Chapter 1

New insights in integral representation theory for the solution of complex canonical diffraction problems

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The problem of scattering by 2D and 3D canonical objects with imperfectly conducting surfaces requires some particular efforts on the representation of scattered and incident fields, and we present here some remarkable aspects of them in complex situations and their applications.

We begin with the study of 2D problems. The Sommerfeld-Maliuzhinets integral and its inversion in the spectral domain of complex angles has opened a new way of investigation on diffraction by a wedge-shaped domain. In this frame, we present this spectral method in a new perspective, by giving some novel exact general expressions and properties of the associated spectral function attached to the total field : novel spectral expression of free space Green function, single face representation and its consequences, uniqueness, existence, reciprocity, spectral causal representation of field in time domain. We then analyze in a distinct section the diffraction by an impedance wedge with curved faces. The diffraction coefficient, when limited to its principal order in curvature, leads us to some miscalculation when the wedge approaches the discontinuity of curvature case for example. Thus, we give here a general asymptotic representation taking into account a development at several orders for arbitrary wedge angle, which is uniformly valid from the discontinuity of curvature to the curved half-plane.

Concerning 3D problems, we begin with a section devoted to an efficient exact solution for the radiation of a point source above an impedance plane in electromagnetism. In classic approaches, we need to consider vertical and horizontal dipole cases separately, and we present here a method, consisting in filling a gap in representation of incident fields by potential theory, to finally permit an efficient global representation of the diffracted field for arbitrary primary sources in a direct manner. We conclude this chapter with the representation of fields for the diffraction by an imperfectly conducting cone. We present a general exact expression of fields with Debye potentials and Kontorovich-Lebedev (KL) integrals, and new general properties of spectral functions attached to them. We then insist on a novel compact

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representation of plane waves, which does not need any use of spherical harmonics series.

1.1 Representations of Spectral Function in Frequency and Time Domain, for the Scattering by a polygonal Region

1.1.1 Basic elements in Sommerfeld-Maliuzhinets representation and properties

1.1.1.1 Basic integral representation

Let us consider the case of diffraction in free space of an incident plane wave

$$u^i(\rho, \varphi) = U^i(\omega) e^{ik\rho \cos(\varphi - \varphi_0)}, \quad (1.1)$$

by a scatterer enclosed in a wedge-shaped region (fig. 1.1), defined in cylindrical coordinates (ρ, φ, z) as the domain outside the free space angular sector with origin O , $-\Phi_r \leq \varphi \leq \Phi_l$. An implicit harmonic dependence on time $e^{i\omega t}$ is understood and henceforth suppressed, and $k = \frac{\omega}{c}$ denotes the wave number of the exterior medium with $|\arg(ik)| < \frac{\pi}{2}$. Physically, $|\arg(ik)| < \frac{\pi}{2}$ means that there are some losses in free space, and $|\arg(ik)| = \frac{\pi}{2}$ is considered as a limit case. The function $U^i(\omega)$ is assumed $O(\omega^{-b})$ for $|\omega| \rightarrow \infty$, $b > 1$ so that the scattered field u is $O(\omega^{-\nu})$ with $\nu > 1$ as $|\operatorname{Re}\omega| \rightarrow \infty$ for $\operatorname{Im}\omega \leq 0$, and is analytic at any ω with $\operatorname{Im}\omega \leq 0$. The characteristics of the scatterer are supposed to be independent of z coordinate. We assume that the total field in the free space region, $u = u_s + u^i$, satisfies the Helmholtz equation,

$$(\Delta + k^2)u(\rho, \varphi) = 0, \quad (1.2)$$

and that u is analytic with respect to ρ , φ and φ_0 , except possibly at the origin, and that there exists a constant s_0 such that $\int_0^\infty |u(\rho, \varphi) e^{-s_0 \rho}| d\rho < \infty$.

The total field u for $-\Phi_r \leq \varphi \leq \Phi_l$ is then represented as a Sommerfeld-Maliuzhinets integral [1],

$$u(\rho, \varphi) = \frac{1}{2\pi i} \int_\gamma f(\alpha + \varphi) e^{ik\rho \cos \alpha} d\alpha, \quad (1.3)$$

which satisfies the Helmholtz equation. In this representation, f is an analytic function and the path γ consists of two branches: one, named γ_+ , going from $(i\infty + \arg(ik) + (a_1 + \frac{\pi}{2}))$ to $(i\infty + \arg(ik) - (a_2 + \frac{\pi}{2}))$ with $0 < a_{1,2} < \pi$, as $\operatorname{Im}\alpha \geq d > 0$, above all the singularities of the integrand, and the other, named γ_- , obtained by inversion of γ_+ with respect to $\alpha = 0$. Stationary phase methods [2],[3] can be applied to (1.3) to find the far field diffraction coefficient $F(\alpha, \varphi_0) = f(\pi + \alpha) - f(-\pi + \alpha)$ [4].

1.1.1.2 Basic properties of the total field and its spectral function f

Some elementary properties can be assumed to hold for the field :

- (a') the only incoming plane wave, from the free space sector with origin O , $-\Phi_r \leq \varphi \leq \Phi_l$, is the incident field;
- (b') the limit of the field u as $\rho \rightarrow 0$ is finite and does not depend on φ , while the derivatives $\partial_\rho u$ and $\partial_\varphi u / \rho$ are locally summable with respect to ρ in the vicinity

Figure 1.1 Geometry of a wedge-shaped region

of the origin. This property applies for an origin taken at any point out of or upon the scatterer;

- (c') the field, except possibly its geometrical optics part when $\text{Im}(k) \neq 0$, does not grow at infinity. In addition, some bounds on the far field are assumed. We consider here that the field is $O(e^{ik\rho \cos(\varphi - \varphi_0)})$ for large ρ , $|\arg(ik)| < \frac{\pi}{2}$, which is a standard assumption in scattering theory.

These properties lead us to the following conditions on f [1],[4],[5] :

- (a) $(f(\alpha) - u^i(O)/(\alpha - \varphi_0))$ is regular at points with $\text{Re}(\alpha)$ belonging to the free space angular sector with origin O , $-\Phi_r \leq \text{Re}(\alpha) \leq \Phi_l$, which ensures (a').

- (b) there exist some constants $g^\pm$, some analytic function h , and some Maluzhinets contour γ such that $|f(\alpha + \varphi) \mp f(-\alpha + \varphi) - g^\pm| < |h(\alpha)|$ on and inside the loop formed by the upper branch γ_+ of γ , when $-\Phi_r \leq \varphi \leq \Phi_l$, the function h being summable on γ_+ , regular on and within it. In this respect, we notice that $(f(i|\ln \rho|) - f(-i|\ln \rho|)) = -iu(0, \varphi) + O(\frac{\rho \partial u}{\partial \rho})$, as $\rho \rightarrow 0$, with $\frac{\rho \partial u}{\partial \rho} = o((\ln \rho)^{-1})$ and $u(0, \varphi) = ig^+$. Since γ is odd, we can add a constant to f without changing u , which implies that we can define f with $f(i\infty) = -f(-i\infty)$. This ensures (b').

- (c) $f(\alpha + \varphi)$ has no singularity, except possibly those associated to incident, reflected or transmitted plane waves not vanishing at infinity, in the zone defined by $\text{Re}(ik \cos \alpha) > 0$ as $|\text{Re}(\alpha)| < \pi$, $-\Phi_r \leq \varphi \leq \Phi_l$, $|\arg(ik)| < \frac{\pi}{2}$. Considering that the far field is $O(e^{ik\rho \cos(\varphi - \varphi_0)})$, $f(\alpha + \varphi)$ has no singularity in this region when $\text{Re}(ik \cos(\varphi - \varphi_0)) < 0$, i.e. $\frac{\pi}{2} < |\varphi - \varphi_0| < \frac{3\pi}{2}$. This ensures (c').

1.1.2 Spectral functions $f_\pm$ attached to the radiation of a single face and simple relation to f

1.1.2.1 Radiation of a single face of a wedge-shaped region

To simplify the notation without losing generality, we take $\Phi_r = \Phi_l = \Phi$ with $0 < \Phi < \pi$. From the properties (b'), (c') on u , the scattered field in free space $u_s = u - u^i$

for $|\varphi| < \Phi$ can be written as the sum $u_+ + u_-$ of the radiations of equivalent surface currents carried by the faces $\varphi = +\Phi$ and $-\Phi$,

$$u_{\pm}(\rho, \varphi) = \frac{-i}{4} \lim_{\rho_0 \rightarrow 0^+} \int_{\rho_0}^{\infty} (u(\rho', \varphi') \frac{\partial H_0^{(2)}(kR)}{\partial n} - \frac{\partial u(\rho', \varphi')}{\partial n} H_0^{(2)}(kR))|_{\varphi' = \pm \Phi} d\rho' \quad (1.4)$$

with $R = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')}$, $\partial(\cdot)/\partial n = \hat{n} \nabla(\cdot) = \mp \partial(\cdot)/\rho' \partial \varphi'$, $\hat{n}$ the outward normal to the face $\varphi' = \pm \Phi$, $|\varphi| < \Phi$, $|\arg(ik)| < \frac{\pi}{2}$. We show that it is possible to simply express the spectral function $f_{\pm}$ attached to $u_{\pm}$ with f as $0 < \Phi < \pi$. For this, we use an original representation of $H_0^{(2)}(kR)$.

1.1.2.2 Spectral function attached to Sommerfeld-Maliuzhinets representation of $H_0^{(2)}(kR)$

As shown in [6], the spectral function for $H_0^{(2)}(kR)$, where $|\varphi'| > \frac{\pi}{2}$, $|\varphi| < |\varphi'|$, is given by

$$f_{H_0^{(2)}}(\alpha) = \frac{-1}{2\pi} \int_{\mathcal{S}} e^{ik\rho' \cos(\alpha' - \varphi')} (\tan(\frac{1}{2}(\alpha - \alpha')) - g_{\circ}(\alpha')) d\alpha', \quad (1.5)$$

for $\alpha \in]\mathcal{S} - \pi, \mathcal{S} + \pi[$ (i.e. the domain limited by $\mathcal{S} - \pi$ and $\mathcal{S} + \pi$), where $\mathcal{S}$ is the path from $-i\infty - \arg(ik)$ to $i\infty + \arg(ik)$ with $\text{Im}(k \sin \alpha) = 0$. By analyticity, the path $\mathcal{S}$ can be deformed continuously, as long as the integrand remains bounded, without changing $f_{H_0^{(2)}}$, and this expression can be continued, for $|\varphi'| \leq \frac{\pi}{2}$ by deforming $\mathcal{S}$, or by $f_{H_0^{(2)}}(\pi + \alpha) - f_{H_0^{(2)}}(-\pi + \alpha) = -2ie^{ik\rho' \cos(\alpha - \varphi')}$, for α outside $]\mathcal{S} - \pi, \mathcal{S} + \pi[$. The terms $g_{\circ}$, normally unnecessary because γ is odd, is chosen as $g_{\circ}(\alpha) = -\tan(\frac{\alpha}{2})$ in order to ameliorate the convergence of the integral.

1.1.2.3 Simple exact expression of single face spectral function $f_{\pm}$

Therefore, we can write the spectral function corresponding to $H_0^{(2)}(kR_{\varepsilon})$, with $R_{\varepsilon} = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varepsilon\Phi)}$, $\varepsilon \equiv +\text{or}-$, in a more general form,

$$f_{H_0^{(2)}}^{\varepsilon}(\alpha) = \frac{-1}{2\pi} \int_{\mathcal{S}_{\varepsilon}} e^{ik\rho' \cos(\alpha' - \varepsilon\Phi)} (\tan(\frac{1}{2}(\alpha - \alpha')) - g_{\circ}(\alpha')) d\alpha', \quad (1.6)$$

where $\mathcal{S}_{\varepsilon} = \mathcal{S} + a_{\varepsilon}$, with a_{ε} a constant satisfying $\Phi - \frac{3\pi}{2} < \varepsilon a_{\varepsilon} < \Phi - \frac{\pi}{2}$ for $0 < \Phi < \pi$.

The expression (1.6) is used for $H_0^{(2)}(kR_{\varepsilon})$ and $\partial_n H_0^{(2)}(kR_{\varepsilon})$ in (1.4), and we then obtain a Sommerfeld-Maliuzhinets representation of $u_{\pm}$ with f [4]-[6] :

$$u_{\pm}(\rho, \varphi) = \frac{1}{2\pi i} \int_{\gamma} f_{\pm}(\alpha + \varphi) e^{ik\rho \cos \alpha} d\alpha, \quad (1.7)$$

where

$$f_{\varepsilon}(\alpha) = \frac{1}{4\pi i} \int_{\mathcal{S}_{\varepsilon}} \varepsilon f(\varepsilon\pi + \alpha') (\tan(\frac{1}{2}(\alpha - \alpha')) - g_{\circ}(\alpha')) d\alpha', \quad (1.8)$$

for α between $\mathcal{S}_\varepsilon - \pi$ and $\mathcal{S}_\varepsilon + \pi$, provided $\frac{\pi}{2} < \Phi - \varepsilon a_\varepsilon < \frac{3\pi}{2}$, $\frac{\pi}{2} < \Phi - \varepsilon \varphi_0 < \frac{3\pi}{2}$ and $g^- = f(i\infty) + f(-i\infty) = 0$, $\varepsilon \equiv +\text{or}-$, $0 < \Phi < \pi$. Let us notice that the conditions (b) and (b'), which concern u and f , don't apply to u_ε and f_ε . However, we remark that $f_\varepsilon(\alpha + \varphi) - f_\varepsilon(-\alpha + \varphi)$ is bounded for large α on γ , even if $f_\varepsilon(\alpha)$ for $\text{Im}\alpha \rightarrow \infty$ (resp. u_ε for $\rho \rightarrow 0$) diverge when $f(\pm i\infty) \neq 0$ (resp. $u \neq 0$ at $\rho = 0$).

