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A note on the simultaneous edge coloring *

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Abstract

Let $G = (V, E)$ be a graph. A (proper) $k$-edge-coloring is a coloring of the edges of $G$ such that any pair of edges sharing an endpoint receive distinct colors. A classical result of Vizing [3] ensures that any simple graph $G$ admits a $(Δ(G) + 1)$-edge coloring where $Δ(G)$ denotes the maximum degree of $G$. Recently, Cabello raised the following question: given two graphs $G_1, G_2$ of maximum degree $Δ$ on the same set of vertices $V$, is it possible to edge-color their (edge) union with $Δ + 2$ colors in such a way the restriction of $G$ to respectively the edges of $G_1$ and the edges of $G_2$ are edge-colorings? More generally, given $ℓ$ graphs, how many colors do we need to color their union in such a way the restriction of the coloring to each graph is proper?

In this short note, we prove that we can always color the union of the graphs $G_1, \ldots, G_ℓ$ of maximum degree $Δ$ with $Ω(√ℓ · Δ)$ colors and that there exist graphs for which this bound is tight up to a constant multiplicative factor. Moreover, for two graphs, we prove that at most $\frac{3}{2}Δ + 4$ colors are enough which is, as far as we know, the best known upper bound.

1 Introduction

All along the paper, we only consider simple loopless graphs. In his seminal paper, Vizing proved in g [3] that any simple graph $G$ can be properly edge-colored using $Δ(G) + 1$ colors (where $Δ(G)$ denotes the maximum degree of $G$). The union of two graphs $G_1$ and $G_2$ on vertex set $V$ is the (simple) graph $G$ with vertex set $V$ and where $uv$ is an edge if and only if $u$ is an edge of $G_1$ or an edge of $G_2$. An edge coloring of $G$ is simultaneous with respect to $G_1$ and $G_2$ if its restrictions to the edge set of $G_1$ and to the edge set of $G_2$ are proper edge-colorings. Recently, Cabello raised the following question$^1$: given two graphs $G_1, G_2$ of maximum degree $Δ$ on the same set of vertices $V$, does it always exist a simultaneous $(Δ + 2)$-edge coloring with respect to $G_1$ and $G_2$? Cabello proved that this property is satisfied if the intersection of $G_1$ and $G_2$ is regular [1]. Using Vizing’s theorem, one can easily notice that there exists a simultaneous $(2Δ + 1)$-edge coloring. From a lower bound perspective, no graph where $Δ + 2$ colors are needed is known.

Cabello introduced a generalization of this notion. Let $ℓ$ graphs $G_1, G_2, \ldots, G_ℓ$ and $G$ be their (edge) union. In other words, $uv$ is an edge of $G$ if and only if $uv$ is an edge of at least one graph $G_i$ with $i ≤ ℓ$. An edge-coloring of $G$ is simultaneous with respect to $G_1, \ldots, G_ℓ$ if its restriction to each graph $G_i$ is a proper edge-coloring. Cabello asked how many colors are needed to ensure the existence of a simultaneous coloring of $G$ with respect to each $G_i$. Let us denote by $χ′(G_1, \ldots, G_ℓ)$ the minimum number of colors needed to obtain a simultaneous coloring. And let $χ′(ℓ, Δ)$ be the largest integer $k$ such that $k = χ′(G_1, \ldots, G_ℓ)$ for some graphs $G_1, \ldots, G_ℓ$ of maximum degree (at most) $Δ$. Vizing’s theorem ensures that $χ′(ℓ, Δ) ≤ ℓΔ + 1$ and Cabello exhibit a graph for which $χ′(3, Δ) ≥ Δ + 5$ (with $Δ = 10$) [1]. In this note, we prove that the order of magnitude of $χ′(ℓ, Δ)$ is $Θ(√ℓΔ)$. More precisely, we prove that the following statement holds: 

