LONG TIME SOLUTIONS FOR QUASI-LINEAR HAMILTONIAN PERTURBATIONS OF SCHRÖDINGER AND KLEIN-GORDON EQUATIONS ON TORI

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ABSTRACT. We consider quasi-linear, Hamiltonian perturbations of the cubic Schrödinger and of the cubic (derivative) Klein-Gordon equations on the d dimensional torus. If $\varepsilon \ll 1$ is the size of the initial datum, we prove that the lifespan of solutions is *strictly* larger than the local existence time ε^{-2}. More precisely, concerning the Schrödinger equation we show that the lifespan is at least of order $O(\varepsilon^{-4})$ while in the Klein-Gordon case, we prove that the solutions exist at least for a time of order $O(\varepsilon^{-8/3^+})$ as soon as $d \geq 3$. Regarding the Klein-Gordon equation, our result presents novelties also in the case of semi-linear perturbations: we show that the lifespan is at least of order $O(\varepsilon^{-10/3^+})$, improving, for cubic non-linearities and $d \geq 4$, the general result in [17], where the time of existence is decreasing with respect to the dimension.

CONTENTS

1. INTRODUCTION

This paper is concerned with the study of the lifespan of solutions of two classes of quasi-linear, Hamiltonian equations on the d-dimensional torus $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$, $d \geq 1$. We study quasi-linear perturbations of the Schrödinger and Klein-Gordon equations.

The Schrödinger equation we consider is the following

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u - V * u + \left[\Delta(h(|u|^2))\right] h'(|u|^2) u - |u|^2 u = 0, \\ u(0,x) = u_0(x) \end{cases} \tag{NLS}$$

Felice Iandoli has been supported by ERC grant ANADEL 757996. Roberto Feola and Benoit Grébert have been supported by the Centre Henri Lebesgue ANR-11-LABX- 0020-01 and by ANR-15-CE40-0001-02 “BEKAM” of the ANR.

where $\mathbb{C} \ni u := u(t,x)$, $x \in \mathbb{T}^d$, $d \geq 1$, $V(x)$ is a real valued potential even with respect to x, $h(x)$ is a function in $C^\infty(\mathbb{R};\mathbb{R})$ such that $h(x) = O(x^2)$ as $x \to 0$. The initial datum u_0 has small size and belongs to the Sobolev space $H^s(\mathbb{T}^d)$ (see (2.2)) with $s \gg 1$.

We examine also the Klein-Gordon equation

$$\begin{cases} \partial_{tt}\psi - \Delta\psi + m\psi + f(\psi) + g(\psi) = 0\,, \\ \psi(0,x) = \psi_0\,, \\ \partial_t\psi(0,x) = \psi_1\,, \end{cases} \tag{KG}$$

where $\psi = \psi(t,x)$, $x \in \mathbb{T}^d$, $d \geq 1$ and $m > 0$. The initial data (ψ_0, ψ_1) have small size and belong to the Sobolev space $H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d)$, for some $s \gg 1$. The nonlinearity $f(\psi)$ has the form

$$f(\psi) := -\sum_{j=1}^{d} \partial_{x_j}\big(\partial_{\psi_{x_j}} F(\psi, \nabla\psi)\big) + (\partial_\psi F)(\psi, \nabla\psi) \tag{1.1}$$

where $F(y_0, y_1, \ldots, y_d) \in C^\infty(\mathbb{R}^{d+1}, \mathbb{R})$, has a zero of order at least 6 at the origin. The non linear term $g(\psi)$ has the form

$$g(\psi) = (\partial_{y_0} G)(\psi, \Lambda_{\rm KG}^{\frac12}\psi) + \Lambda_{\rm KG}^{\frac12}(\partial_{y_1} G)(\psi, \Lambda_{\rm KG}^{\frac12}\psi) \tag{1.2}$$

where $G(y_0, y_1) \in C^\infty(\mathbb{R}^2;\mathbb{R})$ is a homogeneous polynomial of degree 4 and $\Lambda_{\rm KG}$ is the operator

$$\Lambda_{\rm KG} := (-\Delta + m)^{\frac12}\,, \tag{1.3}$$

defined by linearity as

$$\Lambda_{\rm KG} e^{\mathrm{i} j\cdot x} = \Lambda_{\rm KG}(j) e^{\mathrm{i} j\cdot x}\,, \qquad \Lambda_{\rm KG}(j) = \sqrt{|j|^2 + m}\,, \quad \forall\, j \in \mathbb{Z}^d\,. \tag{1.4}$$

Historical introduction for (NLS)**.** Quasi-linear Schrödinger equations of the specific form (NLS) appear in many domains of physics like plasma physics and fluid mechanics [40, 32], quantum mechanics [33], condensed matter theory [41]. They are also important in the study of Kelvin waves in the superfluid turbulence [39]. Equations of the form (NLS) posed in the *Euclidean space* have received the attention of many mathematicians. The first result, concerning the local well-posedness, is due to Poppenberg [43] in the one dimensional case. This has been generalized by Colin to any dimension [12]. A more general class of equations is considered in the pioneering work by Kenig-Ponce-Vega [38]. These results of local well-posedness have been recently optimized, in terms of regularity of the initial condition, by Marzuola-Metcalfe-Tataru [42] (see also references therein). Existence of standing waves has been studied by Colin [13] and Colin-Jeanjean [14]. The global well-posedness has been established by de Bouard-Hayashi-Saut [15] in dimension two and three for small data. This proof is based on dispersive estimates and energy method. New ideas have been introduced in studying the global well-posedness for other quasi-linear equations on the *Euclidean space*. Here the aforementioned tools are combined with *normal form* reductions. We quote Ionescu-Pusateri [35, 36] for the water-waves equation in two dimensions.

Very little is known when the equation (NLS) is posed on a *compact manifold*. The firsts local well-posedness results on the circle are given in the work by Baldi-Haus-Montalto [1] and in the paper [27]. Recently these results have been generalized to the case of tori of any dimension in [28]. Except these local existence results, nothing is known concerning the long time behavior of the solutions. The problem of *global existence/blow-up* is completely opened. In the aforementioned paper [15] it is exploited the dispersive character of the flow of the linear Schrödinger equation. This property is not present on compact manifolds: the solutions of the linear Schrödinger equation do not decay when the time goes to infinity. However in the one dimensional case in [29, 26] it is proven that *small* solutions of quasi-linear Schrödinger equations exist for long, but finite, times. In these works two of us exploit the fact that quasi-linear Schrödinger equations may be reduced to constant coefficients trough a *para-composition* generated by a diffeomorphism of the circle. This powerful tool has been used for the same purpose by other authors in the context of water-waves equations, firstly by Berti-Delort in [6] in a *non resonant* regime, secondly by Berti-Feola-Pusateri in [8, 9] and Berti-Feola-Franzoi [7] in the *resonant* case. We also mention that this feature has been used in other contexts for the same equations, for instance Feola-Procesi [30] prove the existence of a large set of *quasi-periodic* (and hence globally defined) solutions when the problem is posed on the circle. This "reduction to constant coefficients" is a peculiarity of one dimensional problems, in higher dimensions new ideas have to be introduced. For quasi-linear equations on tori of dimension two we quote the paper about long-time solutions for water-waves problem by Ionescu-Pusateri [34], where a different *normal form* analysis has been presented.

Historical introduction for (KG)**.** The local existence for (KG) is classical and we refer to Kato [37]. Many analysis have been done for *global/long time* existence.

When the equation is posed on the Euclidean space we have global existence for small and localized data Delort [16] and Stingo [44], here the authors use dispersive estimates on the linear flow combined with quasi-linear normal forms.

For (KG) on compact manifolds we quote Delort [18, 19] on $\mathbb{S}^d$ and Delort-Szeftel [20] on $\mathbb{T}^d$. The results obtained, in terms of length of the lifespan of solutions, are stronger in the case of the spheres. More precisely in the case of spheres the authors show the following. If m in (KG) is chosen outside of a set of zero Lebesgue measure, then for any natural number N, any initial condition of size ε (small depending on N) produces a solution whose lifespan is at least of magnitude ε^{-N}. In the case of tori in [20] they consider a quasi-linear equation, vanishing quadratically at the origin and they prove that the lifespan of solutions is of order ε^{-2} if the initial condition has size ε small enough. The differences between the two results are due to the different behaviors of the eigenvalues of the square root of the Laplace-Beltrami operator on $\mathbb{S}^d$ and $\mathbb{T}^d$. The difficulty on the tori is a consequence of the fact that the set of differences of eigenvalues of $\sqrt{-\Delta_{\mathbb{T}^d}}$ is dense in $\mathbb{R}$ if $d \geq 2$, this does not happen in the case of spheres. A more general set of manifolds where this does not happen is the Zoll manifolds, in this case we quote the paper by Delort-Szeftel [21] and Bambusi-Delort-Grébert-Szeftel [3] for semi-linear Klein-Gordon equations. For semi-linear Klein-Gordon equations on tori we have the result by Delort [17]. In this paper the author proves that if the non-linearity is vanishing at order k at zero then any initial datum of small size ε produces a solution whose lifespan is at least of magnitude $\varepsilon^{-k(1+\frac{2}{d})}$, up to a logarithmic loss. We improve this result, see Theorem 4, when $k = 2$ and $d \geq 4$.

Statement of the main results. The aim of this paper is to prove, in the spirit of [34], that we may go beyond the trivial time of existence, given by the local well-posedness theorem which is ε^{-2} since we are considering equations vanishing cubically at the origin and initial conditions of size ε.

In order to state our main theorem for (NLS) we need to make some hypotheses on the potential V. We consider potentials having the following form

$$V(x) = \sum_{\xi \in \mathbb{Z}^d} \widehat{V}(\xi) e^{\mathrm{i}\xi \cdot x}, \quad \widehat{V}(\xi) = \frac{x_\xi}{4\langle \xi \rangle^m}, \quad x_\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \mathbb{R}, \ \mathbb{N} \ni m > 1. \tag{1.5}$$

We endow the set $\mathcal{O} := [-1/2, 1/2]^{\mathbb{Z}^d}$ with the standard probability measure on product spaces. Our main theorem is the following.

Theorem 1. **(Long time existence for NLS).** *Consider the* (NLS) *with* $d \geq 2$. *There exists* $\mathcal{N} \subset \mathcal{O}$ *having zero Lebesgue measure such that if* x_ξ *in* (1.5) *is in* $\mathcal{O} \setminus \mathcal{N}$, *we have the following. There exists* $s_0 = s_0(d, m) \gg 1$ *such that for any* $s \geq s_0$ *there are constants* $c_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon \leq \varepsilon_0$ *we have the following. If* $\|u_0\|_{H^s} < c_0\varepsilon$, *there exists a unique solution of the Cauchy problem* (NLS) *such that*

$$u(t, x) \in C^0\big([0, T); H^s(\mathbb{T}^d)\big), \quad \sup_{t \in [0,T)} \|u(t, \cdot)\|_{H^s} \leq \varepsilon, \quad T \geq c_0 \varepsilon^{-4}. \tag{1.6}$$

In the one dimensional case the potential V may be disregarded and we obtain the following.

Theorem 2. *Consider* (NLS) *with* $V \equiv 0$ *and* $d = 1$. *There exists* $s_0 \gg 1$ *such that for any* $s \geq s_0$ *there are constants* $c_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon \leq \varepsilon_0$ *we have the following. If* $\|u_0\|_{H^s} < c_0\varepsilon$, *there exists* *a unique solution of the Cauchy problem* (NLS) *such that*

$$u(t,x) \in C^0\big([0,T); H^s(\mathbb{T}^d)\big), \quad \sup_{t\in[0,T)} \|u(t,\cdot)\|_{H^s} \le \varepsilon, \quad T \ge c_0\varepsilon^{-4}. \tag{1.7}$$

This is, to the best of our knowledge, the first result of this kind for quasi-linear Schrödinger equations posed on compact manifolds of dimension greater than one.

Our main theorem regarding the problem (KG) is the following.

Theorem 3. **(Long time existence for KG).** *Consider the* (KG) *with* $d \ge 2$. *There exists* $\mathcal{N} \subset [1,2]$ *having zero Lebesgue measure such that if* $m \in [1,2]\setminus\mathcal{N}$ *we have the following. There exists* $s_0 = s_0(d,m) \gg 1$ *such that for any* $s \ge s_0$ *there are constants* $c_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon \le \varepsilon_0$ *we have the following. For any initial data* $(\psi_0,\psi_1) \in H^{s+1/2}(\mathbb{T}^d)\times H^{s-1/2}(\mathbb{T}^d)$ *such that*

$$\|\psi_0\|_{H^{s+1/2}} + \|\psi_1\|_{H^{s-1/2}} < c_0\varepsilon,$$

there exists a unique solution of the Cauchy problem (KG) *such that*

$$\begin{aligned} &\psi(t,x) \in C^0\big([0,T); H^{s+\frac12}(\mathbb{T}^d)\big)\bigcap C^1\big([0,T); H^{s-\frac12}(\mathbb{T}^d)\big), \\ &\sup_{t\in[0,T)}\Big(\|\psi(t,\cdot)\|_{H^{s+\frac12}} + \|\partial_t\psi(t,\cdot)\|_{H^{s-\frac12}}\Big) \le \varepsilon, \qquad T \ge c_0\varepsilon^{-\mathtt{a}^+}, \end{aligned} \tag{1.8}$$

where $\mathtt{a} = 3$ *if* $d = 2$ *and* $\mathtt{a} = 8/3$ *if* $d \ge 3$.

We remark that long time existence for quasi-linear Klein-Gordon equations in dimension one are nowadays well known, see for instance [18]. The theorem 2 improves the general result in [17] in the particular case of cubic non-linearities in the following sense. First of all we can consider more general equations containing derivatives in the non-linearity (with “small” quasi-linear term), moreover our time of existence does not depend on the dimension. Furthermore, adapting our proof to the semi-linear case (i.e. when $f = 0$ in (KG) and (1.1) and G in (1.2) does not depend on y_1), we obtain the better time of existence $\varepsilon^{-10/3^+}$ for any $d \ge 4$. In the cases $d = 2,3$ we recover the time of existence in [17]. This is the content of the next Theorem.

Theorem 4. *Consider* (KG) *with* $f = 0$ *and* g *independent of* y_1. *Then the same results of Theorem 3 holds true for* $T \ge c_0\varepsilon^{-\mathtt{a}^+}$, *with* $\mathtt{a} = 4$ *if* $d = 2$, *and* $\mathtt{a} = 10/3$ *if* $d \ge 3$.

Comments on the results. We begin by discussing the (NLS) case. We remark that, beside the mathematical interest, it would be very interesting, from a physical point of view, to be able to deal with the case $h(\tau) \sim \tau$. Indeed, for instance, if we chose $h(\tau) = \sqrt{1+\tau} - 1$; the respective equation (NLS) models the self-channeling of a high power, ultra-short laser pulse in matter, see [11]. Unfortunately we need in our estimates $h(\tau) \sim \tau^{1+\delta}$ with $\delta > 0$, and since h has to be smooth this leads to $h(\tau) \sim \tau^2$.

Our method covers also more general cubic terms. For instance we could replace the term $|u|^2u$ with $g(|u|^2)u$, where $g(\cdot)$ is any analytic function vanishing at least linearly at the origin and having a primitive $G' = g$. We preferred not to write the paper in the most general case since the non-linearity $|u|^2u$ is a good representative for the aforementioned class and allows us to avoid to complicate the notation furtherly. We also remark that we consider a class of potentials V more general than the one we used in [29, 26] and more similar to the one used in [4] in a semi-linear context.

We now make some comments on the result concerning (KG). In this case we use normal forms (the same strategy is used for (NLS) as well) and therefore small divisors' problems arise. The small divisors, coming from the four waves interaction, are of the form

$$\Lambda_{\text{KG}}(\xi-\eta-\zeta) - \Lambda_{\text{KG}}(\eta) + \Lambda_{\text{KG}}(\zeta) - \Lambda_{\text{KG}}(\xi) \tag{1.9}$$

with Λ_{KG} defined in (1.4). In this case we prove the lower bound (see (2.26))

$$|\Lambda_{\text{KG}}(\xi-\eta-\zeta) - \Lambda_{\text{KG}}(\eta) + \Lambda_{\text{KG}}(\zeta) - \Lambda_{\text{KG}}(\xi)| \gtrsim \max{}_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{-N_0}\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{-\beta}, \tag{1.10}$$

for almost any value of the mass m in the interval $[1,2]$ and where β is any real number in the open interval $(3,4)$. The second factor in the r.h.s. of the above inequality represents a loss of derivatives when dividing by the quantity (1.9) which may be transformed in a loss of length of the lifespan through partition of frequencies. This is an extra difficulty, compared with the (NLS) case, which makes the problem challenging already in a semi-linear setting. The novelty in (1.10) is that β does not depend on the dimension d. This is why we can improve the result of [17]. We also quote [5] where Bernier-Faou-Grébert use a control of the small divisors involving only the largest index (and not $\max_2$ as in (1.10)). They obtained, in the semi-linear case, the control of the Sobolev norm as in (1.8), with $\mathtt{a}$ arbitrary large, but assuming that the initial datum satisfies $\|\psi_0\|_{H^{s'+1/2}} + \|\psi_1\|_{H^{s'-1/2}} < c_0\varepsilon$ for some $s' \equiv s'(\mathtt{a}) > s$, i.e. allowing a loss of regularity.

We notice that also in the (KG) case we are not able to deal with the interesting case of cubic quasi-linear term.

Ideas of the proof. In our proof we shall use a *quasi-linear normal forms/modified energies* approach, this seems to be the only successful one in order to improve the time of existence implied by the local theory. We recall, indeed, that on $\mathbb{T}^d$ the dispersive character of the solutions is absent. Moreover, the lack of conservation laws and the quasi-linear nature of the equation prevent the use of *semi-linear* techniques as done by Bambusi-Grébert [4] and Bambusi-Delort-Grébert-Szeftel [3].

The most important feature of equation (NLS) and (KG), for our purposes, is their Hamiltonian structure. This property guarantees some key cancellations in the *energy-estimates* that will be explained later on in this introduction.

The equation (NLS) may be indeed rewritten as follows:

$$\partial_t u = -\mathrm{i}\nabla_{\overline{u}}\mathscr{H}_{\rm NLS}(u,\overline{u}) = \mathrm{i}\big(\Delta u - V * u - p(u)\big),$$

where $\nabla_{\overline{u}} := (\nabla_{\mathrm{Re}(u)} + \mathrm{i}\nabla_{\mathrm{Im}(u)})/2$, ∇ denote the L^2-gradient, the Hamiltonian function $\mathscr{H}_{\rm NLS}$ and the nonlinearity p are

$$\begin{aligned}
&\mathscr{H}_{\rm NLS}(u,\overline{u}) := \int_{\mathbb{T}^d} |\nabla u|^2 + (V * u)\overline{u} + P(u,\nabla u)\,dx\,,\\
&P(u,\nabla u) := \frac{1}{2}\Big(\big|\nabla h(|u|^2)\big|^2 + |u|^4\Big), \quad p(u) := (\partial_{\overline{u}}P)(u,\nabla u) - \sum_{j=1}^{d}\partial_{x_j}\big(\partial_{\overline{u}_{x_j}}P\big)(u,\nabla u)\,.
\end{aligned} \tag{1.11}$$

The equation (KG) is Hamiltonian as well. Thanks to the (1.1), (1.2) we have that also the nonlinear Klein-Gordon in (KG) can be written as

$$\begin{cases}\partial_t\psi = \partial_\phi\mathscr{H}_{\rm KG}(\psi,\phi) = \phi\,,\\ \partial_t\phi = -\partial_\psi\mathscr{H}_{\rm KG}(\psi,\phi) = -\Lambda_{\rm KG}^2\psi - f(\psi) - g(\psi)\,,\end{cases} \tag{1.12}$$

where $\mathscr{H}_{\rm KG}(\psi,\phi)$ is the Hamiltonian

$$\mathscr{H}_{\rm KG}(\psi,\phi) = \int_{\mathbb{T}^d}\frac{\phi^2}{2} + \frac{(\Lambda_{\rm KG}^2\psi)\psi}{2} + F(\psi,\nabla\psi) + G(\psi,\Lambda_{\rm KG}^{\frac{1}{2}}\psi)\,dx\,. \tag{1.13}$$

We describe below our strategy in the case of the (NLS) equation. The strategy for (KG) is similar.

In [28] we prove an energy estimate, without any assumption of smallness on the initial condition, for a more general class of equations. This energy estimate, on the equation (NLS) with small initial datum, would read

$$E(t) - E(0) \lesssim \int_0^t \|u(\tau,\cdot)\|_{H^s}^2 E(\tau)\,d\tau, \tag{1.14}$$

where $E(t) \sim \|u(t,\cdot)\|_{H^s}^2$. An estimate of this kind implies, by a standard bootstrap argument, that the lifespan of the solutions is of order at least $O(\varepsilon^{-2})$, where ε is the size of the initial condition. To increase the time to $O(\varepsilon^{-4})$ one would like to show the improved inequality

$$E(t) - E(0) \lesssim \int_0^t \|u(\tau,\cdot)\|_{H^s}^4 E(\tau)\,d\tau. \tag{1.15}$$

Our main goal is to obtain such an estimate.

PARA-LINEARIZATION OF THE EQUATION (NLS). The first step is the para-linearization, *à la* Bony [10], of the equation as a system of the variables $(u,\overline{u})$, see Prop. 3.1. We rewrite (NLS) as a system of the form (compare with (3.4))

$$\partial_t U = -\mathrm{i}E\big((-\Delta + V*)U + \mathscr{A}_2(U)U + \mathscr{A}_1(U)U\big) + X_{H_4}(U) + R(U), \quad E := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \quad U := \left[\begin{smallmatrix} u \\ \overline{u} \end{smallmatrix}\right],$$

where $\mathscr{A}_2(U)$ is a 2×2 self-adjoint matrix of *para-differential operators* of order two (see (3.3), (3.2)), $\mathscr{A}_1(U)$ is a self-adjoint, diagonal matrix of para-differential operators of order one (see (3.4), (3.2)). These algebraic configuration of the matrices (in particular the fact that $\mathscr{A}_1(U)$ is diagonal) is a consequence of the Hamiltonian structure of the equation. The summand X_{H_4} is the cubic term (coming from the para-linearization of $|u|^2u$, see (3.5)) and $\|R(U)\|_{H^s}$ is bounded from above by $\|U\|_{H^s}^7$ for s large enough. Both the matrices $\mathscr{A}_2(U)$ and $\mathscr{A}_1(U)$ vanish when U goes to 0. Since we assume that the function h, appearing in (NLS), vanishes quadratically at zero, as a consequence of (3.2), we have that

$$\|\mathscr{A}_2(U)\|_{\mathscr{L}(H^s;H^{s-2})}, \|\mathscr{A}_1(U)\|_{\mathscr{L}(H^s;H^{s-1})} \lesssim \|U\|_{H^s}^6,$$

where by $\mathscr{L}(X;Y)$ we denoted the space of linear operators from X to Y. We also remark that the summand X_{H_4} is an *Hamiltonian vector field* with Hamiltonian function $H_4(u) = \int_{\mathbb{T}^d} |u|^4 dx$.

DIAGONALIZATION OF THE SECOND ORDER OPERATOR. The matrix of para-differential operators $\mathscr{A}_2(U)$ is not diagonal, therefore the first step, in order to be able to get at least the weak estimate (1.14), is to diagonalize the system at the maximum order. This is possible since, because of the smallness assumption, the operator $E(-\Delta + \mathscr{A}_2(U))$ is locally elliptic. In section 4.1.1 we introduce a new unknown $W = \Phi_{\rm NLS}(U)U$, where $\Phi_{\rm NLS}(U)$ is a parametrix built from the matrix of the eigenvectors of $E(-\Delta+\mathscr{A}_2(U))$, see (4.4), (4.2). The system in the new coordinates reads

$$\partial_t W = -\mathrm{i}E\big((-\Delta + V*)U + \mathscr{A}_2^{(1)}(U)W + \mathscr{A}_1^{(1)}(U)W\big) + X_{H_4}(W) + R^{(1)}(U),$$

where both $\mathscr{A}_2^{(1)}(U), \mathscr{A}_1^{(1)}(U)$ are diagonal, see (4.11) and where $\|R^{(1)}(U)\|_{H^s} \lesssim \|U\|_{H^s}^7$ for s large enough. We note also that the cubic vector field X_{H_4} remains the same because the map $\Phi_{\rm NLS}(U)$ is equal to the identity plus a term vanishing at order six at zero, see (4.5).

DIAGONALIZATION OF THE CUBIC VECTOR-FIELD. In the second step, in section 4.1.2, we diagonalize the cubic vector-field X_{H_4}. It is fundamental for our purposes to preserve the Hamiltonian structure of this cubic vector-field in this diagonalization procedure. In view of this we perform a (approximatively) *symplectic* change of coordinates generated from the Hamiltonian in (4.22) and (4.21) (note that this is not the case for the diagonalization at order two). Actually the simplecticity of this change of coordinates is one of the most delicate points in our paper. The entire Appendix A is devoted to this. This diagonalization is implemented in order to simplify a *low-high* frequencies analysis. More precisely we prove that the cubic vector field may be conjugated to a diagonal one modulo a smoothing remainder. The diagonal part shall cancel out in the energy estimate due to a symmetrization argument based on its Hamiltonian character. As a consequence the time of existence shall be completely determined by the smoothing reminder. Being this remainder smoothing the contribution coming from high frequencies is already "small", therefore the normal form analysis involves only the *low modes.* This will be explained later on in this introduction.

We explain the result of this diagonalization. We define a new variable $Z = \Phi_{\mathscr{B}_{\rm NLS}}(W)$, see (4.23), and we obtain the new diagonal system (compare with (4.26))

$$\partial_t Z = -\mathrm{i}E\big((-\Delta + V*)Z + \mathscr{A}_2^{(1)}(U)Z + \mathscr{A}_1^{(1)}(U)Z\big) + X_{\mathrm{H}_4}(Z) + R_5^{(2)}(U),$$

where the new vector-field $X_{\mathrm{H}_4}(Z)$ is still Hamiltonian, with Hamiltonian function defined in (4.29), and it is equal to a skew-selfadjoint and diagonal matrix of bounded para-differential operators modulo smoothing reminders, see (4.27). Here $R_5^{(2)}(U)$ satisfies the quintic estimates (4.28).

INTRODUCTION OF THE ENERGY-NORM. Once achieved the diagonalization of the system we introduce an *energy norm* which is equivalent to the Sobolev one. Assume for simplicity $s = 2n$ with n a natural number. Thanks to the smallness condition on the initial datum we prove in Section 5.1.1 that $\|(-\Delta\mathbb{1}+\mathscr{A}_2(U)+\mathscr{A}_1(U))^{s/2}f\|_{L^2}\sim\|f\|_{H^s}$ for any function f in $H^s(\mathbb{T}^d)$. Therefore by setting [1]

$$Z_n := [E(-\Delta\mathbb{1}+\mathscr{A}_2(U)+\mathscr{A}_1(U))]^{s/2}Z\,,$$

we are reduced to study the L^2 norm of the function Z_n. This has been done in Lemma 5.2. Since the system is now diagonalized, we write the scalar equation, see Lemma 5.3, solved by z_n

$$\partial_t z_n = -\mathrm{i}T_{\mathscr{L}}z_n - \mathrm{i}V * z_n - \Delta^n X^+_{\mathtt{H}_4}(Z) + R_n(U)\,,$$

where we have denoted by $T_{\mathscr{L}}$ the element on the diagonal of the self-adjoint operator $-\Delta\mathbb{1}+\mathscr{A}_2(U)+\mathscr{A}_1(U)$, see (5.1), (2.6); $X^+_{\mathtt{H}_4}(Z)$ is the first component of the Hamiltonian vector-field $X_{\mathtt{H}_4}(Z)$ and $R_n(U)$ is a bounded remainder satisfying the quintic estimate (5.12).

CANCELLATIONS AND NORMAL-FORMS. At this point, always in Lemma 5.3, we split the Hamiltonian vector-field $X^+_{\mathtt{H}_4}=X^{+,\mathrm{res}}_{\mathtt{H}_4}+X^{+,\perp}_{\mathtt{H}_4}$, where $X^{+,\mathrm{res}}_{\mathtt{H}_4}$ is the *resonant* part, see (2.48) and Definition 2.47. The first important fact, which is an effect of the Hamiltonian and Gauge preserving structure, is that the resonant term $\Delta^n X^{+,\mathrm{res}}_{\mathtt{H}_4}$ does not give any contribution to the energy estimates. This key cancellation may be interpreted as a consequence of the fact that the *super actions*

$$I_p := \sum_{j\in\mathbb{Z}^d,\,|j|=p}|\widehat{z}(j)|^2\,,\quad p\in\mathbb{N}\,,\quad Z:=\left[\begin{smallmatrix} z\\ \bar{z}\end{smallmatrix}\right],$$

where $\widehat{z}$ is defined in (2.1), are prime integrals of the resonant Hamiltonian vector field $X^{+,\mathrm{res}}_{\mathtt{H}_4}(Z)$ in the same spirit[2] of [24]. This is the content of Lemma 5.4, more specifically equation (5.16).
We are left with the study of the term $-\Delta^n X^{+,\perp}_{\mathtt{H}_4}$. In Lemma 5.3 we prove that $-\Delta^n X^{+,\perp}_{\mathtt{H}_4}=B^{(1)}_n(Z)+B^{(2)}_n(Z)$, where $B^{(1)}_n(Z)$ does not contribute to energy estimates and $B^{(2)}_n(Z)$ is smoothing, gaining one space derivative, see (5.11) and Lemma 2.5. The cancellation for $B^{(1)}_n(Z)$ is again a consequence of the Hamiltonian structure and it is proven in Lemma 5.4, more specifically equation (5.17).
Summarizing we obtain the following energy estimate (see (2.3))

$$\frac{1}{2}\frac{d}{dt}\|z_n(t)\|^2_{L^2} = \mathrm{Re}\,(-\mathrm{i}T_{\mathscr{L}}z_n, z_n)_{L^2} + \mathrm{Re}\,(-\mathrm{i}V*z_n, z_n)_{L^2} \tag{1.16}$$

$$+\,\mathrm{Re}\,(R_n(U), z_n)_{L^2} \tag{1.17}$$

$$+\,\mathrm{Re}\,(-\Delta^n X^{+,\mathrm{res}}_{\mathtt{H}_4}(Z), z_n)_{L^2} \tag{1.18}$$

$$+\,\mathrm{Re}\,(B^{(1)}_n(Z), z_n)_{L^2} \tag{1.19}$$

$$+\,\mathrm{Re}\,(B^{(2)}_n(Z), z_n)_{L^2}. \tag{1.20}$$

The r.h.s. in (1.16) equals to zero because $\mathrm{i}T_{\mathscr{L}}$ is skew-self-adjoint and the Fourier coefficients of V in (1.5) are real valued. The term (1.17) is bounded from above by $\|z_n\|^2_{L^2}\|U\|^6_{H^s}$; (1.18) equals to zero thanks to (5.16), the summand (1.19) equals to zero as well because of (5.17). Setting $E(t)=\|z_n(t)\|^2_{L^2}$, the only term which is still not good in order to obtain an estimate of the form (1.15) is the (1.20).

In order to improve the time of existence we need to reduce the size of this new term $B^{(2)}_n(Z)$ by means of *normal forms/integration by parts*. We note immediately that, thanks to all the reductions we have performed, the term $B^{(2)}_n$ presents two advantages: it is *non-resonant* and *smoothing*. Thanks to the fact that it is smoothing we shall need to perform a normal form only for the *low frequencies* of $B^{(2)}_n(Z)$. More precisely, thanks to (5.9) and (5.11), we prove in Lemma 5.8, see (5.34), that the high frequency part of this vector-field is already small, if N therein is chosen large enough inversely proportional to a power

[1] To be precise the definition of $Z_n=(z_n,\bar{z}_n)$ in 5.1.1 is slightly different than the one presented here, but they coincide modulo smoothing corrections. For simplicity of notation, and in order to avoid technicalities, in this introduction we presented it in this way.

[2] More generally, this cancellation can be viewed as a consequence of the commutation of the linear flow with the resonant part of the nonlinear perturbation which is a key of the Birkhoff normal form theory (see for instance [31]).

of the size of the initial condition. The normal form on the *non-resonant* term, restricted to the low frequencies, is performed in Proposition 5.7. Here we use the lower bound on the small divisor in (5.26) given by Proposition 5.6.

