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Alain Bretto, Alain Faisant, François Hennecart

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Alain Bretto[a], Alain Faisant[b], François Hennecart[b]

[a]*Normandie Univ, UNICAEN, ENSICAEN, CNRS-UMR 6072, GREYC, 14000 Caen, France*
[b] *Univ Lyon, UJM-Saint-Étienne, CNRS, ICJ UMR 5208, 42023 Saint-Étienne, France*

Abstract

A well-known conjecture of Erdős, Faber and Lovász can be stated in the following way: every loopless linear hypergraph $\mathscr{H}$ on n vertices can be n-edge-colored, or equivalently $q(\mathscr{H}) \leq n$, where $q(\mathscr{H})$ is the chromatic index of $\mathscr{H}$, i.e. the smallest number of colors such that intersecting hyperedges of $\mathscr{H}$ are colored with distinct colors. In this article we prove this assertion for Helly hypergraphs, for weakly trianguled hypergraphs, for well ordered hypergraphs and for a certain family of uniform hypergraphs.

Keywords: Hypergraph, hyperedge coloring, Erdős-Faber-Lovász Conjecture, Berge-Füredi Conjecture

1. Introduction

In this paper, we address the problem of coloring hyperedges of a hypergraph, in the case where it is loopless and linear. A good illustration of the problem we are dealing with is given in [14] and can be formulated as a story as follows: *Suppose that, in a university department, there are k committees, each consisting of k faculty members, and that all committees meet in the same room, which has k chairs. Suppose also that at most one person belongs to the intersection of any two committees. Is it possible to assign the committee members to chairs in such a way that each member sits in the same chair for all the different committees to which he or she belongs?*

In a graph modeling of this problem, the faculty members correspond to graph vertices, committees correspond to complete graphs, and chairs correspond to vertex colors. Therefore we can formulate the problem as a graph coloring problem: if k cliques, each having exactly k vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be *properly* colored with k colors: each two adjacent vertices have different colors.

Erdős and his peers did not immediately realize the depth and difficulty of this problem, and originally offered \$50 for its resolution. Erdős has for many years listed

Email addresses: `alain.bretto@unicaen.fr` (Alain Bretto), `faisant@univ-st-etienne.fr` (Alain Faisant), `francois.hennecart@univ-st-etienne.fr` (François Hennecart)

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this as one of his “three favorite combinatorial problems”, and after he realized the difficulty of this problem he increased to \$500 the reward for its resolution (see [11]).

We first recall some basic definitions in Section 2, before giving the formal statement of the conjecture in Section 3. We also mention results in some particular cases. Then in the next sections we propose proofs of the conjecture for weakly trianguled hypergraphs, well ordered hypergraphs and a certain family of uniform hypergraphs.

2. Basic definitions

2.1. General definitions

Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a hypergraph, where $\mathscr{V}$ is the finite set of vertices and $\mathscr{E} = (e_i)_{i\in I}$ is the finite family of hyperedges. We denote by $n = |\mathscr{V}|$ the number of vertices and by $m = |I|$ the number of hyperedges. In the sequel, we will use the following definitions:

