Almost periodic solutions in distribution to affine stochastic differential equations driven by a fractional Brownian motion with Hurst parameter H > 1/2

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Almost periodic solutions in distribution to affine stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$

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Abstract

The few works devoted to the existence of almost periodicity (or almost automorphy) solutions to stochastic differential equations driven by a fractional Brownian motion impose the condition: the coefficient of fractional stochastic part is deterministic. In this work, without this condition, by using the chaos decomposition approach and the representation of the fractional Brownian motion in terms of a standard Brownian motion, we obtain the existence and uniqueness of almost periodic solution in distribution to affine stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

Keywords: Fractional Brownian motion, multiple stochastic integral, almost periodic solution, chaos decomposition.

1 Introduction

Since the introduction of the theory of almost periodic functions by Harald Bohr [9, 10] in the 1920s, it has aroused great interest. In recent years, it has become one of the most attractive topics in the qualitative theory of differential equations because of their significance and applications in physics, mathematical biology, telecommunication net work, finance, control theory, and others related fields. For more details on this theory, see for instance [2, 4, 15, 24].

The extension of almost periodicity theory given by Bochner [7] for functions with values in Polish spaces and especially in Banach spaces, allowed to generalize this theory for a wider class of functions, in particular for the class of random functions (stochastic processes): almost periodicity in p-mean, in probability, in distribution (one-distribution, finite distribution and infinite distribution), almost periodicity of moments, ... Some characterizations of almost periodic functions with values in a Polish space, especially in probability measures space, are given in [3, 28, 33].

It is well known that natural application of the concept of almost periodic random functions is the study of stochastic differential equations, with almost periodic coefficients, driven by different noises (standard Brownian motion, fractional Brownian motion, Lévy, etc.). During the last 30 years, the class of stochastic differential equations driven by standard Brownian motion, $W = \{W_t, t \in \mathbb{R}\}$ is the most studied. The existence and uniqueness of almost periodic solution is carried out by several authors, we quote among others: L. Arnold, C. Tudor, T. Morozan and G. Da Prato (see e.g. [1, 16, 28]). All these authors deal with almost periodicity in distribution. O. Mellah and

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P. Raynaud De Fitte [27] show, by counterexamples, that the almost periodicity in p-mean is a strong property for solutions of stochastic differential equations, which means the nonexistence of almost periodic solution in p-mean in general. However in the class of stochastic differential equations driven by fractional Brownian motion (fBm for short), until now, to our knowledge, there are only few works dedicated to the study of existence and uniqueness of almost periodic solutions. We can cite for example, P. Bezandry [5] has attempted to show the existence and uniqueness of almost periodic solutions. Recently, F. Chen and X. Yang [12] show the existence and uniqueness of almost automorphic (which is a generalization of almost periodicity) solutions in distribution, F. Chen and X. Zhang [13] obtain the same result for another class of differential equations: mean-field stochastic differential equations driven by fBm. Note that in all the works cited above, the authors assume that the coefficient of fractional stochasctic part is deterministic. In this work, without this condition, we will show the existence and uniqueness of almost periodic solution to affine stochastic differential equations driven by fBm.

A fBm, with Hurst parameter $H \in (0,1)$, introduced by Kolmogorov [23] and studied by Mandelbrot and Van Ness [26], is a centered Gaussian process $B^H = \{B^H(t), t \in \mathbb{R}\}$ which has stationary increments and with the covariance function

$$\mathbf{E}(B^H(t)B^H(s)) = \mathrm{R}_H(t,s) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \tag{1}$$

notice that if $H = \frac{1}{2}$, we recover the standard Brownian motion. For $H \neq \frac{1}{2}$ the increments of the fBm are no longer independent (i.e. they are positively correlated if $H > \frac{1}{2}$, and negatively correlated if $H < \frac{1}{2}$). Furthermore, when $H > \frac{1}{2}$, the fBm exhibits long-range dependence: at large time lags, the dependence is so strong, which results the divergence of the series $\sum_{k=1}^{\infty} R_H(1,k)$. Also the fBm is neither a Markov process nor a semimartingale, as result one cannot use the usual Itô stochastic calculus, and alternate methods are required in order to define stochastic integrals with respect to fBm. For this, several different ways of a defining stochastic integral with respect to fBm, $\int_0^t f(s)\, dB^H(s)$, have been suggested:

- pathwise approach, using the results of Young [34], Ciesielski [14] and Zähle [35]. In general, in this case, the property $\mathbf{E}(\int_0^t f(s) dB^H(s)) = 0$ is not satisfied, we cite for example Lin, SJ [25] and Gripenberg and Norros [22].

- Malliavin calculus approach, introduced by Decreusefond and Üstünel [18], Carmona and Coutin [11]. In this approach, the property $\mathbf{E}(\int_0^t f(s) dB^H(s)) = 0$ is satisfied, we see for instance Duncan, Hu and Pasik-Duncan [21] (using Wick calculus), Pérez-Abreu and Tudor [31] (using transfer principle), the two definitions given in these two papers are equivalent. In some cases, this integral is called Skorohod integral with respect to fBm. It is well known that on the case $H = \frac{1}{2}$, the Skorohod integral may be regarded as an extention of the Itô integral to integrands that are not necessarily adapted, see [19, Theorem 2.9].

We consider the following affine stochasctic differential equation:

$$dX(t) = \big(a_0(s) - X(s)\big)ds + \big(b_0(s) + b(s)X(s)\big)dB^H(s), \tag{2}$$

where a_0, b_0 and b are almost periodic continuous real functions. The aim of this paper is to show the existence and uniqueness of almost periodic solution in distribution to (2). For this, to facilitate the understanding, we keep the same notations as in the work of Pérez-Abreu and Tudor [31]. To remedy to the difficulties engender by the fact that integrand part is random function, the basic tools we used in this paper are:

1. The representation of the fractional Brownian motion in terms of standard Brownian motion:

$$B^H(t) = \int_{\mathbb{R}} K(t,s) dW_s, \tag{3}$$

where $K(t,s)$ is a non random fractional kernel. This equality is given in finite distribution and it is shown in [18] and [26] that it holds in trajectorial sense with a fixed standard Brownian motion.

2. The chaos expansion, where each term is given by the multiple fractional stochastic integrals.

3. By using the representation (3) of fBm, we define multiple fractional stochastic integrals in terms of the classical Itô multiple integrals of the standard Brownian motion. What will allow us to recover the interesting properties of the Itô integral, as Itô isometry, change of variable,... .

The organisation of this paper is as follows: in Section 2, we recall some basic definitions related to fBm, almost periodcity and multiple fractional stochastic integrals. In Section 3, we study the existence and uniqueness of almost periodic solution in one dimensional distribution to affine fractional stochastic differential equation (2).

2 Notations and preliminaries

This section is dedicated to the different spaces involved in this work: space of almost periodic functions, the space of integrands with respect to fBm, the sapce of multiple integrands with respect to fBm. the last part of this section is devoted to the chaos expansion.

2.1 Space of almost periodic functions

Let $(\mathbb{E},d)$ be a Polish space. *A continuous function $f:\mathbb{R}\to\mathbb{E}$ is said to be almost periodic if for each $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that any interval of lenght $l(\varepsilon)$ contains at least a number τ for which*

$$\sup_{t\in\mathbb{R}} d\big(f(t+\tau),f(t)\big)<\varepsilon. \tag{4}$$

We denote by $AP(R,\mathbb{E})$ the space of $\mathbb{E}$-valued almost periodic functions.
This definition is not always easy to use in practice. The following theorem gives two characterizations, due to Bochner, of the almost periodicity, which are usually used.

Theorem 2.1 ([8]) *Let $f:\mathbb{R}\to\mathbb{E}$ be a continuous function. The following statements are equivalent*

1. *f is almost periodic.*

2. *The set of translated functions $\{f(t+.);\ t\in\mathbb{R}\}$ is totally bounded in the space of continuous functions $C(\mathbb{R};\mathbb{E})$ endowed with the topology of uniform convergence.*

3. *f satisfies Bochner's double sequences criterion, that is, for every pair of sequences $(\alpha'_n)\subset\mathbb{R}$ and $(\beta'_n)\subset\mathbb{R}$, there are subsequences $(\alpha_n)\subset(\alpha'_n)$ and $(\beta_n)\subset(\beta'_n)$ respectively with same indexes such that, for every $t\in\mathbb{R}$, the limits*

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t+\alpha_n+\beta_m) \quad \textit{and} \quad \lim_{n\to\infty} f(t+\alpha_n+\beta_n), \tag{5}$$

exist and are equal.

