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Amaury Durand, François Roueff

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Abstract

Fractionally integrated autoregressive moving average processes have been widely and successfully used to model univariate time series exhibiting long range dependence. Vector and functional extensions of these processes have also been considered more recently. Here we rely on a spectral domain approach to extend this class of models in the form of a general Hilbert valued processes. In this framework, the usual univariate long memory parameter d is replaced by a long memory *operator* D acting on the Hilbert space. Our approach is compared to processes defined in the time domain that were previously introduced for modeling long range dependence in the context of functional time series.

Contents

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1 Introduction

The study of weakly-stationary time series valued in a separable Hilbert space has been an active field of research in the past decades. For example, ARMA processes have been discussed in [3, 24, 16], a spectral theory is constructed in [20, 21, 25] and several estimation methods have been studied in [11, 13, 12, 14, 17, 2, 18, 7]. However, the literature mainly focuses on short-memory processes and the study of long-memory processes valued in a separable Hilbert space is a more recent topic, see [23, 4, 5, 9, 19]. In particular, in [19, Section 4], the authors propose a generalization of the fractionally integrated autoregressive moving average (often shortened as ARFIMA but we prefer to use the abbreviation FIARMA for reasons that will be made explicit in Remark 3.1) processes to the case of *curve* (or *functional*) time series. In short, they consider the functional case in which the Hilbert space is an L^2 space of real valued functions defined on some compact subset $\mathcal{C}$ of $\mathbb{R}$, and they introduce the time series $(X_t)_{t\in\mathbb{Z}}$ valued in this Hilbert space defined by

$$(1-B)^d X_t(v) = Y_t(v) \ , \quad t \in \mathbb{Z}, v \in \mathcal{C} \ , \tag{1.1}$$

where $-1/2 < d < 1/2$, B is the backshift operator on $\mathbb{R}^{\mathbb{Z}}$, and Y_t is a functional ARMA process. As pointed out in [19, Remark 9], taking the same d for all $v \in \mathcal{C}$ in (1.1) is very restrictive compared to other long memory processes recently introduced, for instance in [4, 5], where they consider long-memory processes of the form

$$X_t(v) = \sum_{k=0}^{\infty} (1+k)^{-\mathrm{n}(v)} \, \epsilon_{t-k}(v) \ , \quad t \in \mathbb{Z} \ , \ v \in \mathsf{V} \ ,$$

where $(\mathsf{V}, \mathcal{V}, \xi)$ is a σ-finite measured space and $(\epsilon_t)_{t\in\mathbb{Z}}$ is a white noise valued in $L^2(\mathsf{V}, \mathcal{V}, \xi)$. A formulation not restricted to an L^2 space was proposed in [9] where the author considers long-memory processes of the form

$$X_t = \sum_{k=0}^{\infty} (1+k)^{-N} \, \epsilon_{t-k} \ , \quad t \in \mathbb{Z} \ , \tag{1.2}$$

where $(\epsilon_t)_{t\in\mathbb{Z}}$ is a white noise valued in a separable Hilbert space $\mathcal{H}_0$ and N is a bounded normal operator on $\mathcal{H}_0$. This suggests to define FIARMA processes in (1.1) with d replaced by a a function $\mathrm{d}(v)$, or in the case where it is valued in an arbitrary separable Hilbert space $\mathcal{H}_0$, by a bounded normal operator D acting on this space.

In this contribution, we fill this gap by providing a definition of FIARMA processes valued in a separable Hilbert space $\mathcal{H}_0$ with a long memory operator D, taken as a bounded linear operator on $\mathcal{H}_0$. If D is normal, then we can rely on its singular value decomposition and find necessary and sufficient conditions to ensure that the FIARMA process with long memory operator D is well defined. This allows us to compare FIARMA processes with the processes defined by (1.2) as in [9]. Our definition relies on linear filtering in the spectral domain. It is a well known fact that linear filtering of (*i.e.* shift-invariant linear transforms on) real valued time series in the time domain is equivalent to pointwise multiplication by a transfer function in the frequency domain. This duality also applies to Hilbert valued time series using a proper spectral representation for them. In this context pointwise multiplication becomes pointwise application of an operator valued transfer function defined on the set of frequencies. A complete account on this topic is provided in the survey paper [8]. We recall in Section 2 the necessary definitions and facts needed for our purpose about operator theory, linear filtering in the spectral domain and some specific tools for functional time series. The construction of FIARMA processes is introduced and discussed in Section 3, where the case of a normal long memory operator is detailed. Proofs are postponed to Section 4 for convenience.

2 Preliminaries and useful notation

2.1 Integration of functions valued in a Banach space

We introduce some notation for useful spaces of functions defined on a measurable space $(\Lambda, \mathcal{A})$ and valued in a separable Banach space $(E, \|\cdot\|_E)$. In this context a function is said to be measurable if it is Borel-measurable, *i.e.* $f^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{B}(E)$, the Borel σ-field on E. We denote by $\mathbb{F}(\Lambda, \mathcal{A}, E)$ the space of measurable functions from Λ to E. For a non-negative measure μ on $(\Lambda, \mathcal{A})$ and $p \in [1, \infty]$, we denote by $\mathcal{L}^p(\Lambda, \mathcal{A}, E, \mu)$ the space of functions $f \in \mathbb{F}(\Lambda, \mathcal{A}, E)$ such that $\int \|f\|_E^p \, \mathrm{d}\mu$ (or μ-essup $\|f\|_E$ for $p = \infty$) is finite and by $L^p(\Lambda, \mathcal{A}, E, \mu)$ its quotient space with respect to μ-a.e. equality. When $E = \mathbb{C}$, we simply omit E in the notation of the $\mathcal{L}^p$ and L^p spaces. The corresponding norms are denoted by $\|\cdot\|_{L^p(\Lambda, \mathcal{A}, E, \mu)}$. For a simple function $f \in \mathrm{Span}\left(\mathbb{1}_A x \ : \ A \in \mathcal{A}, \, \mu(A) < \infty, \, x \in E\right)$ with range $\{\alpha_1, \cdots, \alpha_n\}$, the integral (often referred to as the *Bochner integral*) of the E-valued function f with respect to μ is defined by

$$\int f \, \mathrm{d}\mu = \sum_{k=1}^{n} \alpha_k \, \mu\left(f^{-1}(\{\alpha_k\})\right) \in E \, . \tag{2.1}$$

This integral is extended to $L^1(\Lambda, \mathcal{A}, E, \mu)$ by continuity (and thus also to L^p if μ is finite).

2.2 Banach space valued measure

An E-valued measure is a mapping $\mu : \mathcal{A} \to E$ such that for any sequence $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ of pairwise disjoint sets then $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$ where the series converges in E, that is

$$\lim_{N \to +\infty} \left\| \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) - \sum_{n=0}^{N} \mu(A_n) \right\|_E = 0 \, .$$

We denote by $\mathbb{M}(\Lambda, \mathcal{A}, E)$ the set of E-valued measures. For such a measure μ, the mapping

$$\|\mu\|_E : A \mapsto \sup\left\{ \sum_{i \in \mathbb{N}} \|\mu(A_i)\|_E \ : \ (A_i)_{i \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}} \text{ is a countable partition of } A \right\}$$

defines a non-negative measure on $(\Lambda, \mathcal{A})$ called the *variation measure* of μ. The notation $\|\mu\|_E$ will be adapted to the notation chosen for the norm in E. If $\mu \in \mathbb{M}(\Lambda, \mathcal{A}, E)$ has finite variation, then for a simple function $f : \Lambda \mapsto \mathbb{C}$ with range $\{\alpha_1, \ldots, \alpha_n\}$, the integral of f with respect to μ is defined by the same formula as in (2.1) (but this time the α_k's are scalar and the μ's are E-valued). This definition is extended to $L^1(\Lambda, \mathcal{A}, \|\mu\|_E)$ by continuity.

2.3 Operator theory

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. The inner product and norm, e.g. associated to $\mathcal{H}_0$, are denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ and $\|\cdot\|_{\mathcal{H}_0}$. Let $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ denote the set of all $\mathcal{H}_0 \to \mathcal{G}_0$ continuous operators. We also denote by $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ the set of all compact operators in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and for all $p \in [1, \infty)$, $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ the Schatten-p class. The space $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and the Schatten-p classes are Banach spaces when respectively endowed with the norms

$$\|\mathrm{P}\|_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} := \sup_{\|x\|_{\mathcal{H}_0} \le 1} \|\mathrm{P}x\|_{\mathcal{G}_0} \quad \text{and} \quad \|\mathrm{P}\|_p := \left(\sum_{\sigma \in \mathrm{sing}(\mathrm{P})} \sigma^p \right)^{1/p}$$

where $\mathrm{sing}(\mathrm{P})$ is the set of singular values of P. Following these definitions, we have, for all $1 \le p \le p'$

$$\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{S}_{p'}(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0) \, . \tag{2.2}$$

The space $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ is endowed with the operator norm and the three inclusions in (2.2) are continuous embeddings. If $\mathcal{G}_0 = \mathcal{H}_0$, we omit the $\mathcal{G}_0$ in the notations above.

An operator $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$, is said to be *positive* if for all $x \in \mathcal{H}_0$, $\langle \mathrm{P}x, x \rangle_{\mathcal{H}_0} \ge 0$ and we will use the notations $\mathcal{L}_b^+(\mathcal{H}_0)$, $\mathcal{K}^+(\mathcal{H}_0)$, $\mathcal{S}_p^+(\mathcal{H}_0)$ for positive, positive compact and positive Schatten-p operators. If $\mathrm{P} \in \mathcal{L}_b^+(\mathcal{H}_0)$ then there exists a unique operator of $\mathcal{L}_b^+(\mathcal{H}_0)$, denoted by $\mathrm{P}^{1/2}$, which satisfies $\mathrm{P} = \left(\mathrm{P}^{1/2}\right)^2$. For any $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ we denote its adjoint by P^{H} and for $\mathrm{P} \in \mathcal{S}_1(\mathcal{H}_0)$, we denote its trace by $\mathrm{Tr}(\mathrm{P})$. Schatten-1 and Schatten-2 operators are usually referred to as *trace-class* and *Hilbert-Schmidt* operators respectively.

We conclude this section with the singular value decomposition of a bounded normal operator (see [6, Theorem 9.4.6, Proposition 9.4.7]). Let $N \in \mathcal{L}_b(\mathcal{H}_0)$ be a normal operator, i.e. $NN^{\mathsf{H}} = N^{\mathsf{H}}N$, then, there exists a σ-finite measure space $(\mathsf{V}, \mathcal{V}, \xi)$, a unitary operator $U : \mathcal{H}_0 \to L^2(\mathsf{V}, \mathcal{V}, \xi)$ and $\mathrm{n} \in L^\infty(\mathsf{V}, \mathcal{V}, \xi)$, such that

$$UNU^{\mathsf{H}} = M_{\mathrm{n}} \ , \tag{2.3}$$

where M_{n} denotes the pointwise multiplicative operator on $L^2(\mathsf{V}, \mathcal{V}, \xi)$ associated to n, that is $M_{\mathrm{n}} : f \mapsto \mathrm{n} \times f$. We say that N has singular value function n on $L^2(\mathsf{V}, \mathcal{V}, \xi)$ with decomposition operator U.

2.4 Integration of functions valued in an operator space

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and $(\Lambda, \mathcal{A}, \mu)$ be a measured space. Then, since all the operator spaces displayed in (2.2) are Banach spaces, the integration theory presented in Section 2.1 applies. However, we introduce hereafter a weaker notion of measurability.

Definition 2.1 (Simple measurability)**.** *A function $\Phi : \Lambda \to \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ is said to be* simply measurable *if for all $x \in \mathcal{H}_0$, $\lambda \mapsto \Phi(\lambda)x$ is measurable as a $\mathcal{G}_0$-valued function. The set of such functions is denoted by $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ or simply $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0)$ if $\mathcal{G}_0 = \mathcal{H}_0$.*

Note that for a Banach space $\mathcal{E}$ which is continuously embedded in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ (*e.g.* $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ for $p \geq 1$ or $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$), we have the inclusion $\mathbb{F}(\Lambda, \mathcal{A}, \mathcal{E}) \subset \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$. The reciprocal inclusion holds for $\mathcal{E} = \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ and for $\mathcal{E} = \mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ with $p \in \{1, 2\}$ (see [8, Lemma 2.1]). From this observation, we derive the following useful result.

Lemma 2.1. *Let $\Phi \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1^+(\mathcal{H}_0), \mu)$ and define the function $\Phi^{1/2} : \lambda \mapsto \Phi(\lambda)^{1/2}$. Then $\Phi^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$.*

Proof. Simple measurability of $\Phi^{1/2}$ is given by [15, Lemma 3.4.2] and therefore, by [8, Lemma 2.1], $\Phi^{1/2} \in \mathbb{F}(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0))$. The fact that $\Phi^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$ then follows from the identity $\left\|\Phi^{1/2}(\lambda)\right\|_2^2 = \|\Phi(\lambda)\|_1$. □

2.5 Useful normal Hilbert modules

In this section, we briefly recall the notion of normal Hilbert modules and give some useful examples, see [8, Section 2.3] for details. In short, a normal Hilbert module is a Hilbert space endowed with a *module action* and whose scalar product is inherited from a *gramian*. As mentioned in [8, Example 2.1(3)] the space $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module with module action being the composition of operators, $\mathrm{F} \bullet \mathrm{P} = \mathrm{FP}$, and gramian $[\mathrm{P}, \mathrm{Q}]_{\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)} := \mathrm{PQ}^{\mathsf{H}}$. In particular we have, for all $\mathrm{P}, \mathrm{Q} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathrm{F} \in \mathcal{L}_b(\mathcal{G}_0)$, $\mathrm{FP} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathrm{PQ}^{\mathsf{H}} \in \mathcal{S}_1(\mathcal{G}_0)$, and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ is a Hilbert space when endowed with the scalar product

$$\langle \mathrm{P}, \mathrm{Q} \rangle_{\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)} := \mathrm{Tr}[\mathrm{P}, \mathrm{Q}]_{\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)} = \mathrm{Tr}(\mathrm{PQ}^{\mathsf{H}}) \ .$$

Given a measured space $(\Lambda, \mathcal{A}, \mu)$ where μ is finite and non-negative, we can use the gramian structure of $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ to endow the space $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ with a gramian (see [8, Example 2.1(5)]). Namely, $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module with module action defined for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0)$ and $\Phi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ by $\mathrm{P} \bullet \Phi : \lambda \mapsto \mathrm{P}\Phi(\lambda)$, and gramian defined for all $\Phi, \Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ by

$$[\Phi, \Psi]_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} := \int \Phi\Psi^{\mathsf{H}} \, \mathrm{d}\mu \ .$$

In particular we have, for all $\Phi, \Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0)$,

$$\mathrm{P} \int \Phi\Psi^{\mathsf{H}} \, \mathrm{d}\mu = \mathrm{P}[\Phi, \Psi]_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} = [\mathrm{P} \bullet \Phi, \Psi]_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} = \int \mathrm{P}\Phi\Psi^{\mathsf{H}} \, \mathrm{d}\mu \ ,$$

and $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ endowed with the scalar product

$$\langle \Phi, \Psi \rangle_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} := \mathrm{Tr}[\Phi, \Psi]_\mu = \int \mathrm{Tr}\left(\Phi\Psi^{\mathsf{H}}\right) \, \mathrm{d}\mu$$

is a Hilbert space, whose associated squared norm takes the form

$$\|\Phi\|^2_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} = \int \mathrm{Tr}\left(\Phi\Phi^{\mathsf{H}}\right) \, \mathrm{d}\mu = \int \left\|\Phi\Phi^{\mathsf{H}}\right\|_1 \, \mathrm{d}\mu = \int \|\Phi\|_2 \, \mathrm{d}\mu \ . \tag{2.4}$$

We extend hereafter such module to the case where μ is replaced by a positive operator valued measure μ.

2.6 Positive Operator Valued Measures (p.o.v.m.)

The notion of Positive Operator Valued Measures is widely used in Quantum Mechanics and a good study of such measures can be found in [1]. Here we provide useful definitions and results for our purpose. See [8, Section 2.2] for details.