- The *partial hypergraph* of $\mathscr{H}$ is a hypergraph $\mathscr{H}' = (\mathscr{V}';\mathscr{E}')$ where $\mathscr{E}' \subseteq \mathscr{E}$ and $\bigcup_{e\in\mathscr{E}'} e \subseteq \mathscr{V}'$.
- The *anti-rank* of $\mathscr{H}$ is the minimum cardinality of the hyperedges, i.e. $\mathrm{ar}(H) = \min_{i\in I}\{|e_i|\}$
- A *hyperloop* is a hyperedge e_i of cardinality $|e_i| = 1$. $\mathscr{H}$ is called *loopless* if there is no hyperloop.
- $\mathscr{H}$ is *simple* if for any $i, j \in I$ such that $e_i \subseteq e_j$ one necessarily has $i = j$. In this case the e_i’s are necessarily distinct and the family $\mathscr{E}$ is a set.
- $\mathscr{H}$ is *connected* if for any pair of vertices $x, y \in \mathscr{V}$ there exist distinct hyperedges $e_{j_1}, e_{j_2}, \dots, e_{j_r}$ such that $x \in e_{j_1}$, $y \in e_{j_r}$ and $e_{j_i} \cap e_{j_{i+1}} \neq \emptyset$, $i = 1,\dots,r-1$.
- $\mathscr{H}$ is *linear* if for any $i, j \in I$, $i \neq j$, one has $|e_i \cap e_j| \leq 1$. When $\mathscr{H}$ is moreover loopless then $\mathscr{H}$ is simple. A simple hypergraph is not always linear.
- $\mathscr{H}$ is *k-uniform* or is a *k-hypergraph* if every hyperedge has cardinality k. A (loopless) graph is a special case of k-uniform hypergraph with $k = 2$.
- A family of hyperedges of $\mathscr{H}$ is *intersecting* if any two of its hyperedges intersect.
- A *star* is an intersecting family $\mathscr{E}'$ such that all hyperedges share a same vertex x, i.e. $x \in \cap_{e\in\mathscr{E}'} e$: we say that the star is centered at x.
- If any intersecting family of $\mathscr{H}$ is a star then $\mathscr{H}$ is called to have *the Helly property*. We also say that $\mathscr{H}$ is a *Helly hypergraph*.
- The *degree* of a vertex x in a simple hypergraph $\mathscr{H}$, denoted by $d(x)$, or $d_{\mathscr{H}}(x)$, is the number of hyperedges containing x. We denote $\delta(\mathscr{H}) = \min_{x\in\mathscr{V}} d(x)$ the

minimum degree and $\Delta(\mathscr{H}) = \max_{x\in\mathscr{V}} d(x)$ the maximum degree of $\mathscr{H}$, while the cardinality of the largest intersecting family is denoted by $\Delta_0(\mathscr{H})$. A vertex x is called *isolated* if $d(x) = 0$.

- The *2-section* of $\mathscr{H}$ is the graph denoted by $[\mathscr{H}]_2$ where vertices are the vertices of $\mathscr{H}$ and two distinct vertices form an edge if and only if they belong to a same hyperedge.

- $\mathscr{H}$ is *conformal* if every clique of $[\mathscr{H}]_2$ is contained in some hyperedge.

2.2. Coloring definitions

- A *k-coloring of hyperedges or hyperedge k-coloring* of a hypergraph $\mathscr{H}$ is the assignment of colors from $\{1,2,3,...,k\}$ to all hyperedges of $\mathscr{H}$ in such a way that two intersecting hyperedges have distinct colors.

- The *chromatic index* of a hypergraph $\mathscr{H}$, denoted by $q(\mathscr{H})$, is the smallest k such that there exists a hyperedge k-coloring of $\mathscr{H}$. It is not difficult to see that [4]:
$$\delta(\mathscr{H}) \leq \Delta(\mathscr{H}) \leq \Delta_0(\mathscr{H}) \leq q(\mathscr{H}).$$

- Let $\Gamma = (V;E)$ be a simple graph: a *k-coloring* of the vertices of Γ is an assignment of colors from $\{1,2,3,...,k\}$ to all vertices of the graph such that two adjacent vertices have two distinct colors. The smallest such k, denoted by $\chi(\Gamma)$, is called the *chromatic number* of Γ. Plainly $|V| = n$ implies $\chi(\Gamma) \leq n$.

- For a hyperedge k-coloring of $\mathscr{H}$, we say that a color i is *incident to* a vertex x if x is in some hyperedge with color i.

- A *triangle* in a linear hypergraph is a set of 3 distinct hyperedges $\{e_1,e_2,e_3\}$ and 3 distinct vertices x,y,z such that
$$x \in e_1 \cap e_2,\ y \in e_2 \cap e_3 \text{ and } z \in e_3 \cap e_1.$$

- Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a loopless linear hypergraph. For $e \in \mathscr{E}$ an *inscribed triangle* on e is given by 3 distinct vertices $x,y \in e$, $z \notin e$ and 2 additional hyperedges e', e'' such that $\{x,z\} \subseteq e'$ and $\{z,y\} \subseteq e''$.