Remark 2.2 A striking property of Bochner's double sequence criterion is that the limits in (5) exist in any of the three modes of convergences: pointwise, uniform on compact intervals and uniform on $\mathbb{R}$. This criterion has thus the advantage that it allows to establish uniform convergence by checking pointwise convergence.

2.1.1 Space of probability measures

We denote by $C_b(\mathbb{E})$ the Banach space of continuous and bounded functions $f:\mathbb{E}\to\mathbb{R}$ endowed with the uniform norm

$$\|f\|_\infty = \sup_{x\in\mathbb{E}}|f(x)|,$$

Let $f\in C_b(\mathbb{E})$ we define:

$$\|f\|_L = \sup\left\{\frac{f(x)-f(y)}{d_{\mathbb{E}}(x,y)} : x\neq y\right\},$$

$$|f|_{\mathrm{BL}} = \max\left\{\|f\|_\infty, \|f\|_L\right\},$$

and

$$\mathrm{BL}(\mathbb{E}) = \left\{f\in C_b(\mathbb{E}); \|f\|_{\mathrm{BL}}<\infty\right\}.$$

The space of all probability measures onto σ-Borel field of $\mathbb{E}$, denoted by $\mathcal{P}(\mathbb{E})$ and endowed with the metric: for $\mu,\nu\in\mathcal{P}(\mathbb{E})$,

$$d_{\mathrm{BL}}(\mu,\nu) = \sup_{\|f\|_{\mathrm{BL}}\leq 1}\left|\int_{\mathbb{E}} f d(\mu-\nu)\right|,$$

which generates the weak topology on $\mathcal{P}(\mathbb{E})$, is Polish space [20].

Almost periodcity in distribution

Let $X=(X_t)_{t\in\mathbb{R}}$ be a stochasctic process defined on probability space $(\Omega,\mathcal{F},P)$ with values in $\mathbb{R}$. *We say that X is stochastically continuous if for every $s\in\mathbb{R}$,*

$$\lim_{t\to s}\mathbf{E}|X(t)-X(s)| = 0.$$

We denote by $\mathrm{law}(X)(t)$ the distribution of the random variable $X(t):\Omega\to\mathbb{R}$, which means $\mathrm{law}(X(t))\in\mathcal{P}(\mathbb{R})$. The following defintions are introduced by C. Tudor.

- *A stochastically continuous process X is said to be almost periodic in one-dimensional distributions if the function*

$$\begin{cases}\mathbb{R} & \to & \mathcal{P}(\mathbb{R})\\ t & \mapsto & \mathrm{law}(X(t))\end{cases}$$

  is almost periodic.

- *A stochastically continuous process X is said to be almost periodic in finite dimensional distributions if for any $n\geq 1$ and $t_1<t_2<\ldots<t_n$, the function*

$$\begin{cases}\mathbb{R} & \to & \mathcal{P}(\mathbb{R}^n)\\ t & \mapsto & \mathrm{law}\big(X(t+t_1),X(t+t_2),\ldots,X(t+t_n)\big)\end{cases}$$

  is almost periodic.

- *A stochastic process X with continuous trajectories is said to be almost periodic in distribution (or in infinite dimentional distributions) if the function*

$$\begin{cases}\mathbb{R} & \to & \mathcal{P}(C_k(\mathbb{R},\mathbb{R}))\\ t & \mapsto & \mathrm{law}\big(X(t+.)\big)\end{cases}$$

  is almost periodic (the space of continuous fonctions $C_k(\mathbb{R},\mathbb{R})$ is endowed with the topology of uniform convergence on compact sets).

A comparative study between different almost periodicity of stochastic process is given by C. Tudor [33] and completed by F.Bedouhene et al. [3].

2.2 Integrand space

We fix a fBm $\{B_t^H\}_{t\in\mathbb{R}}$ with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ defined on probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F} = \sigma\left(B_t^H, t \in \mathbb{R}\right)$. We denote by $\mathcal{E}$ the space of elementary (or step) functions defined on real line by all functions of the form

$$f(u) = \sum_{k=1}^{n} f_k 1\!\!1_{[u_k, u_{k+1})}(u), \ \forall u \in \mathbb{R},$$

where $f_k \in \mathbb{R}$. The integral on this space with respect to the fBm is defined by:

$$I^H(f) = \sum_{k=1}^{n} f_k \left(B^H(u_{k+1}) - B^H(u_k)\right) := \int_{\mathbb{R}} f(u) dB^H(u),$$

which is a centred Gaussian random variable. The linear Gaussian space

$$\left\{I^H(f);\ f \in \mathcal{E}\right\}$$

is subset of

$$\mathfrak{I}(B^H) := \{X \in L^2_H(\Omega, \mathcal{F}, P);\ \text{s.t. } \exists (f_n)_{n\in\mathbb{N}} \subset \mathcal{E},\ I^H(f_n) \xrightarrow{L^2} X\}$$

which is also a linear space of centred Gaussian random variables with variance: for every $X \in \mathfrak{I}(B^H)$,

$$var(X) = E(X^2) = \lim_{n\to\infty} E(I^H(f_n)^2) = \lim_{n\to\infty} var(I^H(f_n)).$$

If f is an element of equivalence class of sequences of elementary functions $(f_n)_{n\in\mathbb{N}}$ such that $I^H(f_n) \xrightarrow{L^2} X$ ($X \in \mathfrak{I}(B^H)$), then X can be defined as the integral of f with respect to the fBm:

$$X := \int_{\mathbb{R}} f(u) dB^H(u).$$

For every $X \in \mathfrak{I}(B^H)$, the existence of function (or integrand) f such that $X = \int_{\mathbb{R}} f(u) dB^H(u)$ is assured according to the values of H:

1. In the case $H = \frac{1}{2}$ (which means that $B^{\frac{1}{2}} = W$) the set of of all functions f forms the whole Hilbert space $L^2(\mathbb{R})$ which is isometric to $\mathfrak{I}(W)$.

2. In the case $H \neq \frac{1}{2}$, the problem is dealt by V. Pipiras and M.S. Taqqu [32, Proposition 2.1]: for $H \in (0, \frac{1}{2})$, the authors show the existence of functional space which is a Hilbert space isometric to $\mathfrak{I}(B^H)$. However, for $H \in (\frac{1}{2}, 1)$, they show the existence of functional space which is incomplete and isometric to subspace (strict) of $\mathfrak{I}(B^H)$.

On the following, we will briefly give the idea of the construction of the integrand space, we became interested only in the last case ($H > \frac{1}{2}$) (for more details see [32]). The following representation of fBm will allow us to define the integrand space. But before, let's recall the definition of fractional integral.
For a function $\phi : \mathbb{R} \to \mathbb{R}$, we denote by $I^{\alpha}_{-}(\phi)$ the fractional integral, of order $\alpha > 0$, of ϕ, which is defined by:

$$(I^{\alpha}_{-}\phi)(s) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u) \left(s-u\right)_{-}^{\alpha-1} du,\ s \in \mathbb{R} \tag{6}$$

$$= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u) \left(u-s\right)_{+}^{\alpha-1} du,\ s \in \mathbb{R}, \tag{7}$$

where

$$a_+ = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a \leq 0 \end{cases} \text{ and } a_- = \begin{cases} -a & \text{if } a < 0 \\ 0 & \text{if } a \geq 0 \end{cases}.$$

Remark 2.3 If $\phi \in L^{\frac{1}{H}}(\mathbb{R})$, then $I_-^{\alpha}\phi \in L^2(\mathbb{R})$ with $\alpha = H - \frac{1}{2}$ (see e.g. [31, Theorem 2.3]).

The following representation holds in trajectorial sense.

Proposition 2.4 ([29] Proposition 2.3) *Let W be a standard Brownian motion and $H \in (\frac{1}{2}, 1)$. Then the process $B^H = (B_t^H)_{t\in\mathbb{R}}$, defined as*

$$B_t^H = \frac{1}{c_H}\int_{\mathbb{R}}\Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\Big)dW(s), \tag{8}$$

where

$$c_H = \Big(\int_0^{\infty}\big((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\big)^2 ds + \frac{1}{2H}\Big)^{\frac{1}{2}},$$

is fBm of Hurst parameter H.

