A local Theory of Domains and its (Noncommutative) Symbolic Counterpart.

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Abstract

It is widely accepted nowadays that polyzetas are connected by polynomial relations. One way to obtain relations among polyzetas is to consider their generating series and the relations among these generating series. This leads to the indexation of the generating series of polylogarithms, recently described in [6,5,13]. But, in order to understand the bridge between the extension of this "polylogarithmic calculus" and the world of harmonic sums, a local theory of domains has to be done, preserving quasi-shuffle identities, Taylor expansions and Hadamard products. In this contribution, we present a sketched version of this theory.

As an example of generating series, one can consider the eulerian gamma function,

$$\Gamma(1+z) = \exp\left(-\gamma z + \sum_{n\geq 2} \zeta(n)\frac{(-z)^n}{n}\right)$$

and this may suggest to regularize the divergent zeta value $\zeta(1)$, for the quasi-shuffle structure, as to be Euler's γ constant. In the same vein, in [5], we introduce a family of eulerian functions,

$$\Gamma_{y_k}(1+z) = \exp\left(\sum_{n\geq 1} \zeta(kn)\frac{(-z^k)^n}{n}\right), \text{for } k\geq 2, y_k \in Y = \{y_n\}_{n\geq 1}.$$

This being done, in this work, via their analytical aspects, we establish, on one side, their existence and the fact that their inverses are entire. On the other side, using the same symmetrization technique, we give their distributions of zeroes [1] .

Contents

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1 Introduction

This work is partly the continuation of [6,5,13] where it has been established that the polylogarithms, indexed by the r-tuples $(s_1,\ldots,s_r)\in\mathbb{C}^r$, are well defined locally by

$$\mathrm{Li}_{s_1,\ldots,s_r}(z):=\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}},\text{for }|z|<1, \tag{1}$$

could be extended, in case $(s_1,\ldots,s_r)\in\mathbb{N}_+^r$, to some series, over the alphabet $X=\{x_0,x_1\}$ generating the monoid X^* with the neutral element 1_{X^*} [1]. More precisely,

(i) we start to consider [10]

$$\forall w=x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\in X^*x_1,\ \mathrm{Li}_w=\mathrm{Li}_{s_1,\ldots,s_r}, \tag{2}$$

(ii) then extend [2] $\mathrm{Li}_\bullet$ as the $\sqcup\!\sqcup$-morphism $(\mathbb{C}\langle X\rangle,\sqcup\!\sqcup,1_{X^*})\longrightarrow(\mathbb{C}\{\mathrm{Li}\}_{w\in X^*},\times,1)$ by adding $\mathrm{Li}_{x_0}(z)=\log(z)$. This morphism is injective and satisfies [14]

$$\forall S,T\in\mathbb{C}\langle X\rangle,\ \mathrm{Li}_{S\sqcup\!\sqcup T}=\mathrm{Li}_S\mathrm{Li}_T\,. \tag{3}$$

(iii) For the sake of symbolic calculations, it is important that, on the one hand, these series should belong to some "computable spaces" and, on the other hand, that the new domain (a) be closed by shuffle products and (b) that the $\mathrm{Li}_\bullet$ correspondence should preserve the shuffle identity (3).

To this end a theory of global domains was presented in [6,5,13]. Here we focus on what happens in the neighbourhood of zero, therefore, the aim of this work is manyfold. Let us highligh the many facets of this matter.

(i) Propagate the extension to local Taylor expansions [3] as in (1) and the coefficients of their quotients by $1-z$, namely the harmonic sums, denoted $\mathrm{H}_\bullet$ and defined, for any $w\in X^*x_1$, as follows [4] [12]

$$\frac{\mathrm{Li}_w(z)}{1-z}=\sum_{N\geq 0}\mathrm{H}_{\pi_X(w)}(N)z^N, \tag{4}$$

by a suitable theory of local domains which assures to carry over the computation of these Taylor coefficients and preserves the stuffle indentity, again true for polynomials over the alphabet $Y=\{y_n\}_{n\geq 1}$, *i.e.*

$$\forall S,T\in\mathbb{C}\langle Y\rangle,\ \mathrm{H}_{S\,\uplus\, T}=\mathrm{H}_S\mathrm{H}_T, \tag{5}$$

[2] This paper uses extensively shuffle and stuffle products (noted $\sqcup\!\sqcup$ and $\uplus$ respectively). For readers unfamiliar with these subjects their definitions are recalled at the end of this text, see paragraph 2.2.

[3] Around zero.

[4] Here, the conc-morphism $\pi_X:(\mathbb{C}\langle Y\rangle,\mathtt{conc},1_{Y^*})\longrightarrow(\mathbb{C}\langle X\rangle,\mathtt{conc},1_{X^*})$ is defined by $\pi_X(y_n)=x_0^{n-1}x_1$ and π_Y its inverse on $\mathrm{Im}(\pi_X)$. See [6,5,13] for more details and a full definition of π_Y.

meaning that $\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) \longrightarrow (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, \times, 1)$, mapping any word $w = y_{s_1}\ldots y_{s_r} \in Y^*$ to

$$\mathrm{H}_w = \mathrm{H}_{s_1,\ldots,s_r} = \sum_{N\geq n_1>\ldots>n_r>0} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}, \tag{6}$$

is a injective ⧢-morphism [12].

(ii) Extend these correspondences (*i.e.* $\mathrm{Li}_\bullet, \mathrm{H}_\bullet$) to some series (over X and Y, respectively) in order to preserve the identity [5] [12]

$$\frac{\mathrm{Li}_{\pi_X(S)}(z)}{1-z} \odot \frac{\mathrm{Li}_{\pi_X(T)}(z)}{1-z} = \frac{\mathrm{Li}_{\pi_X(S ⧢ T)}(z)}{1-z}. \tag{7}$$

true for polynomials $S, T \in \mathbb{C}\langle Y\rangle$.

(iii) Taking the definition of polyetas as in (1) at $z = 1$ or in (6) at $+\infty$, one sees that, for any $s_1 > 1$, Abel's theorem, one has

$$\zeta(s_1,\ldots,s_r) = \lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) \tag{8}$$

$$= \lim_{N\to+\infty} \mathrm{H}_{s_1,\ldots,s_r}(N) \tag{9}$$

$$= \sum_{n_1>\ldots>n_r>0} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{10}$$

However, this theorem does not hold in the divergent cases. and we will recall some regularization process based on the computation of a ⧢-character with polynomial values and specialize it to obtain a character [3,4]

$$\gamma_\bullet : (\mathbb{Q}\langle Y\rangle, ⧢, 1_{Y^*}) \longrightarrow (\mathscr{Z}[\gamma], \times, 1), \tag{11}$$

where $\mathscr{Z} := \mathrm{span}_{\mathbb{Q}}\{\zeta(s_1,\ldots,s_r)\}_{r\geq 1, s_1\geq 2, s_2,\ldots,s_r\geq 1}$.

(iv) To this end, we use the explicit parametrization of the conc-characters obtained in [6,5,13] and the fact that, under stuffle products, they form a group. We show the linear independence of the Kleene stars $(z^k y_k)^*$ and show that $\gamma_\bullet$ provides a group morphism between the group of conc-characters (endowed with ⧢) and that of Taylor series g (with radius $R = 1$) such that $g(0) = 1$. This morphism maps each star y_k^* precisely to

$$\frac{1}{\Gamma_{y_k}(1+z)} = \exp\Big(-\sum_{n\geq 1} \zeta(kn)\frac{(-z^k)^n}{n}\Big), \text{for } k \geq 2. \tag{12}$$

and y_1^* to the classical inverse Gamma:

$$\Gamma_{y_1}^{-1}(1+z) = \Gamma^{-1}(1+z). \tag{13}$$

[5] Here $\odot$ stands for the Hadamard product [9].

We will prove that all these “new” functions are entire and linearly independant.

To summarize, the present work concerns the whole project of extending $\mathrm{H}_\bullet$ over a stuffle subalgebra of rational power series on the alphabet Y, in particular the stars of letters and some explicit combinatorial consequences of this extension.

