Recovering an elliptic coated inclusion by an energy gap tracking function

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Abstract. In this paper, we consider the inverse problem of reconstructing coated inclusions from overdetermined boundary data. In a first part, a shape identifiability result from a Cauchy data is presented, i.e. with Neumann and Dirichlet boundary as measurements. Then the inverse geometric problem is reduced into a minimzation of a cost-type functional: energy gap tracking functional. Since the boundary conditions are known, the variable of the functional is the shape of the coated inclusions. The shape sensitivity analysis is rigorously performed by means of a Lagrangian formulation coupled with paramatrization of the shape. Thus we explicit the gradient of the functional by computing the derivative with respect to the missing shape. The optimization problem is numerically solved by means of gradient-based shape strategy then numerical illustrations are presented.

Keywords: Heat conduction, Shape Optimization, Shape Derivative, Coated Inclusions, Inverse problems, Identifiability, Cauchy data.

1 Introduction

In this work, we are dealing with the location of elliptic coated inclusions inside a thermal conductor domain. For the numerical treatment, this ill-posedness constitutes a serious difficulty. Even if the amount of data collected is sufficient to guarantee uniqueness, the unknown coefficient or boundary, respectively, usually does not depend continuously on the measured data. To overcome this difficulty authors often resort to apply regularization techniques to prove an a priori continuous dependency between them[1,2].

The uniquness ussue is crucial in order to perform numerical experiments. In this paper we performed the identifiabilty result using the energy functional [3,4] . From practical point of view a shape optimization approach is used to locate elliptic coated inclusions. In such problem the shape sensitivity analysis plays a central role in the differentiability of the cost function with respect to the shape of the geometric domain on which the partial differential equation is defined. In this work and in order to perform a numerical procedure, we formulate the geometric inverse problem as a shape minimization of a Lagrangian functional associated with an energy gap tracking cost functional $\mathcal{J}$.

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This work is an extension of the problem studied to reconstruct a single circular and elliptic inclusion in[5,6].

In our paper the reconstruction of geometric parameters is based on the minimization of the function error between the Dirichlet and the Neumann solutions. The main result of this paper is to extend the numerical reconstruction in[6] to a coated elliptic shape using a diffrerent theoretical approach. The main difficulty in order to devise such an algorithm is to study the sensitivity of the cost function with respect to the shape.

The main feature of this work is the use of a gradient-type algorithm combined with the boundary expression of the shape gradient. In order to compute the shape derivative of the cost function $\mathcal{J}$, we need to get around the calculus of the material and shape derivative of the variable state and in this context we used a theorem of implicit functions as in [7] which was used for more general variation of the domain. We adapted this technique to a weaker differentiability : a Frechet differentiability enough to express the variation of the boundary of our domain with a small real parameter t and in the direction of a fixed velocity flow h. We will combine this technique with function space parametrization and function space embedding in spirit of Delfour and Zolésio[8] to provide the explicit expression of the shape derivative.

2 Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz boundary $\partial\Omega$ and $\omega \in \mathcal{O}_{ad}$ with

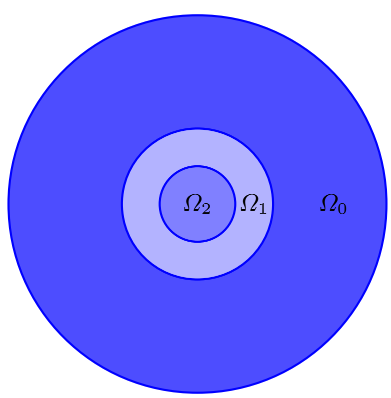
$$\mathcal{O}_{ad} := \{\omega \text{ of classe } C^2 : \quad \omega \subset \Omega,\ \mathrm{dist}(\partial\omega, \partial\Omega) > \nu\},$$

for some $\nu > 0$ Let , $\Omega_0, \Omega_1, \Omega_2$ subsets of Ω such that Ω_2 is surrounded by Ω_1 and the latter is surrounded by Ω_0 and $\omega = \Omega_1 \cup \Omega_2$. We denote by $\Gamma_2 = \partial\Omega_2$ and Γ_1 the external boundary of Ω_1, see Figure 2 for a description of the geometry.

