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ABSTRACT. This paper is concerned with the construction and analysis of fractional differential operators acting on the usual Hilbert space of square integrable functions defined on a bounded interval. That task is attacked thru an abstract approach giving raise to two new objects: differential triplets and boundary restriction operators. This approach allows a systematic study of fractional differential operators supplemented with homogeneous linear boundary conditions. As an application, the regularity of the solution to the one-dimensional fractional Laplace equation with homogeneous Dirichlet boundary conditions is investigated.

1. INTRODUCTION

Caputo's or *Riemann-Liouville's fractional derivatives* are generally meant as operators acting on functions of the *time* variable. However, *Delsarte's representation formula* led us to consider Caputo's derivatives of functions of the *space* variable as well (see [ER19]).

Now the next step is the study of differential operators built upon the fractional derivative D^α, instead of the usual (first order) derivative. Our aim is to define and study *fractional differential operators* acting on the Hilbert space $L^2(0,b)$ with $b \in (0,\infty)$, supplemented with *homogeneous linear boundary conditions*. The basic issues we would like to address are the following.

(i) Are these operators densely defined?
(ii) What can be said about the adjoint of densely defined operators?
(iii) Are these operators self-adjoint?
(iv) Are they onto, one-to-one, bijective? That amounts to solving fractional differential problems with the subsequent issues of existence, uniqueness and well-posedness.

In order to give some answers to these questions, the framework of *operator theory* will be used. That is, we will consider D^α as an instance of an operator A_{M} acting on an abstract Hilbert space $\mathcal{H}$. Then we have to highlight axiomatic properties that allow to define abstract *linear boundary conditions*, and to answer, at least partially, to questions (i)-(iv) at an abstract level. Hence this approach aims to unify the analysis of fractional differential operators, and the one of usual differential operators.

This task gives rise to two new objects featured in Definitions 2.1 and 2.3: *abstract differential triplet* and *boundary restriction operator*. The former encodes the facts that, for any $\alpha \in [\frac{1}{2}, 1]$, D^α admits a right inverse which is related to its adjoint thru the symmetry $S : u \mapsto u(b - \cdot)$. We refer to Example 2.4 and Section 4 for the cases $\alpha = 1$ and $\alpha \in [\frac{1}{2}, 1)$ respectively. *Boundary restriction operator* encapsulates the facts that the adjunction of homogeneous linear boundary conditions to a differential operator conserves the density of its domain, and does not change the regularity of the functions lying in its domains.

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Roughly speaking, the operator A_{M} in a differential triplet (A_{M}, B, S) is meant to be boundary condition free, and consequently is in some sense maximal. On the other hand, a boundary restriction operator of A_{M} is thought as A_{M} supplemented with homogeneous linear boundary conditions. In the case where $A_{\mathrm{M}} := \frac{\mathrm{d}}{\mathrm{d}x}$, our abstract approach gives a new classification of the operators $\frac{\mathrm{d}}{\mathrm{d}x}$ supplemented with homogeneous linear boundary conditions (called here *boundary restriction operators* of $\frac{\mathrm{d}}{\mathrm{d}x}$). We refer to Sub-subsection 2.3.5 and Proposition 2.22 for a detailed presentation.

It turns out that our abstract framework leads to many relationships. Some of them are featured in Section 2 and 3. For instance, let $\mathcal{T} := (A_{\mathrm{M}}, B, S)$ be a differential triplet. As a right inverse of A_{M}, the operator B induces an isomorphism from its range $R(B) := B(\mathcal{H})$ onto $\mathcal{H}$. That isomorphism, denoted by $A_{\mathcal{T}}$, is called the *pivot operator* of $\mathcal{T}$. Then Proposition 2.9 states the formula

$$(A_{\mathcal{T}})^* = A_{\mathcal{T}^*}$$

where $\mathcal{T}^* := (SA_{\mathrm{M}}S, B^*, S)$ is the *adjoint differential triplet of* $\mathcal{T}$ (see Definition 2.2), and $A_{\mathcal{T}^*}$ is the pivot operator of $\mathcal{T}^*$.

Sufficient conditions for a restriction operator to be densely defined are given in Proposition 2.5. That gives rise to a characterization of *boundary restriction operators* by means of the *minimal operator* of an abstract differential triplet (see Corollary 2.14). Roughly speaking, the minimal operator is the restriction of A_{M} with domain consisting in all functions whose boundary values vanish. We refer to Definition 2.6 for a precise statement.

The structure of the domain of a boundary restriction operator is given in Proposition 2.15. The domains of boundary restriction operators are classified by subspaces of a well identified space isomorphic to $\ker A_{\mathrm{M}} \times \ker A_{\mathrm{M}}$. In the case where $\ker A_{\mathrm{M}}$ has finite dimension d, there results that the domain of a given boundary restriction operator is described thru d_E parameters with $0 \leq d_E \leq 2d$: see Subsection 2.3. It turns out that there is no canonical choice for these parameters. However, in applications, these d_E parameters are chosen to be boundary values. Let us notice that our abstract approach highlights the fact that boundary values are closely related to the kernel of A_{M}.

Let us be more specific regarding boundary values. In the standard case where $A_{\mathrm{M}} := \frac{\mathrm{d}}{\mathrm{d}x}$, one has $d = 1$, and any function u in the domain of $\frac{\mathrm{d}}{\mathrm{d}x}$ possess two boundary values: $u(0)$, $u(b)$. However, in the fractional framework featured in Section 4 (where $A_{\mathrm{M}} := \mathrm{D}^\alpha$), one has $d = 2$, and any function u in $D(\mathrm{D}^\alpha)$ possess four boundary values:

$$u(0), \quad g_{1-\alpha} * \big(u - u(0)\big)(0), \quad u(b), \quad g_{1-\alpha} * \big(u - u(0)\big)(b).$$

In view of Subsection 4.5, a straightforward consequence is that the scalar initial value problem with unknown u

$$\mathrm{D}^\alpha u = h \qquad \text{in} \quad L^2(0,b), \quad u(0) = u_0,$$

is never well-posed because uniqueness fails. In order to have well-posedness, two initial conditions must be set (see Proposition 4.4). Let us stress that $A_{\mathrm{M}} := \mathrm{D}^\alpha$ is neither *Caputo's* nor *Riemann-Liouville's derivatives*, but rather one of their extensions (see Definition 4.2).

Once a differential triplet (A_{M}, B, S) is set, one may construct *second order differential operators* like $A_{\mathrm{M}}^2 := A_{\mathrm{M}} \circ A_{\mathrm{M}}$. A convenient way to built a self-adjoint operator from D^α is to start with the symmetric operator $\mathrm{D}^\alpha \circ (\mathrm{D}^\alpha)^*$: see Example 3.1 and Subsection 5.1 for the cases $\alpha = 1$ and $\alpha \in (0,1)$ respectively. That approach allows to define the so-called *one-dimensional Laplace operator with Dirichlet boundary condition*. Our main result regarding that operator is Theorem 5.6 which gives the regularity of the solution to the *one-dimensional fractional Laplace equation with homogeneous Dirichlet boundary condition*.

In the literature, the study on abstract boundary conditions starts at least in 1939 with [Cal39], but seem to focus on symmetric operators. We refer to Example 2.4 and Remark 4.6 for comments related to the work of J.W. Calkin and V. Ryzhov.

The outline is as follows. Abstract differential triplets and boundary restriction operators are considered in the forcoming section. Section 3 deals with abstract second order differential triplets. Sections 4 and 5 are concerned with applications to fractional derivatives. Miscellaneous results are gathered in appendices at the end of this paper.

2. An abstract setting

For all purposes, let us recall some usual definitions and notation. Let $\mathcal{H}$ be a complex Hilbert space, whose inner product, denoted by $(\cdot,\cdot)$, is assumed to be linear w.r.t. the first variable. An *operator* A *on* $\mathcal{H}$ is a linear map defined on a subspace $D(A)$ of $\mathcal{H}$, with values in $\mathcal{H}$. Any operator $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ is equipped with the graph-norm. A^*, $\ker A$ and $R(A)$ stand for the adjoint, the kernel and the range of A. The space of bounded operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. The *identity map of* $\mathcal{H}$ is denoted by $id_{\mathcal{H}}$. If $E \subseteq \mathcal{H}$ then $\langle E \rangle$ is the subspace of $\mathcal{H}$ generated by E, and

$$E^{\perp} := \big\{ f \in \mathcal{H} \mid (f,h) = 0 \ \forall h \in E \big\}.$$

If A, $\mathbb{A}$ are two operators on $\mathcal{H}$ then A is a *restriction of* $\mathbb{A}$, or $\mathbb{A}$ is an *extension of* A if

$$D(A) \subseteq D(\mathbb{A}) \qquad \text{and} \qquad Au = \mathbb{A}u, \quad \forall\, u \in D(A).$$

In symbol, we write $A \subseteq \mathbb{A}$.

2.1. Differential triplets and boundary restriction operators.

Definition 2.1. Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{S}$ be three operators on $\mathcal{H}$. The triplet $(\mathbb{A}, \mathbb{B}, \mathbb{S})$ is called an *(abstract) differential triplet on* $\mathcal{H}$ if

$$\mathbb{B} \in \mathcal{L}(\mathcal{H}) \tag{2.1}$$

$$R(\mathbb{B}) \subseteq D(\mathbb{A}), \quad \mathbb{A} \circ \mathbb{B} = id_{\mathcal{H}} \tag{2.2}$$

$$\mathbb{S} \in \mathcal{L}(\mathcal{H}) \ \text{ satisfies } \ \mathbb{S}^* = \mathbb{S}, \quad \mathbb{S}^2 = id_{\mathcal{H}} \tag{2.3}$$

$$\mathbb{B}^* = \mathbb{S}\mathbb{B}\mathbb{S}. \tag{2.4}$$

By using in particular $\mathbb{B}^* = \mathbb{S}\mathbb{B}\mathbb{S}$, we show easily that $(\mathbb{S}\mathbb{A}\mathbb{S}, \mathbb{B}^*, \mathbb{S})$ is a *differential triplet* on $\mathcal{H}$. Then we set this definition.

Definition 2.2. Let $\mathcal{T} := (\mathbb{A}, \mathbb{B}, \mathbb{S})$ be a differential triplet. The differential triplet $(\mathbb{S}\mathbb{A}\mathbb{S}, \mathbb{B}^*, \mathbb{S})$, denoted by $\mathcal{T}^*$, is called the *adjoint differential triplet of* $\mathcal{T}$. □

Since $\mathbb{B}$ is bounded, we have $\mathbb{B}^{**} = \mathbb{B}$, thus the adjoint differential triplet of $\mathcal{T}^*$ is $\mathcal{T}$.

In the sequel, being given a differential triplet $\mathcal{T} := (A_{\rm M}, B, S)$, our very aim is to describe the *closed densely defined restrictions of* $A_{\rm M}$ whose adjoint operator is a restriction of $SA_{\rm M}S$. In view of Theorem 2.3, Sub-subsection 2.3.5 and Subsection 4.4, these operators are $A_{\rm M}$ supplemented with abstract homogeneous linear boundary conditions. In this respect, $A_{\rm M}$ is meant to be *maximal*. In applications, $A_{\rm M}$ is chosen to be boundary condition free. That justifies this definition.

Definition 2.3. Let $\mathcal{T} = (A_{\rm M}, B, S)$ be a differential triplet, and A be an operator on $\mathcal{H}$. We say that A is a *boundary restriction operator of* $\mathcal{T}$ or, for the sake of simplicity, *of* $A_{\rm M}$ if

(i) A is a densely defined restriction of $A_{\rm M}$;
(ii) $A^* \subseteq SA_{\rm M}S$.

Item (i) is need for A^* to be well defined. Regarding item (ii), we observe that functions in the domain of a differential operator or in the domain of its adjoint have essentially the same regularity, and differ by their boundary values. Item (ii) encodes that observation. Thus a *boundary restriction operator* of A_{M} can be seen as A_{M} *supplemented with some (homogeneous linear) boundary conditions*.

Using the well-known *direct sum* of normed vector spaces (see Definition A.1), we may state this basic result whose importance is fundamental in this work.

Proposition 2.1. *Let* $\mathbb{A} : D(\mathbb{A}) \subseteq \mathcal{H} \to \mathcal{H}$ *and* $\mathbb{B} \in \mathcal{L}(\mathcal{H})$ *be such that* $R(\mathbb{B}) \subseteq D(\mathbb{A})$ *and* $\mathbb{A} \circ \mathbb{B} = id_{\mathcal{H}}$. *Then*

$$D(\mathbb{A}) = \ker \mathbb{A} \oplus R(\mathbb{B}) \qquad \textit{in} \ \ D(\mathbb{A}). \tag{2.5}$$

Moreover, for each $u \in D(\mathbb{A})$, *there exists a unique* $u_0 \in \ker A$ *such that*

$$u = u_0 + \mathbb{B}\mathbb{A}u \qquad \textit{in} \ \ D(\mathbb{A}). \tag{2.6}$$

Proposition 2.1 is an easy consequence of this result.

Proposition 2.2. *Under the assumptions and notation of Proposition 2.1, let, in addition,* $\mathcal{E}$ *be a closed subset of* $\mathcal{H}$. *Then* $\mathbb{B}(\mathcal{E})$ *is closed in* $D(\mathbb{A})$.

Proof. Let $(e_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$ and $u \in D(\mathbb{A})$ be such that

$$\mathbb{B}e_n \longrightarrow u \qquad \text{in} \ \ D(\mathbb{A}). \tag{2.7}$$

Then $e_n = \mathbb{A}\mathbb{B}e_n \longrightarrow \mathbb{A}u$ in $\mathcal{H}$. Since $\mathcal{E}$ is closed in $\mathcal{H}$, $\mathbb{A}u$ lies in $\mathcal{E}$. Moreover, the continuity of $\mathbb{B}$ yields that $\mathbb{B}e_n \to \mathbb{B}\mathbb{A}u$ in $\mathcal{H}$. However, $\mathbb{B}e_n \to u$ in $\mathcal{H}$, according to (2.7). Thus $u = \mathbb{B}\mathbb{A}u$, and u belongs to $\mathbb{B}(\mathcal{E})$ since we have shown that $\mathbb{A}u \in \mathcal{E}$. □

The following basic differential triplet will appear many times in the sequel in order to illustrate the main results of that section. In particular, we will show in Subsubsection 2.3.5, that the class of *boundary restriction operators* of $\frac{\mathrm{d}}{\mathrm{d}x}$ is the set of operators $\frac{\mathrm{d}}{\mathrm{d}x}$ supplemented with *homogeneous linear boundary conditions*.

Example 2.4. Let $\mathcal{H}$ be the standard complex Hilbert space $L^2(0,b)$ (see (4.1)), and g_1 be the function of $L^2(0,b)$ defined for a.e. x in $[0,b]$ by $g_1(x) = 1$. Also, we set

$$A_{\mathrm{M}} := A_{\mathrm{M},1} := \frac{\mathrm{d}}{\mathrm{d}x} : H^1(0,b) \subseteq L^2(0,b) \to L^2(0,b), \tag{2.8}$$

$B_1 f := g_1 * f$ and $Sf := f(b - \cdot)$, for all $f \in L^2(0,b)$. Then $\mathcal{T}_1 := (A_{\mathrm{M},1}, B_1, S)$ is a differential triplet. Moreover,

$$\ker A_{\mathrm{M},1} = \langle g_1 \rangle,$$

and (2.6) implies the basic identity

$$u(x) = u(0) + \int_0^x u'(y)\,\mathrm{d}y, \quad \forall\, x \in [0,b], \quad \forall\, u \in H^1(0,b).$$

Regarding the adjoint differential triplet $\mathcal{T}_1^* := (SA_{\mathrm{M},1}S, {B_1}^*, S)$, one has $SA_{\mathrm{M},1}S = -A_{\mathrm{M},1}$, and (2.6) reads

$$u = u_0 + B_1^* S A_{\mathrm{M},1} S u = u_0 - B_1^* A_{\mathrm{M},1} u,$$

with $u_0 \in \ker SA_{\mathrm{M}}S = \langle g_1 \rangle$. Thus $u_0 = u(b)$ and

$$u(x) = u(b) - \int_x^b u'(y)\,\mathrm{d}y, \quad \forall\, x \in [0,b].$$

Since $SA_{\mathrm{M},1}S = -A_{\mathrm{M},1}$, a boundary restriction operator of $\mathcal{T}_1$ cannot be symmetric. Hence differential triplets and boundary restriction operators are not encapsulated in Calkin's theory of abstract symmetric boundary conditions. See [Cal39, Theorem 1.1]. □

The following result is a step toward the description boundary restriction operators of any differential triplet $\mathcal{T} := (A_{\mathrm{M}}, B, S)$. Let us recall that, in view of Proposition 2.1, for each $u \in D(A_{\mathrm{M}})$, there exists a unique $u_0 \in \ker A_{\mathrm{M}}$ such that

$$u = u_0 + BA_{\mathrm{M}}u \qquad \text{in} \quad D(A_{\mathrm{M}}). \tag{2.9}$$

In a symmetric way, by considering the adjoint triplet of $\mathcal{T}$, each $f \in D(SA_{\mathrm{M}}S)$ has a unique decomposition

$$f = f_0 + B^* SA_{\mathrm{M}}Sf, \tag{2.10}$$

where $f_0 \in \ker A_{\mathrm{M}}S$.

Theorem 2.3. *Let (A_{M}, B, S) be a differential triplet, and A be a boundary restriction operator of A_{M}.*

(i) *If $f \in D(SA_{\mathrm{M}}S)$ then*

$$f \in D(A^*) \quad \Longleftrightarrow \quad (Au, f_0) = (u_0, SA_{\mathrm{M}}Sf), \quad \forall\, u \in D(A),$$

where u_0 and f_0 are given by (2.9) *and* (2.10)*.*

(ii) *If A is closed and $u \in D(A_{\mathrm{M}})$ then*

$$u \in D(A) \quad \Longleftrightarrow \quad (A_{\mathrm{M}}u, f_0) = (u_0, A^*f), \quad \forall\, f \in D(A^*).$$

Proof. For each $f \in D(SA_{\mathrm{M}}S)$ and $u \in D(A_{\mathrm{M}})$, one has

$$\begin{aligned}(A_{\mathrm{M}}u, f) &= (A_{\mathrm{M}}u, f_0) + (BA_{\mathrm{M}}u, SA_{\mathrm{M}}Sf) && \text{(by (2.10))}\\ &= (A_{\mathrm{M}}u, f_0) + (u, SA_{\mathrm{M}}Sf) - (u_0, SA_{\mathrm{M}}Sf) && \text{(by (2.9))}.\end{aligned} \tag{2.11}$$

(i) Assuming $f \in D(A^*)$, we have since $A^* \subseteq SA_{\mathrm{M}}S$ and $A \subseteq A_{\mathrm{M}}$,

$$(u, SA_{\mathrm{M}}Sf) = (u, A^*f) = (Au, f) = (A_{\mathrm{M}}u, f), \quad \forall\, u \in D(A).$$

Thus with (2.11),

$$(Au, f_0) = (u_0, SA_{\mathrm{M}}Sf), \quad \forall\, u \in D(A). \tag{2.12}$$

Conversely, let $f \in D(SA_{\mathrm{M}}S)$. Plugging (2.12) into (2.11), we get, for each $u \in D(A)$,

$$(A_{\mathrm{M}}u, f) = (Au, f) = (u, SA_{\mathrm{M}}Sf).$$

Thus the definition of the adjoint operator gives that $f \in D(A^*)$.

(ii) Let $u \in D(A)$. Arguing as above, we get what we want:

$$(A_{\mathrm{M}}u, f_0) = (u_0, A^*f), \quad \forall\, f \in D(A^*).$$

Conversely, if $u \in D(A_{\mathrm{M}})$ satisfies the above identity, then going back to (2.11), we get

$$(A_{\mathrm{M}}u, f) = (u, SA_{\mathrm{M}}Sf) = (u, A^*f), \quad \forall\, f \in D(A^*).$$

Thus $u \in D(A^{**})$. However, $A^{**} = A$ since A is closed. □

For each differential triplet $\mathcal{T}$, let us denote by $\mathrm{cBRO}(\mathcal{T})$ the set of all closed boundary restriction operators of $\mathcal{T}$.