2 Domains and extensions

All starts with the (multiindexed) polylogarithm defined, for $|z|<1$, by (1). It is (multi-)indexed by a list $(s_1,\ldots,s_r)\in\mathbb{N}_{\geq 1}^r$ which can be reindexed by a word $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\in X^*x_1$. From this, introducing two differential forms

$$\omega_0(z)=z^{-1}dz \text{ and } \omega_1(z)=(1-z)^{-1}dz, \tag{14}$$

we get an integral representation of the functions (1) as follows [6] [10]

$$\mathrm{Li}_w(z)=\begin{cases}1_{\mathscr{H}(\Omega)} & \text{if } w=1_{X^*}\\ \displaystyle\int_0^z\omega_1(s)\,\mathrm{Li}_u(s) & \text{if } w=x_1u\\ \displaystyle\int_1^z\omega_0(s)\,\mathrm{Li}_u(s) & \text{if } w=x_0u \text{ and } |u|_{x_1}=0, w\in x_0^*\\ \displaystyle\int_0^z\omega_0(s)\,\mathrm{Li}_u(s) & \text{if } w=x_0u \text{ and } |u|_{x_1}>0, w\notin x_0^*,\end{cases} \tag{15}$$

where Ω is the simply connected domain $\mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$, over which we consider the algebra of analytic functions, $\mathscr{H}(\Omega)$, with the neutral element $1_{\mathscr{H}(\Omega)}$. This provides not only the analytic continuation of (1) to Ω but also extends the indexation to the whole alphabet X, allowing to study the complete generating series

$$\mathrm{L}(z)=\sum_{w\in X^*}\mathrm{Li}_w(z)w \tag{16}$$

and show that it is the solution of the following first order noncommutative differential equation

$$\begin{cases}\mathbf{d}(S)=(\omega_0(z)x_0+\omega_1(z)x_1)S, & (NCDE)\\ \displaystyle\lim_{z\in\Omega,z\to 0}S(z)e^{-x_0\log(z)}=1_{\mathscr{H}(\Omega)\langle\langle X\rangle\rangle}, & \text{asymptotic initinial condition,}\end{cases} \tag{17}$$

where, for any $S\in\mathscr{H}(\Omega)\langle\langle X\rangle\rangle$, for term by term derivation, one gets [8]

$$\mathbf{d}(S)=\sum_{w\in X^*}\frac{d}{dz}(\langle S\mid w\rangle)w. \tag{18}$$

[6] Given a word $w\in X^*$, we note $|w|_{x_1}$ the number of occurrences of x_1 within w.

This differential system allows to show that L is a $\sqcup\!\sqcup$-character [14], *i.e.*

$$\forall u,v \in X^*, \quad \langle \mathrm{L} \mid u \sqcup\!\sqcup v\rangle = \langle \mathrm{L} \mid u\rangle\langle \mathrm{L} \mid v\rangle \text{ and } \langle \mathrm{L} \mid 1_{X^*}\rangle = 1_{\mathscr{H}(\Omega)}. \tag{19}$$

Note that, in what precedes, we used the pairing $\langle \bullet \mid \bullet\rangle$ between series and polynomials, classically defined by, for $T \in \mathbb{C}\langle\langle X\rangle\rangle$ and $P \in \mathbb{C}\langle X\rangle$ [7]

$$\langle T \mid P\rangle = \sum_{w\in X^*} \langle T \mid w\rangle\langle P \mid w\rangle, \tag{20}$$

where, when w is a word, $\langle S \mid w\rangle$ stands for the coefficient of w in S. With this at hand, we extend at once the indexation of Li from X^* to $\mathbb{C}\langle\langle X\rangle\rangle$ by

$$\mathrm{Li}_P := \sum_{w\in X^*} \langle P \mid w\rangle \mathrm{Li}_w = \sum_{n\geq 0}\Big(\sum_{|w|=n} \langle P \mid w\rangle \mathrm{Li}_w\Big). \tag{21}$$

In [6,5,13], it has been established that the polylogarithm, well defined locally by (1), could be extended to some series (with conditions) by the last part of formula (21) where the polynomial P is replaced by some series.

As was said previously, we focus here on what happens in the neighbourhood of zero. Therefore, the aim of this paragraph concerns the two first points of Section 1. which we summarize here

(i) Propagate the extension to local Taylor expansions [8] of polylogarithms and the coefficients of their quotients by $1-z$, namely the harmonic sums, by a suitable theory of local domains.

(ii) Extend these correspondences (*i.e.* $\mathrm{Li}_\bullet, \mathrm{H}_\bullet$) to some series in order to preserve the identity (7).

2.1 *Polylogarithms: from global to local domains*

The map $\mathrm{Li}_\bullet$ in general has been extended to a subdomain of $\mathbb{C}\langle\langle X\rangle\rangle$, called $\mathrm{Dom}(\mathrm{Li}_\bullet)$ (see [6,5,13]). It is the set of series

$$S = \sum_{n\geq 0} S_n, \text{ where } S_n := \sum_{|w|=n} \langle S \mid w\rangle \tag{22}$$

such that $\sum_{n\geq 0} \mathrm{Li}_{S_n}$ is unconditionally convergent for the standard topology on $\mathscr{H}(\Omega)$ [17].

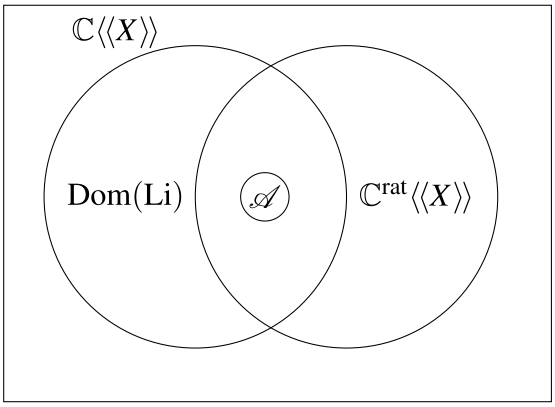
Example 2.1 [[10]] For example, the classical polylogarithms: dilogarithm Li_2, trilogarithm Li_3, etc... are defined and obtained through this coding by

[7] Here R is any commutative ring (like $\mathscr{H}(\Omega), \mathbb{C}, \mathscr{Z}[\gamma]$, ...).

[8] Around zero.

$$\mathrm{Li}_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k} = \mathrm{Li}_{x_0^{k-1}x_1}(z) = \langle \mathrm{L}(z) \mid x_0^{k-1}x_1 \rangle$$

but for $t \geq 0$ (real), the series $(tx_0)^* x_1$ belongs to $\mathrm{Dom}(\mathrm{Li}_\bullet)$ iff $0 \leq t < 1$.



Above $\mathscr{A} = \mathbb{C}\langle X\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ is the set of rational series [6,5,13].

This definition has many merits [9] and can easily be adapted to arbitrary (open and connected) domains. But this definition, based on a global condition of a fixed domain Ω, does not provide a sufficiently clear interpretation of the stable symbolic computations around a point, in particular at $z = 0$. One needs to consider a sort of "symbolic local germ" worked out explicitely. Indeed, as the harmonic sums (or MZV) are the coefficients of the Taylor expansion at zero of the convergent polylogarithms divided by $1-z$, we only need to know locally these functions. In order to gain more indexing series and to describe the local situation at zero, we reshape and define a new domain of Li around zero to $\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$. The first step will be provided by the following theorem.

Theorem 2.2 *Let $S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*}$ such that*

$$S = \sum_{n\geq 0} [S]_n \text{ where } [S]_n = \sum_{w\in X^*, |w|=n} \langle S \mid w\rangle w,$$

($[S]_n$ are the homogeneous components of S), we suppose that $0 < R \leq 1$ and that $\sum_{n\geq 0} \mathrm{Li}_{[S]_n}$ is unconditionally convergent (for the standard topology) within the open disk $|z| < R$. Remarking that $\frac{1}{1-z}\sum_{n\geq 0} \mathrm{Li}_{[S]_n}(z)$ is unconditionally convergent in the same domain, we set

[9] As the fact that, due to special properties of $\mathscr{H}(\Omega)$ (it is a nuclear space [17], see details in [6,5,13]), one can show that $\mathrm{Dom}(\mathrm{Li})$ is closed by shuffle products.

$$\frac{1}{1-z}\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z)=\sum_{N\geq 0}a_N z^N.$$

Then, for all $N\geq 0$,

$$\sum_{n\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N)=a_N.$$

Proof. Let us recall that, for any $w\in X^*$, the function $(1-z)^{-1}\mathrm{Li}_w(z)$ is analytic in the open disk $|z|<R$. Moreover, one has

$$\frac{1}{1-z}\mathrm{Li}_w(z)=\sum_{N\geq 0}\mathrm{H}_{\pi_Y(w)}(N)z^N.$$

Since $[S]_n=\sum_{w\in X^*,|w|=n}\langle S\mid w\rangle w$ and $(1-z)^{-1}\sum_{n\geq 0}\mathrm{Li}_{[S]_n}$ absolutely converges (for the standard topology [10]) within the open disk $|z|<R$, one obtains

$$\begin{aligned}
\frac{1}{1-z}\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z)&=\frac{1}{1-z}\sum_{n\geq 0}\sum_{w\in X^*,|w|=n}\langle S\mid w\rangle w\,\mathrm{Li}_w(z)\\
&=\sum_{n\geq 0}\sum_{w\in X^*,|w|=n}\langle S\mid w\rangle w\frac{\mathrm{Li}_w(z)}{1-z}\\
&=\sum_{n\geq 0}\sum_{w\in X^*,|w|=n}\langle S\mid w\rangle w\sum_{N\geq 0}\mathrm{H}_{\pi_Y(w)}(N)z^N\\
&=\sum_{N\geq 0}\sum_{n\geq 0}\sum_{w\in X^*,|w|=n}\langle S\mid w\rangle w\mathrm{H}_{\pi_Y(w)}(N)z^N\\
&=\sum_{N\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N)z^N.
\end{aligned}$$

