CONSERVATIVITY OF REALIZATION FUNCTORS ON MOTIVES OF ABELIAN TYPE OVER FINITE FIELDS

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Abstract. We show that the ℓ-adic realization functor is conservative when restricted to Chow motives of abelian type over a finite field.

A weak version of this conservativity result extends to mixed motives of abelian type.

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Contents

Introduction

Let k be a field and let $\mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ be Voevodsky's category of mixed motives over k with rational coefficients. Let ℓ be a prime number invertible in k, and consider the ℓ-adic realization functor [Ivo07]

$$R_\ell : \mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} \to D^b(\mathbb{Q}_\ell)$$

to the bounded derived category of $\mathbb{Q}_\ell$-vector spaces.

One of the central conjectures in motives predicts that R_ℓ is conservative (i.e. it detects isomorphisms), see [Ayo15] for an overview on this conjecture (in characteristic zero). This conjecture is deep and still widely open: for instance, it would imply Bloch's conjecture for surfaces.

In this paper, we focus on motives coming from curves and abelian varieties, more precisely we deal with the following categories.

Definition 0.1. For an abelian variety A over k, we write $M(A) \in \mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ for the motive of A. Define $\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}$ to be the smallest rigid and pseudo-abelian full subcategory[1] of $\mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ containing the motives of the form $M(A)$, for all abelian varieties over k. Define $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}} \supset \mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}$ to be the smallest triangulated, rigid and pseudo-abelian full subcategory of $\mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ containing $\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}$.

In characteristic zero, Wildeshaus showed that R_ℓ is conservative when restricted to $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ [Wil15, Theorem 1.12]. He first deals with the subcategory $\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}$ and then treats the whole $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$. Both steps use in a crucial way one of Grothendieck's standard conjectures, namely that numerical and ℓ-adic homological equivalence coincide. This conjecture is known for abelian varieties in characteristic zero [Lie68].

In positive characteristic, homological and numerical equivalence are not known to coincide. The best result in this direction is due to Clozel.

Theorem 0.2. [Clo99] *Given an abelian variety over a finite field, the set of prime numbers ℓ for which numerical and ℓ-adic homological equivalence coincide has positive density.*

Combining Wildeshaus' method with this result one can show the following.

Theorem 0.3. *Suppose that k is finite. Let $f : X \to Y$ be a morphism in $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$. If $R_\ell(f)$ is an isomorphism for almost all primes ℓ, then f itself is an isomorphism.*

Although this result is probably enough for applications over finite fields, it is intellectually unsatisfactory: for instance we cannot deduce, even for a single prime ℓ, that the functor R_ℓ is conservative. To go further we need to restrict to Chow motives.

Theorem 0.4. *Let k be a finite field. For any prime ℓ invertible in k, the ℓ-adic realization functor is conservative when restricted to $\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}$.*

It is amusing to notice how conservativity and the equality between homological and numerical equivalence are related "in the other direction" as well. For instance, we show the following.

Theorem 0.5. *Let k be a finite field and ℓ a prime number invertible in k. Suppose that, for all totally real number fields F and all places λ of F above ℓ, the λ-adic realization functor is conservative when restricted to $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_F$. Then the ℓ-adic homological equivalence coincides with numerical equivalence for abelian varieties over k.*

[1] Recall that the smallest rigid and pseudo-abelian full subcategory of $\mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ containing motives of smooth and projective varieties can be identified with (the opposite of) the classical category of Chow motives $\mathrm{CHM}(k)_{\mathbb{Q}}$, by [Voe00, Proposition 2.1.4] and [Voe02]. Hence, by definition, $\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}} \subset \mathrm{CHM}(k)_{\mathbb{Q}}$.

There are two tools in the proofs of these results. The first, valid over any field, is Kimura finiteness, which is a first approximation to conservativity (for instance, it implies that R_ℓ detects automorphisms among endomorphisms). The other one is the classical fact, due to Tate, that abelian varieties over finite fields have sufficiently many complex multiplications. This allows to decompose their motives in direct factors of dimension one (after extension of the field of coefficients).

Organization of the paper. Section §1 recalls results on motives of abelian type such as Kimura finiteness. In Section §2, we deduce the main technical result (Proposition 2.3), inspired by Hodge Theory, which is valid over any field. Section §3 recalls the theorem of Tate on endomorphisms of abelian varieties over finite fields and the results from [Clo99]. In Section §4, we will combine their results with Proposition 2.3 and deduce Theorem 0.4. Finally, in Section 5, we explain how to prove Theorems 0.3 and 0.5.

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1. The motive of an abelian variety

In this section, we recall classical results on motives of abelian type. Let k be a base field, F a field of coefficients of characteristic zero and $\mathrm{CHM}(k)_F$ the category of Chow motives over k with coefficients in F. For generalities, we refer to [And04] in particular to [And04, Definition 3.3.1.1] for the notion of Weil cohomology and to [And04, Proposition 4.2.5.1] for the associated realization functor.

Note that, following these references, the realization of a motive is a graded vector space.

If not explicitly stated, we will work with general Weil cohomologies (not necessary classical ones).