Proposition 2.4. *Let $\mathcal{T} = (\mathbb{A}, \mathbb{B}, \mathbb{S})$ be a differential triplet, and $\mathcal{T}^*$ be its adjoint triplet in the sense of Definition 2.2. Then, for each A in $\mathrm{cBRO}(\mathcal{T})$, $\mathbb{S}A\mathbb{S}$ lies in $\mathrm{cBRO}(\mathcal{T}^*)$. Thus the map*

$$\mathrm{cBRO}(\mathcal{T}) \to \mathrm{cBRO}(\mathcal{T}^*), \quad A \mapsto \mathbb{S}A\mathbb{S}$$

is well defined, and besides, is a bijection.

Proof. Let A be an element of cBRO($\mathcal{T}$). Clearly, $\mathbb{S}A\mathbb{S}$ is a closed densely defined restriction of $\mathbb{S}\mathbb{A}\mathbb{S}$. Moreover, Theorem C.1 and [DS63, Lemma 6 p. 1189] yield that (this can be proved directly by using (2.3))

$$(\mathbb{S}A\mathbb{S})^* = \mathbb{S}A^*\mathbb{S} \subseteq \mathbb{S}(\mathbb{S}\mathbb{A}\mathbb{S})\mathbb{S},$$

since $A^* \subseteq \mathbb{S}\mathbb{A}\mathbb{S}$. Thus $\mathbb{S}A\mathbb{S}$ lies in cBRO($\mathcal{T}^*$).

Let us denote by F the map $A \mapsto \mathbb{S}A\mathbb{S}$. Since $\mathbb{S}^2 = id_{\mathcal{H}}$, we deduce that F is one-to-one. Moreover, let $\tilde{A} \in$ cBRO($\mathcal{T}^*$). Since $\mathcal{T}$ is the adjoint triplet of $\mathcal{T}^*$, we infer from the first part of that proof that $\mathbb{S}\tilde{A}\mathbb{S}$ belongs to cBRO($\mathcal{T}$). Then $F(\mathbb{S}\tilde{A}\mathbb{S}) = \tilde{A}$, so that F is onto. □

2.2. **Properties of boundary restriction operators.** The following result features sufficient conditions for a restriction of A_M to be densely defined, and is of fundamental importance in that paper.

Proposition 2.5. *Let (A_{M}, B, S) be a differential triplet in the sense of Definition 2.1. Let also F be a closed subspace of $\mathcal{H}$, and A be a restriction of A_{M}. If*

$$B(F^\perp) \subseteq D(A) \tag{2.13}$$

$$SF \cap D(A_{\mathrm{M}}) \subseteq \ker A_{\mathrm{M}}, \tag{2.14}$$

then $D(A)$ is dense in $\mathcal{H}$.

Proof. Let $f \in D(A)^\perp$. Then for each $\psi \in F^\perp$, (2.13) yields that $B\psi \in D(A)$, so that

$$0 = (f, B\psi) = (B^* f, \psi) = (SBSf, \psi).$$

Thus $SBSf \in F^{\perp\perp} = F$, since F is assumed to be closed. Due to $S^2 = id_{\mathcal{H}}$, one has $BSf \in SF$. Using $R(B) \subseteq D(A_{\mathrm{M}})$ and (2.14), we get

$$BSf \in SF \cap D(A_{\mathrm{M}}) \subseteq \ker A_{\mathrm{M}}.$$

Thus (2.2) yields $Sf = 0$, and finally $f = 0$. That proves the density of $D(A)$. □

The choice of $F := \{0\}$ in Proposition 2.5 allows to introduce a special densely defined operator introduced in this definition.

Definition 2.5. Let $\mathcal{T} := (\mathbb{A}, \mathbb{B}, \mathbb{S})$ be a differential triplet. The restriction of $\mathbb{A}$ to the domain $R(\mathbb{B})$ is called the *pivot operator of the differential triplet $\mathcal{T}$*. That-is-to-say, the pivot operator of $\mathcal{T}$, denoted by $\mathbb{A}_{\mathcal{T}}$ satisfies

$$D(\mathbb{A}_{\mathcal{T}}) := R(\mathbb{B}), \quad \mathbb{A}_{\mathcal{T}} u := \mathbb{A}u, \quad \forall\, u \in D(\mathbb{A}_{\mathcal{T}}). \tag{2.15}$$

□

Let a differential triplet $\mathcal{T} := (\mathbb{A}, \mathbb{B}, \mathbb{S})$ be given. It is easily proved that

$$0 \text{ lies in the resolvant set of } \mathbb{A}_{\mathcal{T}}, \quad (\mathbb{A}_{\mathcal{T}})^{-1} = \mathbb{B}, \tag{2.16}$$

and the following *Poincaré inequality* holds

$$\|u\| \le \|\mathbb{B}\| \|\mathbb{A}_{\mathcal{T}} u\|, \quad \forall\, u \in D(\mathbb{A}_{\mathcal{T}}). \tag{2.17}$$

Clearly, any operator $\mathbb{A}$ satisfying $\mathbb{A}_{\mathcal{T}} \subseteq \mathbb{A}$ is densely defined. In the framework of Example 2.4, the domain of the pivot operator of $\mathcal{T}_1$ is equal to

$${}_0H^1(0, b) := \{u \in H^1(0, b) \mid u(0) = 0\}. \tag{2.18}$$

In order to classify *boundary restriction operators,* we need to study the adjoint of restrictions of A_{M}. For, we will first consider *large* restrictions of A_{M} i.e. restrictions of A_{M} that are *extension* of $A_{\mathcal{T}}$. In other words, we will first study operators A satisfying

$$A_{\mathcal{T}} \subseteq A \subseteq A_{\mathrm{M}}. \tag{2.19}$$

Proposition 2.6. *Let $\mathcal{T} := (A_{\mathrm{M}}, B, S)$ be a differential triplet on $\mathcal{H}$, and A be an operator on $\mathcal{H}$ satisfying* (2.19)*. Then $D(A)$ is dense in $\mathcal{H}$, so that A^* exists. Moreover, $B^*A^* = id_{D(A^*)}$.*

Proof. Since $A_{\mathcal{T}} \subseteq A$, the domain of A is dense in $\mathcal{H}$. Moreover, for each $f \in D(A^*)$,

$$(f, Au) = (A^*f, u), \quad \forall\, u \in D(A). \tag{2.20}$$

Let $\psi \in \mathcal{H}$. Then, by assumptions, $B\psi \in D(A)$, and

$$AB\psi = A_{\mathrm{M}}B\psi = \psi,$$

due to (2.2). Thus by choosing $u := B\psi$ in (2.20), one gets

$$(f, \psi) = (B^*A^*f, \psi), \quad \forall\, \psi \in \mathcal{H}.$$

Hence $B^*A^* = id_{D(A^*)}$. □

Proposition 2.7. *Under the assumptions and notation of Proposition 2.6,*

$$D(A^*) = B^*\big(\ker A^{\perp}\big) = \big\{B^*h \mid h \in \ker A^{\perp}\big\}. \tag{2.21}$$

Moreover, for each $f \in D(A^*)$, there is a unique $h \in \ker A^{\perp}$ such that $f = B^*h$. Also $h = A^*f$, thus $A^*f \in \ker A^{\perp}$ and

$$f = B^*A^*f. \tag{2.22}$$

Proof. (2.22) follows from Proposition 2.6. Let us denote for simplicity, $B^*(\ker A^{\perp})$ by D^*. Let $f \in D(A^*)$. Choosing $u \in \ker A$ in (2.20), we get

$$(A^*f, u) = 0.$$

Thus $A^*f \in \ker A^{\perp}$. Then (2.22) leads to $D(A^*) \subseteq D^*$. Conversely, consider $f := B^*h$ any element of D^*. Let $u \in D(A)$. By (2.6), $u = u_0 + BAu$ for some (unique) u_0 in $\ker A$. Thus

$$(f, Au) = (h, BAu) = (h, u - u_0),$$

Since $h \in \ker A^{\perp}$, one has $(h, u_0) = 0$; so that $(f, Au) = (h, u)$. By definition of $D(A^*)$, we infer $f \in D(A^*)$ and $A^*f = h$. That proves (2.21) and the uniqueness of h. □

Proposition 2.8. *Let (A_{M}, B, S) be a differential triplet. Let also $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ be such that*

$$R(B) \subseteq D(A), \quad A \subseteq A_{\mathrm{M}}.$$

Then A is a boundary restriction operator of A_{M}.

Proof. By Proposition 2.6, A is densely defined. Thus there remains to prove that $A^* \subseteq SA_{\mathrm{M}}S$. For, let $f \in D(A^*)$, and $h := A^*f$. By (2.22) and (2.4), (2.3),

$$Sf = BSh.$$

Thus (2.2) yields that $Sf \in D(A_{\mathrm{M}})$ and $A_{\mathrm{M}}Sf = Sh$. Whence $A^* \subseteq SA_{\mathrm{M}}S$. □

Recalling Definition 2.5 of pivot operator, the following result states that the adjoint of the pivot operator is the pivot operator of the adjoint differential triplet.

Proposition 2.9. *Let $\mathcal{T} := (A_{\mathrm{M}}, B, S)$ be a differential triplet and $\mathcal{T}^* := (SA_{\mathrm{M}}S, B^*, S)$ its adjoint triplet. Then the pivot operator $A_{\mathcal{T}}$ of $\mathcal{T}$ is a closed boundary restriction operator of $\mathcal{T}$, and*

$$(A_{\mathcal{T}})^* = A_{\mathcal{T}^*} = SA_{\mathcal{T}}S. \tag{2.23}$$

Proof. Proposition 2.8 entails that $A_{\mathcal{T}}$ is a boundary restriction operator of $A_{\rm M}$. In order to prove the first equality of (2.23), we notice on the one hand that, by definition of $A_{\mathcal{T}^*}$,

$$A_{\mathcal{T}^*}u = SA_{\rm M}Su, \quad \forall\, u \in D(A_{\mathcal{T}^*}) = R(B^*). \tag{2.24}$$

On the other hand, since $A_{\mathcal{T}}$ is a boundary restriction operator of $A_{\rm M}$,

$$(A_{\mathcal{T}})^* \subseteq SA_{\rm M}S. \tag{2.25}$$

By Proposition 2.7,

$$D\big((A_{\mathcal{T}})^*\big) = B^*\big((\ker A_{\mathcal{T}})^{\perp}\big).$$

Since, by (2.16), $A_{\mathcal{T}}$ is one-to-one, we get $D((A_{\mathcal{T}})^*) = R(B^*)$. Hence, with (2.24)

$$D((A_{\mathcal{T}})^*) = D(A_{\mathcal{T}^*}).$$

Besides, for each $u \in D(A_{\mathcal{T}^*})$, one gets by using successively (2.24) and (2.25),

$$A_{\mathcal{T}^*}u = SA_{\rm M}Su = (A_{\mathcal{T}})^*u.$$

We have proved that $(A_{\mathcal{T}})^* = A_{\mathcal{T}^*}$.

Let us show that $A_{\mathcal{T}^*} = SA_{\mathcal{T}}S$. For, since $(A_{\mathcal{T}})^* = A_{\mathcal{T}^*}$, Proposition 2.7 yields that each $f \in D(A_{\mathcal{T}^*})$ reads $f = B^*A_{\mathcal{T}^*}f$. Thus $Sf = BSA_{\mathcal{T}^*}f$. By definition of $A_{\mathcal{T}}$, we deduce $Sf \in D(A_{\mathcal{T}})$ and $A_{\mathcal{T}}Sf = SA_{\mathcal{T}^*}f$; so that $SA_{\mathcal{T}}Sf = A_{\mathcal{T}^*}f$. We have shown that $A_{\mathcal{T}^*} \subseteq SA_{\mathcal{T}}S$. By symmetry, replacing $\mathcal{T}$ by $\mathcal{T}^*$, one has $A_{\mathcal{T}} \subseteq SA_{\mathcal{T}^*}S$. With $S^2 = id_{\mathcal{H}}$, we derive the wished equality.

Finally, since $(A_{\mathcal{T}})^*$ is a closed operator, we deduce from $(A_{\mathcal{T}})^* = SA_{\mathcal{T}}S$ that $A_{\mathcal{T}}$ is closed as well. □

After the study of "large" restrictions of $A_{\rm M}$, we will investigate "small" restrictions A of $A_{\rm M}$. In that case, A^* is not necessarily a restriction of $SA_{\rm M}S$: see (2.29).

Theorem 2.10. *Let A be an operator on $\mathcal{H}$, B in $\mathcal{L}(\mathcal{H})$, and F be a closed subspace of $\mathcal{H}$. Suppose that A is densely defined and*

$$D(A) = B(F^{\perp}) \tag{2.26}$$

$$AB\psi = \psi, \quad \forall\, \psi \in F^{\perp}. \tag{2.27}$$

Then the following assertions hold true.

(i) $R(A) = F^{\perp}$, $\ker A^* = F$.
(ii) $R(B^*) \subseteq D(A^*)$, $A^*B^* = id_{\mathcal{H}}$.
(iii) $D(A^*) = F \oplus R(B^*)$ *in* $D(A^*)$.
(iv) *For each $f \in D(A^*)$, there exists a unique $h_f \in F$ such that*

$$f = h_f + B^*A^*f. \tag{2.28}$$

(v) *Conversely, for each $h_0 \in F$ and each $h \in \mathcal{H}$,*

$$f := h_0 + B^*h \in D(A^*), \quad A^*f = h. \tag{2.29}$$

Proof. (i) Let us show that $R(A) = F^{\perp}$. For, let $y \in R(A)$. By (2.26), there exists $\psi \in F^{\perp}$ such that $y = AB\psi$. Then (2.27) yields that $y = \psi \in F^{\perp}$. Conversely, let $\psi \in F^{\perp}$. Then the vector $u := B\psi$ lies in $D(A)$ and $Au = \psi$ by (2.26)-(2.27). Hence $\psi \in R(A)$, so that $R(A) = F^{\perp}$.

It is well-known that $\ker A^* = R(A)^{\perp}$. Thus, since F is closed in $\mathcal{H}$, we deduce that $\ker A^* = F$.

(ii) Let $u \in D(A)$. Using $R(A) = F^\perp$, we deduce that $\psi := Au$ lies in $F^\perp$. Hence, by (2.26), $B\psi \in D(A)$. Then

$$\begin{aligned} A(B\psi - u) &= \psi - Au && \text{(by (2.27))} \\ &= 0 && \text{(by definition of } \psi). \end{aligned}$$

Since, by (2.26)-(2.27), A is one-to-one, we derive that $B\psi = u$, so that

$$BAu = u, \qquad \forall\, u \in D(A). \tag{2.30}$$

For each $\psi \in \mathcal{H}$, (2.30) yields that

$$(B^*\psi, Au) = (\psi, BAu) = (\psi, u), \qquad \forall\, u \in D(A).$$

Thus, by definition of the *adjoint operator*, $B^*\psi$ belongs to $D(A^*)$, and $A^*B^*\psi = \psi$. That proves the second item.

Items (iii)-(v) in the statement of the theorem follow from Proposition 2.1, by using (i) and (ii). □

The following proposition is a key-step toward characterization of boundary restriction operators.

Proposition 2.11. *Let (A_M, B, S) be a differential triplet. Let also F be a closed subspace of $\mathcal{H}$, and A be a restriction of A_M satisfying $D(A) = B(F^\perp)$. Then*

$$SF \subseteq \ker A_\mathrm{M} \qquad \Longleftrightarrow \qquad \overline{D(A)} = \mathcal{H}, \quad A^* \subseteq SA_\mathrm{M}S.$$

Proof. Assuming $SF \subseteq \ker A_\mathrm{M}$, we deduce that (2.14) holds true. Thus Proposition 2.5 yields that $D(A)$ is dense in $\mathcal{H}$, so that A^* is well defined. Let $f \in D(A^*)$, and $h := A^*f$. By (2.28) and (2.4), there exists $h_f \in F$ such that

$$Sf = Sh_f + BSh.$$

By (2.2), $BSh \in D(A_\mathrm{M})$ and $A_\mathrm{M}BSh = Sh$. Regarding Sh_f, we know by hypothesis that $Sh_f \in \ker A_\mathrm{M}$, that is $A_\mathrm{M}Sh_f = 0$. There results that $Sf \in D(A_\mathrm{M})$ and $A_\mathrm{M}Sf = Sh$, thus $A^* \subseteq SA_\mathrm{M}S$.

Conversely, Theorem 2.10 gives that $F = \ker A^*$. However, $A^* \subseteq SA_\mathrm{M}S$ implies that $\ker A^* \subseteq \ker SA_\mathrm{M}S$. Thus $SF \subseteq \ker A_\mathrm{M}$ due to $S^2 = id_\mathcal{H}$. □

By taking $F := S\ker A_\mathrm{M} = \ker A_\mathrm{M}S$, Proposition 2.11 allows us to introduce a special boundary restriction operator, the so-called *minimal operator* of a differential triplet.

Definition 2.6. The *minimal operator* of a differential triplet $(\mathbb{A}, \mathbb{B}, \mathbb{S})$ is the restriction of $\mathbb{A}$ to the domain $\mathbb{B}\big((\ker \mathbb{A}\mathbb{S})^\perp\big)$.

Proposition 2.12. *If (A_M, B, S) is a differential triplet with minimal operator A_m then $(A_\mathrm{M})^*$ is the minimal operator of the adjoint triplet $(SA_\mathrm{M}S, B^*, S)$. Moreover,*

$$SA_\mathrm{m}S = (A_\mathrm{M})^*.$$

Proof. We start with the equality of the domains of $SA_\mathrm{m}S$ and $(A_\mathrm{M})^*$. By Proposition C.2 and $S^2 = id_\mathcal{H}$,

$$S\big((\ker A_\mathrm{M})^\perp\big) = \big(S(\ker A_\mathrm{M})\big)^\perp = (\ker A_\mathrm{M}S)^\perp. \tag{2.31}$$

Recalling that $D(A_\mathrm{m}) := B\big((\ker A_\mathrm{M}S)^\perp\big)$, we obtain

$$B^*\big((\ker A_\mathrm{M})^\perp\big) = SD(A_\mathrm{m}).$$

Since by Proposition 2.7, $D\big((A_\mathrm{M})^*\big) = B^*\big((\ker A_\mathrm{M})^\perp\big)$, there results that $D\big((A_\mathrm{M})^*\big)$ is equal to $D(SA_\mathrm{m}S)$.

Let $f \in D(SA_{\mathrm{m}}S)$. Then $Sf \in D(A_{\mathrm{m}})$, and, by definition of A_{m}, there exists some $\psi \in (\ker A_{\mathrm{M}}S)^{\perp}$ such that $Sf = B\psi$. Hence, since $B^* = SBS$, one gets $f = B^*(S\psi)$ and

$$SA_{\mathrm{m}}Sf = S\psi. \tag{2.32}$$

However, $f \in D\big((A_{\mathrm{M}})^*\big)$ as well. By (2.31), $S\psi \in (\ker A_{\mathrm{M}})^{\perp}$. Applying Proposition 2.7 with $A := A_{\mathrm{M}}$, we deduce from $f = B^*(S\psi)$ and $S\psi \in (\ker A_{\mathrm{M}})^{\perp}$ that $(A_{\mathrm{M}})^*f = S\psi$. Thus with (2.32), $(A_{\mathrm{M}})^*f = SA_{\mathrm{m}}Sf$. There results that $(A_{\mathrm{M}})^* = SA_{\mathrm{m}}S$. Finally, $(A_{\mathrm{M}})^*$ is minimal since its domain is $B^*\big((\ker A_{\mathrm{M}})^{\perp}\big)$. □

Since A_{m} is minimal, its adjoint is maximal for the adjoint triplet. More precisely, this results holds.