This implies that, for any $N\geq 0$,

$$a_N=\sum_{n\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N).$$

$\square$

We will need the following combinatorial

Lemma 2.3 *For a letter "a", one has*

$$|\,(a^+)^{\sqcup\!\sqcup\, m}\,|\,a^n=m!S_2(n,m) \tag{23}$$

($S_2(n,m)$ being the Stirling numbers of the second kind). The exponential generating series of R.H.S. in equation (23) (w.r.t. n) is given by

$$\sum_{n\geq 0}m!S_2(n,m)\frac{x^n}{n!}=(e^x-1)^m. \tag{24}$$

[10] For this topology, unconditional and absolute convergence coincide [17]

Proof. $(a^+)^{⧢ m}$ is the specialization of

$$L_m = a_1^+ ⧢ a_2^+ ⧢ \ldots ⧢ a_m^+$$

to $a_j \to a$ (for all $j = 1,2\ldots m$). The words of L_m are in bijection with the surjections $[1\ldots n] \to [1\ldots m]$, therefore the coefficient $\langle (a^+)^{⧢ m} | a^n \rangle$ is exactly the number of such surjections namely $m! S_2(n,m)$. A classical formula [11] says that

$$\sum_{n\geq 0} m! S_2(n,m) \frac{x^n}{n!} = (e^x - 1)^m. \tag{25}$$

$\square$

To prepare the construction of the "symbolic local germ" around zero, let us set, in the same manner as in [6,5,13],

$$\begin{aligned} \mathrm{Dom}_R(\mathrm{Li}) := \{ & S \in \mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C} 1_{X^*} \mid \\ & \sum_{n\geq 0} \mathrm{Li}_{[S]_n} \text{ is unconditionally convergent in } \mathscr{H}(D_{<R}) \} \end{aligned} \tag{26}$$

and prove the following:

Proposition 2.4 *With the notations as above, we have:*

(i) *The map* $]0,1] \to \mathbb{C}\langle\langle X \rangle\rangle$ *given by* $R \mapsto \mathrm{Dom}_R(\mathrm{Li})$ *is strictly decreasing*

(ii) *Each* $\mathrm{Dom}_R(\mathrm{Li})$ *is a shuffle subalgebra of* $\mathbb{C}\langle\langle X \rangle\rangle$.

Proof.

(i) It is straightforward that he map $R \longmapsto \mathrm{Dom}_R(\mathrm{Li})$ is decreasing. Set now, with $x_1^+ = x_1 x_1^* = x_1^* - 1$,

$$S(t) = \sum_{m\geq 0} t^m (x_1^+)^{⧢ m}$$

and let $[S]_n(t)$ be its homogeneous components, we have

$$\sum_{n\geq 0} \mathrm{Li}_{[S]_n(t)}(z) = \frac{1-z}{1-(t+1)z}.$$

For $0 < R_1 < R_2 \leq 1$ it is straightforward that

$$\mathrm{Dom}_{R_2}(\mathrm{Li}) \subset \mathrm{Dom}_{R_1}(\mathrm{Li}).$$

Let us prove that the inclusion is strict.

Take $|z| < 1$ and let us, be it finite or infinite, evaluate the sum

$$M(z) = \sum_{n\geq 0} |\mathrm{Li}_{[S]_n(t)}(z)| = \sum_{n\geq 0} \langle S(t) \mid x_1^n \rangle \, |\mathrm{Li}_{x_1^n}(z)|$$

then

[11] See [16], the twelvefold way, formula (1.94b)(pp. 74) for instance.

$$
\begin{aligned}
M(z) &= \sum_{n\geq 0} |\, S(t) \,|\, x_1^n \,|\mathrm{Li}_{x_1^n}(z)\,| \\
&= \sum_{n\geq 0}\sum_{m\geq 0} |\, t^m (x_1^+)^{⧢ m} \,|\, x_1^n \,|\mathrm{Li}_{x_1^n}(z)\,| \\
&= \sum_{m\geq 0} m! t^m \sum_{n\geq 0} S_2(n,m) \frac{|\,\mathrm{Li}_{x_1}(z)\,|^n}{n!} \\
&\leq \sum_{m\geq 0} m! t^m \sum_{n\geq 0} S_2(n,m) \frac{\mathrm{Li}_{x_1}^n(|\,z\,|)}{n!},
\end{aligned}
$$

due to the fact that $|\,\mathrm{Li}_{x_1}(z)\,| \leq \mathrm{Li}_{x_1}(|\,z\,|)$ (Taylor series with positive coefficients). Finally, in view of equation (25), we get, on the one hand, for $|\,z\,| < (t+1)^{-1}$,

$$
M(z) \leq \sum_{m\geq 0} t^m (e^{\mathrm{Li}_{x_1}(|z|)} - 1)^m = \sum_{m\geq 0} t^m \Big(\frac{|\,z\,|}{1-|\,z\,|}\Big)^m = \frac{1-|\,z\,|}{1-(t+1)\,|\,z\,|}.
$$

This proves that, for all $r \in]0, \dfrac{1}{t+1}[$,

$$
\sum_{n\geq 0} \|\,\mathrm{Li}_{[S]_n(t)}(z)\|_r < +\infty.
$$

On the other hand, if $(t+1)^{-1} \leq |\,z\,| < 1$, one has $M(|z|) = +\infty$, and the preceding calculation shows that, with t choosen such that

$$
0 \leq \frac{1}{R_2} - 1 < t < \frac{1}{R_1} - 1,
$$

we have $S(t) \in \mathrm{Dom}_{R_1}(\mathrm{Li})$ but $S(t) \notin \mathrm{Dom}_{R_2}(\mathrm{Li})$ whence, for $0 < R_1 < R_2 \leq 1$, $\mathrm{Dom}_{R_2}(\mathrm{Li}) \subsetneq \mathrm{Dom}_{R_1}(\mathrm{Li})$.

(ii) One has (proofs as in [6])

(a) $1_{X^*} \in \mathrm{Dom}_R(\mathrm{Li})$ (because $1_{X^*} \in \mathbb{C}\langle X\rangle$) and $\mathrm{Li}_{1_{X^*}} = 1_{\mathscr{H}(\Omega)}$.

(b) Taking $S, T \in \mathrm{Dom}_R(\mathrm{Li})$ we have, by absolute convergence, $S ⧢ T \in \mathrm{Dom}_R(\mathrm{Li})$. It is easily seen that $S ⧢ T \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*}$ and, moreover, that $\mathrm{Li}_S \mathrm{Li}_T = \mathrm{Li}_{S ⧢ T}$ [12].

$\square$

In Theorem 2.6 bellow, we study, for series taken in $\mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}.1_{X^*}$, the correspondence $\mathrm{Li}_\bullet$ to some $\mathscr{H}(D_{<R})$, first (point 1) establishes its surjectivity (in a certain sense) and then (points 2 and 3) examine the relation between summability of the functions and that of their Taylor coefficients. For that, let us begin with a very general lemma on sequences of Taylor series which adapts, for our needs, the notion of *normal families* [15].

[12] Proof by absolute convergence as in [6].

Lemma 2.5 *Let $\tau = (a_{n,N})_{n,N\geq 0}$ be a double sequence of complex numbers. Setting*

$$I(\tau) := \{r \in]0,+\infty[\mid \sum_{n,N\geq 0} |a_{n,N} r^N| < +\infty\},$$

one has

(i) *$I(\tau)$ is an interval of $]0,+\infty[$, it is not empty iff there exists $z_0 \in \mathbb{C}\setminus\{0\}$ such that*

$$\sum_{n,N\geq 0} |a_{n,N} z_0^N| < +\infty \tag{27}$$

In this case, we set $R(\tau) := \sup(I(\tau))$, one has

(a) *For all N, the series $\sum_{n\geq 0} a_{n,N}$ converges absolutely (in $\mathbb{C}$). Let us note a_N - with one subscript - its limit*

(b) *For all n, the convergence radius of the Taylor series*

$$T_n(z) = \sum_{N\geq 0} a_{n,N} z^N$$

is at least $R(\tau)$ and $\sum_{n\in\mathbb{N}} T_n$ is summable for the standard topology of $\mathscr{H}(D_{<R(\tau)})$ with sum $T(z) = \sum_{n,N\geq 0} a_N z^N$.

(ii) *Conversely, we suppose that it exists $R > 0$ such that*

(a) *Each Taylor series $T_n(z) = \sum_{N\geq 0} a_{n,N} z^N$ converges in $\mathscr{H}(D_{<R})$.*

(b) *The series $\sum_{n\in\mathbb{N}} T_n$ converges unconditionnally in $\mathscr{H}(D_{<R})$.*

Then $I(\tau) \neq \emptyset$ and $R(\tau) \geq R$.

Proof.

(i) The fact that $I(\tau) \subset]0,+\infty[$ is straightforward from the definition. If it exists $z_0 \in \mathbb{C}$ such that

$$\sum_{n,N\geq 0} | a_{n,N} z_0^N | < +\infty$$

then, for all $r \in]0, |z_0|[$, we have

$$\sum_{n,N\geq 0} | a_{n,N} r^N | = \sum_{n,N\geq 0} | a_{n,N} z_0^N | \left(\frac{r}{|z_0|}\right)^N \leq \sum_{n,N\geq 0} |a_{n,N} z_0^N| < +\infty$$

in particular $I(\tau) \neq \emptyset$ and it is an interval of $]0,+\infty[$ with lower bound zero.