Since f is an analytic function, (1.8) can be analytically continued in the whole complex plane. We note in particular, that, taking into account of the poles of $\tan(\frac{1}{2}(\alpha - \alpha'))$ which can be captured by $\mathcal{S}_\varepsilon$ as α varies, $f_\pm$ satisfies

$$f_\pm(\pi + \alpha) - f_\pm(-\pi + \alpha) = \pm f(\pm\pi + \alpha). \quad (1.9)$$

Concerning the dependence on φ_0 (or Φ), the expression (1.8) has been determined for $\frac{\pi}{2} < \Phi - \varepsilon \varphi_0 < \frac{3\pi}{2}$, but we can consider $f_\varepsilon(\alpha)$ outside this domain by analytical continuation on φ_0 (or Φ), which corresponds to taking account of the contribution of any singularity that would go through $\mathcal{S}_\varepsilon$ as φ_0 (or Φ) goes into these regions.

1.1.2.4 Exact expression of f from diffraction coefficient F and consequences

We have $f(\alpha) = f_+(\alpha) + f_-(\alpha) + f_i(\alpha)$, with $f_i(\alpha) = \frac{\text{Res} f|_{\alpha=\varphi_0}}{2} \cot(\frac{1}{2}(\alpha - \varphi_0))$ and $f_\pm$ verifying (1.8) and (1.9). Thus, we can write, when both conditions $\frac{\pi}{2} < \Phi \mp \varphi_0 < \frac{3\pi}{2}$ (which implies $\Phi > \pi/2$) are satisfied,

$$f(\alpha) = \frac{1}{4\pi i} \int_{\mathcal{S}} F(\alpha', \varphi_0) (\tan(\frac{1}{2}(\alpha - \alpha')) - g_0(\alpha')) d\alpha' + f_i(\alpha), \quad (1.10)$$

for $\alpha \in]\mathcal{S} - \pi, \mathcal{S} + \pi[$. This expression can be extended, by analytical continuation on α , from $f(\pi + \alpha') - f(-\pi + \alpha') = F(\alpha', \varphi_0)$, and on φ_0 and Φ (for $\Phi \mp \varphi_0 < \frac{\pi}{2}$ and $\Phi < \pi/2$ by example) considering suitable deformations of $\mathcal{S}$. As a matter of fact, this expression uniquely defines f from its far field diffraction coefficient, which implies uniqueness and existence. From general properties of F , a continuation on φ_0 adds real singularities (geometrical optics terms), and we can assume that $f(\alpha)$ has only singularities on real axis for $\alpha \in]\mathcal{S} - \pi, \mathcal{S} + \pi[$, $|\varphi_0| < \Phi$. Besides, we remark that, if the system is reciprocal, i.e. $F(\alpha', \varphi_0) = F(\varphi_0, \alpha')$, and if equations on φ_0 variations of F are known, (1.10) implies equations on φ_0 variations of f [6].

1.1.3 Spectral function f from far field radiation of one face with arbitrary shape

1.1.3.1 Simple exact expression of the spectral function f derived from fields on a single planar face and properties

Stationary phase methods [2],[3] can be applied to (1.7) to find the far field radiation of the face $\varphi = \pm\Phi$, also denoted $\varphi = \varepsilon\Phi$. From the regularity of f_ε , we can deform γ to stationary phase points $\alpha = +\pi$ and $-\pi$, when $\frac{\pi}{2} < \Phi - \varepsilon \varphi_0 < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \Phi - \varepsilon \varphi < \frac{3\pi}{2}$, and thus, out of the reflected and shadowed regions $2\Phi - \varepsilon(\varphi + \varphi_0) < \pi$ and $|\varphi - \varphi_0| > \pi$, we can write, for large $k\rho$ and using (1.9),

$$-\sqrt{2\pi k\rho} e^{i(k\rho + \frac{\pi}{4})} u_\pm(\rho, \varphi) \sim f_\pm(\pi + \varphi) - f_\pm(-\pi + \varphi) = \pm f(\pm\pi + \varphi), \quad (1.11)$$

which can be considered as an important generalization of the result obtained by Michaeli for Dirichlet boundary conditions in [7]. In other respects, we can use in (1.4) the formulas [8],

$$H_0^{(2)}(kR) \sim \sqrt{\frac{2}{\pi kR}} e^{-ikR+i\pi/4}, \partial_n H_0^{(2)}(kR) \sim \sqrt{\frac{2}{\pi kR}} e^{-ikR-i\pi/4} \partial_n(kR), \quad (1.12)$$

with $R = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi - \varphi')}$, $\partial_n(kR) = \mp \partial(kR)/\rho' \partial\varphi' = (\pm k\rho \sin(\varphi - \varphi')/R)$ at $\varphi' = \pm\Phi$, and $\cos(\varphi \mp \Phi) < 0$. Taking into account that

$$R = \rho - \rho' \cos(\varphi \mp \Phi) + \frac{(\rho' \sin(\varphi \mp \Phi))^2}{R + \rho - \rho' \cos(\varphi \mp \Phi)}, \quad (1.13)$$

and considering the properties (b')-(c') on u , we then obtain another expression for the far field that, compared with (1.11), gives us :

$$f(\pm\pi + \varphi) = \frac{1}{2} \int_0^\infty (iku(\rho', \pm\Phi) \sin(\varphi \mp \Phi) \pm \frac{\partial u}{\partial n}(\rho', \pm\Phi)) e^{ik\rho' \cos(\varphi \mp \Phi)} d\rho', \quad (1.14)$$

as $\frac{\pi}{2} < \Phi \mp \varphi_0 < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \Phi \mp \varphi < \frac{3\pi}{2}$, $|\arg(ik)| < \frac{\pi}{2}$, $0 < \Phi < \pi$.

By changing Φ for $\Phi_e^\pm$ with $0 < \Phi_e^\pm < \pi$ and letting $\pm\pi + \varphi = \alpha \pm \Phi_e^\pm$, we then notice that,

$$f(\alpha \pm \Phi_e^\pm) = \frac{1}{2} \int_0^\infty (-iku(\rho', \pm\Phi_e^\pm) \sin \alpha \pm \frac{\partial u}{\partial n}(\rho', \pm\Phi_e^\pm)) e^{-ik\rho' \cos \alpha} d\rho', \quad (1.15)$$

when $\frac{\pi}{2} < \Phi_e^\pm \mp \varphi_0 < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \mp\alpha + \pi < \frac{3\pi}{2}$, which is also valid, by analytic continuation, as $\text{Re}(ik(\cos \alpha - \cos(\Phi_e^\pm \mp \varphi_0))) > 0$, $|\text{Re} \alpha| < \pi$ (note: on the straight semi-line $\varphi = \pm\Phi_e^\pm$, we have $\pm \frac{\partial u(\rho', \varphi')}{\partial n} = -\frac{\partial u(\rho', \varphi')}{\rho' \partial \varphi'}$).

Thus, a knowledge of the field and of its normal derivative on one single face (a semi-line) gives a unique definition of the analytical spectral function f attached to the 'total' radiated field (uniqueness and existence). By the way, we also notice that, in accordance with the property (b), we have from (1.15), $(f(ix \pm \Phi_e^\pm) - f(-ix \pm \Phi_e^\pm)) = -iu(0, \pm\Phi_e^\pm) + O(\frac{\partial u}{\partial \ln \rho_e} |_{|\ln \rho_e|=x})$ as $x \rightarrow \infty$.

1.1.3.2 Deformation and simple exact expression of the spectral function f from fields on a piecewise smooth single face

Using Green's theorem, we note that the contour of integration along $\varphi = \pm\Phi$ in (1.4) can be deformed into a new path $L_{0,\infty}^\pm$ from the origin to infinity without changing the field $u_\pm$ (except at points captured by the path during its deformation), provided that the integral remains bounded and no source passes through the path during the deformation. Thus, we can write (1.14) in a more general form, with integration along a piecewise smooth path $L_{0,\infty}^\pm$, following

$$f(\pm\pi + \varphi) = \frac{1}{2} \int_{L_{0,\infty}^\pm} (iku(\rho', \varphi') \sin(\varphi - \varphi') \pm \frac{\partial u}{\partial n}(\rho', \varphi')) e^{ik\rho' \cos(\varphi - \varphi')} dl', \quad (1.16)$$

where dl' and φ'_l , both depending on (ρ', φ') , are respectively the element of length and the tangent angle along the piecewise smooth semi-line $L_{0,\infty}^\pm$.

If we divide the semi-infinite paths $L_{0,\infty}^\pm$ into $L_{0,\Delta^\pm}^\pm$ (i.e. $0 < l' < \Delta^\pm$) and $L_{\Delta,\infty}^\pm$ (i.e. $l' > \Delta^\pm$), we have

$$\begin{aligned} f(\pm\pi + \varphi) = & \frac{1}{2} \int_{L_{0,\Delta^\pm}^\pm} (iku(\rho', \varphi') \sin(\varphi - \varphi'_i) \\ & \pm \frac{\partial u}{\partial n}(\rho', \varphi')) e^{ik\rho' \cos(\varphi - \varphi')} dl'(\rho', \varphi') + f_{L_{\Delta,\infty}^\pm}(\pm\pi + \varphi), \end{aligned} \quad (1.17)$$

where $f_{L_{\Delta,\infty}^\pm}(\alpha) = e^{-ik\rho_{\Delta^\pm} \cos(\alpha - \varphi_{\Delta^\pm})} f_e^\pm(\alpha)$, $f_e^\pm(\alpha)$ is the spectral function related to a shift of the origin at $l' = \Delta^\pm$. This implies, by analytic continuation,

$$\begin{aligned} f(\alpha) = & \frac{1}{2} \int_{L_{0,\Delta^\pm}^\pm} (-iku(\rho', \varphi') \sin(\alpha - \varphi'_i) \\ & \pm \frac{\partial u}{\partial n}(\rho', \varphi')) e^{-ik\rho' \cos(\alpha - \varphi')} dl'(\rho', \varphi') + f_{L_{\Delta,\infty}^\pm}(\alpha). \end{aligned} \quad (1.18)$$

Our expression can then lead us to a suitable solution for complex geometries and boundary conditions, in particular for the scattering by an impedance polygon [4].

1.1.4 Exact causal time domain representation of a field above a dispersive wedge-shaped region

According to the behaviour of f (in particular for ω with large $\text{Im}(\omega) = -\sigma \leq 0$), we can assume that the path γ can be deformed into a contour $\mathcal{D}$ which consists of two branches $\mathcal{D}_\pm$: $\mathcal{D}_+$ with $\text{Im}\alpha > 0$, given by $](i\infty + \pi), \pi + i0^+] \cup [\pi + i0^+, -\pi + i0^+] \cup [-\pi + i0^+, i\infty - \pi[$, above all the singularities of the integrand as $|\text{Re}\alpha| < \pi$, and $\mathcal{D}_-$, obtained by inversion of $\mathcal{D}_+$ with respect to $\alpha = 0$. Let us prove, from the single face expression of the function f , that

$$\mathcal{F}_c(\alpha + \varphi, \tau) = \frac{1}{2\pi} \int_{-\infty - i\sigma}^{+\infty - i\sigma} f(\alpha + \varphi, \omega) e^{i\omega\tau} d\omega, \quad (1.19)$$

vanishes as $\tau < 0$ from field causality, so that $f(\alpha + \varphi)$ is regular and $O(\omega^{-\nu'})$ on $\mathcal{D}$ with $\nu' > 1$ as $\text{Im}\omega \leq 0$. The time domain field $\bar{u}(t)$ deriving from Fourier transform of u , will be then simplified and expressed in an efficient explicitly causal expression on a finite subset $\mathcal{H}$ of $\mathcal{D}$ [6],[9].