Remark 2.5
- The fractional kernel K can be expressed as a function of $I_-^{H-\frac{1}{2}}$, more exactly, we have

$$\begin{aligned} K(t,s) &= \frac{\Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\Big)}{c_H} \\ &= \frac{(H-\frac{1}{2})}{c_H}\int_{\mathbb{R}} 1_{[0,t)}(u)(u-s)_+^{H-\frac{3}{2}}du \\ &= \frac{\Gamma(H+\frac{1}{2})}{c_H} I_-^{H-\frac{1}{2}}\big(1_{[0,t)}\big)(s), \end{aligned}$$

  where $1_{[0,t)}$ is interpreted as $-1_{[t,0)}$, if $t<0$

- Using the representation (8) of fBm, for every $f \in \mathcal{E}$, we have

$$I^H(f) = \int_{\mathbb{R}} f(s)dB^H(s) = \frac{\Gamma(H+\frac{1}{2})}{c_H}\int_{\mathbb{R}}\big(I_-^{H-\frac{1}{2}}(f)\big)(s)dW(s).$$

  (The last integral is the Wiener Itô integral with respect to the standard Brownian motion W).

- For every $f, g \in \mathcal{E}$,

$$E\big(I^H(f)I^H(g)\big) = \frac{\Gamma(H+\frac{1}{2})^2}{(c_H)^2}\int_{\mathbb{R}}\big(I_-^{H-\frac{1}{2}}(f)\big)(s)\big(I_-^{H-\frac{1}{2}}(g)\big)(s)ds.$$

The space of integrands with respect to the fBm B^H, denoted $L_H^2(\mathbb{R})$, is defined as follows:

$$L_H^2(\mathbb{R}) = \Big\{f; I_-^{H-\frac{1}{2}}(f) \in L^2(\mathbb{R})\Big\} = \Big\{f; \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(u)(u-s)_+^{H-\frac{3}{2}}du\Big)^2 ds < \infty\Big\} \tag{9}$$

Proposition 2.6 ([32]Proposition 3.2) *The space $L_H^2(\mathbb{R})$, defined in (9), is linear space with the inner product*

$$\langle f,g\rangle_H = \frac{\Gamma(H+\frac{1}{2})^2}{(c_H)^2}\int_{\mathbb{R}}\big(I_-^{H-\frac{1}{2}}(f)\big)(s)\big(I_-^{H-\frac{1}{2}}(g)\big)(s)ds,\ f,g \in L_H^2(\mathbb{R}). \tag{10}$$

The set of elementary functions $\mathcal{E}$ is dense in $L_H^2(\mathbb{R})$. The space $L_H^2(\mathbb{R})$ is not complete

Remark 2.7 From the defintion of the inner product (10), the following isometry holds

$$\langle f,g\rangle_H = \frac{\Gamma(H+\frac{1}{2})^2}{(c_H)^2}\langle I_-^{H-\frac{1}{2}}(f), I_-^{H-\frac{1}{2}}(g)\rangle_{L^2(\mathbb{R})}.$$

We observe that the elements of the integrand space $L_H^2(\mathbb{R})$ are not very explicitly defined. In what follows, we will see an explicit form of the elements of $L_H^2(\mathbb{R})$.

Lemma 2.8 ([22]Page 404) *We have the following identity*

$$\big(H-\frac{1}{2}\big)^2 d_H^2 \int_{-\infty}^{t\wedge s}(s-u)^{H-\frac{3}{2}}(t-u)^{H-\frac{3}{2}}du = \varphi(s,t), \tag{11}$$

where

$$d_H = \Big(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\Big)^{\frac{1}{2}},$$

and

$$\varphi(s,t) = H(2H-1)|s-t|^{2H-2};\ s,t\in\mathbb{R}.$$

Proposition 2.9 *Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Then $f\in L_H^2(\mathbb{R})$ if and only if*

$$\int_{\mathbb{R}^2} f(t)f(s)\varphi(t,s)dtds < \infty.$$

Proof Let $f\in L_H^2(\mathbb{R})$. We have by Fubini Theorem and Lemma 2.8

$$\begin{aligned}
\int_{\mathbb{R}}\big(\int_{\mathbb{R}} f(u)(u-s)_+^{H-\frac{3}{2}}du\big)^2 ds &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(r)(r-s)_+^{H-\frac{3}{2}}dr\int_{\mathbb{R}} f(t)(t-s)_+^{H-\frac{3}{2}}dt\Big)ds \\
&= \int_{\mathbb{R}}\Big(\iint_{\mathbb{R}^2} f(r)(r-s)_+^{H-\frac{3}{2}} f(t)(t-s)_+^{H-\frac{3}{2}}drdt\Big)ds \\
&= \iint_{\mathbb{R}^2} f(r)f(t)\Big(\int_{-\infty}^{r\wedge t}(r-s)^{H-\frac{3}{2}}(t-s)^{H-\frac{3}{2}}ds\Big)drdt \\
&= \frac{1}{\big(H-\frac{1}{2}\big)^2 d_H^2}\iint_{\mathbb{R}^2} f(r)f(t)\varphi(r,t)drdt.
\end{aligned}$$

□

Remark 2.10 1. The space $L_H^2(\mathbb{R})$ can be endowed with an another scalar product

$$\langle f,g\rangle_{H,\mathbb{R}} = \iint_{\mathbb{R}^2} f(r)g(t)\varphi(r,t)drdt$$

with the associated norm

$$|f|^2_{H,\mathbb{R}} = \langle f,f\rangle_{H,\mathbb{R}}. \tag{12}$$

2. The operator Γ_H defined, from $L_H^2(\mathbb{R})$ to $L^2(\mathbb{R})$, by: for every $f\in L_H^2(\mathbb{R})$,

$$\Gamma_H(f)(u) = d_H\big(H-\frac{1}{2}\big)\int_{\mathbb{R}} f(t)(t-u)_+^{H-\frac{3}{2}}dt = d_H\Gamma(H+\frac{1}{2})\big(I_-^{H-\frac{1}{2}}f\big)(u), \tag{13}$$

is an isometry

3. The stochastic integral with respect to the fBm B^H is defined by:

$$I^H(f) = \int_{\mathbb{R}} f(s)dB^H(s) = \int_{\mathbb{R}}\Gamma_H(f)(s)dW(s) = I^{\frac{1}{2}}(\Gamma_H(f)). \tag{14}$$

4. The fractional kernel, which fits to the definition (14), still denoted by K, is given by:

$$K_H(t,s) = d_H\Gamma(H+\frac{1}{2})\Big(I_-^{H-\frac{1}{2}}1\!\!1_{[0,t[}\Big)(s).$$

   For more details, see [22]

5. The set

$$|L^2_H(\mathbb{R})| := \Big\{f:\mathbb{R}\to\mathbb{R}; \iint_{\mathbb{R}^2}\varphi(r,t)\,|f(r)f(t)|\,drdt<\infty\Big\} \tag{15}$$

   is a strict subspace (dense) of $L^2_H(\mathbb{R})$, endowed with the norm

$$||f||^2_{H,\mathbb{R}} :=< |f|,|f| >_{H,\mathbb{R}} \tag{16}$$

   is Banach space. For more details on this space, see for instance [32].

Proposition 2.11 ([32]) *We have the following spaces inclusions:*

$$L^1(\mathbb{R})\cap L^2(\mathbb{R}) \subset L^{\frac{1}{H}}(\mathbb{R}) \subset |L^2_H(\mathbb{R})| \subset L^2_H(\mathbb{R}).$$

2.3 Multiple integrand space

All results and definitions seen in the below subsection are naturally generalized to the case of multiple integrals with respect to the fBm, but with some difference properties, as the Gaussian property (multiple integral is not gaussian random variable in general). In this subsection we will just recall the most important definitions and results for our work. For more details we refer readers to the work of V. Pérez-Abreu and C. Tudor [31].