(a) Take $r \in I(\tau)$ (hence $r \neq 0$) and $N \in \mathbb{N}$, then we get the expected result as

$$r^N \sum_{n\geq 0} | a_{n,N} | = \sum_{n\geq 0} | a_{n,N} r^N | \leq \sum_{n,N\geq 0} | a_{n,N} r^N | < +\infty.$$

(b) Again, take any $r \in I(\tau)$ and $n \in \mathbb{N}$, then

$$\sum_{N\geq 0} |a_{n,N}r^N| < +\infty$$

which proves that $R(T_n) \geq r$, hence the result [13]. We also have

$$|\sum_{N\geq 0} a_N r^N| \leq \sum_{N\geq 0} r^N |\sum_{n\geq 0} a_{n,N}| \leq \sum_{n,N\geq 0} |a_{n,N}r^N| < +\infty$$

and this proves that $R(T) \geq r$, hence $R(T) \geq R(\tau)$.

(ii) Let $0 < r < r_1 < R$ and consider the path $\gamma(t) = r_1 e^{2i\pi t}$, we have

$$|a_{n,N}| = |\frac{1}{2i\pi}\int_\gamma \frac{T_n(z)}{z^{N+1}}dz| \leq \frac{2\pi}{2\pi}\frac{r_1\|T_n\|_K}{r_1^{N+1}} \leq \frac{\|T_n\|_K}{r_1^N}$$

with $K = \gamma([0,2\pi])$, hence

$$\sum_{n,N\geq 0} |a_{n,N}r^N| \leq \sum_{n,N\geq 0} |T_n|_K \left(\frac{r}{r_1}\right)^N \leq \frac{r_1}{r_1-r}\sum_{n\geq 0}\|T_n\|_K < +\infty.$$

□

Theorem 2.6 (i) *Let $T(z) = \sum_{N\geq 0} a_N z^N$ be a Taylor series i.e. such that*

$$\limsup_{N\to+\infty} |a_N|^{1/n} = B < +\infty,$$

then the series

$$S = \sum_{N\geq 0} a_N(-(-x_1)^+)^{ш N} \tag{28}$$

is summable (see [2]) in $\mathbb{C}\langle\langle X\rangle\rangle$ (with sum in $\mathbb{C}\langle\langle x_1\rangle\rangle$), $S \in \mathrm{Dom}_R(\mathrm{Li})$ with $R = (B+1)^{-1}$ and $\mathrm{Li}_S = T$.

(ii) *Let $S \in \mathrm{Dom}_R(\mathrm{Li})$ and $S = \sum_{\geq 0}[S]_n$ (homogeneous decomposition), we define $N \longmapsto \mathrm{H}_{\pi_Y(S)}(N)$ by* [14]

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0}\mathrm{H}_{\pi_Y(S)}(N)z^N.$$

(iii) *Moreover,*

$$\forall r \in]0,R[, \sum_{n,N\geq 0} |\mathrm{H}_{\pi_Y([S]_n)}(N)r^N| < +\infty, \tag{29}$$

and, for all $N \in \mathbb{N}$, the series (of complex numbers), $\sum_{n\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N)$ converges absolutely to $\mathrm{H}_{\pi_Y(S)}(N)$.

[13] For a Taylor series T, we note $R(T)$ the radius of convergence of T.

[14] This definition is compatible with the old one when S is a polynomial.

(iv) *Conversely, let* $Q \in \mathbb{C}\langle\langle Y \rangle\rangle$ *with* $Q = \sum_{n\geq 0} Q_n$ *(decomposition by weights), we suppose that it exists* $r \in]0,1]$ *such that*

$$\sum_{n,N\geq 0} |\mathrm{H}_{Q_n}(N) r^N| < +\infty, \tag{30}$$

in particular, for all $N \in \mathbb{N}$, $\sum_{n\geq 0} \mathrm{H}_{Q_n}(N) = \ell(N) \in \mathbb{C}$ *unconditionally. Under such circumstances,* $\pi_X(Q) \in \mathrm{Dom}_r(\mathrm{Li})$ *and, for all* $z \in \mathbb{C}, |z| \leq r$,

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \ell(N) z^N, \tag{31}$$

Proof.

(i) The fact that the series (28) is summable comes from the fact that

$$\omega(a_N(-(-x_1)^+)^{ш N}) \geq N$$

(see [2]). Now from the lemma, we get

$$(S)_n = \sum_{N\geq 0} (a_N(-(-x_1)^+)^{ш N})_n = (-1)^{N+n} a_N N! S_2(n,N) x_1^n.$$

Then, with $r = \sup_{z\in K} |z|$ (we have indeed $r = \|Id\|_K$) and taking into account that $\| \mathrm{Li}_{x_1} \|_K \leq \log(1/(1-r))$, we have

$$\begin{aligned}
\sum_{n\geq 0} \| \mathrm{Li}_{(S)_n} \|_K &\leq \sum_{n\geq 0} \sum_{N\geq 0} |a_N| N! S_2(n,N) \| \mathrm{Li}_{x_1^n} \|_K \\
&\leq \sum_{n\geq 0} \sum_{N\geq 0} |a_N| N! S_2(n,N) \frac{\| \mathrm{Li}_{x_1} \|_K^n}{n!} \\
&\leq \sum_{N\geq 0} |a_N| \sum_{n\geq 0} N! S_2(n,N) \frac{| \mathrm{Li}_{x_1} |_K^n}{n!} \\
&\leq \sum_{N\geq 0} |a_N| (e^{\log(\frac{1}{1-r})} - 1)^N \\
&= \sum_{N\geq 0} |a_N| \left(\frac{r}{1-r}\right)^N.
\end{aligned}$$

Now if we suppose that $r \leq (B+1)^{-1}$, we have $r(1-r)^{-1} \leq 1/B$ and this shows that the last sum is finite.

(ii) This point and next point are consequences of Lemma 2.5.

Now, considering the homogeneous decomposition

$$S = \sum_{n\geq 0} [S]_n \in \mathrm{Dom}_R(\mathrm{Li}).$$

we first establish inequation (29). Let $0 < r < r_1 < R$ and consider the path $\gamma(t) = r_1 e^{2i\pi t}$, we have

$$|\mathrm{H}_{\pi_Y([S]_n)}(N)| = \left|\frac{1}{2i\pi}\int_\gamma \frac{\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz\right| \le \frac{2\pi}{2\pi}\frac{\|\mathrm{Li}_{[S]_n}\|_K}{(1-r_1)r_1^{N+1}},$$

$K = \gamma([0,1])$ being the circle of center 0 and radius r_1. Taking into account that, for $K \subset_{comp.} D_{<R}$, we have a decomposition

$$\sum_{n\in\mathbb{N}} |\mathrm{Li}_{[S]_n}|_K = M < +\infty,$$

we get

$$\begin{aligned}
\sum_{n,N\ge 0} |\mathrm{H}_{\pi_Y([S]_n)}(N) r^N| &= \sum_{n,N\ge 0} |\mathrm{H}_{\pi_Y([S]_n)}(N) r_1^N| \left(\frac{r}{r_1}\right)^N \\
&= \sum_{N\ge 0} \left(\frac{r}{r_1}\right)^N \sum_{n\ge 0} |\mathrm{H}_{\pi_Y([S]_n)}(N) r_1^N| \\
&\le \sum_{N\ge 0} \left(\frac{r}{r_1}\right)^N \frac{M}{(1-r_1)r_1} \\
&\le \frac{M}{(1-r_1)(r_1-r)} < +\infty.
\end{aligned}$$

The series $\sum_{n\ge 0} \mathrm{Li}_{[S]_n}(z)$ converges to $\mathrm{Li}_S(z)$ in $\mathscr{H}(D_{<R})$ ($D_{<R}$ is the open disk defined by $|z| < R$). For any $N \ge 0$, by Cauchy's formula, one has,

$$\begin{aligned}
\mathrm{H}_{\pi_Y(S)}(N) &= \frac{1}{2i\pi}\int_\gamma \frac{\mathrm{Li}_S(z)}{(1-z)z^{N+1}}dz \\
&= \frac{1}{2i\pi}\int_\gamma \frac{\sum_{n\ge 0}\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz \\
&= \frac{1}{2i\pi}\sum_{n\ge 0}\int_\gamma \frac{\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz \\
&= \sum_{n\ge 0} \mathrm{H}_{\pi_Y([S]_n)}(N)
\end{aligned}$$

the exchange of sum and integral being due to the compact convergence. The absolute convergence comes from the fact that the convergence of $\sum_{n\ge} \mathrm{Li}_{[S]_n}(z)$ is unconditional [17].