The following theorem summarizes the results about motives of abelian varieties which will be used.

Theorem 1.1. *Let A be an abelian variety ofdimension g. Let* $\mathrm{End}(A)$ *be its ring of endomorphisms (as an abelian variety) and $M(A) \in \mathrm{CHM}(k)_F$ be its motive. Then the following holds:*

(1) [DM91] *The motive $M(A)$ admits a Künneth decomposition*

$$M(A) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(A)$$

natural in $\mathrm{End}(A)$*. Moreover,* $\mathfrak{h}^0(A)$ *is the unit object* $\mathbb{1}$*.*

(2) [Anc15, Proposition 3.5(i)] *For each integer* i *between* 0 *and* $2g$*, and any realization functor* R*, one has*
$$R(\mathfrak{h}^i(A)) = H^i(A).$$

(3) [Kün94] *For each integer* i *between* 0 *and* $2g$ *there is a canonical isomorphism*
$$\mathfrak{h}^i(A) = \mathrm{Sym}^i \mathfrak{h}^1(A).$$

(4) [Kin98, Proposition 2.2.1] *The action of* $\mathrm{End}(A)$ *on* $\mathfrak{h}^1(A)$ *(coming from naturality in (1)) induces an isomorphism of algebras*
$$\mathrm{End}(A) \otimes_{\mathbb{Z}} F = \mathrm{End}_{\mathrm{CHM}(k)_F}(\mathfrak{h}^1(A))$$
and if A *is isogenous to* $B \times C$*, then* $\mathfrak{h}^1(A) = \mathfrak{h}^1(B) \oplus \mathfrak{h}^1(C)$*.*

(5) [Kün93] *The classical isomorphism in* ℓ*-adic cohomology induced by a polarization* $H^1_\ell(A) \cong H^1_\ell(A)^\vee(-1)$ *lifts to an isomorphism*
$$\mathfrak{h}^1(A) \cong \mathfrak{h}^1(A)^\vee(-1).$$

(6) [Kün93] *For each integer* i *between* 0 *and* $2g$*, the Lefschetz decomposition of the* ℓ*-adic cohomology* $H^i_\ell(A)$ *induced by a polarization lifts to a decomposition of the motive* $\mathfrak{h}^i(A)$*.*

Corollary 1.2. *We keep the notation from the theorem above. The following holds:*

(1) *The motive* $\mathbb{1}(-1)$ *is a direct factor of* $\mathfrak{h}^1(A) \otimes \mathfrak{h}^1(A)$*.*

(2) *A map* $f : \mathfrak{h}^1(A) \to \mathfrak{h}^1(A)^\vee(-1)$ *such that* $R_\ell(f) = 0$ *is itself zero.*

Proof. Using the Lefschetz decomposition of Theorem 1.1(6) we have that $\mathbb{1}(-1)$ is a direct factor of $\mathfrak{h}^2(A)$ (recall that, by Theorem 1.1(1) , we have $\mathfrak{h}^0(A) = \mathbb{1}$). On the other hand, $\mathfrak{h}^2(A)$ is a direct factor of $\mathfrak{h}^1(A) \otimes \mathfrak{h}^1(A)$ by Theorem 1.1(3), this proves (1).

To show (2), we compose f with an isomorphism $\mathfrak{h}^1(A) \cong \mathfrak{h}^1(A)^\vee(-1)$ coming from Theorem 1.1(5). This reduces to show that the realization is injective on $\mathrm{End}_{\mathrm{CHM}(k)_F}(\mathfrak{h}^1(A))$, which, by Theorem 1.1(4), is translated to the fact that the realization is injective on $\mathrm{End}(A)$, which is a classical theorem of Weil [Wei48, page 70]. □

Definition 1.3. Define $\mathrm{CHM}^{\mathrm{ab}}(k)_F$ to be the smallest rigid and pseudo-abelian full subcategory of $\mathrm{CHM}(k)_F$ containing motives of abelian varieties. A motive in $\mathrm{CHM}^{\mathrm{ab}}(k)_F$ is called "of abelian type".

A motive X of abelian type is *pure* if there is a realization functor R such that the cohomology groups of $R(M)$ are all zero except in one degree. In this case the degree will be called the weight of X. Moreover such an X is said to be of dimension d if the only non-zero cohomology group of $R(M)$ is of dimension d. For such an X we define
$$\det X = \begin{cases} \wedge^d X & d \text{ even}, \\ \mathrm{Sym}^d X & d \text{ odd}. \end{cases}$$

Similarly, we define $\det f$ for a morphism $f : X \to Y$ between pure motives of same degree and dimension.

Remark 1.4. The above notions do not actually depend on the choice of the realization functor R. For the notion of dimension this is [Jan07, Corollary 3.5]. For the notions of weight, this is Corollary 1.6 (combined with Theorem 1.1(2)).

Note also that an object X of odd weight is of dimension d in our sense, when is of dimension $-d$ in Kimura's sense.