Let $\tau(e)$ the number of triangles inscribed on a given hyperedge. We note that in a linear hypergraph $\mathscr{H} = (\mathscr{V};\mathscr{E})$ we have $\tau(e) \leq \binom{|e|}{2}(n-|e|)$, this bound being linear in the variable n and quadratic in the variable $|e|$. Further we may be convinced that the size of $q(\mathscr{H})$ should be somewhat relied to $\max_{e\in\mathscr{E}} \tau(e)$. We thus introduce a certain family of hypergraphs for which $\tau(e)$ is bounded from above by some appropriate linear function in both variables n and $|e|$.

Definition 2.1. A linear hypergraph $\mathscr{H} = (\mathscr{V};\mathscr{E})$ is called *weakly trianguled* if for any $e \in \mathscr{E}$ we have $\tau(e) < (|e|-2)n + |e|$.

One extremal case occurs when $\tau(e) = 0$ for any e. A triangle $\{e, e', e''\}$ in a linear hypergraph $\mathscr{H}$ defines an intersecting family which is not a star then $\mathscr{H}$ is not a Helly hypergraph. The converse also holds. For sake of completeness we provide a proof.

Lemma 2.2. *A linear hypergraph $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ has the Helly property if and only if it does not contain any triangle.*

Proof. Assume that $\mathscr{H}$ has no triangle. We argue by induction on the cardinality of intersecting families of $\mathscr{H}$. Let $\{e_1, e_2, \ldots, e_k\}$ be an intersecting family of $\mathscr{H}$ with $k \geq 3$ (the cases $k = 1, 2$ yields plainly a star). By induction $\{e_1, e_2, \ldots, e_{k-1}\}$ is a star. Let $x_i = e_i \cap e_k$ for $i = 1, \ldots, k-1$. If $x_i \neq x_j$ for some $i \neq j$, then $\{e_i, e_j, e_k\}$ is a triangle, contradicting the hypothesis. Whence $x_i = x_1$ for any i and $\bigcap_{1 \leq i \leq k-1} e_i = \{x_1\}$. Since $x_1 \in e_k$ anf $\mathscr{H}$ is linear, we conclude that $\{e_1, e_2, \ldots, e_k\}$ is a star centered at x_1. $\square$

Corollary 2.3. *Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a linear hypergraph. Then $\mathscr{H}$ is conformal if and only if it does not contain any triangle.*

Corollary 2.4. *Any Helly linear hypergraph $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ is weakly trianguled.*

3. Erdős-Faber-Lovász Conjecture

One of Erdős' favorite combinatorial problems [10, 17] is given in Conjecture 3.1.

Conjecture 3.1 (Erdős-Faber-Lovász (1972))**.** *Any loopless linear hypergraph $\mathscr{H} = (\mathscr{V}, \mathscr{E})$ with n vertices has a chromatic index at most equal to n, i.e. $q(\mathscr{H}) \leq n$.*

This result is not necessarily true when the hypergraph admits hyperloops, neither when it is not linear.

Example 1. Let $n \geq 2$, $l = \left\lfloor \frac{n}{2} \right\rfloor$ and

$$\mathscr{V} = \{1, 2, \ldots n, n+1\},\ \mathscr{E} = \{A \cup \{n+1\} : A \subset \{1, 2, \ldots n\} \text{ and } |A| = l\}.$$

Then $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ is simple, $|\mathscr{V}| = n+1$ and

$$q(\mathscr{H}) = \binom{n}{l} \geq \frac{2^n}{n}.$$

When n is large $q(\mathscr{H})$ is much larger than n.

In the literature, there are several equivalent forms of conjecture 3.1. For instance we have:

1. Let Γ be a graph that is the union of at most n edge-disjoint copies of K_n, where K_n is the clique with n vertices (two distinct such cliques may have at most one common vertex). Then $\chi(\Gamma) = n$.

2. Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a linear n-uniform hypergraph with n hyperedges. The strong chromatic number of $\mathscr{H}$ is equal to n, i.e. when coloring the vertices each color appears only once in each hyperedge.