We denote by $L^2_H(\mathbb{R}^p)$ the class of all functions $f\in L^2(\mathbb{R}^p)$ such that

$$\int_{\mathbb{R}^{2p}}\prod_{j=1}^{p}\varphi(x_j,y_j)\,|f(x_1,\dots,x_p)f(y_1,\dots,y_p)|\,dx_1\dots dx_pdy_1\dots dy_p<\infty,$$

with an inner product defined, for $f,g\in L^2_H(\mathbb{R}^p)$, by

$$\langle f,g\rangle_{H,\mathbb{R}^p} = \int_{\mathbb{R}^{2p}}\prod_{j=1}^{p}\varphi(x_j,y_j)f(x_1,\dots,x_p)g(y_1,\dots,y_p)dx_1\dots dx_pdy_1\dots dy_p, \tag{17}$$

and the associated norm

$$|f|^2_{H,\mathbb{R}^p} = \langle f,f\rangle_{H,\mathbb{R}^p}. \tag{18}$$

The space $\big(L^2_H(\mathbb{R}^p),\langle .,.\rangle_{H,\mathbb{R}^p}\big)$ is incomplete separable pre-Hilbert space. We have the inclusions

$$L^2(\mathbb{R}^p)\cap L^1(\mathbb{R}^p) \subset L^2_H(\mathbb{R}^p) \subset L^1_{loc}(\mathbb{R}^p).$$

We denote by $L^2_{s,H}(\mathbb{R}^p)$ the subspace of all symetric function $f\in L^2_H(\mathbb{R}^p)$. Every function $f\in L^2_H(\mathbb{R}^p)$ is symetrizable, we denote $sym(f)$ its symetrization. In [31, Lemma 3.4], the following inequality is proved:

$$|sym(f)|_{H,\mathbb{R}^p} \le |f|_{H,\mathbb{R}^p}. \tag{19}$$

The operator

$$\Gamma^{(p)}_H : L^2_H(\mathbb{R}^p)\to L^2(\mathbb{R}^p),$$

defined, for every $f\in L^2_H(\mathbb{R}^p)$, by

$$\Gamma^{(p)}_H(f)(x_1,\dots,x_p) = \Big[d_H\Big(H-\frac{1}{2}\Big)\Big]^p\int_{x_1}^{+\infty}\dots\int_{x_p}^{+\infty}\frac{f(t_1,\dots,t_p)}{\prod_{j=1}^{p}(t_j-x_j)^{\frac{3}{2}-H}}dt_1\dots dt_p$$

$$
\begin{aligned}
&= \left[d_H \Gamma\left(H+\frac{1}{2}\right)\right]^p \frac{1}{\Gamma(H-\frac{1}{2})^p} \int_{x_1}^{+\infty} \cdots \int_{x_p}^{+\infty} \frac{f(t_1,\ldots,t_p)}{\prod_{j=1}^p (t_j-x_j)^{\frac{3}{2}-H}} dt_1 \ldots dt_p \\
&= \left[d_H \Gamma\left(H+\frac{1}{2}\right)\right]^p \left(I_-^{H-\frac{1}{2},p} f\right)(x_1,\ldots,x_p)
\end{aligned}
$$

is an isometry (see [31, Lemma 3.4]).

We denote by $I_p^{\frac{1}{2}}(f)$ the multiple Wiener Itô integral of $f \in L^2(\mathbb{R}^p)$ with respect to the standard Brownian motion W (for more details on multiple Wiener Itô integral see [30] and [19]), that is,

$$I_p^{\frac{1}{2}}(f) = \int_{\mathbb{R}^p} f(t_1,\ldots,t_p) dW(t_1) \ldots dW(t_p). \tag{20}$$

The following definition of the multiple integral with respect to the fBm B^H is one of the most important tools in this work.

If $f \in L_H^2(\mathbb{R}^p)$ then we define the multiple fractional integral of order p with respect to the fBm B^H by

$$I_p^H(f) = I_p^{\frac{1}{2}}(\Gamma_H^{(p)} f), \tag{21}$$

that is

$$
\begin{aligned}
I_p^H(f) &= \int_{\mathbb{R}^p} f(t_1,\ldots,t_p) dB^H(t_1) \ldots dB^H(t_p) \\
&= \int_{\mathbb{R}^p} (\Gamma_H^{(p)} f)(t_1,\ldots,t_p) dW(t_1) \ldots dW(t_p).
\end{aligned}
$$

This definition is the same as that given in [21] and [17], for the proof see [31, Theorem 3.14]. Thus one can define the iterated integral

$$I_p^H(f) := p! \int_{t_1 < \ldots < t_p} f(t_1,\ldots,t_p) dB^H(t_1) \ldots dB^H(t_p). \tag{22}$$

Notice that for $p = 0$ and $f = f_0$ (constant), we set $I_0^H(f_0) = f_0$. For more details on multiple integrals with respect to fBm see [6].

2.4 Chaos Expansion

We denote by $L_H^2(\Omega, \mathcal{F}, P)$ the space of all random variables F which has an orthogonal fractional chaos decomposition of the form

$$F = \mathbf{E}(F) + \sum_{p=1}^{+\infty} I_p^H(f_p), \tag{23}$$

where $f_p \in L_{s,H}^2(\mathbb{R}^p)$ and

$$\sum_{p=1}^{+\infty} p! |f_p|_{H,\mathbb{R}^p}^2 < +\infty. \tag{24}$$

Theorem 2.12 ([31]) *The chaos decomposition (23) is unique, which means:*

$$L_H^2(\Omega, \mathcal{F}, P) = \oplus_{p=0}^{+\infty} I_p^H(L_{s,H}^2(\mathbb{R}^p)), \tag{25}$$

and the subspace $L_H^2(\Omega, \mathcal{F}, P)$ is total in $L^2(\Omega, \mathcal{F}, P)$. Moreover, for every $p \geq 1$, the fractional chaos of order p, $I_p^H(L_{s,H}^2(\mathbb{R}^p))$ is not closed. In particular $L_H^2(\Omega, \mathcal{F}, P)$ is not closed and it is strictly included in $L^2(\Omega, \mathcal{F}, P)$.

2.4.1 Fractional anticipating integral

We denote by $L^2\left(\Omega,\mathcal{F},P,L^2_H(\mathbb{R})\right)$ the space of all measurable processes with trajectories in Banach space $\left(|L^2_H(\mathbb{R})|,||.||_{H,\mathbb{R}}\right)$ (defined in (15) and (16) and still denoted by $L^2_H(\mathbb{R})$), such that for every $X\in L^2\left(\Omega,\mathcal{F},P,L^2_H(\mathbb{R})\right)$, we have: for a.a $t\in\mathbb{R}$, $X(t)\in L^2_H(\Omega,\mathcal{F},P)$, i.e. for each $p\in\mathbb{N}$, there exists $f_p(.,t)\in L^2_{s,H}(\mathbb{R}^p)$ such that

$$X(t)=\sum_{p=0}^{+\infty} I^H_p(f_p(.,t)),$$

with $f_p\in L^2_H(\mathbb{R}^{p+1})$ (t is considered here as variable) , for all $p\geq 1$ and

$$\sum_{p=1}^{+\infty} p!\|f_p\|^2_{H,\mathbb{R}^{p+1}}<+\infty.$$

We consider the operator

$$\delta^{\mathrm{ch}}_H:\begin{cases} L^2_H\left(\Omega,\mathcal{F},\mathbf{P},L^2_H(\mathbb{R})\right) & \to \quad L^2(\Omega,\mathcal{F},\mathbf{P}) \\ X & \mapsto \quad \delta^{\mathrm{ch}}_H(X)=\sum_{p=0}^{+\infty} I^H_{p+1}(\mathrm{sym}(f_p)) \end{cases}.$$

The domain of definition of this operator, denoted by D^{ch}_H, is all processes $X\in L^2\left(\Omega,\mathcal{F},P,L^2_H(\mathbb{R})\right)$ such that the series

$$\sum_{p=0}^{+\infty} I^H_{p+1}(\mathrm{sym}(f_p))$$

converges in $L^2(\Omega,\mathcal{F},P)$.