Proposition 2.13. *Let A_{m} be the minimal operator of a differential triplet (A_{M}, B, S). If A_{M} is a closed operator then $SA_{\mathrm{M}}S = (A_{\mathrm{m}})^*$.*

Proof. Since A_{M} is assumed to be closed, the adjoint of $(A_{\mathrm{M}})^*$ is equal to A_{M}. Since S is invertible and $S^* = S$, one has $(SA_{\mathrm{m}}S)^* = S(A_{\mathrm{m}})^*S$. Thus the result follows from Proposition 2.12. □

A consequence of the latter proposition is that the minimal operator is a boundary restriction operator which is closed according to Proposition 2.12. More generally, we have the following characterization of closed boundary restriction operators.

Corollary 2.14. *Let (A_{M}, B, S) be a differential triplet, and A be a restriction of A_{M}. If A_{M} and A are closed operators then the following assertions are equivalent.*
 (i) *A is a boundary restriction operator of A_{M}.*
 (ii) *$A_{\mathrm{m}} \subseteq A$.*

Proof of Corollary 2.14. Assuming (i), the definition of boundary restriction operators yields that $A^* \subseteq SA_{\mathrm{M}}S$. That is, by Proposition 2.13, $A^* \subseteq (A_{\mathrm{m}})^*$. Moreover, $\ker A_{\mathrm{m}}$ is trivial in view of (2.16) and $A_{\mathrm{m}} \subseteq A_{\mathcal{T}}$. Therefore A_{m} is closed according to Proposition A.1; so that $(A_{\mathrm{m}})^{**} = A_{\mathrm{m}}$. Thus, since A is closed too, we deduce that $A_{\mathrm{m}} \subseteq A$. The converse is then obvious since $(A_{\mathrm{m}})^* = SA_{\mathrm{M}}S$, by Proposition 2.13. □

In order to describe closed *boundary restriction operator* of A_{M}, it is sufficient, in view of Corollary 2.14, to study the closed operators A satisfying $A_{\mathrm{m}} \subseteq A \subseteq A_{\mathrm{M}}$. The following Proposition provides the structure of their domain.

Proposition 2.15. *Let (A_{M}, B, S) be a differential triplet, and A be a restriction of A_{M}. Assume in addition that A_{M} is closed. Then*
 (i) *$D(A_{\mathrm{M}}) = \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}}S) \oplus B\big((\ker A_{\mathrm{M}}S)^{\perp}\big)$ in $D(A_{\mathrm{M}})$;*
 (ii) *the following assertions are equivalent.*
  (ii-a) *A is a closed boundary restriction operator of A_{M}.*
  (ii-b) *There exists a closed subspace E of $D(A_{\mathrm{M}})$ such that $E \subseteq \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}}S)$ and*

$$D(A) = E \oplus B\big((\ker A_{\mathrm{M}}S)^{\perp}\big) \qquad \text{in} \quad D(A_{\mathrm{M}}). \tag{2.33}$$

Proof. (i) Since A_{M} is assumed to be closed, $\ker A_{\mathrm{M}}$ is closed in $\mathcal{H}$, and a fortiori in $D(A_{\mathrm{M}})$. Then using $\ker A_{\mathrm{M}}S = S\ker A_{\mathrm{M}}$ and Proposition 2.2, we prove that $B(\ker A_{\mathrm{M}}S)$ and $B\big((\ker A_{\mathrm{M}}S)^{\perp}\big)$ are closed in $D(A_{\mathrm{M}})$. Finally, (2.5) entails that the sum is direct.

(ii) Assuming (ii-b), let us first show that A is closed. For, Proposition A.2 yields that $D(A)$ is closed in $D(A_{\mathrm{M}})$. Since A_{M} is closed, there results that A is a closed operator in

$\mathcal{H}$. Secondly, since A is a closed restriction of A_{M} satisfying $A_{\mathrm{m}} \subseteq A$, Corollary 2.14 entails (ii-a).

Conversely, let us assume that A is a closed boundary restriction operator of A_{M}. Then Corollary 2.14 entails that $A_{\mathrm{m}} \subseteq A$. Thus, in view of (i), we have

$$B\big((\ker A_{\mathrm{M}}S)^{\perp}\big) \subseteq D(A) \subseteq \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}}S) \oplus B\big((\ker A_{\mathrm{M}}S)^{\perp}\big).$$

Then we set

$$E := D(A) \cap \Big(\ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}}S)\Big). \tag{2.34}$$

We are now in position to prove (2.33). Since A is assumed to be closed and $A \subseteq A_{\mathrm{M}}$, we infer that $D(A)$ is closed in $D(A_{\mathrm{M}})$. Besides applying Proposition A.2 with $E := \ker A_{\mathrm{M}}$ and $F := (\ker A_{\mathrm{M}}S)^{\perp}$, we obtain that $\ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}}S)$ is closed in $D(A_{\mathrm{M}})$. Hence E (defined by (2.34)) is closed in $D(A_{\mathrm{M}})$.

Let $u \in D(A)$. By (i), there exist $u_1 \in \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}}S)$ and $u_2 \in B\big((\ker A_{\mathrm{M}}S)^{\perp}\big)$ such that $u = u_1 + u_2$. Moreover, $u_1 = u - u_2$ belongs to $D(A)$ since

$$u_2 \in B\big((\ker A_{\mathrm{M}}S)^{\perp}\big) = D(A_{\mathrm{m}}) \subseteq D(A).$$

Whence u_1 lies in E. Thus

$$D(A) = E + B\big((\ker A_{\mathrm{M}}S)^{\perp}\big).$$

Since the intersection is trivial, (2.33) follows. □

Corollary 2.16. *Let (A_{M}, B, S) be a differential triplet, and A be a boundary restriction operator of A_{M}. If A_{M} and A are closed operators then*

(i) $D(SA_{\mathrm{M}}S) = \ker A_{\mathrm{M}}S \oplus B^*(\ker A_{\mathrm{M}}) \oplus B^*\big((\ker A_{\mathrm{M}})^{\perp}\big)$;

(ii) *there exists a closed subspace E^{s} of $D(SA_{\mathrm{M}}S)$ such that $E^{\mathrm{s}} \subseteq \ker A_{\mathrm{M}}S \oplus B^*(\ker A_{\mathrm{M}})$ and*

$$D(A^*) = E^{\mathrm{s}} \oplus B^*\big((\ker A_{\mathrm{M}})^{\perp}\big). \tag{2.35}$$

Proof. Let $\mathcal{T}^* := (SA_{\mathrm{M}}S, B^*, S)$ denote the adjoint differential triplet of (A_{M}, B, S). Since A is a boundary restriction operator of A_{M}, one has $A^* \subseteq SA_{\mathrm{M}}S$. Moreover, $A^{**} = A$, hence A^* is a closed boundary restriction operator of $\mathcal{T}^*$. Applying Proposition 2.15 with $\mathcal{T}^*$, we prove easily the assertions of the corollary. □

Example 2.7 (Continuation of Example 2.4)**.** Recalling that $\mathcal{T}_1 := (A_{\mathrm{M},1}, B_1, S)$, $\ker A_{\mathrm{M},1} = \ker(A_{\mathrm{M},1}S) = \langle g_1 \rangle$, we notice that

$$B_1\big((\ker A_{\mathrm{M},1}S)^{\perp}\big) = B_1\big({g_1}^{\perp}\big),$$

and $B_1 g_1 = g_1 * g_1 = g_2$ (see (4.2) for the notation g_α). Item (i) in Proposition 2.15 tells us that each u in $D(A_{\mathrm{M},1})$ reads in a unique way,

$$u = u_1 g_1 + u_1' g_2 + B_1 \psi^{\perp}, \tag{2.36}$$

where u_1, $u_1' \in \mathbb{C}$ and $\psi^{\perp} \in {g_1}^{\perp}$. More explicitly,

$$u(x) = u_1 + u_1' x + \int_0^x \psi^{\perp}(y)\,\mathrm{d}y, \qquad \forall\, x \in [0, b].$$

Since $\int_0^b \psi^{\perp}(y)\,\mathrm{d}y = (\psi^{\perp}, g_1) = 0$, we derive

$$u_1 = u(0), \qquad u_1' = b^{-1}\big(u(b) - u(0)\big). \tag{2.37}$$

Thus the space

$$E_{\mathrm{M},1} := \ker A_{\mathrm{M},1} \oplus B_1(\ker A_{\mathrm{M},1}S) = \langle g_1, g_2 \rangle$$

encapsulates the boundary values of elements of $D(A_{\mathrm{M},1}) = H^1(0, b)$.

Regarding $\mathcal{T}_1^* := (-A_{\mathrm{M},1}, B_1^*, S)$, Corollary 2.16 yields that each f in $D(-A_{\mathrm{M},1})$ reads

$$f = f_1 g_1 + f_1' g_1 *' g_1 + g_1 *' \psi^\perp, \tag{2.38}$$

where f_1, $f_1' \in \mathbb{C}$ and $\psi^\perp \in {g_1}^\perp$. One has $g_1 *' g_1 = S(g_1 * S g_1) = S g_2$. Thus (2.38) entails

$$f(x) = f_1 + f_1'(b - x) + \int_x^b \psi^\perp(y)\,\mathrm{d}y, \quad \forall\, x \in [0, b].$$

We may compute f_1 and f_1' from the latter identity. However, it is simpler to notice that $Sf \in D(A_{\mathrm{M},1})$, so that (2.38) yields that

$$Sf = f_1 g_1 + f_1' g_2 + g_1 * \varphi^\perp,$$

where $\varphi^\perp = S\psi^\perp \in {g_1}^\perp$. By (2.36)-(2.37), we get

$$f_1 = f(b), \quad f_1' = b^{-1}\big(f(0) - f(b)\big). \tag{2.39}$$

That "coordinates trick" will be used in Section 4 for fractional operators.

Again, the space

$$E^s_{\mathrm{M},1} := \ker A_{\mathrm{M},1} \oplus B_1^*(\ker A_{\mathrm{M},1}) = \langle g_1, S g_2 \rangle = \langle g_1, g_2 \rangle \tag{2.40}$$

encapsulates, but in a different way, the boundary values of the elements of $D(-A_{\mathrm{M},1})$.

Recalling Definition 2.6, the minimal operator of $\mathcal{T}_1$, denoted by $A_{\mathrm{m},1}$ has domain $H^1_0(0, b)$. $\square$

2.3. **Boundary restriction operators with finite dimensional kernel.** In this section, being given a differential triplet (A_{M}, B, S) with minimal operator A_{m}, we will assume that A_{M} has non trivial finite dimensional kernel. Hence, there exist an integer $d \geq 1$, and linearly independent vectors $\xi_1, \dots, \xi_d$ of $D(A_{\mathrm{M}})$ such that

$$\ker A_{\mathrm{M}} = \langle \xi_1, \dots, \xi_d \rangle. \tag{2.41}$$

Therefore A_{M} is closed according to Proposition A.1. Let A be a closed boundary restriction operator of A_{M} in the sense of Definition 2.3. In view of Proposition 2.15,

$$D(A) = E \oplus B\big((\ker A_{\mathrm{M}} S)^\perp\big), \tag{2.42}$$

for some subspace E of

$$E_{\mathrm{M}} := \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}} S). \tag{2.43}$$

Assuming first that E is not trivial, we denote by $(e_1, \dots, e_{d_E})$ one of its basis, where $d_E \in [1, 2d]$ is the dimension of E. For each $j \in [1, d_E]$, the embedding

$$E \subseteq \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}} S)$$

and (2.41) yield that

$$e_j = x_{ij}\xi_i + BS y_{ij}\xi_i, \tag{2.44}$$

where x_{ij}, $y_{ij} \in \mathbb{C}$, and the repeated indexes sum convention is used, i.e.

$$x_{ij}\xi_j := \sum_{i=1}^d x_{ij}\xi_i.$$

Setting

$$\omega_j := x_{ij}\xi_i, \quad \omega_j' := y_{ij}\xi_i, \qquad \text{in} \quad \ker A_{\mathrm{M}}, \tag{2.45}$$

we have

$$e_j = \omega_j + BS\omega_j' \in \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}} S), \quad \forall\, 1 \leq j \leq d_E. \tag{2.46}$$

Although the following proposition is known from *Fredholm operator theory*, we give its proof for sake of completeness.

Proposition 2.17. *Under the above assumptions and notations, in particular* (2.41)-(2.46)*, let A be a closed boundary restriction operator of A_{M}. Then the range of A is closed in $\mathcal{H}$.*

Proof. Let $(u_n)_{n\geq 0} \subset D(A)$ and $f \in \mathcal{H}$ be such that $Au_n \to f$ in $\mathcal{H}$. By (2.44), (2.46), for each $n \geq 0$, there exist $x_1^n, \ldots, x_{d_E}^n \in \mathbb{C}$ and $\phi_n^\perp \in (\ker A_{\mathrm{M}} S)^\perp$ such that

$$u_n = x_j^n \omega_j + B\big(x_j^n S\omega_j' + \phi_n^\perp\big).$$

Thus

$$Au_n = x_j^n S\omega_j' + \phi_n^\perp \in \ker A_{\mathrm{M}} S \oplus (\ker A_{\mathrm{M}} S)^\perp.$$

Denoting by $P \in \mathcal{L}(\mathcal{H})$ the orthogonal projection onto $\ker A_{\mathrm{M}} S$, we infer

$$x_j^n S\omega_j' \xrightarrow[n\to\infty]{} Pf \qquad \text{in } \ \mathcal{H}.$$

However, by closedness of the finite dimensional space generated by $S\omega_1', \ldots, S\omega_{d_E}'$, Pf reads

$$Pf = x_j S\omega_j',$$

for some $x_1, \ldots, x_{d_E} \in \mathbb{C}$. Then we set

$$u := x_j e_j + B\big(f - Pf\big).$$

Since $f - Pf$ lies in $(\ker A_{\mathrm{M}} S)^\perp$, and, by (2.42), $D(A)$ contains $B\big((\ker A_{\mathrm{M}} S)^\perp\big)$, we infer that u belongs to $D(A)$, and $Au = f$. The proof of the proposition is now completed. □

Proposition 2.18. *Under the assumptions of Proposition 2.17, and recalling that d denotes the finite dimension of* $\ker A_{\mathrm{M}}$*, the following assertions holds true.*

(i) *If A is onto then* $\dim E \geq d$.
(ii) *If A is one-t-one then* $\dim E \leq d$.
(iii) *If A is a bijection then* $\dim E = d$.

Proof. Let $P \in \mathcal{L}(E_{\mathrm{M}})$ be the projection on $B(\ker A_{\mathrm{M}} S)$ along $\ker A_{\mathrm{M}}$. By (2.42), each u in $D(A)$ reads $u = e + B\phi^\perp$, where $e \in E$ and $\phi^\perp \in (\ker A_{\mathrm{M}} S)^\perp$. Thus, since $e - Pe$ lies in $\ker A_{\mathrm{M}}$,

$$Au = A_{\mathrm{M}} Pe + \phi^\perp.$$

If A is onto then for each h in $\ker A_{\mathrm{M}} S$, there exist $e \in E$ and $\phi^\perp \in (\ker A_{\mathrm{M}} S)^\perp$ such that

$$A_{\mathrm{M}} Pe + \phi^\perp = h.$$

$A_{\mathrm{M}} Pe$ belongs to $\ker A_{\mathrm{M}} S$, thus by orthogonality, $A_{\mathrm{M}} Pe = h$. Thus $A_{\mathrm{M}} P(E) = \ker A_{\mathrm{M}} S$. Since $\dim \ker A_{\mathrm{M}} S = d$, we have $\dim E \geq d$.

If A is one-to-one then $E \cap \ker A_{\mathrm{M}} = \ker A = \{0\}$. Hence *Grassman's formula* leads to

$$\dim E = \dim(E + \ker A_{\mathrm{M}}) - \dim(\ker A_{\mathrm{M}}) \leq d,$$

since E and $\ker A_{\mathrm{M}}$ are subspaces of E_{M}, and $\dim E_{\mathrm{M}} = 2d$. □

2.3.1. *Abstract boundary conditions for $D(A^*)$.* We will describe $D(A^*)$ by linear equations involving only coordinates in E_{M}. The result is stated in Proposition 2.19. These equations are called *abstract boundary conditions* since in application, coordinates in E_{M} are linearly related to boundary values of functions in $D(A^*)$.

Starting from the decomposition

$$D(SA_{\mathrm{M}}S) = \ker A_{\mathrm{M}} S \oplus B^*(\ker A_{\mathrm{M}}) \oplus B^*\big((\ker A_{\mathrm{M}})^\perp\big),$$

given by Corollary 2.16, any $f \in D(SA_{\mathrm{M}}S)$ reads in a unique way (see (2.41))

$$f = f_i S\xi_i + B^* f_i' \xi_i + B^* \varphi^\perp, \tag{2.47}$$

where f_i, $f_i' \in \mathbb{C}$ for all $1 \leq i \leq d$, and $\varphi^\perp \in (\ker A_{\mathrm{M}})^\perp$.

Proposition 2.19. *Under the above assumptions and notations, in particular A is a closed boundary restriction operator of A_{M},* (2.41)-(2.47) *hold and $1 \le d_E \le 2d$, let f belong to $D(SA_{\mathrm{M}}S)$. Then*

$$f \in D(A^*) \iff (\omega_j', f_i\xi_i) = (\omega_j, f_i'\xi_i), \quad \forall\, 1 \le j \le d_E.$$

Proof. Let $f \in D(SA_{\mathrm{M}}S)$ and $u \in D(A)$. Starting from the first equivalence of Theorem 2.3, using (2.42), (2.46), and recalling that $(e_1, \dots, e_{d_E})$ is a basis of E,

$$f \in D(A^*) \tag{2.48}$$

is equivalent to

$$\big(A(\omega_j + BS\omega_j' + BS\psi^\perp), f_0\big) = (\omega_j, SA_{\mathrm{M}}Sf), \quad \forall\, 1 \le j \le d_E,\ \forall \psi^\perp \in (\ker A_{\mathrm{M}})^\perp.$$

However,

$$\big(ABS\psi^\perp, f_0\big) = \big(S\psi^\perp, f_0\big) = \big(\psi^\perp, Sf_0\big) = 0,$$

since $\psi^\perp \in (\ker A_{\mathrm{M}})^\perp$ and $Sf_0 \in \ker A_{\mathrm{M}}$. Thus (2.48) is equivalent to

$$\big(S\omega_j', f_0\big) = (\omega_j, SA_{\mathrm{M}}Sf), \quad \forall\, 1 \le j \le d_E.$$

Moreover, recalling (2.10), and using (2.47), we get $f_0 = f_iS\xi_i$ and $SA_{\mathrm{M}}Sf = f_i'\xi_i + \varphi^\perp$. Since $(\omega_j, \varphi^\perp) = 0$, (2.48) is equivalent to

$$(\omega_j', f_i\xi_i) = (\omega_j, f_i'\xi_i), \quad \forall\, 1 \le j \le d_E.$$

□

2.3.2. *Dimension of E^{s}.* Let us recall that A denotes a closed boundary restriction operator of (A_{M}, B, S). In view of Corollary 2.16, let E^{s} be the subspace of

$$E^{\mathrm{s}}_{\mathrm{M}} := \ker A_{\mathrm{M}}S \oplus B^*(\ker A_{\mathrm{M}}) \tag{2.49}$$

satisfying

$$D(A^*) = E^{\mathrm{s}} \oplus B^*\big((\ker A_{\mathrm{M}})^\perp\big). \tag{2.50}$$

For each $j \in [1, d_E]$, we introduce the following linear form on $E^{\mathrm{s}}_{\mathrm{M}}$

$$F_j : \ker A_{\mathrm{M}}S \oplus B^*(\ker A_{\mathrm{M}}) \to \mathbb{C}, \quad f_0 + f_1 \mapsto (f_0, S\omega_j') - (SA_{\mathrm{M}}Sf_1, \omega_j). \tag{2.51}$$

Then we may compute the dimension of E^{s}.