(iii) Fixing $N \in \mathbb{N}$, from inequation (30), we get $\sum_{n\ge 0} |\mathrm{H}_{Q_n}(N)| < +\infty$ which proves the absolute convergence. Remark now that $(\pi_X(Q))_n = \pi_X(Q_n)$ and $\pi_Y(\pi_X(Q_n)) = Q_n$, one has, for all $|z| \le r$

$$|\mathrm{Li}_{\pi_X(Q_n)}(z)| = \left|\sum_{N\in\mathbb{N}} \mathrm{H}_{Q_n}(N) z^N\right| \le \left|\sum_{N\in\mathbb{N}} \mathrm{H}_{Q_n}(N) r^N\right|,$$

in other words

$$\| \mathrm{Li}_{\pi_X(Q_n)} \|_{D\leq r} \leq \mid \sum_{N\in\mathbb{N}} \mathrm{H}_{Q_n}(N) r^N \mid$$

and

$$\sum_{n\in\mathbb{N}} \| \mathrm{Li}_{\pi_X(Q_n)} \|_{D\leq r} \leq \mid \sum_{n,N\in\mathbb{N}} \mathrm{H}_{Q_n}(N) r^N \mid < +\infty$$

which shows that $\pi_X(Q) \in \mathrm{Dom}_r(\mathrm{Li})$. The equation (31) is a consequence of point 2, taking $S = \pi_X(Q)$.

□

Definition 2.7 We set

$$\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}) = \bigcup_{0<R\leq 1} \mathrm{Dom}_R(\mathrm{Li}); \mathrm{Dom}(\mathrm{H}_\bullet) = \pi_Y(\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}))$$

and, for $S \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$,

$$\mathrm{Li}_S(z) = \sum_{n\geq 0} \mathrm{Li}_{[S]_n}(z) \text{ and } \frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(S)}(N) z^N.$$

Observe that, from this definition, theorem (2.8), will show that $\mathrm{Dom}(\mathrm{H}_\bullet)$ is a stuffle subalgebra of $\mathbb{C}\langle\langle Y\rangle\rangle$.

(i) The series $T = \sum_{n=1}^{\infty} (-1)^{n-1} y_n/n \in \mathbb{C}\langle\langle Y\rangle\rangle$ is not in $\mathrm{Dom}(\mathrm{H}_\bullet)$ because, for all $0 < r < 1$, one has

$$\sum_{n,N} \mid T_n(N) r^N \mid \geq \sum_{n\geq 0} \frac{1}{1-r} = +\infty \tag{32}$$

However one can get unconditional convergence using a sommation by pairs (odd + even).

(ii) For all $s \in]1,+\infty[$, the series $T(s) = \sum_{n=1}^{\infty} (-1)^{n-1} y_n n^{-s} \in \mathbb{C}\langle\langle Y\rangle\rangle$ is in $\mathrm{Dom}(\mathrm{H}_\bullet)$.

We can now state the

Theorem 2.8 *Let $S, T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$, then*

$$S \sqcup\!\sqcup T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}), \pi_X(\pi_Y(S) \uplus \pi_Y(T)) \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$$

and for all $N \geq 0$,

$$\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \mathrm{Li}_T; \quad \mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(\Omega)}, \tag{33}$$

$$\mathrm{H}_{\pi_Y(S) \uplus \pi_Y(T)}(N) = \mathrm{H}_{\pi_Y(S)}(N) \mathrm{H}_{\pi_Y(T)}(N). \tag{34}$$

$$\frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} = \frac{\mathrm{Li}_{\pi_X(\pi_Y(S) \uplus \pi_Y(T))}(z)}{1-z}. \tag{35}$$

Proof. For equation (33), we get, from lemma 2.4 that $Dom^{loc}(\mathrm{Li})$ is the union of an increasing set of shuffle subalgebras of $\mathbb{C}\langle\langle X\rangle\rangle$. It is therefore a shuffle subalgebra of the latter.

For equation (34), suppose $S \in \mathrm{Dom}_0^{R_1}(\mathrm{Li})$ (resp. $T \in \mathrm{Dom}_0^{R_2}(\mathrm{Li})$). By [9] and theorem 2.6, one has

$$\frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} \in \mathrm{Dom}_0^{R_1 R_2}(\mathrm{Li}),$$

where $\odot$ stands for the Hadamard product [9]. Hence, for $|z| < R_1 R_2$, one has

$$f(z) = \frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N)z^N \tag{36}$$

and, due to theorem 2.6 point (iii), for all N, $\sum\limits_{p\geq 0} \mathrm{H}_{\pi_Y(S_p)}(N) = \mathrm{H}_{\pi_Y(S)}(N)$ and $\sum_{q\geq 0} \mathrm{H}_{\pi_Y(T_q)}(N) = \mathrm{H}_{\pi_Y(T)}(N)$ (absolute convergence) then, as the product of two absolutely convergent series is absolutely convergent (w.r.t. the Cauchy product), one has, for all N,

$$\begin{aligned}
\mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N) &= \Big(\sum_{p\geq 0} \mathrm{H}_{\pi_Y(S_p)}(N)\Big)\Big(\sum_{q\geq 0} \mathrm{H}_{\pi_Y(T_q)}(N)\Big)\\
&= \sum_{p,q\geq 0} \mathrm{H}_{\pi_Y(S_p)}(N)\mathrm{H}_{\pi_Y(T_q)}(N)\\
&= \sum_{n\geq 0}\sum_{p+q=n} \mathrm{H}_{\pi_Y(S_p) \,\text{⧢}\, \pi_Y(T_q)}(N)\\
&= \sum_{n\geq 0} \mathrm{H}_{(\pi_Y(S) \,\text{⧢}\, \pi_Y(T))_n}(N).
\end{aligned} \tag{37}$$

Remains to prove that condition of Theorem 2.6, *i.e.* inequation (30) is fulfilled. To this end, we use the well-known fact that if $\sum_{m\geq 0} c_m z^m$ has radius of convergence $R>0$, then $\sum\limits_{m\geq 0} |c_m|\, z^m$ has the same radius of convergence (use $1/R = \limsup_{m\geq 1} |c_m|^{-m}$), then from the fact that $S \in \mathrm{Dom}_0^{R_1}(\mathrm{Li})$ (resp. $T \in Dom_0^{R_2}(\mathrm{Li})$), we have (29) for each of them and, using the Hadamard product of these expressions, we get

$$\forall r \in]0, R_1.R_2[, \sum_{p,q,N\geq 0} |\mathrm{H}_{\pi_Y(S_p)}(N)\mathrm{H}_{\pi_Y(T_q)}(N)\, r^N| < +\infty,$$

and this assures, for $|z| < R_1 R_2$, the convergence of

$$f(z) = \sum_{n,N\geq 0} \mathrm{H}_{(\pi_Y(S) \,\text{⧢}\, \pi_Y(T))_n}(N)z^N \tag{38}$$

applying theorem 2.6 point (iv) to $Q = \pi_Y(S) \,\text{⧢}\, \pi_Y(T)$ (with any $r < R_1 R_2$), we get $\pi_X(Q) = \pi_X(\pi_Y(S) \,\text{⧢}\, \pi_Y(T)) \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$ and

$$f(z)=\sum_{N\ge0}\Big(\sum_{n\ge0}\mathrm{H}_{(\pi_Y(S)\,ш\!\!\!\!-\,\pi_Y(T))_n}(N)\Big)z^N=\frac{\mathrm{Li}_{\pi_X(\pi_Y(S)\,ш\!\!\!\!-\,\pi_Y(T))}(z)}{1-z}.$$

hence (34). $\square$

2.2 *Stuffle product and stuffle characters*

For the some reader's convenience, we recall here the definitions of shuffle and stuffle products. As regards shuffle, the alphabet $\mathscr{X}$ is arbitrary and ш is defined by the following recursion (for $a,b\in\mathscr{X}$ and $u,v\in\mathscr{X}^*$)

$$u\,ш\,1_{\mathscr{X}^*}=1_{\mathscr{X}^*}\,ш\,u=u, \tag{39}$$
$$au\,ш\,bv=a(u\,ш\,bv)+b(au\,ш\,v). \tag{40}$$

As regards stuffle, the alphabet is $Y=Y_{\mathbb{N}_{\ge1}}=\{y_s\}_{s\in\mathbb{N}_{\ge1}}$ and $ш\!\!\!\!-$ is defined by the following recursion

$$u\,ш\!\!\!\!-\,1_{Y^*}=1_{Y^*}\,ш\!\!\!\!-\,u=u, \tag{41}$$
$$y_su\,ш\!\!\!\!-\,y_tv=y_s(u\,ш\!\!\!\!-\,y_tv)+y_t(y_su\,ш\!\!\!\!-\,v)+y_{s+t}(u\,ш\!\!\!\!-\,v). \tag{42}$$

Be it for stuffle or shuffle, the noncommutative [15] polynomials equipped with this product form an associative commutative and unital algebra namely $(\mathbb{C}\langle X\rangle,ш,1_{X^*})$ (resp. $(\mathbb{C}\langle Y\rangle,ш\!\!\!\!-,1_{Y^*})$).