Theorem 1.5. *Let X be a motive of abelian type and R be a realization functor with respect to a fixed Weil cohomology. Then the following holds:*

(1) [Kim05, Corollary 7.3] *If $R(X)$ is zero, then X itself is zero.*
(2) [And05, Corollaire 3.19] *If X is of dimension one, then $X \otimes X^\vee \cong \mathbb{1}$.*
(3) [Jan07, Corollary 3.7] *If X is of dimension one, then*
$$\mathrm{End}_{\mathrm{CHM}^{\mathrm{ab}}(k)_F}(X) = F \cdot \mathrm{id}.$$
(4) [O'S05, Lemma 3.2] *If $R(X)$ is concentrated in even (resp. odd) degree, and of total dimension d, then $X^\vee = \wedge^{d-1} X \otimes (\det X)^\vee$ (respectively $X^\vee = \mathrm{Sym}^{d-1} X \otimes (\det X)^\vee$).*
(5) [Kim05, Corollary 7.8] *Any decomposition of X as homological or numerical motive lifts to a decomposition of X in $\mathrm{CHM}^{\mathrm{ab}}(k)_F$.*
(6) [Kim05, Corollary 7.9] *Let $f : X \to X$ be an endomorphism. If $R(f)$ is an isomorphism then f is an isomorphism too.*
(7) [And05, Corollaire 3.16] *Let Y be another motive of abelian type. If X and Y are isomorphic as homological motives (or numerical motives) then they are isomorphic in $\mathrm{CHM}^{\mathrm{ab}}(k)_F$.*

Corollary 1.6. *Any motive of abelian type can be written as a sum of pure motives. Any pure motive of weight n can be written as a direct factor of $\mathfrak{h}^1(A)^{\otimes n+2m}(m)$, for some abelian variety A and some integer m.*

Proof. From the relation
$$M(A)(m) \otimes M(A')(m') = M(A \times A')(m + m'),$$
we deduce that any motive of abelian type X is a direct factor of a finite sum of the form $Y = \oplus_i M(A_i)(m_i)$. Write the Künneth decompositions for the motives $M(A_i)$ (Theorem 1.1(1)). Hence we have a Künneth decomposition $Y = \bigoplus_n Y_n$.

Fix a decomposition $Y = X \oplus Z$ and consider the maps $X \to Y_n \to X$. These maps may not be projectors modulo rational equivalence but they certainly are projectors modulo homological equivalence inducing the Künneth decomposition for the homological motive associated with X. Indeed the induced decomposition $R(Y) = R(X) \oplus R(Z)$ is a decomposition of graded vector spaces (see reminders at the beginning of the section) and $R(Y_n)$ is the n-th graded piece of $R(Y)$, so that $R(X) \to R(Y_n) \to R(X)$ is nothing else but the projector defining the n-th graded piece of $R(X)$.

Using Theorem 1.5(5), we lift this into a decomposition of $X = \bigoplus_n X_n$ refining the Künneth decomposition of Y. This shows the first part of the statement and moreover that X_n, the pure factor of X of weight n, is a direct factor of $\oplus_i \mathfrak{h}^{n+2m_i}(A_i)(m_i)$.

Now, by Theorem 1.1(3), the motive $\oplus_i \mathfrak{h}^{n+2m_i}(A_i)(m_i)$ is a direct factor of $\oplus_i \mathfrak{h}^1(A_i)^{\otimes n+2m_i}(m_i)$. Take a positive integer m bigger than all the m_i and use Corollary 1.2(1) to deduce that $\oplus_i \mathfrak{h}^1(A_i)^{\otimes n+2m_i}(m_i)$ is a direct factor of $\oplus_i \mathfrak{h}^1(A_i)^{\otimes n+2m}(m)$.

On the other hand, $\mathfrak{h}^1(\times_i A_i) = \oplus_i \mathfrak{h}^1(A_i)$ by Theorem 1.1(4), hence the motive $\oplus_i \mathfrak{h}^1(A_i)^{\otimes n+2m}(m)$ is a direct factor of $\mathfrak{h}^1(\times_i A_i)^{\otimes n+2m}(m)$. Putting all together, we deduce that X_n is a direct factor of $\mathfrak{h}^1(\times_i A_i)^{\otimes n+2m}(m)$. $\square$

2. Autoduality of motives

We keep the notation from the previous section. We prove a criterion to check conservativity of realization functors on Chow motives of abelian type.

By Theorem 1.5(6), we know that the realization functor detects automorphisms among endomorphisms. To show conservativity of realization one has to extend this property to morphisms between two objects that are a priori different. The next proposition is a first step in this direction.