1.1.4.1 Causality of $\mathcal{F}_c(\alpha + \varphi, \tau)$

Let us use in (1.19) the single face expression (1.15) of $f(\alpha \pm \Phi_e^\pm)$ for $\text{Re}\alpha = 0$. Changing the order of the integration (permitted), we can write

$$\begin{aligned} \mathcal{F}_c(\alpha' \pm \Phi_e^\pm, \tau) = & \frac{1}{4\pi} \int_0^\infty d\rho' \int_{-\infty - i\sigma}^{+\infty - i\sigma} d\omega \\ & (-iku(\rho', \pm\Phi_e^\pm) \sin \alpha \pm \frac{\partial u}{\partial n}(\rho', \pm\Phi_e^\pm)) e^{i\omega t'}, \end{aligned} \quad (1.20)$$

for $\text{Re}\alpha' = 0$ with $t' = -\rho \cos \alpha' / c + \tau$, $\frac{\pi}{2} < \Phi_e^\pm \mp \varphi_0 < \frac{3\pi}{2}$, $\Phi_e^\pm \leq \Phi$. The field u is non singular and $O(\omega^{-\nu})$, with $\nu > 1$, as $\text{Im}\omega < 0$ and $\bar{u}(t)$ is causal with respect

to the front of the incident plane wave, so that the contour of integration in ω can be closed at infinity for $\text{Im}(\omega) < 0$ when $t' < -\rho/c$, which implies that $\mathcal{F}_c(\alpha' + \varphi, \tau) = 0$ for $\text{Re}\alpha' = 0$ as $\tau < 0$. Considering the analyticity of f , $\mathcal{F}_c(\alpha' + \varphi, \tau)$ is assumed to be an analytic function of α' as τ is fixed, when $|\text{Re}(\alpha' + \varphi)| \leq \pi + \Phi$ and $|\text{Im}\alpha'| > 0^+$, and we then have,

$$\mathcal{F}_c(\alpha' + \varphi, \tau) = 0, \tau < 0, |\text{Re}(\alpha' + \varphi)| \leq \pi + \Phi. \quad (1.21)$$

1.1.4.2 Spectral causal expression of the field in time domain

From (1.3) and the causality of $\mathcal{F}_c$, the field in time domain is then [6],[9],

$$\bar{u}(t) = \frac{1}{2\pi i} \int_{\mathcal{H}} \mathcal{F}_c(\alpha + \varphi, \tau_c(\alpha)) d\alpha, \quad (1.22)$$

where $\mathcal{H} = \mathcal{H}_+ \cup \mathcal{H}_-$ is a finite subset of $\mathcal{D}$ with $\tau_c(\alpha) = \rho \cos \alpha / c + t > 0$. Let $iv(\rho, t) = \begin{cases} i \ln(ct/\rho + \sqrt{(ct/\rho)^2 - 1}), & ct \geq \rho \\ -\arccos(ct/\rho), & -\rho < ct < \rho \end{cases}$, $\mathcal{H}_+$ is then given by $[\pi + iv + i0^+, -\pi - iv + i0^+]$ as $|ct| < \rho$, and $[\pi + iv(\rho, t), \pi + i0^+] \cup [\pi + i0^+, -\pi + i0^+] \cup [-\pi + i0^+, iv(\rho, t) - \pi]$ as $ct \geq \rho$, above all the real singularities of the integrand as $|\text{Re}\alpha| < \pi$, and $\mathcal{H}_-$ is its inverse relatively to the origin. Considering the generality of (1.21), this expression can be used for a large class of causal dispersive wedge-shaped region, in particular for a wedge with frequency-dependent surface impedances [6],[9].

1.2 Several Orders Asymptotic Representation for Scattering by a Curved Impedance Wedge

1.2.1 Asymptotic representation in a region without creeping waves

1.2.1.1 Maliuzhinets type representation and asymptotic boundary conditions on curved faces

We consider a wedge with tangent planes at the edge defined by $\varphi = \pm\Phi$, in cylindrical coordinates (ρ, φ) , and curved faces of radius $a^\pm$ (fig. 1.2), with $\Phi > \pi/2$ (note: $\Phi = \pi/2$ is obtained by letting $\Phi \rightarrow \pi/2$). The electric and magnetic potentials E_z and H_z , indifferently denoted u , verify outside the wedge $(\Delta + k^2)u = 0$, with $k = \frac{\omega}{c}$ the wave number of the free space, $|\arg(ik)| < \pi/2$, and $e^{i\omega t}$ time dependence. As in previous section, we consider a plane wave illumination, with φ_0 the angle of incidence, and the total field u is represented as an integral of Sommerfeld-Maliuzhinets (1.3) with a contour γ composed of two infinite loops in the complex plane [10],[11],[12],[13], for $|\varphi| < \Phi$, with f to be defined. After deformation of γ to the steepest descent path SDP, we can write

$$u(\rho, \varphi) = u_i + \sum u_r^\pm + \sum u_s^\pm + \frac{e^{-ik\rho}}{2\pi i} \int_{SDP} f(\alpha + \varphi) e^{ik\rho(\cos\alpha + 1)} d\alpha, \quad (1.23)$$

where

- the term u_i is the incident field in the illuminated zone and zero in the shadow zone;

Figure 1.2 Geometry of a curved impedance wedge

•the terms $u_r^\pm$ correspond to geometrical optics field reflected by each faces $\varphi = \pm\Phi$ (note: some secular terms are present in the asymptotic development of this term at high order).

•the terms $u_s^\pm$ are terms of guided waves excited by the edge, which are the contribution of complex pôles $\alpha_s^\pm$ of $f(\alpha + \varphi)$;

•the last term, named u_d , is principally radiated cylindrically when $\rho \rightarrow \infty$. Approximating $f(\alpha + \varphi)$ on $SDP_\pm$ by its value at the stationnary phase points $\alpha = \pm\pi$, we obtain

$$u_d \sim \frac{-e^{-i\pi/4 - ik\rho}}{\sqrt{2\pi k\rho}} [f(\pi + \varphi) - f(-\pi + \varphi)], \quad (1.24)$$

where $F(\varphi) = f(\pi + \varphi) - f(-\pi + \varphi)$ is the diffraction (or far field) coefficient.

We assume impedance boundary conditions on the surface of the wedge,

$$\left(\frac{\partial}{\partial n} - ik \sin \theta^\pm\right)u|_{\text{curved face}\pm} = 0, \quad (1.25)$$

with n the outward normal, and $\sin \theta^\pm$ some constants with $\text{Re} \theta^\pm > 0$.

We approximate (1.25) by asymptotic conditions in entire powers of the curvature $1/a^\pm$ of each face, $\mathcal{B}^\pm u = 0$ on tangent plane at the edge $\varphi = \pm\Phi$, which gives [10],[12], [14],[15] :

$$\left(\mp \frac{\partial}{\rho \partial \varphi} - ik \sin \theta^\pm - \frac{1}{2a^\pm} \left(\frac{\partial^2}{\partial \varphi^2} - \frac{\rho \partial}{\partial \rho} \pm ik \sin \theta^\pm \frac{\rho \partial}{\partial \varphi}\right) + O\left(\frac{\rho}{(a^\pm)^2}\right)\right)u = 0. \quad (1.26)$$

From the Maliuzhinets inversion theorem applied to $\mathcal{B}^\pm u = 0$, we can write,

$$B_{\alpha \mp \frac{\pi}{2}}^\pm f(\alpha \pm \Phi) - B_{-\alpha \mp \frac{\pi}{2}}^\pm f(-\alpha \pm \Phi) = 0, \quad (1.27)$$

where $B_{\alpha+2\pi}^\pm = B_\alpha^\pm$ (which implies $B_{\alpha \pm \frac{\pi}{2}}^\pm F(\alpha \pm \Phi) + B_{-\alpha \pm \frac{\pi}{2}}^\pm F(-\alpha \pm \Phi) = 0$), and,

$$B_{\alpha \mp \frac{\pi}{2}}^\pm = \left(\sin \alpha \pm \sin \theta^\pm \pm \frac{D_\alpha^\pm(\cdot)}{ika^\pm} + O\left(\frac{1}{(ka^\pm)^2}\right)\right),$$

$$D_\alpha^\pm(\cdot) = \frac{1}{2} \left[\frac{\partial^2(\cdot)}{\partial \alpha^2} - \frac{\partial}{\partial \alpha} (\cotg \alpha(\cdot)) \pm \sin \theta^\pm \frac{\partial}{\partial \alpha} \left(\frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} (\cdot) \right) \right]. \quad (1.28)$$

We then let $f = \sum_{n \geq 0} f_n/k^n$ in the functional equation. Considering the terms of same powers in k , we then obtain a succession of functional equations of order m on

the $f_{n \leq m}$, that we can solve, considering $f_\circ$ as the spectral function for a wedge with plane faces [13] (note: the constants $\sin \theta^\pm$ are generally different from the ones for plane faces).

1.2.1.2 The first term of f influenced by the curvatures

Let us detail the integral expression given initially in [10],[11] for f_1 , which is the first term influenced by the curvatures. For this, we let $f_\circ(\alpha) = \Psi(\alpha)\sigma(\alpha)/\Psi(\varphi_\circ)$ where $\sigma(\alpha) = \mu \cos(\mu \varphi_\circ)/(\sin(\mu \alpha) - \sin(\mu \varphi_\circ))$, $\mu = \pi/2\Phi$, $\varphi_\circ$ the angle of incidence, and $f_1(\alpha)/k = \Psi(\alpha)\chi(\alpha)/k$. Taking the initial expression of χ in [10], we can shift the integration path with arbitrary real quantities $\pm d^\pm$ along the real axis, for each term concerning the faces $\pm$, and, considering poles captured during the deformation, we obtain :

$$\begin{aligned} \chi(\alpha') = & \frac{i}{8\Phi\Psi(\varphi_\circ)} \sum_{\pm} \frac{1}{ia^\pm} \left(\int_{-i\infty \pm d^\pm}^{i\infty \pm d^\pm} d\alpha [(-W^\pm \partial_\alpha \ln((- \sin \alpha \pm \sin \theta^\pm) \Psi(-\alpha \pm \Phi)) \right. \\ & + \partial_\alpha W^\pm)(-\sigma(\alpha \pm \Phi) \partial_\alpha \ln(\frac{\Psi(-\alpha \pm \Phi)}{(\sin \alpha \pm \sin \theta^\pm)}) \\ & \left. - \partial_\alpha \sigma(\alpha \pm \Phi)) \frac{1}{\sin \alpha}] \mp 2\pi i \operatorname{sgnd}^\pm \sum_{0 < \pm \operatorname{sgnd}^\pm \operatorname{Re} \alpha'_s < |d^\pm|} \operatorname{Residue}[\dots]_{\alpha=\alpha'_s} \right), \end{aligned} \quad (1.29)$$

where $W^\pm = \operatorname{tg}(\frac{\pi}{4\Phi}(\alpha' \pm \Phi - \alpha))$, Ψ is defined in [13] for wedge with plane faces. We note that this expression is the analytical continuation of the one given in [10] for $\operatorname{Re}(\alpha') \in]-3\Phi + d^+, \Phi + d^+[\cap]-\Phi - d^-, 3\Phi - d^-[$, which allows (also completed by (1.27)) a definition for any α' .

1.2.2 Several orders asymptotic expressions in a region with creeping waves terms

1.2.2.1 Plane wave illumination and observation at infinity

We note, from (1.27) and (1.29), that $f_1(\pm(\pi + \Phi))$ and then the expression (1.24) of u_d is singular for $\varphi = \pm\Phi$. In fact, near the tangents of the faces $\pm\Phi$, it is preferable to consider the radiation function F as the radiation of equivalent surface currents on a cylinder of radius $a^\pm$, that we characterize from the previous asymptotics of F , thereafter named $F_<$. Therefore, we write [10],[11],[12], when $|\pm\varphi - \Phi| < \pi/2$, $|\pm\varphi_\circ - \Phi| > \pi$,

$$F(\varphi) \equiv \frac{e^{ika^\pm \sin(\pm\varphi - \Phi)}}{i\pi ka^\pm} \int_{\mathcal{E}} \frac{e^{-iv(\pm\varphi - \Phi)} \mathcal{M}^\pm(\pm v)}{H_v^{(2)'}(ka^\pm) + Q^\pm H_v^{(2)}(ka^\pm)} dv, \quad (1.30)$$

with $Q^\pm = -i \sin \theta^\pm$, and $\mathcal{E}$ a path from $-\infty - i\varepsilon$ to $+\infty - i\varepsilon$ with $\operatorname{Im} v < 0$. Letting $v = ka^\pm \sin \alpha'$ with $|\operatorname{Re} \alpha'| \leq \pi/2$ and $|\arg(ik)| < \pi/2$, and α' going from $-i\infty - \arg(ik) + 0^+$ to $i\infty + \arg(ik) + 0^+$ on the integration path, an asymptotic expansion of $\mathcal{M}^\pm$ is given by

$$\mathcal{M}^\pm(\pm ka^\pm \sin \alpha') = \frac{i\pi ka^\pm}{2\cos \alpha'} \mathcal{O}_{\pm\alpha'}^\pm(F_<(\pm(\alpha' - \pi/2 + \Phi))), \quad (1.31)$$

where $\mathcal{O}_\alpha^\pm = \pm \frac{2}{i\pi ka^\pm} ((1 + \sum_{p \geq 1} \frac{\mathcal{P}_p^\pm(\cdot)}{(a^\pm)^{p+2}}) B_\alpha^\pm)$, and $\mathcal{P}_p^\pm$ are differential operators of $\sin \alpha$ [12]. We have, letting $\alpha = \alpha' - \pi/2$ in (1.31) and using (1.28),

$$\begin{aligned} \mathcal{M}^\pm(\pm ka^\pm \cos \alpha) &= \frac{1}{\sin \alpha} \left[\frac{-i\pi ka^\pm}{2} \mathcal{O}_{\pm(\alpha+\pi/2)}^\pm(\cdot) \right] F_<(\pm(\alpha + \Phi)) \\ &= \frac{1}{\sin \alpha} \left[(\sin \alpha - \sin \theta^\pm) \cdot (\cdot) - \frac{D_{\pm\alpha+\pi}^\pm(\cdot)}{ika^\pm} + O\left(\frac{1}{(ka^\pm)^2}\right) \right] F_<(\pm(\alpha + \Phi)). \end{aligned} \quad (1.32)$$

where $\mathcal{M}^\pm(\pm ka^\pm \cos \alpha)$ has no singularity at $\alpha = 0$, from (1.27) and $\mathcal{O}_{\alpha \pm \frac{\pi}{2}}^\pm F_<(\alpha \pm \Phi) + \mathcal{O}_{-\alpha \pm \frac{\pi}{2}}^\pm F_<(-\alpha \pm \Phi) = 0$ [12].

The path $\mathcal{E}$ is such that the singularities of the integrand in (1.30), due to $\mathcal{M}^\pm$ and to zeros of $H_V^{(2)'}(ka^\pm) + Q^\pm H_V^{(2)}(ka^\pm)$, are respectively above and below $\mathcal{E}$. Deforming the path $\mathcal{E}$ so that no singularity crosses it, we can analytically continue the edge diffraction coefficient for arbitrary φ_0 . Besides, by application of reciprocity principle, our expression remains valid if we exchange the roles of φ_0 and φ . Furthermore, we will derive an uniform transition expression at finite distance with all radiation terms.