Proposition 2.20. *Under the assumptions and notation of Proposition 2.19, and* (2.49)-(2.51)*, the dimension of E^{s} is $2d - d_E$.*

Proof. We claim that

$$E^{\mathrm{s}} = \bigcap_{j=1}^{d_E} \ker F_j. \tag{2.52}$$

Indeed, $E^{\mathrm{s}} \subseteq \cap_{j=1}^{d_E} \ker F_j$ by Proposition 2.19 and since $E^{\mathrm{s}} \subseteq D(A^*)$. Conversely, let f be in the intersection. Since $f \in E^{\mathrm{s}}_{\mathrm{M}} \subseteq D(SA_{\mathrm{M}}S)$, Proposition 2.19 leads that $f \in D(A^*)$. Besides

$$E^{\mathrm{s}}_{\mathrm{M}} \cap D(A^*) = E^{\mathrm{s}},$$

since

$$\begin{aligned} D(SA_{\mathrm{M}}S) &= E^{\mathrm{s}}_{\mathrm{M}} \oplus B^*\big(\ker A_{\mathrm{M}})^\perp\big) \\ D(A^*) &= E^{\mathrm{s}} \oplus B^*\big((\ker A_{\mathrm{M}})^\perp\big) \\ E^{\mathrm{s}} &\subseteq E^{\mathrm{s}}_{\mathrm{M}}. \end{aligned}$$

Thus $f \in E^{\mathrm{s}}$. That proves the claim.

Since $SA_{\mathrm{M}}SB^* = id_{\mathcal{H}}$, B^* is one-to-one. Hence (see (2.49)) $E^{\mathrm{s}}_{\mathrm{M}}$ has dimension $2d$. Therefore, standard results in linear algebra yield that E^{s} has dimension $2d - d_E$ provided $F_1, \dots, F_{d_E}$ are linearly independent in the dual space of $E^{\mathrm{s}}_{\mathrm{M}}$.

In order to check that independence, let $\lambda_1, \dots, \lambda_{d_E} \in \mathbb{C}$ be such that $\lambda_j F_j = 0$. By setting

$$f_0 := S\overline{\lambda_j}\omega'_j, \quad f_1 := -B^*\overline{\lambda_j}\omega_j,$$

we have (see (2.45))

$$f_0 + f_1 \in \ker A_{\mathrm{M}}S \oplus B^*(\ker A_{\mathrm{M}}) = E^{\mathrm{s}}_{\mathrm{M}}.$$

Thus

$$0 = \langle \lambda_j F_j, f_0 + f_1 \rangle = \|\overline{\lambda_j}\omega'_j\|^2 + \|\overline{\lambda_j}\omega_j\|^2.$$

Going back to (2.46), we infer that $\overline{\lambda_j}e_j = 0$. Thus $\lambda_j = 0$ for all $j \in [1, d_E]$, since $(e_1, \dots, e_{d_E})$ is a basis of E^{s}. The proof of the theorem is now completed. □

2.3.3. *Abstract boundary conditions of $D(A)$.* We assume that E is not maximal, that is $E \neq E_{\mathrm{M}}$, or equivalently

$$0 \leq d_E \leq 2d - 1. \tag{2.53}$$

Then Proposition 2.20 entails that E^{s} is not trivial. Let $(e^s_1, \dots, e^s_{2d-d_E})$ be a basis of E^{s}. In view of (2.49), e^s_j reads

$$e^s_j = S\omega^s_j + B^*\omega^{s\prime}_j, \quad \forall\, 1 \leq j \leq 2d - d_E, \tag{2.54}$$

for some ω^s_j and $\omega^{s\prime}_j$ in $\ker A_{\mathrm{M}}$. By Propositions 2.15 (i) and C.2, each u in $D(A_{\mathrm{M}})$ reads

$$u = u_i\xi_i + BSu'_i\xi_i + BS\psi^\perp, \tag{2.55}$$

where u_i, $u'_i \in \mathbb{C}$ for all $1 \leq i \leq d$, and $\psi^\perp \in (\ker A_{\mathrm{M}})^\perp$. Then arguing as in the proof of Proposition 2.19, we may show the following result.

Proposition 2.21. *Under the above assumptions and notations, in particular* (2.41)-(2.47) *and* (2.53)-(2.55)*, let $u \in D(A_{\mathrm{M}})$. Then*

$$u \in D(A) \iff (u'_i\xi_i, \omega^s_j) = (u_i\xi_i, \omega^{s\prime}_j), \quad \forall\, 1 \leq j \leq 2d - d_E.$$

2.3.4. *Limit cases.* Our analysis is complete when E is a proper subset of E_{M}. There remains to investigate of the cases where $E = E_{\mathrm{M}}$ and E is trivial.

(i) If $E = E_{\mathrm{M}}$ then $d_E = 2d$, $A = A_{\mathrm{M}}$, and

$$E = E_{\mathrm{M}} = \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}}S).$$

Setting

$$\omega_j := \begin{cases} \xi_j & \text{if } 1 \leq j \leq d \\ 0 & \text{if } d+1 \leq j \leq 2d \end{cases}, \quad \omega'_j := \begin{cases} 0 & \text{if } 1 \leq j \leq d \\ \xi_{j-d} & \text{if } d+1 \leq j \leq 2d \end{cases}, \tag{2.56}$$

it is clear that

$$e_j = \omega_j + BS\omega'_j, \quad \forall\, 1 \leq j \leq 2d,$$

constitutes a basis of E. Then Proposition 2.19 yields that

$$f \in D(A^*) \iff \begin{cases} 0 = (\xi_j, f'_i\xi_i) & \forall\, 1 \leq j \leq d \\ (\xi_{j-d}, f_i\xi_i) = 0 & \forall\, d+1 \leq j \leq 2d \end{cases} \iff \begin{cases} f_i = 0 & \\ f'_i = 0 & \forall\, 1 \leq i \leq d. \end{cases} \tag{2.57}$$

Thus $f \in D(A^*)$ iff its component on E_{M} are trivial. We will see in applications, that the above equations are equivalent to the cancellation of all boundary values of f. Also $A^* = (A_{\mathrm{M}})^*$ is the minimal operator of the differential triplet $(SA_{\mathrm{M}}S, B^*, S)$. That result was already proved in Proposition 2.12.

(ii) If E is trivial then $d_E = 0$, $A = A_{\mathrm{m}}$. Starting from the first equivalence of Theorem 2.3, each $u \in D(A_{\mathrm{m}})$ reads $u = BS\psi^{\perp}$, for some $\psi^{\perp} \in \ker A_{\mathrm{M}}^{\perp}$. Thus

$$u_0 = 0, \quad (Au, f_0) = (\psi^{\perp}, Sf_0) = 0, \quad \forall\, f_0 \in \ker A_{\mathrm{M}}S.$$

Thus Item (i) in Theorem 2.3 yields $A^* = SA_{\mathrm{M}}S$, so that there is no boundary condition.

The characterization of $D(A_{\mathrm{m}})$ in terms of boundary conditions follows from Proposition 2.21. Indeed, setting

$$\omega_j^s := \omega_j, \quad \omega_j^{s\prime} := \omega_j{}',$$

where ω_j and $\omega_j{}'$ are defined thru (2.56), it is clear that

$$e_j^s := S\omega_j^s + B^*\omega_j^{s\prime}, \quad \forall\, 1 \le j \le 2d$$

constitutes a basis of $E^{\mathrm{s}} = E^{\mathrm{s}}_{\mathrm{M}} := \ker A_{\mathrm{M}}S \oplus B^*(\ker A_{\mathrm{M}})$. Thus Proposition 2.21 gives

$$u \in D(A_{\mathrm{m}}) \iff \begin{cases} (u_i'\xi_i, \xi_j) = 0 & \forall\, 1 \le j \le d \\ 0 = (u_i\xi_i, \xi_{j-d}) & \forall\, d+1 \le j \le 2d \end{cases} \iff \begin{cases} u_i = 0 \\ u_i' = 0 & \forall\, 1 \le i \le d. \end{cases} \tag{2.58}$$

2.3.5. *Boundary restriction operators of* $\frac{\mathrm{d}}{\mathrm{d}x}$. We will illustrate the latter theory in the case where $A_{\mathrm{M}} := A_{\mathrm{M},1} = \frac{\mathrm{d}}{\mathrm{d}x}$. Starting from Example 2.7, and recalling that $\mathcal{T}_1 := (A_{\mathrm{M},1}, B_1, S)$, one has

$$\ker A_{\mathrm{M}} = \langle g_1 \rangle, \quad d = 1, \quad \xi_1 = g_1, \quad E_{\mathrm{M}} = E_{\mathrm{M},1} = \langle g_1, g_2 \rangle, \quad E^{\mathrm{s}}_{\mathrm{M}} = E^{\mathrm{s}}_{\mathrm{M},1} = \langle g_1, Sg_2 \rangle.$$

In view of (2.36) and (2.38), each $u \in E_{\mathrm{M},1}$ and $f \in E^{\mathrm{s}}_{\mathrm{M},1}$ read

$$u = u_1 g_1 + u_1' g_2, \quad f = f_1 g_1 + f_1' S g_2.$$

We will classify the closed *boundary restriction operators* A of $\frac{\mathrm{d}}{\mathrm{d}x}$ w.r.t. the dimension of E. It will appear that these operators form the set of all operators $\frac{\mathrm{d}}{\mathrm{d}x}$ supplemented with *homogeneous linear boundary conditions*. Also, we will determinate in a systematic way the adjoint of each boundary restriction operator.

(i) If $E = E_{\mathrm{M},1} = \langle g_1, g_2 \rangle$ then, in view of (2.56),

$$d_E = 2, \quad \omega_1 = g_1, \quad \omega_2 = 0, \quad \omega_1' = 0, \quad \omega_2' = g_1.$$

Starting from (2.57) and using (2.39), we obtain

$$f \in D\big((A_{\mathrm{M},1})^*\big) \iff f_1 = f_1' = 0 \iff f(0) = f(b) = 0.$$

(ii) If E is trivial then $A = A_{\mathrm{m},1}$. Starting from (2.58) and using (2.37), we obtain

$$u \in D(A_{\mathrm{m},1}) \iff u_1 = u_1' = 0 \iff u(0) = u(b) = 0.$$

(iii) If E is one-dimensional, that is $d_E = 1$, $E = \langle e_1 \rangle$ where (see (2.45)-(2.46))

$$e_1 = x_{11}g_1 + y_{11}g_2, \quad \omega_1 := x_{11}g_1, \quad \omega_1' := y_{11}g_1,$$

then Proposition 2.19 and (2.39) yield

$$f \in D(A^*) \iff \overline{y}_{11} f_1 = \overline{x}_{11} f_1' \tag{2.59}$$

$$\iff \overline{x}_{11} f(0) = (\overline{x}_{11} + b\overline{y}_{11}) f(b). \tag{2.60}$$

That is the characterization of $D(A^*)$ in terms of boundary conditions.

In order to characterize $D(A)$ in terms of boundary conditions, one needs to find a basis of E^{s}. We know that its dimension is $2d - d_E = 1$. Thus setting

$$\omega_1^{\mathrm{s}} := \overline{x}_{11} g_1, \quad \omega_1^{\mathrm{s}\prime} := \overline{y}_{11} g_1, \quad e_1^{\mathrm{s}} = \omega_1^{\mathrm{s}} + B_1^* \omega_1^{\mathrm{s}\prime},$$

(2.59) yields that $E^{\mathrm{s}} = \langle e_1^{\mathrm{s}} \rangle$, and Proposition 2.21 and (2.37) yield

$$\begin{aligned} u \in D(A) &\iff x_{11} u_1' = y_{11} u_1 \\ &\iff x_{11} u(b) = (x_{11} + b y_{11}) u(0). \end{aligned} \tag{2.61}$$

That is the characterization of $D(A)$ in terms of boundary conditions.

Let us emphasize some standard boundary conditions.

(a) If $x_{11} = 0$ then $D(A) = {}_0H^1(0,b)$, and A is the pivot operator $A_{\mathcal{T}_1}$ of $\mathcal{T}_1$ (see Definition 2.5). Also $D(A^*) = {}_bH^1(0,b)$.
(b) If $x_{11} + b y_{11} = 0$ then $D(A) = {}_bH^1(0,b)$. Also $D(A^*) = {}_0H^1(0,b)$.
(c) If $y_{11} = 0$ then

$$D(A) = D(A^*) = \{u \in H^1(0,b) \mid u(0) = u(b)\}.$$

That corresponds to *periodic boundary conditions.*

Finally, let us show that for each boundary restriction operator A of $A_{\mathrm{M},1}$ satisfies

$$u(b)\overline{f(b)} = u(0)\overline{f(0)}, \quad \forall\, u \in D(A),\ f \in D(A^*). \tag{2.62}$$

Indeed, that relation is clear for Items (i) and (ii). Regarding Item (iii), combining (2.60) and (2.61), one gets

$$x_{11} u \overline{f}(0) = (x_{11} + b y_{11}) \overline{f}(b) u(0) = x_{11} u \overline{f}(b).$$

If $x_{11} \neq 0$ then (2.62) holds. Otherwise, (2.62) follows from the result of Item (a).

Let us notice that (2.62) may be obtained directly by using by-part-integration, indeed

$$0 = (Au, f) - (u, A^* f) = \int_0^b u' \overline{f} + u \overline{f'} \,\mathrm{d}x = u\overline{f}(b) - u\overline{f}(0).$$

We claim that the notion of closed *boundary restriction operator of* $\frac{\mathrm{d}}{\mathrm{d}x}$ fits exactly with the one the "operator $\frac{\mathrm{d}}{\mathrm{d}x}$ supplemented with homogeneous linear boundary conditions". In order to make that statement precise, let us define the latter notion.

Definition 2.8. Let $\frac{\mathrm{d}}{\mathrm{d}x} : H^1(0,b) \subseteq L^2(0,b) \to L^2(0,b)$ and A be an operator on $L^2(0,b)$. We say that A is $\frac{\mathrm{d}}{\mathrm{d}x}$ *supplemented with homogeneous linear boundary conditions* if A is a restriction of $\frac{\mathrm{d}}{\mathrm{d}x}$ and one of the following conditions holds.

(i) $D(A) = \{u \in H^1(0,b) \mid u(0) = u(b) = 0\}$;
(ii) there exist a_1, $a_2 \in \mathbb{C}$ such that

$$D(A) = \big\{u \in H^1(0,b) \mid a_1 u(0) + a_2 u(b) = 0\big\}.$$

Proposition 2.22. *Let A be an operator on $L^2(0,b)$. Then the following assertions are equivalent.*

(i) *A is a closed boundary restriction operator of $(\frac{\mathrm{d}}{\mathrm{d}x}, B_1, S)$.*
(ii) *A is $\frac{\mathrm{d}}{\mathrm{d}x}$ supplemented with homogeneous linear boundary conditions.*

Proof. There results from the above analysis that a boundary restriction operator of $\frac{\mathrm{d}}{\mathrm{d}x}$ is $\frac{\mathrm{d}}{\mathrm{d}x}$ supplemented with homogeneous linear boundary conditions. Conversely, let A be $\frac{\mathrm{d}}{\mathrm{d}x}$ supplemented with some homogeneous linear boundary conditions. Then $D(A)$ contains $H_0^1(0,b)$, thus the minimal operator $A_{\mathrm{m},1}$ of $\mathcal{T}_1$ is a restriction of A. Thus $A^* \subseteq (A_{\mathrm{m},1})^* = SA_{\mathrm{M},1}S$, by Proposition 2.13. Hence A is a boundary restriction operator of $\frac{\mathrm{d}}{\mathrm{d}x}$. ☐

Remark 2.9. For any $c \in (0,b)$, let

$$h_c(x) := \begin{cases} 1 & \text{if } \ x \in [0,c] \\ 0 & \text{otherwise} \end{cases}.$$

Then, by Proposition 2.5, the restriction $\tilde{A}$ of $A_{\mathrm{M},1}$ to $B_1({h_c}^{\perp})$ is still densely defined. However, for each ψ in ${h_c}^{\perp}$, $u := B_1\psi$ satisfies

$$0 = (h_c, \tilde{A}u) = u(c) - u(0) = u(c).$$

Thus, in view of Definition 2.8, $\tilde{A}$ is not $\frac{\mathrm{d}}{\mathrm{d}x}$ supplemented with some homogeneous linear boundary conditions. Hence Proposition 2.22 entails that $\tilde{A}$ is not a boundary restriction operator of $\frac{\mathrm{d}}{\mathrm{d}x}$.

Also, since Sh_c does not belong to $\ker A_{\mathrm{M},1}$, the equivalence of Proposition 2.11 tells us that $\tilde{A}^*$ is not a restriction of $SA_{\mathrm{M},1}S$. By (2.29), we see that $D(\tilde{A}^*)$ contains a non smooth element, namely h_c.

This example highlights the link between homogeneous linear boundary conditions set on $D(A)$, and regularity of functions of $D(A^*)$.

3. Second order differential triplets

In application, the maximal operator A_{M} of a differential triplet is a basic operator like $\frac{\mathrm{d}}{\mathrm{d}x}$ or D^α. Thus A_{M} can be seen as a first order operator. From that point of view, $A_{\mathrm{M}}^2 := A_{\mathrm{M}} \circ A_{\mathrm{M}}$ is a second order operator, like $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ or $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. This section is devoted to a study of boundary restriction operators of *second order differential triplets.*

Being given a differential triplet $\mathcal{T} = (A_{\mathrm{M}}, B, S)$, it is easy to check that

$$\mathcal{T}^2 := (A_{\mathrm{M}}^2, B^2, S), \quad \mathcal{T}\mathcal{T}^* := \big((A_{\mathrm{M}}S)^2, B^*B, id_{\mathcal{H}}\big), \quad \mathcal{T}^*\mathcal{T} := \big((SA_{\mathrm{M}})^2, BB^*, id_{\mathcal{H}}\big) \tag{3.1}$$

are differential triplets. The two latter triplets are equal to their adjoint (see Definition 2.2). However, the adjoint triplet of $\mathcal{T}^2$ is

$$(\mathcal{T}^2)^* := \big(SA_{\mathrm{M}}^2S, (B^*)^2, S\big).$$

Thus Proposition 2.4 yields that the closed boundary restriction operators of $(\mathcal{T}^2)^*$ can be obtained from the ones of $\mathcal{T}^2$. Moreover, if we start from $\mathcal{T}^*$ instead of $\mathcal{T}$, that is if $\mathcal{T} := (SA_{\mathrm{M}}S, B^*, S)$, then

$$\mathcal{T}^*\mathcal{T} = \big((A_{\mathrm{M}}S)^2, B^*B, id_{\mathcal{H}}\big).$$

Thus, among these four so-called *second order differential triplets*, it is enough to study the two first triplets of (3.1).

Let us recall that the *pivot operator* is defined by (2.15), and *minimal operator* by Definition 2.6.

Proposition 3.1. *Let (A_{M}, B, S) be a differential triplet such that A_{M} is a closed operator on $\mathcal{H}$. Then the following assertions hold true.*

(i) $\ker A_{\mathrm{M}}$ *and* $B(\ker A_{\mathrm{M}})$ *are in direct sum in* $D(A_{\mathrm{M}})$ *and*

$$\ker(A_{\mathrm{M}})^2 = \ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}}) \qquad \textit{in} \ \ D(A_{\mathrm{M}}).$$

(ii) $\ker(A_{\mathrm{M}}S)^2 = \ker A_{\mathrm{M}}S \oplus B^*(\ker A_{\mathrm{M}})$.