Proposition 2.1. *Let X and Y be two motives of abelian type and $f : X \to Y$ and $g : Y \to X$ be two morphisms. Let R be a realization functor such that $R(f)$ and $R(g)$ are isomorphisms. Then f and g are isomorphisms too.*

Proof. We do the proof for f (of course the situation is symmetric). The realization of $g \circ f$ is an isomorphism, so, by Theorem 1.5 (6), $g \circ f$ is an isomorphism too. In particular, we can find a morphism $h : X \to X$ such that $(h \circ g) \circ f = \mathrm{id}_X$. This implies that $f \circ (h \circ g) : Y \to Y$ is a projector defining X as a direct factor of Y, hence $Y = X \oplus H$. But the factor H has zero realization, so it is actually zero, which means that f and $(h \circ g)$ are inverse to each other. $\square$

Proposition 2.2. *Let X and Y be two pure motives of abelian type of same weight and dimension. Let $f : X \to Y$ be a morphism such that* $\det f$ *is an isomorphism. Then f is an isomorphism too.*

Proof. We call n the weight and d the dimension and write the proof for n even (the odd case is analogous). Let us fix a realization functor R. As $\det f$ is an isomorphism then $R(f)$ must be an isomorphism. This implies that $R(\wedge^i f)$ is an isomorphism for any i. Then the realization of the map

$$(\wedge^{d-1} f)^\vee \otimes (\det f)^{-1} : (\wedge^{d-1} Y)^\vee \otimes \det Y \to (\wedge^{d-1} X)^\vee \otimes \det X$$

is an isomorphism. Using Theorem 1.5(4), we have constructed a map $g : Y \to X$ whose realization is an isomorphism. We conclude using Proposition 2.1. $\square$

Proposition 2.3. *Suppose that, for all pure motives $X \in \mathrm{CHM}^{\mathrm{ab}}(k)_F$ of even weight n and dimension one, we have an isomorphism*

$$X \cong X^\vee(-n).$$

Then any realization functor is conservative.

Proof. The reader would not be surprised that we will only use the assumption for $n = 0$. Indeed, using Tate twists, the assumption for $n = 0$ is actually equivalent to the assumption for all n even.

Let us fix a realization functor R and $f : X \to Y$ a map of abelian motives such that $R(f)$ is an isomorphism. The aim is to show that f is also an isomorphism.

First, write two (finite) decompositions $X = \oplus_n X_n$ and $Y = \oplus_n Y_n$, where X_n and Y_n are pure of weight n (Corollary 1.6). The map f induces morphisms $f_n : X_n \to Y_n$ (but, in general, f is not just the sum of the f_n). Note that $R(f_n)$ is an isomorphism. It is enough to show that each f_n is an isomorphism. Indeed, the inverses g_n of the f_n induce a morphism $g : Y \to X$ allowing us to apply Proposition 2.1.

We are reduced to the case where X and Y are pure of the same weight and dimension. By Proposition 2.2 it is enough to show that $\det f$ is an isomorphism, in other words we may assume that X and Y are pure of dimension one.

By Proposition 2.1, it is enough to construct a morphism $g : Y \to X$ whose realization is an isomorphism (or equivalently non-zero). It is constructed as follows

$$Y = Y \otimes \mathbb{1} \cong (Y \otimes X^\vee) \otimes X \cong (X \otimes Y^\vee) \otimes X \xrightarrow{\mathrm{id}\otimes f} (X \otimes Y^\vee) \otimes Y \ = X$$

where the first and last isomorphism come from Theorem 1.5(2) and the second comes from the assumption applied to the one dimensional motive of weight zero $X \otimes Y^\vee$. □

3. Abelian varieties over finite fields

We recall here some classical results on abelian varieties over finite fields due to Tate *et al.* and we give some consequences. Throughout the section, we fix a polarized abelian variety A of dimension g over a finite field k. We denote by $\mathrm{End}(A)$ the ring of endomorphisms of A, we write $\mathrm{End}^0(A)$ for $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $*$ for the Rosati involution on it (induced by the fixed polarization).

Theorem 3.1. *With the above notations, the following holds:*

1. [Tat66] *Maximal commutative $\mathbb{Q}$-subalgebras of $\mathrm{End}^0(A)$ have dimension $2g$.*
2. [Yu04, §2.2] *There exists a maximal commutative $\mathbb{Q}$-subalgebra B of $\mathrm{End}^0(A)$ which is $*$-stable.*

(3) [Mum08, pp. 211-212] *An algebra B as above is a finite product of CM number fields $B = L_1 \times \cdots \times L_t$ and $*$ acts as the complex conjugation on each factor.*
(4) [Shi71, Proposition 5.12] *The compositum of CM number fields is itself a CM field. The Galois closure of a CM number field is a CM number field as well.*

We write L for the CM number field which is the Galois closure of the compositum of the fields L_i, see Theorem 3.1(3)-(4). Let Σ_i be the set of embeddings of L_i in L and Σ the disjoint union of the Σ_i (with i varying). Write $\bar{\cdot}$ for the action on Σ induced by composition with the complex conjugation.

Corollary 3.2. *We keep the notations as above. In $\mathrm{CHM}^{\mathrm{ab}}(k)_L$ the motive $\mathfrak{h}^1(A)$ decomposes into a sum of $2g$ motives of dimension one*

$$\mathfrak{h}^1(A) = \bigoplus_{\sigma \in \Sigma} M_\sigma,$$

where the action of $b \in L_i$ on M_σ induced by Theorem 1.1(4) is given by multiplication by $\sigma(b)$ if $\sigma \in \Sigma_i$ and by multiplication by zero otherwise.