1.2.2.2 Plane wave illumination and observation at finite distance

To consider an observation point (ρ, φ) at finite distance, we can then use that,

$$u(\rho, \varphi) = -\frac{1}{2\pi i} \int_{\mathcal{S}_0} F(\alpha \pm (\Phi - \pi/2)) e^{-ik\rho \cos(\alpha \pm (\Phi - \pi/2) - \varphi)} d\alpha, \quad (1.33)$$

is valid for $0 < \pm\varphi < \Phi$ when no source in the zone $\Phi - \pi < \pm\varphi < \Phi$, with correctly chosen $\mathcal{S}_0$ from $-\arg(ik) - i\infty$ to $\arg(ik) + i\infty$. We then use (1.30) in (1.33) with $\mathcal{L}_<^\pm \equiv \frac{\mathcal{M}^\pm}{i\pi ka^\pm}$, following,

$$\begin{aligned} u(\rho, \varphi) &= \frac{1}{2\pi i} \int_{\mathcal{E}} dv \frac{\mathcal{L}_<^\pm(\pm v)}{H_V^{(2)'}(ka^\pm) + Q^\pm H_V^{(2)}(ka^\pm)} \times \\ &\quad \int_{\mathcal{S}_0} d\alpha e^{-ik\rho \cos(\alpha \pm (\Phi - \pi/2) - \varphi) - iv(\pm\alpha - \pi/2) - ika^\pm \cos \alpha} \\ &= \int_{\mathcal{E}} dv \frac{\mathcal{L}_<^\pm(\pm v)}{H_V^{(2)'}(ka^\pm) + Q^\pm H_V^{(2)}(ka^\pm)} \times \\ &\quad \frac{e^{\mp iv\Omega}}{2i} \left(\frac{e^{iv\pi/2}}{\pi} \int_{\mathcal{S}_0} e^{-ikR \cos(\alpha - \Omega) - iv(\pm(\alpha - \Omega))} d\alpha \right), \end{aligned} \quad (1.34)$$

and we obtain,

$$u(\rho, \varphi) = \frac{1}{2i} \int_{\mathcal{E}} \frac{\mathcal{L}_<^\pm(\pm v) H_V^{(2)}(kR) e^{\mp iv\Omega}}{H_V^{(2)'}(ka^\pm) + Q^\pm H_V^{(2)}(ka^\pm)} dv, \quad (1.35)$$

where, relatively to the curvature center of each face, Ω is the angle from the edge to the point (ρ, φ) , and $R = \sqrt{\rho^2 + (a^\pm)^2 + 2\rho a^\pm \cos(\Phi - \pi/2 \mp \varphi)}$ is the distance to (ρ, φ) . The domain of validity of (1.35) is $|\pm\varphi - \Phi| < \pi/2$.

When we go beyond the frontier $\varphi = \pm\Phi$, there is no more stationary phase point for this integral, and considering the residues terms at zeros $v_p^\pm = ka^\pm \cos \alpha_p^\pm$ of $[H_v^{(2)'}(ka^\pm) + Q^\pm H_v^{(2)}(ka^\pm)]$ in the zone $\text{Im}v < 0$, then closing the contour $\mathcal{C}$ at infinity, we obtain, when $\pm\varphi > \Phi$,

$$u(\rho, \varphi) = \sum_p \frac{-\pi \mathcal{L}_<^\pm(\pm v_p^\pm) H_{v_p^\pm}^{(2)}(kR) e^{\mp i v_p^\pm \Omega}}{\frac{\partial}{\partial v} (H_v^{(2)'}(ka^\pm) + Q^\pm H_v^{(2)}(ka^\pm))|_{v_p^\pm}}. \quad (1.36)$$

The expressions of F and u are then uniform in the region $\varphi \sim \pm\Phi$. It permits the continuous transformation of the field into creeping waves (a development in entire powers then changes continuously to one in fractional powers as φ varies), for any wedge angle. This result has been taken up and confirmed for first terms by Borovikov in his paper on impedance wedge with curved sides [14].

We remark that, contrary to expressions of Albertsen and Christiansen [16], the operators go beyond first order. We can then consider a second order development of f without making diverge the expression (1.35), and we can study, for example, at low contrasts, the case of a curved plane with a discontinuity of material 'and' of curvature.

Remark : Noticing that the diffraction coefficient $F_<$ for a cylinder $\mathcal{C}_{a^\pm, \infty}$ of large radius $a^\pm$ without discontinuity is null, we can analytically continue the solution for $\Phi - \pi < \pm\varphi_0 < \Phi$, by adding the exact contribution of $\mathcal{C}_{a^\pm, \infty}$ in (1.36).

Remark : For $v \approx ka^\pm$, $H_v^{(2)'}(ka^\pm) + Q^\pm H_v^{(2)}(ka^\pm) \sim \frac{i}{(m^\pm)^2 \sqrt{\pi}} (w_2'(\tau) - q^\pm w_2(\tau))$, where $m^\pm = (ka^\pm/2)^{1/3}$, $q^\pm = -im^\pm \sin \theta^\pm$, $v = ka^\pm + m^\pm \tau$, $w_2 = \sqrt{\pi}(Bi - iAi)$.

1.2.2.3 Point source illumination and observation at finite distance

We can then derive an expression at finite distance for observation point (ρ', φ') and point source (ρ, φ) radiating $u^i = \frac{i}{2} H_o^{(2)}(k\sqrt{\rho^2 + (\rho')^2 + 2\rho\rho' \cos(\varphi - \varphi')})$. For this, we consider reciprocity, exchange the directions of observation and of incidence in $F_<$, let $\mathcal{L}^\pm(\pm ka^\pm \cos \alpha') = \mathcal{A}_{\alpha'}(\frac{1}{2\pi i} \int_\gamma f(\alpha + \varphi, \pm(\alpha' + \Phi)) e^{ik\rho \cos \alpha} d\alpha)$, with $\mathcal{A}_\alpha = \frac{-\mathcal{O}_{\pm(\alpha+\pi/2)}^\pm}{2 \sin \alpha}$, and obtain, when $|\pm\varphi - \Phi| > \pi$ and $\varphi' \sim \pm\Phi$,

$$u(\rho', \varphi', \rho, \varphi) = \frac{1}{2i} \int_{\mathcal{C}} \frac{\mathcal{L}^\pm(\pm v') H_{v'}^{(2)}(kR') e^{\mp i v' \Omega'}}{H_{v'}^{(2)'}(ka^\pm) + Q^\pm H_{v'}^{(2)}(ka^\pm)} dv', \quad (1.37)$$

where $R' = \sqrt{\rho'^2 + (a^\pm)^2 + 2\rho'a^\pm \cos(\Phi - \pi/2 \mp \varphi')}$, and Ω' is the angle, relatively to the cylinder center, from the edge to the observation point (ρ', φ') .

This expression remains valid, by analytical continuation, for $|\varphi| < \Phi$, and by reciprocity, when the roles of (ρ, φ) and (ρ', φ') , as source and observation points, are exchanged.

When we go beyond the frontier $\varphi' = \pm\Phi$, there is no more stationnary phase point for this integral, and considering the residues terms at zeros $v_p^\pm$ of $[H_v^{(2)'}(ka^\pm) + Q^\pm H_v^{(2)}(ka^\pm)]$ in the zone $\text{Im}v < 0$, we obtain, when $\pm\varphi' > \Phi$,

$$u(\rho', \varphi', \rho, \varphi) = \sum_p \frac{-\pi \mathcal{L}^\pm(\pm v_p^\pm) H_{v_p^\pm}^{(2)}(kR') e^{\mp i v_p^\pm \Omega'}}{\frac{\partial}{\partial v} (H_v^{(2)'}(ka^\pm) + Q^\pm H_v^{(2)}(ka^\pm))|_{v_p^\pm}}, \quad (1.38)$$

where $\mathcal{L}^\pm(\pm ka^\pm \cos \alpha_p) = \mathcal{A}_{\alpha_p}(\frac{1}{2\pi i} \int_\gamma f(\alpha + \varphi, \pm(\alpha_p + \Phi)) e^{ik\rho \cos \alpha} d\alpha)$ can be developed as $\mathcal{A}_{\alpha_p}(u_{i,p} + \sum u_{r,p}^\pm + \sum u_{s,\varepsilon,p}^\pm + \frac{e^{-ik\rho}}{2\pi i} \int_{SDP} f(\alpha + \varphi, \pm(\alpha_p + \Phi)) e^{ik\rho(\cos \alpha + 1)} d\alpha)$.

For $l\varphi > \Phi$ and $l'\varphi' > \Phi$ (l, l' are equal to +or-1), the problem is more complex and we need to express $\mathcal{L}^\pm(\pm v')$ as an integral of previous form. We then obtain for the diffracted part of the field :

$$\begin{aligned} u_{da} = & \pi^2 \sum_{p,q} \mathcal{A}_{\alpha_p}^l \mathcal{A}_{\alpha_q}^{l'} (f(\pi + l(\alpha_p^l + \Phi), l'(\alpha_q^{l'} + \Phi)) \\ & - f(-\pi + l(\alpha_p^l + \Phi), l'(\alpha_q^{l'} + \Phi))) \frac{H_{v_p^l}^{(2)}(kR) e^{-l i v_p^l \Omega}}{\frac{\partial}{\partial v} (H_v^{(2)'}(ka^l) + Q^l H_v^{(2)}(ka^l))|_{v=v_p^l}} \\ & \times \frac{H_{v_q^{l'}}^{(2)}(kR') e^{-l' i v_q^{l'} \Omega'}}{\frac{\partial}{\partial v} (H_v^{(2)'}(ka^{l'}) + Q^{l'} H_v^{(2)}(ka^{l'}))|_{v=v_q^{l'}}}, \end{aligned} \quad (1.39)$$

Let us notice that this expression has a form similar to the one defined by Thompson for the particular case of a discontinuity of materials on a circular cylinder.

Remark : If we denote $f_{a^\pm, \infty}$ the asymptotic spectral function for a cylinder $\mathcal{C}_{a^\pm, \infty}$ of radius $a^\pm$ (resp. for the PO contribution of face $\pm$), we can analytically continue (1.37)-(1.38) for $\Phi - \pi < \pm\varphi < \Phi$ by letting $f(\alpha) - f_{a^\pm, \infty}(\alpha \mp (\Phi - \frac{\pi}{2}))$ in place of $f(\alpha)$ and add the exact contribution of $\mathcal{C}_{a^\pm, \infty}$ (resp. of PO currents on face $\pm$) in (1.37)-(1.38).

1.2.3 Some validations concerning the expression of $f = \sum_{n \geq 0} f_n / k^n$ for arbitrary wedge angle

1.2.3.1 Perfectly conducting case: curved half plane to discontinuity of curvature

We now compare the results obtained from (1.29) with the expressions known for some particular wedge angles, with $\Phi \geq \pi/2$, in the limit case $\sin \theta^\pm \rightarrow \infty$ so that

$\Psi = 1$. Letting $\varepsilon = \text{sgn}(\text{Re}\alpha')$, we choose to take $d^+ = (1 + \varepsilon)\Phi$ and $d^- = (1 - \varepsilon)\Phi$. Denoting $a^\delta = a^+$ or a^- when $\delta = +1$ or -1 , we can write

$$\begin{aligned} \chi(\alpha') = & \left(\frac{1}{8\Phi} \partial_{\alpha'} \partial_{\varphi_0} \int_{-i\infty}^{i\infty} d\alpha \left[\text{tg} \left(\frac{\pi}{4\Phi} (\alpha' - \varepsilon\Phi - \alpha) \right) \frac{\frac{\pi}{2\Phi} \sin \frac{\pi}{2\Phi} \alpha}{\left(\cos \frac{\pi}{2\Phi} \alpha + \varepsilon \sin \frac{\pi}{2\Phi} \varphi_0 \right)} \right. \right. \\ & \times \left(\frac{1}{a^{-\varepsilon} \sin \alpha} + \frac{1}{a^\varepsilon \sin(\alpha + 2\varepsilon\Phi)} \right) \left. \right] + \left(i \frac{\varepsilon}{a^\varepsilon} \frac{\pi}{4\Phi} \partial_{\alpha'} \partial_{\varphi_0} \left[\frac{1}{\sin(\varepsilon\Phi - \varphi_0)} \right. \right. \\ & \times \left. \left. \text{tg} \left(\frac{\pi}{4\Phi} (\alpha' + \varphi_0) \right) + \text{tg} \left(\frac{\pi}{4\Phi} (\alpha' + \varepsilon(\Phi - \pi)) \right) \frac{\frac{\varepsilon\pi}{2\Phi} \sin \frac{\pi}{2\Phi} \pi}{\left(\cos \frac{\pi}{2\Phi} \pi - \varepsilon \sin \frac{\pi}{2\Phi} \varphi_0 \right)} \right] \right), \quad (1.40) \end{aligned}$$

for $\text{Re}(\alpha') \in]-3\Phi, 3\Phi[$ with $\varepsilon = \text{sgn}(\text{Re}\alpha')$.

In the case of a curved half-plane, $\Phi = \pi$ and $\varepsilon a^\varepsilon = a$, and so we obtain

$$\chi(\alpha') = (0) + i \frac{1}{4a} \partial_{\alpha'} \partial_{\varphi_0} \left[\frac{1}{\sin(\varphi_0)} \text{tg} \left(\frac{1}{4} (\alpha' + \varphi_0) \right) - \frac{1}{2} \text{tg} \left(\frac{1}{4} \alpha' \right) \frac{1}{\sin(\frac{1}{2} \varphi_0)} \right]. \quad (1.41)$$

This gives us the second order term for far field function

$$(f_1(\pi + \varphi) - f_1(-\pi + \varphi))/k = \frac{i}{4ka} \partial_{\varphi} \partial_{\varphi_0} \left(\frac{\text{tg}((\varphi + \varphi_0)/2)}{\cos(\varphi/2) \cos(\varphi_0/2)} \right), \quad (1.42)$$

which is conform to the result of Borovikov [17], first developed by V.B. Filippov.