Example 2.9 As examples of characters, we have already seen

- $\mathrm{Li}_\bullet$ from $(\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet),ш,1_{X^*})$ to $\mathscr{H}(\Omega)$
- $\mathrm{H}_\bullet$ from $(\mathrm{Dom}(\mathrm{H}_\bullet),ш\!\!\!\!-,1_{Y^*})$ to $\mathbb{C}^{\mathbb{N}}$ (arithmetic functions $\mathbb{N}\longrightarrow\mathbb{C}$)

In general, a character from a k-algebra [16] $(\mathscr{A},*_1,1_{\mathscr{A}})$ with values in $(\mathscr{B},*_2,1_{\mathscr{B}})$ is none other than a morphism between the k-algebras $\mathscr{A}$ and a commutative algebra [17] $\mathscr{B}$. The algebra $(\mathscr{A},*_1,1_{\mathscr{A}})$ does not have to be commutative for example characters of $(\mathbb{C}\langle\mathscr{X}\rangle,\mathtt{conc},1_{\mathscr{X}^*})$ - *i.e.* conc-characters - where all proved to be of the form [7]

$$\Big(\sum_{x\in\mathscr{X}}\alpha_x x\Big)^* \tag{43}$$

i.e. Kleene stars of the plane [6,5,13]. They are closed under shuffle and stuffle and endowed with these laws, they form a group. Expressions like (43) (*i.e.* homogeneous series of degree 1) form a vector space noted $\widehat{\mathbb{C}.Y}$.

[15] For concatenation.
[16] Here we will use $k=\mathbb{Q}$ or $\mathbb{C}$.
[17] In this context all algebras are associative and unital.

As a consequence, given $P = \sum_{i\geq 1} \alpha_i y_i$ and $Q = \sum_{j\geq 1} \beta_j y_j$, we know in advance that their stuffle is a conc-character *i.e.* of the form $(\sum_{n\geq 1} c_n y_n)^*$. Examining the effect of this stuffle on each letter (which suffices), we get the identity [7]

$$\Big(\sum_{i\geq 1} \alpha_i y_i\Big)^* \uplus \Big(\sum_{j\geq 1} \beta_j y_j\Big)^* = \Big(\sum_{i\geq 1} \alpha_i y_i + \sum_{j\geq 1} \beta_j y_j + \sum_{i,j\geq 1} \alpha_i \beta_j y_{i+j}\Big)^* \tag{44}$$

which suggests to take an auxiliary variable, say q, and code "the plane" $\widehat{\mathbb{C}.Y}$, *i.e.* expressions like (43), like in Umbral calculus by

$$\pi_Y^{\mathrm{Umbra}} : \sum_{n\geq 1} \alpha_n q^n \longmapsto \sum_{n\geq 1} \alpha_n y_n \tag{45}$$

which is linear and bijective [18] from $\mathbb{C}_+[[q]]$ to $\widehat{\mathbb{C}.Y}$.

With this coding at hand and for $S, T \in \mathbb{C}_+[[q]]$, identity (44) reads

$$(\pi_Y^{\mathrm{Umbra}}(S))^* \uplus (\pi_Y^{\mathrm{Umbra}}(T))^* = (\pi_Y^{\mathrm{Umbra}}((1+S)(1+T)-1))^* \tag{46}$$

This shows that if one sets, for $z \in \mathbb{C}$ and $T \in \mathbb{C}_+[[x]]$,

$$G(z) = (\pi_Y^{\mathrm{Umbra}}(e^{zT} - 1))^* \tag{47}$$

we get a one-parameter stuffle group [19], drawn on $1 + \mathbb{C}[z]_+\langle\langle Y \rangle\rangle$ (a Magnus group), i.e. such that every coefficient is polynomial in z. Differentiating it we get

$$\frac{d}{dz}(G(z)) = (\pi_Y^{\mathrm{Umbra}}(T))G(z) \tag{48}$$

and (48) with the initial condition $G(0) = 1_{Y^*}$ integrates as

$$G(z) = \exp_{\uplus}(z\pi_Y^{\mathrm{Umbra}}(T)) \tag{49}$$

where the exponential map for the stuffle product is defined, for any $P \in \mathbb{C}\langle\langle Y \rangle\rangle$ such that $\langle P \mid 1_{Y^*}\rangle = 0$, is defined by

$$\exp_{\uplus}(P) := 1_{Y^*} + \frac{P}{1!} + \frac{P \uplus P}{2!} + \ldots + \frac{P^{\uplus n}}{n!} + \ldots. \tag{50}$$

In particular, from (49), one gets, for $k \geq 1$,

$$(zy_k)^* = \exp_{\uplus}\Big(-\sum_{n\geq 1} y_{nk} \frac{(-z)^n}{n}\Big). \tag{51}$$

This expression and that of

$$\frac{1}{\Gamma(1+z)} = \exp\Big(\gamma z - \sum_{n\geq 2} \zeta(n) \frac{(-z)^n}{n}\Big) \tag{52}$$

[18] Its inverse will be naturally noted π_q^{Umbra}.

[19] *i.e.* $G(z_1 + z_2) = G(z_1) \uplus G(z_2); G(0) = 1_{Y^*}$.

suggests to consider lacunary analogues of the inverse Gamma function together with a character which sends y_1 to γ and $y_n, n \geq 2$ to $\zeta(n)$. This $⧢$-character is provided by asymptotic analysis of the Harmonic Sums. Indeed, one can show that, $w \in Y^*$ being given, the asymptotic expansion of $N \longmapsto \mathrm{H}_w(N)$, along the asymptotic scale $(\log(N)^p N^{-q})_{p,q\in\mathbb{N}}$, at any rate [20], can be written

$$\sum_{q\geq 0} Q_{w,q}(\log(N))N^{-q}, \tag{53}$$

where $Q_{w,q} \in \mathbb{C}[X]$ (univariate polynomials) and, in particular, $Q_{w,0} \in \mathbb{Q}[\gamma][X]$ [3]. From this and the fact that $\mathrm{H}_\bullet$ is a $⧢$-character, one gets that $w \longmapsto Q_{w,q}$ (resp. $\gamma_\bullet : w \longmapsto Q_{w,q}(0)$) is a $⧢$-character with values in $\mathbb{Q}[\gamma][X]$ (resp. $\mathbb{Q}[\gamma][X]$).

Now, a domain [21] Ω being given, it is easy to see that any $⧢$-character χ (with general complex values and in particular $\gamma_\bullet$) classically extends $\mathscr{H}(\Omega)\langle Y\rangle$ by

$$\chi(P) = \sum_{w\in Y^*} \langle P \mid w\rangle\langle \chi \mid w\rangle \tag{54}$$

as a $⧢$-character from $\mathscr{H}(\Omega)\langle Y\rangle$ with values in $\mathscr{H}(\Omega)$.

Now, we can extend χ to some series, over Y. For that, let us set as above and as in [6,5,13],

Definition 2.10 For any $T \in \mathscr{H}(\Omega)\langle\langle Y\rangle\rangle$, we note $[T]_n$ the homogeneous component [22] $\sum_{|w|=n} \langle T \mid w\rangle w$ of T

$$\mathrm{Dom}(\chi,\Omega) = \{T \in \mathscr{H}(\Omega)\langle\langle Y\rangle\rangle | (\chi(T_n))_{n\in\mathbb{N}} \text{ is summable in } \mathscr{H}(\Omega)\} \tag{55}$$

The result, $\sum_{n\geq 0} \chi(T_n)$ will be noted $\hat{\chi}(T)$.

This being defined, we have the following theorem

Theorem 2.11 *Let $\chi : \mathbb{C}\langle Y\rangle \longrightarrow \mathbb{C}$ be a $⧢$-character* [23]

(i) $\mathscr{H}(\Omega)\langle Y\rangle \subset \mathrm{Dom}(\chi,\Omega)$

(ii) *If $S,T \in \mathrm{Dom}(\chi,\Omega)$ then $S ⧢ T \in \mathrm{Dom}(\chi,\Omega)$* [24] *and*

$$\hat{\chi}(S ⧢ T) = \hat{\chi}(S)\hat{\chi}(T) \tag{56}$$

(iii) *If $S \in \mathrm{Dom}(\chi,\Omega)$, then $\exp_{⧢}(S) \in \mathrm{Dom}(\chi,\Omega)$ and*

$$\hat{\chi}(\exp_{⧢}(S)) = e^{\hat{\chi}(S)}$$

[20] This means that the following expression is the limit of all partial asymptotic expansions.
[21] Open, nonempty and connected subset of $\mathbb{C}$.
[22] The weight (w) of $w \in Y^*$ is just the sum of its indices
[23] We will still note its extension to $\mathscr{H}(\Omega)\langle Y\rangle$ by χ.
[24] In fact $\mathrm{Dom}(\chi,\Omega)$ is a subalgebra of $(\mathscr{H}(\Omega)\langle\langle Y\rangle\rangle, ⧢, 1_{Y^*})$

Proof.

(i) By finitely supported sum.