Definition 2.2 (Trace-class Positive Operator Valued Measures)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$ be a separable Hilbert space. A trace-class Positive Operator Valued Measure (p.o.v.m.) on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is an $\mathcal{S}_1(\mathcal{H}_0)$-valued measure (in the sense of Section 2.2) such that for all $A \in \mathcal{A}$, $\nu(A) \in \mathcal{S}_1^+(\mathcal{H}_0)$. In this cas, ν has a finite variation measure denoted by $\|\nu\|_1$.*

Trace class p.o.v.m.'s satisfy the Radon-Nikodym's property which provides a simple representation of the p.o.v.m. through the integral of an operator-valued density function with respect to a finite non-negative measure. The following result, which corresponds to [8, Theorem 2.4], applies for instance with $\mu = \|\nu\|_1$. It will be used repeatedly in this contribution.

Theorem 2.2. *Let $(\Lambda, \mathcal{A})$ be a measure space, $\mathcal{H}_0$ a separable Hilbert space and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Let μ be a finite non-negative measure on $(\Lambda, \mathcal{A})$. Then $\|\nu\|_1 \ll \mu$ (i.e. for all $A \in \mathcal{A}$, $\mu(A) = 0 \Rightarrow \|\nu\|_1(A) = 0$), if and only if there exists $g \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$ such that $\mathrm{d}\nu = g\,\mathrm{d}\mu$, i.e. for all $A \in \mathcal{A}$,*

$$\nu(A) = \int_A g\,\mathrm{d}\mu\;. \tag{2.5}$$

In this case, g is unique and is called the density of ν with respect to μ and denoted as $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$.

The theorem above implies the existence of some operator-valued density function of a trace-class p.o.v.m. ν with respect to its total variation measure $\|\nu\|_1$ thus allowing us to use the results of Section 2.1 to define left and right integration of operator valued functions with respect to (the operator valued) ν.

Definition 2.3 (Left/right integration of operator valued functions with respect to a p.o.v.m.)**.** *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ with density $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Let $\Phi, \Psi \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$, then the pair (Φ, Ψ) is said to be ν-integrable if $\Phi f \Psi^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0), \|\nu\|_1)$ and in this case we define*

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} := \int \Phi f \Psi^{\mathsf{H}}\,\mathrm{d}\|\nu\|_1 \in \mathcal{S}_1(\mathcal{G}_0)\;. \tag{2.6}$$

If (Φ, Φ) is ν-integrable we say that Φ is square ν-integrable and we denote by $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ the space of functions in $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ which are square ν-integrable.

Following Definition 2.3, the quotient space

$$\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu) := \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu) \Big/ \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}\right\}\;. \tag{2.7}$$

is a normal pre-Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module endowed with the gramian

$$[\Phi, \Psi]_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)} := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}}\;.$$

We refer to [8, Definition 2.4 and Proposition 2.8] for details. Here we will principally use that the trace of gramian defines a scalar product on the (pre-Hilbert) space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ that will be denoted in the following by

$$\langle \Phi, \Psi \rangle_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)} := \mathrm{Tr}[\Phi, \Psi]_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)}$$

with associated norm

$$\|\Phi\|_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)} := \left(\langle \Phi, \Phi \rangle_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)}\right)^{1/2}\;.$$

To check that some $\Phi \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ is square ν-integrable in the sense of Definition 2.3, we can replace $\|\nu\|_1$ by an arbitrary dominating measure μ (often taken as Lebesgue's measure, as in [25]). Namely, by [8, Proposition 2.7], if μ is a finite non-negative measure on $(\Lambda,\mathcal{A})$ which dominates $\|\nu\|_1$ and $g=\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then, for all $\Phi\in\mathbb{F}_s(\Lambda,\mathcal{A},\mathcal{H}_0,\mathcal{G}_0)$, we have

$$\Phi\in\mathscr{L}^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)\Leftrightarrow\Phi g\Phi^{\mathsf{H}}\in\mathcal{L}^1(\Lambda,\mathcal{A},\mathcal{S}_1(\mathcal{G}_0),\mu)\Leftrightarrow\Phi g^{1/2}\in\mathcal{L}^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)\;, \tag{2.8}$$

and, if $\Phi,\Psi\in\mathscr{L}^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$, then (Φ,Ψ) is ν-integrable and

$$[\Phi,\Psi]_{\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)}=\int\Phi\mathrm{d}\nu\Psi^{\mathsf{H}}=\int\Phi g\Psi^{\mathsf{H}}\,\mathrm{d}\mu=\int(\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}}\,\mathrm{d}\mu\;. \tag{2.9}$$

In other words, $\Phi\mapsto\Phi g^{1/2}$ is gramian-isometric from the quotient space $\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$ defined in (2.7) to $L^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)$. By (2.4), the squared norm of $\Phi\in\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$ can thus be computed as

$$\|\Phi\|^2_{\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)}=\int\left\|\Phi g\Phi^{\mathsf{H}}\right\|_1\,\mathrm{d}\mu=\int\left\|\Phi g^{1/2}\right\|_2^2\,\mathrm{d}\mu\;. \tag{2.10}$$

2.7 Linear filtering of Hilbert valued time series in the spectral domain

In the following we denote by $\mathbb{T}$ the set $\mathbb{R}/2\pi\mathbb{Z}$ (which can be represented by an interval such as $[0,2\pi)$ or $[-\pi,\pi)$). Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\mathcal{H}_0$ a separable Hilbert space. We recall that the expectation of $X\in L^2(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$ is the unique vector $\mathbb{E}[X]\in\mathcal{H}_0$ satisfying

$$\langle\mathbb{E}[X],x\rangle_{\mathcal{H}_0}=\mathbb{E}\left[\langle X,x\rangle_{\mathcal{H}_0}\right],\quad\text{for all }x\in\mathcal{H}_0\;.$$

We denote by $\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$ the space of all centered random variables in $L^2(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$. The covariance operator between $X,Y\in L^2(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$ is the unique linear operator $\mathrm{Cov}(X,Y)\in\mathcal{L}_b(\mathcal{H}_0)$, satisfying

$$\langle\mathrm{Cov}(X,Y)\,y,x\rangle_{\mathcal{H}_0}=\mathrm{Cov}\left(\langle X,x\rangle_{\mathcal{H}_0},\langle Y,y\rangle_{\mathcal{H}_0}\right),\quad\text{for all }x,y\in\mathcal{H}_0\;.$$

Let $\mathcal{H}=\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$. Then $\mathcal{H}$ is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module for the action $\mathrm{P}\bullet X=\mathrm{P}X$, defined for all $\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0)$ and $X\in\mathcal{H}$, and the gramian $[X,Y]_{\mathcal{H}}=\mathrm{Cov}(X,Y)$. See [8, Section 2.3] for details.

A process $X:=(X_t)_{t\in\mathbb{Z}}$ is said to be an $\mathcal{H}_0$-valued, weakly-stationary process if

(i) For all $t\in\mathbb{Z}$, $X_t\in L^2(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$.

(ii) For all $t\in\mathbb{Z}$, $\mathbb{E}[X_t]=\mathbb{E}[X_0]$. We say that X is centered if $\mathbb{E}[X_0]=0$.

(iii) For all $t,h\in\mathbb{Z}$, $\mathrm{Cov}(X_{t+h},X_t)=\mathrm{Cov}(X_h,X_0)$.

This corresponds to [8, Definition 1.3] in the case $\mathbb{G}=\mathbb{Z}$.

Let $X=(X_t)_{t\in\mathbb{Z}}\in\mathcal{H}^{\mathbb{Z}}$ be a centered, weakly-stationary, $\mathcal{H}_0$-valued time series. By analogy to the univariate case, and taking into account the module structure of $\mathcal{H}$, let us define the *modular time domain* of X as the submodule of $\mathcal{H}$ generated by the X_t's, that is

$$\mathcal{H}^X:=\overline{\mathrm{Span}}^{\mathcal{H}}\left(\mathrm{P}X_t\;:\;\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0),\,t\in\mathbb{Z}\right)\;.$$

Similarly, given another separable Hilbert space $\mathcal{G}_0$, we define

$$\mathcal{H}^{X,\mathcal{G}_0}:=\overline{\mathrm{Span}}^{\mathcal{G}}\left(\mathrm{P}X_t\;:\;\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\,t\in\mathbb{Z}\right)$$

which is a submodule of $\mathcal{G}:=\mathcal{M}(\Omega,\mathcal{F},\mathcal{G}_0,\mathbb{P})$.

As explained in [8, Section 3], a spectral representation for X amounts to define a random countably additive gramian orthogonally scattered measure (c.a.g.o.s.) $\hat{X}$ on $(\mathbb{T},\mathcal{B}(\mathbb{T}))$ such that

$$X_t=\int\mathrm{e}^{\mathrm{i}\lambda\,t}\,\hat{X}(\mathrm{d}\lambda)\quad\text{for all}\quad t\in\mathbb{Z}\;. \tag{2.11}$$

The intensity measure ν_X of $\hat{X}$ is a trace-class p.o.v.m. and is called the *spectral operator measure* of X. It is the unique regular p.o.v.m. satisfying

$$\Gamma_X(h)=\int\mathrm{e}^{\mathrm{i}\lambda\,h}\,\nu_X(\mathrm{d}\lambda)\quad\text{for all}\quad h\in\mathbb{Z}\;. \tag{2.12}$$

Moreover, for any separable Hilbert space $\mathcal{G}_0$, one can define a a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ of $\mathcal{H}_0 \to \mathcal{G}_0$-operator valued functions defined on $\mathbb{T}$, called the *spectral domain*, which is gramian-isomorphic with the time domain $\mathcal{H}^{X,\mathcal{G}_0}$ through the mapping $\Phi \mapsto \int \Phi \, \mathrm{d}\hat{X}$. The gramian isomorphism between $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ and $\mathcal{H}^{X,\mathcal{G}_0}$ reads

$$[\Phi, \Psi]_{\widehat{\mathcal{H}}^{X,\mathcal{G}_0}} := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \operatorname{Cov}\left(\int \Phi \, \mathrm{d}\hat{X}, \int \Psi \, \mathrm{d}\hat{X}\right) , \qquad \Phi, \Psi \in \widehat{\mathcal{H}}^{X,\mathcal{G}_0} .$$

A time-shift invariant linear filter with $\mathcal{H}_0$-valued input and $\mathcal{G}_0$-valued output can be defined through its *transfer operator function*, which is a $\mathcal{H}_0 \to \mathcal{G}_0$–operator valued function Φ defined on $\mathbb{T}$. Applying the filter to the input $X \in \mathcal{H}^{\mathbb{Z}}$ to produce the output $Y \in \mathcal{G}^{\mathbb{Z}}$ can then be written as

$$Y_t = \int \mathrm{e}^{\mathrm{i}t\lambda} \Phi \, \hat{X}(\mathrm{d}\lambda) , \qquad t \in \mathbb{Z} . \tag{2.13}$$

Note that Φ is the spectral representant of Y_0, which is well defined if and only if $Y_0 \in \mathcal{H}^{X,\mathcal{G}_0}$, or, equivalently $\Phi \in \widehat{\mathcal{H}}^{X,\mathcal{G}_0}$.

In the following, we only consider the case where $\Phi \in \mathbb{F}_s(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{H}_0, \mathcal{G}_0)$. For such a Φ, we have

$$\Phi \in \widehat{\mathcal{H}}^{X,\mathcal{G}_0} \quad \text{if and only if} \quad \Phi \in \mathscr{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu_X) . \tag{2.14}$$

For convenience, in the following, we write

$$X \in \mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad Y = F_\Phi(X) \ (\text{or } \hat{Y}(\mathrm{d}\lambda) = \Phi(\lambda)\hat{X}(\mathrm{d}\lambda)) , \tag{2.15}$$

for the assertions $\Phi \in \mathscr{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu_X)$ and $Y = (Y_t)_{t\in\mathbb{Z}}$ with Y_t defined by (2.13). In this case, the spectral operator measure of Y is related to the one of X by the following lemma.

Lemma 2.3. *Let $\mathcal{H}_0, \mathcal{G}_0$ be two separable Hilbert spaces and $\Phi \in \mathbb{F}_s(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{H}_0, \mathcal{G}_0)$. Assume that $X \in \mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ and let $Y = F_\Phi(X)$. Then Y has spectral operator measure defined by $\mathrm{d}\nu_Y = \Phi \mathrm{d}\nu_X \Phi^{\mathsf{H}}$. Consequently, if ν_X has operator density g_X with respect to some non-negative measure μ on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, then ν_Y has density $\Phi g_X \Phi^{\mathsf{H}}$ with respect to μ.*

The following result is a simplified version of [8, Proposition 3.3] specified to the case where the transfer operators are bounded and $\mathbb{G} = \mathbb{Z}$. It will be sufficient for our purpose.

Proposition 2.4 (Composition of filters on weakly stationary time series)**.** *Let $\mathcal{H}_0$, $\mathcal{G}_0$ and $\mathcal{I}_0$ be three separable Hilbert spaces and pick two transfer bounded operator function $\Phi \in \mathbb{F}_s(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{H}_0, \mathcal{G}_0)$ and $\Psi \in \mathbb{F}_s(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{G}_0, \mathcal{I}_0)$. Let X be a centered weakly stationary $\mathcal{H}_0$-valued process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator measure ν_X. Suppose that $X \in \mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$. Then, we have $X \in \mathcal{S}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $F_\Phi(X) \in \mathcal{S}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$, and in this case, we have*

$$F_\Psi \circ F_\Phi(X) = F_{\Psi\Phi}(X). \tag{2.16}$$

2.8 Functional time series

In this section, we consider $\mathcal{H}_0 = L^2(\mathsf{V}, \mathcal{V}, \xi)$ for some measure space $(\mathsf{V}, \mathcal{V}, \xi)$, and we assume that ξ is non-negative and σ-finite and that $\mathcal{H}_0$ is separable. We will denote by $(\phi_n)_{n\in\mathbb{N}}$ an arbitrary Hilbert basis of $\mathcal{H}_0$. This framework is known as *functional* (or *curve*) data analysis. In this case, it is common to consider an $\mathcal{H}_0$-valued random variable Y defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as the realization of some continuous time process $\{Y(v) \,:\, v \in \mathsf{V}\}$. This representation holds in the sense that there always exists a version of Y which is jointly measurable in $\mathsf{V} \times \Omega$ as will be stated in Proposition 2.7. This allows us to define a *cross-spectral density function* containing the spectral information of the functional time series evaluated at two points $v, v' \in \mathsf{V}$. In order to get to this point, we need some results on Hilbert-Schmidt integral operators.

2.8.1 Hilbert-Schmidt integral operators

For $\mathscr{K} \in L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$, we define the *integral operator* with kernel $\mathscr{K}$ as the unique operator K on $\mathcal{H}_0$ satisfying

$$Kf : v \mapsto \int_{\mathsf{V}} \mathscr{K}(v, v') f(v') \, \xi(\mathrm{d}v') , \quad \text{for all } f \in \mathcal{H}_0 .$$

In this case, $K \in \mathcal{S}_2(\mathcal{H}_0)$ with $\|K\|_2^2 = \int_{\mathsf{V}^2} |\mathscr{K}|^2 \, \mathrm{d}\xi^{\otimes 2}$ and, through this relation, the spaces $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ and $\mathcal{S}_2(\mathcal{H}_0)$ are isometrically isomorphic. In particular, any Hilbert-Schmidt operator on $\mathcal{H}_0$ is an integral operator and is associated to a unique kernel in $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$. It is easy to check that the kernel associated to K^{H} is the adjoint kernel $(v, v') \mapsto \overline{\mathscr{K}(v', v)}$ and that $\mathscr{K} \in L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ is the kernel of a Hilbert-Schmidt operator K if and only if

$$\phi_i^{\mathsf{H}} K \phi_j = \int \mathscr{K}(v, v') \, \overline{\phi_i(v)} \phi_j(v') \, \xi(\mathrm{d}v) \xi(\mathrm{d}v') \;, \quad \text{for all } i, j \in \mathbb{N} \;.$$

A special case of interest is when we consider an operator $G \in \mathcal{S}_1^+(\mathcal{H}_0)$. In this case, G is also a Hilbert-Schmidt operator and therefore is also associated to a kernel, say $\mathscr{G} \in L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$. However, because we can write $G = HH^{\mathsf{H}}$ for some well (non-uniquely) chosen $H \in \mathcal{S}_2(\mathcal{H}_0)$, we can be more precise in describing the kernel, as stated in the following lemma, in which, for instance, one can choose $H = H^{\mathsf{H}} = G^{1/2}$.