As said before the strategy for (KG) is similar except for the control of the small divisor (1.10) which implies some extra difficulties that we already talk about. Let us just describe how the paper is organized concerning (KG): In Section 3.2 we paralinearize the equation obtaining, passing to the complex variables (3.11), the system of equations of order one (3.31). In Section 4.2 we diagonalize the system: the operator of order one is treated in Prop. 4.11 and the order zero in Prop. 4.13. As done for (NLS) in the diagonalization of the operator of order zero we preserve its Hamiltonian structure. The energy estimates are given in Section 5.2. The non degeneracy of the linear frequencies is studied in Appendix B.

ACKNOWLEDGEMENTS. We would like to warmly thank prof. Fabio Pusateri for the inspiring discussions.

2. PRELIMINARIES

We denote by $H^s(\mathbb{T}^d;\mathbb{C})$ (respectively $H^s(\mathbb{T}^d;\mathbb{C}^2)$) the usual Sobolev space of functions $\mathbb{T}^d \ni x \mapsto u(x) \in \mathbb{C}$ (resp. $\mathbb{C}^2$). We expand a function $u(x)$, $x \in \mathbb{T}^d$, in Fourier series as

$$u(x) = \frac{1}{(2\pi)^{d/2}} \sum_{n\in\mathbb{Z}^d} \widehat{u}(n) e^{\mathrm{i} n\cdot x}, \qquad \widehat{u}(n) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} u(x) e^{-\mathrm{i} n\cdot x}\, dx. \tag{2.1}$$

We set $\langle j\rangle := \sqrt{1+|j|^2}$ for $j \in \mathbb{Z}^d$. We endow $H^s(\mathbb{T}^d;\mathbb{C})$ with the norm

$$\|u(\cdot)\|_{H^s}^2 := \sum_{j\in\mathbb{Z}^d} \langle j\rangle^{2s} |u_j|^2. \tag{2.2}$$

For $U = (u_1, u_2) \in H^s(\mathbb{T}^d;\mathbb{C}^2)$ we set $\|U\|_{H^s} = \|u_1\|_{H^s} + \|u_2\|_{H^s}$. Moreover, for $r \in \mathbb{R}^+$, we denote by $B_r(H^s(\mathbb{T}^d;\mathbb{C}))$ (resp. $B_r(H^s(\mathbb{T}^d;\mathbb{C}^2))$) the ball of $H^s(\mathbb{T}^d;\mathbb{C})$ (resp. $H^s(\mathbb{T}^d;\mathbb{C}^2)$) with radius r centered at the origin. We shall also write the norm in (2.2) as $\|u\|_{H^s}^2 = (\langle D\rangle^s u, \langle D\rangle^s u)_{L^2}$, where $\langle D\rangle e^{\mathrm{i} j\cdot x} = \langle j\rangle e^{\mathrm{i} j\cdot x}$, for any $j \in \mathbb{Z}^d$, and $(\cdot,\cdot)_{L^2}$ denotes the standard complex L^2-scalar product

$$(u, v)_{L^2} := \int_{\mathbb{T}^d} u \cdot \overline{v}\, dx, \qquad \forall\, u, v \in L^2(\mathbb{T}^d;\mathbb{C}). \tag{2.3}$$

Notation. We shall use the notation $A \lesssim B$ to denote $A \le CB$ where C is a positive constant depending on parameters fixed once for all, for instance d and s. We will emphasize by writing $\lesssim_q$ when the constant C depends on some other parameter q.

Basic Paradifferential calculus. We follow the notation of [28]. We introduce the symbols we shall use in this paper. We shall consider symbols $\mathbb{T}^d \times \mathbb{R}^d \ni (x,\xi) \to a(x,\xi)$ in the spaces $\mathcal{N}_s^m$, $m, s \in \mathbb{R}$, defined by the norms

$$|a|_{\mathcal{N}_s^m} := \sup_{|\alpha|+|\beta|\le s} \sup_{|\xi|>1/2} \langle\xi\rangle^{-m+|\beta|} \|\partial_\xi^\beta \partial_x^\alpha a(x,\xi)\|_{L^\infty}. \tag{2.4}$$

The constant $m \in \mathbb{R}$ indicates the *order* of the symbols, while s denotes its differentiability. Let $0 < \epsilon < 1/2$ and consider a smooth function $\chi : \mathbb{R} \to [0,1]$

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 5/4 \\ 0 & \text{if } |\xi| \ge 8/5 \end{cases} \qquad \text{and define} \qquad \chi_\epsilon(\xi) := \chi(|\xi|/\epsilon). \tag{2.5}$$

For a symbol $a(x,\xi)$ in $\mathcal{N}_s^m$ we define its (Weyl) quantization as

$$T_a h := \frac{1}{(2\pi)^d} \sum_{j\in\mathbb{Z}^d} e^{\mathrm{i} j\cdot x} \sum_{k\in\mathbb{Z}^d} \chi_\epsilon\Big(\frac{|j-k|}{\langle j+k\rangle}\Big) \widehat{a}\Big(j-k, \frac{j+k}{2}\Big) \widehat{h}(k) \tag{2.6}$$

where $\widehat{a}(\eta,\xi)$ denotes the Fourier transform of $a(x,\xi)$ in the variable $x\in\mathbb{T}^d$. Thanks to the choice of χ_ϵ in (2.5) we have that, if $j=0$ then $\chi_\epsilon(|j-k|/\langle j+k\rangle)\equiv 0$ for any $k\in\mathbb{Z}^d\setminus\{0\}$. Moreover, the function $T_a h-\widehat{(T_a h)}(0)$ depends only on the values of $a(x,\xi)$ for $|\xi|\geq 1$. In view of this fact, if $a(x,\xi)=b(x,\xi)$ for $|\xi|>1/2$ then T_a-T_b is a finite rank operator. Therefore, without loss of generality, we write $a=b$ if $a(x,\xi)=b(x,\xi)$ for $|\xi|>1/2$. Moreover the definition of the operator T_a is independent of the choice of the cut-off function χ_ϵ up to smoothing terms, see, for instance, Lemma 2.1 in [28].

Notation. Given a symbol $a(x,\xi)$ we shall also write

$$T_a[\cdot]:=Op^{\scriptscriptstyle BW}(a(x,\xi))[\cdot]\,, \tag{2.7}$$

to denote the associated para-differential operator.
We now collects some fundamental properties of para-differential operators. For details we refer the reader to section 2 in [28].

Lemma 2.1. *(Lemma 2.1 in [28]) The following holds.*
(i) Let $m_1,m_2\in\mathbb{R}$, $s>d/2$ and $a\in\mathcal{N}_s^{m_1}$, $b\in\mathcal{N}_s^{m_2}$. One has

$$|ab|_{\mathcal{N}_s^{m_1+m_2}}+|\{a,b\}|_{\mathcal{N}_{s-1}^{m_1+m_2-1}}+\lesssim |a|_{\mathcal{N}_s^{m_1}}|b|_{\mathcal{N}_s^{m_2}} \tag{2.8}$$

where

$$\{a,b\}:=\sum_{j=1}^{d}\Big((\partial_{\xi_j}a)(\partial_{x_j}b)-(\partial_{x_j}a)(\partial_{\xi_j}b)\Big). \tag{2.9}$$

(ii) Let $\mathbb{N}\ni s_0>d$, $m\in\mathbb{R}$ and $a\in\mathcal{N}_{s_0}^m$. Then, for any $s\in\mathbb{R}$, one has

$$\|T_a h\|_{H^{s-m}}\lesssim |a|_{\mathcal{N}_{s_0}^m}\|h\|_{H^s}\,,\qquad \forall h\in H^s(\mathbb{T}^d;\mathbb{C})\,. \tag{2.10}$$

Proposition 2.2. *(Prop. 2.4 in [28]). Fix $\mathbb{N}\ni s_0>d$ and $m_1,m_2\in\mathbb{R}$, then we have the following. For $a\in\mathcal{N}_{s_0+2}^{m_1}$ and $b\in\mathcal{N}_{s_0+2}^{m_2}$ we have (recall (2.9))*

$$T_a\circ T_b=T_{ab}+R_1(a,b)\,,\quad T_a\circ T_b=T_{ab}+\frac{1}{2\mathrm{i}}T_{\{a,b\}}+R_2(a,b)\,, \tag{2.11}$$

where $R_j(a,b)$ are remainders satisfying, for any $s\in\mathbb{R}$,

$$\|R_j(a,b)h\|_{H^{s-m_1-m_2+j}}\lesssim \|h\|_{H^s}|a|_{\mathcal{N}_{s_0+j}^{m_1}}|b|_{\mathcal{N}_{s_0+j}^{m_2}}\,. \tag{2.12}$$

Moreover, if $a,b\in H^{\rho+s_0}(\mathbb{T}^d;\mathbb{C})$ are functions (independent of $\xi\in\mathbb{R}^n$) then, $\forall s\in\mathbb{R}$,

$$\|(T_aT_b-T_{ab})h\|_{H^{s+\rho}}\lesssim \|h\|_{H^s}\|a\|_{H^{\rho+s_0}}\|b\|_{H^{\rho+s_0}}\,. \tag{2.13}$$

Lemma 2.3. *Fix $s_0>d/2$ and let $f,g,h\in H^s(\mathbb{T};\mathbb{C})$ for $s\geq s_0$. Then*

$$fgh=T_{fg}h+T_{gh}f+T_{fh}g+\mathscr{R}(f,g,h)\,, \tag{2.14}$$

where

$$\begin{aligned}\widehat{\mathscr{R}(f,g,h)}(\xi)&=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}a(\xi,\eta,\zeta)\widehat{f}(\xi-\eta-\zeta)\widehat{g}(\eta)\widehat{h}(\zeta)\,,\\ |a(\xi,\eta,\zeta)|&\lesssim_\rho\frac{\max_2(|\xi-\eta-\zeta|,|\eta,||\zeta|)^\rho}{\max_1(|\xi-\eta-\zeta|,|\eta|,|\zeta|)^\rho}\,,\ \forall\rho\geq 0.\end{aligned} \tag{2.15}$$

Proof. We start by proving the following claim: the term

$$T_{fg}h-\sum_{\xi\in\mathbb{Z}^d}e^{\mathrm{i}\xi\cdot x}\sum_{\eta,\zeta\in\mathbb{Z}^d}\chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\Big)\widehat{f}(\xi-\eta-\zeta)\widehat{g}(\eta)\widehat{h}(\zeta)$$

is a remainder of the form (2.15). By (2.6) this is actually true with coefficients $a(\xi,\eta,\zeta)$ of the form

$$a(\xi,\eta,\zeta):=\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)-\chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\Big).$$

In order to prove this, we consider the following partition of the unity:

$$1 = \Theta_\epsilon(\xi,\eta,\zeta) + \chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|+|\zeta|}{\langle\eta\rangle}\Big) + \chi_\epsilon\Big(\tfrac{|\eta|+|\zeta|}{\langle\xi-\eta-\zeta\rangle}\Big) + \chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\Big). \tag{2.16}$$

Then we can write

$$\begin{aligned} a(\xi,\eta,\zeta) &= \Big(\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)-1\Big)\chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\Big) + \chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|+|\zeta|}{\langle\eta\rangle}\Big) \\ &\quad + \chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\tfrac{|\eta|+|\zeta|}{\langle\xi-\eta-\zeta\rangle}\Big) + \chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\Theta_\epsilon(\xi,\eta,\zeta). \end{aligned}$$

Using (2.5) one can prove that each summand in the r.h.s. of the equation above is non-zero only if $\max_2(|\xi-\eta-\zeta|,|\eta,||\zeta|) \sim \max_1(|\xi-\eta-\zeta|,|\eta,||\zeta|)$. This implies that each summand defines a smoothing remainder as in (2.15). A similar property holds also for $T_{gh}f$ and $T_{fh}g$. At this point we write

$$\begin{aligned} fgh = \sum_{\xi\in\mathbb{Z}^d} e^{i\xi\cdot x} \sum_{\eta,\zeta\in\mathbb{Z}^d} &\Big[\Theta_\epsilon(\xi,\eta,\zeta) + \chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|+|\zeta|}{\langle\eta\rangle}\Big) \\ &+ \chi_\epsilon\Big(\tfrac{|\eta|+|\zeta|}{\langle\xi-\eta-\zeta\rangle}\Big) + \chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\Big)\Big]\widehat{f}(\xi-\eta-\zeta)\widehat{g}(\eta)\widehat{h}(\zeta). \end{aligned}$$

One concludes by using the claim at the beginning of the proof. □

Matrices of symbols and operators. Let us consider the subspace $\mathcal{U}$ defined as

$$\mathcal{U} := \big\{(u^+,u^-)\in L^2(\mathbb{T}^d;\mathbb{C})\times L^2(\mathbb{T}^d;\mathbb{C}) \,:\, u^+ = \overline{u^-}\big\}. \tag{2.17}$$

Along the paper we shall deal with matrices of linear operators acting on $H^s(\mathbb{T}^d;\mathbb{C}^2)$ preserving the subspace $\mathcal{U}$. Consider two operators R_1, R_2 acting on $C^\infty(\mathbb{T}^d;\mathbb{C})$. We define the operator $\mathfrak{F}$ acting on $C^\infty(\mathbb{T}^d;\mathbb{C}^2)$ as

$$\mathfrak{F} := \big[\begin{smallmatrix} R_1 & R_2 \\ \overline{R_2} & \overline{R_1}\end{smallmatrix}\big], \tag{2.18}$$

where the linear operators $\overline{R_i}[\cdot]$, $i=1,2$ are defined by the relation $\overline{R_i}[v] := \overline{R_i[\overline{v}]}$. We say that an operator of the form (2.18) is *real-to-real*. It is easy to note that real-to-real operators preserves $\mathcal{U}$ in (2.17). Consider now a symbol $a(x,\xi)$ of order m and set $A := T_a$. Using (2.6) one can check that

$$\overline{A}[h] = \overline{A[\overline{h}]}, \qquad \Rightarrow \quad \overline{A} = T_{\tilde{a}}, \qquad \tilde{a}(x,\xi) = \overline{a(x,-\xi)}; \tag{2.19}$$

$$(\textbf{Ajdoint})\ \ (Ah,v)_{L^2} = (h, A^* v)_{L^2}, \qquad \Rightarrow \quad A^* = T_{\overline{a}}. \tag{2.20}$$

By (2.20) we deduce that the operator A is self-adjoint with respect to the scalar product (2.3) if and only if the symbol $a(x,\xi)$ is real valued. We need the following definition. Consider two symbols $a,b\in\mathcal{N}_s^m$ and the matrix

$$A := A(x,\xi) := \begin{pmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,-\xi)} & \overline{a(x,-\xi)}\end{pmatrix}.$$

Define the operator (recall (2.7))

$$M := Op^{\mathrm{BW}}(A(x,\xi)) := \begin{pmatrix} Op^{\mathrm{BW}}(a(x,\xi)) & Op^{\mathrm{BW}}(b(x,\xi)) \\ Op^{\mathrm{BW}}(\overline{b(x,-\xi)}) & Op^{\mathrm{BW}}(\overline{a(x,-\xi)})\end{pmatrix}. \tag{2.21}$$

The matrix of paradifferential operators defined above have the following properties:

- *Reality*: by (2.19) we have that the operator M in (2.21) has the form (2.18), hence it is real-to-real;
- *Self-adjointeness*: using (2.20) the operator M in (2.21) is self-adjoint with respect to the scalar product on (2.17)

$$(U,V)_{L^2} := \int_{\mathbb{T}^d} U\cdot\overline{V}\,dx, \qquad U = \big[\begin{smallmatrix} u \\ \overline{u}\end{smallmatrix}\big],\ V = \big[\begin{smallmatrix} v \\ \overline{v}\end{smallmatrix}\big]. \tag{2.22}$$

  if and only if

$$a(x,\xi) = \overline{a(x,\xi)}, \qquad b(x,-\xi) = b(x,\xi). \tag{2.23}$$

Non-homogeneous symbols. In this paper we deal with symbols satisfying (2.4) which depends nonlinearly on an extra function $u(t,x)$ (which in the application will be a solution either of (NLS) or a solution of (KG)). We are interested in providing estimates of the semi-norms (2.4) in terms of the Sobolev norms of the function u.

Consider a function $F(y_0,y_1,\dots,y_d)$ in $C^{\infty}(\mathbb{C}^{d+1};\mathbb{R})$ in the *real* sense, i.e. F is C^{∞} as function of $\mathrm{Re}(y_i)$, $\mathrm{Im}(y_i)$. Assume that F has a zero of order at least $p+2\in\mathbb{N}$ at the origin. Consider a symbol $f(\xi)$, independent of $x\in\mathbb{T}^d$, such that $|f|_{\mathcal{N}_s^m}\le C<+\infty$, for some constant C. Let us define the symbol

$$a(x,\xi) := \big(\partial_{z_j^{\alpha} z_k^{\beta}} F\big)(u,\nabla u) f(\xi)\,,\quad z_j^{\alpha} := \partial_{x_j}^{\alpha} u^{\sigma}, z_k^{\beta} := \partial_{x_k}^{\beta} u^{\sigma'} \tag{2.24}$$

for some $j,k=1,\dots,d$, $\alpha,\beta\in\{0,1\}$ and $\sigma,\sigma'\in\{\pm\}$ where we used the notation $u^+=u$ and $u^-=\overline{u}$. The following lemma is proved in section 2 of [28].

Lemma 2.4. *Fix $s_0>d/2$. For $u\in B_R(H^{s+s_0+1}(\mathbb{T}^d;\mathbb{C}))$ with $0<R<1$, we have*

$$|a|_{\mathcal{N}_s^m}\lesssim \|u\|^p_{H^{s+s_0+1}}\,,$$

where a is the symbol in (2.24). Moreover, for any $h\in H^{s+s_0+1}$, the map $h\to(\partial_u a)(u;x,\xi)h$ extends as a linear form on H^{s+s_0+1} and satisfies

$$|(\partial_u a)h|_{\mathcal{N}_s^m}\lesssim \|h\|_{H^{s+s_0+1}}\|u\|^p_{H^{s+s_0+1}}\,.$$

The same holds for $\partial_{\overline{u}}a$. Moreover if the symbol a does not depend on ∇u, then the same results are true with $s_0+1\rightsquigarrow s_0$.

Trilinear operators. Along the paper we shall deal with trilinear operators on the Sobolev spaces. We shall adopt a combination of notation introduced in [6] and [34]. In particular we are interested in studying properties of operators of the form

$$\begin{aligned} &Q=Q[u_1,u_2,u_3]:(C^{\infty}(\mathbb{T}^d;\mathbb{C}))^3\to C^{\infty}(\mathbb{T}^d;\mathbb{C})\,,\\ &\widehat{Q}(\xi)=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} q(\xi,\eta,\zeta)\,\widehat{u}_1(\xi-\eta-\zeta)\,\widehat{u}_2(\eta)\,\widehat{u}_3(\zeta)\,,\qquad \forall\,\xi\in\mathbb{Z}^d\,, \end{aligned}\tag{2.25}$$

where the coefficients $q(\xi,\eta,\zeta)\in\mathbb{C}$ for any $\xi,\eta,\zeta\in\mathbb{Z}^d$. We introduce the following notation: given $j_1,\dots,j_p\in\mathbb{R}^+$, $p\ge 2$ we define

$$\max{}_i\{j_1,\dots,j_p\} = i\text{–th largest among } j_1,\dots,j_p\,. \tag{2.26}$$

We now prove that, under certain conditions on the coefficients, the operators of the form (2.25) extend as continuous maps on the Sobolev spaces.

Lemma 2.5. *Let $\mu\ge 0$ and $m\in\mathbb{R}$. Assume that for any $\xi,\eta,\zeta\in\mathbb{Z}^d$ one has*

$$|q(\xi,\eta,\zeta)|\lesssim \frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^{\mu}}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^{m}}\,. \tag{2.27}$$

Then, for $s\ge s_0>d/2+\mu$, the map Q in (2.25) with coefficients satisfying (2.27) extends as a continuous map form $(H^s(\mathbb{T}^d;\mathbb{C}))^3$ to $H^{s+m}(\mathbb{T}^d;\mathbb{C})$. Moreover one has

$$\|Q(u_1,u_2,u_3)\|_{H^{s+m}}\lesssim \sum_{i=1}^{3}\|u_i\|_{H^s}\prod_{i\neq k}\|u_k\|_{H^{s_0}}\,. \tag{2.28}$$

Proof. By (2.2) we have

$$
\begin{aligned}
\|Q(u_1,u_2,u_3)\|^2_{H^{s+m}} &\le \sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{2(s+m)}\Big(\sum_{\eta,\zeta\in\mathbb{Z}^d}|q(\xi,\eta,\zeta)||\widehat{u}_1(\xi-\eta-\zeta)||\widehat{u}_2(\eta)||\widehat{u}_3(\zeta)|\Big)^2\\
&\overset{(2.27)}{\lesssim} \sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\eta,\zeta\in\mathbb{Z}^d}\langle\xi\rangle^{s}\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^{\mu}|\widehat{u}_1(\xi-\eta-\zeta)||\widehat{u}_2(\eta)||\widehat{u}_3(\zeta)|\Big)^2\\
&:= I+II+III,
\end{aligned}
\tag{2.29}
$$

where I,II,III are the terms in (2.29) which are supported respectively on indexes such that $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}=\langle\xi-\eta-\zeta\rangle$, $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}=\langle\eta\rangle$ and $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}=\langle\zeta\rangle$. Consider for instance the term III. By using the Young inequality for sequences we deduce

$$
III\lesssim \|(\langle p\rangle^{\mu}\widehat{u}_1(p))*(\langle\eta\rangle^{\mu}\widehat{u}_2(\eta))*(\langle\zeta\rangle^{s}\widehat{u}_3(\zeta))\|_{\ell^2}\lesssim \|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s},
$$

which is the (2.28). The bounds of I and II are similar. □

In the following lemma we shall prove that a class of "para-differential" trilinear operators, having some decay on the coefficients, satisfies the hypothesis of the previous lemma.

Lemma 2.6. *Let $\mu\ge 0$ and $m\in\mathbb{R}$. Consider a trilinear map Q as in* (2.25) *with coefficients satisfying*

$$
q(\xi,\eta,\zeta)=f(\xi,\eta,\zeta)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big),\qquad |f(\xi,\eta,\zeta)|\lesssim\frac{|\xi-\zeta|^{\mu}}{\langle\zeta\rangle^{m}}
\tag{2.30}
$$

for any $\xi,\eta,\zeta\in\mathbb{Z}^d$ and $0<\epsilon\ll 1$. Then the coefficients $q(\xi,\eta,\zeta)$ satisfy the (2.27) *with $\mu\rightsquigarrow\mu+m$.*

Proof. First of all we write $q(\xi,\eta,\zeta)=q_1(\xi,\eta,\zeta)+q_2(\xi,\eta,\zeta)$ with

$$
q_1(\xi,\eta,\zeta)=f(\xi,\eta,\zeta)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\frac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\Big),
\tag{2.31}
$$

$$
q_2(\xi,\eta,\zeta)=f(\xi,\eta,\zeta)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\Big[\chi_\epsilon\Big(\frac{|\xi-\eta-\zeta|+|\zeta|}{\langle\eta\rangle}\Big)+\chi_\epsilon\Big(\frac{|\eta|+|\zeta|}{\langle\xi-\eta-\zeta\rangle}\Big)+\Theta_\epsilon(\xi,\eta,\zeta)\Big],
\tag{2.32}
$$

where $\Theta_\epsilon(\xi,\eta,\zeta)$ is defined in (2.16). Recalling (2.5) one can check that if $q_1(\xi,\eta,\zeta)\neq 0$ then $|\xi-\eta-\zeta|+|\eta|\ll|\zeta|\sim|\xi|$. Together with the bound on $f(\xi,\eta,\zeta)$ in (2.30) we deduce that the coefficients in (2.31) satisfy the (2.27). The coefficients in (2.32) satisfy the (2.27) because of the support of the cut off function in (2.5). □

Hamiltonian formalism in complex variables. Given a Hamiltonian function $H: H^1(\mathbb{T}^d;\mathbb{C}^2)\to\mathbb{R}$, its Hamiltonian vector field has the form

$$
X_H(U):=-\mathrm{i}J\nabla H(U)=-\mathrm{i}\begin{pmatrix}\nabla_{\overline{u}}H(U)\\ -\nabla_u H(U)\end{pmatrix},\quad J=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\quad U=\begin{bmatrix}u\\ \overline{u}\end{bmatrix}.
\tag{2.33}
$$

Indeed one has

$$
dH(U)[V]=-\Omega(X_H(U),V),\qquad \forall U=\begin{bmatrix}u\\ \overline{u}\end{bmatrix},\ V=\begin{bmatrix}v\\ \overline{v}\end{bmatrix},
\tag{2.34}
$$

where Ω is the non-degenerate symplectic form

$$
\Omega(U,V)=-\int_{\mathbb{T}^d}U\cdot\mathrm{i}JV\,dx=-\int_{\mathbb{T}^d}\mathrm{i}(u\overline{v}-\overline{u}v)\,dx.
\tag{2.35}
$$

The Poisson brackets between two Hamiltonians H,G are defined as

$$
\{G,H\}:=\Omega(X_G,X_H)\overset{(2.35)}{=}-\int\mathrm{i}J\nabla G\cdot\nabla H\,dx=-\mathrm{i}\int\nabla_u H\nabla_{\overline{u}}G-\nabla_{\overline{u}}H\nabla_u G\,dx.
\tag{2.36}
$$

The nonlinear commutator between two Hamiltonian vector fields is given by

$$
[X_G,X_H](U)=dX_G(U)\big[X_H(U)\big]-dX_H(U)\big[X_G(U)\big]=-X_{\{G,H\}}(U).
\tag{2.37}
$$

Hamiltonian formalism in real variables. Given a Hamiltonian function $H_{\mathbb{R}} : H^1(\mathbb{T}^d;\mathbb{R}^2) \to \mathbb{R}$, its hamiltonian vector field has the form

$$X_{H_{\mathbb{R}}}(\psi,\phi) := J\nabla H_{\mathbb{R}}(\psi,\phi) = \begin{pmatrix} \nabla_\phi H_{\mathbb{R}}(\psi,\phi) \\ -\nabla_\psi H_{\mathbb{R}}(\psi,\phi) \end{pmatrix}, \tag{2.38}$$

where J is in (2.33). Indeed one has

$$dH_{\mathbb{R}}(\psi,\phi)[h] = -\tilde{\Omega}(X_{H_{\mathbb{R}}}(\psi,\phi),h), \qquad \forall \left[\begin{smallmatrix} \psi \\ \phi \end{smallmatrix}\right],\ h = \left[\begin{smallmatrix} \widehat{\psi} \\ \widehat{\phi} \end{smallmatrix}\right], \tag{2.39}$$

where $\tilde{\Omega}$ is the non-degenerate symplectic form

$$\widetilde{\Omega}(\left[\begin{smallmatrix} \psi_1 \\ \phi_1 \end{smallmatrix}\right],\left[\begin{smallmatrix} \psi_2 \\ \phi_2 \end{smallmatrix}\right]) := \int_{\mathbb{T}^d} \left[\begin{smallmatrix} \psi_1 \\ \phi_1 \end{smallmatrix}\right]\cdot J^{-1}\left[\begin{smallmatrix} \psi_2 \\ \phi_2 \end{smallmatrix}\right] dx = \int_{\mathbb{T}^d} -(\psi_1\phi_2 - \phi_1\psi_2)\,dx, \tag{2.40}$$

We introduce the complex symplectic variables

$$\begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \mathscr{C}\begin{pmatrix} \psi \\ \phi \end{pmatrix} := \frac{1}{\sqrt{2}}\begin{pmatrix} \Lambda_{\rm KG}^{\frac12}\psi + \mathrm{i}\Lambda_{\rm KG}^{-\frac12}\phi \\ \Lambda_{\rm KG}^{\frac12}\psi - \mathrm{i}\Lambda_{\rm KG}^{-\frac12}\phi \end{pmatrix}, \qquad \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \mathscr{C}^{-1}\begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \Lambda_{\rm KG}^{-\frac12}(u+\overline{u}) \\ -\mathrm{i}\Lambda_{\rm KG}^{\frac12}(u-\overline{u}) \end{pmatrix}, \tag{2.41}$$

where $\Lambda_{\rm KG}$ is in (1.3). The symplectic form in (2.40) transforms, for $U = \left[\begin{smallmatrix} u \\ \overline{u} \end{smallmatrix}\right]$, $V = \left[\begin{smallmatrix} v \\ \overline{v} \end{smallmatrix}\right]$, into $\Omega(U,V)$ where Ω is in (2.35). In these coordinates the vector field $X_{H_{\mathbb{R}}}$ in (2.38) assumes the form X_H as in (2.33) with $H := H_{\mathbb{R}} \circ \mathscr{C}^{-1}$.

We now study some algebraic properties enjoyed by the Hamiltonian functions previously defined. Let us consider a homogeneous Hamiltonian $H : H^1(\mathbb{T}^d;\mathbb{C}^2) \to \mathbb{R}$ of degree four of the form

$$H(U) = (2\pi)^{-d} \sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} \mathtt{h}_4(\xi,\eta,\zeta)\,\widehat{u}(\xi-\eta-\zeta)\widehat{\overline{u}}(\eta)\widehat{u}(\zeta)\widehat{\overline{u}}(-\xi), \qquad U = \left[\begin{smallmatrix} u \\ \overline{u} \end{smallmatrix}\right], \tag{2.42}$$

for some coefficients $\mathtt{h}_4(\xi,\eta,\zeta) \in \mathbb{C}$ such that

$$\begin{aligned} \mathtt{h}_4(\xi,\eta,\zeta) &= \mathtt{h}_4(-\eta,-\xi,\zeta) = \mathtt{h}_4(\xi,\eta,\xi-\eta-\zeta), \\ \mathtt{h}_4(\xi,\eta,\zeta) &= \overline{\mathtt{h}_4(\zeta,\eta+\zeta-\xi,\xi)}, \qquad \forall\, \xi,\eta,\zeta \in \mathbb{Z}^d. \end{aligned} \tag{2.43}$$

By (2.43) one can check that the Hamiltonian H is real valued and symmetric in its entries. Recalling (2.33) we have that its Hamiltonian vector field can be written as

$$X_H(U) = \begin{pmatrix} -\mathrm{i}\nabla_{\overline{u}} H(U) \\ \mathrm{i}\nabla_u H(U) \end{pmatrix} = \begin{pmatrix} X_H^+(U) \\ \overline{X_H^+(U)} \end{pmatrix} \tag{2.44}$$

$$\widehat{X_H^+(U)}(\xi) = (2\pi)^{-d} \sum_{\eta,\zeta\in\mathbb{Z}^d} f(\xi,\eta,\zeta)\,\widehat{u}(\xi-\eta-\zeta)\widehat{\overline{u}}(\eta)\widehat{u}(\zeta), \tag{2.45}$$

where the coefficients $f(\xi,\eta,\zeta)$ have the form

$$f(\xi,\eta,\zeta) = 2\mathrm{i}\mathtt{h}_4(\xi,\eta,\zeta), \qquad \xi,\eta,\zeta \in \mathbb{Z}^d. \tag{2.46}$$

We need the following definition.

Definition 2.7. (Resonant set). *We define the following set of* resonant *indexes:*

$$\begin{aligned} \mathscr{R} := \big\{(\xi,\eta,\zeta) \in \mathbb{Z}^{3d} : |\xi| = |\zeta|, |\eta| = |\xi-\eta-\zeta|\big\} \\ \cup \big\{(\xi,\eta,\zeta) \in \mathbb{Z}^{3d} : |\xi| = |\xi-\eta-\zeta|, |\eta| = |\zeta|\big\}. \end{aligned} \tag{2.47}$$

Consider the vector field in (2.45). *We define the field $X_H^{+,\rm res}(U)$ by*

$$\widehat{X_H^{+,\rm res}}(\xi) = (2\pi)^{-d} \sum_{\eta,\zeta\in\mathbb{Z}^d} f^{(\rm res)}(\xi,\eta,\zeta)\,\widehat{u}(\xi-\eta-\zeta)\widehat{\overline{u}}(\eta)\widehat{u}(\zeta), \tag{2.48}$$

where

$$f^{(\rm res)}(\xi,\eta,\zeta) := f(\xi,\eta,\zeta)\mathtt{1}_{\mathscr{R}}(\xi,\eta,\zeta), \tag{2.49}$$

where $\mathtt{1}_{\mathscr{R}}$ is the characteristic function of the set $\mathscr{R}$.