For the discontinuity of curvature, $\Phi = \frac{\pi}{2}$ and then the integral term for χ is

$$\begin{aligned} & \frac{1}{4\pi} \partial_{\alpha'} \partial_{\varphi_0} \int_{-i\infty}^{i\infty} d\alpha \left[\frac{\cos \alpha'}{(\cos \alpha + \varepsilon \sin \alpha')} \frac{1}{(\cos \alpha + \varepsilon \sin \varphi_0)} \left(\frac{1}{a^+} - \frac{1}{a^-} \right) \right] \\ & = \frac{i}{2\pi} \partial_{\alpha'} \partial_{\varphi_0} \left(\frac{\varepsilon \cos \alpha'}{(\sin \varphi_0 - \sin \alpha')} \left(\frac{\pi/2 - \varepsilon \alpha'}{\cos \alpha'} - \frac{\pi/2 - \varepsilon \varphi_0}{\cos \varphi_0} \right) \left(\frac{1}{a^+} - \frac{1}{a^-} \right) \right), \quad (1.43) \end{aligned}$$

for $-\pi/2 < \varepsilon \text{Re}\alpha' < 3\pi/2$. The expression for χ then gives us

$$(f_1(\pi + \varphi) - f_1(-\pi + \varphi))/k = -2i \frac{\cos \varphi \cos \varphi_0}{(\sin \varphi + \sin \varphi_0)^3} \left(\frac{1}{ka^+} - \frac{1}{ka^-} \right), \quad (1.44)$$

which is conform to the diffraction coefficient of Kaminetzky and Keller [15].

1.2.3.2 Discontinuity of curvature in an impedance surface

In this case, we have $\Phi = \pi/2$ et $\sin \theta^\pm = \sin \theta$, and $F(\varphi) = (f_1(\pi + \varphi) - f_1(-\pi + \varphi))/k$. Instead of directly using (1.29), we prefer here to rewrite the functional equation in f_1 , letting $\alpha = \pm\pi/2 + \varphi$,

$$\begin{aligned} & f_1(\pm\pi + \varphi) \frac{\cos \varphi + \sin \theta}{-\cos \varphi + \sin \theta} - f_1(-\varphi) = \\ & = \frac{D_{\pm\pi/2+\varphi}^\pm (f_\circ(\pm\pi + \varphi)) - D_{\mp\pi/2-\varphi}^\pm (f_\circ(-\varphi))}{ia^\pm (\cos \varphi - \sin \theta)}. \quad (1.45) \end{aligned}$$

From $\sin \theta^\pm = \sin \theta$ and $f_\circ(\pi + \varphi) = f_\circ(-\pi + \varphi)$, we have,

$$\begin{aligned} & D_{\pi/2+\varphi}^+ (f_\circ(\pi + \varphi)) - D_{-\pi/2-\varphi}^+ (f_\circ(-\varphi)) \\ & = D_{-\pi/2+\varphi}^- (f_\circ(\pi + \varphi)) - D_{\pi/2-\varphi}^- (f_\circ(-\varphi)) \quad (1.46) \end{aligned}$$

and thus,

$$f_1(\pi + \varphi) - f_1(-\pi + \varphi) = i\left(\frac{1}{a^+} - \frac{1}{a^-}\right) \times \frac{D_{\pi/2+\varphi}^+(f_\circ(\pi + \varphi)) - D_{-\pi/2-\varphi}^+(f_\circ(-\varphi))}{(\cos \varphi + \sin \theta)}. \quad (1.47)$$

Using $f_\circ(\alpha) = \frac{\cos \alpha + \sin \theta}{\cos \varphi_\circ + \sin \theta} \frac{\cos \varphi_\circ}{\sin \alpha - \sin \varphi_\circ}$, and expanding D , we obtain

$$f_1(\pi + \varphi) - f_1(-\pi + \varphi) = \left(\frac{1}{a^+} - \frac{1}{a^-}\right) \times \frac{2i \cos \varphi \cos \varphi_\circ (1 + \sin \varphi \sin \varphi_\circ - \sin^2 \theta)}{(\cos \varphi_\circ + \sin \theta)(\cos \varphi + \sin \theta)(\sin \varphi + \sin \varphi_\circ)^3}, \quad (1.48)$$

which is perfectly in agreement with the result of Kaminetzky and Keller [15].

1.3 A Novel Expression of the Field for Arbitrary Bounded Sources above a Passive or Active Impedance Plane

1.3.1 Formulation of the problem

We consider the scattering by an imperfectly reflective plane when it is illuminated by the field radiated by a bounded source, composed of arbitrary electrical and magnetic currents J and M (see fig. 1.3). The plane is defined by $z = 0$ in Cartesian coordinates (x, y, z) . A harmonic time dependence $e^{i\omega t}$, from now on assumed, is suppressed throughout. Each component of the scattered field is assumed to be regular in the domain $z > 0$, and $O(e^{-\gamma|OP|})$ at $P(x, y, z)$, $\gamma > 0$, as z or $\rho \rightarrow \infty$, when $|\arg(ik)| < \pi/2$.

Following Harrington [18, p.131] (see also Jones [2]), we can write the electric field E and the magnetic field H satisfying the Maxwell equations, with two scalar potentials $\mathcal{E}$ and $\mathcal{H}$, following

$$E = -ik \text{curl}(\mathcal{H} \hat{z}) + (\text{grad}(\text{div}(\cdot)) + k^2)(\mathcal{E} \hat{z})$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H = ik \text{curl}(\mathcal{E} \hat{z}) + (\text{grad}(\text{div}(\cdot)) + k^2)(\mathcal{H} \hat{z}) \quad (1.49)$$

where $(\Delta + k^2)\mathcal{E} = 0$ and $(\Delta + k^2)\mathcal{H} = 0$ outside the sources of radiation (i.e. outside J , M , and the scatterer), with $k = \omega \sqrt{\mu_0 \epsilon_0}$, the constants ϵ_0 and μ_0 being respectively the permittivity and the permeability of the medium above the plane, $|\arg(ik)| \leq \pi/2$. Following the theory of this representation, the constant vector $\hat{z}$ can be chosen regardless of the sources, and $\mathcal{E}$ (or $\mathcal{H}$) $\equiv e^{\pm ikz}$ has no influence on (E, H) . Thereafter, we denote $(\mathcal{E}_{inc}, \mathcal{H}_{inc})$ and $(\mathcal{E}_s, \mathcal{H}_s)$ the potentials corresponding to the incident field (incoming wave) and the scattered field (outgoing wave), and write (1.49) in the compact form $(E, \sqrt{\frac{\mu_0}{\epsilon_0}} H) = \mathcal{L}(\hat{z} \mathcal{E}, \hat{z} \mathcal{H})$.

In [19], we considered multimode boundary conditions on an isotropic plane,

$$\prod_{j=1}^N \left(\frac{\partial}{\partial z} - ik g_j^e \right) E_{z,tot}|_{z=0} = 0, \prod_{j=1}^P \left(\frac{\partial}{\partial z} - ik g_j^h \right) H_{z,tot}|_{z=0} = 0, \quad (1.50)$$

Figure 1.3 Geometry : sources (J, M) and observation point above the plane $z = 0$

which corresponds to the reflection coefficients of a plane wave for the principal polarizations TM (components of E in the plane of incidence) and TE (components of H in the plane of incidence),

$$R_{TM}(\beta) = \prod_{j=1}^N \frac{\cos \beta - g_j^e}{\cos \beta + g_j^e}, R_{TE}(\beta) = \prod_{j=1}^P \frac{\cos \beta - g_j^h}{\cos \beta + g_j^h}, \quad (1.51)$$

where β is the angle of incidence with the normal $\hat{z}$ [19], $g_j^{(e,h)}$ are complex constants, N and P are two positive numbers. This class of problem corresponds to the reflection by a multilayer [19, 21, 22], composed of isotropic media, or more generally, of uniaxial anisotropic media with the principal axis along z , backed with a perfectly reflective plane. From the symmetry at normal incidence, we notice that the condition $R_{TE}(0) = -R_{TM}(0)$ has also to be satisfied. This implies that $g_1^e = 1/g_1^h$ for monomode conditions, when $N = P = 1$.

We now restrict ourselves to this latter case, commonly named the impedance case, and take an arbitrary complex number $g_1^e = g^e$, corresponding to a passive ($\text{Reg}^e > 0$) or active ($\text{Reg}^e < 0$) plane. Using

$$E_z = \frac{\partial^2 \mathcal{E}}{\partial z^2} + k^2 \mathcal{E}, \sqrt{\frac{\mu_0}{\epsilon_0}} H_z = \frac{\partial^2 \mathcal{H}}{\partial z^2} + k^2 \mathcal{H}, \quad (1.52)$$

we will search the potentials $\mathcal{E}_s$ and $\mathcal{H}_s$, satisfying the Helmholtz equation as $z > 0$, regular and vanishing as $z \rightarrow \infty$ when $|\arg(ik)| < \pi/2$, that verify

$$\begin{aligned} \left(\frac{\partial}{\partial z} - ik g^e \right) \mathcal{E}_s(z) &= \left(\frac{\partial}{\partial z} + ik g^e \right) \mathcal{E}_{inc}(-z), \\ \left(\frac{\partial}{\partial z} - ik g^h \right) \mathcal{H}_s(z) &= \left(\frac{\partial}{\partial z} + ik g^h \right) \mathcal{H}_{inc}(-z) \end{aligned} \quad (1.53)$$

as $z \geq 0$, with $g^h = 1/g^e$. Therefore, we need first a definition of $(\mathcal{E}_{inc}, \mathcal{H}_{inc})$ as $z \leq 0$.

Remark :

The boundary conditions on normal components can be simply deduced from boundary conditions on the tangential components after differentiation of them.

1.3.2 An expression of potentials $(\mathcal{E}_{inc}, \mathcal{H}_{inc})$ for bounded sources J and M

Let us consider the incident field (E, H) at r of coordinates (x, y, z) , radiated by arbitrary electric and magnetic bounded sources J and M [2],

$$\begin{aligned} E &= \text{curl}(G * M) + \frac{i}{\omega \epsilon_0} (\text{grad}(\text{div}(\cdot)) + k^2)(G * J) \\ \sqrt{\frac{\mu_0}{\epsilon_0}} H &= -\sqrt{\frac{\mu_0}{\epsilon_0}} \text{curl}(G * J) + \frac{i}{k} (\text{grad}(\text{div}(\cdot)) + k^2)(G * M) \end{aligned} \quad (1.54)$$

where $G(r) = -\frac{e^{-ik|r(x,y,z)|}}{4\pi|r(x,y,z)|}$ with $|r| = \sqrt{x^2 + y^2 + z^2}$, and $*$ is the convolution product. The potentials $(\mathcal{E}_{inc}, \mathcal{H}_{inc})$ for this field, satisfying the Helmholtz equation outside the sources, and vanishing at infinity when $|\arg(ik)| < \pi/2$ as $\pm z \rightarrow \infty$, have a particularly compact expression, that we develop in [19] for arbitrary sources. It is given by,

$$\begin{aligned} (\mathcal{E}_{inc}, \mathcal{H}_{inc}) &= \frac{\hat{z}}{8\pi k^2} \left(\sqrt{\frac{\mu_0}{\epsilon_0}} (\text{grad}(\text{div}(J)) + k^2 J, ik \text{curl}(J)) + \right. \\ &\quad \left. + (-ik \text{curl}(M), \text{grad}(\text{div}(M)) + k^2 M) \right) * \mathcal{W} = \frac{\hat{z}}{8\pi k^2} \mathcal{L} \left(\sqrt{\frac{\mu_0}{\epsilon_0}} J, M \right) * \mathcal{W}, \end{aligned} \quad (1.55)$$

where we can take, to solve (1.53) as $z > 0$,

$$\mathcal{W}(r) = (e^{ik|z|} E_1(ik(|r| + |z|)) + e^{-ik|z|} (E_1(ik(|r| - |z|)) + 2 \ln \rho)) \quad (1.56)$$

with $\rho = \sqrt{x^2 + y^2}$, E_1 being the exponential integral function [20], and the notation $(A, B) * C \equiv (A * C, B * C)$. The reader can verify by inspection our expression, considering that the following conditions are satisfied when $\pm z > 0$,

$$\left(\frac{\partial^2}{\partial z^2} + k^2 \right) \frac{\mathcal{W}(r)}{8\pi ik} = G(r), (\Delta + k^2) \mathcal{W}(r) = 0 \quad (1.57)$$

and that all derivatives of $\mathcal{W}$ are regular in these domains.

Remark :

It is worth noticing that we have $\Delta_{xy} \ln(\rho) = 0$ for $\rho \neq 0$, which implies that $\ln \rho$ in (1.56) has no influence on the field for $z \neq 0$, except by its singularity at $\rho = 0$. So, another way to proceed would be to suppress from the sources J and M a vertical tube enclosing the singularity, with a radius that we let tend to 0 in the field expression.