Lemma 2.5. *Let $\mathcal{H}_0 = L^2(\mathsf{V}, \mathcal{V}, \xi)$ be a separable Hilbert space, $G \in \mathcal{S}_1^+(\mathcal{H}_0)$ and $H \in \mathcal{S}_2(\mathcal{H}_0)$ such that $G = HH^{\mathsf{H}}$. Let $\mathscr{G}, \mathscr{H} \in L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ be the kernels of G and H respectively. Then for $\xi^{\otimes 2} -$ a.e. $(v, v') \in \mathsf{V}^2$,*

$$\mathscr{G}(v, v') = \int \mathscr{H}(v, v'') \, \overline{\mathscr{H}(v', v'')} \, \xi(\mathrm{d}v'') \;. \tag{2.17}$$

Let us now consider an $\mathcal{S}_2(\mathcal{H}_0)$-valued function K defined on a measurable space $(\Lambda, \mathcal{A})$. As explained previously, for any $\lambda \in \Lambda$, $K(\lambda)$ can be seen as an integral operator associated to a kernel $\mathscr{K}(\cdot; \lambda) \in L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$. However it is useful to consider the mapping $(v, v', \lambda) \mapsto \mathscr{K}(v, v'; \lambda)$ and to make this mapping measurable on $(\mathsf{V}^2 \times \Lambda, \mathcal{V}^{\otimes 2} \otimes \mathcal{A})$. For convenience, we introduce the following definition to refer to such a measurable mapping.

Definition 2.4 (Joint kernel function)**.** *Let $\mathcal{H}_0 = L^2(\mathsf{V}, \mathcal{V}, \xi)$ be a separable Hilbert space, with $(\mathsf{V}, \mathcal{V}, \xi)$ a σ-finite measured space. Let K be a measurable function from $(\Lambda, \mathcal{A})$ to $(\mathcal{S}_2(\mathcal{H}_0), \mathcal{B}(\mathcal{S}_2(\mathcal{H}_0)))$ and $\mathscr{K} : (v, v', \lambda) \mapsto \mathscr{K}(v, v'; \lambda)$ be measurable from $(\mathsf{V}^2 \times \Lambda, \mathcal{V}^{\otimes 2} \otimes \mathcal{A})$ to $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that, for all $\lambda \in \Lambda$ and $f \in \mathcal{H}_0$,*

$$K(\lambda) f : v \mapsto \int \mathscr{K}(v, v'; \lambda) \, f(v') \, \xi(\mathrm{d}v') \;. \tag{2.18}$$

Then we call $\mathscr{K}$ the Λ-joint kernel function of K.

In Definition 2.4, the Λ-joint kernel function of K is unique in the sense two Λ-joint kernel functions $\mathscr{K}$ and $\tilde{\mathscr{K}}$ of the same $\mathcal{S}_2(\mathcal{H}_0)$-valued function K must satisfy that, for all $\lambda \in \Lambda$, $\mathscr{K}(\cdot; \lambda) = \tilde{\mathscr{K}}(\cdot; \lambda)$, $\xi^{\otimes 2} -$ a.e. The following lemma asserts that a Λ-joint kernel function of K always exists and provides additional properties in two special cases that will be of interest.

Proposition 2.6. *Let $\mathcal{H}_0$ and K be as in Definition 2.4. Then K admits a Λ-joint kernel function $\mathscr{K}$. Moreover the two following assertions hold for any non-negative measure μ on $(\Lambda, \mathcal{A})$.*

(i) If $K \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$, then $\mathscr{K} \in \mathcal{L}^2(\mathsf{V}^2 \times \Lambda, \mathcal{V}^{\otimes 2} \otimes \mathcal{A}, \xi^{\otimes 2} \otimes \mu)$.

(ii) If $K \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1^+(\mathcal{H}_0), \mu)$, then $\mathscr{K}$ satisfies

$$\int \left(\int \left| \mathscr{K}(v, v'; \lambda) \right|^2 \, \xi(\mathrm{d}v) \xi(\mathrm{d}v') \right)^{1/2} \mu(\mathrm{d}\lambda) < +\infty \;. \tag{2.19}$$

2.8.2 $L^2(\mathsf{V}, \mathcal{V}, \xi)$-valued weakly stationary time series

We first show that we can always find a version of an $\mathcal{H}_0$-valued random variable which is jointly measurable on $\mathsf{V} \times \Omega$.

Proposition 2.7. *Let Y be an $\mathcal{H}_0$-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then Y admits a version $(v, \omega) \mapsto \tilde{Y}(v, \omega)$ jointly measurable on $(\mathsf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F})$.*

Hence, in the following an $\mathcal{H}_0$-valued random variable Y will always be assumed to be represented by a $\mathsf{V} \times \Omega \to \mathbb{C}$-measurable function $\tilde{Y}$. If, moreover, $Y \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, then, by Fubini's theorem, we can see $\tilde{Y}$ as an element of $L^2(\mathsf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F}, \xi \otimes \mathbb{P})$, and we can write

$$\tilde{Y}(v, \omega) = \sum_{k \in \mathbb{N}} \langle Y(\omega), \phi_k \rangle \, \phi_k(v) \;,$$

where the convergence holds in $L^2(\mathsf{V}\times\Omega, \mathcal{V}\otimes\mathcal{F}, \xi\otimes\mathbb{P})$. As expected, in this case, the covariance operator $\operatorname{Cov}(Y)$ is an integral operator with kernel $(v, v') \mapsto \operatorname{Cov}\left(\tilde{Y}(v,\cdot), \tilde{Y}(v',\cdot)\right)$. It is then tempting to write that $\operatorname{Var}\left(\tilde{Y}(v,\cdot)\right)$ is equal to the kernel of the integral operator $\operatorname{Cov}(Y)$ on the diagonal $\{v = v' : (v, v') \in \mathsf{V}^2\}$. However, because this diagonal set has null $\xi^{\otimes 2}$-measure, this "equality" is meaningless in the framework of Hilbert-Schmidt operators. In the following lemma we make this statement rigorous by relying on a decomposition of the form $\operatorname{Cov}(Y) = KK^{\mathsf{H}}$ for some $K \in \mathcal{S}_2(\mathcal{H}_0)$.

Lemma 2.8. *Let Y be a random variable valued in a separable Hilbert space $\mathcal{H}_0 = L^2(\mathsf{V}, \mathcal{V}, \xi)$, with ξ a σ-finite measure on $(\mathsf{V}, \mathcal{V})$. Let $K \in \mathcal{S}_2(\mathcal{H}_0)$ and denote by $\mathscr{K}$ its kernel in $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$. Suppose that $\operatorname{Cov}(Y) = KK^{\mathsf{H}}$. Then, we have, for $\xi -$ a.e. $v \in \mathsf{V}$,*

$$\mathbb{E}\left[\left|\tilde{Y}(v,\cdot)\right|^2\right] = \int \left|\mathscr{K}(v, v')\right|^2 \, \xi(\mathrm{d}v') = \|\mathscr{K}(v,\cdot)\|^2_{\mathcal{H}_0} \,, \tag{2.20}$$

where $\tilde{Y}$ is a version of Y in $L^2(\mathsf{V}\times\Omega, \mathcal{V}\otimes\mathcal{F}, \xi\otimes\mathbb{P})$.

Let now $X = (X_t)_{t\in\mathbb{Z}}$ be an $\mathcal{H}_0$-valued weakly stationary time series defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator measure ν_X and, for each $t \in \mathbb{Z}$, denote by $\tilde{X}_t$ a version of X_t in $L^2(\mathsf{V}\times\Omega, \mathcal{V}\otimes\mathcal{F}, \xi\otimes\mathbb{P})$. Note that, for all $n \in \mathbb{N}$, $\left(\sum_{k=1}^n \langle X_t, \phi_k\rangle \phi_k(v)\right)_{t\in\mathbb{Z}}$ are $\mathbb{C}$-valued sequences which are $(v \in \mathsf{V})$-jointly weakly stationary. Hence, from what precedes, we get that there exists a ξ-full measure set $\mathsf{V}_0 \in \mathcal{V}$ such that $(\tilde{X}_t(v,\cdot))_{t\in\mathbb{Z}}$ are $\mathbb{C}$-valued sequences which are $(v \in \mathsf{V}_0)$-jointly weakly stationary. The next proposition shows that these time series admit spectral densities with respect to any non-negative measure that dominates the spectral measure of X.

Proposition 2.9. *Let $\mathcal{H}_0 = L^2(\mathsf{V}, \mathcal{V}, \xi)$ be a separable Hilbert space, with ξ a σ-finite measure on $(\mathsf{V}, \mathcal{V})$. Let $X = (X_t)_{t\in\mathbb{Z}}$ be an $\mathcal{H}_0$-valued weakly stationary time series defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator measure ν_X. Suppose that μ is a finite non-negative measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ that dominates ν_X. Let $g_X = \frac{\mathrm{d}\nu_X}{\mathrm{d}\mu}$ and $\mathscr{g}_X : (v, v', \lambda) \mapsto \mathscr{g}_X(v, v'; \lambda)$ be its $\mathbb{T}$-joint kernel function as in Definition 2.4. Then for $\xi^{\otimes 2} -$ a.e. $(v, v') \in \mathsf{V}^2$, the cross spectral measure of the time series $(\tilde{X}_t(v,\cdot))_{t\in\mathbb{Z}}$ and $(\tilde{X}_t(v',\cdot))_{t\in\mathbb{Z}}$ admits the density $\lambda \mapsto \mathscr{g}_X(v, v'; \lambda)$ with respect to μ.*

Proposition 2.9 leads to the following.

Definition 2.5 (Cross-spectral density function)**.** *Under the assumptions of Proposition 2.9, we call $\mathscr{g}_X$ the* cross-spectral density function *and with respect to μ.*

3 Hilbert valued FIARMA processes

In this section, we propose a definition of FIARMA processes valued in a separable Hilbert space extending the definition of [19, Section 4] to an operator long memory parameter. First we recall known results on the existence of ARMA processes, then study filtering by a fractional integration operator transfer function. This naturally leads to the definition of FIARMA processes which we compare to the processes introduced in [9].

3.1 ARMA processes

Let p be a positive integer and consider the p-order linear recursive equation

$$Y_t = \sum_{k=1}^p A_k Y_{t-k} + \epsilon_t \,, \quad t \in \mathbb{Z} \,, \tag{3.1}$$

where $\epsilon = (\epsilon_t)_{t\in\mathbb{Z}}$ is an input sequence valued in $\mathcal{H}_0$ and $A_1, \ldots, A_p \in \mathcal{L}_b(\mathcal{H}_0)$. If ϵ is a white noise (that is, it is centered and weakly stationary with a constant spectral density operator), then Equation (3.1) is called a (functional) p-order auto-regressive (AR(p)) equation. If ϵ can be written for some positive integer q as

$$\epsilon_t = Z_t + \sum_{k=1}^q B_k Z_{t-k} \,, \quad t \in \mathbb{Z} \,, \tag{3.2}$$

where $Z = (Z_t)_{t\in\mathbb{Z}}$ is a centered white noise valued in $\mathcal{H}_0$ and $B_1, \dots, B_p \in \mathcal{L}_b(\mathcal{H}_0)$, then ϵ is called a (functional) moving average process of order q (MA(q)) and Eq. (3.1) is called a (functional) (p, q)-order auto-regressive moving average (ARMA(p, q)) equation. Note that (3.2) can be written as the spectral domain filtering

$$\epsilon(\mathrm{d}\lambda) = \mathbb{\theta}(\mathrm{e}^{-\mathrm{i}\lambda})\hat{Z}(\mathrm{d}\lambda) \quad \text{with} \quad \mathbb{\theta}(z) = \mathrm{Id}_{\mathcal{H}_0} + \sum_{k=1}^{p} B_k z^k \; . \tag{3.3}$$

Weakly stationary solutions of AR(p) or ARMA(p,q) equations are called AR(p) or ARMA(p,q) processes. The existence (and uniqueness) of a weakly stationary solution to Eq. (3.1) is given by the following result where $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ is the complex unit circle.

Theorem 3.1. *Let $\epsilon = (\epsilon_t)_{t\in\mathbb{Z}}$ be a centered weakly stationary process valued in $\mathcal{H}_0$ and $A_1, \dots, A_p \in \mathcal{L}_b(\mathcal{H}_0)$ satisfying the condition*

$$\mathbb{\phi}(z) = \mathrm{Id}_{\mathcal{H}_0} - \sum_{k=1}^{p} A_k z^k \quad \textit{is invertible in } \mathcal{L}_b(\mathcal{H}_0) \textit{ for all } z \in \mathbb{U}. \tag{3.4}$$

Then, setting $\Phi(\lambda) = \mathbb{\phi}(\mathrm{e}^{-\mathrm{i}\lambda})$ for all $\lambda \in \mathbb{R}$, the processes $Y = F_{\Phi^{-1}}(\epsilon)$ is well defined and is the unique weakly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ satisfying (3.1). Moreover, the process Y admits the linear representation

$$Y_t = \sum_{k\in\mathbb{Z}} \mathrm{P}_k \epsilon_{t-k} \; , \quad t \in \mathbb{Z} \; , \tag{3.5}$$

where $(\mathrm{P}_k)_{k\in\mathbb{Z}}$ is a sequence valued in $\mathcal{L}_b(\mathcal{H}_0)$ whose operator norms have exponential decays at $\pm\infty$.

Theorem 3.1 is usually proven in the Banach space valued case by constructing the explicit expansion (3.5) from algebraic arguments (see [24, Corollary 2.2] and the references in the proof). In Section 4 we provide a very short proof relying on linear filtering in the spectral domain.

Let us introduce some notation which will be useful in the following.

Definition 3.1 (Polynomial sets $\mathcal{P}_d(\mathcal{H}_0)$ and $\mathcal{P}_d^*(\mathcal{H}_0)$)**.** *For any integer $d \in \mathbb{N}$, let $\mathcal{P}_d(\mathcal{H}_0)$ denote the set of polynomials $\mathbb{p}$ of degree d with coefficients in $\mathcal{L}_b(\mathcal{H}_0)$ and such that $\mathbb{p}(0) = \mathrm{Id}_{\mathcal{H}_0}$. Further denote by $\mathcal{P}_d^*(\mathcal{H}_0)$ the subset of all $\mathbb{p} \in \mathcal{P}_d(\mathcal{H}_0)$ which are invertible on $\mathbb{U}$ (as $\mathbb{\phi}$ in (3.4)).*

An $\mathcal{H}_0$-valued ARMA(p, q) process X can thus be concisely defined as follows.

Definition 3.2 (Hilbert valued ARMA process)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, p, q be non-negative integers, $\mathbb{\theta} \in \mathcal{P}_q(\mathcal{H}_0)$, $\mathbb{\phi} \in \mathcal{P}_p^*(\mathcal{H}_0)$ and Z be a (centered) $\mathcal{H}_0$-valued white noise. The $\mathcal{H}_0$-valued weakly stationary time series with spectral representation given by*

$$\hat{X}(\mathrm{d}\lambda) = [\mathbb{\phi}(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\mathbb{\theta}(\mathrm{e}^{-\mathrm{i}\lambda})\hat{Z}(\mathrm{d}\lambda) \; ,$$

where $\hat{Z}$ is the spectral representation of Z, is called an ARMA(p, q) process.

By Lemma 2.3, in this case, if Σ denotes the covariance operator of Z, then X admits the spectral density

$$g_X(\lambda) = [\mathbb{\phi}(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\mathbb{\theta}(\mathrm{e}^{-\mathrm{i}\lambda})\Sigma[\mathbb{\phi}^{-1}(\mathrm{e}^{-\mathrm{i}\lambda})\mathbb{\theta}(\mathrm{e}^{-\mathrm{i}\lambda})]^{\mathsf{H}}$$

with respect to the Lebesgue measure. The following results will be useful.