In the next lemma we prove a fundamental cancellation.

Lemma 2.8. *For $n \geq 0$ one has (recall* (2.2)*)*

$$\mathrm{Re}(\langle D\rangle^n X_H^{+,\mathrm{res}}(U), \langle D\rangle^n u)_{L^2} \equiv 0\,. \tag{2.50}$$

Proof. Using (2.47)-(2.49) one can check that

$$\widehat{X_H^{+,\mathrm{res}}}(\xi) = (2\pi)^{-d} \sum_{(\eta,\zeta)\in\mathcal{R}(\xi)} \mathcal{F}(\xi,\eta,\zeta)\,\hat{u}(\xi-\eta-\zeta)\,\hat{\overline{u}}(\eta)\,\hat{u}(\zeta)\,,$$

with $\mathcal{R}(\xi) := \{(\eta,\zeta)\in\mathbb{Z}^{2d} : |\xi| = |\zeta|, |\eta| = |\xi-\eta-\zeta|\}$, for $\xi\in\mathbb{Z}^d$, and

$$\mathcal{F}(\xi,\eta,\zeta) := f(\xi,\eta,\zeta) + f(\xi,\eta,\xi-\eta-\zeta)\,. \tag{2.51}$$

By an explicit computation we have

$$\begin{aligned}
&\mathrm{Re}(D^s X_H^{+,\mathrm{res}}(U), D^s u)_{L^2} = \\
&\quad = (2\pi)^{-d} \sum_{\xi\in\mathbb{Z}^d,(\eta,\zeta)\in\mathcal{R}(\xi)} |\xi|^{2s}\Big[\mathcal{F}(\xi,\eta,\zeta) + \overline{\mathcal{F}(\zeta,\zeta+\eta-\xi,\xi)}\Big]\hat{u}(\xi-\eta-\zeta)\hat{\overline{u}}(\eta)\hat{u}(\zeta)\hat{\overline{u}}(-\xi)\,.
\end{aligned}$$

By (2.51), (2.46) and using the symmetries (2.43) we have $\mathcal{F}(\xi,\eta,\zeta) + \overline{\mathcal{F}(\zeta,\zeta+\eta-\xi,\xi)} = 0$. □

Remark 2.9. *We remark that along the paper we shall deal with general Hamiltonian functions of the form*

$$H(W) = (2\pi)^{-d} \sum_{\substack{\sigma_1,\sigma_2,\sigma_3,\sigma_4\in\{\pm\}\\ \xi,\eta,\zeta\in\mathbb{Z}^d}} \mathtt{h}^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(\xi,\eta,\zeta)\,\widehat{u^{\sigma_1}}(\xi-\eta-\zeta)\,\widehat{u^{\sigma_2}}(\eta)\,\widehat{u^{\sigma_3}}(\zeta)\,\widehat{u^{\sigma_4}}(-\xi)\,,$$

where we used the notation

$$\widehat{u^{\sigma}}(\xi) = \hat{u}(\xi)\,,\ \text{if } \sigma = +\,, \quad \text{and} \quad \widehat{u^{\sigma}}(\xi) = \hat{\overline{u}}(\xi)\,,\ \text{if } \sigma = -\,. \tag{2.52}$$

However, by the definition of the resonant set (2.47)*, we can note that the resonant vector field has still the form* (2.48) *and it depends only on the monomials in the Hamiltonian $H(U)$ which are* gauge *invariant, i.e. of the form* (2.42).

3. Para-differential formulation of the problems

In this section we rewrite the equations in a para-differential form by means of the para-linearization formula (*à la* Bony see [10]). In subsection 3.1 we deal with the problem (NLS) and in the 3.2 we deal with (KG).

3.1. **Para-linearization of the NLS.** In the following proposition we para-linearize (NLS), with respect to the variables $(u,\overline{u})$. We shall use the following notation throughout the rest of the paper

$$U := \begin{bmatrix} u \\ \overline{u}\end{bmatrix}, \quad E := \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}, \quad \mathbb{1} := \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}, \quad \mathrm{diag}(b) := b\mathbb{1}\,,\ \ b\in\mathbb{C}\,. \tag{3.1}$$

Define the following *real* symbols

$$\begin{aligned}
&a_2(x) := \big[h'(|u|^2)\big]^2|u|^2, \quad b_2(x) := \big[h'(|u|^2)\big]^2 u^2, \\
&\vec{a}_1(x)\cdot\xi := \big[h'(|u|^2)\big]^2 \sum_{j=1}^d \mathrm{Im}\,(u\overline{u}_{x_j})\xi_j\,, \quad \xi = (\xi_1,\ldots,\xi_d)\,.
\end{aligned} \tag{3.2}$$

We define also the matrix of functions

$$A_2(x) := A_2(U;x) := \begin{bmatrix} a_2(U;x) & b_2(U;x) \\ \overline{b_2(U;x)} & a_2(U;x)\end{bmatrix} = \begin{bmatrix} a_2(x) & b_2(x) \\ \overline{b_2(x)} & a_2(x)\end{bmatrix} \tag{3.3}$$

with $a_2(x)$ and $b_2(x)$ defined in (3.2). We have the following.

Proposition 3.1. (Paralinearization of NLS). *The equation* (NLS) *is equivalent to the following system:*

$$\dot U = -\mathrm{i} E Op^{\mathrm{BW}}\big((\mathbb{1}+A_2(x))|\xi|^2\big)U - \mathrm{i} E V * U - \mathrm{i} Op^{\mathrm{BW}}\big(\mathrm{diag}(\vec a_1(x)\cdot\xi)\big)U + X_{\mathcal{H}^{(4)}_{\mathrm{NLS}}}(U) + R(U)\,, \tag{3.4}$$

where V is the convolution potential in (1.5)*, the matrix $A_2(x)$ is the one in* (3.3)*, the symbol $\vec a_1(x)\cdot\xi$ is in* (3.2) *and the vector field $X_{\mathcal{H}^{(4)}_{\mathrm{NLS}}}(U)$ is defined as follows*

$$X_{\mathcal{H}^{(4)}_{\mathrm{NLS}}}(U) = -\mathrm{i} E\Big[Op^{\mathrm{BW}}\Big(\Big[\begin{smallmatrix} 2|u|^2 & u^2 \\ \overline{u}^2 & 2|u|^2\end{smallmatrix}\Big]\Big)U + Q_3(U)\Big]. \tag{3.5}$$

The semi-norms of the symbols satisfy the following estimates

$$\begin{aligned} |a_2|_{\mathcal{N}^0_p} + |b_2|_{\mathcal{N}^0_p} &\lesssim \|u\|^6_{H^{p+s_0}}\,, \qquad \forall\, p+s_0\le s\,, \quad p\in\mathbb{N}, \\ |\vec a_1\cdot\xi|_{\mathcal{N}^1_p} &\lesssim \|u\|^6_{H^{p+s_0+1}}\,, \quad \forall\, p+s_0+1\le s\,, \quad p\in\mathbb{N}, \end{aligned} \tag{3.6}$$

where we have chosen $s_0>d$. The remainder $Q_3(U)$ has the form $\big(Q_3^+(U),\overline{Q_3^+(U)}\big)^T$ and

$$\widehat{Q_3^+}(\xi) = (2\pi)^{-d}\sum_{\eta,\zeta\in\mathbb{Z}^d}\mathtt{q}(\xi,\eta,\zeta)\,\widehat u(\xi-\eta-\zeta)\,\widehat{\overline u}(\eta)\,\widehat u(\zeta)\,, \tag{3.7}$$

for some $\mathtt{q}(\xi,\eta,\zeta)\in\mathbb{C}$. The coefficients of Q_3^+ satisfy

$$|\mathtt{q}(\xi,\eta,\zeta)| \lesssim \frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^\rho}{\max\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^\rho}\,, \quad \forall\,\rho\ge 0\,. \tag{3.8}$$

The remainder $R(U)$ has the form $(R^+(U),\overline{R^+(U)})^T$. Moreover, for any $s>2d+2$, we have the estimates

$$\|R(U)\|_{H^s}\lesssim\|U\|^7_{H^s}\,, \qquad \|Q_3(U)\|_{H^{s+2}}\lesssim\|U\|^3_{H^s}\,. \tag{3.9}$$

Proof. By Proposition 3.3 in [28] we obtain that at the positive orders the symbols are given by

$$\begin{aligned} a_2(U;x,\xi) &= \sum_{j,k=1}^d (\partial_{\overline{u_{x_k}}u_{x_j}}P)\xi_j\xi_k\,, \quad b_2(U;x,\xi) = \sum_{j,k=1}^d (\partial_{\overline{u_{x_k}}\,\overline{u_{x_j}}}P)\xi_j\xi_k\,, \\ \vec a_1(U;x)\cdot\xi &= \frac{\mathrm{i}}{2}\sum_{j=1}^d\Big((\partial_{\overline{u}u_{x_j}}P) - (\partial_{u\overline{u}_{x_j}}P)\Big)\xi_j = \sum_{j=1}^d \mathrm{Im}\,(\partial_{u\overline{u}_{x_j}}P)\xi_j\,, \end{aligned}$$

then one obtains formulæ (3.2) by direct inspection by using the second line in (1.11). The estimates (3.6) are obtained as consequence of the fact that $h'(s)\sim s$ when s goes to 0 and using Lemma 2.4. The estimate on $R(U)$ in (3.9) may be deduced from (2.10), (2.8), (2.12) and (3.6), for more details one can follow Proposition 3.3 in [28]. Formula (3.5) is obtained by using Lemma 2.3 applied to $|u|^2u$. □

Remark 3.2. • *The cubic term $X_{\mathcal{H}^{(4)}_{\mathrm{NLS}}}(U)$ in* (3.5) *is the Hamiltonian vector field of the Hamiltonian function*

$$\mathcal{H}^{(4)}_{\mathrm{NLS}}(U) := \frac12\int_{\mathbb{T}^d}|u|^4dx\,, \qquad X_{\mathcal{H}^{(4)}_{\mathrm{NLS}}}(U) = -\mathrm{i}|u|^2\big[\begin{smallmatrix} u \\ \overline{u}\end{smallmatrix}\big] \tag{3.10}$$

• *The operators $Op^{\mathrm{BW}}\big((\mathbb{1}+A_2(x))|\xi|^2\big)$, $Op^{\mathrm{BW}}\big(\mathrm{diag}(\vec a_1(x)\cdot\xi)\big)$ and $Op^{\mathrm{BW}}\Big(\Big[\begin{smallmatrix} 2|u|^2 & u^2 \\ \overline{u}^2 & 2|u|^2\end{smallmatrix}\Big]\Big)$ are self-adjoint thanks to* (2.23) *and* (3.2).

3.2. **Para-linearization of the KG.** In this section we rewrite the equation (KG) as a paradifferential system. This is the content of Proposition 3.6. Before stating this result we need some preliminaries. In particular in Lemma 3.3 below we analyze some properties of the cubic terms in the equation (KG). Define the following *real* symbols

$$\begin{aligned} a_2(x,\xi) &:= a_2(u;x,\xi) := \sum_{j,k=1}^d\big(\partial_{\psi_{x_j}\psi_{x_k}}F\big)(\psi,\nabla\psi)\xi_j\xi_k\,, \qquad \psi = \frac{\Lambda_{\mathrm{KG}}^{-\frac12}}{\sqrt2}(u+\overline{u})\,, \\ a_0(x,\xi) &:= a_0(u;x,\xi) := \tfrac12(\partial_{y_1y_1}G)(\psi,\Lambda_{\mathrm{KG}}^{\frac12}\psi) + (\partial_{y_1y_0}G)(\psi,\Lambda_{\mathrm{KG}}^{\frac12}\psi)\Lambda_{\mathrm{KG}}^{-\frac12}(\xi)\,. \end{aligned} \tag{3.11}$$

We define also the matrices of symbols

$$\mathscr{A}_1(x,\xi) := \mathscr{A}_1(u;x,\xi) := \tfrac{1}{2}\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]\Lambda_{\rm KG}^{-2}(\xi)a_2(u;x,\xi)\,, \tag{3.12}$$

$$\mathscr{A}_0(x,\xi) := \mathscr{A}_0(u;x,\xi) := \left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]a_0(u;x,\xi)\,, \tag{3.13}$$

and the Hamiltonian function

$$\mathscr{H}^{(4)}_{\rm KG}(U) := \int_{\mathbb{T}^d} G\big(\psi,\Lambda_{\rm KG}^{\frac{1}{2}}\psi\big)dx\,, \tag{3.14}$$

with G the function appearing in (1.13). First of all we study some properties of the vector field of the Hamiltonian $\mathscr{H}^{(4)}_{\rm KG}$.

Lemma 3.3. *We have that*

$$X_{\mathscr{H}^{(4)}_{\rm KG}}(U) = -\mathrm{i}J\nabla\mathscr{H}^{(4)}_{\rm KG}(U) = -\mathrm{i}EOp^{\rm BW}(\mathscr{A}_0(x,\xi))U + Q_3(u)\,, \tag{3.15}$$

with $\mathscr{A}_0$ in (3.13)*. The remainder $Q_3(u)$ has the form $\big(Q_3^+(u),\overline{Q_3^+(u)}\big)^T$ and (recall* (2.52)*)*

$$\widehat{Q_3^+}(\xi) = \frac{1}{(2\pi)^d}\sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\widehat{u^{\sigma_1}}(\xi-\eta-\zeta)\widehat{u^{\sigma_2}}(\eta)\widehat{u^{\sigma_3}}(\zeta)\,, \tag{3.16}$$

for some $\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\in\mathbb{C}$. The coefficients of Q_3^+ satisfy

$$|\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)| \lesssim \frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}}{\max\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}} \tag{3.17}$$

for any $\sigma_1,\sigma_2,\sigma_3\in\{\pm\}$. Finally, for $s>2d+1$, we have

$$|a_0|_{\mathcal{N}_p^0} \lesssim \|u\|^2_{H^{p+s_0}}\,, \quad p+s_0\le s\,, \quad s_0>d\,, \tag{3.18}$$

$$\|X_{\mathscr{H}^{(4)}_{\rm KG}}(U)\|_{H^s} \lesssim \|u\|^3_{H^s}\,, \qquad \|Q_3(u)\|_{H^{s+1}} \lesssim \|u\|^3_{H^s}\,, \tag{3.19}$$

$$\|d_U X_{\mathscr{H}^{(4)}_{\rm KG}}(U)[h]\|_{H^s} \lesssim \|u\|^2_{H^s}\|h\|_{H^s}\,, \qquad \forall h\in H^s(\mathbb{T}^d;\mathbb{C}^2)\,. \tag{3.20}$$

Proof. By and explicit computation and using (1.2) we get

$$X_{\mathscr{H}^{(4)}_{\rm KG}}(U) = \Big(X^+_{\mathscr{H}^{(4)}_{\rm KG}}(U),\overline{X^+_{\mathscr{H}^{(4)}_{\rm KG}}(U)}\Big)^T\,, \qquad X^+_{\mathscr{H}^{(4)}_{\rm KG}}(U) = -\mathrm{i}\frac{\Lambda_{\rm KG}^{-\frac{1}{2}}}{\sqrt{2}}g\big(\psi\big)\,.$$

The function g is a homogeneous polynomial of degree three. Hence, by using Lemma 2.3, we obtain

$$\mathrm{i}X^+_{\mathscr{H}^{(4)}_{\rm KG}}(U) = A_0 + A_{-\frac{1}{2}} + A_{-1} + Q^{-\rho}(u) \tag{3.21}$$

where

$$A_0 := \frac{1}{2}Op^{\rm BW}(\partial_{y_1y_1}G(\psi,\Lambda_{\rm KG}^{1/2}\psi))[u+\overline{u}]\,, \tag{3.22}$$

$$A_{-\frac{1}{2}} := \frac{1}{2}Op^{\rm BW}(\partial_{y_1y_0}G(\psi,\Lambda_{\rm KG}^{1/2}\psi))[\Lambda_{\rm KG}^{-\frac{1}{2}}(u+\overline{u})] + \frac{\Lambda_{\rm KG}^{-\frac{1}{2}}}{2}Op^{\rm BW}(\partial_{y_1y_0}G(\psi,\Lambda_{\rm KG}^{1/2}\psi))[u+\overline{u}]\,, \tag{3.23}$$

$$A_{-1} := \frac{\Lambda_{\rm KG}^{-\frac{1}{2}}}{2}Op^{\rm BW}(\partial_{y_0y_0}G(\psi,\Lambda_{\rm KG}^{1/2}\psi))[\Lambda_{\rm KG}^{-\frac{1}{2}}(u+\overline{u})]\,, \tag{3.24}$$

and $Q^{-\rho}$ is a cubic smoothing remainder of the form (2.15) whose coefficients satisfy the bound (3.17). The symbols of the the paradifferential operators have the form (using that G is a polynomial)

$$(\partial_{kj}G)\Big(\tfrac{\Lambda_{\rm KG}^{-\frac{1}{2}}(u+\overline{u})}{\sqrt{2}},\tfrac{u+\overline{u}}{\sqrt{2}}\Big) = (2\pi)^{-d}\sum_{\xi\in\mathbb{Z}^d}e^{\mathrm{i}\xi\cdot x}\sum_{\eta\in\mathbb{Z}^d}\mathtt{g}_{k,j}^{\sigma_1,\sigma_2}(\xi,\eta)\widehat{u^{\sigma_1}}(\xi-\eta)\widehat{u^{\sigma_2}}(\eta) \tag{3.25}$$

where $k,j\in\{y_0,y_1\}$ and where the coefficients $\mathtt{g}_{k,j}^{\sigma_1,\sigma_2}(\xi,\eta)\in\mathbb{C}$ satisfy $|\mathtt{g}_{k,j}^{\sigma_1,\sigma_2}(\xi,\eta)|\lesssim 1$.

We claim that the term in (3.24) is a cubic remainder of the form (3.16) with coefficients satisfying (3.17). By (2.6) we have

$$\widehat{A_{-1}}(\xi) = \frac{1}{2(2\pi)^d} \sum_{\zeta\in\mathbb{Z}^d, \sigma\in\{\pm\}} \widehat{\partial_{y_0 y_0} G}(\xi-\zeta)\Lambda_{\rm KG}^{-\frac{1}{2}}(\xi)\Lambda_{\rm KG}^{-\frac{1}{2}}(\zeta)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\widehat{u^\sigma}(\xi)$$
$$\stackrel{(3.25)}{=} \frac{1}{2(2\pi)^d} \sum_{\substack{\sigma_1,\sigma_2,\sigma\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{g}_{y_0,y_0}^{\sigma_1,\sigma_2}(\xi-\zeta,\eta)\Lambda_{\rm KG}^{-\frac{1}{2}}(\xi)\Lambda_{\rm KG}^{-\frac{1}{2}}(\zeta)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\widehat{u^{\sigma_1}}(\xi-\eta-\zeta)\widehat{u^{\sigma_2}}(\eta)\widehat{u^\sigma}(\zeta),$$

which implies that A_{-1} has the form (3.16) with coefficients

$$\mathtt{a}_{-1}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta) = \frac{1}{2}\mathtt{g}_{y_0,y_0}^{\sigma_1,\sigma_2}(\xi-\zeta,\eta)\Lambda_{\rm KG}^{-\frac{1}{2}}(\xi)\Lambda_{\rm KG}^{-\frac{1}{2}}(\zeta)\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big). \tag{3.26}$$

By Lemma 2.6 we have that the coefficients in (3.26) satisfy (3.17). This prove the claim for the operator A_{-1}. We now study the term in (3.23). We remark that, by Proposition 2.2 (see the composition formula (2.11)), we have that $A_{-1/2} = Op^{\rm BW}(\Lambda_{\rm KG}^{-\frac{1}{2}}(\xi)\partial_{y_0 y_1}G)$ up to a smoothing operator of order $-3/2$. Actually to prove that such a remainder has the form (3.16) with coefficients (3.17) it is more convenient to compute the composition operator explicitly. In particular, recalling (2.6), we get

$$A_{-\frac{1}{2}} = Op^{\rm BW}(\Lambda_{\rm KG}^{-\frac{1}{2}}(\xi)\partial_{y_0y_1}G) + R_{-1}, \tag{3.27}$$

where

$$\widehat{R_{-1}}(\xi) = (2\pi)^{-d}\sum_{\substack{\sigma_1,\sigma_2,\sigma\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{r}^{\sigma_1,\sigma_2,\sigma}(\xi-\eta-\zeta,\eta,\zeta)\widehat{u^{\sigma_1}}(\xi-\eta-\zeta)\widehat{u^{\sigma_2}}(\eta)\widehat{u^\sigma}(\zeta),$$

$$\mathtt{r}^{\sigma_1,\sigma_2,\sigma}(\xi-\eta-\zeta,\eta,\zeta) = \tfrac{1}{2}\mathtt{g}_{y_0,y_1}^{\sigma_1,\sigma_2}(\xi-\zeta,\eta)\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\big[\Lambda_{\rm KG}^{-\frac{1}{2}}(\xi)+\Lambda_{\rm KG}^{-\frac{1}{2}}(\zeta)-2\Lambda_{\rm KG}^{-\frac{1}{2}}(\tfrac{\xi+\zeta}{2})\big].$$

We note that

$$\Lambda_{\rm KG}^{-\frac{1}{2}}(\xi) = \Lambda_{\rm KG}^{-\frac{1}{2}}(\tfrac{\xi+\zeta}{2}) - \frac{1}{2}\int_0^1 \Lambda_{\rm KG}^{-\frac{3}{2}}(\tfrac{\xi+\zeta}{2}+\tau\tfrac{\xi-\zeta}{2})d\tau.$$

Then we deduce

$$\Big|\Lambda_{\rm KG}^{-\frac{1}{2}}(\xi)+\Lambda_{\rm KG}^{-\frac{1}{2}}(\zeta)-2\Lambda_{\rm KG}^{-\frac{1}{2}}(\tfrac{\xi+\zeta}{2})\Big| \lesssim |\xi|^{-\frac{3}{2}}+|\zeta|^{-\frac{3}{2}}.$$

Again by Lemma 2.6 one can conclude that $\mathtt{r}^{\sigma_1,\sigma_2,\sigma}(\xi-\eta-\zeta,\eta,\zeta)$ satisfies the (3.17). By (3.27), (3.22), (3.24) and recalling the definition of $a_0(x,\xi)$ in (3.11), we obtain the (3.15). The bound (3.19) for Q_3 follows by (3.17) and Lemma 2.5. Moreover the bound (3.18) follows by Lemma 2.12 in [28] recalling that $G(\psi,\Lambda_{\rm KG}^{\frac{1}{2}}\psi)\sim O(u^4)$. Then the bound (3.19) for $X_{\mathcal{H}^{(4)}_{\rm KG}}$ follows by Lemma 2.1. Let us prove the (3.20). By differentiating (3.15) we get

$$d_U X_{\mathcal{H}^{(4)}_{\rm KG}}(U)[h] = -\mathrm{i}EOp^{\rm BW}(\mathscr{A}_0(x,\xi))h - \mathrm{i}EOp^{\rm BW}(d_U\mathscr{A}_0(x,\xi)h)U + d_U Q_3(u)[h]. \tag{3.28}$$

The first summand in (3.28) satisfies (3.20) by Lemma 2.1 and (3.18). Moreover using (3.25) and (3.11) one can check that

$$|d_U\mathscr{A}_0(x,\xi)h|_{\mathcal{N}_p^0} \lesssim \|u\|_{H^{p+s_0}}\|h\|_{H^{p+s_0}}, \quad p+s_0\le s.$$

Then the second summand in (3.28) verify the bound (3.20) again by Lemma 2.1. The estimate on the third summand in (3.28) follows by (3.16), (3.17) and Lemma 2.5. □

Remark 3.4. *We remark that the symbol $a_0(x,\xi)$ in* (3.11) *is homogenenous of degree two in the variables $u,\overline{u}$. In particular, by* (3.25)*, we have*

$$a_0(x,\xi)=(2\pi)^{-\frac{d}{2}}\sum_{p\in\mathbb{Z}^d}e^{\mathrm{i}p\cdot x}\widehat{a_0}(p,\xi)\,,\qquad \widehat{a_0}(p,\xi)=(2\pi)^{-d}\sum_{\substack{\sigma_1,\sigma_2\in\{\pm\}\\ \eta\in\mathbb{Z}^d}}a_0^{\sigma_1,\sigma_2}(p,\eta,\xi)\widehat{u^{\sigma_1}}(p-\eta)\widehat{u^{\sigma_2}}(\eta)$$
$$a_0^{\sigma_1,\sigma_2}(p,\eta,\xi):=\frac{1}{2}\mathrm{g}^{\sigma_1,\sigma_2}_{y_1,y_1}(p,\eta)+\mathrm{g}^{\sigma_1,\sigma_2}_{y_0,y_1}(p,\eta)\Lambda_{\text{KG}}^{-\frac{1}{2}}(\xi)\,. \tag{3.29}$$

Moreover one has $|a_0^{\sigma_1,\sigma_2}(p,\eta,\xi)|\lesssim 1$. Since the symbol $a_0(x,\xi)$ is real-valued one can check that

$$a_0^{\sigma_1,\sigma_2}(p,\eta,\xi)=\overline{a_0^{-\sigma_1,-\sigma_2}(-p,-\eta,\xi)}\,,\qquad \forall\,\xi,p,\eta\in\mathbb{Z}^d\,,\ \sigma_1,\sigma_2\in\{\pm\}\,. \tag{3.30}$$

Remark 3.5. *Consider the special case when the function G in* (1.2) *is independent of y_1. Following the proof of Lemma 3.3 one could obtain the formula* (3.15) *with symbol $a_0(x,\xi)$ of order -1 given by (see* (3.24)*)*

$$a_0(x,\xi):=\tfrac{1}{2}\partial_{y_0y_0}G(\psi)\Lambda_{\text{KG}}^{-1}(\xi)\,.$$

The remainder Q_3 would satisfy the (3.17) *with better denominator* $\max\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^2$.

The main result of this section is the following.

Proposition 3.6. (Paralinearization of KG). *The system* (1.12) *is equivalent to*

$$\dot{U}=-\mathrm{i}EOp^{\text{BW}}\big((\mathbb{1}+\mathscr{A}_1(x,\xi))\Lambda_{\text{KG}}(\xi)\big)U+X_{\mathscr{H}^{(4)}_{\text{KG}}}(U)+R(u)\,, \tag{3.31}$$

where $U:=\left[\begin{smallmatrix}u\\ \overline{u}\end{smallmatrix}\right]:=\mathscr{C}\left[\begin{smallmatrix}\psi\\ \phi\end{smallmatrix}\right]$ (see (2.41)*), $\mathscr{A}_1(x,\xi)$ is in* (3.12)*, $X_{\mathscr{H}^{(4)}_{\text{KG}}}(U)$ is the Hamiltonian vector field of* (3.14)*. The operator $R(u)$ has the form $(R^+(u),\overline{R^+(u)})^T$. Moreover we have that*

$$|\mathscr{A}_1|_{\mathcal{N}_p^0}+|a_2|_{\mathcal{N}_p^2}+\lesssim\|u\|^4_{H^{p+s_0+1}}\,,\quad \forall\,p+s_0+1\le s\,,\quad p\in\mathbb{N}\,, \tag{3.32}$$

where we have chosen $s_0>d$. Finally there is $\mu>0$ such that, for any $s>2d+\mu$, the remainder $R_{\ge5}(u)$ satisfy

$$\|R(u)\|_{H^s}\lesssim\|u\|^5_{H^s}\,. \tag{3.33}$$

Proof. First of all we note that system (1.12) in the complex coordinates (2.41) reads

$$\partial_t u=-\mathrm{i}\Lambda_{\text{KG}}u-\mathrm{i}\frac{\Lambda_{\text{KG}}^{-\frac{1}{2}}}{\sqrt{2}}(f(\psi)+g(\psi))\,,\qquad \psi=\frac{\Lambda_{\text{KG}}^{-\frac{1}{2}}(u+\overline{u})}{\sqrt{2}}\,, \tag{3.34}$$

with $f(\psi)$, $g(\psi)$ in (1.1), (1.2). The term $-\mathrm{i}/\sqrt{2}\Lambda_{\text{KG}}^{-1/2}g(\psi)$ is the first component of the vector field $X_{\mathscr{H}^{(4)}_{\text{KG}}}(U)$ which has been studied in Lemma 3.3. By using the Bony para-linearization formula (see [10]), passing to the Weyl quantization and (1.1) we get

$$f(\psi)=-\sum_{j,k=1}^{d}\partial_{x_j}\circ Op^{\text{BW}}\Big(\big(\partial_{\psi_{x_j}\psi_{x_k}}F\big)(\psi,\nabla\psi)\Big)\circ\partial_{x_k}\psi \tag{3.35}$$

$$+\sum_{j=1}^{d}\Big[Op^{\text{BW}}\big((\partial_{\psi\psi_{x_j}}F)(\psi,\nabla\psi)\big),\partial_{x_j}\Big]\psi+Op^{\text{BW}}\big((\partial_{\psi\psi}F)(\psi,\nabla\psi)\big)\psi+R^{-\rho}(\psi)\,, \tag{3.36}$$

where $R^{-\rho}(\psi)$ satisfies $\|R^{-\rho}(\psi)\|_{H^{s+\rho}}\lesssim\|\psi\|^5_{H^s}$ for any $s\ge s_0>d+\rho$. By Lemma 2.12 in [28], and recalling that $F(\psi,\nabla\psi)\sim O(\psi^6)$, we have that

$$|\partial_{\psi_{x_k}\psi_{x_j}}F|_{\mathcal{N}_p^0}+|\partial_{\psi\psi_{x_j}}F|_{\mathcal{N}_p^0}+|\partial_{\psi\psi}F|_{\mathcal{N}_p^0}\lesssim\|\psi\|^4_{H^{p+s_0+1}}\,,\quad p+s_0+1\le s\,, \tag{3.37}$$

where $s_0>d$. Recall that $\partial_{x_j}=Op^{\text{BW}}(\xi_j)$. Then, by Proposition 2.2, we have

$$\Big[Op^{\text{BW}}\big(\partial_{\psi\psi_{x_j}}F\big),\partial_{x_j}\Big]=Op^{\text{BW}}(-\mathrm{i}\{\partial_{\psi\psi_{x_j}}F,\xi_j\})+Q(\psi)$$

with (see (2.12)) $\|Q(\psi)\|_{H^{s+1}} \lesssim |\partial_{\psi\psi_{x_j}} F|_{\mathcal{N}^0_{s_0+2}} \|\psi\|_{H^s}$. Then by (2.8), (3.37) and (2.10) (see Lemma 2.1 and Proposition 2.2) we deduce that the terms in (3.36) can be absorbed in a remainder satisfying (3.33) with $s \gg 2d$ large enough. We now consider the first term in the r.h.s. of (3.35). We have

$$-\partial_{x_j} \circ Op^{BW}\Big(\big(\partial_{\psi_{x_j}\psi_{x_k}} F\big)(\psi, \nabla\psi)\Big) \circ \partial_{x_k} = Op^{BW}(\xi_j) Op^{BW}\Big(\big(\partial_{\psi_{x_j}\psi_{x_k}} F\big)(\psi, \nabla\psi)\Big) Op^{BW}(\xi_k) .$$

By using again Lemma 2.1 and Proposition 2.2 we get that

$$f(\psi) = Op^{BW}(a_2(x,\xi))\psi + \tilde{R}(\psi) , \tag{3.38}$$

where a_2 is in (3.11) and $\tilde{R}(\psi)$ is a remainder satisfying (3.33). The symbol $a_2(x,\xi)$ satisfies (3.32) by (3.37). Moreover

$$\tfrac{1}{\sqrt{2}} \Lambda_{KG}^{-\frac{1}{2}} f(\psi) = \tfrac{1}{\sqrt{2}} \Lambda_{KG}^{-\frac{1}{2}} f\big(\tfrac{\Lambda_{KG}^{-\frac{1}{2}}(u+\overline{u})}{\sqrt{2}}\big) \overset{(3.38)}{=} \tfrac{1}{2} Op^{BW}(a_2(x,\xi)\Lambda_{KG}^{-1}(\xi))[u+\overline{u}] \tag{3.39}$$

up to remainders satisfying (3.33). Here we used Proposition 2.2 to study the composition operator $\Lambda_{KG}^{-\frac{1}{2}} Op^{BW}(a_2(x,\xi))\Lambda_{KG}^{-\frac{1}{2}}$. By the discussion above and formula (3.34) we deduce the (3.31). $\square$

Remark 3.7. *In the semi-linear case, i.e. when $f = 0$ and g does not depend on y_1 (see (1.1), (1.2)), the equation (3.31) reads*

$$\dot{U} = -\mathrm{i} E Op^{BW}\big(\mathbb{1}\Lambda_{KG}(\xi)\big) U + X_{\mathcal{H}^{(4)}_{KG}}(U) ,$$

and where the vector field $X_{\mathcal{H}^{(4)}_{KG}}$ has the particular structure described in Remark 3.5.