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Theorem 1
\[ \chi'(\ell, \Delta) \leq 2\sqrt{2\ell \Delta} - \sqrt{2\ell} + 2. \]

We claim that this bound is tight up to a constant multiplicative factor. Let \( \ell \in \mathbb{N} \) and \( \Delta \) be an even value. Let \( G := S_{1,k,\Delta} \) be the star with \( k\Delta \) leaves, where \( k = \lfloor \sqrt{2} \rfloor \). Partition the edges of \( G \) into \( 2k \) sets \( A_1, \ldots, A_{2k} \) of size \( \frac{\Delta}{2} \). For every pair \( i, j \), create the graph \( G_{i,j} \) with edge set \( A_i \cup A_j \). Note that each graph \( G_{i,j} \) has maximum degree \( \Delta \) since by construction the set of edges \( A_i \) induces a graph of maximum degree \( \Delta/2 \). Moreover the total number of graphs \( G_{i,j} \) is \( 2k(2k - 1)/2 \leq \ell \). Finally, by construction, every pair of edges of \( G \) appears in at least one graph \( G_{i,j} \). So in order to obtain a simultaneous coloring, we need to color all the edges of \( G \) with different colors (since all the edges are incident to the center of the star). So:

Proposition 2
\[ \chi'(\ell, \Delta) \geq \left\lfloor \sqrt{\frac{\ell}{2}} \right\rfloor \Delta. \]

Note that, for \( \ell = 3 \) and the graph \( S_{1,3,\lfloor \Delta/2 \rfloor} \), a similar construction ensures that \( \chi'(3, \Delta) \geq 3 \lfloor \frac{\Delta}{2} \rfloor \), improving the lower bound of \( \Delta + 5 \). Indeed, let us partition the edges of the star into three sets \( A_1, A_2, A_3 \) of size \( \lfloor \Delta/2 \rfloor \). We similarly define for every \( i \neq j \) the graph \( G_{i,j} \) with edge set \( A_i \cup A_j \). Each graph \( G_{i,j} \) has maximum degree \( \Delta \) and every pair of edges appears in at least one graph \( G_{i,j} \). So an edge coloring of \( S_{1,3,\lfloor \Delta/2 \rfloor} \) simultaneous with respect to \( G_{1,2}, G_{1,3} \) and \( G_{2,3} \) is a proper edge coloring of \( S_{1,3,\lfloor \Delta/2 \rfloor} \).

When \( \ell = 2 \), a careful reading of the proof of Theorem 1 permits to remark that we can improve the trivial \((2\Delta + 1)\) upper bound into \(2\Delta\). We prove the following much better upper bound with a different technique:

Theorem 3
\[ \chi'(2, \Delta) \leq \left\lfloor \frac{3}{2} \Delta + 4 \right\rfloor. \]

As far as we know, it is the best known upper bound.

2 Proof of Theorem 1

Let \( G_1, \ldots, G_\ell \) be \( \ell \) graphs of maximum degree \( \Delta \). Let us partition the set of edges of \( G = \bigcup_{i=1}^{\ell} G_i \) into two sets (all along the paper, the notation \( \cup \) stands for edge union). The multiplicity of an edge \( e \) is the number of graphs \( G_i \) with \( i \leq \ell \) on which \( e \) appears. For some fixed \( k \), the set \( E_1 \) is the set of edges with multiplicity at least \( k \) and \( E_2 \) is the set of edges with multiplicity less than \( k \). We will optimize the value of \( k \) later. (Note that we do not necessarily assume that \( k \) is an integer). For every \( i \in \{1, 2\} \), let us denote by \( H_i \) the graph \( G \) restricted to the edges of \( E_i \). Note that \( G = H_1 \cup H_2 \).