(ii) $S, T \in \mathrm{Dom}(\chi, \Omega)$ then $(\chi(S_p))_{p\geq 0}, (\chi(T_q))_{p\geq 0}$ are summable. But, as $\sqcup\!\!\sqcup$ is graded for the weight, one has $[S \sqcup\!\!\sqcup T]_n = \sum_{p+q=n} [S]_p \sqcup\!\!\sqcup [T]_q$. Take any K nonempty compact within Ω, then

$$\begin{aligned}
\sum_{n\geq 0} \|\chi([S \sqcup\!\!\sqcup T]_n)\|_K &= \sum_{n\geq 0} \|\chi\Big(\sum_{p+q=n} [S]_p \sqcup\!\!\sqcup [T]_q\Big)\|_K \\
&= \sum_{n\geq 0} \|\sum_{p+q=n} \chi([S]_p)\chi([T]_q)\|_K \\
&\leq \sum_{n\geq 0} \|\sum_{p+q=n} \chi([S]_p)\|_K \|\chi([T]_q)\|_K \\
&= \sum_{p,q\geq 0} \|\chi([S]_p)\|_K \|\chi([T]_q)\|_K < +\infty.
\end{aligned}$$

The same computation without the seminorm proves (56).

(iii) If $S \in \mathrm{Dom}(\chi, \Omega)$ and we have to examine (and prove) the summability of the family $(\chi([\exp_{\sqcup\!\!\sqcup}(S)]_n))_{n\geq 0}$. Setting $S = \sum_{q\geq 0} [S]_q$, we have

$$[\exp_{\sqcup\!\!\sqcup}(S)]_n = \sum_{m\geq 0} \sum_{q_1+2q_2+\cdots mq_m=n} \frac{[S]_1^{\sqcup\!\!\sqcup q_1} \sqcup\!\!\sqcup \cdots \sqcup\!\!\sqcup [S]_m^{\sqcup\!\!\sqcup q_m}}{q_1!q_2!\cdots q_m!}. \tag{57}$$

Hence, with all $q_i > 0$,

$$\begin{aligned}
\sum_{n\geq 0} \|\chi([\exp_{\sqcup\!\!\sqcup}(S)]_n)\|_K &\leq 1 + \sum_{n>0} \sum_{m>0} \sum_{q_1+2q_2+\cdots mq_m=n} \\
&\qquad \frac{\|\chi([S]_1\|_K^{q_1} \cdots \|\chi([S]_m)\|_K^{q_m}}{q_1!q_2!\cdots q_m!} \qquad (58)\\
&\leq \prod_{q\geq 1} e^{\|\chi([S]_q\|_K} \\
&= e^{\sum_{q\geq 1} \|\chi([S]_q\|_K} \\
&= e^M < +\infty. \qquad (59)
\end{aligned}$$

because, as $S \in \mathrm{Dom}(\chi, \Omega)$, we have $\sum_{q\geq 1} \|\chi([S]_q)\|_K = M < +\infty$.

$\square$

2.3 *A remarkable set of exponents*

On the formal side, from (51), we have [7]

$$(z^k y_k)^* = \exp_{\sqcup\!\!\sqcup}\Big(-\sum_{n\geq 1} y_{nk} \frac{(-z)^{nk}}{n}\Big), \text{ for } z \in \mathbb{C}, |z| < 1, \tag{60}$$

and transform it through the $\sqcup\!\sqcup$-character $\hat{\gamma}_\bullet$. First of all, we compute the radius of convergence of the image of the exponent (for coherence with the "bullet-notation", we will note γ_{y_n} the image of y_n by the character $\gamma_\bullet$) which gives [5]

$$\text{for } z \in \mathbb{C}, |z| < 1,\ \ell_k(z) = \begin{cases} \gamma z - \sum_{n\geq 2} \zeta(n) \frac{(-z)^n}{n} & \text{if } k=1, \\ -\sum_{n\geq 1} \zeta(nk) \frac{(-z)^{nk}}{n} & \text{if } k>1. \end{cases} \tag{61}$$

Then, from the fact that $1 < \zeta(n) \leq \zeta(2) = \pi^2/6$ (for $n \geq 2$), we get that the radius of convergence of all $\ell_k(z)$ is $R = 1$. Therefore, we set $\Omega = D_{<1}$, the open disk of radius one centered at zero and get that all $-\sum_{n\geq 1} y_{nk}(-z)^{nk}/n$ belong to $\mathrm{Dom}(\hat{\gamma}_\bullet, \Omega)$. Third point of theorem (2.11) implies at once

- Their exponentials [5,7]

$$(z^k y_k)^* = \exp_{\sqcup\!\sqcup}\left(-\sum_{n\geq 1} y_{nk} \frac{(-z)^{nk}}{n}\right); \text{for } z \in \mathbb{C}, |z| < 1. \tag{62}$$

  are all in $\mathrm{Dom}(\hat{\gamma}_\bullet, \Omega)$ and therefore linearly independent.
- and their transforms through $\hat{\gamma}_\bullet$ follow exponentiation (for $z \in \mathbb{C}, |z| < 1$), *i.e.* [5]

$$\hat{\gamma}_{(zy_k)^*} = \exp(\ell_k(z)) = \begin{cases} \exp\left(\gamma z - \sum_{n\geq 2} \frac{(-z)^n \zeta(n)}{n}\right) & \text{if } k=1, \\ \exp\left(-\sum_{n\geq 1} \zeta(nk) \frac{(-z)^{nk}}{n}\right) & \text{if } k>1. \end{cases} \tag{63}$$

This leads us to set, for all $k \geq 1$ and for $z \in \mathbb{C}, |z| < 1$, [5]

$$\Gamma_{y_k}(1+z) := e^{-\ell_k(z)}, \text{for } z \in \mathbb{C}, |z| < 1. \tag{64}$$

Proposition 2.12 ([5]) *The families $(\ell_r)_{r\geq 1}$ and $(e^{\ell_r})_{r\geq 1}$ are $\mathbb{C}$-linearly free and free from $1_{\mathscr{H}(\Omega)}$.*

Proof. Since $(\ell_r)_{r\geq 1}$ is triangular[25] then $(\ell_r)_{r\geq 1}$ is $\mathbb{C}$-linearly free. So is $(e^{\ell_r} - e^{\ell_r(0)})_{r\geq 1}$, being triangular, we get that $(e^{\ell_r})_{r\geq 1}$ is $\mathbb{C}$-linearly independent and free from $1_{\mathscr{H}(\Omega)}$. $\square$

Now, for any $r \geq 1$, let G_r (resp. $\mathscr{G}_r$) denote the set (resp. group) of solutions, $\{\xi_0, \ldots, \xi_{r-1}\}$, of the equation $z^r = (-1)^{r-1}$ (resp. $z^r = 1$). If r is odd, it is a group

[25] A family $(g_i)_{i\geq 1}$ is said to be *triangular* if the valuation of $g_i, \varpi(g_i)$, equals $i \geq 1$. It is easy to check that such a family is $\mathbb{C}$-linearly free and that is also the case of families such that $(g_i - g(0))_{i\geq 1}$ is triangular.

as $G_r = \mathscr{G}_r$ otherwise it is an orbit as $G_r = \xi\mathscr{G}_r$, where ξ is any solution of $\xi^r = -1$ (this is equivalent to $\xi \in \mathscr{G}_{2r}$ and $\xi \notin \mathscr{G}_r$). For $r, q \geq 1$, we will need also a system $\mathbb{X}$ of representatives of $\mathscr{G}_{qr}/\mathscr{G}_r$, *i.e.* $\mathbb{X} \subset \mathscr{G}_{qr}$ such that

$$\mathscr{G}_{qr} = \biguplus_{\tau \in \mathbb{X}} \tau\mathscr{G}_r. \tag{65}$$

It can also be assumed that $1 \in \mathbb{X}$ as with $\mathbb{X} = \{e^{2ik\pi/qr}\}_{0 \leq k \leq q-1}$.

Proposition 2.13 ([5]) (i) *For $r \geq 1, \chi \in \mathscr{G}_r$ and $z \in \mathbb{C}, |z| < 1$, the functions ℓ_r and e^{ℓ_r} have the symmetry, $\ell_r(z) = \ell_r(\chi z)$ and $e^{\ell_r(z)} = e^{\ell_r(\chi z)}$.*

In particular, for r even, as $-1 \in \mathscr{G}_r$, these functions are even.

(ii) *For $|z| < 1$, we have*

$$\ell_r(z) = -\sum_{\chi \in G_r} \log(\Gamma(1+\chi z)) \textit{ and } e^{\ell_r(z)} = \prod_{\chi \in G_r} e^{\gamma\chi z} \prod_{n \geq 1} (1 + \chi z/n) e^{-\chi z/n}.$$

(iii) *For any odd $r \geq 2$,*

$$\Gamma_{y_r}^{-1}(1+z) = e^{\ell_r(z)} = \Gamma^{-1}(1+z) \prod_{\chi \in G_r \setminus \{1\}} e^{\ell_1(\chi z)}$$

(iv) *and, in general, for any odd or even $r \geq 2$,*

$$\ell_r(z) = \prod_{\chi \in G_r} e^{\ell_1(\chi z)} = \prod_{n \geq 1} (1 + z^r/n^r).$$

(v) *For $r \geq 1$, the function ℓ_r is holomorphic on the open unit disc, $D_{<1}$,*

(vi) *For $r \geq 1$, the function e^{ℓ_r} (resp. $e^{-\ell_r}$) is entire (resp. meromorphic), and admits a countable set of isolated zeroes (resp. poles) on the complex plane which is expressed as $\biguplus_{\chi \in G_r} \chi\mathbb{Z}_{\leq -1}$.*

Proof. The results are known for $r = 1$ (*i.e.* for Γ^{-1}). For $r \geq 2$, we get

(i) By (61), with $\chi \in \mathscr{G}_r$, we get

$$\ell_r(\chi z) = -\sum_{n \geq 1} \zeta(kr) \frac{(-\chi^r z^r)^k}{k} = -\sum_{k \geq 1} \zeta(kr) \frac{(-z^r)^k}{k} = \ell_r(z),$$

thanks to the fact that, for any $\chi \in \mathscr{G}_r$, one has $\chi^r = 1$.