3. If $\mathscr{A} = \{A_1, \ldots, A_n\}$ is a family of distinct sets such that $|\bigcup_{i=1}^{n} A_i| = k$ and $|A_i \cap A_j| \leq 1$ whenever $i \neq j$, then there exists a k-coloring of $\mathscr{A}$, i.e. a function $f : \mathscr{A} \longrightarrow \{1, \ldots, k\}$ such that: $A_i \cap A_j \neq \emptyset$, $i \neq j \Longrightarrow f(A_i) \neq f(A_j)$.

4. Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a linear hypergraph with n vertices such that

   – every two vertices belong together to some hyperedge;

   – there are exactly n distinct hyperloops.

   Then $\mathscr{E}$ is decomposable into n partitions of $\mathscr{V}$: this means i) $\mathscr{E} = \cup_{i=1}^{n} \mathscr{E}_i$, ii) $\mathscr{E}_i \cap \mathscr{E}_j = \emptyset$ if $i \neq j$, iii) for any i, $\mathscr{E}_i \neq \emptyset$, iv) for any i, if $(e, e') \in \mathscr{E}_i \times \mathscr{E}_i$, $e \neq e'$, then $e \cap e' = \emptyset$.
   See [18] for more information about this formulation of the conjecture.

We now summarize partial known results about this conjecture (see [22]).

- The conjecture is true for hypergraphs with $n \leq 10$ vertices (cf. [15]).

- Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a hypergraph that is an intersecting family.
  Then the conjecture holds. (cf. [13]).

- Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a linear hypergraph with n vertices.
  Then $q(\mathscr{H}) \leq n + o(n)$ as n tends to infinity (cf. [16]).

- The conjecture is true for dense hypergraphs, i.e. hypergraphs $\mathscr{H}$ such that $\delta(\mathscr{H}) > \sqrt{n}$ (cf. [20]).

- Some results are also known for Steiner 2-designs which are particular cases of hypergraphs (cf. [6]).

4. Critical hypergraphs

Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a hypergraph. We denote by $q(\mathscr{H}) \geq 1$ its chromatic index. Then $\mathscr{H}$ is said $q(\mathscr{H})$-*critical* or *critical* for short if for any hyperedge $e \in \mathscr{E}$ we have

$$q(\mathscr{H} \setminus e) = q(\mathscr{H}) - 1,$$

where $\mathscr{H} \setminus e$ is the subhypergraph of $\mathscr{H}$ with the same vertices and hyperedge set $\mathscr{E} \setminus \{e\}$.

We summarize in the next lemma some properties satisfied by critical linear hypergraphs.
The *degree* of a hyperedge $e \in \mathscr{E}$ is defined by

$$d(e) = \sum_{x \in e} (d(x) - 1).$$

Lemma 4.1. *Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a linear loopless critical hypergraph with $|\mathscr{V}| = n$ vertices. Then* *a) For any hyperedge $e \in \mathscr{E}$, there is a hyperedge $q(\mathscr{H})$-coloring of $\mathscr{H}$ such that if i is the color of e, then e is the unique hyperedge with this color.*

b) For all $e \in \mathscr{E}$, one has $d(e) \geq q(\mathscr{H}) - 1$.

c) If $\mathscr{H}$ has no isolated vertex then $\mathscr{H}$ is connected.

Proof. a) From any hyperedge $(q(\mathscr{H})-1)$-coloring of $H \setminus e$, we obtain the desired hyperedge coloring of $\mathscr{H}$ by assigning to e a new color.

b) Suppose by contradiction that there is a hyperedge $e \in \mathscr{E}$ such that $d(e) \leq q(\mathscr{H}) - 2$. Since $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ is $q(\mathscr{H})$-critical, we have $q(\mathscr{H} \setminus e) = q(\mathscr{H}) - 1$. We color the hyperedges of $\mathscr{H} \setminus e$ with the set of colors $\mathscr{C} = \{1, 2, 3, \ldots, q(\mathscr{H}) - 1)\}$. Since $d(e) \leq q(\mathscr{H}) - 2$ there is a remaining color $i \in \mathscr{C}$ to be assigned to e. So $q(\mathscr{H}) = q(\mathscr{H} \setminus e)$, a contradiction.