(iii) *The pivot operator $L_{\mathcal{T}^2}$ of $\mathcal{T}^2$ is equal to $(A_{\mathcal{T}})^2$.*

(iv) *The pivot operator $L_{\mathcal{T}\mathcal{T}^*}$ of $\mathcal{T}\mathcal{T}^*$ is equal to $(A_{\mathcal{T}}S)^2$.*

(v) *The minimal operator of $\mathcal{T}^2$ is equal to $(A_{\mathrm{m}})^2$, where A_{m} is the minimal operator of $\mathcal{T}$.*

(vi) *The minimal operator of $\mathcal{T}\mathcal{T}^*$ is equal to $(A_{\mathrm{m}}S)^2$.*

Proof. We will prove only the odd items relative to $\mathcal{T}^2$. The others items, relative to $\mathcal{T}\mathcal{T}^*$ can be established in a same way.

(i) Arguing as in the proof of Proposition 2.15, we may show that $\ker A_{\mathrm{M}}$ and $B(\ker A_{\mathrm{M}})$ are in direct sum in $D(A_{\mathrm{M}})$. Let $u \in \ker(A_{\mathrm{M}})^2$. Then $A_{\mathrm{M}}u = \xi$ for some $\xi \in \ker A_{\mathrm{M}}$. By Proposition 2.1, there exists $u_0 \in \ker A_{\mathrm{M}}$ such that $u = u_0 + B\xi$. Thus u lies in $\ker A_{\mathrm{M}} \oplus B(\ker A_{\mathrm{M}})$. Since the converse is obvious, Item (i) is proved.

(iii) Let us start to show that $L_{\mathcal{T}^2} \subseteq (A_{\mathcal{T}})^2$. For, each $u \in D(L_{\mathcal{T}^2})$ reads $u = B^2\psi$ for some $\psi \in \mathcal{H}$, and satisfies $L_{\mathcal{T}^2}u = \psi$. Since $D(A_{\mathcal{T}}) = R(B)$ and $A_{\mathcal{T}}B = id_{\mathcal{H}}$, we derive $u \in D((A_{\mathcal{T}})^2)$ and $(A_{\mathcal{T}})^2 u = \psi$. Thus $L_{\mathcal{T}^2} \subseteq (A_{\mathcal{T}})^2$.

We will be done if we prove that $D((A_{\mathcal{T}})^2) \subseteq D(L_{\mathcal{T}^2})$. For, each $u \in D((A_{\mathcal{T}})^2)$ satisfies $B^2(A_{\mathcal{T}})^2 u = u$, since $(A_{\mathcal{T}})^{-1} = B$. Thus u belongs to $R(B^2)$, which by definition is the domain of $L_{\mathcal{T}^2}$.

(v) Denoting by L_{m} the minimal operator of $\mathcal{T}^2$, we start to show that $L_{\mathrm{m}} \subseteq {A_{\mathrm{m}}}^2$. For, by definition of L_{m}, each $u \in D(L_{\mathrm{m}})$ reads $u = B^2\psi^\perp$ for some $\psi^\perp \in (\ker {A_{\mathrm{M}}}^2 S)^\perp$, and satisfies $L_{\mathrm{m}}u = \psi^\perp$. Moreover, by Item (i) and Proposition C.2,

$$(\ker {A_{\mathrm{M}}}^2 S)^\perp = (\ker A_{\mathrm{M}} S)^\perp \cap \big(SB(\ker A_{\mathrm{M}})\big)^\perp. \tag{3.2}$$

We claim that $u = B^2\psi^\perp$ lies in $D(A_{\mathrm{m}})$. Indeed, by definition of A_{m}, it is enough to prove that $B\psi^\perp \in (\ker A_{\mathrm{M}} S)^\perp$. For each $\xi \in \ker A_{\mathrm{M}}$, one has

$$(B\psi^\perp, S\xi) = (\psi^\perp, B^* S\xi) = (\psi^\perp, SB\xi) = 0,$$

since $\psi^\perp$ lies in $\big(SB(\ker A_{\mathrm{M}})\big)^\perp$, according to (3.2). Thus $B\psi^\perp \in (\ker A_{\mathrm{M}} S)^\perp$, and the claim is proved.

Then $A_{\mathrm{m}}u = B\psi^\perp$. By (3.2) again, we know that $\psi^\perp \in (\ker A_{\mathrm{M}} S)^\perp$. Whence $A_{\mathrm{m}}u$ belongs to $D(A_{\mathrm{m}})$ and ${A_{\mathrm{m}}}^2 u = \psi^\perp$. Recalling that $L_{\mathrm{m}}u = \psi^\perp$, we get $L_{\mathrm{m}} \subseteq {A_{\mathrm{m}}}^2$.

Finally, let us show that $D({A_{\mathrm{m}}}^2) \subseteq D(L_{\mathrm{m}})$. For, let $u \in D({A_{\mathrm{m}}}^2)$. Since $A_{\mathrm{m}} \subseteq A_{\mathcal{T}}$, one has

$$B^2 {A_{\mathrm{m}}}^2 u = u. \tag{3.3}$$

There remains to prove that ${A_{\mathrm{m}}}^2 u \in (\ker {A_{\mathrm{M}}}^2 S)^\perp$. For, we will use (3.2). Item (i) in Theorem 2.10 tells us that $R(A_{\mathrm{m}}) = (\ker A_{\mathrm{M}} S)^\perp$. Thus ${A_{\mathrm{m}}}^2 u \in (\ker A_{\mathrm{M}} S)^\perp$. Moreover, for each $\xi \in \ker A_{\mathrm{M}}$,

$$\big({A_{\mathrm{m}}}^2 u, SB\xi\big) = \big((SA_{\mathrm{m}}S)SA_{\mathrm{m}}u, B\xi\big).$$

By Proposition 2.12, $SA_{\mathrm{m}}S = (A_{\mathrm{M}})^*$. Thus, since $B\xi$ lies in $D(A_{\mathrm{M}})$, one has

$$\big({A_{\mathrm{m}}}^2 u, SB\xi\big) = (A_{\mathrm{m}}u, S\xi) = 0,$$

since $R(A_{\mathrm{m}}) = (\ker A_{\mathrm{M}} S)^\perp$. Hence ${A_{\mathrm{m}}}^2 u \in \big(SB(\ker A_{\mathrm{M}})\big)^\perp$. There results from (3.2), that ${A_{\mathrm{m}}}^2 u$ belongs to $(\ker {A_{\mathrm{M}}}^2 S)^\perp$. Going back to (3.3), we get $u \in D(L_{\mathrm{m}})$. □

According to Proposition 3.1, ${A_{\mathrm{m}}}^2$ and ${A_{\mathcal{T}}}^2$ are boundary restriction operators of $\mathcal{T}^2$. More generally, the following result investigates the product of two closed boundary restriction operators. The adjoint of the product is computed via a deep result of S. Holland stated in Theorem C.1.

Proposition 3.2. *Let (A_{M}, B, S) be a differential triplet such that A_{M} has a finite dimensional kernel. Then*

(i) *if A_1, A_2 are closed boundary restriction operators of $\mathcal{T}$ then A_2A_1 is a boundary restriction operator of $\mathcal{T}^2$.*
(ii) *If A_1, A_2 are closed boundary restriction operator of $\mathcal{T}^*$ and $\mathcal{T}$ respectively, then A_2A_1 is a boundary restriction operator of $\mathcal{T}\mathcal{T}^*$.*

Proof. We will prove only (i) since (ii) follows in the same way. Let us start to show that A_2A_1 is densely defined. Since A_1 and A_{M} are closed, A_1 is an extension of the minimal operator A_{m} of $\mathcal{T}$, by Corollary 2.14. Thus

$$R(A_1) \supseteq R(A_{\mathrm{m}}) = (\ker A_{\mathrm{M}}S)^{\perp},$$

by Item (i) in Theorem 2.10. Since $R(A_1)$ is closed by Proposition 2.17, we deduce that $R(A_1)$ has finite codimension in $\mathcal{H}$. Then Theorem C.1 yields that A_2A_1 is densely defined, and $(A_2A_1)^* = A_1^*A_2^*$.

Since A_1, A_2 are boundary restriction operators of A_{M}, one has $A_i{}^* \subseteq SA_{\mathrm{M}}S$. Thus $A_1^*A_2^* \subseteq SA_{\mathrm{M}}^2S$. Hence A_2A_1 is a boundary restriction operator of $\mathcal{T}^2$. □

This well-known proposition, whose proof will be omitted, provides self-adjoint restrictions of $(A_{\mathrm{M}}S)^2$.

Proposition 3.3. *If A is closed densely defined operator on $\mathcal{H}$ then the following assertions hold true.*

(i) $L_{\mathrm{op}} := AA^*$ *is a non-negative (i.e. $(L_{\mathrm{op}}u, u) \geq 0$, $\forall u \in D(L_{\mathrm{op}})$) self-adjoint operator on $\mathcal{H}$.*
(ii) *The spectrum of L_{op} is contained in the real interval $[0, \infty)$.*
(iii) $\overline{R(L_{\mathrm{op}})} = \overline{R(A)}$ *and* $\ker L_{\mathrm{op}} = R(A)^{\perp}$.

Example 3.1. With the notation of Example 2.7, we consider the following second order differential triplets

$$\mathcal{T}_1{}^2 := (\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}, B_1{}^2, S), \quad \mathcal{T}_1\mathcal{T}_1^* := (-\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}, B_1{}^*B_1, id_{L^2(0,b)}).$$

We know by Item (a) in Subsubsection 2.3.5, that the domain of the pivot operator of $\mathcal{T}_1$ is ${}_0H^1(0,b)$. Thus Proposition 3.1 (iii) tells us that the domain of the pivot operator $L_{\mathcal{T}_1^2}$ of $\mathcal{T}_1{}^2$ is

$$D\big(L_{\mathcal{T}_1{}^2}\big) = \big\{u \in H^2(0,b) \mid u(0) = u'(0) = 0\big\}.$$

In a same way, the pivot operator $L_{\mathcal{T}_1\mathcal{T}_1^*}$ of $\mathcal{T}_1\mathcal{T}_1^*$ has domain

$$D\big(L_{\mathcal{T}_1\mathcal{T}_1^*}\big) = \big\{u \in H^2(0,b) \mid u(b) = u'(0) = 0\big\}.$$

Moreover, using also Proposition 2.9, we get

$$L_{\mathcal{T}_1\mathcal{T}_1^*} = A_{\mathcal{T}_1}(A_{\mathcal{T}_1})^*.$$

Hence Proposition 3.3 entails that $L_{\mathcal{T}_1\mathcal{T}_1^*}$ is self-adjoint.

Regarding the minimal operators of $\mathcal{T}_1{}^2$ and $\mathcal{T}_1\mathcal{T}_1^*$, their common domain is

$$\begin{aligned} D\big(L_{\mathrm{m},1}\big) &= \big\{u \in H_0^1(0,b) \mid u' \in H_0^1(0,b)\big\} \\ &= \big\{u \in H^2(0,b) \mid u(0) = u(b) = u'(0) = u'(b) = 0\big\}. \end{aligned}$$

Also, Proposition 3.1 gives

$$\ker(-\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}) = \ker(\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}) = \langle g_1, g_2\rangle.$$

Finally, let us introduce $L_{\mathrm{op},D,1}$, the well-known opposite of the *one-dimensional Laplace operator with Dirichlet boundary condition*. In our framework, that operator is defined by

$$L_{\mathrm{op},D,1} := \frac{\mathrm{d}}{\mathrm{d}x} \circ \big(\frac{\mathrm{d}}{\mathrm{d}x}\big)^*, \tag{3.4}$$

where $\frac{\mathrm{d}}{\mathrm{d}x} := A_{\mathrm{M},1}$. Thus $L_{\mathrm{op},D,1}u = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}u$ for each u in

$$D\big(L_{\mathrm{op},D,1}\big) = \big\{u \in H^2(0,b) \mid u(0) = u(b) = 0\big\}.$$

Proposition 3.3 entails that $L_{\mathrm{op},D,1}$ is a positive self-adjoint operator on $\mathcal{H}$, a well-known result.

4. First order fractional differential operators

Let us turn our attention to the application of the theoretical results of Section 2 to fractional differential operators. Applications of these theoretical results to the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ has been achieved in Subsubsection 2.3.5. Keeping the same approach, we will study the first order fractional differential operator D^α and its boundary restriction operators.

4.1. **A differential triplet.** Let $b \in (0,\infty)$, and $\mathcal{H} := L^2(0,b)$. The standard inner product of the complex Hilbert space $\mathcal{H}$ is

$$(f,h) := \int_0^b f(x)\overline{h(x)}\,\mathrm{d}x, \qquad \forall\, f,h \in \mathcal{H}. \tag{4.1}$$

The related norm of $L^2(0,b)$ is denoted by $\|\cdot\|$. Also, for each $\beta > 0$, let g_β be the function of $L^1(0,b)$ defined for a.e. x in $[0,b]$ by

$$g_\beta(x) = \frac{1}{\Gamma(\beta)} x^{\beta-1}. \tag{4.2}$$

The domain of the *fractional differential operator* D^α is a follows.

Definition 4.1. For each $\alpha \in (0,1)$, we denote by $D(A_{\mathrm{M},\alpha})$ the space of functions u in $L^2(0,b)$ for which there exists $u_0 \in \mathbb{C}$ such that

$$g_{1-\alpha} * (u - u_0) \in H^1(0,b).$$

The following result whose proof is elementary, is a particular case of [ER19, Prop. 2.1].

Proposition 4.1. *Let $b \in (0,\infty)$, $\alpha \in [\frac{1}{2},1)$ and $u \in D(A_{\mathrm{M},\alpha})$. Then there exists a unique $u_0 \in \mathbb{C}$ such that*

$$g_{1-\alpha} * (u - u_0) \in H^1(0,b).$$

Although elements of $D(A_{\mathrm{M},\alpha})$ are not continuous in general (see Remark 4.3 (ii) below), we will set, in view of Proposition 4.1, $u(0) := u_0$ for each $u \in D(A_{\mathrm{M},\alpha})$.

We are now in position to define a *(maximal) fractional differential operator* on $L^2(0,b)$.

Definition 4.2. Let $b \in (0,\infty)$ and $\alpha \in [\frac{1}{2},1)$. Then we define the operator

$$A_{\mathrm{M},\alpha} : D(A_{\mathrm{M},\alpha}) \subset L^2(0,b) \to L^2(0,b), \qquad u \mapsto \frac{\mathrm{d}}{\mathrm{d}x}\big\{g_{1-\alpha} * (u - u_0)\big\}, \tag{4.3}$$

where u_0 is the unique complex number given by Proposition 4.1. For each $u \in D(A_{\mathrm{M},\alpha})$, $A_{\mathrm{M},\alpha}u$ is called the *fractional derivative of u of order α* in $L^2(0,b)$. Also we put

$$\mathrm{D}^\alpha := A_{\mathrm{M},\alpha}.$$

If $\alpha = 1$ then $A_{\mathrm{M},1}$ is defined according to Example 2.4.

Remark 4.3. Let $\beta > 0$. If $\alpha = \frac{1}{2}$ then

$$g_\beta \in D\big(A_{\mathrm{M},\frac{1}{2}}\big) \quad \Longleftrightarrow \quad \beta \geq 1.$$

If $\alpha \in (\frac{1}{2},1)$ then

$$g_\beta \in D(A_{\mathrm{M},\alpha}) \quad \Longleftrightarrow \quad \beta > \alpha + \frac{1}{2} \quad \text{or} \quad \beta \in \{1,\alpha\}.$$

Let $\alpha \in [\frac{1}{2},1)$ and $g_\beta \in D(A_{\mathrm{M},\alpha})$.

(i) If $\beta \neq \alpha$ then g_β admits a continuous representative on $[0, b]$, still labeled g_β. Moreover, if $\beta \neq 1$ then $A_{\mathrm{M},\alpha} g_\beta = g_{\beta-\alpha}$ and $g_\beta(0) = 0$. On the other hand, $A_{\mathrm{M},\alpha} g_1 = 0$, $g_1(0) = 1$.
(ii) If $\beta = \alpha$ then $\alpha > \frac{1}{2}$, $A_{\mathrm{M},\alpha} g_\alpha = 0$ and $g_\alpha(0) = 0$. Let us notice that g_α is not continuous at $x = 0$, although it lies in $D(A_{\mathrm{M},\alpha})$.

Finally,

$$\ker A_{\mathrm{M},\alpha} = \begin{cases} \langle g_1 \rangle & \text{if } \alpha = \frac{1}{2} \\ \langle g_1, g_\alpha \rangle & \text{if } \frac{1}{2} < \alpha < 1 \end{cases}. \tag{4.4}$$

□

For each $\alpha \in [\frac{1}{2}, 1]$, let

$$B_\alpha : L^2(0, b) \to L^2(0, b), \quad f \mapsto g_\alpha * f. \tag{4.5}$$

$$S : L^2(0, b) \to L^2(0, b), \quad f \mapsto f(b - \cdot) \tag{4.6}$$

$$\mathcal{T}_\alpha := (A_{\mathrm{M},\alpha}, B_\alpha, S). \tag{4.7}$$

Clearly, for all $f \in L^2(0, b)$, $B_\alpha f \in D(A_{\mathrm{M},\alpha})$ with $B_\alpha f(0) = 0$. Using also the fact that g_α is real-valued, we check easily that $\mathcal{T}_\alpha$ is a *differential triplet*. We put

$$g_\alpha *' u := (B_\alpha)^* u = S\big(g_\alpha * (Su)\big), \quad \forall\, \alpha \in (0, \infty), \quad \forall\, u \in L^2(0, b). \tag{4.8}$$

Moreover, in view of (4.4), we infer from Propositions 2.1 and A.1 that $A_{\mathrm{M},\alpha}$ is a closed operator on $L^2(0, b)$.

4.2. **Restriction operators.** For $\alpha \in [\frac{1}{2}, 1)$, let $A_{\mathrm{m},\alpha}$ denote the minimal operator of $\mathcal{T}_\alpha$. Then

$$D(A_{\mathrm{m},\alpha}) = B_\alpha\big((\ker A_{\mathrm{M},\alpha} S)^\perp\big). \tag{4.9}$$

Definition 4.4. For $\alpha \in [\frac{1}{2}, 1)$, *Caputo's operator* ${}^{\mathrm{C}}\mathrm{D}^\alpha$ is the restriction of $A_{\mathrm{M},\alpha}$ to the domain

$$D({}^{\mathrm{C}}\mathrm{D}^\alpha) := \big\{u \in D(A_{\mathrm{M},\alpha}) \mid g_{1-\alpha} * \big(u - u(0)\big)(0) = 0\big\}.$$

□

Let us focus on eigenmodes of ${}^{\mathrm{C}}\mathrm{D}^\alpha$. Since, for all $f \in L^2(0, b)$, $k \in \mathbb{N}$,

$$\|(B_\alpha)^k f\| = \|g_{\alpha k} * f\| \leq \|g_{\alpha k}\|_{L^1(0,b)} \|f\| = g_{\alpha k+1}(b) \|f\|,$$

and the kernel of ${}^{\mathrm{C}}\mathrm{D}^\alpha$ is generated by $g_1 = 1$, we recover, thanks to Proposition C.3, the following well-known result. For each $\lambda \in \mathbb{C}$, the kernel of ${}^{\mathrm{C}}\mathrm{D}^\alpha - \lambda$ is the one-dimensional space generated by the function

$$e_{\mathrm{C},\alpha,\lambda} := \sum_{k \geq 0} \lambda^k g_{\alpha k+1} \qquad \text{in } \ L^2(0, b). \tag{4.10}$$

Of course, $e_{\mathrm{C},\alpha,\lambda}$ may be expressed by means of the *Mittag Leffler function* E_α thru the identity

$$e_{\mathrm{C},\alpha,\lambda}(x) = E_\alpha(\lambda x^\alpha) \qquad \text{a.e } \ x \in [0, b].$$

Let us now introduce the another famous fractional operator.