Proposition 2.13 ([31][Proposition 3.35) *1. Assuming that the space $L^2_H(\mathbb{R})$ is endowed with the norm $|.|_{H,\mathbb{R}}$ and $X\in D^{ch}_H$. Then*

$$E\left(|X|^2_{H,\mathbb{R}}\right)=\sum_{1}^{\infty} p!\,|f|^2_{H,\mathbb{R}^{p+1}}.$$

2. The operator $\delta^{\mathrm{ch}}_H:D^{ch}_H\to L^2(\Omega,\mathcal{F},\mathbf{P})$ is closable.

The closure of the operator defined above is denoted again by δ^{ch}_H. If the process $X1\!\!1_{]-\infty,a]}\in D^{ch}_H$ then we define:

$$\int_{-\infty}^{a} X(s)dB^H(s)=\delta^{\mathrm{ch}}_H(X1\!\!1_{]-\infty,a]}).$$

3 Main result

In what follows, we fix a normalized fBm $(B^H(t))_{t\in\mathbb{R}}$ ($\frac{1}{2}<H<1$ and $(W(t))_{t\in\mathbb{R}}$ a standard Brownian motion defined on the same stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}},\mathbf{P})$ (i.e $(B^H(t))_{t\in\mathbb{R}}$ and $(W(t))_{t\in\mathbb{R}}$ generate the same filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$). It is known that there exists the following representation formula:

$$B^H(t)=\int_{\mathbb{R}} K_H(t,s)dW(s),\ \text{where } K_H(t,s)=d_H\Gamma(H+\frac{1}{2})\left(I_-^{H-\frac{1}{2}}1\!\!1_{[0,t[}\right)(s).$$

We consider the fractional affine equation

$$dX(t)=[a_0(t)-X(t)]dt+[b_0(t)+b(t)X(t)]dB^H(t),\tag{26}$$

where, $a_0,b_0,b:\mathbb{R}\to\mathbb{R}$ are continuous functions. In the sequel we suppose that

1. The mappings a_0, b_0, b are almost periodic.

2. The functions

$$(t,s) \in \mathbb{R}^2 \mapsto g(t,s) := 1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} b_0(s),$$

and

$$(t,s) \in \mathbb{R}^2 \mapsto h(t,s) := 1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} b(s)$$

are elements of $L^2_H(\mathbb{R}^2)$.

Theorem 3.1 *Under the conditions (1) and (2), the equation (26) has a unique evolution solution $X \in \mathrm{D}^{\mathrm{ch}}_H$, which can be expressed as follows:*

$$X(t) = \int_{-\infty}^{t} e^{-(t-s)} a_0(s) ds + \int_{-\infty}^{t} e^{-(t-s)} \left[b_0(s) + b(s) X(s) \right] dB^H(s), \tag{27}$$

and with the chaos decomposition

$$X(t) = \sum_{p=0}^{+\infty} I_p^H(f_p^t), \tag{28}$$

where

$$f_0^t = \int_{-\infty}^{t} e^{-(t-s)} a_0(s) ds; \tag{29}$$

$$f_1^t(s) = 1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} [b_0(s) + b(s) f_0^s]; \tag{30}$$

$$f_p^t(t_1, \ldots, t_p) = \frac{1}{p} \sum_{j=1}^{p} 1\!\!1_{]-\infty,t]}(t_j) e^{-(t-t_j)} b(t_j) f_{p-1}^{t_j}(\hat{t}_j);\ \ p \geq 2. \tag{31}$$

and $f_{p-1}^{t_j}(\hat{t}_j)$ denote the function of $(p-1)$ variables without the variable t_j, more precisely:

$$\begin{aligned} f_p^t(t_1, \ldots, t_p) = sym\{ & 1\!\!1_{-\infty < t_1 < \ldots < t_p < t} e^{-(t-t_1)} b_0(t_1) b(t_2) \ldots b(t_p) \} \\ & + \frac{1}{p!} 1\!\!1_{]-\infty,t]^p}(t_1, \ldots, t_p) b(t_1) \ldots b(t_p) \int_{-\infty}^{t_1 \wedge \ldots \wedge t_p} e^{-(t-s)} a_0(s) ds. \end{aligned} \tag{32}$$

Furthermore X is almost periodic in one-dimensional distribution.

To prove this result, we use the following three lemmas. The first is obvious.

Lemma 3.2 *The following two-variable function*

$$F(t,s) = \int_{-\infty}^{t} \int_{-\infty}^{s} \varphi(r,u) e^{-\left(t+s-(r+u)\right)} dr du$$

is bounded in $\mathbb{R}^2$.

Before introducing the two others lemmas, let us denote by:

$$M = \max\big(\sup_{\mathbb{R}}(b(t), \sup_{\mathbb{R}}(b_0(t))\big),\ m = \sup_{\mathbb{R}}(a_0(t)) \text{ and } L = \sup_{R^2}(F(t,s). \tag{33}$$

Lemma 3.3 *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a real sequence, $l : \mathbb{R} \to \mathbb{R}$ a continuous bounded function such that*

$$1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} l(s) \in L^2_H(\mathbb{R}^2).$$

If the sequence $(l(.+\alpha_n))_{n\in\mathbb{N}}$ converges uniformly to $l^*(.)$, then

$$1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} l^*(s) \in L^2_H(\mathbb{R}^2)$$

Proof By dominated convergence theorem and Fatou lemma we get:

$$\iint_{\mathbb{R}^2} \varphi(t_2,s_2) \int_{-\infty}^{t_2}\int_{-\infty}^{s_2} \varphi(t_1,s_1) e^{-(t_2-t_1)} e^{-(s_2-s_1)} |l^*(t_1)l^*(s_1)| dt_1 ds_1 dt_2 ds_2 =$$
$$\iint_{\mathbb{R}^2} \varphi(t_2,s_2) \lim_{n\to\infty} \int_{-\infty}^{t_2}\int_{-\infty}^{s_2} \varphi(t_1,s_1) e^{-(t_2-t_1)} e^{-(s_2-s_1)} |l(t_1+\alpha_n)l(s_1+\alpha_n)| dt_1 ds_1 dt_2 ds_2 \leq$$
$$\underline{\lim} \iint_{\mathbb{R}^2} \varphi(t_2,s_2) \int_{-\infty}^{t_2+\alpha_n}\int_{-\infty}^{s_2+\alpha_n} \varphi(t_1,s_1) e^{-(t_2+\alpha_n-t_1)} e^{-(s_2+\alpha_n-s_1)} |l(t_1)l(s_1)| dt_1 ds_1 dt_2 ds_2 < \infty,$$

thus

$$1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} l^*(s) \in L^2_H(\mathbb{R}^2).$$

□

Lemma 3.4 *The operator $\Im_H : D_H^{ch} \to L^2(\Omega,\mathcal{F},P)$, defined by: for every $X \in D_H^{ch}$, we have*

$$\Im_H(X)(t) = \int_{-\infty}^t e^{-(t-s)} a_0(s)ds + \int_{-\infty}^t e^{-(t-s)} \left[b_0(s)+b(s)X(s)\right] dB^H(s);\;\; t\in\mathbb{R}$$

is well-defined. Moreover, if $X \in D_H^{ch}$ with the chaos decomposition

$$X(t) = \sum_{p=0}^{+\infty} I_p^H(f_p^t), \tag{34}$$

then X is a fixed point of $\Im_H$ if and only if f_p^t satisfy (29), (30) and (31).

Proof Firstly, let us show that the operator $\Im_H$ is well-defined. Let $X \in D_H^{ch}$, therefore, for each $p\in\mathbb{N}$, there exist $f_p(.,t) \in L^2_{s,H}(\mathbb{R}^p)$ such that

$$X(t) = \sum_{p=0}^{+\infty} I_p^H(f_p(.,t)), \tag{35}$$

with $f_p \in L^2_H(\mathbb{R}^{p+1})$, for all $p\geq 1$ and

$$\sum_{p=1}^{+\infty} p!\|f_p\|^2_{H,\mathbb{R}^{p+1}} < +\infty. \tag{36}$$

We denote by (Y_t) the image of (X_t) by $\Im_H$, that is

$$Y(t) = \int_{-\infty}^t e^{-(t-s)} a_0(s)ds + \int_{-\infty}^t e^{-(t-s)} \left[b_0(s)+b(s)X(s)\right] dB^H(s),\; \forall t\in\mathbb{R}. \tag{37}$$