Assume that thermal conductivity in Ω is

$$\sigma = \sigma_0\chi_{\Omega_0} + \sigma_1\chi_{\Omega_1} + \sigma_2\chi_{\Omega_2},$$

where $\sigma_0, \sigma_1, \sigma_2 > 0$, are a positive constants,$\sigma_i \neq \sigma_j, i \neq j$ and χ denotes the indicator function.



For a given source term $f \in L^2(\Omega)$ then the scalar potential u_ω, generated by the Neumann data $g \in H^{-1/2}(\partial\Omega)$ (in the prescence of the coated inclusion ω), is the solution to the following Neumann problem

$$\begin{cases} -\operatorname{div}(\sigma\nabla u_\omega) = f & \text{in } \Omega, \\ [\![u_\omega]\!] = 0 & \text{on } \Gamma_i, i = 1, 2, \\ [\![\sigma\partial_n u_\omega]\!] = 0 & \text{on } \Gamma_i, i = 1, 2, \\ -\sigma_0\partial_n u_\omega = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, n is the outward unit normal vector to $\partial\Omega$.
The solvability of this elliptic boundary value problem is well known, when

$$g \in H^{-1/2}(\partial\Omega) \text{ and } \int_{\partial\Omega} g(x)\, ds(x) = 0$$

, there exits a unique solution $u \in H^1(\Omega)$. For a proof, we refer to[**?**].
The problem under consideration is the following.

Given the Neumann data g and the potential $u|_{\partial\Omega} := q$ measured on the boundary of Ω, we want to find the location of the coated inclusion $\omega \subset \Omega$.

(2)

In order to approximate the inclusion ω, we introduce the following optimization problem:

$$\begin{cases} \text{minimize } J(\omega, u_N, u_D) \\ \text{subject to } \omega \in \mathcal{A} \text{ and } u_N, u_D \text{ the solutions of 5 and 6 respectively,} \end{cases} \tag{3}$$

where J is the energy gap tracking functional, defined by :

$$J(u_N, u_D) := \frac{1}{2}\int_\Omega \sigma|\nabla(u_N - u_D)|^2\, dx. \tag{4}$$

Here

$$\mathcal{A} = \{\omega \subset \Omega : P(\omega, \Omega) < \infty\},$$

where $P(\omega, \Omega)$, is the relative perimeter of ω in Ω. For the sake of completeness, we give the definition of the relative perimeter.

$$P(\omega, \Omega) := \sup\left\{\int_\omega \operatorname{div}\phi\, dx : \phi \in C_c^1(\Omega, \mathbb{R}^2), \|\phi\|_\infty \leq 1\right\}.$$

The states u_N and u_D are respectively solutions of the following problems.

$$-\operatorname{div}(\sigma\nabla u_N) = f \text{ in } \Omega, \quad -\sigma_0\partial_n u_N = g \text{ in } \partial\Omega, \quad \int_\Omega u_N\, ds(x) = 0. \tag{5}$$

$$-\operatorname{div}(\sigma\nabla u^D) = f \text{ in } \Omega, \quad u_D = q \text{ in } \partial\Omega. \tag{6}$$

3 Identifiability in the case of monotonous inclusions

In this section, we discuss the uniqueness issue for the geometric inverse problem considered using the energy functional [4]. For $i = 1, 2$, let $\Omega^{(i)}$ be a domain defined in the same manner as in section 2 and let u_i be the solution of the problem

$$\begin{cases} -\operatorname{div}(\sigma \nabla u_i) = 0 & \text{in} \quad \Omega^{(i)}, \\ u_i = q & \text{on} \quad \Gamma \setminus \Upsilon, \\ \sigma \partial_n u_i = g & \text{on} \quad \Upsilon, \\ [\![u_i]\!] = 0 & \text{on} \quad \Gamma_j^{(i)}, j = 1, 2, \\ [\![\sigma \partial_n u_i]\!] = 0 & \text{on} \quad \Gamma_j^{(i)}, j = 1, 2, \end{cases} \tag{7}$$

where $\Upsilon \subset \Gamma := \partial \Omega$. We consider the following notation $\omega^{(i)} := \Omega_1^{(i)} \cup \Omega_2^{(i)}$ then, $\partial \omega^{(i)} = \Gamma_1^{(i)} \cup \Gamma_2^{(i)}$. We consider the particular case when $\Omega_1^{(2)} \subset \Omega_1^{(1)}$ as depicted in Figure 1.