1.3.3 Expression of the potentials $(\mathcal{E}_s, \mathcal{H}_s)$ for an impedance plane

Using our expression of $(\mathcal{E}_{inc}, \mathcal{H}_{inc})$ for the radiation of J and M , we can express the potentials $\mathcal{E}_s$ and $\mathcal{H}_s$ which satisfy the impedance boundary conditions (1.53), from

the method developed in [19]. So, letting $N = P = 1$ in [19, prop.(5.2)], we obtain, as $z \geq 0$,

$$\begin{aligned}
 \mathcal{E}_s(x, y, z) &= \mathcal{E}_{inc}(x, y, -z) + \left(\left(\frac{\hat{z}}{\omega \epsilon_0} \frac{\text{grad}(\text{div}(J)) + k^2 J}{8\pi k} + \right. \right. \\
 &\quad \left. \left. + \frac{\hat{z}}{k} \frac{(-ik \text{curl}(M))}{8\pi k} \right) * \sum_{\epsilon'=-1,1} \frac{a^e}{(g^e - \epsilon')} (\mathcal{V}_{\epsilon'} + \epsilon' \mathcal{K}_{g^e}) \right)(x, y, -z) \\
 &= \mathcal{E}_{inc}(x, y, -z) + \left(\left(\frac{\hat{z}}{\omega \epsilon_0} \frac{\text{grad}(\text{div}(J)) + k^2 J}{8\pi k} + \frac{\hat{z}}{k} \frac{(-ik \text{curl}(M))}{8\pi k} \right) * \right. \\
 &\quad \left. * \sum_{\epsilon'=-1,1} \left(\left(\frac{\epsilon' + g^e}{\epsilon' - g^e} - 1 \right) \mathcal{V}_{\epsilon'} + \frac{\epsilon' a^e \mathcal{K}_{g^e}}{(g^e - \epsilon')} \right) \right)(x, y, -z) \tag{1.58}
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{H}_s(x, y, z) &= \mathcal{H}_{inc}(x, y, -z) + \left(\left(\frac{\hat{z}}{\omega \epsilon_0} \frac{(ik \text{curl}(J))}{8\pi k} \right. \right. \\
 &\quad \left. \left. + \frac{\hat{z}}{k} \frac{(\text{grad}(\text{div}(M)) + k^2 M)}{8\pi k} \right) * \sum_{\epsilon'=-1,1} \frac{a^h}{(g^h - \epsilon')} (\mathcal{V}_{\epsilon'} + \epsilon' \mathcal{K}_{g^h}) \right)(x, y, -z) \\
 &= -\mathcal{H}_{inc}(x, y, -z) + \left(\left(\frac{\hat{z}}{\omega \epsilon_0} \frac{(ik \text{curl}(J))}{8\pi k} + \frac{\hat{z}}{k} \frac{(\text{grad}(\text{div}(M)) + k^2 M)}{8\pi k} \right) * \right. \\
 &\quad \left. * \sum_{\epsilon'=-1,1} \left(\left(\frac{\epsilon' + g^h}{\epsilon' - g^h} + 1 \right) \mathcal{V}_{\epsilon'} + \frac{\epsilon' a^h \mathcal{K}_{g^h}}{(g^h - \epsilon')} \right) \right)(x, y, -z) \tag{1.59}
 \end{aligned}$$

where $g^e = 1/g^h$, $a^{e,h} = -2g^{e,h}$. In these expressions, the functions $\mathcal{V}_{\epsilon'}$, $\mathcal{K}_g$ satisfy

$$\begin{aligned}
 \mathcal{V}_{\epsilon'}(x, y, -z) &= e^{\epsilon' ikz} (E_1(ik(|r| + \epsilon' z)) + (1 - \epsilon') \ln \rho), \\
 \mathcal{K}_g(x, y, -z) &= e^{ikgz} \mathcal{J}_g(\rho, -z) \tag{1.60}
 \end{aligned}$$

where $\rho = \sqrt{x^2 + y^2}$, $g = g^e$ or $g = g^h$, and $\mathcal{J}_g(\rho, -z)$ is given by the integral

$$\mathcal{J}_g(\rho, -z) = \frac{e^{-ikgz}}{2} \int_{\mathcal{D}} \frac{H_0^{(2)}(k\rho \sin \beta) e^{-ikz \cos \beta}}{\cos \beta + g} \sin \beta d\beta, \tag{1.61}$$

with $\text{Re}(ik \sin \beta) = 0$ on $\mathcal{D}$ from $-i\infty - \arg(ik)$ to $i\infty + \arg(ik)$, which is a Fourier-Bessel integral commonly encountered in scattering theory [21, p.234], also called a Sommerfeld-type integral. Letting $g = \sin \theta_1$ with $|\text{Re}(\theta_1)| \leq \pi/2$, we notice the presence of a cut with $\text{Re}(ik \cos \theta_1) = 0$ in active case ($\text{Reg} < 0$), which is due to poles of $(\cos \beta + g)^{-1}$ that can go through $\mathcal{D}$. Thus, as discussed in [19], any transformations that does not consider the presence of this branch cut can be wrong in active case.

To avoid any restriction, we have developed in [19],[26], [27], a novel expression for arbitrary $g = \sin \theta_1$ with $|\text{Re}(\theta_1)| \leq \pi/2$ as $|\arg(ik)| \leq \pi/2$,

$$\mathcal{J}_g(\rho, -z) = i \int_b^{i\infty} e^{-a \cos \alpha} d\alpha \tag{1.62}$$

where the parameters a and b , with $|Reb| \leq \pi$, $Rea > 0$, are defined following,

$$\begin{aligned} e^{\mp ib} &= \frac{ikR}{a}(1 \pm \sin \theta_1)(1 \pm \cos \varphi), \quad a = \varepsilon ikR \sin \varphi \cos \theta_1, \\ -ia \sin b &= ikR(\cos \varphi + \sin \theta_1), \quad a \cos b = ikR(1 + \sin \theta_1 \cos \varphi) \end{aligned} \quad (1.63)$$

and $z = R \cos \varphi$, $\rho = R \sin \varphi$, $R = \sqrt{\rho^2 + z^2}$, $\varepsilon = \text{sgn}(\text{Re}(ikR \sin \varphi \cos \theta_1))$ ($\text{Re}(a) = 0$ being a limit case), $0 < \varphi < \pi/2$. So defined, we have $\text{sgn}(Reb) = -\varepsilon \text{sgn}(\text{Im}(\sin \theta_1))$, and $\text{sgn}(Ima) = -\text{sgn}(\arg(ik))$ when $\varepsilon = -1$.

We notice that, as g varies in the complex plane, this expression has a correct cut as ε changes of sign for $\text{Reg} \leq 0$, is singular for $g = -1$, and is regular elsewhere (note: for $\text{Reg} > 0$, the change of sign of ε does not induce a cut as g varies).

Let us remark that this integral was also given in [23] for passive impedance case but it was with a definition of parameters which restricts its application (see details in section 2 of [19]). It is intimately related to the incomplete cylindrical function in the Poisson form [24] and to the leaky aquifer function [25]. The reader can refer to [19],[26],[27] for its calculus in passive or active impedance cases.

Remark : The reader can verify by inspection that, when $z > 0$,

$$\frac{\partial \mathcal{J}_g(\rho, -z)}{\partial z} = \frac{e^{-ik(R+gz)}}{R}, \quad (\Delta + k^2)(e^{ikgz} \mathcal{J}_g(\rho, -z)) = 0. \quad (1.64)$$

Remark : Let us notice that the solution for an impedance plane in acoustics given by Ingard [28] presents, for some values of g with $\text{Re}(g) > 0$ (passive), non physical discontinuities as φ varies (due to errors in surface waves contributions reported by Wenzel [29]), that Thomasson [30] has corrected.

1.4 Spectral Representation of the Field for 3D Conical Scatterers

We describe here some new aspects of integral representations in problems with conical boundary conditions, encountered in electromagnetic scattering theory. We carefully analyse some new points in the general expression of Debye potentials in electromagnetism, and give an novel efficient representation of incident plane waves for solving the problems of scattering with conical geometries.

1.4.1 Formulation

We consider the diffraction of an electromagnetic plane wave by an imperfectly conducting cone C , in the domain Ω of the three-dimensional Euclidean space R^3 bounded by C . A dependence on time $e^{i\eta t}$ is assumed and suppressed throughout. The Maxwell's equations for the scattered wave field are given in Ω by

$$ik\vec{E} = \text{rot}(Z_0\vec{H}), \quad ikZ_0\vec{H} = -\text{rot}\vec{E}, \quad (1.65)$$

where $k = \eta\sqrt{\varepsilon_0\mu_0}$ and Z_0 are respectively the wave number and the impedance of the free space, $\vec{E}$ and $\vec{H}$ are respectively the electric and the magnetic field, and k is

complex with $|\arg(ik)| \leq \pi/2$. We introduce the representation of fields outside any sources with Debye potentials $(\mathcal{E}, \mathcal{H})$ [2],

$$\vec{E} = \text{rot rot}(\vec{r}\mathcal{E}) - ik\text{rot}(\vec{r}\mathcal{H}), \quad Z_0\vec{H} = \text{rot rot}(\vec{r}\mathcal{H}) + ik\text{rot}(\vec{r}\mathcal{E}), \quad (1.66)$$

where $\text{rot rot} = \nabla \text{div} - \Delta$, and the potentials $\mathcal{E}, \mathcal{H}$ satisfy the Helmholtz equations. We then have the expressions

$$\begin{aligned} \vec{E} &= \vec{e}_r(\partial_r^2(r\mathcal{E}) + k^2r\mathcal{E}) + r^{-1}\nabla_\omega\partial_r(r\mathcal{E}) - ik\nabla_\omega\mathcal{H} \wedge \vec{e}_r, \\ Z_0\vec{H} &= \vec{e}_r(\partial_r^2(r\mathcal{H}) + k^2r\mathcal{H}) + r^{-1}\nabla_\omega\partial_r(r\mathcal{H}) + ik\nabla_\omega\mathcal{E} \wedge \vec{e}_r, \end{aligned} \quad (1.67)$$

where $\nabla_\omega = \vec{e}_\theta\partial_\theta + \frac{\vec{e}_\varphi}{\sin\theta}\partial_\varphi$, (r, θ, φ) are the spherical coordinates, and $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ are the unit vectors basis of the spherical coordinate system, which gives us,

$$\begin{aligned} E_r &= k^2r\mathcal{E} + \frac{\partial^2}{\partial r^2}(r\mathcal{E}), \quad E_\theta = \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(r\mathcal{E}) - \frac{ik}{\sin\theta}\frac{\partial\mathcal{H}}{\partial\varphi}, \\ E_\varphi &= \frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\frac{\partial\mathcal{E}}{\partial\varphi}) + ik\frac{\partial\mathcal{H}}{\partial\theta}, \end{aligned} \quad (1.68)$$

and

$$\begin{aligned} Z_0H_r &= k^2r\mathcal{H} + \frac{\partial^2}{\partial r^2}(r\mathcal{H}), \quad Z_0H_\theta = \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(r\mathcal{H}) + \frac{ik}{\sin\theta}\frac{\partial\mathcal{E}}{\partial\varphi}, \\ Z_0H_\varphi &= \frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\frac{\partial\mathcal{H}}{\partial\varphi}) - ik\frac{\partial\mathcal{E}}{\partial\theta}. \end{aligned} \quad (1.69)$$

Let the two-dimensional unit sphere S^2 be centered at the vertex T of the convex cone C (fig. 1.4). We denote $\sigma = S^2 \cap C$ the boundary of the domain $\Sigma \subset S^2$ which is cut out on the unit sphere by the cone C . We assume that the curve σ is smooth and that Σ is geodesically convex and belongs to a hemisphere of S^2 . The vector $\vec{r} = (r, \omega)$ refers to the observation point, $r = |\vec{r}|$ is the distance from T to this point, and ω is the point of the unit sphere defined by $\vec{\omega} = \vec{r}/r$, described by two angular coordinates (θ, φ) which have the traditional meaning as in the spherical coordinate system. In what follows, $\hat{\theta}(\omega, \omega_0) = \text{dist}(\omega, \omega_0) = \arccos \langle \vec{\omega}, \vec{\omega}_0 \rangle_{R^3}$, is the geodesic distance between the points ω and ω_0 on the unit sphere S^2 .

The conditions at infinity and at the vertex are those given in [33, sect.2]. To exhibit the properties of scattered fields for large kr , we rather choose to consider illumination by a point source at $\vec{r}_0$ that we let tend to ∞ , kr being large but fixed. Denoting $\hat{\theta}'(\omega, \omega_0) = \inf_{\omega' \in \sigma} (\hat{\theta}(\omega, \omega') + \hat{\theta}(\omega', \omega_0))$ the length of the geodesics (broken for the reflected wave), we note that the reflection and shadow regions, in the sense of geometrical optics, are defined by the inequality $\hat{\theta}'(\omega, \omega_0) \leq \pi$, while $\hat{\theta}'(\omega, \omega_0) > \pi$ is the zone where the behaviour for large $|kr|$ and $|kr_0|$ is dictated by the spherically diffracted wave excited by the tip of the cone. We can then assume that the scattered field in Ω , verifies for large kr as $r_0 \rightarrow \infty$, when $|\arg(ik)| < \pi/2$ [33],

$$ikr_0 \exp(ikr_0)(\vec{E}, Z_0\vec{H}) = O(\exp(ikr \cos(\pi/2 + \varepsilon(\omega, \omega_0)))), \quad (1.70)$$

Figure 1.4 Geometry of a conical region

where $\varepsilon(\omega, \omega_0) = \hat{\theta}'(\omega, \omega_0) - \pi/2$ when $\hat{\theta}'(\omega, \omega_0) \leq \pi$, with $\hat{\theta}'(\omega, \omega_0) = \hat{\theta}(\omega, \omega_*)$ in reflection zone and $\hat{\theta}'(\omega, \omega_0) = \hat{\theta}(\omega, \omega_0)$ in shadow zone, and $\varepsilon(\omega, \omega_0) = \pi/2$ as $\hat{\theta}'(\omega, \omega_0) > \pi$.