By replacing (35) in (37) we get

$$Y(t) = \int_{-\infty}^t e^{-(t-s)} a_0(s)ds + \delta_H^{ch}(Z^t),\; \forall t\in\mathbb{R}, \tag{38}$$

where

$$Z^t(s) = \sum_{p=0}^{\infty} I_p^H(g_p^t(.,s))$$

and

$$g_0^t(s) := 1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} [b_0(s)+b(s)f_0(s)];$$

$$g_p^t(.,s) := 1\!\!1_{]-\infty,t]}(s)e^{-(t-s)}b(s)f_p(.,s), \ p \geq 1.$$

As $X \in D_H^{ch}$, b_0 and b are bounded (almost periodic), it is straightforward to show that, for every $t \in R$, $Z^t \in D_H^{ch}$ and thus

$$\delta_H^{ch}(Z^t) = \sum_{p=0}^{\infty} I_{p+1}^H\big(sym(g_p^t)\big),$$

converges in $L^2(\Omega, \mathcal{F}, P)$. Then, for every $t \in \mathbb{R}$, $Y(t) \in L^2(\Omega, \mathcal{F}, P)$ with the chaos decomposition:

$$Y(t) = \sum_{p=0}^{\infty} I_p^H(h_p^t),$$

where

$$\begin{aligned} h_0^t &:= \int_{-\infty}^t e^{-(t-s)}a_0(s)ds; \\ h_1^t &:= 1\!\!1_{]-\infty,t]}(s)e^{-(t-s)}[b_0(s)+b(s)f_0(s)]; \\ h_p^t &:= \frac{1}{p}\sum_{j=1}^{p} 1\!\!1_{]-\infty,t]}(t_j)e^{-(t-t_j)}b(t_j)f_{p-1}^{t_j}(\hat{t}_j). \end{aligned}$$

With the same arguments as used to prove that $Z^t \in D_H^{ch}$, we can also show that for every $t \in \mathbb{R}$, the kernels $h_p^t \in L_{s,H}^2(\mathbb{R}^p)$.
Secondly, let $X \in D_H^{ch}$ such that $\Im_H(X)(t) = X(t)$ for every $t \in \mathbb{R}$. By identifying the kernels we get identities (29)-(31).
Conversely, assume that f_p^t satisfy (29), (30) and (31), it's straightforward to cheek that X is a fixed point of $\Im_H$. □

Proof of Theorem 3.1. ***Step 1: existence and uniqueness of evolution in D_H^{ch}.*** For the existence, by using Lemma 3.4, it suffices to show that the process X defined in (28)-(32) is an element of D_H^{ch}. Clearly, by using (19), (22) and Lemma 3.2, for every $p \in \mathbb{N}$ and $t \in \mathbb{R}$, we have $f_p^t \in L_{s,H}^2(\mathbb{R}^p)$ and

$$\sum_{p=1}^{+\infty} p!\|f_p^t\|_{H,\mathbb{R}^p}^2 < +\infty,$$

which means that for every $t \in \mathbb{R}$, $X(t) \in L_H^2(\Omega, \mathcal{F}, \mathbf{P})$. Let us show that $f_p \in L_H^2(\mathbb{R}^{p+1})$ for every $p \geq 1$. We denote by g_p the multiple function

$$g_p(t_1,\ldots,t_{p+1}) = 1\!\!1_{-\infty<t_1<\ldots<t_p<t_{p+1}}e^{-(t_{p+1}-t_1)}b_0(t_1)b(t_2)\ldots b(t_p)$$

and by h_p the multiple function

$$h_p(t_1,\ldots,t_{p+1}) = 1\!\!1_{]-\infty,t_{p+1}]^p}(t_1,\ldots,t_p)b(t_1)\ldots b(t_p)\int_{-\infty}^{t_1\wedge\ldots\wedge t_p} e^{-(t_{p+1}-s)}a_0(s)ds.$$

Thanks to Lemma 3.2 and the condition (2), we can show that $g_p, h_p \in L_H^2(\mathbb{R}^{p+1})$, therefore $f_p \in L_H^2(\mathbb{R}^{p+1})$ and

$$\|f_p\|_{H,\mathbb{R}^{p+1}}^2 \leq \frac{2}{(p!)^2}(M^2L)^{p-1}\big(\|g\|_{H,\mathbb{R}^2}^2 + m\|h\|_{H,\mathbb{R}^2}^2\big), \ \forall p \geq 1,$$

thus

$$\sum_{p=1}^{+\infty} p!\|f_p\|_{H,\mathbb{R}^{p+1}}^2 < +\infty.$$

It remains to show that

$$\sum_{p=0}^{\infty} I_{p+1}^{H}\big(sym(f_p)\big)$$

converges in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. For this purpose, it's enough to show that

$$\sum_{p=0}^{+\infty} (p+1)!\|f_p\|^2_{H,\mathbb{R}^{p+1}} < +\infty.$$

We have

$$(p+1)!\|f_p\|^2_{H,\mathbb{R}^{p+1}} \le \frac{4}{(p-1)!}(M^2L)^{p-1}\big(\|g\|^2_{H,\mathbb{R}^2} + m\|h\|^2_{H,\mathbb{R}^2}\big),\ \forall p \ge 1,$$

which gives the convergence in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ of the series

$$\sum_{p=0}^{\infty} I_{p+1}^{H}\big(sym(f_p)\big).$$

Thus $X \in D_H^{ch}$. The unicity is deduced from the explicit form (32).
Step 2: almost periodicity. To prove that X is almost periodic in distribution, we use Bochner's double sequences criterion. Let (α'_n) and (β'_n) be two sequences in $\mathbb{R}$, we show that there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that, for every $t \in \mathbb{R}$, the limits

$$\lim_{n\to+\infty}\lim_{m\to+\infty} \mu(t+\alpha_n+\beta_m) \quad \text{ and } \lim_{n\to+\infty} \mu(t+\alpha_n+\beta_n), \tag{39}$$

exist and are equal, where $\mu(t) := \text{law}(X)(t)$ is the distribution of $X(t)$.
We have asssumed that the functions a_0, b_0 and b are almost peridiodic, then there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that

$$\lim_{n\to\infty}\lim_{m\to\infty} a_0(t+\alpha_n+\beta_m) = \lim_{n\to\infty} a_0(t+\alpha_n+\beta_n) =: a_0^*(t), \tag{40}$$

$$\lim_{n\to\infty}\lim_{m\to\infty} b_0(t+\alpha_n+\beta_m) = \lim_{n\to\infty} b_0(t+\alpha_n+\beta_n) =: b_0^*(t), \tag{41}$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty} b(t+\alpha_n+\beta_m) = \lim_{n\to\infty} b(t+\alpha_n+\beta_n) =: b^*(t), \tag{42}$$

and these limits exist in any of the three modes of convergences: pointwise, uniform on compact intervals and uniform on $\mathbb{R}$. We now denote by $(\gamma_n)_{n\in\mathbb{N}}$ the sequence $(\alpha_n+\beta_n)_{n\in\mathbb{N}}$. From step 1, we can deduce that, for each fixed integer n,

$$X^n(t) = \int_{-\infty}^{t} e^{-(t-s)} a_0(s+\gamma_n)ds + \int_{-\infty}^{t} e^{-(t-s)}\left[b_0(s+\gamma_n) + b(s+\gamma_n)X^n(s)\right]dB^H(s)$$

is evolution solution of

$$dX^n(t) = [a_0(t+\gamma_n) - X^n(t)]dt + [b_0(t+\gamma_n) + b(t+\gamma_n)X^n(t)]dB^H(t), \tag{43}$$

with chaos decomposition

$$X^n(t) = \sum_{p=0}^{+\infty} I_p^H(f_p^{t,n}) = \sum_{p=0}^{+\infty} I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,n})\right), \tag{44}$$

where

$$f_p^{t,n}(t_1,\dots,t_p) = sym\{1\!\!1_{-\infty<t_1<\dots<t_p<t} e^{-(t-t_1)} b_0(t_1+\gamma_n) b(t_2+\gamma_n)\dots b(t_p+\gamma_n)\}$$
$$+\frac{1}{p!} 1\!\!1_{]-\infty,t]^p}(t_1,\dots,t_p) b(t_1+\gamma_n)\dots b(t_p+\gamma_n) \int_{-\infty}^{t_1\wedge\dots\wedge t_p} e^{-(t-s)} a_0(s+\gamma_n) ds. \quad (45)$$