We claim that the graph \( H_1 \) has degree at most \( \ell \Delta/k \). Indeed, let \( u \) be a vertex and \( E_1(u) \) be the set of edges of \( H_1 \) incident to it. Since every edge of \( H_1 \) has multiplicity at least \( k \) and at most \( \ell \Delta \) edges (with multiplicity) are incident to \( u \) in \( G \), at most \( \frac{\ell}{k} \Delta \) different edges are in \( E_1(u) \). So \( H_1 \) has maximum degree \( \frac{\ell}{k} \Delta \). By Vizing’s theorem, \( H_1 \) can be properly edge-colored with \( (\frac{\ell}{k} \Delta + 1) \) colors.

Let us now prove that \( H_2 \) can be simultaneously edge-colored with \( 2k(\Delta - 1) + 1 \) colors. Let us prove it by induction on the number of edges of \( H_2 \). The empty graph can indeed be edge-colored with \( 2k(\Delta - 1) + 1 \) colors. Let \( e = uv \) be an edge of \( H_2 \). By induction, there exists a simultaneous coloring \( c' \) of \( H_2 \setminus e \) with \( 2k(\Delta - 1) + 1 \) colors. Let us prove that \( c' \) can be extended into a simultaneous coloring of \( H_2 \). Without loss of generality, we can assume that \( e \) is an edge of the graphs \( G_1, \ldots, G_r \) with \( r < k \) and is not an edge of \( G_{r+1}, \ldots, G_\ell \). Let \( F \) be the set of edges of \( G_1, \ldots, G_r \) incident to \( u \) or to \( v \) distinct from \( e \). By assumption, the set of \( e \) is an edge of \( G_i \) with \( i \leq r \), and there are at most \( 2r(\Delta - 1) \) such edges \( (2(\Delta - 1) \) in each graph \( G_i \). Since \( r < k \), at most \( 2k(\Delta - 1) \) edges are in \( F \). So there exists a color \( a \) that does not appear in \( F \). The edge \( e \) can be colored with \( a \) without violating any constraints. It holds by choice of \( a \) for \( G_i \) with \( i > r \) and it holds since \( e \notin G_i \) for \( i > r \).
So \( \chi'(\ell, \Delta) \leq \frac{\ell}{2} \Delta + 2k(\Delta - 1) + 2 \) colors. We finally optimize the integer \( k \) which minimize the number of colors. We want to minimize \( \frac{\ell}{2} + 2k \) which is minimal when \( k = \sqrt{\frac{\ell}{2}} \). It finally ensures that \( \chi'(\ell, \Delta) \leq 2\sqrt{2\ell}\Delta - \sqrt{2\ell} + 2 \), which completes the proof of Theorem 1.

### 3 Proof of Theorem 3

Let \( G_1, G_2 \) be two graphs of maximum degree \( \Delta \) and let \( G \) be their union. Let \( E_2 \) be the edges that appear in both graphs and, for every \( i \in \{1, 2\} \), let \( E_i \) be the set of edges that appears only in \( G_i \). Let us denote by \( H_2 \) (resp. \( H_1 \)) the graph restricted to the edges of \( E_2 \) (resp. \( E_1 \)).

For every vertex \( v \) and every graph \( H \), we denote by \( \text{deg}_H(v) \) the degree of \( v \) in \( H \). Let \( H \) be a graph and \( f,g \) be two functions from \( V(H) \) to \( \mathbb{R}^+ \). A \((g,f)\)-factor of \( H \) is an edge-subgraph \( H' \) of \( H \) such that every vertex \( v \) satisfies \( g(v) \leq \text{deg}_{H'}(v) \leq f(v) \). Kano and Saito proved in [2] that the graph \( H \) admits a \((g,f)\)-factor if

(i) \( f \) and \( g \) are two integer valued functions, and

(ii) for every vertex \( v, g(v) < f(v) \), and

(iii) there exists a real number \( \theta \) such that \( 0 \leq \theta \leq 1 \) and for every vertex \( v, g(v) \leq \theta \cdot \text{deg}_H(v) \leq f(v) \).