Moreover, the isomorphism $p : \mathfrak{h}^1(A) \cong \mathfrak{h}^1(A)^\vee(-1)$ of Theorem 1.1(5) restricts to an isomorphism

$$M_\sigma \cong M_{\bar{\sigma}}^\vee(-1)$$

for all σ, and to the zero map

$$M_\sigma \xrightarrow{0} M_{\sigma'}^\vee(-1)$$

for all $\sigma' \neq \bar{\sigma}$.

Proof. Consider the injection $L_1 \times \cdots \times L_t \hookrightarrow \mathrm{End}^0(A)$. By Theorem 1.1(4), we deduce an injection $(\prod_i L_i) \otimes L \hookrightarrow \mathrm{End}_{\mathrm{CHM}^{\mathrm{ab}}(k)_L}(\mathfrak{h}^1(A))$. Each projector of $(\prod_i L_i) \otimes L \cong \prod_i L^{[L_i:\mathbb{Q}]}$ defines a factor M_σ.

The last part of the statement can be checked after realization because of Corollary 1.2(2). It is then a consequence of Theorem 3.1(3). □

Definition 3.3. We keep notations from the theorem above and define L_0 to be $L \cap \mathbb{R}$.

Following Clozel, we define a set of prime numbers $\mathrm{Clo}(A, *, B)$ as those primes ℓ (different from the characteristic of k), such that there is a place λ of L_0 above ℓ such that the λ-adic completion of L_0 does not contain L.

If there are several $B \subset \mathrm{End}^0(A)$ as in the theorem above we can let B vary and consider the union of the $\mathrm{Clo}(A, *, B)$. We will call it $\mathrm{Clo}(A, *)$ or simply $\mathrm{Clo}(A)$.

Proposition 3.4. [Clo99, §3] *Given a totally real number field F and an imaginary quadratic extension E, the set of primes ℓ such that there is a place λ of F above ℓ such that the λ-adic completion of F does not contain E is of positive density.*

In particular, $\mathrm{Clo}(A, *, B)$ *is of positive density.*

Theorem 3.5. [Clo99] *Given a prime number* ℓ *in* $\mathrm{Clo}(A)$*, numerical and* ℓ*-adic homological equivalence on* A *(and all powers of* A*) coincide.*

The improvement on powers of A is due to Milne [Mil01, Proposition B.2].

4. Conservativity on Chow motives

In all this section the base field k is finite. We show here Theorem 0.4 from the Introduction. By Proposition 2.3, it is enough to show the following.

Theorem 4.1. *Suppose that the base field* k *is finite and that the field of coefficients* F *verifies that* $F \cap \overline{\mathbb{Q}}$ *is totally real. Then for any*

$$X \in \mathrm{CHM}^{\mathrm{ab}}(k)_F$$

of even weight n *and dimension one we have an isomorphism*

$$X \cong X^{\vee}(-n).$$

Proof. Let us start with some reduction steps. First, note that it is enough to have such an isomorphism in the category of numerical motives (by Theorem 1.5(7)). If Z is a Chow motive we will write $\overline{Z}$ for the corresponding numerical motive. Recall that the category of numerical motives is semisimple [Jan92] and notice that $\overline{X}$ is simple as it if of dimension one.

We claim that the numerical motive $\overline{X}$ exists already with coefficients in $F \cap \overline{\mathbb{Q}}$. To show this claim it is enough to show that there are no more simple objects with coefficients in F than with coefficients in $F \cap \overline{\mathbb{Q}}$. As the endomorphisms algebra of a simple object is a division algebra, it suffices to prove that if D is a division algebra over $F \cap \overline{\mathbb{Q}}$, then $D \otimes_{F \cap \overline{\mathbb{Q}}} F$ is also a division algebra. This is certainly classical, but we do not know a reference. It is for example a direct consequence of [Gro95, Théorème 6.1].

The claim reduces the question whether X and $X^{\vee}(-n)$ are isomorphic to the case $F \subset \overline{\mathbb{Q}}$. As the projectors defining these two motives (as algebraic cycles) have finitely many coefficients, we are allowed to suppose that F is a (totally real) number field.

Consider two totally real number fields $F \subset K$. We claim that the statement for K implies the statement for F. To show this claim we work again with numerical motives. Let X be a motive as in the statement, with coefficients in F. Note that $\mathrm{Hom}(\overline{X}, \overline{X}^{\vee}(-n))$ and $\mathrm{Hom}(\overline{X}^{\vee}(-n), \overline{X})$ are at most one-dimensional. Moreover, passing to coefficients in K corresponds to apply $\otimes_F K$ to these Hom (as numerical equivalence commutes with extension of scalars). Hence, if the relation $f \circ g = \mathrm{id}$ can be satisfied with coefficients in K then it can be satisfied also with coefficients in F.

We can now show the statement. We are reduced to the case where F is a totally real number field as big as we want. Any motive X as in the statement can be written as a direct factor of $\mathfrak{h}^1(A)^{\otimes n+2m}(m)$, for some abelian variety A and some integer m, by Corollary 1.6. After twist, we can suppose that X is a direct factor of $\mathfrak{h}^1(A)^{\otimes n}$, with n even.