Proposition 3.2. *Let $\mathcal{H}_0$ be a separable Hilbert space and X be an ARMA(p,q) process defined by $\hat{X}(\mathrm{d}\lambda) = [\mathbb{\phi}(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\mathbb{\theta}(\mathrm{e}^{-\mathrm{i}\lambda})\hat{Z}(\mathrm{d}\lambda)$ with $\mathbb{\theta} \in \mathcal{P}_q(\mathcal{H}_0)$, $\mathbb{\phi} \in \mathcal{P}_p^*(\mathcal{H}_0)$ and Z an $\mathcal{H}_0$-valued white noise with covariance operator Σ. Then there exists $\eta \in (0, \pi)$ and $k : (-\eta, \eta) \to \mathcal{S}_2(\mathcal{H}_0)$ continuous and bounded such that, for* Leb − a.e. *$\lambda \in (-\eta, \eta)$, we have*

$$g_X(\lambda) = h(\lambda)\,[h(\lambda)]^{\mathsf{H}} \quad \textit{with} \quad h(\lambda) = [\mathbb{\phi}(1)]^{-1}\mathbb{\theta}(1)\Sigma^{1/2} + \lambda\, k(\lambda) \; . \tag{3.6}$$

In the case where $\mathcal{H}_0 = L^2(\mathsf{V}, \mathcal{V}, \xi)$ for some σ-finite measure space $(\mathsf{V}, \mathcal{V})$, then the $(-\eta, \eta)$-joint kernel function $\hbar$ in $L^2(\mathsf{V}^2 \times (-\eta, \eta), \mathcal{V}^{\otimes 2} \otimes \mathcal{B}(-\eta, \eta), \xi^{\otimes 2} \otimes \mathrm{Leb})$ associated to k also satisfies

$$\int_{\mathsf{V}^2} \underset{\lambda\in(-\eta,\eta)}{\text{Leb-essup}} \left|\hbar(v, v'; \lambda)\right|^2 \, \xi(\mathrm{d}v)\, \xi(\mathrm{d}v') < +\infty \; . \tag{3.7}$$

The following lemma indicates that an invertible linear transform of an ARMA process is still an ARMA process. It will be useful in particular in the case where U is an isometry mapping $\mathcal{H}_0$ to a space of the form $\mathcal{G}_0 = L^2(\mathsf{V}, \mathcal{V}, \xi)$.

Lemma 3.3. *Let ξ be a σ-finite measure on $(\mathsf{V}, \mathcal{V})$, $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Let X be an ARMA(p, q) process defined by $\hat{X}(\mathrm{d}\lambda) = [\phi(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\theta(\mathrm{e}^{-\mathrm{i}\lambda})\hat{Z}(\mathrm{d}\lambda)$ with $\theta \in \mathcal{P}_q(\mathcal{H}_0)$, $\phi \in \mathcal{P}_p^*(\mathcal{H}_0)$ and Z an $\mathcal{H}_0$-valued white noise. Then, for any invertible operator $U \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, the process $UX = (UX_t)_{t\in\mathbb{Z}}$ is the $\mathcal{G}_0$-valued ARMA(p, q) process defined by $\widehat{UX}(\mathrm{d}\lambda) = [\tilde{\phi}(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\,\tilde{\theta}(\mathrm{e}^{-\mathrm{i}\lambda})\,\widehat{UZ}(\mathrm{d}\lambda)$, where $\tilde{\theta} := U\theta U^{-1} \in \mathcal{P}_q(\mathcal{G}_0)$ and $\tilde{\phi} := U\phi U^{-1} \in \mathcal{P}_p^*(\mathcal{G}_0)$, and $UZ = (UZ_t)_{t\in\mathbb{Z}}$ is a $\mathcal{G}_0$-valued white noise.*

3.2 Fractional operator integration of weakly stationary processes

In the following, we use the notation $(1-z)^D$ for some $D \in \mathcal{L}_b(\mathcal{H}_0)$ and $z \in \mathbb{C} \setminus [1, \infty)$. This must be understood as

$$(1-z)^D = \exp(D\ln(1-z)) = \sum_{k=0}^{\infty} \frac{1}{k!}\,(D\ln(1-z))^k\,,$$

where ln denotes the principal complex logarithm, so that $z \mapsto \ln(1-z)$ is holomorphic on $\mathbb{C} \setminus [1, \infty)$, and so is $z \mapsto (1-z)^D$, as a $\mathcal{L}_b(\mathcal{H}_0)$-valued function, see [10, Chapter 1] for an introduction on this subject.

Definition 3.3 (Fractional integration operator transfer function)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space and $D \in \mathcal{L}_b(\mathcal{H}_0)$. We define the D-order fractional integration operator transfer function FI_D by*

$$\mathrm{FI}_D(\lambda) = \begin{cases} \left(1-\mathrm{e}^{-\mathrm{i}\lambda}\right)^{-D} & \text{if } \lambda \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using the properties of $z \mapsto (1-z)^D$ recalled previously, we get that FI_D is a mapping from $\mathbb{T}$ to $\mathcal{L}_b(\mathcal{H}_0)$, continuous on $\mathbb{T} \setminus \{0\}$. Then we have $\mathrm{FI}_D \in \mathbb{F}_s(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{H}_0)$ and we can define the filter F_{FI_D} as in (2.15) whose domain of definition are the centered weakly stationary $\mathcal{H}_0$-valued processes $X \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$. Then a *fractionaly integrated autoregressive moving average* (FIARMA) process is simply the output of the filter in the case where X is an ARMA process, as defined in the following.

Definition 3.4 (FIARMA processes)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space and p, q be two non-negative integers. Let $D \in \mathcal{L}_b(\mathcal{H}_0)$, $\theta \in \mathcal{P}_q(\mathcal{H}_0)$, $\phi \in \mathcal{P}_p^*(\mathcal{H}_0)$ and Z be an $\mathcal{H}_0$-valued centered white noise. Let X be the ARMA(p, q) process defined by $\hat{X}(\mathrm{d}\lambda) = [\phi(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\theta(\mathrm{e}^{-\mathrm{i}\lambda})\hat{Z}(\mathrm{d}\lambda)$ and suppose that $X \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$. Then the process defined by $Y = F_{\mathrm{FI}_D}(X)$, or, in the spectral domain, by*

$$\hat{Y}(\mathrm{d}\lambda) = \mathrm{FI}_D(\lambda)\phi^{-1}(\mathrm{e}^{-\mathrm{i}\lambda})\theta(\mathrm{e}^{-\mathrm{i}\lambda})\hat{Z}(\mathrm{d}\lambda)\,, \tag{3.8}$$

is called a FIARMA process of order (p, q) with long memory operator D, shortened as FIARMA(D, p, q).

Remark 3.1. *Definition 3.4 extends the usual definition of univariate ($\mathbb{C}$ or $\mathbb{R}$-valued) ARFIMA(p, d, q) processes to the Hilbert-valued case. In the general case we use the acronym FIARMA to indicate the order of the operators in the definition (3.8), where the fractional integration operator appears on the left of the autoregressive operator, itself appearing on the left of the moving average operator. We also respected this order in the list of parameters (D, p, q). Of course, one can also define an ARFIMA(p, D, q) process as the solution of (3.1) with ϵ defined as a FIARMA$(0, D, q)$ process but the ARFIMA(p, D, q) process do not coincide with the FIARMA(p, D, q) process (this is already the case in finite dimension larger than 1). Observe that in the univariate case all the operators commute and FIARMA and ARFIMA boils down to the same definition. Note also that Definition 3.4 extends the definition of ARFIMA curve time series proposed in [19] in the case where $\mathcal{H}_0$ is an $L^2(\mathcal{C}, \mathcal{B}(\mathcal{C}), \mathrm{Leb})$ for some compact set $\mathcal{C} \subset \mathbb{R}$ and where D is a scalar operator, that is $D = M_{\mathrm{d}}$ for some constant function $\mathrm{d} \equiv d$ with $-1/2 < d < 1/2$. We will see below that, in this case, we have $X \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$ for any ARMA process X, see Remark 3.2 below.*

Since FI_D has a singularity at the null frequency, we want to provide conditions to ensure that, given a weakly stationary process X, the filter with transfer function FI_D applies to X, or, as explained in Section 2.7, we look for conditions implying that $X \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$. For instance in the scalar case, it is well known that if X has a positive and continuous spectral density at the null frequency, then $F_{\mathrm{FI}_d}(X)$ is well defined if and only if $d < 1/2$. The Hilbert valued case relies on the singular value decomposition of D, which we will assume to be normal. Based on the spectral decomposition of a normal operator, we derive, in the following result, a necessary and sufficient condition involving the spectral operator density of X and the spectral decomposition of D.

Theorem 3.4. *Let $\mathcal{H}_0$ be a separable Hilbert space, $D \in \mathcal{L}_b(\mathcal{H}_0)$ and $X = (X_t)_{t\in\mathbb{Z}}$ be an $\mathcal{H}_0$-valued weakly stationary time series defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator measure ν_X. Suppose that D is normal, with singular value function* d *on $\mathcal{G}_0 := L^2(\mathsf{V}, \mathcal{V}, \xi)$ and decomposition operator U. Let μ be a nonnegative measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ which dominates ν_X and let $h \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{S}_2(\mathcal{G}_0), \mu)$ such that $\lambda \mapsto h(\lambda)[h(\lambda)]^{\mathsf{H}}$ is the spectral operator density function of $UX = (UX_t)_{t\in\mathbb{Z}}$ with respect to μ, that is,*

$$h(\lambda)[h(\lambda)]^{\mathsf{H}} = U\, g_X(\lambda)\, U^{\mathsf{H}} \quad \text{for } \mu - \text{a.e. } \lambda \in \mathbb{T} ,$$

where $g_X = \frac{\mathrm{d}\nu_X}{\mathrm{d}\mu}$. Let $\hbar$ denote the $\mathbb{T}$-joint kernel function of h. Then the three following assertions are equivalent.

(i) We have $X \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) There exists $\eta \in (0, \pi)$ arbitrarily small such that

$$\int_{\mathsf{V}^2\times(-\eta,\eta)} |\lambda|^{-2\Re(\mathrm{d}(v))} \left|\hbar(v, v'; \lambda)\right|^2 \xi(\mathrm{d}v)\xi(\mathrm{d}v')\mu(\mathrm{d}\lambda) < \infty . \tag{3.9}$$

(iii) Equation (3.9) holds for all $\eta \in (0, \pi)$.

Remark 3.2. *If the dominating measure μ is the Lebesgue measure and if* d *is a constant function, $\mathrm{d} \equiv d$ for some $d < 1/2$ then the integral in (3.9) is bounded from above by*

$$\frac{2\eta^{1-2d}}{1-2d} \operatorname*{Leb-essup}_{\lambda\in(-\eta,\eta)} \int_{\mathsf{V}^2} \left|\hbar(v, v'; \lambda)\right|^2 \xi(\mathrm{d}v)\xi(\mathrm{d}v') = \frac{2\eta^{1-2d}}{1-2d} \operatorname*{Leb-essup}_{\lambda\in(-\eta,\eta)} \left\|g_X(\lambda)\right\|_1 .$$

Thus, in this case, a sufficient condition for $X \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$ is to have that $\left\|g_X\right\|_1$ is locally bounded around the null frequency. This is always the case if X is an ARMA process as in Definition 3.2.

Theorem 3.5. *Let $\mathcal{H}_0$ be a separable Hilbert space and $D \in \mathcal{L}_b(\mathcal{H}_0)$. Suppose that X is an $\mathcal{H}_0$-valued ARMA(p,q) process defined by $\hat{X}(\mathrm{d}\lambda) = \left[\phi(\mathrm{e}^{-\mathrm{i}\lambda})\right]^{-1} \theta(\mathrm{e}^{-\mathrm{i}\lambda})\hat{Z}(\mathrm{d}\lambda)$ with $\theta \in \mathcal{P}_q(\mathcal{H}_0)$, $\phi \in \mathcal{P}^*_p(\mathcal{H}_0)$ and Z a white noise with covariance operator Σ. Suppose that D is normal, with singular value function* d *on $\mathcal{G}_0 := L^2(\mathsf{V}, \mathcal{V}, \xi)$ and decomposition operator U. Let $\sigma_W : v \mapsto \left(\mathbb{E}\left[\left|\tilde{W}(v, \cdot)\right|^2\right]\right)^{1/2}$ where $\tilde{W}$ is a jointly measurable version of the L^2 $\mathcal{G}_0$-valued variable $W = U[\phi(1)]^{-1}\theta(1)Z_0$. Consider the following assertions.*

(i) We have $X \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) We have $\Re(\mathrm{d}) < 1/2$, $\xi -$ a.e. on $\{\sigma_W > 0\}$.

(iii) We have $\displaystyle\int_{\{\Re(\mathrm{d})<1/2\}} \frac{\sigma_W^2(v)}{1 - 2\Re(\mathrm{d}(v))} \xi(\mathrm{d}v) < +\infty$.

(iv) We have $\Re(\mathrm{d}) < 1$, $\xi -$ a.e.

(v) We have $\phi(z) = \theta(z) = \mathrm{Id}$ for all $z \in \mathbb{C}$ (i.e. $X = Z$).

Then (i) implies (ii) and (iii). Conversely, if (iv) or (v) hold, then (i) is implied by (ii) and (iii).

Remark 3.3. *If $\Re(\mathrm{d}) < 1/2$, $\xi -$ a.e. then both (ii) and (iv) hold, and (iii) simplifies to $\displaystyle\int \frac{\sigma_W^2(v)}{1 - 2\Re(\mathrm{d}(v))} \xi(\mathrm{d}v) < +\infty$. Hence, applying our result, we get that (i) is implied by* *(vi)* $\Re(\mathrm{d}) < 1/2$, $\xi-$a.e. *and* $\displaystyle\int \frac{\sigma_W^2(v)}{1-2\Re(\mathrm{d}(v))}\,\xi(\mathrm{d}v) < +\infty$,

which we think is the most useful consequence of this theorem. However it is important to note that (vi) is not necessary as our result says that, under assertion (v) (X is a white noise), only the sufficient conditions (ii) and (iii) are necessary (and it is easy to find D and Σ such that (ii) and (iii) holds but (vi) does not). Observe also that since

$$\int \sigma_W^2(v)\,\xi(\mathrm{d}v) = \mathbb{E}\left[\|W\|_{\mathcal{G}_0}^2\right] \leq \left\|[\phi(1)]^{-1}\theta(1)\right\|_{\mathcal{L}_b(\mathcal{H}_0)} \mathbb{E}\left[\|Z_0\|_{\mathcal{H}_0}^2\right] < +\infty\ ,$$

Condition (ii) is immediately satisfied if $\Re(\mathrm{d})$ *uniformly stay away from* $1/2$ *on* $\{\sigma_W > 0\}$*, that is,* $\Re(\mathrm{d}) \leq 1/2 - \eta\ \xi-$a.e. *on* $\{\sigma_W > 0\}$ *for some* $\eta > 0$*. In the n-dimensional case with n finite, we have* $\mathsf{V} = \{1,\dots,n\}$*,* ξ *is the counting measure on* V *and U can be interpreted as a* $n\times n$ *unitary matrix, and* d *and* σ_W *as n-dimensional vectors. Condition (ii) then says that* $\Re(\mathrm{d}(k)) < 1/2$ *on the components* $k \in \{1,\dots,n\}$ *such that* $\sigma_W(k) > 0$*, and Condition (iii) always follows from (ii). For the real univariate case (*$n = 1$*,* $D = d \in \mathbb{R}$*), Condition (ii) says that* $d < 1/2$ *or* $\sigma_W = 0$ *and the latter happens if and only if* $\Sigma = 0$ *(Z is the null process) or* $\theta(1) = 0$ *(the MA operator contains a difference operator of order larger than or equal to 1). In particular we find the usual* $d < 1/2$ *condition for the existence of a weakly stationary ARFIMA*(p,d,q) *model in the case where the underlying ARMA*(p,q) *process is invertible (*θ *does not vanish on the unit circle).*

3.3 Other long-memory processes

Several non-equivalent definitions of long rang dependence or long memory are available in the literature for time series. Some approaches focus on the behavior of the auto-covariance function at large lags, others on the spectral density at low frequencies, see [22, Section 2.1] and the references therein. Separating short range from long range dependence is often made more natural within a particular class of models. For instance, for a Hilbert-valued process $Y = (Y_t)_{t\in\mathbb{Z}}$, one may rely on a causal linear representation, namely

$$Y_t = \sum_{k=0}^{\infty} \mathrm{P}_k \epsilon_{t-k}\ , \quad t \in \mathbb{Z} \quad \text{i.e.} \quad \hat{Y}(\mathrm{d}\lambda) = \left(\sum_{k=0}^{\infty} \mathrm{P}_k \mathrm{e}^{-\mathrm{i}\lambda k}\right)\hat{\epsilon}(\mathrm{d}\lambda)\ , \tag{3.10}$$

where $\epsilon = (\epsilon_t)_{t\in\mathbb{Z}}$ is a centered white noise valued in the separable Hilbert space $\mathcal{H}_0$ and $(\mathrm{P}_k)_{k\in\mathbb{Z}}$ is a sequence of $\mathcal{L}_b(\mathcal{H}_0)$ operators. Then, by isometry, the first infinite sum appearing in (3.10) converges in $\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$ if and only if the second one converges in $\mathsf{L}^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathcal{L}_b(\mathcal{H}_0),\nu_\epsilon)$. A sufficient condition for these convergences to hold is $\sum_{k=0}^{\infty}\|\mathrm{P}_k\|_{\mathcal{L}_b(\mathcal{H}_0)} < +\infty$, and this assumption is referred to as the *short-range dependence* (or short memory) case (for example ARMA processes), in contrast to *long range dependence* (long-memory) case, for which $\sum_{k=0}^{\infty}\|\mathrm{P}_k\|_{\mathcal{L}_b(\mathcal{H}_0)} = +\infty$, under which the convergences in (3.10) are no longer granted. In [9], the case where $\mathrm{P}_k = (k+1)^{-N}$ for some normal operator $N \in \mathcal{L}_b(\mathcal{H}_0)$ is investigated and the following result is obtained.