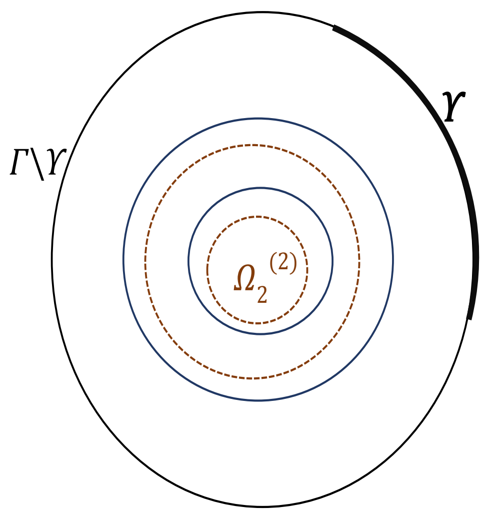


Fig. 1. The case of monotonuous inclusions.

Theorem 1. *Let $\omega^{(1)} = \Omega_1^{(1)} \cup \Omega_2^{(1)}$ and $\omega^{(2)} = \Omega_1^{(2)} \cup \Omega_2^{(2)}$ be two coated inclusions such that $\omega^{(2)} \subset \omega^{(1)}$. For $i = 1, 2$, and $\Omega_2^{(i)} = \alpha \omega^{(i)}, \alpha \in]0, 1[$, let $u^{(i)}$ be the solution of the direct problem (7) defined in $\Omega^{(i)}$. Then, if $\omega^{(1)}$ and $\omega^{(2)}$ lead to the same measured temperature on Υ, we have $\omega^{(1)} = \omega^{(2)}$ $\forall \sigma_2 < \sigma_1 < \sigma_0$ or $\sigma_0 < \sigma_1 < \sigma_2$ or if $\alpha \neq \dfrac{\sigma_1 - \sigma_0}{\sigma_1 - \sigma_2} \quad \forall \sigma_1 > \sigma_0 > \sigma_2$ or $\sigma_1 < \sigma_2 < \sigma_0$.*

4 Shape sensitivity analysis

In this section, our goal is to determine the variation of the energy-gap functional J due to the modifications in the configuration Ω. We firstly presente some concepts related to the shape optimisation theorie.

4.1 Perturbation of domains

Let $\Omega \subset \mathbb{R}^2$ be the domain such as defined in section 2. For $t \in \mathbb{R}$, we introduce a family of perturbations $\{\Omega_t\}$ of a fixed domain Ω, we define by

$$F_t = I_d + th, \tag{8}$$

the perturbation of the identity operator I_d, for a deformation field h belonging to the space

$$H = \left\{ h \in C^{1,1}(\overline{U}) : h = 0 \text{ on } \partial\Omega, h.\tau = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \right\},$$

where U is an open and bounded domain such that $\overline{\Omega} \subset U$ and τ is the tangent vector to the interface $\Gamma_i, i = 1, 2$. We note that for t sufficiently small, F_t is a diffeomorphism from Ω into its image. Then the family $\{\Omega_t\}$ and $\{\Gamma_{i,t}\}$ are defined by

$$\Omega_t = F_t(\Omega), \quad \Gamma_{i,t} = F_t(\Gamma_i), i = 1, 2.$$

4.2 Shape derivative

To obtain the expression of the shape derivative of J, it is necessery firstly to recall the following lemma.

Theorem 2. *Let $\phi : [0, T] \to W^{1,\infty}(\mathbb{R}^d)$ differentiable at $t = 0$ with $\phi(0) = I_d$ and $\phi'(0) = h$ and let $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. Assume $[0, T] \ni t \to G(t, .) \in L^1(\mathbb{R}^d)$ is differentiable at 0 and $G'(0, .) \in W^{1,1}(\mathbb{R}^d)$. Then $\int\limits_{\phi(t)(\Omega)} G(t, x)dx$ is differentiable at 0 and we have*

$$\frac{d}{dt}\left(\int\limits_{\phi(t)(\Omega)} G(t, x)dx \right)\Bigg|_{t=0} = \int\limits_{\partial\Omega} G(0, x)h.nds + \int\limits_{\Omega} \frac{\partial G(t, x)}{\partial t}\Bigg|_{t=0} dx.$$

We come now to the main result of this section.