c) Assume by contradiction that $\mathscr{H}$ is not connected. Then $\mathscr{H} = \bigsqcup_{1 \leq i \leq k} \mathscr{H}_i$ must be decomposed into $k \geq 2$ connected components. There is a i_0 such that $q(\mathscr{H}) = q(\mathscr{H}_{i_0})$. Since $\mathscr{H}$ has no isolated vertex there exists at least one hyperedge e in $\mathscr{H}$ such that e does not fall into $\mathscr{H}_{i_0}$. We have $q(\mathscr{H} \setminus e) = q(\mathscr{H}_{i_0}) = q(\mathscr{H})$, contrary to the assumption that $\mathscr{H}$ is critical. $\square$

We now provide an upper bound for $q(\mathscr{H})$ when $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ is a linear loopless critical hypergraph and a corollary.

Proposition 4.2. *Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a linear loopless critical hypergraph. Then for any $e \in \mathscr{E}$ such that $|e| = \mathrm{ar}(\mathscr{H})$,*

$$q(\mathscr{H}) \leq \Delta([\mathscr{H}]_2) + \Delta(\mathscr{H}) - |e| + 1.$$

Proof. Let $e \in \mathscr{E}$ and $x_0 \in e$ such that $d(x_0) = \max_{x \in e} d(x)$. By Lemma 4.1 b), we have

$$q(\mathscr{H}) \leq 1 + \sum_{x \in e} (d(x) - 1) \leq 1 + |e|(d(x_0) - 1) = 1 + (|e| - 1)d(x_0) + d(x_0) - |e|.$$

Since $(|e| - 1)d(x_0) \leq d_{[\mathscr{H}]_2}(x_0) \leq \Delta([\mathscr{H}]_2)$, we get the result. $\square$

Corollary 4.3. *Let $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ be a linear loopless hypergraph with $n = |\mathscr{V}|$ vertices and such that $\Delta([\mathscr{H}]_2) \leq \frac{n+1}{2}$. Then $q(\mathscr{H}) \leq n$.*

Proof. If $\mathscr{H}$ is critical this follows from the previous proposition and the plain inequality $\Delta(\mathscr{H}) \leq \Delta([\mathscr{H}]_2)$. In the general case we remove hyperedges from $\mathscr{H}$ until we get a critical hypergraph $\mathscr{H}'$ satisfying $q(\mathscr{H}) = q(\mathscr{H}')$ and we conclude from above using the obvious inequality $\Delta([\mathscr{H}']_2) \leq \Delta([\mathscr{H}]_2)$. $\square$

The following theorem allows us to restrict our attention to critical linear hypergraphs when dealing with the chromatic index of an hypergraph.

Theorem 4.4. *Both following assertions are equivalent:*

a) Any linear loopless hypergraph $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ with $|\mathscr{V}| = n$ satisfies $q(\mathscr{H}) \leq n$.

b) *Any critical linear loopless hypergraph* $\mathscr{H} = (\mathscr{V};\mathscr{E})$ *with* $|\mathscr{V}| = n$ *satisfies* $q(\mathscr{H}) \leq n$.

Proof. Assume that b) is true. We argue by induction on $m = |\mathscr{E}|$.

- The result is clear for $m = 1$ and $m = 2$.
- Suppose the assertion a) is true for any hypergraph with $m-1$ hyperedges where $m > 2$.
  Let $\mathscr{H}$ be a hypergraph with m hyperedges and n vertices. Two cases arise:
  – if there exists $e \in \mathscr{E}$ such that $q(\mathscr{H} \setminus e) = q(\mathscr{H})$, we obtain by induction that $q(\mathscr{H} \setminus e) \leq n$, hence $q(\mathscr{H}) \leq n$.
  – if for every hyperedge e, $q(\mathscr{H} \setminus e) = q(\mathscr{H}) - 1$, $\mathscr{H}$ is critical, we conclude by b) that $q(\mathscr{H}) \leq n$. $\square$