1.4.2 Expression of the potentials with Kontorovich-Lebedev integrals

1.4.2.1 Integral and non-integral terms

Considering the general conditions at the apex given in [33], the wave field is $O((kr)^m)$ with $m > -3/2$ as r tends to 0. Thus, assuming the Debye potentials and their angular derivatives are $O(r^l)$ in vicinity of the tip of cone, we obtain, from the expressions of fields with potentials, that $(l+1)r^{l-1} = O((kr)^m)$, $m > -3/2$. The latter condition is satisfied if $l > -1/2$ or $l = -1$. We then look for the representations of the scattered wave field potentials in the form,

$$(\mathcal{E}, \mathcal{H}) = \frac{2}{k} \sqrt{\frac{2}{\pi}} \left(\int_{-i\infty}^{i\infty} \frac{v \sin \pi v}{v^2 - 1/4} \frac{K_v(ikr)}{\sqrt{ikr}} g_{e,h}(\omega, \omega_0, v) dv \right. \\ \left. + i\pi \frac{K_{1/2}(ikr)}{\sqrt{ikr}} w_{e,h}(\omega, \omega_0) \right), \quad (1.71)$$

where K_v is a modified Bessel function, and $K_{1/2}(ikr) = \sqrt{\frac{\pi}{2ikr}} \exp(-ikr)$. The integral terms are of Kontorovich-Lebedev type, and $g_{e,h}(\omega, \omega_0, v)$ are even analytical functions of v that we assume definite at $v = \pm 1/2$. We can derive, from the theory of KL transform [33, sect.3.3 and app.A], that these expressions are valid as $|\arg(ik)| < \varepsilon(\omega, \omega_0)$ when $\hat{\theta}'(\omega, \omega_0) \leq \pi$, and as $|\arg(ik)| \leq \varepsilon(\omega, \omega_0)$ when $\hat{\theta}'(\omega, \omega_0) > \pi$, which is equivalent to $\hat{\theta}'(\omega, \omega_0) > \pi/2 + |\arg(ik)|$.

We can write the Helmholtz equation for the potentials expressed with (1.71). Multiplying this equation by r^3 , and assuming that angular derivatives of the integral term in (1.71) remain $O(r^{-1/2})$ as r tends to 0, we notice that $\triangle_\omega w_{e,h}(\omega, \omega_0) = O(r^{1/2})$ as $r \rightarrow 0$, where $\triangle_\omega$ is the Laplacian on the unit sphere (also called the Laplace-Beltrami operator), and thus $\triangle_\omega w_{e,h}(\omega, \omega_0) = 0$. In consequence, the KL

integral term and the non-integral term in (1.71) satisfies the Helmholtz equation independently, and, we obtain that,

$$(\triangle_\omega + \nu^2 - 1/4)g_{e,h}(\omega, \omega_0, \nu) = 0, \triangle_\omega w_{e,h}(\omega, \omega_0) = 0. \quad (1.72)$$

1.4.2.2 The equality $w_{e,h}(\omega, \omega_0) = g_{e,h}(\omega, \omega_0, 1/2)$, and the compatibility conditions on $g_{e,h}(\omega, \omega_0, 1/2)$

Considering that the energy flux through a small sphere surrounding the vertex vanishes as its surface collapses [2], we can shown [33, sect.2] that we have no pure term of the form $p(\omega, \omega_0) \exp(-ikr)/ikr$ which satisfies the Maxwell equations in the expression of the field. To express this condition on $g_{e,h}$, we begin to introduce the linear combinations $\vec{C}_\pm = \vec{E} \pm iZ_0 \vec{H}$ and $\mathcal{M}_\pm = \mathcal{E} \pm i\mathcal{H}$ which satisfy

$$\vec{C}_\pm = \text{rot rot}(\vec{r}\mathcal{M}_\pm) \mp k \text{rot}(\vec{r}\mathcal{M}_\pm), \quad (1.73)$$

or, developed in the spherical coordinates,

$$\begin{aligned} C_{\pm,r} &= k^2 r \mathcal{M}_\pm + \frac{\partial^2}{\partial r^2}(r \mathcal{M}_\pm), \quad C_{\pm,\theta} = \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta}(r \mathcal{M}_\pm) \mp \frac{k}{\sin \theta} \frac{\partial \mathcal{M}_\pm}{\partial \varphi}, \\ C_{\pm,\varphi} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial r}(r \frac{\partial \mathcal{M}_\pm}{\partial \varphi}) \pm k \frac{\partial \mathcal{M}_\pm}{\partial \theta}. \end{aligned} \quad (1.74)$$

Considering combinations of the terms in (1.74), we obtain

$$\begin{aligned} C_{\pm,\theta} \pm iC_{\pm,\varphi} &= \left(\frac{\partial}{\partial \theta} \pm i \frac{\partial}{\sin \theta \partial \varphi} \right) \left(\frac{\partial}{r \partial r}(r \mathcal{M}_\pm) + ik(\mathcal{M}_\pm) \right), \\ C_{\pm,\theta} \mp iC_{\pm,\varphi} &= \left(\frac{\partial}{\partial \theta} \mp i \frac{\partial}{\sin \theta \partial \varphi} \right) \left(\frac{\partial}{r \partial r}(r \mathcal{M}_\pm) - ik(\mathcal{M}_\pm) \right). \end{aligned} \quad (1.75)$$

Using (1.71), we can derive field terms with strict $\exp(-ikr)/ikr$ dependence in (1.75). Concerning integral terms, we exploit the relations $2\nu K_\nu(z)/z = K_{\nu+1}(z) - K_{\nu-1}(z)$ and $2K'_\nu(z) = -K_{\nu+1}(z) - K_{\nu-1}(z)$. We then deform the contours of integration so that we capture residue terms attached to simple poles of $K_{\nu\pm 1}(z)/(\nu^2 - 1/4)$ at $\nu = \mp 1/2$. We add to these terms the contribution of non integral terms of (1.71), and isolate the complete field term with strict $\exp(-ikr)/ikr$ dependence in (1.75). Writing that it must vanish, we derive the compatibility conditions,

$$\frac{\partial g_e(\omega, \omega_0, 1/2)}{\partial \theta} = \frac{\partial g_h(\omega, \omega_0, 1/2)}{\sin \theta \partial \varphi}, \quad \frac{\partial g_h(\omega, \omega_0, 1/2)}{\partial \theta} = -\frac{\partial g_e(\omega, \omega_0, 1/2)}{\sin \theta \partial \varphi}, \quad (1.76)$$

and

$$\begin{aligned} \frac{\partial (g_e(\omega, \omega_0, 1/2) - w_e(\omega, \omega_0))}{\partial \theta} &= -\frac{\partial (g_h(\omega, \omega_0, 1/2) - w_h(\omega, \omega_0))}{\sin \theta \partial \varphi}, \\ \frac{\partial (g_h(\omega, \omega_0, 1/2) - w_h(\omega, \omega_0))}{\partial \theta} &= \frac{\partial (g_e(\omega, \omega_0, 1/2) - w_e(\omega, \omega_0))}{\sin \theta \partial \varphi}. \end{aligned} \quad (1.77)$$

All choices of $w_{e,h}$ satisfying this condition and Laplace-Beltrami equation gives the same fields $\vec{E}$ and $\vec{H}$, and we can choose $w_{e,h}(\omega, \omega_0) = g_{e,h}(\omega, \omega_0, 1/2)$. We then obtain for the potentials,

$$\begin{aligned} (\mathcal{E}, \mathcal{H}) = & \frac{2}{k} \sqrt{\frac{2}{\pi}} \left(\int_{-i\infty}^{i\infty} \frac{v \sin \pi v}{v^2 - 1/4} \frac{K_v(ikr)}{\sqrt{ikr}} g_{e,h}(\omega, \omega_0, v) dv \right. \\ & \left. + i\pi \frac{K_{1/2}(ikr)}{\sqrt{ikr}} g_{e,h}(\omega, \omega_0, 1/2) \right), \end{aligned} \quad (1.78)$$

with (1.76) which ensures that $\vec{E} = (\vec{C}_+ + \vec{C}_-)/2$ and $Z_0 \vec{H} = (\vec{C}_+ - \vec{C}_-)/2i$ do not contain any non-integral terms with strict $\exp(-ikr)/r$ radial dependence.

It is useful to note that the equalities in (1.76) can be rewritten as

$$(\vec{e}_\varphi \mp i\vec{e}_\theta) \nabla_\omega (g_e(\omega, \omega_0, 1/2) \pm ig_h(\omega, \omega_0, 1/2)) = 0, \quad (1.79)$$

or

$$\begin{aligned} & \left(\frac{\partial}{\partial \varphi} \mp i \frac{\sin \theta \partial}{\partial \theta} \right) (g_e(\omega, \omega_0, 1/2) \pm ig_h(\omega, \omega_0, 1/2)) \\ & = 2 \frac{\partial}{\partial z_\pm} (g_e(\omega, \omega_0, 1/2) \pm ig_h(\omega, \omega_0, 1/2)) = 0, \end{aligned} \quad (1.80)$$

where $z_\pm = \varphi \pm i \ln(\tan(\theta/2))$, $z_+ = z_-^*$. We can generalize in a simple manner the conditions (1.79) and (1.80) on $g_{e,h}(\omega, \omega_0, 1/2)$ for a new orthogonal vectors basis $(\vec{e}_1, \vec{e}_2 = \hat{r} \wedge \vec{e}_1)$ on the unit sphere, either by noticing that $(\vec{e}_\theta \pm i\vec{e}_\varphi) = \frac{(\vec{e}_1 \pm i\vec{e}_2)}{\vec{e}_1 \cdot (\vec{e}_\theta \mp i\vec{e}_\varphi)}$ or by using a conformal transformation in (1.80), and we have in particular,

$$(\vec{e}_2 \mp i\vec{e}_1) \nabla_\omega (g_e(\omega, \omega_0, 1/2) \pm ig_h(\omega, \omega_0, 1/2)) = 0. \quad (1.81)$$

Because of the complexity of the expressions of the fields with potentials defined with (1.78), the properties of the spectral functions are more difficult to derive than in acoustics case [31, 32]. For details, the reader can refer to [33],[34]. We now insist on an efficient expression of the potentials attached to an incident electromagnetic plane wave.

Remark : notice that $\triangle_\omega g_{e,h}(\omega, \omega_0, 1/2) = 0$ can be written, with the variables $z_\pm$, $\partial_{z_+} \partial_{z_-} g_{e,h}(\omega, \omega_0, 1/2) = 0$.

Remark : We easily deduce from (1.76) that the fields, respectively attached to the non-integral terms of $\mathcal{E}$ and $\mathcal{H}$ in (1.71), are equal. Another remarkable consequence of (1.76) is the fact that changing $\exp(-ikr)$ into $\exp(ikr)$ in the non-integral terms of the potentials lead their contributions to vanish. This implies that we can take $-2i \sin(kr)$ in place of $\exp(-ikr)$ in the expressions of non-integral terms.

Remark : Functional equations on $g_{e,h}$ can be derived from (1.78) and boundary conditions on the cone, for some domain of complex k , then be used for arbitrary k from analyticity. Once $g_{e,h}$ determined, we can calculate (1.78), directly in the oasis region with $\hat{\theta}'(\omega, \omega_0) > \pi$, or more generally after subtracting some approximation

of $g_{e,h}$ as $v \rightarrow i\infty$ due to geometrical optics contributions (with Physical Optics approximation for example) (see [34, p.63]).

1.4.3 Potentials and properties for an incident plane wave

1.4.3.1 An efficient expression

We here show how to obtain simple explicit expressions of potentials ($\mathcal{E}^i, \mathcal{H}^i$) for an incident plane wave electromagnetic field [33], without usual series of spherical harmonics, and we detail some of their complex properties.