Theorem 3. *The mapping $t \to J(\Omega_t)$ is C^1 in a neighborhood of 0 and its derivative at $t = 0$ is given by*

$$L'(\Omega, h) = \sum_{i=1}^{i=2} \frac{(\sigma_{i-1} - \sigma_i)}{2} \int\limits_{\Gamma_i} \left(\frac{\sigma_i}{\sigma_{i-1}} \left((\partial_n u_N^-)^2 - (\partial_n u_D^-)^2 \right) + |\nabla_\tau u_N|^2 - |\nabla_\tau u_D|^2 \right) h.nd\Gamma_i.$$

5 Numerical results

We take Ω the unit disk. The conductivity values are set $\sigma_0 = 10, \sigma_1 = 2.7$ and $\sigma_2 = 1.9$. The term source values are $f_0 = 0, f_1 = 0$ and $f_2 = 0$. without loss of generality.

5.1 Example 1

We first reconstruct the shape of three coated elliptic $\omega = \Omega_1 \cup \Omega_2$, parametrized as

$$\begin{aligned} \Gamma_1 &= \{(0.4\cos(\theta) - 0.35, 0.2\sin(\theta) - 0.1) \quad \theta \in [0, 2\pi]\} \\ \Gamma_2 &= \{(0.3\cos(\theta) - 0.35, 0.1\sin(\theta) - 0.1) \quad \theta \in [0, 2\pi]\}. \end{aligned} \tag{9}$$

For the reconstruction of ω we impose five fluxes :
$g_1 = (1/x^{10}, 0, 0, 0, 0)$, $g_2 = (0, 1/y^4, 0, 0, 0)$, $g_3 = (0, 0, 1/x^5, 0)$, $g_4 = (0, 0, 0, 1/y^3, 0)$ and $g_5 = (0, 0, 0, 0, \cos(\theta))$.

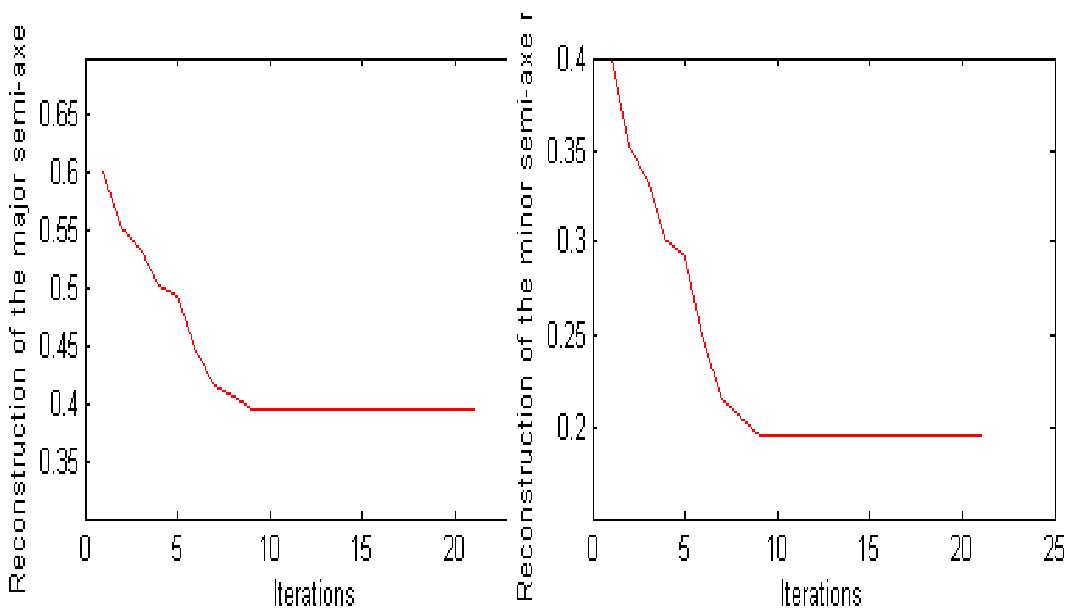


Fig. 2. Reconstruction of the semi-major axe r_1 and the semi-minor axe r_{11} of the external ellipse Ω_1 using the Newton method.