Lemma 3.6. *Let* $\mathcal{H}_0$ *be a separable Hilbert space,* $N \in \mathcal{L}_b(\mathcal{H}_0)$ *be a normal operator with singular value function* n *on* $\mathcal{G}_0 := L^2(\mathsf{V},\mathcal{V},\xi)$ *and decomposition operator U. Let* $\mathrm{h} : v \mapsto \Re(\mathrm{n}(v))$*. Let* $\epsilon := (\epsilon_t)_{t\in\mathbb{Z}}$ *be a white noise in* $\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$ *and* $\sigma_W^2 : s \mapsto \mathbb{E}\left[\left|\tilde{W}(v,\cdot)\right|^2\right]$*, where* $\tilde{W}$ *is a jointly measurable version of* $W = U\epsilon_0$*. Suppose that*

$$\mathrm{h} > \frac{1}{2}\ \xi\text{-a.e.} \quad \text{and} \quad \int_{\mathsf{V}} \frac{\sigma_W^2(v)}{2\mathrm{h}(v)-1}\,\xi(\mathrm{d}v) < +\infty\ . \tag{3.11}$$

Then, for all $t \in \mathbb{Z}$*,*

$$Y_t = \sum_{k=0}^{+\infty}(k+1)^{-N}\epsilon_{t-k} \tag{3.12}$$

converges in $\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$*. If, moreover,* $(\epsilon_k)_{k\in\mathbb{Z}}$ *is an i.i.d. sequence, then the convergence also holds a.s.*

In [9, Theorem 2.1], the author also studies the partial sums of the process (3.12) and exhibits asymptotic properties which naturally extend the usual behavior observed for univariate long-memory processes. In the following, we explain how the process (3.12) can be related to a FIARMA(D,0,0) process. First we prove the analogous of Lemma 3.6, namely, that that Condition (3.11) implies the existence of this FIARMA process.

Lemma 3.7. *Let N, ϵ, h and σ_W be as in Lemma 3.6. Set $D = \mathrm{Id}_{\mathcal{H}_0} - N$. Then Condition (3.11) implies $\epsilon \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$.*

We can now state a result which shows that the two process defined by Lemmas 3.6 and 3.7 (3.12) are closely related up to a bounded operator C and to an additive short-memory process.

Proposition 3.8. *Under the assumptions of Lemma 3.6, defining $Y = (Y_t)_{t\in\mathbb{Z}}$ by (3.12), there exists $C \in \mathcal{L}_b(\mathcal{H}_0)$ and $(\Delta_k)_{k\in\mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0)^{\mathbb{N}}$ with $\sum_{k=0}^{+\infty} \|\Delta_k\|_{\mathcal{L}_b(\mathcal{H}_0)} < +\infty$ such that*

$$F_{\mathrm{FI}_D}(\epsilon) = C\,Y + Z \;,$$

where Z is the short-memory process defined, for all $t \in \mathbb{Z}$, by $Z_t = \sum_{k=0}^{\infty} \Delta_k \epsilon_{t-k}$.

4 Postponed proofs

4.1 Proofs of Section 2.8.1

Proof of Lemma 2.5. We first prove that $(v, v') \mapsto \int \mathscr{H}(v, v'')\,\overline{\mathscr{H}(v', v'')}\ \xi(\mathrm{d}v'')$ is in $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$. By the Cauchy-Schwartz inequality, we have, for all $(v, v') \in \mathsf{V}^2$,

$$\left(\int \left|\mathscr{H}(v, v'')\,\overline{\mathscr{H}(v', v'')}\right|\ \xi(\mathrm{d}v'')\right)^2 \le \left(\int \left|\mathscr{H}(v, v'')\right|^2\ \xi(\mathrm{d}v'')\right)\left(\int \left|\mathscr{H}(v', v'')\right|^2\ \xi(\mathrm{d}v'')\right)$$

and, integrating the right-hand side with respect to $\xi(\mathrm{d}v)$ and $\xi(\mathrm{d}v')$ and using the fact that $\displaystyle\int |\mathscr{H}|^2\ \mathrm{d}\xi^{\otimes 2} = \|h\|_2^2$ we get that

$$\int \left(\int \left|\mathscr{H}(v, v'')\,\overline{\mathscr{H}(v', v'')}\right|\ \xi(\mathrm{d}v'')\right)^2\ \xi(\mathrm{d}v')\,\xi(\mathrm{d}v) \le \|h\|_2^4 < +\infty\;. \tag{4.1}$$

Hence $(v, v') \mapsto \int \mathscr{H}(v, v'')\,\overline{\mathscr{H}(v', v'')}\ \xi(\mathrm{d}v'')$ is well defined and is in $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$.

Now, for all $f \in \mathcal{H}_0$ and $v \in \mathsf{V}$,

$$\begin{aligned}\left(\int \left|\mathscr{H}(v, v'')\,\overline{\mathscr{H}(v', v'')}\,f(v')\right|\ \xi(\mathrm{d}v'')\xi(\mathrm{d}v')\right)^2 \\ \le \|f\|_{\mathcal{H}_0}^2 \int \left(\int \left|\mathscr{H}(v, v'')\,\overline{\mathscr{H}(v', v'')}\right|\ \xi(\mathrm{d}v'')\right)^2\ \xi(\mathrm{d}v')\end{aligned}$$

which is is finite for $\xi-$a.e. $v \in \mathsf{V}$ by (4.1). Hence, by Fubini's theorem, for $\xi-$a.e. $v \in \mathsf{V}$,

$$\begin{aligned}Gf(v) = HH^{\mathsf{H}}f(v) &= \int \mathscr{H}(v, v'')\left(\int \overline{\mathscr{H}(v', v'')}f(v')\,\xi(\mathrm{d}v')\right)\ \xi(\mathrm{d}v'') \\ &= \int \left(\int \mathscr{H}(v, v'')\overline{\mathscr{H}(v', v'')}\,\xi(\mathrm{d}v'')\right)f(v')\,\xi(\mathrm{d}v')\end{aligned}$$

which implies (2.17) by uniqueness of the kernel associated to G. □

Proof of Proposition 2.6. Define, for all $v, v' \in \mathsf{V}$ and $\lambda \in \mathbb{T}$,

$$\mathscr{K}_n(v, v'; \lambda) := \sum_{0\le i,j\le n} \phi_i^{\mathsf{H}} K(\lambda)\phi_j\ \phi_i(v)\bar{\phi}_j(v')\;,$$

and, for all $\epsilon > 0$,

$$N_\epsilon(\lambda) = \inf\left\{n \in \mathbb{N} :\ \sum_{i \text{ or } j>n} \left|\phi_i^{\mathsf{H}} K(\lambda)\phi_j\right|^2 \le \epsilon\right\}\;.$$

Note that since $\sum_{i,j\in\mathbb{N}} \left|\phi_i^{\mathsf{H}} K(\lambda)\phi_j\right|^2 = \|K(\lambda)\|_2 < \infty$, $N_\epsilon(\lambda)$ is well defined and finite. Now let us define, for all $v, v' \in \mathsf{V}$ and $\lambda \in \mathbb{T}$,

$$\mathscr{K}(v, v'; \lambda) := \lim_{n\to\infty} \mathscr{K}_{N_{2^{-n}}(\lambda)}(v, v'; \lambda) , \tag{4.2}$$

whenever this limit exists in $\mathbb{C}$ and set $\mathscr{K}(v, v'; \lambda) = 0$ otherwise. Since $(\phi_k \otimes \bar{\phi}_{k'})_{k,k'\in\mathbb{N}}$ is a Hilbert basis of $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$, we immediately have that, for any $\lambda \in \Lambda$, $\mathscr{K}_{N_{2^{-n}}(\lambda)}(\cdot; \lambda)$ converges in the sense of this L^2 space to $\sum_{i,j\in\mathbb{N}} \phi_i^{\mathsf{H}} K(\lambda)\phi_j \; \phi_i \otimes \bar{\phi}_j$, and so this limit must be equal to $\mathscr{K}(\cdot; \lambda)$ $\xi^{\otimes 2}$ – a.e.. It follows that, that for any $\lambda \in \Lambda$, for all $i, j \in \mathbb{N}$,

$$\int \mathscr{K}(v, v'; \lambda)\bar{\phi}_i(v)\phi_j(v') \; \xi(\mathrm{d}v)\xi(\mathrm{d}v') = \phi_i^{\mathsf{H}} K(\lambda)\phi_j ,$$

which gives that $K(\lambda)$ is an integral operator associated to the kernel $\mathscr{K}(\cdot; \lambda)$. Since $(v, v', \lambda) \mapsto \mathscr{K}(v, v'; \lambda)$ is measurable by definition, we get that it is the Λ-joint kernel of K as in Definition 2.4. Assertion (i) follows by observing that, if $K \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$, then $(v, v', \lambda) \mapsto \mathscr{K}_n(v, v'; \lambda)$ converges in $L^2(\mathsf{V}^2 \times \Lambda, \mathcal{V}^{\otimes 2} \otimes \mathcal{A}, \xi^{\otimes 2} \otimes \mu)$ and the limit must be equal to $\mathscr{K}$ $\xi^{\otimes 2} \otimes \mu$ – a.e. since for each $\lambda \in \Lambda$, $(v, v') \mapsto \mathscr{K}_n(v, v'; \lambda)$ converges to $\mathscr{K}(\cdot; \lambda)$ in $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$.
It only remains to prove Assertion (ii). Assume that $K \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1^+(\mathcal{H}_0), \mu)$ as in this assertion and let $H \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$ be such that for all $\lambda \in \Lambda$, $K(\lambda) = H(\lambda)H(\lambda)^{\mathsf{H}}$ (for example, by Lemma 2.1, we can take $H(\lambda) = K(\lambda)^{1/2}$) . Then by Assertion (i), the Λ-joint kernel of H satisfies $\mathscr{H} \in L^2(\mathsf{V}^2\times\Lambda, \mathcal{V}^{\otimes 2}\otimes\mathcal{A}, \xi^{\otimes 2}\otimes\mu)$. Using Lemma 2.5 and the Cauchy-Schwartz inequality the integral in (2.19) is bounded from above by $\int |\mathscr{H}(v, v'; \lambda)|^2 \; \xi(\mathrm{d}v)\xi(\mathrm{d}v')\mu(\mathrm{d}\lambda)$ which is finite. ☐

4.2 Proofs of Section 2.8.2

Proof of Proposition 2.7. Decomposing Y on $(\phi_n)_{n\in\mathbb{N}}$, we can define $\tilde{Y}$ on $\mathsf{V} \times \Omega$ by

$$\tilde{Y}(v, \omega) = \begin{cases} \lim\limits_{n\to\infty} S^Y_{N^Y_{2^{-n}}(\omega)}(v, \omega) & \text{if the limit exists in } \mathbb{C} , \\ 0 & \text{otherwise,} \end{cases}$$

where we set, for all $n \in \mathbb{N}$, $\omega \in \Omega$, $v \in \mathsf{V}$ and $\epsilon > 0$,

$$S^Y_n(v, \omega) = \sum_{k=0}^{n} \langle Y(\omega), \phi_k\rangle \, \phi_k(v) \quad \text{and}$$

$$N^Y_\epsilon(\omega) = \inf\left\{n \in \mathbb{N} \;:\; \left\|S^Y_n(\cdot, \omega) - Y(\omega)\right\|^2_{\mathcal{H}_0} \le \epsilon\right\} .$$

It is easy to show that the following assertions hold for all $\omega \in \Omega$:

(i) $N^Y_\epsilon(\omega)$ is well defined in $\mathbb{N}$ for all $\epsilon > 0$,

(ii) $(N^Y_{2^{-n}}(\omega))_n$ is a non-decreasing sequence,

(iii) as $n \to \infty$, $S^Y_{N^Y_{2^{-n}}(\omega)}(\cdot, \omega)$ converges to Y in $\mathcal{H}_0$;

(iv) $S^Y_{N^Y_{2^{-n}}(\omega)}(v, \omega)$ converges to $\tilde{Y}(v, \omega)$ for ξ – a.e. $v \in \mathsf{V}$,

(v) $\tilde{Y}(\cdot, \omega) = Y(\omega)$ (as elements of $\mathcal{H}_0$).

Since S^Y_n is jointly measurable on $\mathsf{V} \times \Omega$ for all $n \in \mathbb{N}$ and N^Y_ϵ is measurable on Ω for all $\epsilon > 0$, we get the result. ☐

Proof of Lemma 2.8. As explained before the statement of the lemma, we have that

$$(v, \omega) \mapsto \tilde{Y}_N(v, \omega) := \sum_{n=0}^{N} \langle Y(\omega), \phi_n\rangle_{\mathcal{H}_0} \; \phi_n(v)$$

converges to $\tilde{Y}$ as $N \to \infty$ in $L^2(\mathsf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F}, \xi \otimes \mathbb{P})$. Let us define, for all $v, v' \in \mathsf{V}$,

$$\mathscr{K}_N(v, v') = \sum_{n=0}^{N} \left\langle \mathscr{K}(\cdot, v'), \phi_n\right\rangle_{\mathcal{H}_0} \phi_n(v)$$

Using that $\mathscr{K} \in L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$, it is easy to show that $\mathscr{K}_N$ converges to $\mathscr{K}$ in $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ as $N \to +\infty$.

By the Cauchy-Schwartz inequality, the mappings $(g,h) \mapsto [v \mapsto \mathbb{E}\left[g(v,\cdot)\,\overline{h(v,\cdot)}\right]]$ and $(g,h) \mapsto [v \mapsto \langle g(v,\cdot), h(v,\cdot)\rangle_{\mathcal{H}_0}]$ sesquilinear continuous from $L^2(\mathsf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F}, \xi \otimes \mathbb{P})$ to $L^1(\mathsf{V}, \mathcal{V}, \xi)$ and from $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ to $L^1(\mathsf{V}, \mathcal{V}, \xi)$, respectively. This, with the two previous convergence result shows that $[v \mapsto \mathbb{E}\left[\left|\tilde{Y}_N(v,\cdot)\right|^2\right]]$ and $[v \mapsto \|\mathscr{K}_N(v,\cdot)\|^2_{\mathcal{H}_0}]$ both converge in $L^1(\mathsf{V}, \mathcal{V}, \xi)$, to $\mathbb{E}\left[\left|\tilde{Y}(v,\cdot)\right|^2\right]$ and $\|\mathscr{K}(v,\cdot)\|^2_{\mathcal{H}_0}$, respectively, that is to the left-hand side and right-hand side of (2.20).

