[7] J. Moro, J.V. Burke and M.L. Overton, On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure. *SIAM Journal on Matrix Analysis and Applications*, 18(4):793–817, 1997.

[8] C. Rakotonirina, Matrices de commutation tensorielle: de l'équation de Dirac vers une application en physique des particules. Comments: 68 pages, Dissertation, Mémoire d'Habilitation à Diriger des Recherches, *arXiv:1405.7668,* May 2014. *https://cds.cern.ch/record/1705445*

[9] J. Royer, Limiting absorption principle for the dissipative Helmholtz equation, *Communication in Partial Differential Equations* 35(8) (2010), 1458-1489.

[10] X.P. Wang, Time-decay of semigroups generated by dissipative schrödinger operators. *Journal of Differential Equations*, 253(12):3523–3542, 2012.

[11] X.P. Wang, Time-decay of solutions to dissipative Schrödinger equations, *Séminaire & Congrès*, 30 (2017), 141-153.

[12] X.P. Wang, Gevrey estimates of the resolvent and sub-exponential time-decay for the heat and Schrödinger semigroups. *J. Math. Pures Appl.*, 135(2020), 284-338.

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