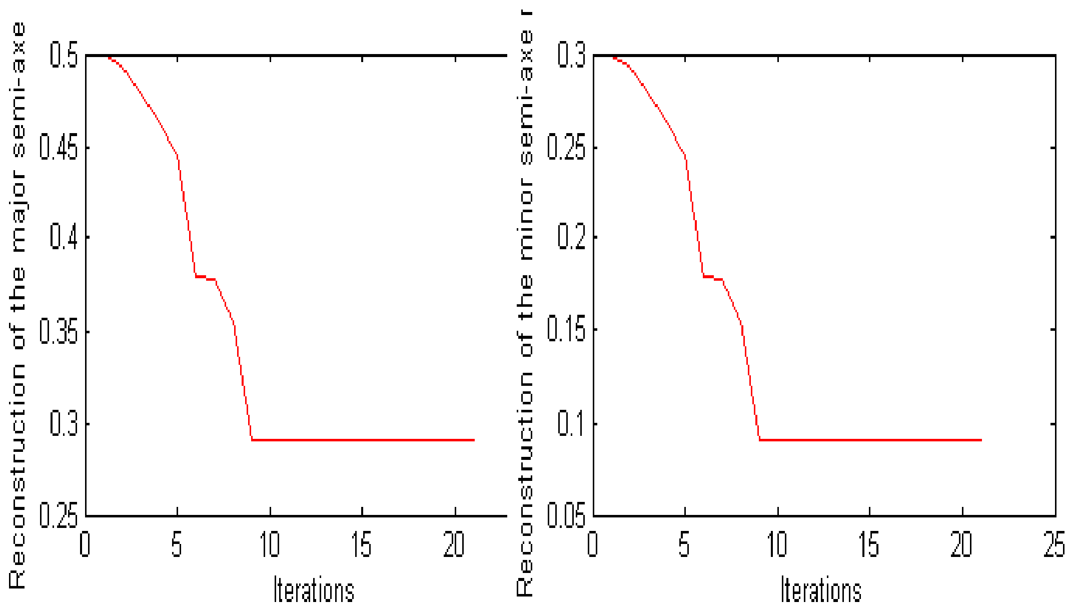


Fig. 3. Reconstruction of the semi-major axe r_2 and the semi-minor axe r_{22} of the internal ellipse Ω_2 using the Newton method

Table 1. Table of convergence of the center and the semi-major and semi-minor axes of the external and the internal ellipses

	x-centre	y-centre	r_1	r_{11}	r_2	r_{22}	The norm error $\| . \|_2$
Figure (5),(6)	-0.3509	-0.0997	0.3946	0.1946	0.2910	0.0910	0.0148

5.2 Example 2

In this example we reconstruct the shape of a three coated elliptic inclusions $\omega_i = \Omega_{1,i} \cup \Omega_{2,i}$ $i = 1..3$ which have the same size and whose parametrized as

$$\begin{aligned} \Gamma_{1,i} &= \{(0.25\cos(\theta) - x_i, 0.125\sin(\theta) - y_i) \quad \theta \in [0, 2\pi]\} \\ \Gamma_{2,i} &= \{(0.15\cos(\theta) - x_i, 0.06\sin(\theta) - y_i) \quad \theta \in [0, 2\pi]\}. \end{aligned} \tag{10}$$

Where $X_i = [x_i, y_i]$ are the centers and $X_1 = [-0.5, -0.3]$, $X_2 = [0, 0.56], X_3 = [0.6, -0.3]$.