Consider the decomposition explained in Corollary 3.2

$$\mathfrak{h}^1(A)=\bigoplus_{\sigma\in\Sigma} M_\sigma,$$

with the notation fixed above that statement. We can suppose that F contains the maximal totally real subfield in L. In particular, we can decompose the motive $\mathfrak{h}^1(A)^{\otimes n}$ in $\mathrm{CHM}^{\mathrm{ab}}(k)_F$ into a sum of motives of dimension two of the form

$$(M_{\sigma_1}\otimes\cdots\otimes M_{\sigma_n})\oplus(M_{\overline{\sigma_1}}\otimes\cdots\otimes M_{\overline{\sigma_n}}),$$

where $\sigma_i\in\Sigma$ and barring denotes the action of complex conjugation.

Again we can work with numerical motives. By semisimplicity, the isomorphism class of $\overline{X}$ appears in a motive of the form

$$\overline{Y}=(\overline{M}_{\sigma_1}\otimes\cdots\otimes\overline{M}_{\sigma_n})\oplus(\overline{M}_{\overline{\sigma_1}}\otimes\cdots\otimes\overline{M}_{\overline{\sigma_n}}),$$

hence we can then suppose that $\overline{X}$ is a direct factor of $\overline{Y}$. Moreover, we can see X as a direct factor of Y also in the category of Chow motives because of Theorem 1.5(5)

By Corollary 3.2, the morphism $p^{\otimes n}$ induces an isomorphism between Y and $Y^\vee(-n)$. As X is a direct factor of Y, $X^\vee(-n)$ is a direct factor of $Y^\vee(-n)$, hence we have maps between X and $X^\vee(-n)$ in both directions. We want to show that these maps are isomorphisms between X and $X^\vee(-n)$. This can be checked after realization by Proposition 2.1. In practice: we have a pairing on $R(Y)$ and we have to check that $R(X)$ is not an isotropic line. The pairing is perfect and symmetric on $R(Y)$ so at most two lines are isotropic. By Corollary 3.2, $R(M_{\sigma_1}\otimes\cdots\otimes M_{\sigma_n})$ and $R(M_{\overline{\sigma_1}}\otimes\cdots\otimes M_{\overline{\sigma_n}})$ are isotropic lines, so we have to check that $R(X)$ is not one of these two lines.

We can choose R to be the λ-adic realization, with λ one of the primes of F as in the Proposition 3.4 (to be applied to E the compositum of F and L). In this way the complex conjugation acts on the coefficients sending $R(M_{\sigma_1}\otimes\cdots\otimes M_{\sigma_n})$ to $R(M_{\overline{\sigma_1}}\otimes\cdots\otimes M_{\overline{\sigma_n}})$ and fixing $R(X)$. This implies that they are not the same line and concludes the proof. □

Corollary 4.2. *Suppose that the base field k is finite and that the field of coefficients F verifies that $F\cap\overline{\mathbb{Q}}$ is totally real. Then any realization functor is conservative on* $\mathrm{CHM}^{\mathrm{ab}}(k)_F$.

Proof. Combine the previous theorem with Proposition 2.3. □

Remark 4.3. The condition on F is a necessary hypothesis in the theorem. Indeed, if E is an elliptic curve with CM multiplication by a field L, then, by Corollary 3.2, $\mathfrak{h}^1(A)\in\mathrm{CHM}^{\mathrm{ab}}(k)_L$ decomposes as $V\oplus W$ with $V\cong W^\vee(-1)$. On the other hand $V^{\otimes 2}$ and $W^{\otimes 2}$ are not isomorphic as their realizations are not isomorphic (except if E^2 is supersingular). In particular $X=V^{\otimes 2}$

is a motive with coefficients in L, even weight (two) and dimension one but the isomorphism stated in the theorem cannot exist.

Instead, the corollary on conservativity should hold without any assumptions on the field of coefficients, but we are not able to show it. Note that this would have deep consequences as the following proposition shows.

Proposition 4.4. *Let k be a finite field and let ℓ be a prime number (invertible in k). Suppose that the ℓ-adic realization functor R_ℓ is conservative on $\mathrm{CHM}^{\mathrm{ab}}(k)_{\overline{\mathbb{Q}}}$. Then numerical equivalence coincides with ℓ-adic homological equivalence on abelian varieties over k.*

Proof. Consider a cycle Z on an abelian variety A of codimension i and suppose that it has a non-zero ℓ-adic homological class. By definition of Chow motives Z is a map living in $\mathrm{Hom}_{\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}}(M(A), \mathbb{L}^{\otimes i})$, where $\mathbb{L}$ is the Lefschetz motive. We want to show that it is not numerically trivial, for this it suffices to construct a map Y in $\mathrm{Hom}_{\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}}(\mathbb{L}^{\otimes i}, M(A))$ such that $Z \circ Y$ is not zero. As numerical equivalence commutes with extension of scalars we can work with coefficients in $\overline{\mathbb{Q}}$ and we will construct Y there.