4. DIAGONALIZATION

4.1. **Diagonalization of the NLS.** In this section we diagonalize the system (3.4). We first diagonalize the matrix $E(\mathbb{1} + A_2(x))$ in (3.4) by means of a change of coordinates as the ones made in the papers [28, 29]. After that we diagonalize the matrix of symbols of order 0 at homogeneity 3, by means of an approximatively *symplectic* change of coordinates. Throughout the rest of the section we shall assume the following.

Hypothesis 4.1. *We restrict the solution of* (NLS) *on the interval of times* $[0, T)$, *with T such that*

$$\sup_{t\in[0,T)} \|u(t,x)\|_{H^s} \le \varepsilon , \quad \|u_0(x)\|_{H^s} \le c_0(s)\varepsilon \ll 1 ,$$

for some $0 < c_0(s) < 1$.

Note that such a time $T > 0$ exists thanks to the local existence theorem in [28].

4.1.1. *Diagonalization at order 2.* We consider the matrix $E(\mathbb{1} + A_2(x))$ in (3.4). We define

$$\lambda_{NLS}(x) := \lambda_{NLS}(U;x) := \sqrt{1 + 2|u|^2[h'(|u|^2)]^2} , \qquad a_2^{(1)}(x) := \lambda_{NLS}(x) - 1 , \tag{4.1}$$

and we note that $\pm\lambda_{NLS}(x)$ are the eigenvalues of the matrix $E(\mathbb{1} + A_2(x))$. We denote by S matrix of the eigenvectors of $E(\mathbb{1} + A_2(x))$, more explicitly

$$\begin{aligned} &S = \begin{pmatrix} s_1 & s_2 \\ \overline{s}_2 & s_1 \end{pmatrix} , \qquad S^{-1} = \begin{pmatrix} s_1 & -s_2 \\ -\overline{s}_2 & s_1 \end{pmatrix} , \\ &s_1(x) := \frac{1 + |u|^2[h'(|u|^2)]^2 + \lambda_{NLS}(x)}{\sqrt{2\lambda_{NLS}(x)(1 + |u|^2 + \lambda_{NLS}(x))}} , \qquad s_2(x) := \frac{-u^2[h'(|u|^2)]^2}{\sqrt{2\lambda_{NLS}(x)(1 + |u|^2 + \lambda_{NLS}(x))}} . \end{aligned} \tag{4.2}$$

Since $\pm\lambda_{NLS}(x)$ are the eigenvalues and $S(x)$ is the matrix eigenvectors of $E(\mathbb{1} + A_2(x))$ we have that

$$S^{-1} E(\mathbb{1} + A_2(x)) S = E\mathrm{diag}(\lambda_{NLS}(x)) , \quad s_1^2 - |s_2|^2 = 1 , \tag{4.3}$$

where we have used the notation (3.1). In the following lemma we estimate the semi-norms of the symbols defined above.

Lemma 4.2. *Let $\mathbb{N}\ni s_0>d$. The symbols $a_2^{(1)}$ defined in* (4.1)*, s_1-1 and s_2 defined in* (4.2) *satisfy the following estimate*

$$|a_2^{(1)}|_{\mathcal{N}_p^0}+|s_1-1|_{\mathcal{N}_p^0}+|s_2|_{\mathcal{N}_p^0}\lesssim \|u\|_{p+s_0}^6\,,\qquad p+s_0\le s\,,\quad p\in\mathbb{N}.$$

Proof. The proof follows by using the estimate (3.6) on the symbols in (3.2), the fact that $h'(s)\sim s$ when $s\sim 0$, $\|u\|_s\ll 1$, and the explicit expression (4.1), (4.2). □

We now study how the system (3.4) transforms under the maps

$$\Phi_{\rm NLS}:=\Phi_{\rm NLS}(U):=Op^{\rm BW}(S^{-1}(U;x))\,,\qquad \Psi_{\rm NLS}:=\Psi_{\rm NLS}(U):=Op^{\rm BW}(S(U;x))\,. \tag{4.4}$$

Lemma 4.3. *Let $U=\left[\begin{smallmatrix}u\\ \bar u\end{smallmatrix}\right]$ be a solution of* (3.4) *and assume Hyp. 4.1. Then for any $s\ge 2s_0+2$, $\mathbb{N}\ni s_0>d$, we have the following.*
(i) One has the upper bound

$$\begin{aligned}&\|\Phi_{\rm NLS}(U)W\|_{H^s}+\|\Psi_{\rm NLS}(U)W\|_{H^s}\le \|W\|_{H^s}\big(1+C\|u\|^6_{H^{2s_0}}\big)\,,\\ &\|(\Phi_{\rm NLS}(U)-\mathbb{1})W\|_{H^s}+\|(\Psi_{\rm NLS}(U)-\mathbb{1})W\|_{H^s}\lesssim \|W\|_{H^s}\|u\|^6_{H^{2s_0}}\,,\qquad \forall\, W\in H^s(\mathbb{T}^d;\mathbb{C})\,,\end{aligned} \tag{4.5}$$

where the constant C depends on s;
(ii) one has $\Psi_{\rm NLS}(U)\circ\Phi_{\rm NLS}(U)=\mathbb{1}+R(u)$ where R is a real-to-real remainder of the form (2.18) *satisfying*

$$\|R(u)W\|_{H^{s+2}}\lesssim \|W\|_{H^s}\|u\|^6_{H^{2s_0+2}}\,. \tag{4.6}$$

The map $\mathbb{1}+R(u)$ is invertible with inverse $(\mathbb{1}+R(u))^{-1}:=(1+\tilde R(u))$ with $\tilde R(u)$ of the form (2.18) *and*

$$\|\tilde R(u)W\|_{H^{s+2}}\lesssim \|W\|_{H^s}\|u\|^6_{H^{2s_0+2}}\,, \tag{4.7}$$

as a consequence the map $\Phi_{\rm NLS}$ is invertible and $\Phi^{-1}_{\rm NLS}=(1+\tilde R)\Psi_{\rm NLS}$ with estimates

$$\|\Phi^{-1}_{\rm NLS}(U)W\|_{H^s}\le \|W\|_{H^s}(1+C\|u\|^6_{H^{2s_0+2}})\,, \tag{4.8}$$

where the constant C depends on s;
(iii) for any $t\in[0,T)$, one has $\partial_t\Phi_{\rm NLS}(U)[\cdot]=Op^{\rm BW}(\partial_t S^{-1}(U;x))$ and

$$|\partial_t S^{-1}(U;x)|_{\mathcal{N}^0_{s_0}}\lesssim \|u\|^6_{H^{2s_0+2}}\,,\qquad \|\partial_t\Phi_{\rm NLS}(U)V\|_{H^s}\lesssim \|W\|_{H^s}\|u\|^6_{H^{2s_0+2}}\,. \tag{4.9}$$

Proof. (i) The bounds (4.5) follow by (2.10) and Lemma 4.2.
(ii) We apply Proposition 2.2 to the maps in (4.4), in particular the first part of the item follows by using the expansion (2.13) and recalling that symbols $s_1(x)$ and $s_2(x)$ do not depend on ξ. The (4.7) is obtained by Neumann series by using that (see Hyp. 4.1) $\|u\|_{H^s}\ll 1$.
(iii) We note that $\partial_t s_1(x,\xi)=(\partial_u s_1)(u;x,\xi)[\dot u]+(\partial_{\bar u}s_1)(u;x,\xi)[\dot{\bar u}]$. Since u solves (3.4) and satisfies Hypothesis 4.1, then using Lemma 2.1 and (3.9) we deduce that $\|\dot u\|_{H^s}\lesssim\|u\|_{H^{s+2}}$. Hence the estimates (4.9) follow by direct inspection by using the explicit structure of the symbols s_1, s_2 in (4.2), Lemma 2.4 and (2.10). □

We are now in position to state the following proposition.

Proposition 4.4 (**Diagonalization at order** 2)**.** *Consider the system* (3.4) *and set*

$$W=\Phi_{\rm NLS}(U)U\,, \tag{4.10}$$

with $\Phi_{\rm NLS}$ defined in (4.4)*. Then W solves the equation*

$$\begin{aligned}\dot W=&-\mathrm{i}EOp^{\rm BW}\big(\mathrm{diag}\big(1+a_2^{(1)}(U;x)\big)|\xi|^2\big)W-\mathrm{i}EV*W\\ &-\mathrm{i}Op^{\rm BW}\big(\mathrm{diag}\big(\vec a_1^{(1)}(U;x)\cdot\xi\big)\big)W+X_{\mathcal{H}^{(4)}_{\rm NLS}}(W)+R^{(1)}(U)\,,\end{aligned} \tag{4.11}$$

where the vector field $X_{\mathcal{H}^{(4)}_{\rm NLS}}$ is defined in (3.5). *The symbols $a_2^{(1)}$ and $\vec{a}_1^{(1)}\cdot\xi$ are* real *valued and satisfy the following estimates*

$$
\begin{aligned}
|a_2^{(1)}|_{\mathcal{N}_p^0} &\lesssim \|u\|^6_{H^{p+s_0}}, &&\forall\, p+s_0\le s, \quad p\in\mathbb{N},\\
|\vec{a}_1^{(1)}\cdot\xi|_{\mathcal{N}_p^1} &\lesssim \|u\|^6_{H^{p+s_0+1}}, &&\forall\, p+s_0+1\le s, \quad p\in\mathbb{N},
\end{aligned}
\tag{4.12}
$$

where we have chosen $s_0>d$. The remainder $R^{(1)}$ has the form $(R^{(1,+)},\overline{R^{(1,+)}})^T$. Moreover, for any $s>2d+2$, it satisfies the estimate

$$
\|R^{(1)}(U)\|_{H^s}\lesssim \|U\|^7_{H^s}. \tag{4.13}
$$

Proof. The function W defined in (4.10) satisfies

$$
\begin{aligned}
\dot W =\ & [\partial_t\Phi_{\rm NLS}(U)]U+\Phi_{\rm NLS}(U)\dot U\\
=\ & -\Phi_{\rm NLS}(U){\rm i}EOp^{\rm BW}\big((\mathbb{1}+A_2(U))|\xi|^2\big)\Psi_{\rm NLS}(U)W-\Phi_{\rm NLS}(U){\rm i}EV*\Psi_{\rm NLS}W && (4.14)\\
& -{\rm i}\Phi_{\rm NLS}(U)Op^{\rm BW}\big({\rm diag}(\vec{a}_1(U)\cdot\xi)\big)\Psi_{\rm NLS}(U)W && (4.15)\\
& +\Phi_{\rm NLS}(U)X_{\mathcal{H}^{(4)}_{\rm NLS}}(U) && (4.16)\\
& +\Phi_{\rm NLS}(U)R(U)+Op^{\rm BW}(\partial_t S^{-1}(U))U && (4.17)\\
& -\Phi_{\rm NLS}(U){\rm i}\Big[EOp^{\rm BW}\big((\mathbb{1}+A_2(U))|\xi|^2\big)+Op^{\rm BW}\big({\rm diag}(\vec{a}_1\cdot\xi)\big)+\Phi_{\rm NLS}(U){\rm i}EV*\Big]\tilde R(U)\Psi_{\rm NLS}(U), && (4.18)
\end{aligned}
$$

here we have used items (ii) and (iii) of Lemma 4.3.

We are going to analyze each term in the r.h.s. of the equation above. Because of estimates (4.7), (4.5) (applied for the map $\Phi_{\rm NLS}$), Lemma 4.2 (applied for the symbols a_2, b_2 and $\vec{a}_1\cdot\xi$) and finally item (ii) of Lemma 2.1 we may absorb term (4.18) in the remainder $R^{(1)}(U)$ verifying (4.13). The term in (4.17) may be absorbed in $R^{(1)}(U)$ as well because of (3.9) and (4.5) for the first addendum, because of (4.9) and item (ii) of Lemma 2.1 for the second one.

We study the first addendum in (4.14). We recall (4.4) and (4.2), we apply Proposition 2.2 and we get, by direct inspection, that the new term, modulo contribution that may be absorbed in $R^{(1)}(U)$, is given by

$$
-{\rm i}EOp^{\rm BW}\Big({\rm diag}(\lambda_{\rm NLS})\Big)W-2{\rm i}Op^{\rm BW}\Big({\rm diag}\big({\rm Im}\big\{(s_2\overline{b}_2)\nabla s_1+(s_1b_2+s_2(1+a_2))\nabla\overline{s}_2\big\}\cdot\xi\big)\Big)W,
$$

where by ${\rm Im}\{\vec{b}\}$, with $\vec{b}=(b_1,\dots,b_d)$, we denoted the vector $({\rm Im}(b_1),\dots,{\rm Im}(b_d))$. The second addendum in (4.14) is equal to $-{\rm i}EV*W$ modulo contributions to $R^{(1)}(U)$ thanks to (1.5) and (4.5).

Reasoning analogously one can prove that the term in (4.15) equals to $-{\rm i}Op^{\rm BW}\big({\rm diag}(\vec{a}_1(U)\cdot\xi)\big)W$, modulo contributions to $R^{(1)}(U)$. We are left with studying (4.16). First of all we note that $X_{\mathcal{H}^{(4)}_{\rm NLS}}(U)=-{\rm i}E|u|^2U$, then we write

$$
X_{\mathcal{H}^{(4)}_{\rm NLS}}(U)=X_{\mathcal{H}^{(4)}_{\rm NLS}}(W)+X_{\mathcal{H}^{(4)}_{\rm NLS}}(U)-X_{\mathcal{H}^{(4)}_{\rm NLS}}(W).
$$

Lemma 4.2 and item (ii) of Lemma 2.1 (recall also (4.2)), imply $\|\Phi_{\rm NLS}(U)U-U\|_{H^s}\lesssim\|U\|^7_{H^s}$, therefore it is a contribution to $R^{(1)}(U)$. We have obtained $\Phi_{\rm NLS}(U)X_{\mathcal{H}^{(4)}_{\rm NLS}}(U)=X_{\mathcal{H}^{(4)}_{\rm NLS}}(W)$ modulo $R^{(1)}(U)$.

Summarizing we obtained the (4.11) with symbols $a_2^{(1)}$ defined in (4.1) and

$$
\vec{a}_1^{(1)}=\vec{a}_1+2{\rm Im}\big\{(s_2\overline{b}_2)\nabla s_1+(s_1b_2+s_2(1+a_2))\nabla\overline{s}_2\big\}\in\mathbb{R}, \tag{4.19}
$$

with $\vec{a}_1$ in (3.2). □

4.1.2. *Diagonalization of cubic terms at order 0.* The aim of this section is to diagonalize the cubic vector field $X_{\mathcal{H}^{(4)}_{\rm NLS}}$ in (4.11) (see also (3.5)) up to smoothing remainder. In order to do this we will consider a change of coordinates which is *symplectic* up to high degree of homogeneity. We reason as follows. Define the following frequency localization:

$$
S_\xi w:=\sum_{k\in\mathbb{Z}^d}\widehat{w}(k)\chi_\epsilon\Big(\frac{|k|}{\langle\xi\rangle}\Big)e^{{\rm i}k\cdot x}, \qquad \xi\in\mathbb{Z}^d, \tag{4.20}
$$

for some $0<\epsilon<1$, where χ_ϵ is defined in (2.5). Consider the matrix of symbols

$$B_{\rm NLS}(W;x,\xi):=B_{\rm NLS}(x,\xi):=\begin{pmatrix}0 & b_{\rm NLS}(x,\xi)\\ \overline{b_{\rm NLS}(x,-\xi)} & 0\end{pmatrix},\quad b_{\rm NLS}(x,\xi)=w^2\frac{1}{2|\xi|^2}\,, \tag{4.21}$$

and the Hamiltonian function

$$\mathscr{B}_{\rm NLS}(W):=\frac{1}{2}\int_{\mathbb{T}^d}{\rm i}EOp^{\rm BW}(B_{\rm NLS}(S_\xi W;x,\xi))W\cdot\overline{W}dx\,, \tag{4.22}$$

where $S_\xi W:=(S_\xi w,S_\xi\overline{w})^T$. The presence of truncation on the high modes (S_ξ) will be decisive in obtaining Lemma A.1 (see comments in the proof of this Lemma).
Let

$$Z:=\left[\begin{smallmatrix}z\\ \overline{z}\end{smallmatrix}\right]:=\Phi_{\mathscr{B}_{\rm NLS}}(W):=W+X_{\mathscr{B}_{\rm NLS}}(W) \tag{4.23}$$

where $X_{\mathscr{B}_{\rm NLS}}$ is the Hamiltonian vector field of (4.22). We note that $\Phi_{\mathscr{B}_{\rm NLS}}$ is not symplectic, nevertheless it is close to the flow of $\mathscr{B}_{\rm NLS}(W)$ which is symplectic. The properties of $X_{\mathscr{B}_{\rm NLS}}$ and the estimates of $\Phi_{\mathscr{B}_{\rm NLS}}$ are discussed in Lemma A.1 and in Proposition A.2.

Remark 4.5. *Recall* (4.10) *and* (4.23)*. One can note that, owing to Hypothesis 4.1, for $s>2d+2$, we have*

$$\|U\|_{H^s}\sim_s\|W\|_{H^s}\sim_s\|Z\|_{H^s}\,. \tag{4.24}$$

This is a consequence of the estimates (4.5), (4.8), (A.7), (A.4), (A.9).

We introduce the following notation. We define the operator $\Lambda_{\rm NLS}$ as the Fourier multiplier acting on periodic functions as follows:

$$\Lambda_{\rm NLS}e^{{\rm i}\xi\cdot x}=\Lambda_{\rm NLS}(\xi)e^{{\rm i}\xi\cdot x}\,,\qquad \mathbb{R}\ni\Lambda_{\rm NLS}(\xi):=|\xi|^2+\widehat{V}(\xi)\,,\quad \xi\in\mathbb{Z}^d\,, \tag{4.25}$$

where $\widehat{V}(\xi)$ are the *real* Fourier coefficients of the convolution potential $V(x)$ given in (1.5). We prove the following.

Proposition 4.6 (Diagonalization at order 0). *Let $U=(u,\overline{u})$ be a solution of* (3.4) *and assume Hyp. 4.1. Define $W:=\Phi_{\rm NLS}(U)U$ where $\Phi_{\rm NLS}(U)$ is the map in* (4.4) *given in Lemma 4.3. Then the function $Z=\left[\begin{smallmatrix}z\\ \overline{z}\end{smallmatrix}\right]$ defined in* (4.23) *satisfies (recall* (4.25)*)*

$$\begin{aligned}\partial_t Z=-{\rm i}E\Lambda_{\rm NLS}Z-{\rm i}EOp^{\rm BW}\Big({\rm diag}\big(a_2^{(1)}(x)|\xi|^2\big)\Big)Z\\ -{\rm i}Op^{\rm BW}\Big({\rm diag}\big(\vec{a}_1^{(1)}(x)\cdot\xi\big)\Big)Z+X_{{\rm H}^{(4)}_{\rm NLS}}(Z)+R_5^{(2)}(U)\,,\end{aligned} \tag{4.26}$$

where $a_2^{(1)}(x)$, $\vec{a}_1^{(1)}(x)$ are the real valued symbols appearing in Proposition 4.4, the cubic vector field $X_{{\rm H}^{(4)}_{\rm NLS}}(Z)$ has the form (see (A.16)*)*

$$X_{{\rm H}^{(4)}_{\rm NLS}}(Z):=-{\rm i}EOp^{\rm BW}\begin{pmatrix}2|z|^2 & 0\\ 0 & 2|z|^2\end{pmatrix}Z+Q_{{\rm H}^{(4)}_{\rm NLS}}(Z)\,, \tag{4.27}$$

the remainder $Q_{{\rm H}^{(4)}_{\rm NLS}}$ is given by Lemma A.4 and satisfies (A.17)*-*(A.18)*. The remainder $R_5^{(2)}(U)$ has the form $(R_5^{(2,+)},\overline{R_5^{(2,+)}})^T$. Moreover, for any $s>2d+4$,*

$$\|R_5^{(2)}(U)\|_{H^s}\lesssim\|U\|^5_{H^s}\,. \tag{4.28}$$

The vector field $X_{{\rm H}^{(4)}_{\rm NLS}}(Z)$ in (4.27) *is Hamiltonian, i.e. (see* (2.33), (2.36)*) $X_{{\rm H}^{(4)}_{\rm NLS}}(Z):=-{\rm i}J\nabla{\rm H}^{(4)}_{\rm NLS}(Z)$ with*

$${\rm H}^{(4)}_{\rm NLS}(Z):=\mathscr{H}^{(4)}_{\rm NLS}(Z)-\{\mathscr{B}_{\rm NLS}(Z),\mathscr{H}^{(2)}_{\rm NLS}(Z)\}\,,\quad \mathscr{H}^{(2)}_{\rm NLS}(Z)=\int_{\mathbb{T}^d}\Lambda_{\rm NLS}z\cdot\overline{z}dx \tag{4.29}$$

where $\mathscr{H}^{(4)}_{\rm NLS}$ is in (3.10)*, and $\mathscr{B}_{\rm NLS}$ is in* (4.22), (4.21).

Proof. Recalling (3.10) and (4.29) we set

$$\mathcal{H}^{(\le 4)}_{\rm NLS}(W) := \mathcal{H}^{(2)}_{\rm NLS}(W) + \mathcal{H}^{(4)}_{\rm NLS}(W)\,. \tag{4.30}$$

Then we have that the equation (4.11) reads

$$\partial_t W = X_{\mathcal{H}^{(\le 4)}_{\rm NLS}}(W) - {\rm i}Op^{\rm BW}\big(A(U;x,\xi)\big)W + R^{(1)}(U)$$

where we set

$$A(U;x,\xi) := E{\rm diag}(a_2^{(1)}(U;x)|\xi|^2) + {\rm diag}(\vec{a}_1^{(1)}(U;x)\cdot\xi)\,. \tag{4.31}$$

Hence by (4.23) we get

$$\begin{aligned}\partial_t Z = {}& (d_W\Phi_{\mathscr{B}_{\rm NLS}}(W))\Big[-{\rm i}Op^{\rm BW}\big(A(U;x,\xi)\big)W\Big] + (d_W\Phi_{\mathscr{B}_{\rm NLS}})(W)\big[X_{\mathcal{H}^{(\le 4)}_{\rm NLS}}(W)\big]\\ &+ (d_W\Phi_{\mathscr{B}_{\rm NLS}})(W)\big[R^{(1)}(U)\big]\,.\end{aligned} \tag{4.32}$$

We study each summand separately. First of all we have that

$$\|d_W\Phi_{\mathscr{B}_{\rm NLS}}(W)\big[R^{(1)}(U)\big]\|_{H^s} \overset{(A.4),(4.13)}{\lesssim} \|u\|^7_{H^s}(1+\|w\|^2_{H^s}) \overset{(4.24)}{\lesssim} \|u\|^7_{H^s}. \tag{4.33}$$

Let us now analyze the first summand in the r.h.s. of (4.32). We write

$$\begin{aligned}&(d_W\Phi_{\mathscr{B}_{\rm NLS}}(W))\Big[{\rm i}Op^{\rm BW}\big(A(U;x,\xi)\big)W\Big] = {\rm i}Op^{\rm BW}\big(A(U;x,\xi)\big)Z + P_1 + P_2\,,\\ &P_1 := {\rm i}Op^{\rm BW}\big(A(U;x,\xi)\big)\big[W-Z\big]\,,\\ &P_2 := \Big((d_W\Phi_{\mathscr{B}_{\rm NLS}}(W)) - \mathbb{1}\Big)\big[{\rm i}Op^{\rm BW}\big(A(U;x,\xi)\big)W\big]\,.\end{aligned} \tag{4.34}$$

Fix $s_0 > d$, we have that, for $s \ge 2s_0+4$,

$$\|P_2\|_{H^s} \overset{(A.4)}{\lesssim} \|w\|^2_{H^s}\|Op^{\rm BW}\big(A(U;x,\xi)\big)W\|_{H^{s-2}} \overset{(4.12),(2.10),(4.24)}{\lesssim} \|u\|^9_{H^s}\,. \tag{4.35}$$

By (4.23), (A.4) we get $\|W-Z\|_{H^s} \lesssim \|w\|^3_{H^{s-2}}$. Therefore, by (4.34), (4.31), (4.12), (2.10) and (4.24) we get

$$\|P_1\|_{H^s} \lesssim \|u\|^6_{H^{2s_0+1}}\|W-Z\|_{H^{s+2}} \lesssim \|u\|^6_{H^{2s_0+1}}\|w\|^3_{H^s} \lesssim \|u\|^9_{H^s}\,. \tag{4.36}$$

The estimates (4.33), (4.35), (4.36) imply that the term P_1, P_2 and $d_W\Phi_{\mathscr{B}_{\rm NLS}}(W)\big[R^{(1)}(U)\big]$ can be absorbed in a remainder satisfying (4.28). Finally we consider the second summand in (4.32). By Lemma A.3 we deduce

$$d_W\Phi_{\mathscr{B}_{\rm NLS}}(W)\big[X_{\mathcal{H}^{(\le 4)}_{\rm NLS}}(W)\big] = X_{\mathcal{H}^{(\le 4)}_{\rm NLS}}(Z) + \big[X_{\mathscr{B}_{\rm NLS}}(Z), X_{\mathcal{H}^{(2)}_{\rm NLS}}(Z)\big] + R_5(Z)$$

where R_5 is a remainder satisfying the quintic estimate (A.11). By Lemma A.4 we also have that

$$X_{\mathcal{H}^{(\le 4)}_{\rm NLS}}(Z) + \big[X_{\mathscr{B}_{\rm NLS}}(Z), X_{\mathcal{H}^{(2)}_{\rm NLS}}(Z)\big] = -{\rm i}E\Lambda_{\rm NLS}Z + X_{\mathtt{H}^{(4)}_{\rm NLS}}(Z)\,,$$

with $X_{\mathtt{H}^{(4)}_{\rm NLS}}$ as in (4.27). Moreover it is Hamiltonian with Hamiltonian as in (4.29) by formulæ (A.16) and (2.37). This concludes the proof. □

Remark 4.7. *The Hamiltonian function in* (4.29) *may be rewritten, up to symmetrizations, as in* (2.42) *with coefficients* $\mathtt{h}_4(\xi,\eta,\zeta)$ *satisfying* (2.43). *The coefficients of its Hamiltonian vector field have the form* (2.46) *(see also* (2.45)*). Moreover, by* (4.27), (2.6), (A.16), (A.17)*, we deduce that*

$$2{\rm i}\mathtt{h}_4(\xi,\eta,\zeta) = 2{\rm i}\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big) + \mathtt{q}_{\mathtt{H}^{(4)}_{\rm NLS}}(\xi,\eta,\zeta)\,. \tag{4.37}$$

4.2. **Diagonalization of the KG.** In this section we diagonalize the system (3.31) up to a smoothing remainder. This will be done into two steps. We first diagonalize the matrix $E(\mathbb{1}+\mathscr{A}_1(x,\xi))$ in (3.31) by means of a change of coordinates similar to the one made in the previous section for the (NLS) case. After that we diagonalize the matrix of symbols of order 0 at homogeneity 3, by means of an *approximatively symplectic* change of coordinates. Consider the Cauchy problem associated to (KG). Throughout the rest of the section we shall assume the following.

Hypothesis 4.8. *We restrict the solution of* (KG) *on the interval of times* $[0,T)$*, with* T *such that*

$$\sup_{t\in[0,T)}\big(\|\psi(t,\cdot)\|_{H^{s+\frac{1}{2}}}+\|\partial_t\psi(t,\cdot)\|_{H^{s-\frac{1}{2}}}\big)\leq\varepsilon\,,\quad \|\psi_0(\cdot)\|_{H^{s+\frac{1}{2}}}+\|\psi_1(\cdot)\|_{H^{s-\frac{1}{2}}}\leq c_0(s)\varepsilon\ll 1\,,$$

for some $0<c_0(s)<1$ *with* $\psi(0,x)=\psi_0(x)$ *and* $(\partial_t\psi)(0,x)=\psi_1(x)$.

Note that such a T exists thanks to the local well-posedness proved in [37].

Remark 4.9. *Recall the* (2.41)*. Then one can note that* $\|\psi\|_{H^{s+\frac{1}{2}}}+\|\partial_t\psi\|_{H^{s-\frac{1}{2}}}\sim\|u\|_{H^s}$.

4.2.1. *Diagonalization at order 1.* Consider the matrix of symbols (see (3.11), (3.12))

$$E(\mathbb{1}+\mathscr{A}_1(x,\xi))\,,\qquad \mathscr{A}_1(x,\xi):=\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\widetilde{a}_2(x,\xi)\,,\quad \widetilde{a}_2(x,\xi):=\tfrac{1}{2}\Lambda_{\rm KG}^{-2}(\xi)a_2(x,\xi)\,. \tag{4.38}$$

Define

$$\lambda_{\rm KG}(x,\xi):=\sqrt{(1+\widetilde{a}_2(x,\xi))^2-(\widetilde{a}_2(x,\xi))^2},\qquad \widetilde{a}_2^+(x,\xi):=\lambda_{\rm KG}(x,\xi)-1. \tag{4.39}$$

Notice that the symbol $\lambda_{\rm KG}(x,\xi)$ is well-defined by taking $\|u\|_{H^s}\ll 1$ small enough. The matrix of eigenvectors associated to the eigenvalues of $E(\mathbb{1}+\mathscr{A}_1(x,\xi))$ is

$$\begin{gathered}S(x,\xi):=\begin{pmatrix}s_1(x,\xi)&s_2(x,\xi)\\s_2(x,\xi)&s_1(x,\xi)\end{pmatrix},\qquad S^{-1}(x,\xi):=\begin{pmatrix}s_1(x,\xi)&-s_2(x,\xi)\\-s_2(x,\xi)&s_1(x,\xi)\end{pmatrix},\\ s_1:=\frac{1+\widetilde{a}_2+\lambda_{\rm KG}}{\sqrt{2\lambda_{\rm KG}\big(1+\widetilde{a}_2+\lambda_{\rm KG}\big)}},\qquad s_2:=\frac{-\widetilde{a}_2}{\sqrt{2\lambda_{\rm KG}\big(1+\widetilde{a}_2+\lambda_{\rm KG}\big)}}.\end{gathered} \tag{4.40}$$

By a direct computation one can check that

$$S^{-1}(x,\xi)E(\mathbb{1}+\mathscr{A}_1(x,\xi))S(x,\xi)=E{\rm diag}(\lambda_{\rm KG}(x,\xi)),\qquad s_1^2-|s_2|^2=1\,. \tag{4.41}$$

We shall study how the system (3.31) transforms under the maps

$$\Phi_{\rm KG}=\Phi_{\rm KG}(U)[\cdot]:=Op^{\rm BW}(S^{-1}(x,\xi))\,,\qquad \Psi_{\rm KG}=\Psi_{\rm KG}(U)[\cdot]:=Op^{\rm BW}(S(x,\xi))\,. \tag{4.42}$$

We prove the following result.