Also from step 1 and Lemma 3.3, we deduce that

$$X^*(t) = \int_{-\infty}^{t} e^{-(t-s)} a_0^*(s) ds + \int_{-\infty}^{t} e^{-(t-s)} \left[b_0^*(s) + b^*(s) X^*(s)\right] dB^H(s)$$

is evolution solution of

$$dX^*(t) = [a_0^*(t) - X^*(t)]dt + [b_0^*(t) + b^*(t) X^*(t)] dB^H(t), \quad (46)$$

with chaos decomposition

$$X^*(t) = \sum_{p=0}^{+\infty} I_p^H(f_p^{t,*}) = \sum_{p=0}^{+\infty} I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,*})\right), \quad (47)$$

where

$$f_p^{t,*}(t_1,\dots,t_p) = sym\{1\!\!1_{-\infty<t_1<\dots<t_p<t} e^{-(t-t_1)} b_0^*(t_1) b^*(t_2)\dots b^*(t_p)\}$$
$$+\frac{1}{p!} 1\!\!1_{]-\infty,t]^p}(t_1,\dots,t_p) b^*(t_1)\dots b^*(t_p) \int_{-\infty}^{t_1\wedge\dots\wedge t_p} e^{-(t-s)} a_0^*(s) ds. \quad (48)$$

We have:

$$X(t+\gamma_n) = \sum_{p=0}^{+\infty} I_p^H(f_p^{t+\gamma_n}) = \sum_{p=0}^{+\infty} I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t+\gamma_n})\right). \quad (49)$$

The changes of variables, $\sigma = t+\gamma_n$ and $\sigma_j = t_j + \gamma_n$ for all $j = 1, p$, transform

$$\sum_{p=0}^{+\infty} I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t+\gamma_n})\right)$$

to

$$\sum_{p=0}^{+\infty} \hat{I}_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,n})\right), \quad (50)$$

where

$$\hat{I}_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,n})\right) = \int_{\mathbb{R}^p} \Gamma_H^{(p)}(f_p^{t,n})(t_1,\dots,t_p) d\tilde{W}(t_1)\dots d\tilde{W}(t_p)$$

and $\tilde{W}_n(t_j) = W(t_j+\gamma_n) - W(\gamma_n)$ is a standard Brownian motion with the same distribution as $W(t_j)$.

Let us show that, for each fixed $t\in\mathbb{R}$, $I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,n})\right)$ converges in quadratic mean to $I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,*})\right)$. Note that for $p=0$ the process is deterministic and we get the convergence easily. Let $M^* = \max\left(M, \sup(b_0^*), \sup(b^*)\right)$. For $p>0$, using the Itô isometry and the fact that the operator $\Gamma_H^{(p)}$ is an isometry, we obtain

$$\mathbf{E}\left|I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,n})\right) - I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,*})\right)\right|^2 = \left\|\Gamma_H^{(p)}\left(f_p^{t,n} - f_p^{t,*}\right)\right\|^2_{L^2(\mathbb{R}^p)} = \left|f_p^{t,n} - f_p^{t,*}\right|^2_{H,\mathbb{R}^p}.$$

By replacing $f_p^{t,n}$ and $f_p^{t,*}$ with their explicit forms (45) and (48) respectively, we get

$$\left|f_p^{t,n} - f_p^{t,*}\right|^2_{H,\mathbb{R}^p} \le 2|sym\{1\!\!1_{-\infty<t_1<\dots<t_p<t} e^{-(t-\cdot_1)} b_0(\cdot_1+\gamma_n) b(\cdot_2+\gamma_n)\dots b(\cdot_p+\gamma_n)\}$$

$$
\begin{aligned}
&- sym\{1\!\!1_{-\infty<t_1<\ldots<t_p<t} e^{-(t-\cdot_1)} b_0^*(\cdot_1) b^*(\cdot_2)\ldots b^*(\cdot_p)\}|^2_{H,\mathbb{R}^p}\\
&+2|\frac{1}{p!}1\!\!1_{]-\infty,t]^p}(\cdot_1,\ldots,\cdot_p) b(\cdot_1+\gamma_n)\ldots b(\cdot_p+\gamma_n)\int_{-\infty}^{\cdot_1\wedge\ldots\wedge\cdot_p} e^{-(t-s)} a_0(s+\gamma_n)ds\\
&-\frac{1}{p!}1\!\!1_{]-\infty,t]^p}(\cdot_1,\ldots,\cdot_p) b^*(\cdot_1)\ldots b^*(\cdot_p)\int_{-\infty}^{\cdot_1\wedge\ldots\wedge\cdot_p} e^{-(t-s)} a_0^*(s)ds|^2_{H,\mathbb{R}^p}\\
&\le 2p\big(I_1+I_2+\ldots I_k+\ldots+I_p\big)\\
&+2(p+1)\big(J_1+J_2+\ldots+J_k+\ldots+J_{p+1}\big),
\end{aligned}
$$

where $I_1, I_2, \ldots I_k, \ldots I_p$ are such that:

$$
\begin{aligned}
I_1 &= \left| sym\{1\!\!1_{-\infty<t_1<\ldots<t_p<t} e^{-(t-\cdot_1)}\big(b_0(\cdot_1+\gamma_n)-b_0^*(\cdot_1)\big) b(\cdot_2+\gamma_n)\ldots b(\cdot_p+\gamma_n)\}\right|^2_{H,\mathbb{R}^p}\\
&\le \sup_{\mathbb{R}} |b_0(t_1+\gamma_n)-b_0^*(t_1)|^2 \frac{L}{(p!)^2}\big((M^*)^2L\big)^{p-1},
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \left| sym\{1\!\!1_{-\infty<t_1<\ldots<t_p<t} e^{-(t-\cdot_1)} b_0^*(\cdot_1)\big(b(\cdot_2+\gamma_n)-b^*(\cdot_2)\big) b(\cdot_3+\gamma_n)\ldots b(\cdot_p+\gamma_n)\}\right|^2_{H,\mathbb{R}^p}\\
&\le \sup_{\mathbb{R}} |b(t_2+\gamma_n)-b^*(t_2)|^2 \frac{L}{(p!)^2}\big((M^*)^2L\big)^{p-1},
\end{aligned}
$$

$$
\begin{aligned}
&I_k =\\
&\left| sym\{1\!\!1_{-\infty<t_1<\ldots<t_p<t} e^{-(t-\cdot_1)} b_0^*(\cdot_1) b^*(\cdot_2)\ldots b^*(\cdot_{k-1})\big(b(\cdot_k+\gamma_n)-b^*(\cdot_k)\big) b(\cdot_{k+1}+\gamma_n)\ldots b(\cdot_p+\gamma_n)\}\right|^2_{H,\mathbb{R}^p}\\
&\le \sup_{\mathbb{R}} |b(t_k+\gamma_n)-b^*(t_k)|^2 \frac{L}{(p!)^2}\big((M^*)^2L\big)^{p-1},
\end{aligned}
$$

and

$$
\begin{aligned}
I_p &= \left| sym\{1\!\!1_{-\infty<t_1<\ldots<t_p<t} e^{-(t-\cdot_1)} b_0^*(\cdot_1) b^*(\cdot_2)\ldots b^*(\cdot_{p-1}\big(b(\cdot_p+\gamma_n)-b^*(\cdot_p)\big))\}\right|^2_{H,\mathbb{R}^p}\\
&\le \sup_{\mathbb{R}} |b(t_p+\gamma_n)-b^*(t_p)|^2 \frac{L}{(p!)^2}\big((M^*)^2L\big)^{p-1}.
\end{aligned}
$$

$$
\begin{aligned}
&J_1 =\\
&\left|\frac{1}{p!}1\!\!1_{]-\infty,t]^p}\big(b(\cdot_1+\gamma_n)-b^*(\cdot_1)\big) b(\cdot_2+\gamma_n)\ldots b(\cdot_p+\gamma_n)\int_{-\infty}^{\cdot_1\wedge\ldots\wedge\cdot_p} e^{-(t-s)} a_0(s+\gamma_n)ds\right|^2_{H,\mathbb{R}^p}\\
&\le \sup_{\mathbb{R}} |b(t_1+\gamma_n)-b^*(t_1)|^2 \frac{Lm^2}{(p!)^2}\big((M^*)^2L\big)^{p-1},
\end{aligned}
$$

$$
\begin{aligned}
&J_2 =\\
&\left|\frac{1}{p!}1\!\!1_{]-\infty,t]^p} b^*(\cdot_1)\big(b(\cdot_2+\gamma_n)-b^*(\cdot_2)\big) b(\cdot_3+\gamma_n)\ldots b(\cdot_p+\gamma_n)\int_{-\infty}^{\cdot_1\wedge\ldots\wedge\cdot_p} e^{-(t-s)} a_0(s+\gamma_n)ds\right|^2_{H,\mathbb{R}^p}\\
&\le \sup_{\mathbb{R}} |b(t_2+\gamma_n)-b^*(t_2)|^2 \frac{Lm^2}{(p!)^2}\big((M^*)^2L\big)^{p-1},
\end{aligned}
$$