In particular, if r is even then $\ell_r(z) = \ell_r(-z)$, *i.e.* ℓ_r is even.

(ii) If r is odd, as $G_r = \mathscr{G}_r$ and, applying the symmetrization principle [26], we get

[26] Within the same disk of convergence as f, one has,

$$f(z) = \sum_{n \geq 1} a_n z^n \text{ and } \sum_{\chi \in \mathscr{G}_r} f(\chi z) = r \sum_{k \geq 1} a_{rk} z^{rk}.$$

$$-\sum_{\chi\in G_r}\ell_1(\chi z)=-\sum_{\chi\in\mathscr{G}_r}\ell_1(\chi z)=r\sum_{k\geq 1}\zeta(kr)\frac{(-z)^{kr}}{kr}=\sum_{k\geq 1}\zeta(kr)\frac{(-z^r)^k}{k}.$$

The last term being due to the fact that, precisely, r is odd.

If r is even, we have the orbit $G_r=\xi\mathscr{G}_r$ (still with $\xi^r=-1$) and then, by the same principle,

$$-\sum_{\chi\in\mathscr{G}_r}\ell_1(\chi\xi z)=r\sum_{k\geq 1}\zeta(kr)\frac{(-\xi z)^{kr}}{kr}=\sum_{k\geq 1}\zeta(kr)\frac{\big((-\xi z)^r\big)^k}{k}=\sum_{k\geq 1}\zeta(kr)\frac{(-z^r)^k}{k}.$$

(iii) Straightforward.

(iv) Due to the fact that the external product is finite, we can distribute it on each factor and get

$$e^{\ell_r(z)}=\overbrace{\Big(\prod_{\chi\in\mathscr{G}_r}e^{\gamma\chi z}\Big)}^{=1}\prod_{\substack{n\geq 1\\ \chi\in\mathscr{G}_r}}\Big(1+\frac{\chi z}{n}\Big)e^{-\frac{\chi z}{n}}=\overbrace{\Big(\prod_{\substack{n\geq 1\\ \chi\in\mathscr{G}_r}}e^{-\frac{\chi z}{n}}\Big)}^{=1}\prod_{\substack{n\geq 1\\ \chi\in\mathscr{G}_r}}\Big(1+\frac{\chi z}{n}\Big).$$

Using the elementary symmetric functions of G_r, we get the expected result.

(v) One has $e^{\ell_1(z)}=\Gamma^{-1}(1+z)$ which proves the claim for $r=1$. For $r\geq 2$, note that $1\leq\zeta(r)\leq\zeta(2)$ which implies that the radius of convergence of the exponent is 1 and means that ℓ_r is holomorphic on the open unit disc. This proves the claim.

(vi) The function $e^{\ell_r(z)}=\Gamma_{y_r}^{-1}(1+z)$ (resp. $e^{-\ell_r(z)}=\Gamma_{y_r}(1+z)$) is entire (resp. meromorphic) as finite product of entire (resp. meromorphic) functions, for $r\geq 1$. The factorization in Proposition 2.13 yields the set of zeroes (resp. poles).

$\square$

As an example of projection, through $\hat{\gamma}_\bullet$ of an algebraic identity let us mention, for any $z\in\mathbb{C},|z|<1$, one has, from (51), [7]

$$(z^k y_k)^* \sqcup\!\sqcup (-z^k y_k)^*=(-z^{2k}y_{2k})^*, \tag{66}$$

which, transformed by $\hat{\gamma}_\bullet$ and for $||z|<1$ and $k\geq 1$, amounts to Euler's reflection formula (generalized to arbitrary k) [5]

$$\Gamma_{y_{2r}}(1+z)=\Gamma_{y_r}(1+\rho z)\Gamma_{y_r}(1+\rho\xi z), \tag{67}$$

where ρ is a $2r^{th}-$root of (-1) and ξ a primitive $2r^{th}$ root of unity.

It is well known that the function $e^{\ell_1(z)}=\Gamma^{-1}(1+z)$ is entire. In fact, all functions (64) are so (see Proposition 2.12). From this, ones get that (67) holds on the whole plane.

Example 2.14 [[12,11]] Let us give examples relating to polyzetas. For that, we use the following identities, for $z \in \mathbb{C}, |z| < 1$, (see [7])

$$(-zx_1)^* ш (zx_1)^* = 1 \text{ and } (-zy_1)^*)^* ⧢ (zy_1)^* = (-z^2y_2)^*,$$
$$(-z^2x_0x_1)^* ш (z^2x_0x_1)^* = (-z^4x_0^2x_1^2)^* \text{ and } (-z^2y_2)^* ⧢ (z^2y_2)^* = (-z^4y_4)^*.$$

- From (67) and for $r = 1$, we have

$$\begin{aligned} \hat{\gamma}_{(-z^2y_2)^*} &= \hat{\gamma}_{(zy_1)^*}\hat{\gamma}_{(-zy_1)^*} \\ \Gamma_{y_2}^{-1}(1+\mathrm{i}z) &= \Gamma_{y_1}^{-1}(1+z)\Gamma_{y_1}^{-1}(1-z) \\ e^{-\sum\limits_{n\geq 2}\zeta(2n)z^{2n}/n} &= \frac{\sin(z\pi)}{z\pi} = \sum_{k\geq 1}\frac{(z\mathrm{i}\pi)^{2k}}{(2k)!}. \end{aligned}$$

  One can show (with a suitable extension of ζ, see [12,11]) that [27] $\hat{\gamma}_{(-z^2y_2)^*} = \zeta((-z^2x_0x_1)^*)$. Then, identifying the coeffients of z^{2k}, we get

$$\frac{\zeta(\overbrace{2,\ldots,2}^{k\texttt{times}})}{\pi^{2k}} = \frac{1}{(2k+1)!} \in \mathbb{Q}.$$

- Now, with $r = 2$, letting $\rho^4 = -1$, we have

$$\begin{aligned} \hat{\gamma}_{(-z^4y_4)^*} &= \hat{\gamma}_{(z^2y_2)^*}\hat{\gamma}_{(-z^2y_2)^*} \\ \Gamma_{y_4}^{-1}(1+z) &= \Gamma_{y_2}^{-1}(1+\rho z)\Gamma_{y_2}^{-1}(1+\mathrm{i}\rho z), \\ e^{-\sum\limits_{k\geq 1}\zeta(4k)(-1)^k z^{4k}/k} &= \frac{\sin(\mathrm{i}\rho z\pi)}{\mathrm{i}\rho z\pi}\frac{\sin(\rho\pi z)}{z\pi\rho}. \end{aligned}$$

  Again, with a suitable extension of [28] ζ (see [12,11])

$$\hat{\gamma}_{(-z^4y_4)^*} = \zeta((-z^4y_4)^*), \hat{\gamma}_{(-z^2y_2)^*} = \zeta((-z^2y_2)^*), \hat{\gamma}_{(z^2y_2)^*} = \zeta((z^2y_2)^*)$$

  then, using the poly-morphism ζ, we obtain

$$\begin{aligned} \zeta((-z^4y_4)^*) &= \zeta((-z^2y_2)^*)\zeta((z^2y_2)^*) \\ &= \zeta((-z^2x_0x_1)^*)\zeta((z^2x_0x_1)^*)) \\ &= \zeta((-4z^4x_0^2x_1^2)^*). \end{aligned}$$

  It follows then, by identification the coeffients of z^{4k}, that

$$\frac{\zeta(\overbrace{3,1,\ldots,3,1}^{k\texttt{times}})}{\pi^{4k}} = \frac{4^k\zeta(\overbrace{4,\ldots,4}^{k\texttt{times}})}{\pi^{4k}} = \frac{2}{(4k+2)!} \in \mathbb{Q}.$$

[27] Recall that, for any $w \in Y^* \setminus y_1Y^*$, one has $\gamma_w = \zeta(\pi_X(w))$ [13] and then it can be extended over series.

[28] *idem*.

3 Conclusion

Noncommutative symbolic calculus allows to get identities easy to check and to implement. With some amount of complex and functional analysis, it is possible to bridge the gap between symbolic, functional and number theoretic worlds. This was the case already for polylogarithms and polyzetas. This is the project of this paper and will be pursued in forthcoming works.

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