Definition 4.5. For $\alpha \in [\frac{1}{2}, 1)$, *Riemann-Liouville's operator* ${}^{\mathrm{RL}}\mathrm{D}^\alpha$ is the restriction of $A_{\mathrm{M},\alpha}$ to the domain

$$D({}^{\mathrm{RL}}\mathrm{D}^\alpha) := \big\{u \in D(A_{\mathrm{M},\alpha}) \mid u(0) = 0\big\}.$$

□

Let us focus on eigenmodes of ${}^{\mathrm{RL}}\mathrm{D}^{\alpha}$. In the case where $\alpha \in (\frac{1}{2}, 1)$, the kernel of ${}^{\mathrm{RL}}\mathrm{D}^{\alpha}$ is generated by g_α. Thus, arguing as for Caputo's operator, we get that, for each $\lambda \in \mathbb{C}$, the kernel of ${}^{\mathrm{RL}}\mathrm{D}^{\alpha} - \lambda$ is generated by the function

$$e_{\mathrm{RL},\alpha,\lambda} := \sum_{k\geq 0} \lambda^k g_{\alpha(k+1)}. \tag{4.11}$$

On the over hand, if $\alpha = \frac{1}{2}$ then ${}^{\mathrm{RL}}\mathrm{D}^{\alpha}$ has a trivial kernel, thus Proposition C.3 yields that ${}^{\mathrm{RL}}\mathrm{D}^{\alpha}$ has no eigenmode.

4.3. **Analysis of $A_{\mathrm{M},\alpha}$ for $\frac{1}{2} < \alpha < 1$.** In view of Subsection 2.3, one has $d = 2$, and (see (2.43), (2.42))

$$E_{\mathrm{M},\alpha} := \ker A_{\mathrm{M},\alpha} \oplus B_\alpha(\ker A_{\mathrm{M},\alpha} S) = \langle g_1, g_\alpha, B_\alpha g_1, B_\alpha S g_\alpha \rangle. \tag{4.12}$$

$$D(A_{\mathrm{M},\alpha}) = E_{\mathrm{M},\alpha} \oplus B_\alpha\big(S(\ker A_{\mathrm{M},\alpha})^\perp\big) \tag{4.13}$$

Regarding the eigenmodes of D^α, in view of the above analysis, and in particular of (4.10)-(4.11), we derive that, for each $\lambda \in \mathbb{C}$, the kernel of $\mathrm{D}^\alpha - \lambda$ is generated by $e_{\mathrm{C},\alpha,\lambda}$ and $e_{\mathrm{RL},\alpha,\lambda}$.

Let us focus on boundary values of functions in $D(\mathrm{D}^\alpha)$. There results from (4.12)-(4.13) that u lies in $D(A_{\mathrm{M},\alpha})$ if and only if there exists $u_1, u_2, u_1', u_2' \in \mathbb{C}$ and $\psi^\perp \in \langle g_1, g_\alpha\rangle^\perp$ such that

$$u = u_1 g_1 + u_2 g_\alpha + B_\alpha u_1' g_1 + B_\alpha S u_2' g_\alpha + B_\alpha S \psi^\perp. \tag{4.14}$$

We claim that u admits a continuous representative on $(0, b]$, and that continuous representative, still labelled u satisfies

$$\begin{aligned} u_1 &= u(0) \\ u_2 &= g_{1-\alpha} * \big(u - u(0)\big)(0) \\ u_1' b + u_2' g_{1+\alpha}(b) &= g_{1-\alpha} * \big(u - u(0)\big)(b) - g_{1-\alpha} * \big(u - u(0)\big)(0) \\ u_1' g_{1+\alpha}(b) + u_2' \|g_\alpha\|^2 &= u(b) - u(0) - g_{1-\alpha} * \big(u - u(0)\big)(0) g_\alpha(b). \end{aligned} \tag{4.15}$$

Indeed, regarding u_1, we have already observed that $g_1(0) = 1$, $g_\alpha(0) = 0$, and $B_\alpha h(0) = 0$ for each $h \in L^2(0, b)$. Thus $u_1 = u(0)$. For u_2, convoluting (4.14) with $g_{1-\alpha}$, we derive

$$g_{1-\alpha} * \big(u - u(0)\big) = u_2 g_1 + g_1 * (u_1' g_1 + u_2' S g_\alpha + S\psi^\perp).$$

Thus the second equation of (4.15) follows. Regarding u_j', we start from the identity

$$A_{\mathrm{M},\alpha} u = u_1' g_1 + u_2' S g_\alpha + S\psi^\perp.$$

Using $(S g_\alpha, g_1) = g_{1+\alpha}(b)$, and $(S\psi^\perp, g_1) = (\psi^\perp, g_1) = 0$, we get

$$(A_{\mathrm{M},\alpha} u, g_1) = u_1' b + u_2' g_{1+\alpha}(b). \tag{4.16}$$

Then the definition of $A_{\mathrm{M},\alpha}$ entails the third equation. In the same way,

$$(A_{\mathrm{M},\alpha} u, S g_\alpha) = u_1' g_{1+\alpha}(b) + u_2' \|g_\alpha\|^2. \tag{4.17}$$

Moreover, since $\alpha > \frac{1}{2}$, $g_\alpha * A_{\mathrm{M},\alpha} u$ is continuous on $[0, b]$, according to Theorem B.1. Thus,

$$g_\alpha * A_{\mathrm{M},\alpha} u(b) = (A_{\mathrm{M},\alpha} u, S g_\alpha) = u_1' g_{1+\alpha}(b) + u_2' \|S g_\alpha\|^2.$$

Hence, starting from

$$u = u_1 g_1 + u_2 g_\alpha + g_\alpha * A_{\mathrm{M},\alpha} u,$$

and noticing that the right hand side is continuous on $(0, b]$, we deduce that u admits a continuous representative on $(0, b]$ satisfying the last equation of (4.15). That proves the claim.

Regarding $SA_{\mathrm{M},\alpha}S$, in view of (2.49) and Corollary 2.16, one has

$$E^{\mathrm{s}}_{\mathrm{M},\alpha} := \ker A_{\mathrm{M},\alpha}S \oplus {B_\alpha}^*(\ker A_{\mathrm{M},\alpha}) = \langle g_1, Sg_\alpha, {B_\alpha}^* g_1, {B_\alpha}^* g_\alpha\rangle$$
$$D(SA_{\mathrm{M},\alpha}S) = E^{\mathrm{s}}_{\mathrm{M},\alpha} \oplus {B_\alpha}^*\big((\ker A_{\mathrm{M},\alpha})^\perp\big).$$

Thus f lies in $D(SA_{\mathrm{M},\alpha}S)$ if and only if there exists $f_1, f_2, f_1', f_2' \in \mathbb{C}$ and $\varphi^\perp \in (\ker A_{\mathrm{M},\alpha})^\perp$ such that

$$f = f_1 g_1 + f_2 Sg_\alpha + {B_\alpha}^*\big(f_1' g_1 + f_2' g_\alpha + \varphi^\perp\big). \tag{4.18}$$

Thus

$$Sf = f_1 g_1 + f_2 g_\alpha + B_\alpha\big(f_1' g_1 + f_2' Sg_\alpha + S\varphi^\perp\big). \tag{4.19}$$

Let $P_{E_{\mathrm{M},\alpha}} : D(A_{\mathrm{M},\alpha}) \to D(A_{\mathrm{M},\alpha})$ be the projection on $E_{\mathrm{M},\alpha}$ along $B\big((\ker A_{\mathrm{M}}S)^\perp\big)$. By comparison with (4.14), there results that (f_1, f_2, f_1', f_2') are the coordinates of $P_{E_{\mathrm{M},\alpha}}(Sf)$ in the basis $(g_1, g_\alpha, B_\alpha g_1, B_\alpha S g_\alpha)$ of $E_{\mathrm{M},\alpha}$. Hence f has a continuous representative on $[0, b)$, still labelled f, and (4.15) yields

$$\begin{aligned} f_1 &= f(b) \\ f_2 &= g_{1-\alpha} * \big(Sf - f(b)\big)(0) \\ f_1' b + f_2' g_{1+\alpha}(b) &= g_{1-\alpha} * \big(Sf - f(b)\big)(b) - g_{1-\alpha} * \big(Sf - f(b)\big)(0) \\ f_1' g_{1+\alpha}(b) + f_2' \|g_\alpha\|^2 &= f(0) - f(b) - g_{1-\alpha} * \big(Sf - f(b)\big)(0) g_\alpha(b). \end{aligned} \tag{4.20}$$

For all purposes, let us mention that $f(b) := Sf(0)$ where $Sf(0)$ is defined thru Proposition 4.1.

4.4. **Boundary restriction operators for** $\alpha \in (\frac{1}{2}, 1)$**.**

4.4.1. *Minimal operator.* In view of (4.9) and combining (4.14) and (4.15), this result is obtained easily.

Proposition 4.2. *Let $\alpha \in (\frac{1}{2}, 1)$ and $A_{\mathrm{m},\alpha}$ be the minimal operator of $\mathcal{T}_\alpha$. Then*

$$\begin{aligned} D(A_{\mathrm{m},\alpha}) &= \big\{ g_\alpha * \psi^\perp \mid \psi^\perp \in \langle g_1, Sg_\alpha\rangle^\perp \big\} \\ &= \big\{ u \in D(A_{\mathrm{M},\alpha}) \mid u(0) = u(b) = (g_{1-\alpha} * u)(0) = (g_{1-\alpha} * u)(b) = 0 \big\}. \end{aligned}$$

Remark 4.6. We claim that there is no symmetric boundary restriction operator of D^α. Thus as far as one is concerned with the use of hilbertian operator theory to study boundary value problems, our approach is different from the one of V. Ryzhov, since the latter relies on self-adjoint operators (see [Ryz07, Assumption 2, Section 1].

In order to prove the claim, we proceed by contradiction assuming that A is a symmetric boundary restriction operator of D^α. Then $A_{\mathrm{m},\alpha} \subseteq A$ according to Corollary 2.14; so that $A_{\mathrm{m},\alpha}$ is a symmetric operator. For each $\psi^\perp, \varphi^\perp \in \langle g_1, Sg_\alpha\rangle^\perp$, Proposition 4.2 entails that the functions $g_\alpha * \psi^\perp$ and $g_\alpha * \varphi^\perp$ belong to $D(A_{\mathrm{m},\alpha})$. Using the symmetry of $A_{\mathrm{m},\alpha}$, we infer

$$\big(\varphi^\perp, ({B_\alpha}^* - B_\alpha)\psi^\perp\big) = 0.$$

Hence there exist k_1, $k_2 \in \mathbb{C}$ such that

$$S(g_\alpha * S\psi^\perp) - g_\alpha * \psi^\perp = k_1 g_1 + k_2 S g_\alpha. \tag{4.21}$$

By Theorem B.1(i), the functions

$$g_\alpha * S\psi^\perp,\ g_\alpha * \psi^\perp,\ g_1$$

are continuous at $x = b$, unlike Sg_α. Thus $k_2 = 0$. Then, evaluating (4.21) at $x = b$, we get

$$k_1 = -(S\psi^\perp, g_\alpha) = -(\psi^\perp, Sg_\alpha) = 0,$$

since $\psi^\perp \in \langle g_1, Sg_\alpha\rangle^\perp$ and $g_\alpha * S\psi^\perp$ vanishes at $x = 0$ due to Theorem B.1(i). Thus

$$B_\alpha{}^*\psi^\perp = B_\alpha\psi^\perp, \quad \forall\, \psi^\perp \in \langle g_1, Sg_\alpha\rangle^\perp. \tag{4.22}$$

Now, let $\psi \in C^1[0,b]$ be such that $\psi(0) = \psi(b) = 0$. Clearly, there exist k_1, $k_2 \in \mathbb{C}$ such that

$$\psi^\perp := \psi - k_1 g_1 - k_2 Sg_\alpha \in \langle g_1, Sg_\alpha\rangle^\perp.$$

Then (4.22) yields that

$$S(g_\alpha * S\psi) - g_\alpha * \psi \in \langle g_{1+\alpha}, Sg_{1+\alpha}, Sg_{2\alpha}, g_\alpha * Sg_\alpha\rangle.$$

By Theorem B.1(ii-b), the left hand side lies in $C^1[0,b]$. However, $g_{1+\alpha}$, $Sg_{1+\alpha}$, $Sg_{2\alpha}$ and $g_\alpha * Sg_\alpha$ do not belong to $C^1[0,b]$ since

$$g_\alpha * Sg_\alpha = g_\alpha * \big(Sg_\alpha - Sg_\alpha(0)\big) + Sg_\alpha(0)g_{1+\alpha}$$

and $g_\alpha * \big(Sg_\alpha - Sg_\alpha(0)\big)$ lies in $C^1[0, \frac{b}{2}]$. Thus, by density, $B_\alpha{}^* = B_\alpha$, which is false as we can see by testing with g_1. That proves the claim.

4.4.2. *Caputo's operator.* In view of Definition 4.4, setting

$$E_{\mathrm{C}} := \langle g_1, B_\alpha g_1, B_\alpha Sg_\alpha\rangle,$$

(4.15) yields that

$$D({}^{\mathrm{C}}\mathrm{D}^\alpha) = E_{\mathrm{C}} \oplus B_\alpha\big(S(\ker A_{\mathrm{M},\alpha})^\perp\big).$$

Thus Proposition 2.15 tells us that ${}^{\mathrm{C}}\mathrm{D}^\alpha$ is a closed boundary restriction operator of $A_{\mathrm{M},\alpha}$. With the notations of Subsection 2.3, one has $d_{E_{\mathrm{C}}} = 3$ and

$$\begin{array}{lll} \xi_1 = g_1 & \xi_2 = g_\alpha & \\ e_1 = g_1 & \omega_1 = g_1 & \omega_1' = 0 \\ e_2 = B_\alpha g_1 & \omega_2 = 0 & \omega_2' = g_1 \\ e_3 = B_\alpha S g_\alpha & \omega_3 = 0 & \omega_3' = g_\alpha. \end{array}$$

Then Proposition 2.19 yields that, in view of the notation (4.19),

$$\begin{aligned} f \in D({}^{\mathrm{C}}\mathrm{D}^{\alpha *}) &\iff 0 = (g_1, f_i'\xi_i),\ (g_1, f_i\xi_i) = (g_\alpha, f_i\xi_i) = 0 \\ &\iff (f_1' g_1 + f_2' g_\alpha, g_1) = 0,\ f_1 = f_2 = 0. \end{aligned}$$

Moreover, by the third equation of (4.20),

$$(f_1' g_1 + f_2' g_\alpha, g_1) = g_{1-\alpha} * \big(Sf - f(b)\big)(b) - g_{1-\alpha} * \big(Sf - f(b)\big)(0).$$

Accordingly, we obtain the following characterization of elements of $D({}^{\mathrm{C}}\mathrm{D}^{\alpha *})$ in terms of their boundary values.

$$f \in D({}^{\mathrm{C}}\mathrm{D}^{\alpha *}) \quad \iff \quad f(b) = \big(g_{1-\alpha} * Sf\big)(0) = \big(g_{1-\alpha} * Sf\big)(b) = 0. \tag{4.23}$$

Also, we have

$$\begin{aligned} D({}^{\mathrm{C}}\mathrm{D}^{\alpha *}) &= \big\{ f \in L^2(0,b) \mid g_{1-\alpha} * Sf \in H_0^1(0,b) \big\} \\ &= B_\alpha{}^*(g_1{}^\perp). \end{aligned} \tag{4.24}$$

4.4.3. *Riemann-Liouville operator.* Proceeding in the same way, we obtain that the Riemann-Liouville operator is a closed boundary restriction operator of D^α, and

$$f \in D\big({}^{\mathrm{RL}}\mathrm{D}^{\alpha*}\big) \iff 0 = (g_\alpha, f_1' g_1 + f_2' g_\alpha), \ f_1 = f_2 = 0.$$

Since $(f_1' g_1 + f_2' g_\alpha, g_\alpha) = f_1' g_{1+\alpha}(b) + f_2' \|g_\alpha\|^2$, (4.20) gives

$$f \in D\big({}^{\mathrm{RL}}\mathrm{D}^{\alpha*}\big) \iff f(0) = f(b) = g_{1-\alpha} * Sf(0) = 0.$$

4.4.4. *the operator* $A_{\mathcal{T}_\alpha}$. Let us focus on the *pivot operator* $A_{\mathcal{T}_\alpha}$ defined by (see (2.15))

$$D(A_{\mathcal{T}_\alpha}) := R(B_\alpha), \qquad A_{\mathcal{T}_\alpha} u := A_{\mathrm{M},\alpha} u, \qquad \forall\, u \in D(A_{\mathcal{T}_\alpha}). \tag{4.25}$$

This result is easily proved thanks (4.14)-(4.15) and Proposition 2.9.

Proposition 4.3. *For* $\alpha \in (\frac{1}{2}, 1)$,

$$\begin{aligned} D(A_{\mathcal{T}_\alpha}) &= \big\{u \in L^2(0,b) \mid g_{1-\alpha} * u \in {}_0H^1(0,b)\big\} \\ &= \big\{u \in D(A_{\mathrm{M},\alpha}) \mid u(0) = g_{1-\alpha} * u(0) = 0\big\}. \end{aligned}$$

In a symmetric way, one has

$$D\big((A_{\mathcal{T}_\alpha})^*\big) = \big\{f \in D(SA_{\mathrm{M},\alpha}S) \mid f(b) = (g_{1-\alpha} * Sf)(0) = 0\big\}.$$

4.5. **First order fractional boundary value problems.** For $\alpha \in (\frac{1}{2}, 1)$, let $\mathcal{T}_\alpha := (\mathrm{D}^\alpha, B_\alpha, S)$ be the differential triplet defined by (4.7). There results from the invertibility of the system (4.15) that the boundary values of a function u in $D(\mathrm{D}^\alpha)$ are

$$u(0), \ u_\alpha(0), \ u(b), \ u_\alpha(b),$$

where $u_\alpha := g_{1-\alpha} * \big(u - u(0)\big) \in H^1(0,b)$. For sake of concision, we will set

$$C(u) := \begin{pmatrix} u(0) \\ u_\alpha(0) \\ u(b) \\ u_\alpha(b) \end{pmatrix}.$$

For $h \in L^2(0,b)$, $M \in \mathcal{M}(4,\mathbb{C})$, a 4×4 matrix with coefficients in $\mathbb{C}$, and $h_{\mathrm{b.c}} \in \mathbb{C}^4$ a vector belonging to the range of M, we will consider this boundary value problem

$$\begin{cases} \text{Find } u \in D(\mathrm{D}^\alpha) \ \text{ such that} \\ \mathrm{D}^\alpha u = h, \quad MC(u) = h_{\mathrm{b.c}}. \end{cases} \tag{4.26}$$

The second equation of (4.26) is clearly *non homogeneous linear boundary conditions.* In the sequel, we will discuss existence, uniqueness and well-posedness for Problem (4.26). We consider the cases where M has rank 1 or 2.

(i) The case where M has rank 1.