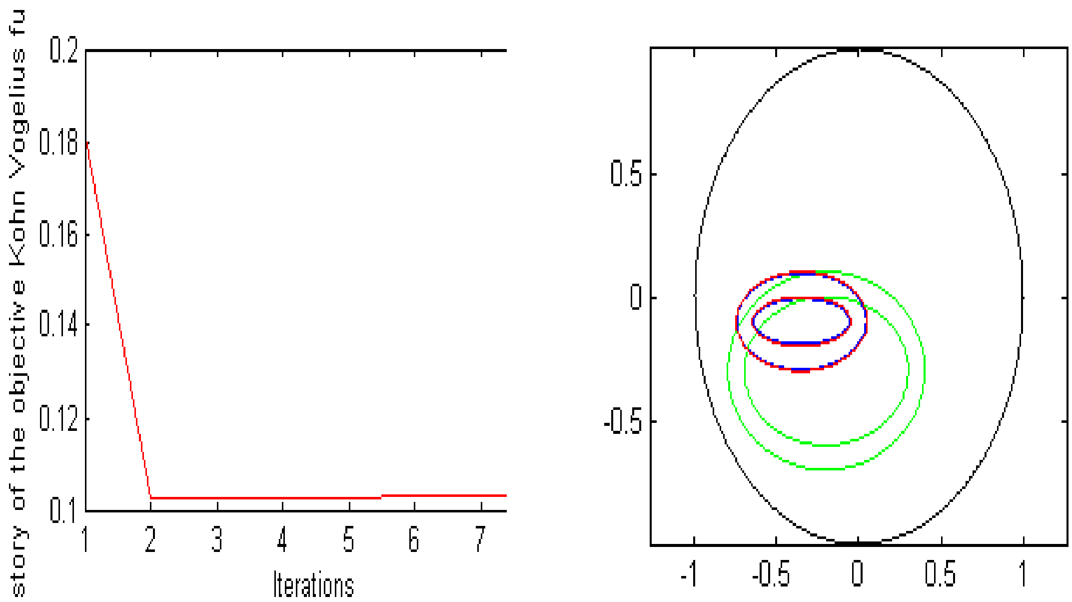


Fig. 4. The history of the energy gap cost functional and reconstruction of the coated ellipse(the intial shape(green), the approximated shape(blue), the exact shape(red))

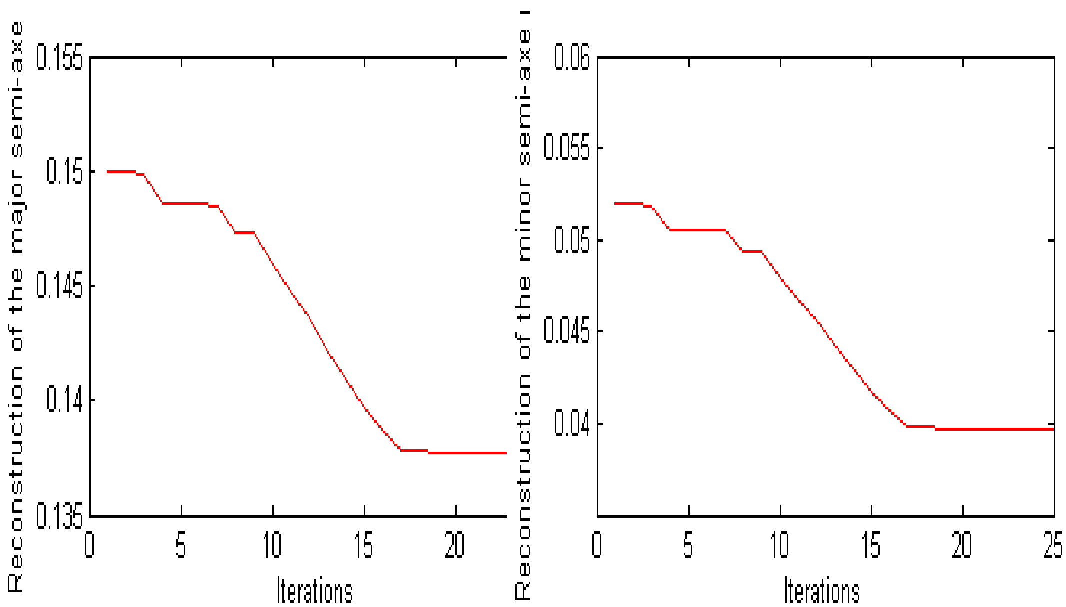


Fig. 6. Reconstruction of the semi-major axe r_2 and the semi-minor axe r_{22} of the internal ellipse Ω_2 using the Newton method