5. Erdős-Faber-Lovász Conjecture for weakly trianguled hypergraphs

Lemma 5.1. *Let* $\mathscr{H} = (\mathscr{V};\mathscr{E})$ *be a loopless linear hypergraph with* $|\mathscr{V}| = n$*. For any* $e \in \mathscr{E}$*, let* $\tau(e)$ *be the number of triangles inscribed on* e*. Then for any hyperedge* e *such that* $|e| = \mathrm{ar}(\mathscr{H})$*, we have*

$$d(e) \leq \frac{n - \mathrm{ar}(\mathscr{H}) + \tau(e)}{\mathrm{ar}(\mathscr{H}) - 1}.$$

Proof. Let $e \in \mathscr{E}$ with $e = \{x_1, x_2, \dots, x_{|e|}\}$ and $|e| = \mathrm{ar}(\mathscr{H})$. We have

$$(\mathrm{ar}(\mathscr{H})-1)d(e) = (|e|-1)\sum_{i=1}^{|e|}(d(x_i)-1) = \sum_{i=1}^{|e|}\sum_{\substack{e' \in \mathscr{E}\setminus\{e\} \\ x_i \in e'}}(|e|-1) \leq \sum_{i=1}^{|e|}\sum_{\substack{e' \in \mathscr{E}\setminus\{e\} \\ x_i \in e'}}(|e'|-1).$$

The right-hand side of the above inequality is the cardinality of the set S of all pairs (x_i, x) of distinct vertices for which there exists $e' \in E \setminus \{e\}$ with $x_i, x \in e'$.
If $(x_i, x) \in S$ then $x \in \mathscr{V} \setminus \{e\}$. Furthermore the number of triangles on e is exactly the number of *collisions* $(x_i, x), (x_j, x)$, $i \neq j$, in S. Hence $|S| \leq n - |e| + \tau(e)$. This gives the desired bound. $\square$

Theorem 5.2. *Let* $\mathscr{H} = (\mathscr{V};\mathscr{E})$ *be a loopless weakly trianguled linear hypergraph. Then* $q(\mathscr{H}) \leq n$.

Proof. We prove this property by induction on $|\mathscr{E}|$.

- If $|\mathscr{E}| \leq n$ the result is plain.
- Assume now that the property is true for any loopless weakly trianguled linear hypergraph with $k \geq n$ hyperedges.
  Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a loopless weakly trianguled linear hypergraph with $|\mathscr{E}| = k+1$. Let $e \in \mathscr{E}$ with $|e| = \mathrm{ar}(\mathscr{H})$. By removing e we obtain a hypergraph

$\mathscr{H}' = \mathscr{H} \setminus e$ with $|\mathscr{E}| - 1 = k$ hyperedges. Moreover $\mathscr{H}'$ is loopless, weakly trianguled and linear. By the induction hypothesis we get $q(\mathscr{H}') \leq n$. Since $|e| = \mathrm{ar}(\mathscr{H})$ and $\tau(e) \leq (|e|-2)n+1$, we get from Lemma 5.1

$$d(e) \leq \frac{n - \mathrm{ar}(\mathscr{H}) + \tau(e)}{\mathrm{ar}(\mathscr{H}) - 1} < \frac{n - \mathrm{ar}(\mathscr{H}) + (|e|-2)n + |e|}{\mathrm{ar}(\mathscr{H}) - 1} = n.$$

We conclude by assigning to e the free nth color. □

Corollary 5.3. *Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a loopless linear Helly hypergraph with $|\mathscr{V}| = n$. Then $q(\mathscr{H}) \leq n$.*

Proof. Straightforward from Corollary 2.4. □

6. Erdős-Faber-Lovász Conjecture for uniform linear hypergraphs

Theorem 6.1. *Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a linear hypergraph with $n = |\mathscr{V}| \geq 2$ vertices such that $\mathrm{ar}(\mathscr{H}) \geq \sqrt{n} + 1/2$. Then $q(\mathscr{H}) \leq n$.*