(i-a) Regarding uniqueness issue, we may assume that $h = 0$ and $h_{\mathrm{b.c}} = 0$. Let A be the restriction of D^α with domain

$$D(A) := \big\{u \in D(\mathrm{D}^\alpha) \mid MC(u) = 0\big\}.$$

Clearly, (4.26) is uniquely solvable iff A is one-to-one. Moreover, it turns out that A is not one-to-one. Indeed, setting

$$E := D(A) \cap E_{\mathrm{M},\alpha} = \big\{u \in E_{\mathrm{M},\alpha} \mid MC(u) = 0\big\},$$

we claim that

$$D(A) = E \oplus B\big((\ker A_{\mathrm{M}} S)^\perp\big).$$

Indeed, since $B\big((\ker A_{\mathrm{M}}S)^{\perp}\big) = D(A_{\mathrm{m},\alpha})$, Proposition 4.2 entails

$$B\big((\ker A_{\mathrm{M}}S)^{\perp}\big) \subseteq D(A),$$

Then the claim may be easily proved by using (4.13). Thus Proposition 2.15 yields that A is a closed boundary restriction operator of D^{α}. Moreover, by (4.14)-(4.15), $\dim E = 3$, hence Proposition 2.18 tells us that A is not one-to-one, so that (4.26) is not uniquely solvable. □

(i-b) Existence issue. Since M has rank 1, we may assume that the boundary condition of (4.26) reads

$$a_1 u(0) + a_2 u_{\alpha}(0) + a_3 u(b) + a_4 u_{\alpha}(b) = h_{\mathrm{b.c}},$$

where $a_i \in \mathbb{C}$ and $h_{\mathrm{b.c}} \in \mathbb{C}$. Using the splitting

$$L^2(0,b) = \ker A_{\mathrm{M}}S \oplus (\ker A_{\mathrm{M}}S)^{\perp},$$

h reads $h = h_1' g_1 + h_2' S g_{\alpha} + \phi$, for some (unique) h_1', $h_2' \in \mathbb{C}$ and $\phi \in (\ker A_{\mathrm{M}}S)^{\perp}$. Thus setting

$$\begin{aligned} c_1(h) &:= h_1' g_{1+\alpha}(b) + h_2' \|g_{\alpha}\|^2 \\ c_2(h) &:= h_1' b + h_2' g_{1+\alpha}(b), \end{aligned}$$

we prove easily with (4.14)-(4.15), that if

$$a_1 + a_3 \neq 0 \qquad \text{or} \qquad a_2 + g_{\alpha}(b) a_3 + a_4 \neq 0$$

then, for each h and $h_{\mathrm{b.c}}$, (4.26) has at least a solution. Otherwise, (4.26) is solvable iff

$$a_3 c_1(h) + a_4 c_2(h) = h_{\mathrm{b.c}}.$$

It could be surprising that (4.26) with one boundary condition may have no solution. However the same thing appears also in the case $\alpha = 1$. Indeed, the problem

$$\frac{\mathrm{d}}{\mathrm{d}x} u = h, \quad u(b) - u(0) = h_{\mathrm{b.c}} \in \mathbb{C},$$

has no solution if $\int_0^b h \,\mathrm{d}x \neq h_{\mathrm{b.c}}$.

(ii) A case where M has rank 2. Assuming, again with a slight abuse of notation, that $h_{\mathrm{b.c}} \in \mathbb{C}^2$, we will show that

$$\mathrm{D}^{\alpha} u = h, \quad \left(\begin{smallmatrix} u(0) \\ u_{\alpha}(0) \end{smallmatrix}\right) = h_{\mathrm{b.c}} \tag{4.27}$$

is well-posed. Let $|\cdot|$ be any norm on $\mathbb{C}^2$.

Proposition 4.4. *For each $h_{\mathrm{b.c}} \in \mathbb{C}^2$ and $h \in L^2(0,b)$, the boundary value problem* (4.27) *admits a unique solution u. Moreover*

$$\|u\|_{D(\mathrm{D}^{\alpha})} \leq C(\|h\| + |h_{\mathrm{b.c}}|),$$

for some constant C independent of u, $h_{\mathrm{b.c}}$ and h. Hence (4.27) *is well-posed.*

Proof. Uniqueness: with the notation of Item (i-a), we have $A = A_{\mathcal{T}_{\alpha}}$ where $A_{\mathcal{T}_{\alpha}}$ is defined by (4.25). Thus (2.16) gives that (4.27) has at most a solution. Regarding existence, we consider the problems of finding functions v and w such that

$$v \in D(A_{\mathcal{T}_{\alpha}}), \quad A_{\mathcal{T}_{\alpha}} v = h, \tag{4.28}$$

$$w \in \ker \mathrm{D}^{\alpha}, \quad \left(\begin{smallmatrix} w(0) \\ w_{\alpha}(0) \end{smallmatrix}\right) = h_{\mathrm{b.c}}. \tag{4.29}$$

By (2.16), one has $v = (\mathbb{A}_{\mathcal{T}})^{-1}h$. Moreover, since $\ker \mathrm{D}^\alpha = \langle g_1, g_\alpha \rangle$, we show easily that (4.29) has a unique solution w, and that

$$\|w\| \le C|h_{\mathrm{b.c}}|. \tag{4.30}$$

Thus $u := v + w$ solves (4.27). Finally, using (4.30) and (2.17), we get the estimate of u in $D(\mathrm{D}^\alpha)$.

□

5. Second order fractional differential operators

Starting from first order fractional differential operators, we will built *second order* fractional differential operators. More precisely, starting from $\mathcal{T}_\alpha := (\mathrm{D}^\alpha, B_\alpha, S)$ given by (4.7), we will study some boundary restriction operators of $\mathcal{T}_\alpha {\mathcal{T}_\alpha}^*$ by using the abstract setting of Section 3. According to (3.1),

$$\mathcal{T}_\alpha {\mathcal{T}_\alpha}^* = \big((\mathrm{D}^\alpha S)^2, {B_\alpha}^* B_\alpha, id_{L^2(0,b)}\big).$$

The domain representation of $(\mathrm{D}^\alpha S)^2$ featured in this proposition is a consequence of the basic splitting (2.5). Let us recall that $g_\alpha *' f := S(g_\alpha * Sf)$.

Proposition 5.1. *For $\alpha \in (\frac{1}{2}, 1)$, each u in $D((\mathrm{D}^\alpha S)^2)$ reads*

$$u = x_1 g_1 + x_2 S g_\alpha + x_3 S g_{1+\alpha} + x_4 g_\alpha *' g_\alpha + g_\alpha *' \big(g_\alpha * (\mathrm{D}^\alpha S)^2 u\big), \tag{5.1}$$

where

$$\begin{cases} x_1 = u(b) \\ x_2 = g_{1-\alpha} * \big(Su - u(b)\big)(0) \\ x_3 = \mathrm{D}^\alpha S u(b) \\ x_4 = g_{1-\alpha} * \big(S\mathrm{D}^\alpha S u - \mathrm{D}^\alpha S u(b)\big)(0). \end{cases} \tag{5.2}$$

Conversely, for all $h \in L^2(0,b)$, and $x_1, \dots, x_4 \in \mathbb{C}$, the function

$$u := x_1 g_1 + x_2 S g_\alpha + x_3 S g_{1+\alpha} + x_4 g_\alpha *' g_\alpha + g_\alpha *' (g_\alpha * h),$$

belongs to $D((\mathrm{D}^\alpha S)^2)$ and satisties $(\mathrm{D}^\alpha S)^2 u = h$.

Proof. Each u in $D((\mathrm{D}^\alpha S)^2)$ satisfies $S\mathrm{D}^\alpha S u \in D(\mathrm{D}^\alpha)$. Thus Proposition 2.1 and (4.14)-(4.15) yield

$$S\mathrm{D}^\alpha S u = x_3 g_1 + x_4 g_\alpha + g_\alpha * (\mathrm{D}^\alpha S)^2 u, \tag{5.3}$$

with x_3, x_4 as in (5.2). In the same way, since $u \in D(S\mathrm{D}^\alpha S)$, we derive from (4.18)-(4.20), that

$$u = x_1 g_1 + x_2 S g_\alpha + g_\alpha *' (S\mathrm{D}^\alpha S u), \tag{5.4}$$

with x_1, x_2 as in (5.2). Using $g_\alpha *' g_1 = S g_{1+\alpha}$, we obtain the wished representation of u. The converse is obvious. □

This proposition allows to describe the domain of the minimal operator $L_{\mathrm{m},\alpha}$ of $\mathcal{T}_\alpha {\mathcal{T}_\alpha}^*$.

Proposition 5.2. *For $\alpha \in (\frac{1}{2}, 1)$,*

$$D(L_{\mathrm{m},\alpha}) = \big\{ g_\alpha *' (g_\alpha * \psi^\perp) \mid \psi^\perp \in \langle g_1, S g_\alpha, S g_{1+\alpha}, g_\alpha *' g_\alpha \rangle^\perp \big\}.$$

Alternatively, $D(L_{\mathrm{m},\alpha})$ is the set of all functions $u \in D((\mathrm{D}^\alpha S)^2)$ satisfying

$$u(0) = u(b) = g_{1-\alpha} * Su(0) = g_{1-\alpha} * Su(b) = 0$$
$$\mathrm{D}^\alpha S u(0) = \mathrm{D}^\alpha S u(b) = g_{1-\alpha} * S\mathrm{D}^\alpha S u(0) = g_{1-\alpha} * S\mathrm{D}^\alpha S u(b) = 0.$$

Since $\ker(\mathrm{D}^\alpha S)^2$ has dimension four, a function u in $(\mathrm{D}^\alpha S)^2$ has eight boundary conditions. They are featured in Proposition 5.2.

Proof of Proposition 5.2. By Proposition 3.1 (ii),

$$\ker(\mathrm{D}^\alpha S)^2 = \langle g_1, S g_\alpha, S g_{1+\alpha}, g_\alpha *' g_\alpha \rangle. \tag{5.5}$$

Thus the first representation follows from Definition 2.6 of minimal operators. The second one is a straightforward consequence of Proposition 3.1 (vi) and Proposition 4.2. □

Let us now investigate *regularity issues*. Among other things, we will show that the more regular functions of $D((\mathrm{D}^\alpha S)^2)$ are contained in the domain of the restriction L_1 of $(\mathrm{D}^\alpha S)^2$, defined by

$$\begin{aligned} D(L_1) := \big\{ & u \in D\big((\mathrm{D}^\alpha S)^2\big) \mid \\ & g_{1-\alpha} * \big(Su - u(b)\big)(0) = g_{1-\alpha} * \big(S\mathrm{D}^\alpha S u - \mathrm{D}^\alpha S u(b)\big)(0) = \mathrm{D}^\alpha S u(0) = 0 \big\}. \end{aligned} \tag{5.6}$$

In other words, with the coordinates of (5.2), $D(L_1)$ is the set of all functions $u \in D((\mathrm{D}^\alpha S)^2)$ such that

$$x_2 = x_4 = \mathrm{D}^\alpha S u(0) = 0. \tag{5.7}$$

This result gives a representation of elements of $D(L_1)$.

Proposition 5.3. *For $\alpha \in (\frac{1}{2}, 1)$, each u in $D(L_1)$ reads*

$$u = x_1 g_1 + x_3 S g_{1+\alpha} + g_\alpha *' (g_\alpha * L_1 u),$$

where $x_1 \in \mathbb{C}$ and $x_3 := -(L_1 u, S g_\alpha)$. Conversely, for all $h \in L^2(0,b)$ and $x_1 \in \mathbb{C}$, the function

$$u := x_1 g_1 - (h, S g_\alpha) S g_{1+\alpha} + g_\alpha *' (g_\alpha * h),$$

belongs to $D(L_1)$, and $L_1 u = h$.

Proof. Let u be in $D(L_1)$. According to (5.7), one has $x_2 = x_4 = 0$. Thus, evaluating (5.3) at $x = b$, we end up with

$$0 = \mathrm{D}^\alpha S u(0) = x_3 + (g_\alpha * L_1 u)(b).$$

Since $(g_\alpha * L_1 u)(b) = (L_1 u, S g_\alpha)$, the representation of elements of $D(L_1)$ follows. The converse is obvious. □

The domain of L_1 is designed so that Theorem B.1 is fully applicable, due to cancellation of singular terms. That gives the following regularity result.

Proposition 5.4. *For $\alpha \in (\frac{1}{2}, 1)$, let $D(L_1)$ be defined by (5.6).*

(i) *If $\alpha < \frac{3}{4}$ then $D(L_1) \subset C^{0, 2\alpha - \frac{1}{2}}[0,b]$, with continuous injection.*
(ii) *If $\frac{3}{4} < \alpha$ then $D(L_1) \subset C^{1, 2\alpha - \frac{3}{2}}[0,b]$, with continuous injection.*

Proof. Let $u \in D(L_1)$. Recalling that $x_2 = x_4 = 0$, (5.4) reads

$$S u = x_1 g_1 + g_\alpha * \mathrm{D}^\alpha S u. \tag{5.8}$$

On the other hand, by (5.3),

$$S \mathrm{D}^\alpha S u = x_3 g_1 + g_\alpha * (\mathrm{D}^\alpha S)^2 u. \tag{5.9}$$

Let us estimate x_1 and x_3. Taking $\mathbb{A} := (\mathrm{D}^\alpha S)^2$ in Proposition 2.1, (2.5) reads

$$D\big((\mathrm{D}^\alpha S)^2\big) = \ker(\mathrm{D}^\alpha S)^2 \oplus R\big({B_\alpha}^* B_\alpha\big) \qquad \text{in } \ D\big((\mathrm{D}^\alpha S)^2\big).$$

Thus the projection onto $\ker(\mathrm{D}^\alpha S)^2$ along $R\big(B_\alpha{}^* B_\alpha^{\mathbb{B}}\big)$ is a well defined element of $\mathcal{L}\big(D\big((\mathrm{D}^\alpha S)^2\big)\big)$. Thus in view of (5.1) and (5.5), we derive

$$\|x_1 g_1 + x_3 S g_{1+\alpha}\| \leq C\|u\|_{D(L_1)}.$$

Thus, since g_1 and $Sg_{1+\alpha}$ are linearly independent in $L^2(0,b)$, we get

$$|x_1| + |x_3| \leq C'\|u\|_{D(L_1)}. \tag{5.10}$$

With Theorem B.1 (i) and (5.10), we derive from (5.9) that $\mathrm{D}^\alpha Su$ lies in $C^{0,\alpha-\frac{1}{2}}[0,b]$, and

$$\|\mathrm{D}^\alpha Su\|_{0,\alpha-\frac{1}{2}} \leq C\|u\|_{D(L_1)}.$$

Thus, using also $\mathrm{D}^\alpha Su(0) = 0$, Theorem B.1 (ii-a) yields that, if $\alpha < \frac{3}{4}$,

$$\|g_\alpha * \mathrm{D}^\alpha Su\|_{0,2\alpha-\frac{1}{2}} \leq C\|u\|_{D(L_1)}.$$

Going back to (5.8), and using (5.10), we get

$$\|u\|_{0,2\alpha-\frac{1}{2}} \leq C\|u\|_{D(L_1)}.$$

In the case where $\alpha > \frac{3}{4}$, the same computation gives the assertion (ii) in the statement of the current proposition. □

More generally, let us discuss the regularity of any function u in $D((\mathrm{D}^\alpha S)^2)$ in terms of the coordinates $x_1, \dots, x_4$ appearing in (5.2). For that, Theorem B.1 will be often used, although not always cited. First of all, we claim that

$$u \in C^{0,\beta}[0,b), \quad \forall\, \beta < 2\alpha - 1. \tag{5.11}$$

Indeed, starting from (5.1) and setting $f := S\big(g_\alpha * (\mathrm{D}^\alpha S)^2 u\big)$, we notice that the function

$$S\big(g_\alpha *' \big(g_\alpha * (\mathrm{D}^\alpha S)^2 u\big)\big) = g_\alpha * f = g_\alpha * \big(f - f(0)\big) + f(0) g_{1+\alpha}$$

lies in $C^{0,\alpha}[0,b] \subset C^{0,2\alpha-1}[0,b]$. Moreover, g_α belongs to $L^p(0,b)$ for any $p < \frac{1}{1-\alpha}$. Thus Theorem B.1 (i) yields that

$$g_\alpha *' g_\alpha \in C^{0,\beta}[0,b], \quad \forall\, \beta < 2\alpha - 1.$$

Then the claim follows by returning to (5.1).

(i) If $x_2 := g_{1-\alpha} * \big(Su - u(b)\big)(0) \neq 0$ then u has a singularity at $x = b$. Let us notice that even if $|u(x)| \to \infty$ when $x \to b$, $u(b)$ is a well defined finite number. Indeed, by definition, it is equal to $Su(0)$, the latter being given by Proposition 4.1, owing to the fact $Su \in D(\mathrm{D}^\alpha)$. For instance, the function $v := x_1 g_1 + S g_\alpha$ lies in $D((\mathrm{D}^\alpha S)^2)$ with $Sv(0) = x_1$, and has a singularity at $x = b$.

(ii) If $x_2 = 0$ then we deduce from the proof of (5.11) that

$$u \in C^{0,\beta}[0,b], \quad \forall\, \beta < 2\alpha - 1. \tag{5.12}$$

(ii-a) if $x_4 := g_{1-\alpha} * \big(S\mathrm{D}^\alpha Su - \mathrm{D}^\alpha Su(b)\big)(0) \neq 0$ then (5.3) entails that $\mathrm{D}^\alpha Su$ has a singularity at $x = b$. That result implies that u cannot be more than C^α-regular, that is to say $u \notin C^{0,\beta}[0,b]$, for all $\beta \in (\alpha, 1]$. Indeed, otherwise $g_{1-\alpha} * \big(Su - Su(0)\big)$ would belong to $C^{1,\beta-\alpha}[0,b]$, by Theorem B.1 (ii-b). Whence $\mathrm{D}^\alpha Su \in C^{0,\beta-\alpha}[0,b]$, which is impossible since $\mathrm{D}^\alpha Su$ is not continuous at $x = b$;

(ii-b) if $x_4 = 0$ then (5.3) yields that
$$\mathrm{D}^\alpha Su \in C^{0,\alpha-\frac{1}{2}}[0,b].$$
Accordingly,
$$g_\alpha * \big(\mathrm{D}^\alpha Su - \mathrm{D}^\alpha Su(0)\big)u \in \begin{cases} C^{0,2\alpha-\frac{1}{2}}[0,b] & \text{if } \alpha < \frac{3}{4} \\ C^{1,2\alpha-\frac{3}{2}}[0,b] & \text{if } \frac{3}{4} < \alpha \end{cases}.$$
Hence, for each $\alpha \in (\frac{1}{2}, 1)$, there exists $\varepsilon \in (0, 1-\alpha)$ such that
$$g_\alpha * \big(\mathrm{D}^\alpha Su - \mathrm{D}^\alpha Su(0)\big)u \in C^{0,\alpha+\varepsilon}[0,b]; \tag{5.13}$$
(ii-b-1) if $\mathrm{D}^\alpha Su(0) \neq 0$ then (5.4) entails
$$Su = x_1 g_1 + \mathrm{D}^\alpha Su(0) g_{1+\alpha} + g_\alpha * \big(\mathrm{D}^\alpha Su - \mathrm{D}^\alpha Su(0)\big).$$
Thus $u \in C^{0,\alpha}[0,b]$, but owing to (5.13), $u \notin C^{0,\beta}[0,b]$, for all $\beta \in (\alpha, 1]$;

(ii-b-2) if $\mathrm{D}^\alpha Su(0) = 0$ then u lies in $D(L_1)$, hence its regularity is given by Proposition 5.4.

5.1. **One-dimensional fractional Laplace operator with Dirichlet boundary condition.** Roughly speaking, instead of constructing differential operators starting from the basic operator $\frac{\mathrm{d}}{\mathrm{d}x}$, we construct differential operators starting from D^α. More precisely, instead of building differential operators in the framework of the differential triplet $\mathcal{T}_1 := (\frac{\mathrm{d}}{\mathrm{d}x}, B_1, S)$, we will use $\mathcal{T}_\alpha := (A_{\mathrm{M},\alpha}, B_\alpha, S)$, by keeping the same process.

Let $\alpha \in [\frac{1}{2}, 1)$. In view of (3.4), the *opposite of the one-dimensional fractional Laplace operator with Dirichlet boundary condition* is
$$L_{\mathrm{op},D,\alpha} := \mathrm{D}^\alpha \circ (\mathrm{D}^\alpha)^*, \tag{5.14}$$
where $\mathrm{D}^\alpha := A_{\mathrm{M},\alpha}$ is given in Definition 4.2. Of course, *the one-dimensional fractional Laplace operator with Dirichlet boundary condition* is $-\mathrm{D}^\alpha \circ (\mathrm{D}^\alpha)^*$.

Proposition 3.3 entails that $L_{\mathrm{op},D,\alpha}$ is a positive self-adjoint operator on $L^2(0,b)$. By Proposition 2.12,
$$L_{\mathrm{op},D,\alpha} = \mathrm{D}^\alpha \circ S A_{\mathrm{m},\alpha} S, \tag{5.15}$$
where, according to (4.9), $A_{\mathrm{m},\alpha}$ is the minimal operator of $\mathcal{T}_\alpha := (A_{\mathrm{M},\alpha}, B_\alpha, S)$.