Lemma 4.10. *Assume Hypothesis 4.8. We have the following:*

(i) *if* $s_0>d$*, then*

$$|\widetilde{a}_2^+|_{\mathcal{N}_p^0}+|\widetilde{a}_2|_{\mathcal{N}_p^0}+|s_1-1|_{\mathcal{N}_p^0}+|s_2|_{\mathcal{N}_p^0}\lesssim\|u\|^4_{H^{p+s_0+1}}\,,\quad p+s_0+1\leq s; \tag{4.43}$$

(ii) *for any* $s\in\mathbb{R}$ *one has*

$$\|\Phi_{\rm KG}(U)V-V\|_{H^s}+\|\Psi_{\rm KG}(U)V-V\|_{H^s}\lesssim\|V\|_{H^s}\|u\|^4_{H^{2s_0+1}}\,,\qquad \forall\, V\in H^s(\mathbb{T}^d;\mathbb{C}^2); \tag{4.44}$$

(iii) *one has* $\Psi_{\rm KG}(U)\circ\Phi_{\rm KG}(U)=\mathbb{1}+Q(U)$ *where* Q *is a real-to-real remainder satisfying*

$$\|Q(U)V\|_{H^{s+1}}\lesssim\|V\|_{H^s}\|u\|^4_{H^{2s_0+3}}\,; \tag{4.45}$$

(iv) *for any* $t\in[0,T)$*, one has* $\partial_t\Phi_{\rm KG}(U)[\cdot]=Op^{\rm BW}(\partial_tS^{-1}(x,\xi))$ *and*

$$|\partial_tS^{-1}(x,\xi)|_{\mathcal{N}_{s_0}^0}\lesssim\|u\|^4_{H^{2s_0+3}}\,,\qquad \|\partial_t\Phi_{\rm KG}(U)V\|_{H^s}\lesssim\|V\|_{H^s}\|u\|^4_{H^{2s_0+3}}\,. \tag{4.46}$$

Proof. (i) The (4.43) follows by (3.32) using the explicit formulæ (4.40), (4.39).
(ii) It follows by using (4.43) and item (ii) in Lemma 2.1.
(iii) By formula (2.11) in Proposition 2.2 one gets

$$\Psi_{\rm KG}(U)\circ\Phi_{\rm KG}(U)=\mathbb{1}+Op^{\rm BW}\begin{pmatrix}0 & {\rm i}\{s_1,s_2\}\\ -{\rm i}\{s_1,s_2\} & 0\end{pmatrix}+R(s_1,s_2)\,,$$

for some remainder satisfying (2.12) with $a\rightsquigarrow s_1$ and $b\rightsquigarrow s_2$. Therefore the (4.45) follows by using (2.8), (2.10) and (4.43).
(iv) It is similar to the proof of item (iii) of Lemma 4.3. □

Proposition 4.11 (**Diagonalization at order** 1)**.** *Consider the system* (3.31) *and set*

$$W=\Phi_{\rm KG}(U)U\,, \tag{4.47}$$

with $\Phi_{\rm KG}$ *defined in* (4.42). *Then W solves the equation (recall* (3.1)*)*

$$\dot W=-{\rm i}EOp^{\rm BW}\big({\rm diag}\big(1+\widetilde a_2^{+}(x,\xi)\big)\Lambda_{\rm KG}(\xi)\big)W+X_{\mathscr{H}^{(4)}_{\rm KG}}(W)+R^{(1)}(u)\,, \tag{4.48}$$

where the vector field $X_{\mathscr{H}^{(4)}_{\rm KG}}$ *is defined in* (3.15). *The symbol* $\tilde a_2^{+}$ *is defined in* (4.39). *The remainder* $R^{(1)}$ *has the form* $(R^{(1,+)},\overline{R^{(1,+)}})^T$. *Moreover, for any* $s>2d+\mu$, *for some* $\mu>0$, *it satisfies the estimate*

$$\|R^{(1)}(u)\|_{H^s}\lesssim\|u\|_{H^s}^5\,. \tag{4.49}$$

Proof. By (4.47) and (3.31) we get

$$\begin{aligned}\partial_t W&=\Phi_{\rm KG}(U)\dot U+(\partial_t\Phi_{\rm KG}(U))[U]\\ &=-{\rm i}\Phi_{\rm KG}(U)Op^{\rm BW}\big(E(\mathbb{1}+\mathscr{A}_1(x,\xi))\Lambda_{\rm KG}(\xi)\big)\Psi_{\rm KG}(U)W\\ &+\Phi_{\rm KG}(U)X_{\mathscr{H}^{(4)}_{\rm KG}}(U)\\ &+\Phi_{\rm KG}(U)R(u)+(\partial_t\Phi_{\rm KG}(U))[U]\\ &+{\rm i}\Phi_{\rm KG}(U)Op^{\rm BW}\big(E(\mathbb{1}+\mathscr{A}_1(x,\xi))(\xi)\big)Q(U)U\,,\end{aligned} \tag{4.50}$$

where we used items (ii), (iii) in Lemma 4.10. We study the first summand in the r.h.s of (4.50). By direct inspection, using Lemma 2.1 and Proposition 2.2 we get

$$\begin{aligned}-{\rm i}\Phi_{\rm KG}(U)Op^{\rm BW}\big(E(\mathbb{1}+\mathscr{A}_1(x,\xi))\Lambda_{\rm KG}(\xi)\big)\Psi_{\rm KG}(U)&=-{\rm i}Op^{\rm BW}\Big(S^{-1}E(\mathbb{1}+\mathscr{A}_1(x,\xi))S\Big)+R(u)\\ &\overset{(4.41)}{=}-{\rm i}EOp^{\rm BW}({\rm diag}(\lambda_{\rm KG}(x,\xi)))+R(u)\end{aligned}$$

where $R(u)$ is a remainder satisfying (4.49). Thanks to the discussion above and (4.39) we obtain the highest order term in (4.48). All the other summands in the r.h.s. of (4.50) may be analyzed as done in the proof of Prop. 4.4 by using Lemma 4.10. □

4.2.2. *Diagonalization of cubic terms at order 0.* In the previous section we showed that if the function U solves (3.31) then W in (4.47) solves (4.48). The cubic terms in the system (4.48) are the same appearing in (3.31) and have the form (3.15). The aim of this section is to diagonalize the matrix of symbols of order zero $\mathscr{A}_0(x,\xi)$. We must preserve the Hamiltonian structure of the cubic terms in performing this step. In order to do this, in analogy with the (NLS) case, we reason as follows. Consider the matrix of symbols

$$B_{\rm KG}(W;x,\xi):=B_{\rm KG}(x,\xi):=\begin{pmatrix}0 & b_{\rm KG}(x,\xi)\\ \overline{b_{\rm KG}(x,-\xi)} & 0\end{pmatrix},\quad b_{\rm KG}(W;x,\xi)=\frac{a_0(x,\xi)}{2\Lambda_{\rm KG}(\xi)}\,, \tag{4.51}$$

with $a_0(x,\xi)$ in (3.11), and define the Hamiltonian function

$$\mathscr{B}_{\rm KG}(W):=\frac12\int_{\mathbb{T}^d}{\rm i}EOp^{\rm BW}(B_{\rm KG}(S_\xi W;x,\xi))W\cdot\overline{W}dx\,, \tag{4.52}$$

where $S_\xi W := (S_\xi w, S_\xi \overline{w})^T$ where S_ξ is in (4.20). Let us define

$$Z := \left[\begin{smallmatrix} z \\ \bar{z} \end{smallmatrix}\right] := \Phi_{\mathscr{B}_{\rm KG}}(W) := W + X_{\mathscr{B}_{\rm KG}}(W) \tag{4.53}$$

where $X_{\mathscr{B}_{\rm KG}}$ is the Hamiltonian vector field of (4.52) and W is the function in (4.47). The properties of $X_{\mathscr{B}_{\rm KG}}$ and the estimates of $\Phi_{\mathscr{B}_{\rm KG}}$ are discussed in Lemma A.1 and in Proposition A.2.

Remark 4.12. *Recall* (4.47) *and* (4.53). *One can note that, owing to Hypothesis 4.8, for $s > 2d+3$, we have*

$$\|U\|_{H^s} \sim_s \|W\|_{H^s} \sim_s \|Z\|_{H^s}. \tag{4.54}$$

This is a consequence of the estimates (4.44), (4.45) (A.7), (A.5), (A.9).

Proposition 4.13 (Diagonalization at order 0). *Let U be a solution of* (3.31) *and assume Hyp. 4.8 (see also Remark 4.9). Then the function Z defined in* (4.53), *with W given in* (4.47), *satisfies*

$$\partial_t Z = -\mathrm{i} E Op^{\rm BW}\big(\mathrm{diag}\big(1+\tilde{a}_2^+(x,\xi)\big)\Lambda_{\rm KG}(\xi)\big) Z + X_{\mathtt{H}^{(4)}_{\rm KG}}(Z) + R_5^{(2)}(u), \tag{4.55}$$

where $\tilde{a}_2^+(x,\xi)$ is the real valued symbol in (4.39), *the cubic vector field $X_{\mathtt{H}^{(4)}_{\rm KG}}(Z)$ has the form*

$$X_{\mathtt{H}^{(4)}_{\rm KG}}(Z) := -\mathrm{i} E Op^{\rm BW}\big(\mathrm{diag}(a_0(x,\xi))\big) Z + Q_{\mathtt{H}^{(4)}_{\rm KG}}(Z) \tag{4.56}$$

the symbol $a_0(x,\xi)$ is in (3.11), *the remainder $Q_{\mathtt{H}^{(4)}_{\rm KG}}(Z)$ is the cubic remainder given in Lemma A.5. The remainder $R_5^{(2)}(u)$ has the form $(R_5^{(2,+)}(u), \overline{R_5^{(2,+)}}(u))^T$. Moreover, for any $s > 2d+\mu$, for some $\mu > 0$, we have the estimate*

$$\|R_5^{(2)}(u)\|_{H^s} \lesssim \|u\|^5_{H^s}. \tag{4.57}$$

Finally the vector field $X_{\mathtt{H}^{(4)}_{\rm KG}}(Z)$ in (4.56) *is Hamiltonian, i.e. $X_{\mathtt{H}^{(4)}_{\rm KG}}(Z) := -\mathrm{i} J \nabla \mathtt{H}^{(4)}_{\rm KG}(Z)$ with*

$$\mathtt{H}^{(4)}_{\rm KG}(Z) := \mathscr{H}^{(4)}_{\rm KG}(Z) - \{\mathscr{B}_{\rm KG}(Z), \mathscr{H}^{(2)}_{\rm KG}(Z)\}, \quad \mathscr{H}^{(2)}_{\rm KG}(Z) = \int_{\mathbb{T}^d} \Lambda_{\rm KG} z \cdot \bar{z} dx \tag{4.58}$$

where $\mathscr{H}^{(4)}_{\rm KG}$ is in (3.14), *and $\mathscr{B}_{\rm KG}$ is in* (4.52), (4.51).

Proof. Recalling (3.14) and (the second equation in) (4.58) we define

$$\mathscr{H}^{(\leq 4)}_{\rm KG}(W) := \mathscr{H}^{(2)}_{\rm KG}(W) + \mathscr{H}^{(4)}_{\rm KG}(W), \tag{4.59}$$

and we rewrite the equation (4.48) as

$$\partial_t W = X_{\mathscr{H}^{(\leq 4)}_{\rm KG}}(W) - \mathrm{i} E Op^{\rm BW}\big(\tilde{a}_2^+(x,\xi)\Lambda_{\rm KG}(\xi)\big) W + R^{(1)}(u).$$

Then, using (4.53), we get

$$\begin{aligned} \partial_t Z &= d_W \Phi_{\mathscr{B}_{\rm KG}}(W)\big[\partial_t W\big] \\ &= d_W \Phi_{\mathscr{B}_{\rm KG}}(W)\big[X_{\mathscr{H}^{(\leq 4)}_{\rm KG}}(W)\big] && (4.60) \\ &\quad + d_W \Phi_{\mathscr{B}_{\rm KG}}(W)\big[-\mathrm{i} E Op^{\rm BW}\big(\mathrm{diag}(\tilde{a}_2^+(x,\xi)\Lambda_{\rm KG}(\xi))\big) W\big] && (4.61) \\ &\quad + d_W \Phi_{\mathscr{B}_{\rm KG}}(W)\big[R^{(1)}(u)\big]. && (4.62) \end{aligned}$$

By estimates (A.5) and (4.49) we have that the term in (4.62) can be absorbed in a remainder satisfying the (4.57). Consider the term in (4.61). We write

$$\begin{aligned} &(4.61) = -\mathrm{i} E Op^{\rm BW}\big(\mathrm{diag}(\tilde{a}_2^+(x,\xi)\Lambda_{\rm KG}(\xi))\big) Z + P_1 + P_2, \\ &P_1 := -\mathrm{i} E Op^{\rm BW}\big(\mathrm{diag}(\tilde{a}_2^+(x,\xi)\Lambda_{\rm KG}(\xi))\big)\big[W - Z\big], \\ &P_2 := \Big((d_W \Phi_{\mathscr{B}_{\rm KG}}(W)) - \mathbb{1}\Big)\big[-\mathrm{i} E Op^{\rm BW}\big(\mathrm{diag}(\tilde{a}_2^+(x,\xi)\Lambda_{\rm KG}(\xi))\big) W\big]. \end{aligned} \tag{4.63}$$

We have that, for $s \geq 2s_0 + 2$,

$$\|P_2\|_{H^s} \overset{(A.5)}{\lesssim} \|u\|^2_{H^s} \|Op^{\rm BW}\big(\tilde{a}_2^+(x,\xi)\Lambda_{\rm KG}(\xi)\big) w\|_{H^{s-1}} \overset{(4.43),(2.10),(4.54)}{\lesssim} \|u\|^7_{H^s},$$

which implies the (4.57). By (A.9) in Lemma A.2 and estimate (A.5) we deduce $\|W-Z\|_{H^{s+1}}\lesssim\|u\|^3_{H^s}$. Hence using again (4.43), (2.10), (4.54) we get P_1 satisfies (4.57). It remains to discuss the structure of the term in (4.60). By Lemma A.3 we obtain

$$d_W\Phi_{\mathscr{B}_{\rm KG}}(W)\big[X_{\mathscr{H}^{(\leq4)}_{\rm KG}}(W)\big]=X_{\mathscr{H}^{(\leq4)}_{\rm KG}}(Z)+\big[X_{\mathscr{B}_{\rm KG}}(Z),X_{\mathscr{H}^{(2)}_{\rm KG}}(Z)\big]\,, \tag{4.64}$$

modulo remainders that can be absorbed in $R_5^{(2)}$ satisfying (4.57). The (4.64), (4.60)-(4.62) and the discussion above imply the (4.55) where the cubic vector field has the form

$$X_{\mathtt{H}^{(4)}_{\rm KG}}(Z)=X_{\mathscr{H}^{(4)}_{\rm KG}}(Z)+\big[X_{\mathscr{B}_{\rm KG}}(Z),X_{\mathscr{H}^{(2)}_{\rm KG}}(Z)\big]\,. \tag{4.65}$$

Using (2.37), (2.36), we conclude that $X_{\mathtt{H}^{(4)}_{\rm KG}}$ is the Hamiltonian vector field of $\mathtt{H}^{(4)}_{\rm KG}$ in (4.58). The (4.56) follows by Lemma A.5. $\square$

Remark 4.14. *In view of Remarks 3.5, 3.7, following the same proof of Proposition 4.13, in the semi-linear case we obtain that equation* (4.55) *reads*

$$\partial_t Z=-\mathrm{i}EOp^{\rm BW}\big(\mathrm{diag}(\Lambda_{\rm KG}(\xi))\big)Z+X_{\mathtt{H}^{(4)}_{\rm KG}}(Z)+R_5^{(2)}(u)\,,$$

where $X_{\mathtt{H}^{(4)}_{\rm KG}}$ *has the form* (4.56) *with* $a_0(x,\xi)$ *a symbol of order* -1 *and* $Q_{\mathtt{H}^{(4)}_{\rm KG}}$ *a remainder of the form* (A.23) *with coefficients satisfying* (A.24) *with the better denominator* $\max\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^2$.

5. ENERGY ESTIMATES

5.1. **Estimates for the NLS.** In this section we prove *a priori* energy estimates on the Sobolev norms of the variable Z in (4.23). In subsection 5.1.1 we introduce a convenient energy norm on $H^s(\mathbb{T}^d;\mathbb{C})$ which is equivalent to the classic H^s-norm. This is the content of Lemma 5.2. In subsection 5.1.2, using the non-resonance conditions of Proposition 5.6, we provide bounds on the non-resonant terms appearing in the energy estimates. We deal with resonant interactions in Lemma 5.4.

5.1.1. *Energy norm.* Let us define the symbol

$$\mathscr{L}=\mathscr{L}(x,\xi):=|\xi|^2+\Sigma\,,\quad \Sigma=\Sigma(x,\xi):=a_2^{(1)}(x)|\xi|^2+\vec{a}_1^{(1)}(x)\cdot\xi\,, \tag{5.1}$$

where the symbols $a_2^{(1)}(x)$, $\vec{a}_1^{(1)}(x)$ are given in Proposition 4.4. We have the following.

Lemma 5.1. *Assume the Hypothesis 4.1 and let* $\gamma>0$. *Then for* $\varepsilon>0$ *small enough we have the following.*
(i) *One has*

$$\begin{aligned}&|\mathscr{L}|_{\mathscr{N}^2_{s_0}}+|\mathscr{L}^\gamma|_{\mathscr{N}^{2\gamma}_{s_0}}\leq1+C\|u\|^6_{H^{2s_0+1}}\,,\\&|\Sigma|_{\mathscr{N}^2_{s_0}}+|\mathscr{L}^\gamma-|\xi|^{2\gamma}|_{\mathscr{N}^{2\gamma}_{s_0}}\lesssim\|u\|^6_{H^{2s_0+1}}\end{aligned} \tag{5.2}$$

for some $C>0$ *depending on* s_0.
(ii) *For any* $s\in\mathbb{R}$ *and any* $h\in H^s(\mathbb{T}^d;\mathbb{C})$, *one has*

$$\begin{aligned}&\|T_{\mathscr{L}}h\|_{H^{s-2}}+\|T_{\mathscr{L}^\gamma}h\|_{H^{s-2\gamma}}\leq\|h\|_{H^s}(1+C\|u\|^6_{H^{2s_0+1}})\,,\\&\|T_\Sigma h\|_{H^{s-2}}+\|T_{\mathscr{L}^\gamma-|\xi|^{2\gamma}}h\|_{H^{s-2\gamma}}\lesssim\|h\|_{H^s}\|u\|^6_{H^{2s_0+1}}\,,\end{aligned} \tag{5.3}$$

for some $C>0$ *depending on* s.
(iii) *For any* $t\in[0,T)$ *one has* $|\partial_t\Sigma|_{\mathscr{N}^2_{s_0}}\lesssim\|u\|^6_{H^{2s_0+3}}$. *Moreover*

$$\|(T_{\partial_t\mathscr{L}^\gamma})h\|_{H^{s-2\gamma}}\lesssim\|h\|_{H^s}\|u\|^6_{H^{2s_0+3}}\,,\qquad\forall\,h\in H^s(\mathbb{T}^d;\mathbb{C})\,. \tag{5.4}$$

(iv) *The operators* $T_{\mathscr{L}}$, $T_{\mathscr{L}^\gamma}$ *are self-adjoint with respect to the* L^2*-scalar product* (2.3).

Proof. Items (i)-(ii). The (5.2) follows by using (5.1), the bounds (4.12) on the symbols $a_2^{(1)}$ and $\vec{a}_1^{(1)}\cdot\xi$. The (5.3) follows by Lemma 2.1.
Item (iii). The bound on $\partial_t\Sigma$ follows by reasoning as in item (iii) of Lemma 4.3 using the explicit formula of $a_2^{(1)}$ in (4.1) and the formula for $a_1^{(1)}\cdot\xi$ in (4.19) (see also (4.2)). Then the (2.10) implies the (5.4).
Item (iv). This follows by (2.20) since the symbol $\mathscr{L}$ in (5.1) is real-valued. □

In the following we shall construct the *energy norm*. By using this norm we are able to achieve the *energy estimates* on the previously diagonalized system. For $s\in\mathbb{R}$ we define

$$z_n := T_{\mathscr{L}^n} z\,, \qquad Z_n = \left[\begin{smallmatrix} z_n \\ \overline{z_n}\end{smallmatrix}\right] := T_{\mathscr{L}^n}\mathbb{1}Z\,, \quad Z = \left[\begin{smallmatrix} z \\ \overline{z}\end{smallmatrix}\right], \quad n := s/2\,. \tag{5.5}$$

Lemma 5.2. (Equivalence of the energy norm). *Assume Hypothesis 4.1 with $s > 2d+4$. Then, for $\varepsilon > 0$ small enough enough, one has*

$$\|z\|_{L^2} + \|z_n\|_{L^2} \sim \|z\|_{H^s}\,. \tag{5.6}$$

Proof. Let $s = 2n$. Then by (5.3) and (5.5) we have $\|z_n\|_{L^2} \le \|z\|_{H^s}(1 + C\|u\|^6_{H^{2s_0+1}})$, with $s_0 > d$. Moreover

$$\|z\|_{H^s} \sim \|z\|_{L^2} + \|T_{|\xi|^{2n}} z\|_{L^2} \overset{(5.3)}{\le} \|z\|_{L^2} + \|z_n\|_{L^2} + C\|z\|_{H^s}\|u\|^6_{H^{2s_0+1}}$$

which implies $(1 - C\|u\|^6_{H^{2s_0+1}})\|z\|_{H^s} \le \|z\|_{L^2} + \|z_n\|_{L^2}$, for some constant C depending on s. The discussion above implies the (5.6) by taking $\varepsilon > 0$ in Hyp. 4.1 small enough. □

Recalling (4.26), (4.25) and (5.1) we have

$$(\partial_t + \mathrm{i}\Lambda_{\rm NLS}) z = -\mathrm{i}T_\Sigma z + X^+_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) + R_5^{(2,+)}(U)\,, \qquad Z = \left[\begin{smallmatrix} z \\ \overline{z}\end{smallmatrix}\right], \tag{5.7}$$

where $X_{\mathtt{H}_4^{(4)}}$ is given in (4.27) (see also Remark 4.7) and $R_5^{(2,+)}$ is the remainder satisfying (4.28).

Lemma 5.3. *Fix $s > 2d+4$ and recall* (5.7). *One has that the function z_n defined in* (5.5) *solves the problem*

$$\partial_t z_n = -\mathrm{i}T_{\mathscr{L}} z_n - \mathrm{i}V * z_n + T_{|\xi|^{2n}} X^{+,\rm res}_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) + B_n^{(1)}(Z) + B_n^{(2)}(Z) + R_{5,n}(U)\,, \tag{5.8}$$

where $X^{+,\rm res}_{\mathtt{H}^{(4)}_{\rm NLS}}$ is defined as in Def. 2.7,

$$\begin{aligned} \widehat{B_n^{(1)}(Z)}(\xi) &= \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} b^{(1)}(\xi,\eta,\zeta)\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}_n(\zeta)\,, \\ \widehat{B_n^{(2)}(Z)}(\xi) &= \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} b_n^{(2)}(\xi,\eta,\zeta)\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}(\zeta)\,, \end{aligned} \tag{5.9}$$

with

$$b^{(1)}(\xi,\eta,\zeta) := -2\mathrm{i}\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta)\,, \tag{5.10}$$

$$|b_n^{(2)}(\xi,\eta,\zeta)| \lesssim \frac{\langle\xi\rangle^{2n}\max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^4}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}}\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta)\,, \tag{5.11}$$

and where the remainder $R_{5,n}$ satisfies

$$\|R_{5,n}(U)\|_{L^2} \lesssim \|u\|^5_{H^s}\,. \tag{5.12}$$

Proof. Recalling (2.48) we define

$$X^{+,\perp}_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) := X^+_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) - X^{+,\rm res}_{\mathtt{H}^{(4)}_{\rm NLS}}(Z)\,. \tag{5.13}$$

By differentiating (5.5) and using the (5.1) and (5.7) we get

$$\begin{aligned} \partial_t z_n &= T_{\mathscr{L}^n}\partial_t z + T_{\partial_t\mathscr{L}^n} z \\ &= -\mathrm{i}T_{\mathscr{L}} z_n - \mathrm{i}T_{\mathscr{L}^n}(V*z) + T_{\mathscr{L}^n} X^+_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) + T_{\mathscr{L}^n} R_5^{(2,+)}(U) + T_{\partial_t\mathscr{L}^n} z - \mathrm{i}[T_{\mathscr{L}^n}, T_{\mathscr{L}}]z\,. \end{aligned} \tag{5.14}$$

By using Lemmata 2.1, 5.1 and Proposition 2.2, and the (5.6), (4.24) one proves that the last summand gives a contribution to $R_{5,n}(U)$ satisfying (5.12). By using (5.4), (4.24), (4.28) we deduce that

$$\|T_{\mathscr{L}^n} R_5^{(2,+)}(U)\|_{L^2} + \|T_{\partial_t \mathscr{L}^n} z\|_{L^2} \lesssim \|u\|_{H^s}^5 .$$

Secondly we write

$$\mathrm{i} T_{\mathscr{L}^n}(V * z) = \mathrm{i} V * z_n + \mathrm{i} V * \big(T_{|\xi|^{2n} - \mathscr{L}^n} z\big) + \mathrm{i} T_{\mathscr{L}^n - |\xi|^{2n}}(V * z) .$$

By (5.3), (4.24), and recalling (1.5) we conclude $\|T_{\mathscr{L}^n}(V*z) - V * z_n\|_{L^2} \lesssim \|u\|_{H^s}^7$. We now study the third summand in (5.14). We have (see (5.13))

$$T_{\mathscr{L}^n} X^+_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) = T_{|\xi|^{2n}} X^{+,\rm res}_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) + T_{|\xi|^{2n}} X^{+,\perp}_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) + T_{\mathscr{L}^n - |\xi|^{2n}} X^+_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) .$$

By (5.3), (4.27), (2.10), Lemma 2.5 and using the estimate (A.18), one obtains

$$\|T_{\mathscr{L}^n - |\xi|^{2n}} X^+_{\mathtt{H}^{(4)}_{\rm NLS}}(Z)\|_{L^2} \lesssim \|u\|_{H^s}^9 .$$

Recalling (4.37) and (5.13) we write

$$\begin{aligned}
& T_{|\xi|^{2n}} X^{+,\perp}_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) = \mathscr{C}_1 + \mathscr{C}_2 + \mathscr{C}_3 , \qquad \widehat{\mathscr{C}_i}(\xi) = \tfrac{1}{(2\pi)^d} \sum_{\eta,\zeta \in \mathbb{Z}^d} \mathtt{c}_i(\xi,\eta,\zeta) \widehat{z}(\xi-\eta-\zeta) \widehat{\overline{z}}(\eta) \widehat{z}(\zeta) , \\
& \mathtt{c}_1(\xi,\eta,\zeta) := -2\mathrm{i} \chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle \xi+\zeta\rangle}\Big) |\zeta|^{2n} 1_{\mathscr{R}^c}(\xi,\eta,\zeta) \\
& \mathtt{c}_2(\xi,\eta,\zeta) := -2\mathrm{i} \chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle \xi+\zeta\rangle}\Big) \Big[|\xi|^{2n} - |\zeta|^{2n}\Big] 1_{\mathscr{R}^c}(\xi,\eta,\zeta) \\
& \mathtt{c}_3(\xi,\eta,\zeta) := \mathtt{q}_{\mathtt{H}^{(4)}_{\rm NLS}}(\xi,\eta,\zeta) |\xi|^{2n} 1_{\mathscr{R}^c}(\xi,\eta,\zeta)
\end{aligned} \tag{5.15}$$

We now consider the operator $\mathscr{C}_1$ with coefficients $\mathtt{c}_1(\xi,\eta,\zeta)$. First of all we remark that it can be written as $\mathscr{C}_1 = M(z,\overline{z},z)$ where M is a trilinear operator of the form (2.25). Moreover, setting

$$z_n = T_{|\xi|^{2n}} z + h_n , \quad h_n := T_{\mathscr{L}^n - |\xi|^{2n}} z ,$$

we can write $\mathscr{C}_1 = B_n^{(1)}(Z) - M(z,\overline{z},h_n)$, where $B_n^{(1)}$ has the form (5.9) with coefficients as in (5.10). Using that $|\mathtt{c}_1(\xi,\eta,\zeta)| \lesssim 1$, Lemma 2.5 (with $m = 0$) and (5.3) we deduce that $\|M(z,\overline{z},h_n)\|_{L^2} \lesssim \|u\|_{H^s}^9$. Therefore this is a contribution to $R_{5,n}(U)$ satisfying (5.12). The discussion above implies formula (5.8) by setting $B_n^{(2)}$ as the operator of the form (5.9) with coefficients $b_n^{(2)}(\xi,\eta,\zeta) := \mathtt{c}_2(\xi,\eta,\zeta) + \mathtt{c}_3(\xi,\eta,\zeta)$. The coefficient $\mathtt{c}_3(\xi,\eta,\zeta)$ satisfies the (5.11) by (A.18). For the coefficient $\mathtt{c}_2(\xi,\eta,\zeta)$ one has to apply Lemma 2.6 with $\mu = m = 1$ and $f(\xi,\eta,\zeta) := (|\xi|^{2n} - |\zeta|^{2n})|\xi|^{-2n}$. This concludes the proof. □

In the following lemma we prove a key cancellation due to the fact that the *super actions* are prime integrals of the resonant Hamiltonian vector field $X^{+,\rm res}_{\mathtt{H}_4}(Z)$ in the same spirit of [24]. We also prove an important algebraic property of the operator $B_n^{(1)}$ in (5.8).

Lemma 5.4. *For any $n \geq 0$ we have*

$$\mathrm{Re}(T_{|\xi|^n} X^{+,\rm res}_{\mathtt{H}^{(4)}_{\rm NLS}}(Z), T_{|\xi|^n} z)_{L^2} = 0 , \tag{5.16}$$

$$\mathrm{Re}(B_n^{(1)}(Z), z_n)_{L^2} = 0 , \tag{5.17}$$

where $X^{+,\rm res}_{\mathtt{H}^{(4)}_{\rm NLS}}$ is defined in Lemma 5.3 and $B_n^{(1)}$ in (5.9), (5.10).

Proof. The (5.16) follows by Lemma 2.8. Let us check the (5.17). By an explicit computation using (2.3), (5.9) we get

$$\begin{aligned}\mathrm{Re}(B_n^{(1)}(Z), z_n)_{L^2} &= \frac{1}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} b^{(1)}(\xi,\eta,\zeta)\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}_n(\zeta)\widehat{\overline{z}}_n(-\xi)\\ &+\frac{1}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d}\overline{b^{(1)}(\xi,\eta,\zeta)}\widehat{\overline{z}}(-\xi+\eta+\zeta)\widehat{z}(-\eta)\widehat{\overline{z}}_n(-\zeta)\widehat{z}_n(\xi)\\ &=\frac{1}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d}\Big[b^{(1)}(\xi,\eta,\zeta)+\overline{b^{(1)}(\zeta,\zeta+\eta-\xi,\xi)}\Big]\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}_n(\zeta)\widehat{\overline{z}}_n(-\xi)\,.\end{aligned}$$

By (5.10) we have

$$b^{(1)}(\xi,\eta,\zeta)+\overline{b^{(1)}(\zeta,\zeta+\eta-\xi,\xi)} = 2\mathrm{i}\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\big[1_{\mathscr{R}^c}(\xi,\eta,\zeta)-1_{\mathscr{R}^c}(\zeta,\zeta+\eta-\xi,\xi)\big]=0\,,$$

where we used the form of the resonant set $\mathscr{R}$ in (2.47). This proves the lemma. □

We conclude the section with the following proposition.