$J_k =$

$$\left|\frac{1}{p!}1\!\!1_{]-\infty,t]^p}b^*(.{}_1)\dots b^*(.{}_{k-1})\big(b(.{}_k+\gamma_n)-b_{(.{}_k)}\big)b(.{}_{k+1}+\gamma_n)\dots b(.{}_p+\gamma_n)\int_{-\infty}^{.{}_1\wedge\dots\wedge .{}_p}e^{-(t-s)}a_0(s+\gamma_n)ds\right|^2_{H,\mathbb{R}^p}$$
$$\leq \sup_{\mathbb{R}}|b(t_k+\gamma_n)-b^*(t_k)|^2\frac{Lm^2}{(p!)^2}\left((M^*)^2L\right)^{p-1}$$

and

$$\begin{aligned}J_{p+1}&=\left|\frac{1}{p!}1\!\!1_{]-\infty,t]^p}b^*(.{}_1)\dots b^*(.{}_p)\int_{-\infty}^{.{}_1\wedge\dots\wedge .{}_p}e^{-(t-s)}\left(a_0(s+\gamma_n)-a_0^*(s)\right)ds\right|^2_{H,\mathbb{R}^p}\\&\leq \sup_{\mathbb{R}}|a_0(s+\gamma_n)-a_0^*(s)|^2\frac{L(M^*)^2}{(p!)^2}\left((M^*)^2L\right)^{p-1}.\end{aligned}$$

From (40)-(42) and the above inequalities, we deduce that

$$\lim_{n\to\infty}\mathbf{E}\left|I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,n})\right)-I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,*})\right)\right|^2=0,$$

hence $I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,n})\right)$ converges in quadratic mean to $I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^{t,*})\right)$. Using the previous inequalities again, we get the convergence of $X^n(t)$ in quadratic mean to $X^*(t)$ uniformly with respect to $t\in\mathbb{R}$, which gives the convergence in distribution of $X^n(t)$ to $X^*(t)$. It remains to show that $X^n(t)$ and $X(t+\gamma_n)$ have the same distribution for every $t\in\mathbb{R}$. From the transfer principle given by (21), we are able to define the Skorohod integral with respect to fBm $(B^H(t))$ by Skorohod integral with respect to standard Brownian motion $(W(t))$ as follows:

$$\begin{aligned}X(t)&=\int_{-\infty}^t e^{-(t-s)}a_0(s)ds+\int_{-\infty}^t e^{-(t-s)}\left[b_0(s)+b(s)X(s)\right]dB^H(s)=\sum_{p=0}^{+\infty}I_p^H(f_p^t)\\&=\sum_{p=0}^{+\infty}I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^t)\right)=f_0^t+\sum_{p=1}^{+\infty}I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^t)\right)=f_0^t+\sum_{p=0}^{+\infty}I_{p+1}^{\frac{1}{2}}\left(\Gamma_H^{(p+1)}(f_{p+1}^t)\right)\\&=\int_{-\infty}^t e^{-(t-s)}a_0(s)ds+\int_{-\infty}^t U^t(s)dW(s),\end{aligned}$$

where $(U^t(s))_{s\in\mathbb{R}}$, for each $t\in\mathbb{R}$, is a process with chaos decomposition

$$U^t(s)=\sum_{p=0}^{+\infty}I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^t(.,s))\right).$$

If the integrand process $(U^t(s))_{s\in\mathbb{R}}$ is adapted, we get the classical Itô integral and the change of variable $\sigma=s+\gamma_n$ gives another Itô integral with respect to $\tilde{W}_n(t)$, where $\tilde{W}_n(t)=W(t+\gamma_n)-W(\gamma_n)$ is standard Brownian motion with the same distribution as $W(t)$. From the independence of the increments of standard Brownian motion, we deduce that the process $X(t+\gamma_n)$ has the same distribution as $X^n(t)$. Thus it suffices to show that the solution X is an adapted process. First, let us show that for every $p\in\mathbb{N}$, $\left(I_p^{\frac{1}{2}}\left(\Gamma_H^{(p)}(f_p^t)\right)\right)$ is an adapted process. It is straightforward to show that for $p=0$ and $p=1$, the processes are adapted. For $p=2$, by using the iterated Itô integral, we get:

$$\begin{aligned}I_2^{\frac{1}{2}}\left(\Gamma_H^{(2)}(f_2^t)\right)&=\int_{-\infty}^t\Gamma_H^{(1)}\left[e^{-(t-.{}_2)}b(.{}_2)\Big(\int_{-\infty}^{.{}_2}\Gamma_H^{(1)}\left(e^{-(.{}_2-.{}_1)}b_0(.{}_1)\right)(t_1)dW(t_1)\Big)\right](t_2)dW(t_2)\\&+\int_{-\infty}^t\Gamma_H^{(1)}\left[e^{-(t-.{}_2)}b(.{}_2)\Big(\int_{-\infty}^{.{}_2}\Gamma_H^{(1)}\left(e^{-(.{}_2-.{}_1)}b(.{}_1)\int_{-\infty}^{.{}_1}e^{-(.{}_1-s)}a_0(s)ds\right)(t_1)dW(t_1)\Big)\right](t_2)dW(t_2),\end{aligned}$$

where:

- $$\Gamma_H^{(1)}\big(e^{-(.2-.1)}b_0(.1)\big)(t_1) = d_H(H-\frac{1}{2})\int_{t_1}^{.2}\frac{e^{-(.2-x_1)}b_0(x_1)}{(x_1-t_1)^{(\frac{3}{2}-H)}}dx_1,$$

  since this process is deterministic and square mean integrable in $]-\infty,.2]\times\Omega$, we deduce that the process obtained by Itô integral, $\int_{-\infty}^{.2}\Gamma_H^{(1)}\big(e^{-(.2-.1)}b_0(.1)\big)(t_1)dW(t_1)$, is adapted and we denote it by $Y(.2)$.

- $$\Gamma_H^{(1)}\Big(e^{-(t-.2)}b(.2)\big)Y(.2)\Big)(t_2) = d_H(H-\frac{1}{2})\int_{t_2}^{t}\frac{e^{-(t-x_2)}b(x_2)Y(x_2)}{(x_2-t_2)^{(\frac{3}{2}-H)}}dx_2,$$

  since $Y(.)$ is adapted and the process $\int_{t_2}^{t}\frac{e^{-(t-x_2)}b(x_2)Y(x_2)}{(x_2-t_2)^{(\frac{3}{2}-H)}}dx_2$ belongs to $L^2(]-\infty,t]\times\Omega)$, we deduce that the process obtained by Itô integral, $\int_{-\infty}^{t}\Gamma_H^{(1)}\big(e^{-(t-.2)}b(.2)Y(.2)\big)(t_2)dW(t_2)$, is adapted.

Similarly we can show that the process

$$\int_{-\infty}^{t}\Gamma_H^{(1)}\Big[e^{-(t-.2)}b(.2)\Big(\int_{-\infty}^{.2}\Gamma_H^{(1)}\big(e^{-(.2-.1)}b(.1)\int_{-\infty}^{.1}e^{-(.1-s)}a_0(s)ds\big)(t_1)dW(t_1)\Big)\Big](t_2)dW(t_2)$$

is adapted process. Thus $(I_2^{\frac{1}{2}}\big(\Gamma_H^{(2)}(f_2^t)\big))$ is an adapted process. The proof for a general p follows by induction as above.

The square mean convergence of chaos decomposition series implies that the solution X is adapted. Thus, for every $t\in\mathbb{R}$, the sequence $X(t+\gamma_n)$ converges in distribution to $X^*(t)$, i.e.

$$\lim_{n\to+\infty}\mu(t+\alpha_n+\beta_n) = \mu^*(t) := \text{law}(X^*(t)),$$

By analogy we can prove that

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\mu(t+\alpha_m+\beta_n) = \mu^*(t).$$

We have thus proved that X has almost periodic one-dimensional distributions. $\square$

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