Let \( 1 \leq i \leq 2 \). We will extract from \( H_i \) a \((g,f)\)-factor where \( g(v) = \left\lceil \frac{\text{deg}_{H_i}(v)}{2} \right\rceil - 1 \) and \( f(v) = \left\lceil \frac{\text{deg}_{H_i}(v)}{2} \right\rceil \). The points (i) and (ii) are satisfied. Moreover, by choosing \( \theta = \frac{1}{2} \), (iii) is also satisfied. Thus by [2], the graphs \( H_1 \) and \( H_2 \) admit \((g,f)\)-factors. For \( i \leq 2 \), let \( K'_i \) be a \((g,f)\)-factor of \( H'_i \). For every \( i \), let \( L_i = H'_1 \setminus K'_1 \). Let \( L = L_1 \cup L_2 \) and \( R = H_2 \cup K'_1 \cup K'_2 \). Note that \( G = L \cup R \). Let us now color these two graphs.

Let us first prove that \( L \) can be colored with \( \left\lceil \frac{\Delta}{2} \right\rceil + 2 \) colors. For every \( i \), the graph \( L_i \) has maximum degree at most \( \left\lceil (\Delta/2) + 1 \right\rceil \) since every vertex \( v \) of \( K'_i \) has degree at least \( \left\lfloor (\text{deg}_{H'_i}(v)/2) - 1 \right\rfloor \). By Vizing’s theorem, the graph \( L_i \) can be colored with at most \( \left\lceil \frac{\Delta}{2} \right\rceil + 1 = \left\lceil \frac{\Delta}{2} \right\rceil + 2 \) colors. Since the edges of \( L_1 \) and \( L_2 \) are disjoint, \( L = L_1 \cup L_2 \) can be simultaneously colored with \( \left\lceil \frac{\Delta}{2} \right\rceil + 2 \) colors (the same set of colors can be re-used for each graph).

Let us now color the graph \( R \). Let \( v \) be a vertex of \( R \). Let us denote by \( d \) the degree of \( v \) in \( H_2 \). Since edges of \( H_2 \) are in both \( G_1 \) and \( G_2 \), the vertex \( v \) has degree at most \( \Delta - d \) in both graphs \( H_1 \) and \( H_2 \). Since the graphs \( K'_1 \) and \( K'_2 \) are \((g,f)\)-factors of respectively \( H'_1 \) and \( H'_2 \), the degree of the vertex \( v \) is at most \( \left\lfloor \frac{\Delta - d}{2} \right\rfloor \) in each graph. So the degree of \( v \) in the graph \( R \) is at most \( d + 2 \left\lfloor \frac{\Delta - d}{2} \right\rfloor \leq \Delta + 1 \). By Vizing’s theorem, the graph \( R \) can be colored using at most \( \Delta + 2 \) colors.

Since \( G = L \cup R \), we can find a simultaneous edge-coloring with respect to \( G_1 \) and \( G_2 \) using at most \( \left\lceil \frac{\Delta}{2} \right\rceil + 2 + \Delta + 2 = \left\lceil \frac{\Delta}{2} \right\rceil + 4 \) colors.

### 4 Conclusion

Theorem 1 and Proposition 2 ensures that the following holds:

\[
\left\lceil \frac{\sqrt{\ell}}{2} \right\rceil \Delta \leq \chi'(\ell, \Delta) \leq 2\sqrt{2\ell}\Delta - \sqrt{2\ell} + 2.
\]

Closing the multiplicative gap of 4 between lower and upper bound is an interesting open problem. For \( \ell = 2 \), we still do not know any graph for which \( \chi'(2, \Delta) > \Delta + 1 \). Cabello asked the following question that is still widely open despite the progress obtained in Theorem 3:

**Question 4 (Cabello)** Is it true that

\[
\chi'(2, \Delta) \leq \Delta + 2?
\]

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