Proposition 5.5. *Let $u(t,x)$ be a solution of* (NLS) *satisfying Hypothesis 4.1 and consider the function z_n in* (5.5) *(see also* (4.23), (4.10)*). Then, setting $s=2n>2d+4$ we have $\|z_n(t)\|_{L^2}\sim\|u(t)\|_{H^s}$ and*

$$\partial_t\|z_n(t)\|_{L^2}^2 = \mathscr{B}(t)+\mathscr{B}_{>5}(t)\,,\qquad t\in[0,T)\,, \tag{5.18}$$

where

- *the term $\mathscr{B}(t)$ has the form*

$$\begin{aligned}\mathscr{B}(t) &= \frac{2}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d}|\xi|^{2n}\mathtt{b}(\xi,\eta,\zeta)\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}(\zeta)\widehat{\overline{z}}(-\xi)\,,\\ \mathtt{b}(\xi,\eta,\zeta) &= b_n^{(2)}(\xi,\eta,\zeta)+\overline{b_n^{(2)}(\zeta,\zeta+\eta-\xi,\xi)}\,,\qquad \xi,\eta,\zeta\in\mathbb{Z}^d\,,\end{aligned} \tag{5.19}$$

where $b_n^{(2)}(\xi,\eta,\zeta)$ are the coefficients in (5.9), (5.11)*;*

- *the term $\mathscr{B}_{>5}(t)$ satisfies*

$$|\mathscr{B}_{>5}(t)|\lesssim\|u\|_{H^s}^6\,,\quad t\in[0,T)\,. \tag{5.20}$$

Proof. The norm $\|z_n\|_{L^2}$ is equivalent to $\|u\|_{H^s}$ by using Lemma 5.2 and Remark 4.5. By using (5.8) we get

$$\frac12\partial_t\|z_n(t)\|_{L^2}^2 = \mathrm{Re}(T_{|\xi|^{2n}}X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{NLS}}}(Z),z_n)_{L^2} \tag{5.21}$$

$$+\mathrm{Re}(-\mathrm{i}T_{\mathscr{L}}z_n,z_n)_{L^2}+\mathrm{Re}(B_n^{(1)}(Z),z_n)_{L^2}+\mathrm{Re}(-\mathrm{i}V*z_n,z_n)_{L^2} \tag{5.22}$$

$$+\mathrm{Re}(B_n^{(2)}(Z),z_n)_{L^2} \tag{5.23}$$

$$+\mathrm{Re}(R_{5,n}(Z),z_n)_{L^2}\,. \tag{5.24}$$

Recall that $T_{\mathscr{L}}$ is self-adjoint (see item (iv) in Lemma 5.1) and the convolution potential V has real Fourier coefficients. Then by using also Lemma 5.4 (see (5.17)) we deduce (5.22) = 0. Moreover by Cauchy-Schwarz inequality, estimates (5.12), (5.6) and (4.24) we obtain that the term in (5.24) is bounded form above by $\|u\|_{H^s}^6$. Consider the terms in (5.21) and (5.23). Recalling (5.5) and (5.1) we write

$$\begin{aligned}\mathrm{Re}(T_{|\xi|^{2n}}X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{NLS}}}(Z),z_n)_{L^2} &= \mathrm{Re}(T_{|\xi|^{2n}}X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{NLS}}}(Z),T_{|\xi|^{2n}}z)_{L^2}+\mathrm{Re}(T_{|\xi|^{2n}}X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{NLS}}}(Z),T_{\mathscr{L}^n-|\xi|^{2n}}z)_{L^2}\,,\\ &\overset{(5.16)}{=}\mathrm{Re}(T_{|\xi|^{2n}}X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{NLS}}}(Z),T_{\mathscr{L}^n-|\xi|^{2n}}z)_{L^2}\,.\end{aligned}$$

Moreover we write

$$\mathrm{Re}(B_n^{(2)}(Z),z_n)_{L^2} = \mathrm{Re}(B_n^{(2)}(Z),T_{|\xi|^{2n}}z)_{L^2}+\mathrm{Re}(B_n^{(2)}(Z),T_{\mathscr{L}^n-|\xi|^{2n}}z)_{L^2}\,.$$

Using the bound (5.3) in Lemma 5.1 to estimate the operator $T_{\mathscr{L}^n-|\xi|^{2n}}$, Lemma 2.5 and (5.11) to estimate the operator $B_n^{(2)}(Z)$, we get

$$|\mathrm{Re}(T_{|\xi|^{2n}}X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{NLS}}}(Z), T_{\mathscr{L}^n-|\xi|^{2n}}z)_{L^2}| + |\mathrm{Re}(B_n^{(2)}(Z), T_{\mathscr{L}^n-|\xi|^{2n}}z)_{L^2}| \lesssim \|u\|_{H^s}^{10},$$

which means that these remainders can be absorbed in the term $\mathscr{B}_{>5}(t)$. Then we set

$$\mathscr{B}(t) := 2\mathrm{Re}(B_n^{(2)}(Z), T_{|\xi|^{2n}}z)_{L^2}.$$

Formulæ (5.19) follow by an explicit computation using (5.9), (5.11). □

5.1.2. *Estimates of non-resonant terms.* In this subsection we provide estimates on the term $\mathscr{B}(t)$ appearing in (5.18).

Proposition 5.6. (Non-resonance conditions). *Consider the phase $\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)$ defined as*

$$\omega_{\mathrm{NLS}}(\xi,\eta,\zeta) := \Lambda_{\mathrm{NLS}}(\xi-\eta-\zeta) - \Lambda_{\mathrm{NLS}}(\eta) + \Lambda_{\mathrm{NLS}}(\zeta) - \Lambda_{\mathrm{NLS}}(\xi), \qquad (\xi,\eta,\zeta)\in\mathbb{Z}^{3d} \tag{5.25}$$

where Λ_{NLS} is in (4.25) *and the potential V is in* (1.5). *We have the following.*
(i) *Let $d\geq 2$. There exists $\mathcal{N}\subset\mathcal{O}$ with zero Lebesgue measure such that, for any $(x_i)_{i\in\mathbb{Z}^d}\in\mathcal{O}\setminus\mathcal{N}$, there exist $\gamma>0$, $N_0 := N_0(d,m)>0$ such that for any $(\xi,\eta,\zeta)\notin\mathscr{R}$ (see* (2.47)*) one has*

$$|\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)| \geq \gamma \max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{-N_0}. \tag{5.26}$$

(ii) *Let $d=1$ and assume that $V\equiv 0$. Then one has $|\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)|\gtrsim 1$ unless*

$$\xi=\zeta,\ \ \eta=\xi-\eta-\zeta, \quad \text{or} \quad \xi=\xi-\eta-\zeta,\ \ \eta=\zeta, \qquad \xi,\eta,\zeta\in\mathbb{Z}. \tag{5.27}$$

Proof. Item (i) follows by Proposition 2.8 in [25]. Item (ii) is classical. □

We are now in position to state the main result of this section.

Proposition 5.7. *Let $N>0$. Then there is $s_0 = s_0(N_0)$, where $N_0>0$ is given by Proposition 5.6, such that, if Hypothesis 4.1 holds with $s\geq s_0$, one has*

$$\left|\int_0^t \mathscr{B}(\sigma)d\sigma\right| \lesssim \|u\|^{10}_{L^\infty H^s}TN + \|u\|^6_{L^\infty H^s}T + \|u\|^4_{L^\infty H^s}TN^{-1} + \|u\|^4_{L^\infty H^s}, \tag{5.28}$$

where $\mathscr{B}(t)$ is in (5.19).

We need some preliminary results. We consider the following trilinear maps:

$$\mathscr{B}_i = \mathscr{B}_i[z_1,z_2,z_3], \quad \widehat{\mathscr{B}_i}(\xi) = \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}\mathtt{b}_i(\xi,\eta,\zeta)\widehat{z}_1(\xi-\eta-\zeta)\widehat{z}_2(\eta)\widehat{z}_3(\zeta), \ \ i=1,2, \tag{5.29}$$

$$\mathscr{T}_< = \mathscr{T}_<[z_1,z_2,z_3], \quad \widehat{\mathscr{T}_<}(\xi) = \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}\mathtt{t}_<(\xi,\eta,\zeta)\widehat{z}_1(\xi-\eta-\zeta)\widehat{z}_2(\eta)\widehat{z}_3(\zeta), \tag{5.30}$$

where

$$\mathtt{b}_1(\xi,\eta,\zeta) = \mathtt{b}(\xi,\eta,\zeta)\mathbb{1}_{\{\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}\leq N\}}, \tag{5.31}$$

$$\mathtt{b}_2(\xi,\eta,\zeta) = \mathtt{b}(\xi,\eta,\zeta)\mathbb{1}_{\{\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}> N\}}, \tag{5.32}$$

$$\mathtt{t}_<(\xi,\eta,\zeta) = \tfrac{-1}{\mathrm{i}\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)}\mathtt{b}_1(\xi,\eta,\zeta), \tag{5.33}$$

where $\mathtt{b}(\xi,\eta,\zeta)$ are the coefficients in (5.19), and ω_{NLS} is the phase in (5.25). We remark that if $(\xi,\eta,\zeta)\in\mathscr{R}$ then the coefficients $\mathtt{b}(\xi,\eta,\zeta)$ are equal to zero (see (5.19), (5.9), (5.11)). Therefore, since ω_{NLS} is non-resonant (see Proposition 5.6), the coefficients in (5.33) are well-defined. We now prove an abstract results on the trilinear maps introduced in (5.29)-(5.30).

Lemma 5.8. *One has that, for $s = 2n > d/2+4$,*

$$\|\mathscr{B}_2[z_1,z_2,z_3]\|_{L^2} \lesssim N^{-1}\sum_{i=1}^{3}\|z_i\|_{H^s}\prod_{i\neq k}\|z_k\|_{H^{d/2+4+\epsilon}}\,, \qquad \forall \epsilon>0\,. \tag{5.34}$$

There is $s_0(N_0)>0$ ($N_0>0$ given by Proposition 5.6) such that for $s\geq s_0(N_0)$ one has

$$\|\mathscr{T}_<[z_1,z_2,z_3]\|_{H^p} \lesssim N\sum_{i=1}^{3}\|z_i\|_{H^{s+p-2}}\prod_{i\neq k}\|z_k\|_{H^{s_0}}\,, \quad p\in\mathbb{N}, \tag{5.35}$$

$$\|\mathscr{T}_<[z_1,z_2,z_3]\|_{L^2} \lesssim \sum_{i=1}^{3}\|z_i\|_{H^s}\prod_{i\neq k}\|z_k\|_{H^{s_0}}\,. \tag{5.36}$$

Proof. Using (5.32), (5.19), (5.11) we get that

$$\begin{aligned}\|\mathscr{B}_2[z_1,z_2,z_3]\|_{L^2}^2 &\lesssim \sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\eta,\zeta\in\mathbb{Z}^d}|\mathtt{b}_2(\xi,\eta,\zeta)||\widehat{z}_1(\xi-\eta-\zeta)||\widehat{z}_2(\eta)||\widehat{z}_3(\zeta)|\Big)^2\\ &\lesssim N^{-2}\sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\eta,\zeta\in\mathbb{Z}^d}\langle\xi\rangle^{2n}\max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^4|\widehat{z}_1(\xi-\eta-\zeta)||\widehat{z}_2(\eta)||\widehat{z}_3(\zeta)|\Big)^2.\end{aligned}$$

Then, by reasoning as in the proof of Lemma 2.5, one obtains the (5.34). Let us prove the bound (5.35) for $p=0$, the others are similar. Using (5.33), (5.26), (5.19), (5.11) we have

$$\begin{aligned}\|\mathscr{T}_<[z_1,z_2,z_3]\|_{L^2}^2 &\lesssim \sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\eta,\zeta\in\mathbb{Z}^d}|\mathtt{t}_<(\xi,\eta,\zeta)||\widehat{z}_1(\xi-\eta-\zeta)||\widehat{z}_2(\eta)||\widehat{z}_3(\zeta)|\Big)^2\\ &\lesssim_\gamma N^2\sum_{\xi\in\mathbb{Z}^d}\left(\sum_{\eta,\zeta\in\mathbb{Z}^d}\frac{\langle\xi\rangle^{2n}\max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{N_0+4}}{\max_1\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^2}|\widehat{z}_1(\xi-\eta-\zeta)||\widehat{z}_2(\eta)||\widehat{z}_3(\zeta)|\right)^2.\end{aligned}$$

Again, by reasoning as in the proof of Lemma 2.5, one obtains the (5.35). The (5.36) follows similarly. $\square$

Proof of Proposition 5.7. By (5.29), (5.31), (5.32), and recalling the definition of $\mathscr{B}$ in (5.19), we can write

$$\int_0^t \mathscr{B}(\sigma)d\sigma = \int_0^t(\mathscr{B}_1[z,\overline{z},z],T_{|\xi|^{2n}}z)_{L^2}d\sigma + \int_0^t(\mathscr{B}_2[z,\overline{z},z],T_{|\xi|^{2n}}z)_{L^2}d\sigma\,. \tag{5.37}$$

By Lemma 5.8 we have

$$\left|\int_0^t(\mathscr{B}_2[z,\overline{z},z],T_{|\xi|^{2n}}z)_{L^2}d\sigma\right| \overset{(5.34)}{\lesssim} N^{-1}\int_0^t\|z\|_{H^s}^4d\sigma \overset{(4.24)}{\lesssim} N^{-1}\int_0^t\|u\|_{H^s}^4\,. \tag{5.38}$$

Consider now the first summand in the r.h.s. of (5.37). We claim that we have the following identity:

$$\begin{aligned}\int_0^t(\mathscr{B}_1[z,\overline{z},z],T_{|\xi|^{2n}}z)_{L^2}d\sigma &= \int_0^t(\mathscr{T}_<[z,\overline{z},z],T_{|\xi|^{2n}}(\partial_t+\mathrm{i}\Lambda_{\rm NLS})z)_{L^2}d\sigma\\ &+\int_0^t(\mathscr{T}_<[(\partial_t+\mathrm{i}\Lambda_{\rm NLS})z,\overline{z},z],T_{|\xi|^{2n}}z)_{L^2}d\sigma\\ &+\int_0^t(\mathscr{T}_<[z,\overline{z},(\partial_t+\mathrm{i}\Lambda_{\rm NLS})z],T_{|\xi|^{2n}}z)_{L^2}d\sigma\\ &+\int_0^t(\mathscr{T}_<[z,\overline{(\partial_t+\mathrm{i}\Lambda_{\rm NLS})z},z],T_{|\xi|^{2n}}z)_{L^2}d\sigma + O(\|u\|_{H^s}^4)\,.\end{aligned} \tag{5.39}$$

We use the claim, postponing its proof. Consider the first summand in the r.h.s. of (5.39). Using the self-adjointness of $T_{|\xi|^2}$ and the (5.7) we write

$$\begin{aligned}(\mathscr{T}_<[z,\overline{z},z],T_{|\xi|^{2n}}(\partial_t+\mathrm{i}\Lambda_{\rm NLS})z)_{L^2} &= (T_{|\xi|^2}\mathscr{T}_<[z,\overline{z},z],-T_{|\xi|^{2n-2}}\mathrm{i}T_\Sigma z)_{L^2}\\ &+\big(\mathscr{T}_<[z,\overline{z},z],T_{|\xi|^{2n}}(X^+_{\mathtt{H}^{(4)}_{\rm NLS}}(Z)+R_5^{(2,+)}(U))\big)_{L^2}\,.\end{aligned}$$

We estimate the first summand in the r.h.s. by means of the Cauchy-Schwarz inequality, the (5.35) with $p=2$ and the (5.3); analogously we estimate the second summand by means of the Cauchy-Schwarz inequality, (5.36), the (4.27) and the (4.28), obtaining

$$\Big|\int_0^t (\mathscr{T}_<[z,\overline{z},z], T_{|\xi|^{2n}}(\partial_t + \mathrm{i}\Lambda_{\rm NLS})z)_{L^2}\, d\sigma\Big| \le \int_0^t \|u(\sigma)\|_{H^s}^{10} N + \|u(\sigma)\|_{H^s}^6\, d\sigma\,.$$

The other terms in (5.39) are estimated in a similar way. We eventually obtain the (5.28).
We now prove the claim (5.39). Recalling (5.7) we have that

$$\partial_t \widehat{z}(\xi) = -\mathrm{i}\Lambda_{\rm NLS}(\xi)\widehat{z}(\xi) + \widehat{\mathscr{Q}}(\xi)\,,\qquad \xi\in\mathbb{Z}^d\,,\qquad \mathscr{Q} := -\mathrm{i}T_\Sigma z + X^+_{\mathtt{H}^{(4)}_{\rm NLS}}(Z) + R_5^{(2,+)}(U)\,.$$

We define $\widehat{g}(\xi) := e^{\mathrm{i}t\Lambda_{\rm NLS}(\xi)}\widehat{z}(\xi)$, $\forall \xi\in\mathbb{Z}^d$. One can note that $\widehat{g}(\xi)$ satisfies

$$\partial_t \widehat{g}(\xi) = e^{\mathrm{i}t\Lambda_{\rm NLS}(\xi)}\widehat{\mathscr{Q}}(\xi) = e^{\mathrm{i}t\Lambda_{\rm NLS}(\xi)}(\partial_t + \mathrm{i}\Lambda_{\rm NLS})\widehat{z}(\xi)\,,\qquad \forall\,\xi\in\mathbb{Z}^d\,. \tag{5.40}$$

According to this notation and using (5.29) and (5.25) we have

$$\int_0^t (\mathscr{B}_1[z,\overline{z},z], T_{|\xi|^{2n}}z)_{L^2}\, d\sigma = \int_0^t \sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} \tfrac{1}{(2\pi)^d}\mathtt{b}_1(\xi,\eta,\zeta)e^{-\mathrm{i}\sigma\omega_{\rm NLS}(\xi,\eta,\zeta)}\widehat{g}(\xi-\eta-\zeta)\widehat{\overline{g}}(\eta)\widehat{g}(\zeta)\widehat{\overline{g}}(-\xi)|\xi|^{2n}\, d\sigma\,.$$

By integrating by parts in σ and using (5.40) one gets the (5.39) with

$$O(\|u\|_{H^s}^4) = (\mathscr{T}_<[z(t),\overline{z}(t),z(t)], T_{|\xi|^{2n}}z(t))_{L^2} - (\mathscr{T}_<[z(0),\overline{z}(0),z(0)], T_{|\xi|^{2n}}z(0))_{L^2}\,.$$

The remainder above is bounded from above by $\|u\|_{L^\infty H^s}^4$ using Cauchy-Schwarz and the (5.36). $\square$

5.2. **Estimates for the KG.** In this section we provide *a priori* energy estimates on the variable Z solving (4.55). This implies similar estimates on the solution U of the system (3.31) thanks to the equivalence (4.54). In subsection 5.2.1 we introduce an equivalent energy norm and we provide a first energy inequality. This is the content of Proposition 5.11. Then in subsection 5.2.2 we give improved bounds on the non-resonant terms.

5.2.1. *First energy inequality.* We recall that the system (4.55) is diagonal up to smoothing terms plus some higher degree of homogeneity remainder. Hence, for simplicity, we pass to the scalar equation

$$\partial_t z + \mathrm{i}\Lambda_{\rm KG} z = -\mathrm{i}Op^{\rm BW}\big(\widetilde{a}_2^+(x,\xi)\Lambda_{\rm KG}(\xi)\big)z + X^+_{\mathtt{H}^{(4)}_{\rm KG}}(Z) + R_5^{(2,+)}(u) \tag{5.41}$$

where (recall (4.56)) $X^+_{\mathtt{H}^{(4)}_{\rm KG}}(Z) = -\mathrm{i}Op^{\rm BW}(a_0(x,\xi))z + Q^+_{\mathtt{H}^{(4)}_{\rm KG}}(Z)$. For $n\in\mathbb{R}$ we define

$$z_n := \Lambda_{\rm KG}^n z\,,\qquad Z_n = \big[\begin{smallmatrix} z_n\\ \overline{z_n}\end{smallmatrix}\big] := \mathbb{1}\Lambda_{\rm KG}^n Z\,,\quad Z = \big[\begin{smallmatrix} z\\ \overline{z}\end{smallmatrix}\big]\,. \tag{5.42}$$

We have the following.

Lemma 5.9. *Fix $n := n(d) \gg 1$ large enough and recall* (5.41). *One has that the function z_n defined in* (5.42) *solves the problem*

$$\partial_t z_n = -\mathrm{i}Op^{\rm BW}\big((1+\widetilde{a}_2^+(x,\xi))\Lambda_{\rm KG}(\xi)\big)z_n + \Lambda_{\rm KG}^n X^{+,\rm res}_{\mathtt{H}^{(4)}_{\rm KG}}(Z) + B_n^{(1)}(Z) + B_n^{(2)}(Z) + R_{5,n}(U)\,, \tag{5.43}$$

where the resonant vector field $X^{+,\rm res}_{\mathtt{H}^{(4)}_{\rm KG}}$ is defined as in Def. 2.7 (see also Rmk. 2.9), the cubic terms $B_n^{(i)}$, $i=1,2$, have the form

$$\widehat{B_n^{(1)}(Z)}(\xi) = \frac{1}{(2\pi)^d}\sum_{\substack{\sigma_1,\sigma_2\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} b_1^{\sigma_1,\sigma_2}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z}_n(\zeta)\,, \tag{5.44}$$

$$\widehat{B_n^{(2)}(Z)}(\xi) = \frac{1}{(2\pi)^d}\sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} b_{2,n}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z^{\sigma_3}}(\zeta)\,, \tag{5.45}$$

with (recall Rmk. 3.4)

$$b_1^{\sigma_1,\sigma_2}(\xi,\eta,\zeta) := -\mathrm{i}a_0^{\sigma_1,\sigma_2}\big(\xi-\zeta,\eta,\tfrac{\xi+\zeta}{2}\big)\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta)\,, \tag{5.46}$$

$$|b_{2,n}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)| \lesssim \tfrac{\langle\xi\rangle^n \max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{\mu}}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}}\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta)\,, \tag{5.47}$$

for some $\mu>1$. The remainder satisfies

$$\|R_{5,n}(U)\|_{L^2} \lesssim \|u\|_{H^n}^5\,. \tag{5.48}$$

Proof. Recalling the definition of resonant vector fields in Def. 2.7 we set

$$X^{+,\perp}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) := X^{+}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) - X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)\,, \tag{5.49}$$

which represents the non resonant terms in the cubic vector field of (5.41). By differentiating in t the (5.42) and using the (5.41) we get

$$\begin{aligned}
\partial_t z_n = &-\mathrm{i}Op^{\mathrm{BW}}\big((1+\widetilde{a}_2^{+}(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)z_n + \Lambda^n_{\mathrm{KG}}X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)\\
&-\mathrm{i}\big[\Lambda^n_{\mathrm{KG}}, Op^{\mathrm{BW}}\big((1+\widetilde{a}_2^{+}(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)\big]z \qquad (5.50)\\
&+\Lambda^n_{\mathrm{KG}}X^{+,\perp}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) \qquad (5.51)\\
&+\Lambda^n_{\mathrm{KG}}R_5^{(2,+)}(u)\,, \qquad (5.52)
\end{aligned}$$

We analyse each summand above separately. First of all we remark that we have the equivalence between the two norms (see (2.2)) $\|u\|^2_{H^n}\sim(\Lambda^n_{\mathrm{KG}}u,\Lambda^n_{\mathrm{KG}}u)_{L^2}$. By estimate (4.57) we deduce $\|(5.52)\|_{L^2}\lesssim\|u\|^5_{H^n}$. Let us now consider the commutator term in (5.50). By Lemma 2.1, Proposition 2.2 and the estimate on the semi-norm of the symbol $\widetilde{a}_2^+(x,\xi)$ in (4.43), we obtain that $\|(5.50)\|_{L^2}\lesssim\|u\|^4_{H^n}\|z\|_{H^n}\lesssim\|u\|^5_{H^n}$, we have used also the (4.54). The term in (5.51) is the most delicate. By (4.56) and (5.49) (recall also Rmk. 3.4 and (2.6))

$$\Lambda^n_{\mathrm{KG}}X^{+,\perp}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) = B_n^{(1)}(Z) + \mathscr{C}_1 + \mathscr{C}_2\,, \tag{5.53}$$

with $B_n^{(1)}(Z)$ as in (5.44) and coefficients as in (5.46), the term $\mathscr{C}_1$ has the form

$$\begin{aligned}
\widehat{\mathscr{C}}_1(\xi) &= \tfrac{1}{(2\pi)^d}\sum_{\substack{\sigma_1,\sigma_2\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{c}_1^{\sigma_1,\sigma_2}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z}(\zeta)\,,\\
\mathtt{c}_1^{\sigma_1,\sigma_2}(\xi,\eta,\zeta) &= -\mathrm{i}a_0^{\sigma_1,\sigma_2}\big(\xi-\zeta,\eta,\tfrac{\xi+\zeta}{2}\big)\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\Big[\Lambda^n_{\mathrm{KG}}(\xi)-\Lambda^n_{\mathrm{KG}}(\zeta)\Big]\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta)\,,
\end{aligned} \tag{5.54}$$

and the term $\mathscr{C}_2$ has the form (5.45) with coefficients (see (A.23))

$$\mathtt{c}_2^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta) := \mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(\xi,\eta,\zeta)\Lambda^n_{\mathrm{KG}}(\xi)\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta)\,. \tag{5.55}$$

In order to conclude the proof we need to show that the coefficients in (5.54), (5.55) satisfy the bound (5.47). This is true for the coefficients in (5.55) thanks to the bound (A.24). Moreover notice that

$$|\Lambda^n_{\mathrm{KG}}(\xi)-\Lambda^n_{\mathrm{KG}}(\zeta)| \lesssim |\xi-\zeta|\max\{\langle\xi\rangle,\langle\zeta\rangle\}^{n-1}\,.$$

Then the coefficients in (5.54) satisfy (5.47) by using Remark 3.4 and Lemma 2.6. ☐

Remark 5.10. *In view of Remarks 3.5, 3.7, 4.14 if* (KG) *is semi-linear then the symbol $\tilde{a}_2^+$ in* (5.43) *is equal to zero, the coefficients $b_{2,n}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)$ in* (5.45) *satisfies the bound* (5.47) *the the better denominator* $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^2$.

In view of Lemma 5.9 we deduce the following.

Proposition 5.11. *Let $\psi(t,x)$ be a solution of* (KG) *satisfying Hypothesis 4.8 and consider the function z_n in* (5.42) *(see also* (4.53), (4.47)*). Then, setting $s = n = n(d) \gg 1$ we have $\|z_n\|_{L^2} \sim \|\psi\|_{H^{s+1/2}} + \|\dot\psi\|_{H^{s-1/2}}$ and*

$$\partial_t \|z_n(t)\|^2_{L^2} = \mathscr{B}(t) + \mathscr{B}_{>5}(t), \qquad t \in [0,T), \tag{5.56}$$

where

- *the term $\mathscr{B}(t)$ has the form*

$$\mathscr{B}(t) = \sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \xi,\eta,\zeta\in\mathbb{Z}^d}} \Lambda^{2n}_{\mathrm{KG}}(\xi)\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z^{\sigma_3}}(\zeta)\widehat{\overline{z}}(-\xi), \tag{5.57}$$

where $\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta) \in \mathbb{C}$ satisfy, for $\xi,\eta,\zeta \in \mathbb{Z}^d$,

$$|\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)| \lesssim \tfrac{\max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{\mu}}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}} \mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta) \tag{5.58}$$

for some $\mu > 1$;

- *the term $\mathscr{B}_{>5}(t)$ satisfies*

$$|\mathscr{B}_{>5}(t)| \lesssim \|u\|^6_{H^s}, \quad t \in [0,T). \tag{5.59}$$

Proof. By using (5.43) we get

$$\tfrac{1}{2}\partial_t \|z_n(t)\|^2_{L^2} = \mathrm{Re}\Big(-\mathrm{i}Op^{\mathrm{BW}}\big((1+\widetilde{a}^+_2(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)z_n, z_n\Big)_{L^2} \tag{5.60}$$

$$+ \mathrm{Re}(\Lambda^n_{\mathrm{KG}} X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z), z_n)_{L^2} \tag{5.61}$$

$$+ \mathrm{Re}(B^{(1)}_n(Z), z_n)_{L^2} \tag{5.62}$$

$$+ \mathrm{Re}(B^{(2)}_n(Z), z_n)_{L^2} \tag{5.63}$$

$$+ \mathrm{Re}(R_{5,n}(Z), z_n)_{L^2}. \tag{5.64}$$

By (4.39), (4.38) and (3.11) we have that the symbol $(1+\widetilde{a}^+_2(x,\xi))\Lambda_{\mathrm{KG}}(\xi)$ is real-valued. Hence the operator $\mathrm{i}Op^{\mathrm{BW}}\big((1+\widetilde{a}^+_2(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)$ is skew-self-adjoint. We deduce (5.60)$\equiv 0$. By Lemma 2.8 (see also Remark 2.9) we also have that (5.61)$\equiv 0$. We also have that (5.62) $\equiv 0$, to see this one can reason as done in the proof of Prop. 5.4, by using Remark 3.4, in particular (3.30). By formula (5.45) and estimates (5.47) we have that the term in (5.63) has the form (5.57) with coefficients satisfying (5.58). By Cauchy-Schwarz inequality and estimate (5.48) we get that the term in (5.64) satisfies the bound (5.59). □

Remark 5.12. *In view of Remark 5.10, if* (KG) *is semi-linear, then the coefficients $\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)$ of the energy in* (5.57) *satisfy the bound* (5.58) *with the better denominator* $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^2$.

5.2.2. *Estimates of non-resonant terms.* In Proposition 5.11 we provide a precise structure of the term $\mathscr{B}(t)$ of degree 4 in (5.56). In this section we show that, actually, $\mathscr{B}(t)$ satisfies better bounds with respect to a general quartic multilinear maps by using that it is *non-resonant.* We need the following.

Proposition 5.13. (Non-resonance conditions). *Consider the phase $\omega^{\vec\sigma}_{\mathrm{KG}}(\xi,\eta,\zeta)$ defined as*

$$\omega^{\vec\sigma}_{\mathrm{KG}}(\xi,\eta,\zeta) := \sigma_1\Lambda_{\mathrm{KG}}(\xi-\eta-\zeta) + \sigma_2\Lambda_{\mathrm{KG}}(\eta) + \sigma_3\Lambda_{\mathrm{KG}}(\zeta) - \Lambda_{\mathrm{KG}}(\xi), \qquad (\xi,\eta,\zeta) \in \mathbb{Z}^{3d}, \tag{5.65}$$

where $\vec\sigma := (\sigma_1,\sigma_2,\sigma_3) \in \{\pm\}^3$, Λ_{KG} is in (1.4)*. Let $0 < \sigma \ll 1$ and set $\beta := 2+\sigma$ if $d = 2$, and $\beta := 3+\sigma$ if $d \geq 3$. There exists $\mathscr{C}_\beta \subset [1,2]$ with Lebesgue measure 1 such that, for any $m \in \mathscr{C}_\beta$, there exist $\gamma > 0$, $N_0 := N_0(d,m) > 0$ such that for any $(\xi,\eta,\zeta) \notin \mathscr{R}$ (see* (2.47)*) one has*

$$|\omega^{\vec\sigma}_{\mathrm{KG}}(\xi,\eta,\zeta)| \geq \gamma \max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{-N_0}\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{-\beta}. \tag{5.66}$$

Proof. The case $d = 2$ follows by Theorem 2.1.1 in [17]. We postpone the proof for $d \geq 3$ to the Appendix B. □

We are now in position to state the main result of this section.

Proposition 5.14. *Let $N>0$ and let β be as in Proposition 5.13. Then there is $s_0 = s_0(N_0)$, where $N_0>0$ is given by Proposition 5.13, such that, if Hypothesis 4.8 holds with $s \geq s_0$, one has*

$$\left|\int_0^t \mathscr{B}(\sigma)d\sigma\right| \lesssim \|u\|^6_{L^\infty H^s} TN^{\beta-1} + \|u\|^8_{L^\infty H^s} N^\beta T + \|u\|^4_{L^\infty H^s} TN^{-1} + N^{\beta-1}\|u\|^4_{L^\infty H^s}, \tag{5.67}$$

where $\mathscr{B}(t)$ is in (5.57).

We firstly introduce some notation. Let $\vec{\sigma} := (\sigma_1,\sigma_2,\sigma_3) \in \{\pm\}^3$ and consider the following trilinear maps:

$$\mathscr{B}_i^{\vec{\sigma}} = \mathscr{B}_i^{\vec{\sigma}}[z_1,z_2,z_3], \quad \widehat{\mathscr{B}}_i^{\vec{\sigma}}(\xi) = \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} \mathtt{b}_i^{\vec{\sigma}}(\xi,\eta,\zeta)\widehat{z_1^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z_2^{\sigma_2}}(\eta)\widehat{z_3^{\sigma_3}}(\zeta), \tag{5.68}$$

$$\mathscr{T}_<^{\vec{\sigma}} = \mathscr{T}_<^{\vec{\sigma}}[z_1,z_2,z_3], \quad \widehat{\mathscr{T}}_<^{\vec{\sigma}}(\xi) = \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} \mathtt{t}_<^{\vec{\sigma}}(\xi,\eta,\zeta)\widehat{z_1^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z_2^{\sigma_2}}(\eta)\widehat{z_3^{\sigma_3}}(\zeta), \tag{5.69}$$

where

$$\mathtt{b}_1^{\vec{\sigma}}(\xi,\eta,\zeta) = \mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\mathbb{1}_{\{\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}\leq N\}}, \tag{5.70}$$

$$\mathtt{b}_2^{\vec{\sigma}}(\xi,\eta,\zeta) = \mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\mathbb{1}_{\{\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}> N\}}, \tag{5.71}$$

$$\mathtt{t}_<^{\vec{\sigma}}(\xi,\eta,\zeta) = \frac{-1}{\mathrm{i}\omega^{\vec{\sigma}}_{\text{KG}}(\xi,\eta,\zeta)}\mathtt{b}_1^{\vec{\sigma}}(\xi,\eta,\zeta), \tag{5.72}$$

where $\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)$ are the coefficients in (5.57), and $\omega^{\vec{\sigma}}_{\text{KG}}$ is the phase in (5.65). We remark that if $(\xi,\eta,\zeta)\in\mathscr{R}$ then the coefficients $\mathtt{b}(\xi,\eta,\zeta)$ are equal to zero (see (5.57), (5.45), (5.47)). Therefore, since $\omega^{\vec{\sigma}}_{\text{KG}}$ is non-resonant (see Proposition 5.13), the coefficients in (5.72) are well-defined. We now state an abstract results on the trilinear maps introduced in (5.68)-(5.69).