This proposition gives a description of the domain of $L_{\mathrm{op},D,\alpha}$ and an inversion formula for that operator, namely (5.17)-(5.18).

Proposition 5.5. *For $\alpha \in (\frac{1}{2}, 1)$, each u in $D(L_{\mathrm{op},D,\alpha})$ reads*
$$u = x_3 S g_{1+\alpha} + x_4 g_\alpha *' g_\alpha + g_\alpha *' (g_\alpha * L_{\mathrm{op},D,\alpha} u),$$
where $(x_3, x_4) \in \mathbb{C}^2$ is the unique solution to this linear system
$$\begin{cases} x_3 b + x_4 g_{1+\alpha}(b) + (L_{\mathrm{op},D,\alpha} u, S g_{1+\alpha}) = 0 \\ x_3 g_{1+\alpha}(b) + x_4 \|g_\alpha\|^2 + (L_{\mathrm{op},D,\alpha} u, g_\alpha *' g_\alpha) = 0. \end{cases} \tag{5.16}$$
Conversely, for all $h \in L^2(0,b)$ and $(x_3, x_4) \in \mathbb{C}^2$ satisfying
$$\begin{cases} x_3 b + x_4 g_{1+\alpha}(b) + (h, S g_{1+\alpha}) = 0 \\ x_3 g_{1+\alpha}(b) + x_4 \|g_\alpha\|^2 + (h, g_\alpha *' g_\alpha) = 0, \end{cases} \tag{5.17}$$
the function
$$u := x_3 S g_{1+\alpha} + x_4 g_\alpha *' g_\alpha + g_\alpha *' (g_\alpha * h), \tag{5.18}$$
belongs to $D(L_{\mathrm{op},D,\alpha})$, and satisfies $L_{\mathrm{op},D,\alpha} u = h$.

Proof. Let u be in $D(L_{\text{op},D,\alpha})$. By (5.15), Su lies in $D(A_{\text{m},\alpha})$, so that Proposition 4.2 entails $u(b) = (g_{1-\alpha} * Su)(0) = 0$. Then Proposition 5.1 yields that

$$Su = g_\alpha * \big(x_3 g_1 + x_4 S g_\alpha + S(g_\alpha * L_{\text{op},D,\alpha} u)\big).$$

Since, by Proposition 4.2 again,

$$D(A_{\text{m},\alpha}) = \big\{ g_\alpha * \psi^\perp \mid \psi^\perp \in \langle g_1, S g_\alpha \rangle^\perp \big\},$$

we derive (5.16). The converse is obvious. □

For all $h \in L^2(0,b)$, let us consider the following *fractional Laplace equation with homogeneous Dirichlet boundary condition*:

$$L_{\text{op},D,\alpha} u = h, \quad u \in D(L_{\text{op},D,\alpha}). \tag{5.19}$$

This theorem gives well-posedness and regularity results for Problem (5.19). For all purposes, let us recall that the function g_β is defined by (4.2), the symmetry S by (4.6), and that $g_\alpha *' g_\alpha := S\big(g_\alpha * (S g_\alpha)\big)$.

Theorem 5.6. *Let $\alpha \in (\frac{1}{2}, 1)$ and $h \in L^2(0,b)$. Then the following assertions hold.*

(i) *Problem (5.19) has a unique solution u. Moreover, $\|u\| \le C\|h\|$, where C is independent of u and h.*

Let E_2 be the two-dimensional subspace of $L^2(0,b)$ generated by $S g_{\alpha+1} - b S g_\alpha$ and $g_\alpha *' g_\alpha - g_{\alpha+1}(b) S g_\alpha$.

(ii) *If $h \in {E_2}^\perp$ then*

$$u \in \begin{cases} C^{0,2\alpha-\frac{1}{2}}[0,b] & \text{if } \alpha < \frac{3}{4} \\ C^{1,2\alpha-\frac{3}{2}}[0,b] & \text{if } \frac{3}{4} < \alpha \end{cases}. \tag{5.20}$$

(iii) *If $h \notin {E_2}^\perp$ then*

$$u \in C^{0,\beta}[0,b], \quad \forall\, \beta < 2\alpha - 1.$$

Moreover, for all $\beta \in (\alpha, 1]$, $u \notin C^{0,\beta}[0,b]$. The regularity of u is analyzed after the proof of Proposition 5.4.

Let us notice that there is a regularity gap between cases (ii) and (iii). Indeed, if $h \in {E_2}^\perp$ then there exists $\beta \in (\alpha, 1]$ such that $u \in C^{0,\beta}[0,b]$.

In the standard $\alpha = 1$, (5.19) reads

$$-u'' = h, \quad u \in H^2(0,b) \cap H_0^1(0,b).$$

It is well know that the solution u of that problem lies in $C^{1,\frac{1}{2}}[0,b]$, for all $h \in L^2(0,b)$. That corresponds to the limit case $\alpha \to 1$ in (5.20). However, E_2 reduces to $\langle g_2 \rangle$. Thus E_2 does not induce any dichotomy in that case.

Proof of Theorem 5.6. (i) By Proposition 5.5, (5.19) is uniquely solvable. Since $L_{\text{op},D,\alpha}$ is self-adjoint, it is a closed operator. Whence it is a Banach isomorphism between its domain and $L^2(0,b)$. Hence the estimate follows.

(ii) According to Proposition 5.3, the function

$$u := -(h, S g_\alpha) S g_{1+\alpha} + g_\alpha *' (g_\alpha * h) \tag{5.21}$$

lies in $D(L_1)$ and satisfies $L_1 u = h$. For each $h \in {E_2}^\perp$, we readily check that $x_3 := -(h, S g_\alpha)$, $x_4 := 0$ satisfies (5.17). Thus u is the solution to (5.19). Therefore (5.20) follows from Proposition 5.4.

(iii)Let us assume that $h \notin {E_2}^{\perp}$. By the above regularity analysis (in particular (5.12) and (5.13)), it remains to show that the solution u to (5.19) does not belong to $D(L_1)$. In view of Proposition 5.5, u reads

$$u = x_3 S g_{1+\alpha} + x_4 g_\alpha *' g_\alpha + g_\alpha *' (g_\alpha * h).$$

If $x_4 \neq 0$ then (5.7) entails that $u \notin D(L_1)$. If $x_4 = 0$ then we claim that $\mathrm{D}^\alpha S u(0) \neq 0$. Indeed, one has

$$\mathrm{D}^\alpha S u(0) = x_3 + (g_\alpha * h)(b) = x_3 + (h, S g_\alpha).$$

If $\mathrm{D}^\alpha S u(0) = 0$ then, since $u \in D(L_{\mathrm{op},D,\alpha})$, Proposition 5.5 entails that $x_3 = -(h, S g_\alpha)$ and $x_4 = 0$ solve (5.16). Hence $h \in {E_2}^{\perp}$. That proves the claim. There results from (5.7) that $u \notin D(L_1)$. □

Appendix A. Direct sums decomposition

Definition A.1. Let X be a normed vector space, and E, F be subspaces of X. If $E \cap F = \{0\}$ and E, F are closed in X, then E and F are said to be *in direct sum in* X, and the set

$$E \oplus F := \{e + f \mid e \in E,\ f \in F\}$$

is called the *direct sum of* E *and* F *in* X.

Proposition A.1. *Let A be an operator on $\mathcal{H}$, $B \in \mathcal{L}(\mathcal{H})$, and $\mathcal{E}$ be a closed subspace of $\mathcal{H}$. Assume in addition that*

(i) $D(A) = \ker A \oplus B(\mathcal{E})$ *in* $D(A)$*;*
(ii) $AB\psi = \psi$ *for all* $\psi \in \mathcal{E}$.

Then the following assertions are equivalent.

(a) $\ker A$ *is closed in* $\mathcal{H}$.
(b) A *is a closed operator on* $\mathcal{H}$.

Proof. It is obvious that (b) implies (a). Conversely, let us consider $(u_n)_{n\in\mathbb{N}} \subset D(A)$, and u, f in $\mathcal{H}$ such that

$$u_n \xrightarrow{\mathcal{H}} u, \quad Au_n \xrightarrow{\mathcal{H}} f.$$

By Assumption (i), for each $n \in \mathbb{N}$, there exists $h_n \in \ker A$ and $\psi_n \in \mathcal{E}$ such that

$$u_n = h_n + B\psi_n.$$

With (ii), we get $Au_n = \psi_n \to f$. Thus $f \in \mathcal{E}$ since $\mathcal{E}$ is assumed to be closed in $\mathcal{H}$. Moreover, since B is continuous,

$$h_n = u_n - B\psi_n \to u - Bf.$$

Then (a) yields that $u - Bf$ lies in $\ker A$. Thus recalling that $f \in \mathcal{E}$, we derive

$$u = (u - Bf) + Bf \in \ker A \oplus B(\mathcal{E}).$$

Whence, $u \in D(A)$, and $Au = f$ (thanks (ii)), so that A is a closed. □

The (direct) sum of two closed subspaces is not always closed. The following result gives a sufficent condition for a direct sum to be closed. For all purposes, let us recall that $D(\mathbb{A})$ is always endowed with the graph-norm, and accordingly, a *closed* subspace of $D(\mathbb{A})$ is meant to be closed w.r.t. the norm of $D(\mathbb{A})$.

Proposition A.2. *Let $\mathbb{A}$ be a closed operator on $\mathcal{H}$ and $\mathbb{B} \in \mathcal{L}(\mathcal{H})$ be such that $R(\mathbb{B}) \subseteq D(\mathbb{A})$ and $\mathbb{A} \circ \mathbb{B} = id_{\mathcal{H}}$. Let also F be a closed subspace of $\mathcal{H}$, and E be a closed subspace of $D(\mathbb{A})$ satisfying*

$$E \subseteq \ker \mathbb{A} \oplus \mathbb{B}(F). \tag{A.1}$$

Then $E \oplus \mathbb{B}(F^{\perp})$ is closed in $D(\mathbb{A})$. In particular, $\ker \mathbb{A} \oplus \mathbb{B}(F^{\perp})$ is closed in $D(\mathbb{A})$.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset E \oplus \mathbb{B}(F^\perp)$ and $u \in D(\mathbb{A})$ be such that

$$u_n \longrightarrow u \qquad \text{in } \; D(\mathbb{A}). \tag{A.2}$$

In order to show that u lies in $E \oplus \mathbb{B}(F^\perp)$, we write, for all $n \geq 0$,

$$u_n = e_n + \mathbb{B}\psi_n^\perp \tag{A.3}$$

$$= h_n + \mathbb{B}\psi_n + \mathbb{B}\psi_n^\perp, \tag{A.4}$$

where $e_n = h_n + \mathbb{B}\psi_n \in E$, $h_n \in \ker \mathbb{A}$, $\psi_n \in F$ and $\psi_n^\perp \in F^\perp$.

By (A.2),

$$\mathbb{A}u_n = \psi_n + \psi_n^\perp \longrightarrow \mathbb{A}u.$$

Thus the continuity of the orthogonal projections on F and $F^\perp$ yields that

$$\psi_n \longrightarrow \psi_\infty, \quad \psi_n^\perp \longrightarrow \psi_\infty^\perp,$$

for some $\psi_\infty \in F$, $\psi_\infty^\perp \in F^\perp$. Now going back to (A.4) and using the continuity of $\mathbb{B}$, we arrive at

$$h_n \longrightarrow u - \mathbb{B}\psi_\infty - \mathbb{B}\psi_\infty^\perp.$$

Since $\mathbb{A}$ is closed, its kernel is closed in $\mathcal{H}$, so that

$$h_\infty := u - \mathbb{B}\psi_\infty - \mathbb{B}\psi_\infty^\perp$$

lies in $\ker \mathbb{A}$. Regarding e_n which is equal to $h_n + \mathbb{B}\psi_n$, setting $e_\infty := h_\infty + \mathbb{B}\psi_\infty$, one has

$$e_n \longrightarrow e_\infty, \quad \mathbb{A}e_n = \psi_n \longrightarrow \psi_\infty.$$

Since $\mathbb{A}$ is a closed operator, we get $\mathbb{A}e_\infty = \psi_\infty$, so that (e_n) converges toward e_∞ in $D(\mathbb{A})$. By assumption, E is closed in $D(\mathbb{A})$, hence e_∞ belongs to E. Finally, letting $n \to \infty$ in (A.3), we end up with

$$u = e_\infty + \mathbb{B}\psi_\infty^\perp \in E \oplus \mathbb{B}(F^\perp).$$

Hence $E \oplus \mathbb{B}(F^\perp)$ is closed in $D(\mathbb{A})$.

As a particular case, the choice of $E = \ker \mathbb{A}$ yields that $\ker \mathbb{A} \oplus \mathbb{B}(F^\perp)$ is closed in $D(\mathbb{A})$. □

Appendix B. Regularity results for fractional integration

Let us denote by $C[0,b]$ the space of all complex continuous functions defined on $[0,b]$ equipped with the norm $\|u\|_{0,0} := \sup_{x\in[0,b]} |u(x)|$. For each $\beta \in (0,1]$, we denote by $C^{0,\beta}[0,b]$ the space of functions $u \in C[0,b]$ such that

$$|u|_{0,\beta} := \sup_{\substack{x,y\in[0,b] \\ x\neq y}} \frac{|u(x)-u(y)|^\beta}{|x-y|} < \infty.$$

We equip that space with the norm

$$\|u\|_{0,\beta} := \sup_{x\in[0,b]} |u(x)| + |u|_{0,\beta}.$$

In other words, if $\beta < 1$ then $C^{0,\beta}[0,b]$ is the space of Hölder continuous functions on $[0,b]$. If $\beta = 1$ then $C^{0,1}[0,b]$ is the space of Lipschitz continuous functions on $[0,b]$. We denote by ${}_0C^{0,\beta}(0,b)$ the subspace of $C^{0,\beta}[0,b]$ whose elements vanish at $x = 0$. Besides, we set $C^{0,0}[0,b] := C[0,b]$.

For each $\beta \in [0,1]$, $C^{1,\beta}[0,b]$ is the space of all complex functions $u \in C^1[0,b]$ such that the derivative u' of u belongs to $C^{0,\beta}[0,b]$. The space $C^{1,\beta}[0,b]$ is endowed with the norm

$$\|u\|_{1,\beta} := \sup_{x\in[0,b]} |u(x)| + \|u'\|_{0,\beta}.$$

Based on the seminal work of Hardy and Littlewood ([HL28]), we may state this result.

Theorem B.1. (i) *If* $\beta \in (0,1]$, $p \in (\frac{1}{\beta}, \infty]$, *and* $f \in L^p(0,b)$ *then* $g_\beta * f$ *belongs to* ${}_0C^{0,\beta-\frac{1}{p}}[0,b]$. *Moreover, the linear map*

$$L^p(0,b) \to {}_0C^{0,\beta-\frac{1}{p}}[0,b], \quad f \mapsto g_\beta * f$$

is continuous on $L^p(0,b)$.

(ii) *Let* $\alpha \in (0,1)$, $\beta \in [0,1]$, *and* $f \in {}_0C^{0,\beta}[0,b]$.

(ii-a) *If* $\alpha + \beta < 1$ *then* $g_\alpha * f$ *lies in* $C^{0,\alpha+\beta}[0,b]$ *and the map*

$${}_0C^{0,\beta}[0,b] \to C^{0,\alpha+\beta}[0,b], \quad f \mapsto g_\alpha * f$$

is continuous on ${}_0C^{0,\beta}[0,b]$.

(ii-b) *If* $1 < \alpha + \beta$ *then* $g_\alpha * f$ *lies in* $C^{1,\alpha+\beta-1}[0,b]$ *and the map*

$${}_0C^{0,\beta}[0,b] \to C^{1,\alpha+\beta-1}[0,b], \quad f \mapsto g_\alpha * f$$

is continuous on ${}_0C^{0,\beta}[0,b]$.

(iii) *Limit case* $\alpha = \frac{1}{2}$. *If* $p \in [1,\infty)$ *and* $f \in L^2(0,b)$ *then* $g_{1/2} * f$ *belongs to* $L^p(0,b)$. *Moreover, the map*

$$L^2(0,b) \to L^p(0,b), \quad f \mapsto g_{1/2} * f$$

is continuous on $L^2(0,b)$.

Proof. Item (i) is proved in [SKM93, Theorem 3.6 p67]; see also [HL28, Theorem 12]. Item (ii) follows from [SKM93, Corollary 1 of Theorem 3.1, p56]. Item (iii) follows from Young inequality in the case $1 \le p \le 2$. If $p > 2$ then a simple modification of the proof of [SKM93, Theorem 3.5 p66] yields the result. Indeed, just take $q = \infty$ in that proof. □

Appendix C. Miscellaneous results

Let $\mathcal{H}$ be a complex Hilbert space. By using the notations featured at the end of the introduction, we start with this deep result of S. Holland.

Theorem C.1. *(*[Hol68, Theorem 1]*) Let* A_1, A_2 *be closed densely defined operators on* $\mathcal{H}$. *If* $R(A_1)$ *is a closed subspace of finite codimension in* $\mathcal{H}$ *then* A_2A_1 *is densely defined, and especially,* $(A_2A_1)^* = A_1^*A_2^*$.

Proposition C.2. *Let* $\mathcal{E} \subseteq \mathcal{H}$ *and* $\mathbb{S} \in \mathcal{L}(\mathcal{H})$ *be such that* $\mathbb{S}^* = \mathbb{S}$ *and* $\mathbb{S}^2 = id_{\mathcal{H}}$. *Then* $\mathbb{S}(\mathcal{E}^\perp) = (\mathbb{S}\mathcal{E})^\perp$.

Proof. Let $f \in \mathcal{H}$. Then

$$\begin{aligned} f \in \mathbb{S}(\mathcal{E}^\perp) &\iff \mathbb{S}f \in \mathcal{E}^\perp && \text{(since } \mathbb{S}^2 = id_{\mathcal{H}}) \\ &\iff (\mathbb{S}f, h) = 0, \quad \forall\, h \in \mathcal{E} \\ &\iff (f, \mathbb{S}h) = 0, \quad \forall\, h \in \mathcal{E} && \text{(since } \mathbb{S}^* = \mathbb{S}) \\ &\iff f \in (\mathbb{S}\mathcal{E})^\perp. \end{aligned}$$

□

Proposition C.3. *Let* A *be an operator on* $\mathcal{H}$ *and* $B \in \mathcal{L}(\mathcal{H})$ *be such that* $R(B) \subseteq D(A)$ *and* $A \circ B = id_{\mathcal{H}}$. *Let also* $\lambda \in \mathbb{C}$. *If* $|\lambda| < \frac{1}{\|B\|}$ *or*

$$\|B^k\|^{\frac{1}{k}} \xrightarrow[k\to\infty]{} 0 \tag{C.1}$$

then the following assertions are equivalent.

(i) $u \in D(A)$, $Au = \lambda u$.
(ii) *There exists* $u_0 \in \ker A$ *such that* $u = \sum_{k\geq 0} \lambda^k B^k u_0$.

Proof. Let us assume (C.1), and in a first step, Item (i). Then Proposition 2.1 yields the existence of some $u_0 \in \ker A$ such that $u = u_0 + \lambda Bu$. Using the latter relation, we show by induction that

$$u = \sum_{k=0}^{n} \lambda^k B^k u_0 + \lambda^{n+1} B^{n+1} u, \quad \forall\, n \in \mathbb{N}.$$

Then assumption (C.1) allows us to pass to the limit w.r.t. n, so that Item (ii) holds true.

Conversely, assuming (ii) and using (C.1), we derive that the series

$$u := \sum_{k\geq 0} \lambda^k B^k u_0$$

converges in H and, by continuity of B, that $u = u_0 + B(\lambda u)$. Hence the assumption $A \circ B = id_{\mathcal{H}}$ yields Item (i).

The proof under assumption $|\lambda| < \frac{1}{\|B\|}$ is similar and even simpler to the one under (C.1). So we will skip it. □

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