Hence to conclude we only have to show that, for all $v \in \mathsf{V}$,

$$\mathbb{E}\left[\left|\tilde{Y}_N(v,\cdot)\right|^2\right] = \|\mathscr{K}_N(v,\cdot)\|^2_{\mathcal{H}_0} . \tag{4.3}$$

Indeed we can write

$$\begin{aligned}\mathbb{E}\left[\left|\tilde{Y}_N(v,\cdot)\right|^2\right] &= \mathbb{E}\left[\sum_{n,m=0}^{N} \langle Y, \phi_n\rangle_{\mathcal{H}_0} \langle \phi_m, Y\rangle_{\mathcal{H}_0} \phi_n(v)\overline{\phi_m(v)}\right] \\ &= \sum_{n,m=0}^{N} \phi_n^{\mathsf{H}} \operatorname{Cov}(Y)\phi_m \; \phi_n(v)\overline{\phi_m(v)} .\end{aligned}$$

Using $\operatorname{Cov}(Y) = KK^{\mathsf{H}}$ and Fubini's theorem leads to

$$\phi_n^{\mathsf{H}} \operatorname{Cov}(Y)\phi_m = \int \langle \mathscr{K}(\cdot, v''), \phi_n\rangle_{\mathcal{H}_0} \overline{\langle \mathscr{K}(\cdot, v''), \phi_m\rangle_{\mathcal{H}_0}} \, \xi(\mathrm{d}v'') .$$

Inserting this in the previous display, the double sum, put inside the integral in $\xi(\mathrm{d}v'')$, separates into a product of two conjugate terms and we get

$$\mathbb{E}\left[\left|\tilde{Y}_N(v,\cdot)\right|^2\right] = \int \left|\sum_{n=0}^{N} \langle \mathscr{K}(\cdot, v''), \phi_n\rangle_{\mathcal{H}_0} \; \phi_n(v)\right|^2 \xi(\mathrm{d}v'') .$$

so that (4.3) is proven, which concludes the proof. ☐

Proof of Proposition 2.9. For all $n, n' \in \mathbb{N}$ and $\lambda \in \mathbb{T}$, by the Cauchy-Schwartz inequality and since $\|\phi_n\|_{\mathcal{H}_0} = \|\phi_{n'}\|_{\mathcal{H}_0} = 1$, we have

$$\int \left|\mathscr{g}_X(v, v'; \lambda)\bar{\phi}_n(v)\phi_{n'}(v')\right| \, \xi(\mathrm{d}v)\xi(\mathrm{d}v') \leq \left(\int \left|\mathscr{g}_X(v, v'; \lambda)\right|^2 \, \xi(\mathrm{d}v)\xi(\mathrm{d}v')\right)^{1/2} .$$

By Proposition 2.6(ii), we get that

$$\int \left|\mathscr{g}_X(v, v'; \lambda)\bar{\phi}_n(v)\phi_{n'}(v')\right| \, \xi(\mathrm{d}v)\xi(\mathrm{d}v') \, \mu(\mathrm{d}\lambda) < \infty . \tag{4.4}$$

Therefore we can apply Fubini's theorem which gives, for all $n, n' \in \mathbb{N}$, and $s, t \in \mathbb{Z}$,

$$\begin{aligned}\int \mathrm{e}^{\mathrm{i}\lambda(s-t)} \, \mathscr{g}_X(v, v'; \lambda)\bar{\phi}_n(v)\phi_{n'}(v') \, \xi(\mathrm{d}v)\xi(\mathrm{d}v') \, \mu(\mathrm{d}\lambda) &= \int \mathrm{e}^{\mathrm{i}\lambda(s-t)} \, \phi_n^{\mathsf{H}} g_X(\lambda)\phi_{n'} \, \mu(\mathrm{d}\lambda) \\ &= \operatorname{Cov}\left(\phi_n^{\mathsf{H}} X_s, \phi_{n'}^{\mathsf{H}} X_t\right) .\end{aligned}$$

On the other hand, by Fubini's theorem, we have that, for all $n, n' \in \mathbb{N}$, and $s, t \in \mathbb{Z}$,

$$\operatorname{Cov}\left(\phi_n^{\mathsf{H}} X_s, \phi_{n'}^{\mathsf{H}} X_t\right) = \int \operatorname{Cov}\left(\tilde{X}_s(v,\cdot), \tilde{X}_t(v',\cdot)\right) \bar{\phi}_n(v)\phi_{n'}(v') \, \xi(\mathrm{d}v)\xi(\mathrm{d}v') .$$

This is also $\phi_n^{\mathsf{H}}\operatorname{Cov}(X_s, X_t)\,\phi_{n'}$ and since $\operatorname{Cov}(X_s, X_t)$ is a trace class hence Hilbert Schmidt operator the previous display says that this operator is associated with the $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ kernel $(v, v') \mapsto \operatorname{Cov}\left(\tilde{X}_s(v,\cdot), \tilde{X}_t(v',\cdot)\right)$.

The last two displays now imply that, for all $n, n' \in \mathbb{N}$, and $s, t \in \mathbb{Z}$,

$$\int \mathrm{Cov}\left(\tilde{X}_s(v,\cdot), \tilde{X}_t(v',\cdot)\right) \bar{\phi}_n(v)\phi_{n'}(v')\,\xi(\mathrm{d}v)\xi(\mathrm{d}v')$$

$$= \int \mathrm{e}^{\mathrm{i}\lambda(s-t)}\, g_X(v,v',\lambda)\,\bar{\phi}_n(v)\phi_{n'}(v')\,\xi(\mathrm{d}v)\xi(\mathrm{d}v')\,\mu(\mathrm{d}\lambda)$$

$$= \int \left(\int \mathrm{e}^{\mathrm{i}\lambda(s-t)}\, g_X(v,v',\lambda)\,\mu(\mathrm{d}\lambda)\right) \bar{\phi}_n(v)\phi_{n'}(v')\,\xi(\mathrm{d}v)\xi(\mathrm{d}v')\;,$$

where we used Fubini's theorem (justified by (4.4) as above). Since the kernel $(v,v') \mapsto \mathrm{Cov}\left(\tilde{X}_s(v,\cdot), \tilde{X}_t(v',\cdot)\right)$ is in $L^2(\mathsf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ of which $(\phi_k \otimes \bar{\phi}_{k'})_{k,k'\in\mathbb{N}}$ is a Hilbert basis, the last display shows that, for all $s,t \in \mathbb{Z}$,

$$\mathrm{Cov}\left(\tilde{X}_s(v,\cdot), \tilde{X}_t(v',\cdot)\right) = \int \mathrm{e}^{\mathrm{i}\lambda(s-t)}\, g_X(v,v';\lambda)\,\mu(\mathrm{d}\lambda) \quad \text{for } \xi^{\otimes 2} - \text{a.e. } (v,v'),$$

which concludes the proof. □

4.3 Proofs of Section 3.1

Proof of Theorem 3.1. Denote $\Phi(\lambda) = \phi(\mathrm{e}^{-\mathrm{i}\lambda})$ for all $\lambda \in \mathbb{R}$. As a trigonometric polynomial with $\mathcal{L}_b(\mathcal{H}_0)$-valued coefficients, Φ belongs to $\mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0))$. Moreover, (3.4) directly implies that $\Phi^{-1} \in \mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0))$. By Proposition 2.4, it follows that

(i) $Y = F_{\Phi^{-1}}(\epsilon)$ satisfies $F_\Phi(Y) = \epsilon$, and thus is a solution of (3.1);

(ii) for any centered weakly stationary process Y such that $F_\Phi(Y) = \epsilon$, we have $Y = F_{\Phi^{-1}} \circ F_\Phi(Y) = F_{\Phi^{-1}}(\epsilon)$.

We thus conclude that $Y = F_{\Phi^{-1}}(\epsilon)$ is the unique weakly stationary solution of (3.1).

Then the representation (3.5) holds as an immediate consequence of the fact that $z \mapsto \phi(z)^{-1}$ is $\mathcal{L}_b(\mathcal{H}_0)$-valued holomorhic on a ring containing the unit circle, so that

$$[\Phi(\lambda)]^{-1} = [\phi(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1} = \sum_{k\in\mathbb{Z}} \mathrm{P}_k \mathrm{e}^{-\mathrm{i}\lambda k}\;,$$

where $(\mathrm{P}_k)_{k\in\mathbb{Z}}$ are the Laurent series coefficients of ϕ^{-1} (see [10, Theorem 1.9.1], hence the series in the displayed equation converges absolutely in $\mathcal{L}_b(\mathcal{H}_0)$) and it can be shown that they have exponential decay at $\pm\infty$ (as a consequence of Eq. (1.9.4) in [10, Theorem 1.9.1]). □

Proof of Proposition 3.2. Since $z \mapsto [\phi(z)]^{-1}\,\theta(z)$ is holomorphic in an open ring containing $\mathbb{U}$, by [10, Theorem 1.8.5], there exists $\rho > 0$ and $(\mathrm{P}_n)_{n\in\mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0)$ such that $\sum_{n=0}^{\infty} \rho^n \|\mathrm{P}_n\|_{\mathcal{L}_b(\mathcal{H}_0)} < \infty$ and $[\phi(z)]^{-1}\,\theta(z)$ coincides with the $\mathcal{L}_b(\mathcal{H}_0)$-valued power series $\sum_{n=0}^{\infty}(z-1)^n \mathrm{P}_n$ on the set $\{z \in \mathbb{C} : |z-1| \le \rho\}$. Let $\eta > 0$ such that $\left\{\mathrm{e}^{-\mathrm{i}\lambda} : \lambda \in (-\eta,\eta)\right\} \subset \{z \in \mathbb{C} : |z-1| \le \rho\}$. Then we have, for all $\lambda \in (-\eta,\eta)$,

$$\sum_{n=0}^{\infty} \left|\mathrm{e}^{-\mathrm{i}\lambda} - 1\right|^n \|\mathrm{P}_n\|_{\mathcal{L}_b(\mathcal{H}_0)} \le \sum_{n=0}^{\infty} \rho^n \|\mathrm{P}_n\|_{\mathcal{L}_b(\mathcal{H}_0)} < \infty\;. \tag{4.5}$$

Thus we can write $[\phi(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\,\theta(\mathrm{e}^{-\mathrm{i}\lambda}) = \mathrm{P}_0 + \lambda\,\Psi(\lambda)$ by setting $\Psi(0) = 0$ and, for all $\lambda \in (-\eta,\eta)$,

$$\Psi(\lambda) = \frac{\mathrm{e}^{-\mathrm{i}\lambda} - 1}{\lambda} \sum_{n=1}^{\infty} (\mathrm{e}^{-\mathrm{i}\lambda} - 1)^{n-1} \mathrm{P}_n\;,$$

where the sum is absolutely convergent in $\mathcal{L}_b(\mathcal{H}_0)$ and where we used the standard convention $(\mathrm{e}^{-\mathrm{i}\lambda}-1)/\lambda = 1$ for $\lambda = 0$ (hence $\Psi(0) = 0$). Since $\mathrm{P}_0 = [\phi(1)]^{-1}\theta(1)$, it follows by Lemma 2.3 that (3.6) holds with $k(\lambda) := \Psi(\lambda)\Sigma^{1/2}$. Since Ψ is $\mathcal{L}_b(\mathcal{G}_0)$-valued, continuous and bounded on $(-\eta,\eta)$, we get that k is continuous and bounded from $(-\eta,\eta)$ to $\mathcal{S}_2(\mathcal{H}_0)$.

Suppose now that $\mathcal{H}_0 = L^2(\mathsf{V}, \mathcal{V}, \xi)$. For all $n \in \mathbb{N}$, let $\mathscr{k}_n$ denote the kernel associated to the operator $k_n := \mathrm{P}_n\Sigma^{1/2} \in \mathcal{S}_2(\mathcal{H}_0)$. Let us introduce the following notation for all $\mathsf{V}^2 \times (-\eta,\eta) \to \mathbb{C}$-measureable function f,

$$\|f\|_* = \left(\int_{\mathsf{V}^2} \operatorname*{Leb-essup}_{\lambda\in(-\eta,\eta)} \left|f(v,v';\lambda)\right|^2\,\xi(\mathrm{d}v)\xi(\mathrm{d}v')\right)^{1/2}\;,$$

which allows to define a Banach space L^* endowed with $\|\cdot\|_*$ as a norm. Note that, for all $n \in \mathbb{N}$, $(\int |\mathscr{k}_n|^2 \mathrm{d}\xi^{\otimes 2})^{1/2} = \left\|\mathrm{P}_n \Sigma^{1/2}\right\|_2 = \|\mathrm{P}_n\|_{\mathcal{L}_b(\mathcal{H}_0)} \|\Sigma\|_1^{1/2}$. By (4.5) and since $\lambda \mapsto (\mathrm{e}^{-\mathrm{i}\lambda} - 1)^n/\lambda$ is bounded by ρ^{n-1} on $(-\eta, \eta)$, we get that

$$\sum_{n=1}^{\infty} \left\|(v, v'\lambda) \mapsto \frac{(\mathrm{e}^{-\mathrm{i}\lambda} - 1)^n}{\lambda} \mathscr{k}_n(v, v')\right\|_* = \sum_{n=1}^{\infty} \underset{\lambda \in (-\eta,\eta)}{\text{Leb-essup}} \left|\frac{(\mathrm{e}^{-\mathrm{i}\lambda} - 1)^n}{\lambda}\right| \left\|\mathrm{P}_n \Sigma^{1/2}\right\|_2$$
$$\leq \|\Sigma\|_1^{1/2} \sum_{n=1}^{+\infty} \rho^{n-1} \|\mathrm{P}_n\|_{\mathcal{L}_b(\mathcal{H}_0)} < \infty \;.$$

To conclude the proof, we observe that L^* is continuously embedded in $L^2(\mathsf{V}^2 \times (-\eta, \eta), \mathcal{V}^{\otimes 2} \otimes \mathcal{B}(-\eta, \eta), \xi^{\otimes 2} \otimes \mathrm{Leb})$, which gives that the above series also converges in the latter space to the $(-\eta, \eta)$-joint kernel function $\mathscr{k}$ of k and satifies (3.7). ☐

Proof of Lemma 3.3. For any $\mathrm{P}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$ such that P is invertible, we have that $U\mathrm{P}^{-1}\mathrm{Q} = [U\mathrm{P}U^{-1}]^{-1}[U\mathrm{Q}U^{-1}]U$. Thus, we obtain, defining $\tilde{\theta}$ and $\tilde{\phi}$ as above,

$$\widehat{UX}(\mathrm{d}\lambda) = U[\phi(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\, \theta(\mathrm{e}^{-\mathrm{i}\lambda})\, \widehat{Z}(\mathrm{d}\lambda) = [\tilde{\phi}(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\, \tilde{\theta}(\mathrm{e}^{-\mathrm{i}\lambda})\, \widehat{UZ}(\mathrm{d}\lambda) \;.$$

It is then immediate to check that $\tilde{\theta} \in \mathcal{P}_q(\mathcal{G}_0)$ and $\tilde{\phi} \in \mathcal{P}_p^*(\mathcal{G}_0)$, and that $UZ = (UZ_t)_{t\in\mathbb{Z}}$ is a $\mathcal{G}_0$-valued white noise. ☐

4.4 Proofs of Section 3.2

The proof of Theorem 3.4 relies on the following lemma.