Lemma 5.15. *Let $\mu>1$ as in* (5.58). *One has that, for $s > d/2+\mu$,*

$$\|\mathscr{B}_2^{\vec{\sigma}}[z_1,z_2,z_3]\|_{L^2} \lesssim N^{-1}\sum_{i=1}^3 \|z_i\|_{H^s}\prod_{i\neq k}\|z_k\|_{H^{d/2+\mu+\epsilon}}, \tag{5.73}$$

for any $\vec{\sigma}\in\{\pm\}^3$ and any $\epsilon>0$. There is $s_0(N_0)>0$ ($N_0>0$ given by Proposition 5.13) such that for $s\geq s_0(N_0)$ one has

$$\|\mathscr{T}_<[z_1,z_2,z_3]\|_{H^p} \lesssim N^\beta \sum_{i=1}^3 \|z_i\|_{H^{s+p-1}}\prod_{i\neq k}\|z_k\|_{H^{s_0}}, \quad p\in\mathbb{N}, \tag{5.74}$$

$$\|\mathscr{T}_<[z_1,z_2,z_3]\|_{L^2} \lesssim N^{\beta-1} \sum_{i=1}^3 \|z_i\|_{H^{s}}\prod_{i\neq k}\|z_k\|_{H^{s_0}}. \tag{5.75}$$

where β is defined in Proposition 5.13.

Proof. The proof is similar to the one of Lemma 5.8. □

Remark 5.16. *In view of Remark 5.12, if* (KG) *is semi-linear we may improve* (5.75) *with*

$$\|\mathscr{T}_<[z_1,z_2,z_3]\|_{L^2} \lesssim N^{\beta-2} \sum_{i=1}^3 \|z_i\|_{H^{s}}\prod_{i\neq k}\|z_k\|_{H^{s_0}}. \tag{5.76}$$

We are now in position to prove the main Proposition 5.14.

Proof of Proposition 5.14. By (5.68), (5.70), (5.71), and recalling the definition of $\mathscr{B}$ in (5.57), we can write

$$\int_0^t \mathscr{B}(\tau)d\tau = \sum_{\vec{\sigma}\in\{\pm\}^3}\int_0^t (\mathscr{B}_1^{\vec{\sigma}}[z,z,z],\Lambda^s_{\text{KG}}z)_{L^2}d\tau + \sum_{\vec{\sigma}\in\{\pm\}^3}\int_0^t (\mathscr{B}_2^{\vec{\sigma}}[z,z,z],\Lambda^s_{\text{KG}}z)_{L^2}d\tau. \tag{5.77}$$

By Lemma 5.15 we have

$$\left|\int_0^t (\mathscr{B}_2^{\vec{\sigma}}[z,z,z],\Lambda^s_{\text{KG}}z)_{L^2}d\sigma\right| \overset{(5.73)}{\lesssim} N^{-1}\int_0^t \|z\|^4_{H^s}d\tau \overset{(4.54)}{\lesssim} N^{-1}\int_0^t \|u\|^4_{H^s}d\tau. \tag{5.78}$$

Consider now the first summand in the r.h.s. of (5.77). Integrating by parts as done in the proof of Prop. 5.7 we have

$$\begin{aligned}\int_0^t (\mathscr{B}_1^{\vec{\sigma}}[z,\overline{z},z],\Lambda_{\rm KG}^s z)_{L^2}d\tau &= \int_0^t (\mathscr{T}_<^{\vec{\sigma}}[z,z,z],\Lambda_{\rm KG}^s(\partial_t+{\rm i}\Lambda_{\rm KG})z)_{L^2}d\tau\\ &+\int_0^t (\mathscr{T}_<^{\vec{\sigma}}[(\partial_t+{\rm i}\Lambda_{\rm KG})z,z,z],\Lambda_{\rm KG}^s z)_{L^2}d\tau\\ &+\int_0^t (\mathscr{T}_<^{\vec{\sigma}}[z,\overline{z},(\partial_t+{\rm i}\Lambda_{\rm KG})z],\Lambda_{\rm KG}^s z)_{L^2}d\tau\\ &+\int_0^t (\mathscr{T}_<^{\vec{\sigma}}[z,(\partial_t+{\rm i}\Lambda_{\rm KG})z,z],\Lambda_{\rm KG}^s z)_{L^2}d\tau+R,\end{aligned}\tag{5.79}$$

where

$$R=(\mathscr{T}_<^{\vec{\sigma}}[z(t),z(t),z(t)],\Lambda_{\rm KG}^s z(t))_{L^2}-(\mathscr{T}_<^{\vec{\sigma}}[z(0),z(0),z(0)],\Lambda_{\rm KG}^s z(0))_{L^2}.$$

The remainder R above is bounded from above by $N^{\beta}\|u\|^4_{L^\infty H^s}$ using Cauchy-Schwarz and the (5.74). Let us now consider the first summand in the r.h.s. of (5.79). Using that the operator $\Lambda_{\rm KG}$ is self-adjoint and recalling the equation (5.41) we have

$$\begin{aligned}(\mathscr{T}_<^{\vec{\sigma}}[z,z,z],\Lambda_{\rm KG}^s(\partial_t+{\rm i}\Lambda_{\rm KG})z)_{L^2} &= (\Lambda_{\rm KG}\mathscr{T}_<^{\vec{\sigma}}[z,z,z],\Lambda_{\rm KG}^{s-1}(\partial_t+{\rm i}\Lambda_{\rm KG})z)_{L^2}\\ &=(\Lambda_{\rm KG}\mathscr{T}_<^{\vec{\sigma}}[z,z,z],\Lambda_{\rm KG}^{s-1}Op^{BW}\big(-{\rm i}\widetilde{a}_2^+(x,\xi)\Lambda_{\rm KG}(\xi)\big)z)_{L^2} \qquad (5.80)\\ &+(\mathscr{T}_<^{\vec{\sigma}}[z,z,z],\Lambda_{\rm KG}^s\big(X^+_{{\rm H}^{(4)}_{\rm KG}}(Z)+R^{(2,+)}_{\geq5}(u)\big))_{L^2}. \qquad (5.81)\end{aligned}$$

By Cauchy-Schwarz inequality, estimate (5.74) with $p=1$, estimate (4.43) on the semi-norm of the symbol $\widetilde{a}_2^+(x,\xi)$ Lemma 2.1 and the equivalence (4.54), we get $|(5.80)|\lesssim\|u\|^8_{H^s}N^{\beta}$. Consider the term in (5.81). First of all notice that, by (3.18) and Lemma 2.1, and by (A.24) and Lemma 2.5, the field $X_{{\rm H}^{(4)}_{\rm KG}}(Z)$ in (4.56) satisfies the same estimates (3.19) as the field $X_{\mathscr{H}^{(4)}_{\rm KG}}$. Therefore, using (5.75) and (4.57), we obtain $|(5.81)|\lesssim\|u\|^6_{H^s}N^{\beta-1}$. Using that (see Hyp. 4.8) $\|u\|_{H^s}\ll1$, we conclude that the first summand in the r.h.s. of (5.79) is bounded from above by $N^{\beta}\int_0^t\|u(\tau)\|^8d\tau+N^{\beta-1}\int_0^t\|u(\tau)\|^6d\tau$. The other terms in (5.79) are estimated in a similar way. We eventually obtain the (5.67). ☐

Remark 5.17. *In view of Remarks 3.5, 3.7, 4.14, 5.10, 5.12 and 5.16, if* (KG) *is semi-linear we have the better (w.r.t.* (5.67)*) estimate*

$$\left|\int_0^t\mathscr{B}(\sigma)d\sigma\right|\lesssim\|u\|^6_{L^\infty H^s}TN^{\beta-2}+\|u\|^4_{L^\infty H^s}TN^{-2}+N^{\beta-2}\|u\|^4_{L^\infty H^s}.\tag{5.82}$$

6. Proof of the main results

In this section we conclude the proof of our main theorems.

Proof of Theorem 1. Consider (NLS) and let u_0 as in the statement of Theorem 1. By the result in [28] we have that there is $T>0$ and a unique solution $u(t,x)$ of (NLS) with $V\equiv0$ such that Hypothesis 4.1 is satisfied. To recover the result when $V\neq0$ one can argue as done in [27]. Consider a potential V as in (1.5) with $\vec{x}\in\mathscr{O}\setminus\mathscr{N}$ with $\mathscr{N}$ is the zero measure set given in Proposition 5.6. We claim that we have the following *a priori* estimate: fix any $0<N$, then for any $t\in[0,T)$, with T as in Hyp. 4.1, one has

$$\|u(t)\|^2_{H^s}\lesssim\|u_0\|^2_{H^s}+\|u\|^{10}_{L^\infty H^s}TN+\|u\|^6_{L^\infty H^s}T+\|u\|^4_{L^\infty H^s}TN^{-1}+\|u\|^4_{L^\infty H^s}.\tag{6.1}$$

To prove the claim we reason as follows. By Proposition 3.1 we have that (NLS) is equivalent to the system (3.4). By Propositions 4.4, 4.6 and Lemma 5.3 we can construct a function z_n with $2n=s$ such that if $u(t,x)$ solves the (NLS) then z_n solves the equation (5.8). Moreover by Lemma 5.2 and Remark 4.5 we also have that $\|z\|_{L^2}+\|z_n\|_{L^2}\sim\|u\|_{H^s}$. By Proposition 5.5 we get

$$\|u(t)\|^2_{H^s}\lesssim\|z(t)\|^2_{L^2}+\|z_n(t)\|^2_{L^2}\lesssim\|u_0\|^2_{H^s}+\left|\int_0^t\mathscr{B}(\sigma)d\sigma\right|+\left|\int_0^t\mathscr{B}_{>5}(\sigma)d\sigma\right|.\tag{6.2}$$

Propositions 5.5 and 5.7 apply, therefore, by (5.28) and (5.20), we obtain the (6.1). The thesis of Theorem 1 follows from the following lemma.

Lemma 6.1. (Main Bootstrap). *Let $u(t,x)$ be a solution of* (NLS) *with $t\in[0,T)$ and initial condition $u_0\in H^s(\mathbb{T}^d;\mathbb{C})$. Then, for $s\gg 1$ large enough, there exist $\varepsilon_0, c_0>0$ such that, for any $0<\varepsilon\le\varepsilon_0$, if*

$$\|u_0\|_{H^s}\le c_0\varepsilon\,,\qquad \sup_{t\in[0,T)}\|u(t)\|_{H^s}\le\varepsilon\,,\qquad T\le c_0\varepsilon^{-4}\,, \tag{6.3}$$

then we have the improved bound $\sup_{t\in[0,T)}\|u(t)\|_{H^s}\le\varepsilon/2$.

Proof. For ε small enough the bound (6.1) holds true, and we fix $N:=\varepsilon^{-3}$. Therefore, there is $C=C(s)>0$ such that, for any $t\in[0,T)$,

$$\begin{aligned}\|u(t)\|_{H^s}^2&\le C\big(\|u_0\|_{H^s}^2+\|u\|_{L^\infty H^s}^4+\|u\|_{L^\infty H^s}^{10}T\varepsilon^{-3}+\|u\|_{L^\infty H^s}^6T+\|u\|_{L^\infty H^s}^4T\varepsilon^3\big)\\&\overset{(6.3)}{\le}C\big(c_0^2\varepsilon^2+\varepsilon^4+2\varepsilon^7T+\varepsilon^6T\big)\\&\le C\tfrac{\varepsilon^2}{4}(4c_0^2+4\varepsilon^2+5c_0)\le\varepsilon^2/4\end{aligned} \tag{6.4}$$

where in the last inequality we have chosen c_0 and ε sufficiently small. This implies the thesis. □

Proof of Theorem 2. One has to follow almost word by word the proof of Theorem 1. The only difference relies on the estimates on the small divisors which in this case are given by item (ii) of Proposition 5.6.

Proof of Theorem 3. Consider (KG) and let (ψ_0,ψ_1) as in the statement of Theorem 3. Let $\psi(t,x)$ be a solution of (KG) satisfying the condition in Hyp. 4.8. By Proposition 3.6, recall (2.41), the function $U:=\big[\begin{smallmatrix}u\\ \bar{u}\end{smallmatrix}\big]$ solves (3.4) with initial condition $u_0=\frac{1}{\sqrt{2}}(\Lambda_{\mathrm{KG}}^{\frac12}\psi_0+\mathrm{i}\Lambda_{\mathrm{KG}}^{-\frac12}\psi_1)$. Moreover, by Hyp. 4.8 one has $\sup_{t\in[0,T)}\|u\|_{H^s}\le\varepsilon$. By Remark 4.9, in order to get the (1.8), we have to show that the bound on the function u above holds for a longer time $T\gtrsim\varepsilon^{-3^+}$ if $d=2$ and $T\gtrsim\varepsilon^{-8/3^+}$ if $d\ge3$. Fix β as in Proposition 5.13 and let $m\in\mathscr{C}_\beta$. By Propositions 4.11, 4.13 and Lemma 5.9 we can construct a function z_n with $n=s$ such that if $\psi(t,x)$ solves the (KG) then z_n solves the equation (5.43). By Proposition 5.11 we get

$$\|u(t)\|_{H^s}^2\lesssim\|z(t)\|_{L^2}^2+\|z_n(t)\|_{L^2}^2\lesssim\|u_0\|_{H^s}^2+\left|\int_0^t\mathscr{B}(\sigma)d\sigma\right|+\left|\int_0^t\mathscr{B}_{>5}(\sigma)d\sigma\right|. \tag{6.5}$$

Propositions 5.11 and 5.14 apply, therefore, by (5.67) and (5.59), we obtain the following *a priori* estimate: fix any $0<N$, then for any $t\in[0,T)$, with T as in Hyp. 4.8, one has

$$\|u(t)\|_{H^s}^2\lesssim\|u_0\|_{H^s}^2+\|u\|_{L^\infty H^s}^6TN^{\beta-1}+\|u\|_{L^\infty H^s}^8TN^{\beta}+\|u\|_{L^\infty H^s}^6T+\|u\|_{L^\infty H^s}^4TN^{-1}+N^{\beta-1}\|u\|_{L^\infty H^s}^4\,. \tag{6.6}$$

The thesis of Theorem 3 follows from the following lemma.

Lemma 6.2. (Main bootstrap). *Let $u(t,x)$ be a solution of* (3.31) *with $t\in[0,T)$ and initial condition $u_0\in H^s(\mathbb{T}^d;\mathbb{C})$. Define $\mathsf{a}=3$ if $d=2$ and $\mathsf{a}=8/3$ if $d\ge3$. Then, for $s\gg1$ large enough, there exist $\varepsilon_0,c_0>0$ such that, for any $0<\varepsilon\le\varepsilon_0$, if*

$$\|u_0\|_{H^s}\le c_0\varepsilon\,,\qquad \sup_{t\in[0,T)}\|u(t)\|_{H^s}\le\varepsilon\,,\qquad T\le c_0\varepsilon^{-\mathsf{a}^+}\,, \tag{6.7}$$

then we have the improved bound $\sup_{t\in[0,T)}\|u(t)\|_{H^s}\le\varepsilon/2$.

Proof. We start with $d\ge3$.For ε small enough the bound (6.6) holds true. Let $0<\sigma\ll1$. Define

$$\beta:=3+\sigma\,,\quad N:=\varepsilon^{-\frac{2}{3+\sigma}}\,. \tag{6.8}$$

By (6.6), (6.7), (6.8), there is $C=C(s)>0$ such that, for any $t\in[0,T)$,

$$\|u(t)\|_{H^s}^2\le C\varepsilon^2(\varepsilon^{\frac{2}{3+\sigma}}+c_0^2)+2CT\varepsilon^2(\varepsilon^4+\varepsilon^{2+\frac{2}{3+\sigma}})\le\varepsilon^2/4 \tag{6.9}$$

where in the last inequality we have chosen c_0 and ε sufficiently small and we used the choice of T in (6.7) and that σ is arbitrary small. This implies the thesis. In the case $d = 2$ the proof is similar setting $\beta = 2+\sigma$ and $N = \varepsilon^{-2/(2+\sigma)}$. □

Proof of Theorem 4. Using the Remarks 3.5, 3.7, 4.14, 5.10, 5.12, 5.16, 5.17 one deduces the result by reasoning as in the proof of Theorem 3 and using in particular the estimate (5.82).

APPENDIX A. APPROXIMATELY SYMPLECTIC MAPS

A.1. **Para-differential Hamiltonian vector fields.** In this section we study some properties of the maps generated by the Hamiltonians $\mathscr{B}_{\rm NLS}(W)$ in (4.22) and $\mathscr{B}_{\rm KG}(W)$ in (4.52). In the next lemma we show that their Hamiltonian vector fields are given by $Op^{\rm BW}(B_{\rm NLS}(W;x,\xi))W$ and $Op^{\rm BW}(B_{\rm KG}(W;x,\xi))W$ respectively, modulo smoothing remainders. More precisely we have the following.

Lemma A.1. *Consider the Hamiltonian function $\mathscr{B}(W)$ equals to $\mathscr{B}_{\rm NLS}$ in (4.22) or $\mathscr{B}_{\rm KG}$ in (4.52). One has that the Hamiltonian vector field of $\mathscr{B}(W)$ has the form*

$$X_{\mathscr{B}}(W) = -\mathrm{i}J\nabla\mathscr{B}(W) = Op^{\rm BW}(B(W;x,\xi))W + Q_{\mathscr{B}}(W)\,, \tag{A.1}$$

where $Q_{\mathscr{B}}(W)$ is a smoothing remainder of the form $(Q^+_{\mathscr{B}}(W),\overline{Q^+_{\mathscr{B}}(W)})^T$ and the symbol $B(W;x,\xi)$ is respectively equal to $B_{\rm NLS}(W;x,\xi)$ in (4.21) or $B_{\rm KG}(W;x,\xi)$ in (4.51). In particular the cubic remainder $Q_{\mathscr{B}}(W)$ has the form

$$(\widehat{Q^+_{\mathscr{B}}(W)})(\xi) = \frac{1}{(2\pi)^d}\sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{q}_{\mathscr{B}}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\,\widehat{w^{\sigma_1}}(\xi-\eta-\zeta)\,\widehat{w^{\sigma_2}}(\eta)\,\widehat{w^{\sigma_3}}(\zeta)\,,\quad \xi\in\mathbb{Z}^d\,, \tag{A.2}$$

where $\mathtt{q}_{\mathscr{B}}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\in\mathbb{C}$ satisfy, for any $\xi,\eta,\zeta\in\mathbb{Z}^d$, a bound like (2.15). In the case that $\mathscr{B}=\mathscr{B}_{\rm NLS}$ we have that $\sigma_1=+,\sigma_2=-,\sigma_3=+$. Moreover, for $s>d/2+\rho$, we have the following

$$\|d_W^k Q_{\mathscr{B}}(W)[h_1,\ldots,h_k]\|_{H^{s+\rho}} \lesssim \|w\|_{H^s}^{3-k}\prod_{i=1}^k\|h_i\|_{H^s}\,,\qquad \forall\, h_i\in H^s(\mathbb{T}^d;\mathbb{C}^2)\,,\ i=1,2,3\,, \tag{A.3}$$

for $k=0,1,2,3$. Moreover, for any $s>2d+2$, one has

$$\|d_W^k X_{\mathscr{B}_{\rm NLS}}(W)[h_1,\ldots,h_k]\|_{H^{s+2}} \lesssim \|w\|_{H^s}^{3-k}\prod_{i=1}^k\|h_i\|_{H^s}\,,\qquad \forall\, h_i\in H^s(\mathbb{T}^d;\mathbb{C}^2)\,,\ i=1,2,3\,, \tag{A.4}$$

$$\|d_W^k X_{\mathscr{B}_{\rm KG}}(W)[h_1,\ldots,h_k]\|_{H^{s+1}} \lesssim \|w\|_{H^s}^{3-k}\prod_{i=1}^k\|h_i\|_{H^s}\,,\qquad \forall\, h_i\in H^s(\mathbb{T}^d;\mathbb{C}^2)\,,\ i=1,2,3\,, \tag{A.5}$$

with $k=0,1,2,3$.

Proof. We prove the statement in the case $\mathscr{B}=\mathscr{B}_{\rm NLS}$, the other case is similar. Using the formulæ (4.21), (4.22) we obtain $\mathscr{B}_{\rm NLS}(W) = -G_1(W)-G_2(W)$ with

$$G_1(W) := -\frac{\mathrm{i}}{2}\int_{\mathbb{T}^d} Op^{\rm BW}(b_{\rm NLS}(S_\xi w))\overline{w}\,\overline{w}\,dx\,,\qquad G_2(W) := \frac{\mathrm{i}}{2}\int_{\mathbb{T}^d} Op^{\rm BW}(\overline{b_{\rm NLS}(S_\xi w)})\,w\,w\,dx\,,$$

where we recall (4.20). By (4.21) we obtain that $\nabla_{\overline{w}} G_1(W) = -\mathrm{i}Op^{\mathrm{BW}}(b_{\mathrm{NLS}}(S_\xi w))\overline{w}$. We compute the gradient with respect $\overline{w}$ of the term $G_2(W)$. We have

$$
\begin{aligned}
d_{\overline{w}} G_2(W)(\overline{h}) &= \frac{\mathrm{i}}{2}\int_{\mathbb{T}^d} Op^{\mathrm{BW}}(S_\xi(\overline{w}) S_\xi(\overline{h}) \tfrac{1}{|\xi|^2}) w\, w\, dx \\
&\overset{(2.6)}{=} \frac{\mathrm{i}}{2}\frac{1}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} \widehat{S_{\frac{\xi+\zeta}{2}}(\overline{w})}(\xi-\eta-\zeta)\widehat{S_{\frac{\xi+\zeta}{2}}(\overline{h})}(\eta)\,\widehat{w}(\zeta)\,\tfrac{4}{|\zeta+\xi|^2}\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\widehat{w}(-\xi) \\
&\overset{(4.20)}{=} 2\mathrm{i}\frac{1}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} \tfrac{1}{|\zeta+\xi|^2}\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\tfrac{2|\xi-\eta-\zeta|}{\langle\xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\tfrac{2|\eta|}{\langle\xi+\zeta\rangle}\Big)\widehat{\overline{w}}(\xi-\eta-\zeta)\,\widehat{\overline{h}}(\eta)\,\widehat{w}(\zeta)\,\widehat{w}(-\xi) \\
&= 2\mathrm{i}\frac{1}{(2\pi)^d}\sum_{\eta\in\mathbb{Z}^d}\widehat{\overline{h}}(-\eta)\sum_{\xi,\zeta\in\mathbb{Z}^d}\tfrac{1}{|\zeta+\xi|^2}\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big) \\
&\quad\times\chi_\epsilon\Big(\tfrac{2|\xi+\eta-\zeta|}{\langle\xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\tfrac{2|\eta|}{\langle\xi+\zeta\rangle}\Big)\widehat{\overline{w}}(\xi+\eta-\zeta)\,\widehat{w}(\zeta)\,\widehat{w}(-\xi)\,.
\end{aligned}
$$

Recalling (2.33) and the computations above, after some changes of variables in the summations, we obtain

$$
X_{\mathscr{B}_{\mathrm{NLS}}}(W) = Op^{\mathrm{BW}}(B_{\mathrm{NLS}}(S_\xi W;x,\xi))W + R_1(W)
$$

where the remainder $R_1(W)$ has the form $(R_1^+(W), \overline{R_1^+(W)})^T$ where (recall (2.5))

$$
\begin{aligned}
(\widehat{R_1^+(W)})(\xi) &= \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} r_1(\xi,\eta,\zeta)\,\widehat{w}(\xi-\eta-\zeta)\,\widehat{\overline{w}}(\eta)\,\widehat{w}(\zeta)\,, \qquad \xi\in\mathbb{Z}^d\,, \\
r_1(\xi,\eta,\zeta) &= -\tfrac{2}{|2\zeta-\xi+\eta|^2}\chi_\epsilon\Big(\tfrac{|\eta-\xi|}{\langle 2\zeta-\xi+\eta\rangle}\Big)\chi_\epsilon\Big(\tfrac{2|\xi|}{\langle\xi-\eta-2\zeta\rangle}\Big)\chi_\epsilon\Big(\tfrac{2|\eta|}{\langle\xi-\eta-2\zeta\rangle}\Big)\,.
\end{aligned}
$$

One can check, for $0<\epsilon<1$ small enough, $|\xi|+|\eta| \ll |\xi-\eta-\zeta|\sim|\zeta|$. Therefore the coefficients $r_1(\xi,\eta,\zeta)$ satisfies the (2.15). Here we really need the truncation operator S_ξ: if you don't insert it in the definition of $\mathscr{B}_{\mathrm{NLS}}$ (see (4.22)) then R_1 is not a regularizing operator. Furthermore this truncation does not affect the leading term: define the operator

$$
R_2(W) = \begin{pmatrix} R_2^+(W) \\ \overline{R_2^+(W)} \end{pmatrix} := Op^{\mathrm{BW}}\Big(B_{\mathrm{NLS}}(S_\xi W;x,\xi) - B_{\mathrm{NLS}}(W;x,\xi)\Big)W\,,
$$

we are going to prove that R_2 is also a regularizing operator. By an explicit computation using (2.6), (4.20) and (4.21) one can check that

$$
\begin{aligned}
(\widehat{R_2^+(W)})(\xi) &= \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} r_2(\xi,\eta,\zeta)\,\widehat{w}(\xi-\eta-\zeta)\,\widehat{\overline{w}}(\eta)\,\widehat{w}(\zeta)\,, \qquad \xi\in\mathbb{Z}^d\,, \\
r_2(\xi,\eta,\zeta) &= -\tfrac{1}{|\xi+\zeta|^2}\chi_\epsilon\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\Big(1-\chi_\epsilon\Big(\tfrac{|\xi-\eta-\zeta|}{\langle\xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\tfrac{|\eta|}{\langle\xi+\zeta\rangle}\Big)\Big)\,.
\end{aligned}
$$

We write $1\cdot r_2(\xi,\eta,\zeta)$ and we use the partition of the unity in (2.16). Hence using the (2.5) one can check that each summand satisfies the bound in (2.15). Therefore the operator $Q_G := R_1+R_2$ has the form (A.2) and (A.1) is proved. The estimates (A.3) follow by Lemma 2.5. We note that

$$
d_W\Big(Op^{\mathrm{BW}}(B_{\mathrm{NLS}}(W;x,\xi))W\Big)[h] = Op^{\mathrm{BW}}(B_{\mathrm{NLS}}(W;x,\xi))h + Op^{\mathrm{BW}}(d_W B_{\mathrm{NLS}}(W;x,\xi)[h])W\,.
$$

Then the estimates (A.4) with $k=0,1$, follow by using (A.3), the explicit formula of $B(W;x,\xi)$ in (4.21) and Lemma 2.1. Reasoning similarly one can prove the (A.4) with $k=2,3$. ☐

In the next proposition we define the changes of coordinates generated by the Hamiltonian vector fields $X_{\mathscr{B}_{\mathrm{NLS}}}$ and $X_{\mathscr{B}_{\mathrm{KG}}}$ and we study their properties as maps on Sobolev spaces.

Proposition A.2. *For any $s\geq s_0>2d+2$ there is $r_0>0$ such that for $0\leq r\leq r_0$, the following holds. Define*

$$
Z := \Phi_{\mathscr{B}_\star}(W) := W + X_{\mathscr{B}_\star}(W)\,, \tag{A.6}
$$

where $\star \in \{\mathrm{NLS}, \mathrm{KG}\}$ *(recall* (4.22), (4.52)*) and assume, respectively, Hypothesis 4.1 or Hypothesis 4.8. Then one has*

$$\|Z\|_{H^s} \leq 2\|w\|_{H^s}, \tag{A.7}$$

and

$$W = Z - X_{\mathcal{B}_\star}(Z) + r(w), \tag{A.8}$$

where

$$\|r(w)\|_{H^s} \lesssim \|w\|^5_{H^s}. \tag{A.9}$$

Proof. By (A.6) we can write

$$W = Z - X_{\mathcal{B}_\star}(W) = Z - X_{\mathcal{B}_\star}(Z) + \big[X_{\mathcal{B}_\star}(W) - X_{\mathcal{B}_\star}(Z)\big].$$

By using estimates (A.4) or (A.5) one can deduce that $X_{\mathcal{B}_\star}(W) - X_{\mathcal{B}_\star}(Z)$ satisfies the bound (A.9). The bound (A.7) follows by Lemma A.1. □

A.2. **Conjugations.** In the following lemma we study how the Hamiltonian vector fields $X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(W)$, see (4.30), and $X_{\mathcal{H}^{(\leq 4)}_{\mathrm{KG}}}(W)$, see (4.59), transform under the change of variables given by the previous lemma.

Lemma A.3. *Let* $s_0 > 2d+4$. *Then for any* $s \geq s_0$ *there is* $r_0 > 0$ *such that for all* $0 < r \leq r_0$ *and* $Z = \big[\begin{smallmatrix} z \\ \bar{z} \end{smallmatrix}\big] \in B_r(H^s(\mathbb{T}^d;\mathbb{C}^2))$ *the following holds. Consider the Hamiltonian* $\mathcal{B}_\star$ *with* $\star \in \{\mathrm{NLS}, \mathrm{KG}\}$ *(recall* (4.22), (4.52)*) and the Hamiltonian* $\mathcal{H}^{(\leq 4)}_\star$ *(see* (4.30), (4.59)*). Then*

$$d_W \Phi_{\mathcal{B}_\star}(W)\big[X_{\mathcal{H}^{(\leq 4)}_\star}(W)\big] = X_{\mathcal{H}^{(\leq 4)}_\star}(Z) + \big[X_{\mathcal{B}_\star}(Z), X_{\mathcal{H}^{(2)}_\star}(Z)\big] + R_5(Z), \tag{A.10}$$

where $\mathcal{H}^{(2)}_\star$ *is in* (4.29) *or* (4.58) *and where the remainder* R_5 *satisfies*

$$\|R_5(Z)\|_{H^s} \lesssim \|z\|^5_{H^s}, \tag{A.11}$$

and $[\cdot,\cdot]$ *is the nonlinear commutator defined in* (2.37).

Proof. We prove the statement in the case $\mathcal{B}_\star = \mathcal{B}_{\mathrm{NLS}}$ and $\mathcal{H}^{(\leq 4)}_\star = \mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}$, the KG-case is similar. One can check that (A.10) follows by setting

$$R_5 := d_W X_{\mathcal{B}_{\mathrm{NLS}}}(W)\big[X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(W) - X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(Z)\big] \tag{A.12}$$

$$+ \big(d_W X_{\mathcal{B}_{\mathrm{NLS}}}(W) - d_W X_{\mathcal{B}_{\mathrm{NLS}}}(Z)\big)\big[X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(Z)\big] \tag{A.13}$$

$$+ X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(W) - X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(Z) + d_W X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(Z)\big[X_{\mathcal{B}_{\mathrm{NLS}}}(Z)\big], \tag{A.14}$$

$$+ \big[X_{\mathcal{B}_{\mathrm{NLS}}}(Z), X_{\mathcal{H}^{(4)}_{\mathrm{NLS}}}(Z)\big]. \tag{A.15}$$

We are left to prove that R_5 satisfies (A.11). We start from the term in (A.12). First of all we note that

$$X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(W) - X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(Z) = -\mathrm{i}E\Lambda_{\mathrm{NLS}}(W-Z) + X_{\mathcal{H}^{(4)}_{\mathrm{NLS}}}(W) - X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(Z),$$

where we used that $X_{\mathcal{H}^{(2)}_{\mathrm{NLS}}}(W) = -\mathrm{i}E\Lambda_{\mathrm{NLS}}W$. By Proposition A.2, the (3.10) and (A.4) we deduce that

$$\|X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(W) - X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(Z)\|_{H^s} \lesssim \|w\|^3_{H^s}.$$

Hence using again the bounds (A.4) we obtain

$$\|d_W X_{\mathcal{B}_{\mathrm{NLS}}}(W)\big[X_{\mathcal{H}^{(\leq 4)}_{\mathrm{NLS}}}(W) - X_{\mathcal{H}^{(\leq)}_{\mathrm{NLS}}}(Z)\big]\|_{H^s} \lesssim \|w\|^5_{H^s}.$$

Reasoning in the same way, using also (A.8), one can check that the terms in (A.13), (A.14), (A.15) satisfies the same quintic estimates. □

In the next lemma we study the structure of the the cubic terms in the vector field in (A.10) in the NLS case.