By Corollary 3.2 (combined with Theorem 1.1) the motive $M(A)$ decomposes into a sum of motives of dimension one $M(A) = \bigoplus_i M_i$. Hence also $Z = \bigoplus_i Z_i$ can be decomposed. At least one component $Z_i : M_i \to \mathbb{L}^{\otimes i}$ has non-zero realization, hence $R_\ell(Z_i)$ is an isomorphism. Then by assumption Z_i is also an isomorphism of motives. Take Y to be the inverse of Z_i (on that one component and zero on the others). □

5. Conservativity on mixed motives

In all this section the base field k is finite. We study the conservativity of the realization functors on the category $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ (Definition 0.1). The results are weaker than the previous section.

Theorem 5.1. *Let X and Y be two motives in $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$. There exists a set of prime numbers $P_{X,Y}$ of positive density such that, for any $f : X \to Y$ and any $\ell \in P_{X,Y}$, if $R_\ell(f)$ is an isomorphism then f itself is an isomorphism.*

In particular, if $R_\ell(f)$ is an isomorphism for almost all primes ℓ, then f itself is an isomorphism.

Proof. First note that our category $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ coincides with the one that Wildeshaus studies, by [Anc16, Remark 5.6]. Now, $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ has a canonical weight structure (in the sens of [Bon10, §6]), whose heart is $\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}$ [Wil15, Proposition 1.2 and its proof]. Moreover, this weight structure is finite, hence only finitely many abelian varieties are needed to generate X and Y. Let A be the product of those and fix ℓ a prime number in $\mathrm{Clo}(A)$ (Definition 3.3).

Define $\mathcal{C}$ to be the smallest triangulated, rigid and pseudoabelian category containing the motive of A. Note that $X, Y \in \mathcal{C}$.

By Theorem 3.4, numerical and ℓ-adic homological equivalence coincide on powers of A (for the fixed ℓ), hence we can now apply Wildeshaus's methods [Wil15, proofs of 1.10-1.12] to $\mathcal{C}$, to conclude that the ℓ-adic realization (again for the fixed ℓ) is conservative on $\mathcal{C}$. $\square$

Theorem 5.2. *Let k be a finite field and ℓ be a prime number invertible in k. Suppose that, for all totally real number fields F and all places λ of F above ℓ, the λ-adic realization functor is conservative when restricted to $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_F$. Then the ℓ-adic homological equivalence coincides with numerical equivalence on abelian varieties over k.*

Proof. We start arguing as in Proposition 4.4. Suppose that there is an algebraic cycle Z of codimension i on an abelian variety A which is numerically trivial but has non-trivial ℓ-adic class. By definition of Chow motives Z is a map living in $\mathrm{Hom}_{\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}}(M(A), \mathbb{L}^{\otimes i})$, where $\mathbb{L}$ is the Lefschetz motive. Use the decomposition of Theorem 1.1(1) and consider at the component $Z_{2i} \in \mathrm{Hom}_{\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}}(\mathfrak{h}^{2i}(A), \mathbb{L}^{\otimes i})$. Notice that the realization of Z_{2i} is non-zero (this as the same cohomology class as Z) and the corresponding cycle must be numerically trivial.

Using Theorem 1.1(3), we have that $\mathfrak{h}^{2i}(A)$ is a direct factor of $\mathfrak{h}^1(A)^{\otimes 2i}$, hence we can look at Z_{2i} as an map $\alpha \in \mathrm{Hom}_{\mathrm{CHM}^{\mathrm{ab}}(k)_{\mathbb{Q}}}(\mathfrak{h}^1(A)^{\otimes 2i}, \mathbb{L}^{\otimes i})$. Its realization is still non-zero and the corresponding cycle is still numerically trivial.

Arguing as in the proof of Theorem 4.1, we can decompose the motive $\mathfrak{h}^1(A)^{\otimes 2i}$ in $\mathrm{CHM}^{\mathrm{ab}}(k)_F$ into a sum of motives of dimension two of the form

$$(M_{\sigma_1} \otimes \cdots \otimes M_{\sigma_{2i}}) \oplus (M_{\overline{\sigma_1}} \otimes \cdots \otimes M_{\overline{\sigma_{2i}}}).$$

This induces a decomposition of the morphism α. The assumption is that there exists one of its components which is numerically trivial but whose realization is non-zero. We call

$$f \in \mathrm{Hom}_{\mathrm{CHM}^{\mathrm{ab}}(k)_F}(Y, \mathbb{L}^{\otimes i})$$

such a component. Recall that Y is a motive of dimension two.

Consider now the isomorphism p from Corollary 3.2 and define

$$g = f^{\vee}(-2i) \circ p^{\otimes 2i} \in \mathrm{Hom}_{\mathrm{CHM}^{\mathrm{ab}}(k)_F}(\mathbb{L}^{\otimes i}, Y).$$

As f is numerically trivial, we must have $f \circ g = 0$.

On the other hand, in the category[2] $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_F$, we can complete f into a triangle

$$C \longrightarrow Y \xrightarrow{f} \mathbb{L}^{\otimes i}$$

and g must factorise into a morphism

$$h : \mathbb{L}^{\otimes i} \longrightarrow C.$$

[2]For simplicity, we take the embedding of $\mathrm{CHM}^{\mathrm{ab}}(k)_F$ into $\mathbf{DM}^{\mathrm{ab}}_{\mathrm{gm}}(k)_F$ to be covariant.