Let us consider a plane wave, described in spherical coordinates, following

$$\begin{aligned}\vec{E}^i &= (\hat{\theta}_0 \sin \beta + \hat{\varphi}_0 \cos \beta) e^{ikr \cos \hat{\theta}(\omega, \omega_0)}, \\ Z_0 \vec{H}^i &= (\hat{\theta}_0 \cos \beta - \hat{\varphi}_0 \sin \beta) e^{ikr \cos \hat{\theta}(\omega, \omega_0)},\end{aligned}\tag{1.82}$$

with $\omega_0 = (\theta_0, \varphi_0)$ the point on the unit sphere S^2 attached to the direction of incidence (i.e. the direction from which the wave comes), $(\hat{\theta}_0, \hat{\varphi}_0)$ the spherical vectors associated to ω_0 , and $\cos \hat{\theta}(\omega, \omega_0) = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\varphi - \varphi_0)$. Concerning the exponential term in (1.82), we notice [32], that

$$u^i := e^{ikr \cos \hat{\theta}(\omega, \omega_0)} = \frac{4}{i\sqrt{2\pi}} \int_{-i\infty}^{+i\infty} v \sin(\pi v) u_v^i(\omega, \omega_0) \frac{K_v(ikr)}{\sqrt{ikr}} dv, \tag{1.83}$$

where $u_v^i(\omega, \omega_0) = -\frac{P_{v-1/2}(-\cos \hat{\theta}(\omega, \omega_0))}{4\cos(\pi v)}$, with $(\Delta_\omega + v^2 - 1/4)u_v^i(\omega, \omega_0) = \delta(\omega - \omega_0)$ [32],[35],[36] and the integral in (1.83) converging provided $\hat{\theta}(\omega, \omega_0) > \pi/2 + |\arg(ik)|$. We then derive the radial components of the plane wave field from (1.82), and express them in the form,

$$E_r^i = \mathcal{D}_e \left(\frac{e^{ikr \cos \hat{\theta}(\omega, \omega_0)}}{ikr} \right), Z_0 H_r^i = \mathcal{D}_h \left(\frac{e^{ikr \cos \hat{\theta}(\omega, \omega_0)}}{ikr} \right), \tag{1.84}$$

with $\mathcal{D}_e = \sin \beta \frac{\partial}{\partial \theta_0} + \cos \beta \frac{\partial}{\sin \theta_0 \partial \varphi_0}$ and $\mathcal{D}_h = \cos \beta \frac{\partial}{\partial \theta_0} - \sin \beta \frac{\partial}{\sin \theta_0 \partial \varphi_0}$. Denoting $C_{e,h}^i = (E_r^i, Z_0 H_r^i)$ and $\mathcal{D}_{e,h} = (\mathcal{D}_e, \mathcal{D}_h)$, we can exploit the representation for the modified Bessel function $\frac{K_v(ikr)}{ikr} = \frac{1}{4iv \sin \pi v} \int_{\gamma} e^{ikr \cos \alpha} \sin \alpha \cos(v\alpha) d\alpha$, in (1.83) and express (1.84) with Sommerfeld integrals. By means of an inverse Sommerfeld-Maliuzhinets transformation [1], we then obtain

$$\begin{aligned}& \frac{-ik \sin \alpha}{2} \int_{-i\infty}^{+i\infty} \sqrt{ikr} C_{e,h}^i e^{-ikr \cos \alpha} dr \\ &= \frac{2}{i\sqrt{2\pi}} \int_{-i\infty}^{+i\infty} \mathcal{D}_{e,h}(u_v^i(\omega, \omega_0)) \pi \sin \alpha \cos v\alpha dv.\end{aligned}\tag{1.85}$$

On the other hand, we consider the Debye potentials $(\mathcal{E}^i, \mathcal{H}^i)$, attached to a representation of the incident field with (1.68)-(1.69), in a form similar to (1.78),

$$\begin{aligned} (\mathcal{E}^i, \mathcal{H}^i) &= \frac{4}{k\sqrt{2\pi}} \left(\int_{-i\infty}^{+i\infty} \frac{v \sin(\pi v)}{v^2 - 1/4} g_{e,h}^i(\omega, \omega_0, v) \frac{K_v(ikr)}{\sqrt{ikr}} dv \right. \\ &\quad \left. + i\pi \frac{K_{1/2}(ikr)}{\sqrt{ikr}} g_{e,h}^i(\omega, \omega_0, 1/2) \right), \end{aligned} \quad (1.86)$$

where the functions $g_{e,h}^i$ are even functions of v . Exploiting the expressions (1.68-1.69) for E_r, H_r and substituting the expressions (1.86), we verify that

$$\begin{aligned} C_{e,h}^i &= (E_r^i, Z_0 H_r^i) = (k^2 + \frac{\partial^2}{\partial r^2})(r \mathcal{E}^i, r \mathcal{H}^i) \\ &= \frac{4i}{\sqrt{2\pi}} \int_{-i\infty}^{+i\infty} v \sin \pi v g_{e,h}^i(\omega, \omega_0, v) \frac{K_v(ikr)}{(ikr)^{3/2}} dv, \end{aligned} \quad (1.87)$$

noticing that the contribution of $\frac{K_{1/2}(ikr)}{\sqrt{ikr}}$ term for the radial component is zero. We then apply an inverse Sommerfeld-Maliuzhinets transformation [1] to the expression of $\sqrt{ikr} C_{e,h}^i$ derived from (1.87), and obtain

$$\begin{aligned} &\frac{-ik \sin \alpha}{2} \int_0^\infty \sqrt{ikr} C_{e,h}^i e^{-ikr \cos \alpha} dr \\ &= \frac{2}{i\sqrt{2\pi}} \int_{-i\infty}^{+i\infty} (-g_{e,h}^i(\omega, \omega_0, v)) \pi \sin \alpha \cos v \alpha dv. \end{aligned} \quad (1.88)$$

Comparing the expressions (1.85) and (1.88), we have

$$\int_{-i\infty}^{+i\infty} (-g_{e,h}^i(\omega, \omega_0, v)) \cos v \alpha dv = \int_{-i\infty}^{+i\infty} \mathcal{D}_{e,h}(u_v^i(\omega, \omega_0)) \cos v \alpha dv. \quad (1.89)$$

To reduce (1.89), we have to consider the elementary properties of functions involved in it. They are derived from (1.83)-(1.84) and (1.87) according to the behaviour of the field at the apex and at infinity : the even functions $\mathcal{D}_{e,h}(u_v^i(\omega, \omega_0))$ and $g_{e,h}^i(\omega, \omega_0, v)$ are regular in the strip $|\text{Re}(v)| < 3/2$ and are $O(1/\cos(v\hat{\theta}(\omega, \omega_0)))$ as $\text{Im}v$ tends to infinity. These properties permit to use Fourier transformation for (1.89) and to obtain, $g_{e,h}^i(\omega, \omega_0, v) = -\mathcal{D}_{e,h}(u_v^i(\omega, \omega_0))$, which can be also written,

$$\begin{aligned} g_e^i(\omega, \omega_0, v) &= -(\sin \beta \frac{\partial}{\partial \theta_0} + \cos \beta \frac{\partial}{\sin \theta_0 \partial \varphi_0})(u_v^i(\omega, \omega_0)), \\ g_h^i(\omega, \omega_0, v) &= -(\cos \beta \frac{\partial}{\partial \theta_0} - \sin \beta \frac{\partial}{\sin \theta_0 \partial \varphi_0})(u_v^i(\omega, \omega_0)). \end{aligned} \quad (1.90)$$

It is worth noticing that the linear operator $\mathcal{D}_{e,h}$ applies only on the variables θ_0 and φ_0 , which signifies a direct and easy work on u_v^i for the integral representation of the

incident field, and that the condition $\hat{\theta}(\omega, \omega_0) > \pi/2 + |\arg(ik)|$, also applies for the convergence of integrals in the representation (1.86) of $(\mathcal{E}^i, \mathcal{H}^i)$.

1.4.3.2 Complex properties of $g_{e,h}^i$

Letting $\mathcal{D}_0^\pm \equiv (\frac{\partial}{\partial \varphi_0} \pm i \frac{\sin \theta_0 \partial}{\partial \theta_0})$, the functions $g_{e,h}^i(\omega, \omega_0, \nu)$ verify,

$$\begin{aligned} g_e^i(\omega, \omega_0, \nu) \pm i g_h^i(\omega, \omega_0, \nu) &= -\frac{e^{\mp i\beta}}{\sin \theta_0} \mathcal{D}_0^\pm(u_v^i(\omega, \omega_0)) \\ &= -\frac{e^{\mp i\beta}}{\sin \theta_0} \mathcal{D}_0^\pm(\cos(\hat{\theta}(\omega, \omega_0))) \frac{\nu^2 - 1/4}{\cos(\pi\nu)} \frac{(1-x^2)^{1/2} P_{\nu-1/2}^{-1}(x)}{(x^2-1)}, \\ g_e^i(\omega, \omega_0, 1/2) \pm i g_h^i(\omega, \omega_0, 1/2) &= \frac{e^{\mp i\beta} / \sin \theta_0}{\pi(1 - \cos(\hat{\theta}(\omega, \omega_0)))} \mathcal{D}_0^\pm(\cos \hat{\theta}(\omega, \omega_0)), \end{aligned} \quad (1.91)$$

with $x = -\cos(\hat{\theta}(\omega, \omega_0))$, $0 \leq \hat{\theta}(\omega, \omega_0) \leq \pi$. We can then consider the transform by the differential operator $\mathcal{D}^\pm \equiv (\frac{\partial}{\partial \varphi} \pm i \frac{\sin \theta \partial}{\partial \theta})$, following,

$$\begin{aligned} \sin \theta_0 e^{\pm i\beta} \mathcal{D}^\mp(g_e^i(\omega, \omega_0, \nu) \pm i g_h^i(\omega, \omega_0, \nu)) &= \mathcal{D}^\mp \mathcal{D}_0^\pm u_v^i(\omega, \omega_0) \\ &= \frac{1}{4\pi} (\mathcal{D}^\mp \mathcal{D}_0^\pm(\cos(\hat{\theta}(\omega, \omega_0)))) \frac{\partial P_{\nu-1/2}(x)}{\partial x} \\ &\quad - \mathcal{D}^\mp(\cos(\hat{\theta}(\omega, \omega_0))) \mathcal{D}_0^\pm(\cos(\hat{\theta}(\omega, \omega_0))) \frac{\partial^2 P_{\nu-1/2}(x)}{\cos(\pi\nu) \partial x^2} \\ &= \frac{\mathcal{D}^\mp \mathcal{D}_0^\pm(\cos(\hat{\theta}(\omega, \omega_0)))}{4\pi} \left(\frac{\partial P_{\nu-1/2}(x)}{\partial x} + (1 - \cos(\hat{\theta}(\omega, \omega_0))) \frac{\partial^2 P_{\nu-1/2}(x)}{\cos(\pi\nu) \partial x^2} \right), \end{aligned} \quad (1.92)$$

where we have used that,

$$\begin{aligned} \mathcal{D}_0^\pm(\cos(\hat{\theta}(\omega, \omega_0))) \mathcal{D}^\mp(\cos(\hat{\theta}(\omega, \omega_0))) \\ = -(1 - \cos(\hat{\theta}(\omega, \omega_0))) \mathcal{D}^\mp \mathcal{D}_0^\pm(\cos(\hat{\theta}(\omega, \omega_0))). \end{aligned} \quad (1.93)$$

and recover the compatibility condition (1.80) at $\nu = 1/2$.

Remark: In the equations on $g_e^i(\omega, \omega_0, \nu) \pm i g_h^i(\omega, \omega_0, \nu)$, we have used that,

$$\begin{aligned} \mathcal{D}_0^\pm(\cos(\hat{\theta}(\omega, \omega_0))) &= \sin \theta_0 (\sin \theta \sin(\varphi - \varphi_0) \\ &\quad \pm i(\sin \theta \cos \theta_0 \cos(\varphi - \varphi_0) - \cos \theta \sin \theta_0)), \\ \mathcal{D}^\mp(\cos(\hat{\theta}(\omega, \omega_0))) &= -\sin \theta (\sin \theta_0 \sin(\varphi - \varphi_0) \\ &\quad \pm i(\sin \theta_0 \cos \theta \cos(\varphi - \varphi_0) - \cos \theta_0 \sin \theta)), \\ \mathcal{D}^\mp \mathcal{D}_0^\pm(\cos(\hat{\theta}(\omega, \omega_0))) &= \sin \theta \sin \theta_0 [\cos(\varphi - \varphi_0)(1 + \cos \theta \cos \theta_0) \\ &\quad + \sin \theta \sin \theta_0 \mp i(\cos \theta_0 + \cos \theta) \sin(\varphi - \varphi_0)], \end{aligned} \quad (1.94)$$

and that,

$$\begin{aligned}
 \left(\frac{1-x}{1+x}\right)^{\mu/2} P_{\nu-1/2}^{\mu}(x) &= \left(\frac{z-1}{1+z}\right)^{\mu/2} P_{\nu-1/2}^{\mu}(z)|_{z=x \pm i0^+}, \\
 \frac{1}{\cos(\pi\nu)} \frac{\partial P_{\nu-1/2}(z)}{\partial z} &= \frac{\nu^2 - 1/4}{\cos(\pi\nu)} \frac{(z^2 - 1)^{1/2} P_{\nu-1/2}^{-1}(z)}{z^2 - 1}, \\
 \frac{(1-z^2)}{\cos(\pi\nu)} \frac{\partial^2 P_{\nu-1/2}(z)}{\partial z^2} &= \frac{2z}{\cos(\pi\nu)} \frac{\partial P_{\nu-1/2}(z)}{\partial z} - \frac{\nu^2 - 1/4}{\cos(\pi\nu)} P_{\nu-1/2}(z) \\
 &= \frac{\nu^2 - 1/4}{\cos(\pi\nu)} \left(\frac{2z(z^2 - 1)^{1/2}}{z^2 - 1} P_{\nu-1/2}^{-1}(z) - P_{\nu-1/2}(z) \right), \\
 P_0^{-1}(z) &= \frac{z-1}{(z^2 - 1)^{1/2}}, \quad \lim_{\nu \rightarrow 1/2} \frac{\partial P_{\nu-1/2}(z)}{\cos(\pi\nu) \partial z} = \frac{-1}{\pi(z+1)}, \\
 \lim_{\nu \rightarrow 1/2} \frac{\partial^2 P_{\nu-1/2}(z)}{\cos(\pi\nu) \partial z^2} &= \frac{1}{\pi(z^2 - 1)} \left(\frac{2z}{z+1} - 1 \right) = \frac{1}{\pi} \frac{1}{(z+1)^2}.
 \end{aligned} \tag{1.95}$$

with $z = x \pm i0^+$, $x = -\cos(\hat{\theta}(\omega, \omega_0))$.

Remark: The function $u_v^i(\omega, \omega_0)$ satisfies the Helmholtz equation on the unit sphere with Dirac source [33]-[36], and we can write that the spectral functions attached to the incident field satisfy

$$(\triangle_{\omega} + \nu^2 - 1/4)(g_{e,h}^i)(\omega, \omega_0, \nu) = -\mathcal{D}_{e,h} \delta(\omega - \omega_0). \tag{1.96}$$

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