Lemma A.4. *Consider the Hamiltonian* $\mathscr{B}_{\rm NLS}(W)$ *in* (4.22) *and recall* (3.10), (4.29). *Then we have that*

$$X_{\mathscr{H}^{(4)}_{\rm NLS}}(Z)+\big[X_{\mathscr{B}_{\rm NLS}}(Z),X_{\mathscr{H}^{(2)}_{\rm NLS}}(Z)\big]=-{\rm i}EOp^{\rm BW}\begin{pmatrix}2|z|^2 & 0\\ 0 & 2|z|^2\end{pmatrix}Z+Q_{{\rm H}^{(4)}_{\rm NLS}}(Z)\,, \tag{A.16}$$

where the remainder $Q_{{\rm H}^{(4)}_{\rm NLS}}$ *has the form* $Q_{{\rm H}^{(4)}_{\rm NLS}}(Z)=(Q^{+}_{{\rm H}^{(4)}_{\rm NLS}}(Z),\overline{Q^{+}_{{\rm H}^{(4)}_{\rm NLS}}(Z)})^{T}$ *and*

$$(\widehat{Q^{+}_{{\rm H}^{(4)}_{\rm NLS}}(Z)})(\xi)=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}{\tt q}_{{\rm H}^{(4)}_{\rm NLS}}(\xi,\eta,\zeta)\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}(\zeta)\,,\qquad \xi\in\mathbb{Z}^d\,, \tag{A.17}$$

with symbol satisfying

$$|{\tt q}_{{\rm H}^{(4)}_{\rm NLS}}(\xi,\eta,\zeta)|\lesssim\frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^4}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^2}\,. \tag{A.18}$$

Proof. We start by considering the commutator between $X_{\mathscr{B}_{\rm NLS}}$ and $X_{\mathscr{H}^{(2)}_{\rm NLS}}$. First of all notice that (see (A.1), (4.21))

$$X_{\mathscr{B}_{\rm NLS}}(Z)=\begin{pmatrix}X^{+}_{\mathscr{B}_{\rm NLS}}(Z)\\ \overline{X^{+}_{\mathscr{B}_{\rm NLS}}(Z)}\end{pmatrix}\,,\qquad X^{+}_{\mathscr{B}_{\rm NLS}}(Z):=Op^{\rm BW}\Big(\frac{z^2}{2|\xi|^2}\Big)[\overline{z}]+Q^{+}_{\mathscr{B}_{\rm NLS}}(Z)\,,$$

and hence (recall (2.6)), for $\xi\in\mathbb{Z}^d$

$$(\widehat{X^{+}_{\mathscr{B}_{\rm NLS}}(Z)})(\xi)=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}(\zeta)\Big[\frac{2}{|\xi+\eta|^2}\chi_{\epsilon}\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)+q_{\mathscr{B}_{\rm NLS}}(\xi,\eta,\zeta)\Big] \tag{A.19}$$

where $q_{\mathscr{B}_{\rm NLS}}(\zeta,\eta,\zeta)$ satisfies the bound in (2.15). Hence, by using formulæ (4.25), (A.19), (2.37), one obtains

$$X_{\mathscr{H}^{(4)}_{\rm NLS}}(Z)+\big[X_{\mathscr{B}_{\rm NLS}}(Z),X_{\mathscr{H}^{(2)}_{\rm NLS}}(Z)\big]=\begin{pmatrix}\mathscr{C}^{+}(Z)\\ \overline{\mathscr{C}^{+}(Z)}\end{pmatrix}\,,$$
$$(\widehat{\mathscr{C}^{+}(Z)})(\xi)=\frac{-1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}{\rm i}{\tt c}(\xi,\eta,\zeta)\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}(\zeta)$$

where

$${\tt c}(\xi,\eta,\zeta)=1+\Big[\frac{2}{|\xi+\eta|^2}\chi_{\epsilon}\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)+q_{\mathscr{B}_{\rm NLS}}(\xi,\eta,\zeta)\Big]\Big[\Lambda_{\rm NLS}(\xi-\eta-\zeta)-\Lambda_{\rm NLS}(\eta)+\Lambda_{\rm NLS}(\zeta)-\Lambda_{\rm NLS}(\xi)\Big]\,. \tag{A.20}$$

We need to prove that this can be written as the r.h.s. of (A.16). First we note that the term in (A.20)

$$q_{\mathscr{B}_{\rm NLS}}(\xi,\eta,\zeta)\Big[\Lambda_{\rm NLS}(\xi-\eta-\zeta)-\Lambda_{\rm NLS}(\eta)+\Lambda_{\rm NLS}(\zeta)-\Lambda_{\rm NLS}(\xi)\Big] \tag{A.21}$$

can be absorbed in ${\tt R}_1$ since the (A.21) satisfy the same bound as in (A.18). Moreover, using the (4.25) and the (1.5), we have that the coefficients

$$\frac{2}{|\xi+\eta|^2}\chi_{\epsilon}\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\Big[\widehat{V}(\xi-\eta-\zeta)-\widehat{V}(\eta)+\widehat{V}(\zeta)-\widehat{V}(\xi)\Big]$$

satisfy the bound in (A.18) by using also Lemma 2.6. Therefore the corresponding operator contributes to ${\tt R}_1$. The same holds for the operator corresponding to the coefficients

$$\frac{2}{|\xi+\eta|^2}\chi_{\epsilon}\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\Big[|\xi-\eta-\zeta|^2+|\zeta|^2\Big]\,.$$

We are left with the most relevant terms in (A.20) containing the highest frequencies η and ξ. We have that

$$\frac{-2(|\xi|^2+|\eta|^2)}{|\xi+\eta|^2}\chi_{\epsilon}\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)=-\chi_{\epsilon}\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)-r_1(\xi,\eta,\zeta)\,,\quad r_1(\xi,\eta,\zeta)=\frac{|\xi-\eta|^2}{|\xi+\eta|^2}\chi_{\epsilon}\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\,.$$

Again we note that the coefficients $r_1(\xi,\eta,\zeta)$, using Lemma 2.6, satisfy (A.18). Then it remains to study the operator $\mathscr{R}^{+}(Z)$ with

$$(\widehat{\mathscr{R}^{+}(Z)})(\xi):=\frac{-1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}{\rm i}\Big(1-\chi_{\epsilon}\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\Big)\widehat{z}(\xi-\eta-\zeta)\widehat{\overline{z}}(\eta)\widehat{z}(\zeta)\,.$$

By formula (3.5) and (2.6) we get $\mathscr{R}^+(Z) = -\mathrm{i}Op^{\mathrm{BW}}(2|z|^2)z + Q_3^+(U)$, where Q_3 satisfies (3.7), (3.8). This concludes the proof. $\square$

In the next lemma we study the structure of the the cubic terms in the vector field in (A.10) in the KG case.

Lemma A.5. *Consider the Hamiltonian* $\mathscr{B}_{\mathrm{KG}}(W)$ *in* (4.52) *and recall* (3.14), (4.58). *Then we have that*

$$X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(Z) + \big[X_{\mathscr{B}_{\mathrm{KG}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)\big] = -\mathrm{i}EOp^{\mathrm{BW}}\big(\mathrm{diag}(a_0(x,\xi))\big)Z + Q_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) \tag{A.22}$$

the symbol $a_0(x,\xi) = a_0(u,x,\xi)$ *is in* (3.11), *the remainder* $Q_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z)$ *has the form* $(Q^+_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z), \overline{Q^+_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z)})^T$ *with*

$$\widehat{Q^+_{\mathtt{H}^{(4)}_{\mathrm{KG}}}}(\xi) = (2\pi)^{-d} \sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z^{\sigma_3}}(\zeta), \tag{A.23}$$

for some $\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(\xi,\eta,\zeta) \in \mathbb{C}$ *satisfying*

$$|\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(\xi,\eta,\zeta)| \lesssim \frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^{\mu}}{\max\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}} \tag{A.24}$$

for some $\mu > 1$.

Proof. Using (A.1) (with $\mathscr{B} = \mathscr{B}_{\mathrm{KG}}$) we can note that

$$\big[X_{\mathscr{B}_{\mathrm{KG}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)\big] = \big[Op^{\mathrm{BW}}(B_{\mathrm{KG}}(Z;x,\xi)), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)\big] + R_2(Z) \tag{A.25}$$

where $R_2(Z) = (R_2^+(Z), \overline{R_2^+(Z)})^T$ with

$$\begin{aligned} \widehat{(R_2^+(Z))}(\xi) &= (2\pi)^{-d} \sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{r}_2^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z^{\sigma_3}}(\zeta), \quad \xi\in\mathbb{Z}^d, \\ \mathtt{r}_2^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta) &:= \mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathscr{B}_{\mathrm{KG}}}(\xi,\eta,\zeta)\Big[\sigma_1\Lambda_{\mathrm{KG}}(\xi-\eta-\zeta) + \sigma_2\Lambda_{\mathrm{KG}}(\eta) + \sigma_3\Lambda_{\mathrm{KG}}(\zeta) - \Lambda_{\mathrm{KG}}(\xi)\Big], \end{aligned} \tag{A.26}$$

where the coefficients are defined in (A.2). The remainder R_2 has the form (A.23) and we have that the coefficients $\mathtt{r}_2^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)$ satisfy the bound (A.24). On the other hand, recalling (4.51), (2.37), we have

$$\big[Op^{\mathrm{BW}}(B_{\mathrm{KG}}(Z;x,\xi)), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)\big] = R_3(Z) + R_4(Z), \qquad R_j(Z) = \begin{pmatrix} R_j^+(Z) \\ \overline{R_j^+(Z)} \end{pmatrix}, \ j = 3,4, \tag{A.27}$$

where

$$R_3^+(Z) := Op^{\mathrm{BW}}(b_{\mathrm{KG}}(Z;x,\xi))[\mathrm{i}\Lambda_{\mathrm{KG}}\overline{z}] + \mathrm{i}\Lambda_{\mathrm{KG}}Op^{\mathrm{BW}}(b_{\mathrm{KG}}(Z;x,\xi))[\overline{z}], \tag{A.28}$$

$$R_4^+(Z) := Op^{\mathrm{BW}}\big((d_Z b_{\mathrm{KG}})(z;x,\xi)[X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)]\big)[\overline{z}]. \tag{A.29}$$

By Remark 3.4 and (2.6) we get

$$\begin{aligned} \widehat{R_4^+}(\xi) = (2\pi)^{-d} \sum_{\substack{\sigma_1,\sigma_2\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} a_0^{\sigma_1,\sigma_2}(\xi-\zeta,\eta,\tfrac{\xi+\zeta}{2})\frac{1}{2\Lambda_{\mathrm{KG}}(\frac{\xi+\zeta}{2})}\chi_{\epsilon}\Big(\tfrac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\Big[-\mathrm{i}\sigma_1\Lambda_{\mathrm{KG}}(\xi-\eta-\zeta) - \mathrm{i}\sigma_2\Lambda_{\mathrm{KG}}(\eta)\Big]\times \\ \times \widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{\overline{z}}(\zeta). \end{aligned}$$

Using the explicit form of the coefficients of R_4^+ and Lemma 2.6 one can conclude that the operator R_4^+ has the form (A.23) with coefficients satisfying (A.24). To summarize, by (4.65), (A.25) and (A.27), we have obtained (recall also (3.15), (3.13))

$$\text{l.h.s. of (A.22)} = Op^{\mathrm{BW}}\begin{pmatrix} -\mathrm{i}a_0(x,\xi) & 0 \\ 0 & \mathrm{i}a_0(x,\xi) \end{pmatrix} Z + F_3(Z) + Q_3(Z) + R_2(Z) + R_4(Z) \tag{A.30}$$

where R_4 is in (A.29), R_2 is in (A.26), $Q_3(Z)$ is in (3.15) and

$$F_3(Z) = \begin{pmatrix} F_3^+(Z) \\ \overline{F_3^+(Z)} \end{pmatrix}, \qquad F_3^+(Z) = -\mathrm{i}Op^{\mathrm{BW}}(a_0(x,\xi))[\overline{z}] + R_3^+(Z) \tag{A.31}$$

where R_3^+ is in (A.28). By the discussion above and by Lemma 3.3 we have that the remainders R_2, R_4 and Q_3 have the form (A.23) with coefficients satisfying (A.24). To conclude the prove we need to show that F_3 has the same property. This will be a consequence of the choice of the symbol $b_{\mathrm{KG}}(W;x,\xi)$ in (4.51). Indeed, by (4.51), Remark 3.4, (A.31), (A.28), we have

$$\widehat{F_3^+}(\xi) = (2\pi)^{-d} \sum_{\substack{\sigma_1,\sigma_2\in\{\pm\} \\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{f}_3^{\sigma_1,\sigma_2,-}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{\overline{z}}(\zeta)$$

where

$$\mathtt{f}_3^{\sigma_1,\sigma_2,-}(\xi,\eta,\zeta) := a_0^{\sigma_1,\sigma_2}(\xi-\zeta,\eta,\tfrac{\xi+\zeta}{2})\mathrm{i}\Big[\frac{\Lambda_{\mathrm{KG}}(\xi)+\Lambda_{\mathrm{KG}}(\zeta)}{2\Lambda_{\mathrm{KG}}(\frac{\xi+\zeta}{2})} - 1\Big]\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\zeta+\zeta\rangle}\Big). \tag{A.32}$$

By Taylor expanding the symbol $\Lambda_{\mathrm{KG}}(\xi)$ in (1.4) (see also Remark 3.4) one deduces that

$$\Big|a_0^{\sigma_1,\sigma_2}(\xi-\zeta,\eta,\tfrac{\xi+\zeta}{2})\mathrm{i}\Big[\frac{\Lambda_{\mathrm{KG}}(\xi)+\Lambda_{\mathrm{KG}}(\zeta)}{2\Lambda_{\mathrm{KG}}(\frac{\xi+\zeta}{2})} - 1\Big]\Big| \lesssim \frac{|\xi-\zeta|}{(\langle\xi\rangle+\langle\zeta\rangle)^{3/2}}\,.$$

Therefore, using Lemma 2.6, we have that the coefficients $\mathtt{f}_3^{\sigma_1,\sigma_2,-}(\xi,\eta,\zeta)$ in (A.32) satisfy the (A.24). This implies the (A.22). $\square$

Appendix B. Non-resonance conditions for (KG)

In this section we prove Proposition 5.13 providing lower bounds on the phase in (5.25). Recall the symbol $\Lambda_{\mathrm{KG}}(j)$ in (1.4). Throughout this subsection, in order to lighten the notation, we shall write $\Lambda_{\mathrm{KG}}(j) \rightsquigarrow \Lambda_j$ for any $j\in\mathbb{Z}^d$. The main result of this section is the following.

Proposition B.1. *Let $4>\beta>3$, there exist $\alpha>0$ and $\mathscr{C}_\beta\subset[1,2]$ a set of Lebesgue measure 1 and for $m\in\mathscr{C}_\beta$ there exists $\kappa(m)>0$ such that*

$$|\sigma_1\Lambda_{j_1}+\sigma_2\Lambda_{j_2}+\sigma_3\Lambda_{j_3}+\sigma_4\Lambda_{j_4}| \geq \frac{\kappa(m)}{|j_3|^\alpha|j_1|^\beta} \tag{B.1}$$

for all $\sigma_1,\sigma_2,\sigma_3,\sigma_4\in\{-1,1\}$, $j_1,j_2,j_3,j_4\in\mathbb{Z}^d$ satisfying $|j_1|\geq|j_2|\geq|j_3|\geq|j_4|$ and $\sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3+\sigma_4 j_4=0$, except when $\sigma_1=\sigma_4=-\sigma_2=-\sigma_3$ and $|j_1|=|j_2|\geq|j_3|=|j_4|$.

The Proposition B.1 will implies Proposition 5.6. Its proof is done in three steps.

Step 1: control with respect to the highest index.

Lemma B.2. *There exist $\nu>0$ and $\mathscr{M}_\nu\subset[1,2]$ a set of Lebesgue measure 1 and for $m\in\mathscr{M}_\nu$ there exists $\gamma(m)>0$ such that*

$$|\sigma_1\Lambda_{j_1}+\sigma_2\Lambda_{j_2}+\sigma_3\Lambda_{j_3}+\sigma_4\Lambda_{j_4}| \geq \gamma(m)|j_1|^{-\nu} \tag{B.2}$$

for all $\sigma_1,\sigma_2,\sigma_3,\sigma_4\in\{-1,1\}$, $j_1,j_2,j_3,j_4\in\mathbb{Z}^d$ satisfying $|j_1|\geq|j_2|\geq|j_3|\geq|j_4|$, except when $\sigma_1=\sigma_4=-\sigma_2=-\sigma_3$ and $|j_1|=|j_2|\geq|j_3|=|j_4|$.

The proof of this Lemma is standard and follows the line of Theorem 6.5 in [2], see also [4] or [23]. We briefly repeat the steps.
Let us assume that $j_1,j_2,j_3,j_4\in\mathbb{Z}^d$ satisfy $|j_1|>|j_2|>|j_3|>|j_4|$. First of all, by reasoning as in Lemma 3.2 in [23], one can deduce the following.

Lemma B.3. *Consider the matrix D whose entry at place (p,q) is given by $\frac{d^p}{dm^p}\Lambda_{j_q}$, $p,q=1,\dots,4$. The modulus of the determinant of D is bounded from below: one has $|\det(D)|\geq C|j_1|^{-\mu}$ where $C>0$ and $\mu>0$ are universal constants.*

From Lemma 3.3 in [23] we learn

Lemma B.4. *Let $u^{(1)},...,u^{(4)}$ be 4 independent vectors in $\mathbb{R}^4$ with $\|u^{(i)}\|_{\ell^1} \le 1$. Let $w \in \mathbb{R}^4$ be an arbitrary vector, then there exist $i \in [1,\cdots,4]$, such that $|u^{(i)} \cdot w| \ge C\|w\|_{\ell^1} \det(u^{(1)},\dots,u^{(4)})$.*

Let us define

$$\psi_{\rm KG}(m) = \sigma_1\Lambda_{j_1}(m) + \sigma_2\Lambda_{j_2}(m) + \sigma_3\Lambda_{j_3}(m) + \sigma_4\Lambda_{j_4}(m).$$

Combining Lemmata B.3 and B.4 we deduce the following.

Corollary B.5. *For any $m \in [1,2]$ there exists an index $i \in [1,\cdots,4]$ such that $\left|\frac{d^i\psi_{\rm KG}}{dm^i}(m)\right| \ge C|j_1|^{-\mu}$.*

Now we need the following result (see Lemma B.1 in [22]):

Lemma B.6. *Let $\mathfrak{g}(x)$ be a C^{n+1}-smooth function on the segment $[1,2]$ such that*

$$|\mathfrak{g}'|_{C^n} = \beta \quad \text{and} \qquad \max_{1\le k\le n} \min_x |\partial^k \mathfrak{g}(x)| = \sigma.$$

Then

$$\text{meas}(\{x \,|\, |\mathfrak{g}(x)| \le \rho\}) \le C_n \left(\frac{\beta}{\sigma}+1\right)\left(\frac{\rho}{\sigma}\right)^{1/n}.$$

Define

$$\mathscr{E}_j(\kappa) := \left\{m \in [1,2] \,|\, |\sigma_1\Lambda_{j_1} + \sigma_2\Lambda_{j_2} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}| \le \kappa|j_1|^{-\nu}\right\}.$$

By combining Corollary B.5 and Lemma B.6 we get

$$\text{meas}\,(\mathscr{E}_j(\kappa)) \le C|j_1|^{\mu}\left(\kappa|j_1|^{\mu-\nu}\right)^{1/4} \le C\kappa^{\frac14}|j_1|^{\frac{5\mu-\nu}{4}}. \tag{B.3}$$

Define

$$\mathscr{E}(\kappa) = \bigcup_{|j_1|>|j_2|>|j_3|>|j_4|} \mathscr{E}_j(\kappa),$$

and set $\nu = 5\mu + 4(d+1)$. Then the (B.3) implies meas$(\mathscr{E}(\kappa)) \le C\kappa^{\frac14}$. Then taking $m \in \bigcup_{\kappa>0}([1,2]\setminus\mathscr{E}(\kappa))$ we obtain (B.2) for any $|j_1| > |j_2| > |j_3| > |j_4|$. Furthermore $\bigcup_{\kappa>0}([1,2]\setminus\mathscr{E}(\kappa))$ has measure 1. Now if for instance $|j_1| = |j_2|$ then we are left with a small divisor of the type $|2\Lambda_{j_1} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}|$ or $|\Lambda_{j_3} + \sigma_4\Lambda_{j_4}|$, i.e. involving 2 or 3 frequencies. So following the same line we can also manage this case.

Step 2: control with respect to the third highest index. In this subsection we show that small dividers can be controlled by a smaller power of $|j_1|$ even if it means transferring part of the weight to $|j_3|$.

Proposition B.7. *Let $4 > \beta > 3$, there exists $\mathscr{N}_\beta \subset [1,2]$ a set of Lebesgue measure 1 and for $m \in \mathscr{N}_\beta$ there exists $\kappa(m) > 0$ such that*

$$|\Lambda_{j_1} - \Lambda_{j_2} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}| \ge \frac{\kappa(m)}{|j_3|^{2d+3}|j_1|^{\beta}}$$

for all $\sigma_3,\sigma_4 \in \{-1,+1\}$, for all $j_1, j_2, j_3, j_4 \in \mathbb{Z}^d$ satisfying $|j_1| > |j_2| \ge |j_3| > |j_4|$, the momentum condition $j_1 - j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$ and

$$|j_1| \ge J(\kappa,|j_3|) := \left(\tfrac{C}{\kappa}\right)^{\frac{1}{4-\beta}}|j_3|^{\frac{2d+8}{4-\beta}}$$

where C is an universal constant.

We begin with two elementary lemmas

Lemma B.8. *Let $\sigma = \pm1$, $j,k \in \mathbb{Z}^d$, with $|j| > |k| > 0$ and $|j| \ge 8$, and $[1,2] \ni m \mapsto \mathfrak{g}(m)$ a C^1 function satisfying $|\mathfrak{g}'(m)| \le \frac{1}{10|j|^3}$ for $m \in [1,2]$. For all $\kappa > 0$ there exists $\mathscr{D} \equiv \mathscr{D}(j,k,\sigma,\kappa,\mathfrak{g}) \subset [1,2]$ such that for $m \in \mathscr{D}$*

$$|\Lambda_j + \sigma\Lambda_k - \mathfrak{g}(m)| \ge \kappa$$

and

$$\textit{meas}([1,2]\setminus\mathscr{C}) \le 10\kappa|j|^3.$$

Proof. Let $f(m)=\Lambda_j+\sigma\Lambda_k-\mathfrak{g}(m)$. In the case $\sigma=-1$, which is the worst, we have

$$\begin{aligned} f'(m) &= \frac{1}{2}\Big(\frac{1}{\sqrt{|j|^2+m}}-\frac{1}{\sqrt{|k|^2+m}}\Big)-\mathfrak{g}'(m) \\ &= \frac{|k|^2-|j|^2}{2(\sqrt{|j|^2+m}+\sqrt{|k|^2+m})\sqrt{|j|^2+m}\sqrt{|k|^2+m}}-\mathfrak{g}'(m)\,. \end{aligned}$$

We want to estimate $|f'(m)|$ from above. By using that $4(|j|^2+2)^{\frac{3}{2}}\le 5|j|^3$ for $|j|\ge 8$ we get

$$|f'(m)|\ge \frac{1}{5|j|^3}-\frac{1}{10|j|^3}\ge\frac{1}{10|j|^3}.$$

In the case $\sigma=1$, the same bound holds true. Then we conclude by a standard argument that

$$\mathrm{meas}\{m\in[1,2]\,|\,|f(m)|\le\kappa\}\le 10\kappa|j|^3\,,$$

which is the thesis. □

Lemma B.9. *Let $j,k\in\mathbb{Z}^d$ with $|j|\ge|k|$ and $|j-k|\le|j|^{\frac{1}{2}}$ then*

$$\big|\Lambda_j-\Lambda_k-\mathfrak{g}(|j|,|j-k|,(j,j-k),m)\big|\le C\frac{|j-k|^5}{|j|^4} \tag{B.4}$$

for some explicit rational function $\mathfrak{g}$ and some universal constant $C>0$.
Furthermore one has

$$|\partial_m\mathfrak{g}(|j|,|j-k|,(j,j-k),m)|\le\frac{1}{|j|^{3/2}}\,.$$

Proof. By Taylor expansion we have for $|j|$ large

$$\Lambda_j=|j|(1+\frac{m}{|j|^2})^{\frac{1}{2}}=|j|+\frac{m}{2|j|}-\frac{m^2}{8|j|^3}+O(\frac{1}{|j|^5})$$

and

$$\begin{aligned}\Lambda_k &= |j|(1+\frac{2(k-j,j)+|j-k|^2+m}{|j|^2})^{\frac{1}{2}} \\ &= |j|+\frac{2(k-j,j)+|j-k|^2+m}{2|j|}-\frac{(2(k-j,j)+|j-k|^2+m)^2}{8|j|^3} \\ &\quad+\frac{3}{48}\frac{(2(k-j,j)+|j-k|^2+m)^3}{|j|^5}-\frac{15}{16}\frac{1}{4!}\frac{(2(k-j,j)+|j-k|^2+m)^4}{|j|^7}+O(\frac{|j-k|^5}{|j|^4})\end{aligned}$$

which leads to (B.4) where with (we use that $|(k-j,j)|\le|j-k||j|$ and $|j-k|\le|j|^{\frac{1}{2}}$)

$$\mathfrak{g}(x,y,z,m)=\frac{2z+y^2}{2x}-\frac{(2z+y^2+m)^2-m^2}{8x^3}+\frac{3}{48}\frac{8z^3+12z^2(y^2+m)}{x^5}-\frac{1}{4!}\frac{15}{16}\frac{16z^4}{x^7}\,.$$

□

We are now in position to prove the main result of this subsection.

Proof of Proposition B.7. Let $\mathfrak{g}$ be the rational function introduced in Lemma B.9. We write, with $\sigma=\sigma_3\sigma_4$,

$$\begin{aligned}|\Lambda_{j_1}-\Lambda_{j_2}+\sigma_3\Lambda_{j_3}+\sigma_4\Lambda_{j_4}|\ge&|\Lambda_{j_3}+\sigma\Lambda_{j_4}+\sigma_3\mathfrak{g}(|j_1|,|j_1-j_2|,(j_1,j_1-j_2),m)| \\ &-|\Lambda_{j_1}-\Lambda_{j_2}-\mathfrak{g}(|j_1|,|j_1-j_2|,(j_1,j_1-j_2),m)|\,.\end{aligned}$$

By assumption $j_1 - j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$ and thus $|j_1 - j_2| \le 2|j_3|$. Choosing $\kappa = \frac{\gamma}{|j_3|^{2d+3}|j_1|^\beta}$ in Lemma B.8 and assuming $2|j_3| \le |j_1|^{\frac12}$ we have by Lemma B.8 and Lemma B.9

$$|\Lambda_{j_1} - \Lambda_{j_2} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}| \ge \frac{\gamma}{|j_3|^{2d+3}|j_1|^\beta} - C\frac{|j_3|^5}{|j_1|^4} \ge \frac{\gamma}{2|j_3|^{2d+3}|j_1|^\beta}$$

as soon as

$$|j_1| \ge \left(\tfrac{C}{\gamma}\right)^{\frac{1}{4-\beta}}|j_3|^{\frac{2d+8}{4-\beta}} =: J(\gamma,|j_3|) \ge 10|j_3|^3$$

and $m \in \mathscr{D}(j_3, j_4, \sigma, \kappa, \sigma_3\mathfrak{g}(|j_1|, |j_1 - j_2|, (j_1, j_1 - j_2), \cdot))$ (the set $\mathscr{D}$ is defined in Lemma B.8). Then denoting

$$\mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4) := \{m \in [1,2] : |\Lambda_{j_1} - \Lambda_{j_2} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}| \ge \frac{\gamma}{2|j_3|^{2d+3}|j_1|^\beta},$$
$$\forall (j_1, j_2) \text{ such that } |j_1| \ge \max(|j_2|, J(\gamma,|j_3|)),\ j_1 - j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0\}$$

we have

$$\mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4) = \bigcap_{\mathfrak{g}} \mathscr{D}(j_3, j_4, \sigma, \frac{\gamma}{|j_3|^{2d+4}|j_1|^\beta}, \sigma_3\mathfrak{g}(|j_1|, |j_1 - j_2|, (j_1, j_1 - j_2), \cdot))$$

where the intersection is taken over all functions $\mathfrak{g}$ generated by $(j_1, j_2) \in (\mathbb{Z}^d)^2$ such that

$$|j_1| \ge \max(|j_2|, J(\gamma, |j_3|))$$

and $j_1 - j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$. Thus by Lemma B.8

$$\text{meas}\left([1,2] \setminus \mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4)\right) \le$$
$$\sum_{n\ge1} \frac{10\gamma}{|j_3|^{2d+3} n^{\frac{\beta}{2}}} \#\{(|j_1|, |\sigma_3 j_3 + \sigma_4 j_4|, (j_1, \sigma_3 j_3 + \sigma_4 j_4)) \mid j_1 \in \mathbb{Z}^d,\ |j_1|^2 = n\}.$$

But, the scalar product $(j_1, \sigma_3 j_3 + \sigma_4 j_4))$ takes only integer values no larger in modulus than $2|j_1||j_3|$ thus

$$\text{meas}\, \mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4) \le \frac{20\gamma}{|j_3|^{2d+2}} \sum_{n\ge1} \frac{1}{n^{\frac{\beta-1}{2}}} \le C_\beta \frac{\gamma}{|j_3|^{2d+2}}.$$

Then it remains to define

$$\mathscr{N}_\beta = \cup_{\gamma>0} \bigcap_{\substack{(j_3,j_4)\in(\mathbb{Z}^d)^2 \\ |k_4|\le|k_3| \\ \sigma_3,\sigma_4\in\{-1,1\}}} \mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4)$$

to conclude the proof. □

Step 3: proof of Proposition B.1 We are now in position to prove Proposition B.1. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{-1, 1\}$, $j_1, j_2, j_3, j_4 \in \mathbb{Z}^d$ satisfying $|j_1| \ge |j_2| \ge |j_3| \ge |j_4|$ and $\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$. If $\sigma_1 = \sigma_2$, then, since $|j_1| \ge |j_2| \ge |j_3| \ge |j_4|$, we conclude that the associated small divisor cannot be small except if $|j_1| = |j_2| = |j_3| = |j_4|$ and $\sigma_1 = \sigma_2 = -\sigma_3 = -\sigma_4$ but this case is excluded in Proposition B.1. Thus we can assume $\sigma_1 = -\sigma_2$ and we can apply Proposition B.7 which implies the control (B.1) for $m \in \mathscr{N}_\beta$ with $\alpha = 2d + 3$ under the additional constrain $|j_1| \ge J(\kappa(m), |j_3|)$. Now if $|j_1| \le J(\kappa(m), |j_3|)$ we can apply Lemma B.2 to obtain that there exists $\nu > 0$ and full measure set $\mathscr{M}_\nu$ such that for $m \in \mathscr{M}_\nu \cap \mathscr{N}_\beta := \mathscr{C}_\beta$ we have

$$|\sigma_1\Lambda_{j_1} + \sigma_2\Lambda_{j_2} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}| \ge \frac{\gamma(m)}{|j_1|^\nu} \ge \frac{\gamma(m)}{J(|j_3|, \kappa(m))^\nu} = C\frac{\gamma(m)\kappa(m)^{4-\beta}}{|j_3|^\alpha}$$

with $\alpha = \nu\frac{2d+8}{4-\beta}$ which, of course, implies (B.1).

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