Note that the realization of f is a non-zero map between two graded vector spaces concentrated in the same degree, one of dimension one and the other of dimension two. Hence C is a vector space of dimension one concentrated in the same degree.

As the realization of f is non-zero, the realization of h is a non-zero map between vector spaces of dimension one, hence it is an isomorphism. Conservativity implies that h is an isomorphism too, hence $C \cong \mathbb{L}^{\otimes i}$. This means that the triangle above is a triangle between Chow motives. By [Voe00, Corollary 4.2.6], the triangle splits, hence $Y \cong \mathbb{L}^{\otimes i} \oplus \mathbb{L}^{\otimes i}$. In particular, numerical and homological equivalence coincide on $\mathrm{Hom}_{\mathrm{CHM}^{\mathrm{ab}}(k)_F}(Y, \mathbb{L}^{\otimes i})$, which gives a contradiction. $\square$

References

[Anc15] Giuseppe Ancona. Décomposition de motifs abéliens. *Manuscripta Math.*, 146(3-4):307–328, 2015.

[Anc16] Giuseppe Ancona. Numerical functors on Voevodsky's motives. *Research in Mathematical Sciences*, DOI: 10.1186/s40687-016-0061-1, 2016.

[And04] Yves André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2004.

[And05] Yves André. Motifs de dimension finie (d'après S.-I. Kimura, P. O'Sullivan...). *Astérisque*, (299):Exp. No. 929, viii, 115–145, 2005. Séminaire Bourbaki. Vol. 2003/2004.

[Ayo15] Joseph Ayoub. Motives and algebraic cycles: a selection of conjectures and open questions. *preprint*, 2015.

[Bon10] Mikhail V. Bondarko. Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general). *J. K-Theory*, 6(3):387–504, 2010.

[Clo99] Laurent Clozel. Equivalence numérique et équivalence cohomologique pour les variétés abéliennes sur les corps finis. *Ann. of Math. (2)*, 150(1):151–163, 1999.

[DM91] Christopher Deninger and Jacob Murre. Motivic decomposition of abelian schemes and the Fourier transform. *J. Reine Angew. Math.*, 422:201–219, 1991.

[Gro95] Alexander Grothendieck. Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 290, 199–219. Soc. Math. France, Paris, 1995.

[Ivo07] Florian Ivorra. Réalisation l-adique des motifs triangulés géométriques. I. *Doc. Math.*, 12:607–671, 2007.

[Jan92] Uwe Jannsen. Motives, numerical equivalence, and semi-simplicity. *Invent. Math.*, 107(3):447–452, 1992.

[Jan07] Uwe Jannsen. On finite-dimensional motives and Murre's conjecture. In *Algebraic cycles and motives. Vol. 2*, volume 344 of *London Math. Soc. Lecture Note Ser.*, pages 112–142. Cambridge Univ. Press, Cambridge, 2007.

[Kim05] Shun-Ichi Kimura. Chow groups are finite dimensional, in some sense. *Math. Ann.*, 331(1):173–201, 2005.

[Kin98] Guido Kings. Higher regulators, Hilbert modular surfaces, and special values of L-functions. *Duke Math. J.*, 92(1):61–127, 1998.

[Kün93] K. Künnemann. A Lefschetz decomposition for chow motives of abelian schemes. *Invent. Math.*, 113(4):85–102, 1993.

[Kün94] Klaus Künnemann. On the Chow motive of an abelian scheme. In *Motives*, volume 55.1 of *Proceedings of Symposia in pure mathematics*, pages 189–205. American mathematical society, 1994.

[Lie68] David I. Lieberman. Numerical and homological equivalence of algebraic cycles on Hodge manifolds. *Amer. J. Math.*, 90:366–374, 1968.

[Mil01] James Stuart Milne. The Tate conjecture for certain abelian varieties over finite fields. *Acta Arith.*, 100(2):135–166, 2001.

[Mum08] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[O'S05] Peter O'Sullivan. The structure of certain rigid tensor categories. *C. R. Math. Acad. Sci. Paris*, 340(8):557–562, 2005.

[Shi71] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. Kanô Memorial Lectures, No. 1.

[Tat66] John Tate. Endomorphisms of abelian varieties over finite fields. *Invent. Math.*, 2:134–144, 1966.

[Voe00] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[Voe02] Vladimir Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.*, (7):351–355, 2002.

[Wei48] André Weil. *Sur les courbes algébriques et les variétés qui s'en déduisent*. Actualités Sci. Ind., no. 1041 = Publ. Inst. Math. Univ. Strasbourg **7** (1945). Hermann et Cie., Paris, 1948.

[Wil15] Jörg Wildeshaus. On the interior motive of certain Shimura varieties: the case of Picard surfaces. *Manuscripta Math.*, 148(3-4):351–377, 2015.

[Yu04] Chia-Fu Yu. The isomorphism classes of abelian varieties of CM-type. *J. Pure Appl. Algebra*, 187(1-3):305–319, 2004.

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