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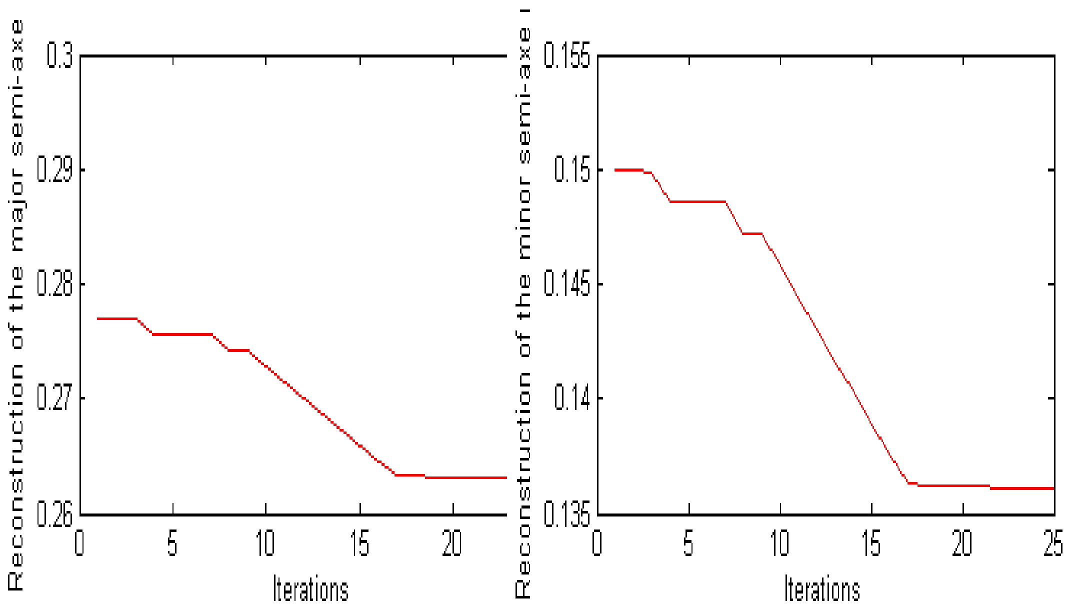

Fig. 5. Reconstruction of the semi-major axe r_1 and the semi-minor axe r_{11} of the external ellipse Ω_1 using the Newton method.

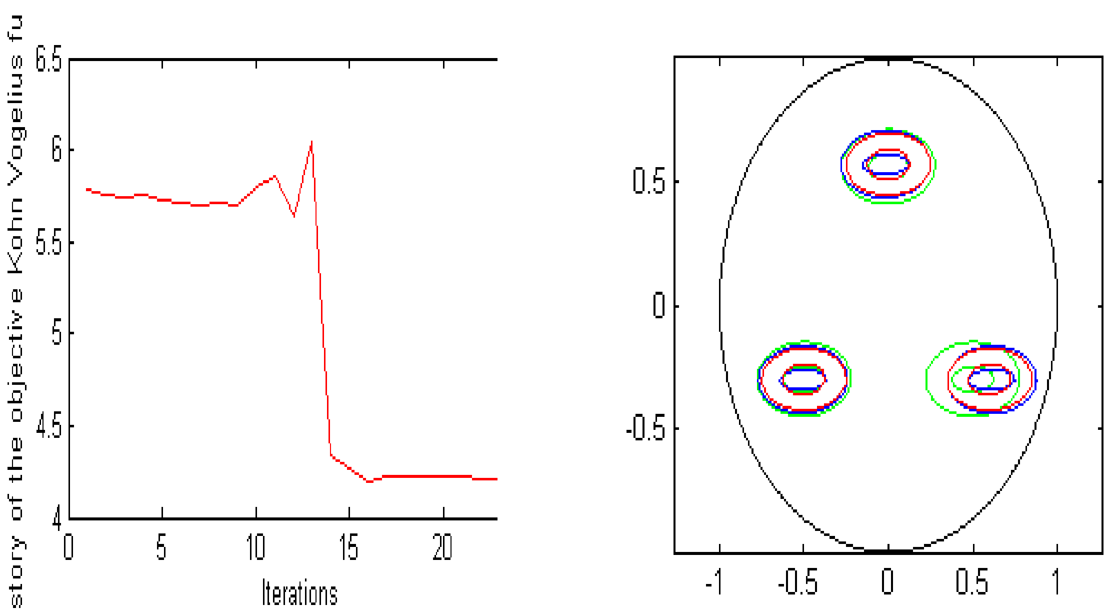

Fig. 7. The history of the energy gap cost functional and reconstruction of three coated ellipses(the intial shape(green), the approximated shape(blue), the exact shape(red))

6 Conclusion

In this study, a reconstruction of coated inclusions from over-determined boundary data is obtained by minimizing an energy gap tracking functional. Using the shape sensitivity tools, the explicit formula of the shape derivative for the focused problem is presented. Then, the numerical results show that such identification process produced reasonable location of the unknown coated inclusions. A complete theoritical analysis for more general shape is delivered while we only considered the specific elliptic shape for numerical illustration, this leave space for further numerical reconstruction of more complicated shape as well as its extension to three-dimensional situations. It is also of great interest to examine this identification procedure from partial boundary measurements and study the case of time-dependent heat conduction problem.

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