Lemma 4.1. *For all $z \in \mathbb{C}$ and $\lambda \in [-\pi, \pi]$, we have*

$$(2/\pi)^{2[\Re(z)]_+}\; |\lambda|^{2\Re(z)}\; \mathrm{e}^{-\pi|\Im(z)|} \leq \left|(1 - \mathrm{e}^{-\mathrm{i}\lambda})^z\right|^2 \leq (\pi/2)^{2[\Re(z)]_-}\; |\lambda|^{2\Re(z)}\; \mathrm{e}^{\pi|\Im(z)|} \;. \tag{4.6}$$

Proof. Let $z \in \mathbb{C}$ with $\Re(z)$, then it can be shown that, for all $\lambda \in (-\pi, \pi] \setminus \{0\}$,

$$\left|(1 - \mathrm{e}^{-\mathrm{i}\lambda})^z\right|^2 = \left|1 - \mathrm{e}^{-\mathrm{i}\lambda}\right|^{2\Re(z)} \mathrm{e}^{-2\Im(z) b(\mathrm{e}^{-\mathrm{i}\lambda})} \;,$$

where $b(\mathrm{e}^{-\mathrm{i}\lambda})$ denotes the argument of $1 - \mathrm{e}^{-\mathrm{i}\lambda}$ that belongs to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It follows that

$$\mathrm{e}^{-\pi|\Im(z)|} \leq \mathrm{e}^{-2\Im(z) b(\mathrm{e}^{-\mathrm{i}\lambda})} \leq \mathrm{e}^{\pi|\Im(z)|} \;.$$

Using that $\frac{|\lambda|}{\pi} \leq |\sin(\lambda/2)| \leq \frac{|\lambda|}{2}$ for all $\lambda \in (-\pi, \pi)$ and separating the cases where $\Re(z) \geq 0$ and where $\Re(z) < 0$, we easily get (4.6). ☐

Proof of Theorem 3.4. Recall that ξ is a σ-finite measure and $L^2(\mathsf{V}, \mathcal{V}, \xi)$ is separable since $\mathcal{H}_0$ is by assumption and they are isomorphic. As defined in Section 2.7, $X \in \mathcal{S}_{\mathrm{FI}_D}(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $\mathrm{FI}_D \in \mathsf{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0), \nu_X)$, which, by (2.8), is equivalent to have

$$\int_{\mathbb{T}} \left\|(1 - \mathrm{e}^{-\mathrm{i}\lambda})^{-D} g_X(\lambda) \left[(1 - \mathrm{e}^{-\mathrm{i}\lambda})^{-D}\right]^{\mathsf{H}}\right\|_1 \mu(\mathrm{d}\lambda) < +\infty \;. \tag{4.7}$$

We have, for all $\lambda \in \mathbb{T} \setminus \{0\}$, since U is unitary from $\mathcal{H}_0$ to $L^2(\mathsf{V}, \mathcal{V}, \xi)$,

$$\begin{aligned}\left\|(1 - \mathrm{e}^{-\mathrm{i}\lambda})^{-D} g_X(\lambda) \left[(1 - \mathrm{e}^{-\mathrm{i}\lambda})^{-D}\right]^{\mathsf{H}}\right\|_1 &= \left\|U^{\mathsf{H}} M_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-d}} U g_X(\lambda) U^{\mathsf{H}} M^{\mathsf{H}}_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-d}} U\right\|_1 \\ &= \left\|M_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-\mathrm{d}}} U\, g_X(\lambda)\, U^{\mathsf{H}} M^{\mathsf{H}}_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-\mathrm{d}}}\right\|_1 \\ &= \left\|M_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-\mathrm{d}}} g_{UX}(\lambda) M^{\mathsf{H}}_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-\mathrm{d}}}\right\|_1 \\ &= \left\|M_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-\mathrm{d}}} h(\lambda)\right\|_2^2 \;.\end{aligned}$$

Hence (4.7) holds if and only if

$$\int_{\mathbb{T}} \left\|M_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-\mathrm{d}}} h(\lambda)\right\|_2^2 \mu(\mathrm{d}\lambda) < +\infty \;,$$

which, using the $\mathbb{T}$-joint kernel $\hbar$ of h reads

$$\int \left|(1-\mathrm{e}^{-\mathrm{i}\lambda})^{-\mathrm{d}(v)}\hbar(v,v';\lambda)\right|^2 \xi(\mathrm{d}v)\xi(\mathrm{d}v')\,\mu(\mathrm{d}\lambda) < +\infty \;.$$

Applying Lemma 4.1 to $z = -\mathrm{d}(v)$, since d is a μ-essentially bounded function, we get that Assertion (i) is equivalent to

$$\int_{\mathsf{V}^2\times(-\pi,\pi]} |\lambda|^{-2\Re(\mathrm{d}(v))} \left|\hbar(v,v';\lambda)\right|^2 \xi(\mathrm{d}v)\xi(\mathrm{d}v')\mu(\mathrm{d}\lambda) < \infty \;.$$

This of course implies Assertion (iii), which implies Assertion (ii). Now, if Assertion (ii) holds, since $|\lambda|^{-2\Re(\mathrm{d}(v))}$ is bounded independently of v on $\lambda \in (-\pi,\pi]\setminus(-\eta,\eta)$ and

$$\int \left|\hbar(v,v';\lambda)\right|^2 \xi(\mathrm{d}v)\xi(\mathrm{d}v')\mu(\mathrm{d}\lambda) = \int \|h(\lambda)\|_2^2\, \mu(\mathrm{d}\lambda) < \infty \;,$$

we get back the above condition involving an integration over $\mathsf{V}^2\times(-\pi,\pi]$. □

Proof of Theorem 3.5. Before proving the claimed implications, we start with some preliminary facts that are obtained from Lemma 3.3, Proposition 3.2, Lemma 2.8 and Theorem 3.4.

By Lemma 3.3, the process $UX = (UX_t)_{t\in\mathbb{Z}}$ is the $\mathcal{G}_0$-valued ARMA(p,q) process defined by $\widehat{UX}(\mathrm{d}\lambda) = [\tilde{\Phi}(\mathrm{e}^{-\mathrm{i}\lambda})]^{-1}\,\tilde{\Theta}(\mathrm{e}^{-\mathrm{i}\lambda})\,\widehat{UZ}(\mathrm{d}\lambda)$, where $\tilde{\Theta} := U\Theta U^{-1} \in \mathcal{P}_q(\mathcal{G}_0)$ and $\tilde{\Phi} := U\Phi U^{-1} \in \mathcal{P}_p^*(\mathcal{G}_0)$, and $UZ = (UZ_t)_{t\in\mathbb{Z}}$ is a $\mathcal{G}_0$-valued white noise. Applying Proposition 3.2 with μ as the Lebesgue measure, we get that, for some $\eta>0$, ν_{UX} has density $h(\lambda)[h(\lambda)]^{\mathsf{H}}$ on $(-\eta,\eta)$ with h valued in $\mathcal{S}_2(\mathcal{G}_0)$ satisfying, for all $\lambda\in(-\eta,\eta)$,

$$h(\lambda) = [\tilde{\Phi}(1)]^{-1}\tilde{\Theta}(1)(U\Sigma U^{\mathsf{H}})^{1/2} + \lambda\, k(\lambda) \;, \tag{4.8}$$

where k is continuous from $(-\eta,\eta)$ to $\mathcal{S}_2(\mathcal{G}_0)$. Moreover, since $\mathcal{G}_0 = L^2(\mathsf{V},\mathcal{V},\xi)$, Proposition 3.2 also gives that the joint kernel $\hbar$ of k satisfies (3.7), which implies

$$\int \mathrm{s}^2(v)\xi(\mathrm{d}v) \le \int \underset{\lambda\in(-\eta,\eta)}{\text{Leb-essup}} \left|\hbar(v,v';\lambda)\right|^2 \xi(\mathrm{d}v')\xi(\mathrm{d}v) < +\infty \;, \tag{4.9}$$

where we defined, for all $v\in\mathsf{V}$,

$$\mathrm{s}(v) = \underset{\lambda\in(-\eta,\eta)}{\text{Leb-essup}} \|\hbar(v,\cdot;\lambda)\|_{\mathcal{G}_0} \;.$$

Define, for any $\eta'\in(0,\eta)$,

$$\begin{aligned} I(\eta') &:= \int_{\mathsf{V}^2\times(-\eta',\eta')} |\lambda|^{-2\Re(\mathrm{d}(v))} \left|\hbar(v,v';\lambda)\right|^2 \xi(\mathrm{d}v)\xi(\mathrm{d}v')\frac{\mathrm{d}\lambda}{2\pi} \\ &= \int_{\mathsf{V}^2\times(-\eta',\eta')} |\lambda|^{-2\Re(\mathrm{d}(v))} \left|\hbar_0(v,v') + \lambda\,\hbar(v,v';\lambda)\right|^2 \xi(\mathrm{d}v)\xi(\mathrm{d}v')\frac{\mathrm{d}\lambda}{2\pi} \;, \end{aligned} \tag{4.10}$$

where $\hbar$ is the kernel of h in (4.8) and $\hbar_0$ is the kernel of $k_0 := U[\Phi(1)]^{-1}\Theta(1)\Sigma^{1/2}U^{\mathsf{H}} \in \mathcal{S}_2(\mathcal{G}_0)$. Integrating w.r.t. v', by the Minkowski inequality, we get that

$$I(\eta') \ge \int_{\mathsf{V}\times(-\eta',\eta')} |\lambda|^{-2\Re(\mathrm{d}(v))} \left|\sigma_W(v) - |\lambda|\,\|\hbar(v,\cdot;\lambda)\|_{\mathcal{G}_0}\right|^2 \xi(\mathrm{d}v)\frac{\mathrm{d}\lambda}{2\pi} \;, \tag{4.11}$$

where we used that $\sigma_W(v) = \|\hbar_0(v,\cdot)\|_{\mathcal{G}_0}$ for ξ − a.e. $v\in\mathsf{V}$, which holds as a consequence of Lemma 2.8 since $\mathrm{Cov}(W) = k_0k_0^{\mathsf{H}}$. Similarly, using the definition of s above, we can upper bound $I(\eta')$ by

$$\begin{aligned} &I(\eta') \le 2(I_1(\eta') + I_2(\eta')) \;, \quad \text{where} \\ &I_1(\eta') = \int_{\mathsf{V}\times(-\eta',\eta')} |\lambda|^{-2\Re(\mathrm{d}(v))}\sigma_W^2(v)\;\xi(\mathrm{d}v)\frac{\mathrm{d}\lambda}{2\pi} \;, \text{ and} \\ &I_2(\eta') = \int_{\mathsf{V}\times(-\eta',\eta')} |\lambda|^{2-2\Re(\mathrm{d}(v))}\mathrm{s}^2(v)\;\xi(\mathrm{d}v)\frac{\mathrm{d}\lambda}{2\pi} \;. \end{aligned} \tag{4.12}$$

To conclude these preliminaries, by Theorem 3.4, we have that Assertion (i) of Theorem 3.5 is equivalent to the two following assertions:

(vii) for all $\eta' \in (0,\eta)$, we have $I(\eta') < \infty$;

(viii) there exists $\eta' \in (0,\eta)$ such that $I(\eta') < \infty$.

We are now ready to prove the claimed implications.

Proof of (i) ⇒(ii). Let us define, for any $n \in \mathbb{N}$,

$$A_n = \left\{\sigma_W > 2^{-n}\right\} \cap \{\mathrm{s} \leq 2^n\} \ .$$

Then, if $\eta' \in (0, 2^{-2n-1}]$, we have, for $\mathrm{Leb}-$a.e. $\lambda \in (-\eta', \eta')$ and all $v \in A_n$,

$$|\lambda| \, \|\not{k}(v,\cdot;\lambda)\| \leq \eta' \mathrm{s}(v) < 2^{-n-1} < 2^{-n} < \sigma_W(v) \ ,$$

which implies that $\sigma_W(v) - |\lambda| \, \|\not{k}(v,\cdot;\lambda)\|_{\mathcal{G}_0} \geq 2^{-n} - 2^{-n-1} = 2^{-n-1}$ and thus, with (4.11),

$$I(\eta') \geq 2^{-2n-2} \int_{A_n \times (-\eta',\eta')} |\lambda|^{-2\Re(\mathrm{d}(v))} \, \xi(\mathrm{d}v) \frac{\mathrm{d}\lambda}{2\pi} \ . \tag{4.13}$$

Suppose that (i) holds. Then so does (vii) and thus, for all $n \in \mathbb{N}$, the integral in (4.13) must be finite which implies $\Re(\mathrm{d}) < 1/2$, $\xi-$a.e. on A_n (since $\int_{(-\eta',\eta')} |\lambda|^{-2d} \mathrm{d}\lambda = \infty$ for $d \geq 1/2$). On the other hand, we have $\bigcup_n A_n = \{\sigma_W > 0\} \cup \{\mathrm{s} < \infty\}$ and, by (4.9), $\mathrm{s} < \infty$ $\xi-$a.e.; hence we get (ii).

Proof of (i) ⇒(iii). Note that, for all $(v,\lambda) \in \mathsf{V} \times (-\eta,\eta)$,

$$\left|\sigma_W(v) - |\lambda| \, \|\not{k}(v,\cdot;\lambda)\|_{\mathcal{G}_0}\right|^2 \geq \sigma_W^2(v) - 2\,|\lambda|\,\sigma_W(v)\,\|\not{k}(v,\cdot;\lambda)\|_{\mathcal{G}_0} \geq \sigma_W^2(v) - 2\,|\lambda|\,\sigma_W(v)\,\mathrm{s}(v) \ .$$

and thus, using (4.11), we get that, for all $\eta' \in (0,\eta)$,

$$\begin{aligned}
I(\eta') &\geq \int_{\{\Re(\mathrm{d})<1/2\} \times (-\eta',\eta')} |\lambda|^{-2\Re(\mathrm{d}(v))} \left|\sigma_W(v) - |\lambda| \, \|\not{k}(v,\cdot;\lambda)\|_{\mathcal{G}_0}\right|^2 \, \xi(\mathrm{d}v) \frac{\mathrm{d}\lambda}{2\pi} \\
&\geq \int_{\{\Re(\mathrm{d})<1/2\} \times (-\eta',\eta')} |\lambda|^{-2\Re(\mathrm{d}(v))} \sigma_W^2(v) \, \xi(\mathrm{d}v) \frac{\mathrm{d}\lambda}{2\pi} \\
&\quad - 2 \int_{\{\Re(\mathrm{d})<1/2\} \times (-\eta',\eta')} |\lambda|^{1-2\Re(\mathrm{d}(v))} \sigma_W(v)\,\mathrm{s}(v) \, \xi(\mathrm{d}v) \frac{\mathrm{d}\lambda}{2\pi} \\
&= \int_{\{\Re(\mathrm{d})<1/2\}} \frac{\eta'^{1-2\Re(\mathrm{d}(v))}}{2\pi} \, \frac{\sigma_W^2(v)}{1 - 2\Re(\mathrm{d}(v))} \, \xi(\mathrm{d}v) &\quad (4.14) \\
&\quad - \int_{\{\Re(\mathrm{d})<1/2\}} \frac{\eta'^{2-2\Re(\mathrm{d}(v))}}{2\pi} \, \frac{\sigma_W(v)\,\mathrm{s}(v)}{1 - \Re(\mathrm{d}(v))} \, \xi(\mathrm{d}v) \ . &\quad (4.15)
\end{aligned}$$

Since d is bounded on V, we have that $\eta'^{2-2\Re(\mathrm{d}(v))}$ is upper bounded on $v \in \mathsf{V}$ and since $(1 - \Re(\mathrm{d}(v)))^{-1} \leq 1/2$ on $\{\Re(\mathrm{d}) < 1/2\}$, we get that the integral in (4.15) is bounded from above, up to a multiplicative constant, by

$$\int_{\{\Re(\mathrm{d})<1/2\}} \sigma_W(v)\,\mathrm{s}(v) \, \xi(\mathrm{d}v) \leq \|\sigma_W\|_{\mathcal{G}_0} \, \|\mathrm{s}\|_{\mathcal{G}_0} \ ,$$

which is finite using (4.9) and $\|\sigma_W\|_{\mathcal{G}_0}^2 = \mathbb{E}\left[\|W\|_{\mathcal{G}_0}^2\right] < \infty$. Using again that d is bounded on V, we have that $\eta'^{1-2\Re(\mathrm{d}(v))}$ is lower bounded by a positive constant on V. Hence, we finally get that, if (i) holds, then (vii) holds as well and what precedes yields Assertion (iii).

Proof of (ii) and (iii)⇒(i) under (iv) or (v). To obtain (i), it is sufficient to show that Assertion (viii) holds, which, by (4.12), follows from $I_1(\eta') < \infty$ and $I_2(\eta') < \infty$. Under Assertion (ii), we have, for all $\eta' \in (0,\eta)$,

$$I_1(\eta') = \int_{\{\Re(\mathrm{d})<1/2\}} \frac{\eta'^{1-2\Re(\mathrm{d}(v))}}{\pi} \, \frac{\sigma_W^2(v)}{1 - 2\Re(\mathrm{d}(v))} \, \xi(\mathrm{d}v) \ ,$$

and since d is bounded, this integral is finite under (iii). Thus (ii) and (iii) imply that $I_1(\eta') < \infty$ for all $\eta' \in (0,\eta)$. To conclude the proof it only remains to show that $I_2(\eta') < \infty$ for some $\eta' \in (0,\eta)$ whenever (iv) or (v) holds. We have in fact $I_2(\eta') < \infty$ for all $\eta' \in (0,\eta)$ under (iv) by using (4.9) while under (v), we have $I_2(\eta') = 0$ for all $\eta' \in (0,\eta)$ since in this case $h(\lambda) = h(0)$ so that, in (4.8), $k(\lambda) = 0$ for all $\lambda \in (-\eta,\eta)$ (thus implying $\mathrm{s} = 0$). This concludes the proof. □

4.5 Proofs of Section 3.3

Proof of Lemma 3.6. The statement in i.i.d. case is exactly [9, Lemma A.1]. The convergence in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ follows from the proof of [9, Lemma A.1], which continues to hold under the weaker assumption that $(\epsilon_k)_{k\in\mathbb{Z}}$ is a white noise. □

Proof of Lemma 3.7. Since ϵ is a white noise, Assertion (v) of Theorem 3.5 holds. The result follows since the conditions in (3.11) imply Assertions (ii) and (iii) of Theorem 3.5 with and $D = \mathrm{Id}_{\mathcal{H}_0} - N$. □

The proof of Proposition 3.8 relies on the two following lemmas where the open and closed complex unit discs of $\mathbb{C}$ are denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ respectively.

Lemma 4.2. *Let E be a Banach space and $(a_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ such that $\|a_n\|_E \xrightarrow[n\to+\infty]{} 0$ and the series $\sum \|a_n - a_{n+1}\|_E$ converges. Then for all $z_0 \in \overline{\mathbb{D}} \setminus \{1\}$, the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges in E and the mapping $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ is uniformly continuous on $[0, z_0]$.*

Proof. By assumption on (a_n), $\sum a_n z^n$ is a power series valued in E with convergence radius at least equal to 1, hence is uniformly continuous on the open disk with radius 1. When $|z_0| = 1$, the result follows using Abel's transform. □

Lemma 4.3. *Let $\mathcal{H}_0$ be a separable Hilbert space, $N \in \mathcal{L}_b(\mathcal{H}_0)$ be a normal operator with singular value function n on $\mathcal{G}_0 := L^2(\mathsf{V}, \mathcal{V}, \xi)$ and decomposition operator U. Define*

$$\varrho = \xi\text{-}\operatorname*{essinf}_{v\in\mathsf{V}} \Re(\mathrm{n}(v)) \; .$$

Then there exist $C \in \mathcal{L}_b(\mathcal{H}_0)$ and $(\Delta_k)_{k\in\mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0)^{\mathbb{N}}$ with $\|\Delta_k\|_{\mathcal{L}_b(\mathcal{H}_0)} = O\left(k^{-1-\varrho}\right)$ such that, for all $z \in \mathbb{D}$,

$$(1-z)^{N-\mathrm{Id}} = C\left(\sum_{k=0}^{\infty} (k+1)^{-N} z^k\right) + \sum_{k=0}^{\infty} \Delta_k z^k \; , \tag{4.16}$$

where the two infinite sums on the right-hand side are $\mathcal{L}_b(\mathcal{H}_0)$-valued power series with convergence radius at least equal to 1. Moreover, if $\varrho > 0$, then Eq. (4.16) continues to hold for all $z \in \overline{\mathbb{D}} \setminus \{1\}$ with the two infinite sums converging in $\mathcal{L}_b(\mathcal{H}_0)$.

Proof. The proof is three steps. We first show Relation (4.16) for all $z \in \mathbb{D}$, then that $\|\Delta_k\|_{\mathcal{L}_b(\mathcal{H}_0)} = O\left(k^{-1-\varrho}\right)$ and finally extend the relation to $z \in \overline{\mathbb{D}} \setminus \{1\}$ when $\varrho > 0$.
Step 1. Let $z \in \mathbb{D}$, then

$$(1-z)^{N-\mathrm{Id}} = \mathrm{Id} + \sum_{k\geq 1} N_k z^k \; ,$$

where for all $k \geq 1$, $N_k = \prod_{j=1}^{k}\left(\mathrm{Id} - \frac{N}{j}\right)$. Let $k_0 \geq 1$, such that $\|N\|_{\mathcal{L}_b(\mathcal{H}_0)}/k_0 < 1$ and take $k \geq k_0$, then

$$\mathrm{Id} - \frac{N}{k} = \exp\left(\ln\left(\mathrm{Id} - \frac{N}{k}\right)\right) = \exp\left(-\sum_{j\geq 1} \frac{N^j}{k^j j}\right) \; ,$$

and therefore,

$$N_k = \prod_{j=1}^{k_0-1}\left(\mathrm{Id} - \frac{N}{j}\right) \exp\left(-\sum_{j\geq 1} \frac{N^j}{j} \sum_{t=k_0}^{k} \frac{1}{t^j}\right) \; .$$

Moreover, we have the following asymptotic expansions,

$$\sum_{t=k_0}^{k} \frac{1}{t} = \sum_{t=1}^{k} \frac{1}{t} - \sum_{t=1}^{k_0-1} \frac{1}{t} = \ln(k+1) + \gamma_e - \sum_{t=1}^{k_0-1} \frac{1}{t} + \frac{\alpha_k}{k}$$

and for all $j \geq 2$,

$$\sum_{t=k_0}^{k} \frac{1}{t^j} = \sum_{k=k_0}^{+\infty} \frac{1}{t^j} - \sum_{k=k+1}^{+\infty} \frac{1}{t^j} = \frac{\beta_j}{k_0^j} + \frac{\eta_{k,j}}{(j-1)k^{j-1}}$$

where γ_e is Euler's constant and $(\alpha_k)_{k\geq 1}$, $(\eta_{k,j})_{k\geq 1, j\geq 2}$ such that $\sup_{k\geq 1} |\alpha_k| < +\infty$ and $\sup_{k\geq 1, j\geq 2} |\eta_{k,j}| < +\infty$ and $\beta_j = \sum_{t=k_0}^{+\infty} \left(\frac{k_0}{t}\right)^j$ satisfies $\sup_{j\geq 2} \beta_j < +\infty$. This gives, for all $k \geq k_0$,

$$N_k = C(k+1)^{-N} \exp\left(-N\frac{\alpha_k}{k} - \sum_{j\geq 2} \frac{N^j \eta_{k,j}}{(j-1)k^{j-1}}\right)$$

where

$$C = \prod_{j=1}^{k_0-1} \left(\mathrm{Id} - \frac{N}{j}\right) \exp\left(-N\left(\gamma_e - \sum_{t=1}^{k_0-1} \frac{1}{t}\right)\right) \exp\left(-\sum_{j\geq 2} \left(\frac{N}{k_0}\right)^j \frac{\beta_j}{j}\right) .$$

Combining everything, we get

$$\begin{aligned}(1-z)^{N-\mathrm{Id}} &= \mathrm{Id} + \sum_{k=1}^{k_0-1} \prod_{j=1}^{k} \left(\mathrm{Id} - \frac{N}{j}\right) z^k \\ &\quad + C \sum_{k\geq k_0} (k+1)^{-N} \exp\left(-N\frac{\alpha_k}{k} - \sum_{j\geq 2} \frac{N^j \eta_{k,j}}{(j-1)k^{j-1}}\right) z^k\end{aligned}$$

which leads to Relation (4.16) with

$$\begin{aligned}\Delta_0 &= \mathrm{Id} - C , \\ \Delta_k &= \prod_{j=1}^{k} \left(\mathrm{Id} - \frac{N}{j}\right) - C(k+1)^{-N} , && 1 \leq k \leq k_0 - 1, \\ \Delta_k &= C(k+1)^{-N} \left[\exp\left(-N\frac{\alpha_k}{k} - \sum_{j\geq 2} \frac{N^j \eta_{k,j}}{(j-1)k^{j-1}}\right) - \mathrm{Id}\right] , && k \geq k_0.\end{aligned}$$

Step 2. For all $k \geq k_0$, denoting by $\Phi_k := -N\frac{\alpha_k}{k} - \sum_{j\geq 2} \frac{N^j \eta_{k,j}}{(j-1)k^{j-1}}$, we get

$$\begin{aligned}\|\Delta_k\|_{\mathcal{L}_b(\mathcal{H}_0)} &= \left\|C(k+1)^{-N} \left(\mathrm{e}^{\Phi_k} - \mathrm{Id}\right)\right\|_{\mathcal{L}_b(\mathcal{H}_0)} \\ &\leq \|C\|_{\mathcal{L}_b(\mathcal{H}_0)} \left\|(k+1)^{-N}\right\|_{\mathcal{L}_b(\mathcal{H}_0)} \sum_{t\geq 1} \frac{\|\Phi_k\|^t_{\mathcal{L}_b(\mathcal{H}_0)}}{t!} = O\left(k^{-1-\varrho}\right) ,\end{aligned}$$

where we used that

$$\begin{aligned}\|\Phi_k\|_{\mathcal{L}_b(\mathcal{H}_0)} &\leq \|N\|_{\mathcal{L}_b(\mathcal{H}_0)} \frac{|\alpha_k|}{k} + \sum_{j\geq 2} \frac{\|N\|^j_{\mathcal{L}_b(\mathcal{H}_0)} \eta_{k,j}}{(j-1)k^{j-1}} \\ &= \|N\|_{\mathcal{L}_b(\mathcal{H}_0)} \left(\frac{|\alpha_k|}{k} + \sum_{j\geq 1} \frac{\|N\|^j_{\mathcal{L}_b(\mathcal{H}_0)}}{jk^j} \eta_{k,j+1}\right) = O\left(k^{-1}\right) ,\end{aligned}$$

and that $\left\|(k+1)^{-N}\right\|_{\mathcal{L}_b(\mathcal{H}_0)} = \left\|(k+1)^{-M_{\mathrm{n}}}\right\|_{\mathcal{L}_b(\mathcal{H}_0)} = \left\|M_{(k+1)^{-\mathrm{n}}}\right\|_{\mathcal{L}_b(\mathcal{H}_0)} = \xi\text{-}\mathrm{essup}_{v\in\mathsf{V}} \left|(k+1)^{-\mathrm{n}(v)}\right| = (k+1)^{-\varrho}$.

Step 3. We now assume $\varrho > 0$ and extend (4.16) to $\overline{\mathbb{D}} \setminus \{1\}$, that is to the case $z = \mathrm{e}^{-\mathrm{i}\lambda}$ for some $\lambda \in \mathbb{T} \setminus \{0\}$. For such a λ, we already have, for all $0 < a < 1$,

$$(1 - a\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} = C \sum_{k\geq 0} (k+1)^{-N} a^k \mathrm{e}^{-\mathrm{i}\lambda k} + \sum_{k\geq 0} \Delta_k a^k \mathrm{e}^{-\mathrm{i}\lambda k} .$$

Moreover, $(1-\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} = \lim_{a\uparrow 1}(1-a\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}}$ by continuity of $z \mapsto (1-z)^{N-\mathrm{Id}}$ in $\overline{\mathbb{D}}\setminus\{1\}$ and $\sum_{k\geq 0}\Delta_k \mathrm{e}^{-\mathrm{i}\lambda k} = \lim_{a\uparrow 1}\sum_{k\geq 0}\Delta_k a^k \mathrm{e}^{-\mathrm{i}\lambda k}$ because $\sum_{k\geq 0}\|\Delta_k\|_{\mathcal{L}_b(\mathcal{H}_0)} < +\infty$. It remains to show that $\sum_{k\geq 0}(k+1)^{-N}z$ is well defined on $\mathbb{U}\setminus\{1\}$ and that, for $\lambda\in\mathbb{T}\setminus\{0\}$, $\sum_{k\geq 0}(k+1)^{-N}a^k\mathrm{e}^{-\mathrm{i}\lambda k}$ converges to $\sum_{k\geq 0}(k+1)^{-N}\mathrm{e}^{-\mathrm{i}\lambda k}$ as $a\uparrow 1$, which we prove at once by applying Lemma 4.2. For all $k\in\mathbb{N}$, we have

$$\left\|(k+1)^{-N}\right\|_{\mathcal{L}_b(\mathcal{H}_0)} = \xi\text{-}\underset{v\in\mathsf{V}}{\mathrm{essup}}\left|(k+1)^{-\mathrm{n}(v)}\right| = (k+1)^{-\varrho}\ ,$$

Since $\varrho>0$, we get that $\|(k+1)^{-N}\|_{\mathcal{L}_b(\mathcal{H}_0)}\to 0$ as $k\to\infty$. Hence, to apply Lemma 4.2 it only remains to show

$$\sum_{k\in\mathbb{N}}\left\|(k+1)^{-N}-(k+2)^{-N}\right\|_{\mathcal{L}_b(\mathcal{H}_0)} < \infty\ . \tag{4.17}$$

Note that we have, for all $k\in\mathbb{N}$,

$$\left\|(k+1)^{-N}-(k+2)^{-N}\right\|_{\mathcal{L}_b(\mathcal{H}_0)} = \xi\text{-}\underset{v\in\mathsf{V}}{\mathrm{essup}}\left|(k+1)^{-\mathrm{n}(v)}-(k+2)^{-\mathrm{n}(v)}\right|\ . \tag{4.18}$$

Moreover, for all $k\in\mathbb{N}$, and ξ − a.e. $v\in\mathsf{V}$, since $\Re(\mathrm{n}(v))\geq\varrho>0$, we have

$$\begin{aligned}\left|(k+1)^{-\mathrm{n}(v)}-(k+2)^{-\mathrm{n}(v)}\right| &= |k+1|^{-\Re(\mathrm{n}(v))}\left|1-\exp\left(-\ln\left(1+\frac{1}{k+1}\right)\mathrm{n}(v)\right)\right| \\ &\leq \varsigma\ \alpha(\varsigma\ln(2))\ (k+1)^{-\varrho}\ \ln\left(1+\frac{1}{k+1}\right)\ ,\end{aligned}$$

where we set $\varsigma := \xi\text{-essup}\,|\mathrm{n}|$ and, for any $r>0$,

$$\alpha(r) := \sup\left\{\left|\frac{1-\mathrm{e}^{-z}}{z}\right|\ :\ z\in\mathbb{C}\ 0<|z|\leq r\right\}\ .$$

This leads to the asymptotic bound, as $k\to\infty$,

$$\xi\text{-}\underset{v\in\mathsf{V}}{\mathrm{essup}}\left|(k+1)^{-\mathrm{n}(v)}-(k+2)^{-\mathrm{n}(v)}\right| = O\left((k+1)^{-\varrho-1}\right)\ .$$

Hence, with (4.18) and the assumption $\varrho>0$, we obtain (4.17) and Step 3 is completed. □

Proof of Proposition 3.8. The processes Y and $F_{\mathrm{FI}_D}(\epsilon)$ are well defined by Lemma 3.6 and Lemma 3.7 respectively. Moreover, the first condition in (3.11) gives $\varrho\geq 1/2$ in Lemma 4.3 which therefore implies that there exists $C\in\mathcal{L}_b(\mathcal{H}_0)$ and $(\Delta_k)_{k\in\mathbb{N}}\in\mathcal{L}_b(\mathcal{H}_0)^{\mathbb{N}}$ with $\|\Delta_k\|_{\mathcal{L}_b(\mathcal{H}_0)} = O(k^{-3/2})$ (in particular $\sum_{k=0}^{+\infty}\|\Delta_k\|_{\mathcal{L}_b(\mathcal{H}_0)} < +\infty$) such that

$$(1-\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} = C\sum_{k=0}^{\infty}(k+1)^{-N}\mathrm{e}^{-\mathrm{i}\lambda k}+\sum_{k=0}^{\infty}\Delta_k\mathrm{e}^{-\mathrm{i}\lambda k}\quad \text{in } \mathcal{L}_b(\mathcal{H}_0) \text{ for all } \lambda\in\mathbb{T}\setminus\{0\}\ , \tag{4.19}$$

thus concluding the proof. □

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