Proof. Our hypothesis on $\mathrm{ar}(\mathscr{H})$ implies that $\mathscr{H}$ is loopless.
We argue by induction. If $n = 2$ the result is clear.
Suppose now that $n \geq 3$. We remove hyperedges from $\mathscr{H}$ until we get a partial critical hypergraph $\mathscr{H}'$ such that we still have $q(\mathscr{H}') = q(\mathscr{H})$. Clearly $\mathscr{H}'$ is linear and loopless.
Let $r := \mathrm{ar}(\mathscr{H}')$ and $e \in \mathscr{E}$ with $e = \{x_1, x_2, \ldots, x_r\}$. From Lemma 4.1 we get

$$(r-1)(q(\mathscr{H}')-1) \leq (r-1)d(e) = (r-1)\sum_{i=1}^{r}(d(x_i)-1) = \sum_{i=1}^{r}(r-1)(d(x_i)-1).$$

Thus

$$(r-1)(q(\mathscr{H}')-1) \leq \sum_{i=1}^{r}(d_{[\mathscr{H}]_2}(x_i) - (r-1)) \leq r(\Delta([\mathscr{H}]_2) - (r-1)).$$

We obtain

$$q(\mathscr{H}) = q(\mathscr{H}') \leq \frac{r(\Delta([\mathscr{H}]_2) - (r-1))}{r-1} + 1 = \frac{r\Delta([\mathscr{H}]_2) - (r-1)^2}{r-1}.$$

Since $\Delta([\mathscr{H}]_2) \leq n-1$ we get $q(\mathscr{H}) \leq n$ whenever

$$\frac{r(n-1) - (r-1)^2}{r-1} \leq n$$

that is $(r-1)^2 + r - n \geq 0$. The above condition is valid when $r \geq \sqrt{n} + 1/2$. □

Corollary 6.2. *Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a linear r-uniform hypergraph with $n = |\mathscr{V}| \geq 2$ vertices such that $r \geq \sqrt{n} + 1/2$. Then $q(\mathscr{H}) \leq n$.*

7. Erdős-Faber-Lovász Conjecture for well ordered hypergraphs

7.1. Hyperedge Colouring Conjecture

A famous theorem due to Vizing on the edge coloring of a graph reads as follows:

Theorem 7.1 (Vizing's theorem)**.** *Let* Γ *be a simple graph of maximum degree* $\Delta(\Gamma)$*. Then* $q(\Gamma) \leq \Delta(\Gamma) + 1$*.*

This has prompted Berge (see [3]), Füredi (see [13]) to conjecture a generalized version of Vizing's theorem (cf. [8]).

Conjecture 7.2. *(Generalized Vizing's theorem) For any loopless linear hypergraph* $\mathscr{H} = (\mathscr{V}, \mathscr{E})$*, we have* $q(\mathscr{H}) \leq \Delta([\mathscr{H}]_2) + 1$*.*

The above bound has also been conjectured for dual hypergraphs of Steiner triple system by Colbourn and Colbourn (cf. [6]). The conjecture has been proved for r-uniform hypergraphs with $\Delta(\mathscr{H}) \leq r$ by Berge (see [3]) and for intersecting families by Füredi (see [13]). To our knowledge, these two partial results are the only cases for which the conjecture is established.

Remark 1. Observe that since $\Delta([\mathscr{H}]_2) \leq n-1$, Conjecture 7.2 implies Conjecture 3.1.

7.2. Well ordered hypergraphs

We first recall that

$$d_{[H]_2}(x) = \sum_{\substack{e \in \mathscr{E} \\ e \ni x}} (|e|-1) \leq \Delta([\mathscr{H}]_2).$$

Definition 7.3. A loopless hypergraph $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ is *well ordered* if for any vertex $x \in \mathscr{V}$, all incident hyperedges at x have distinct cardinalities:

Lemma 7.4. *Let* $\mathscr{H} = (\mathscr{V}; \mathscr{E})$ *a loopless critical linear hypergraph. Fix* $e_1 \in \mathscr{E}$ *such that* $|e_1| = \operatorname{ar}(\mathscr{H})$ *and* $y \in e_1$ *such that* $d(y) = \max\{d(x) : x \in e_1\}$*. Denote by* $e_2, \ldots, e_k \in \mathscr{E}$ *all the other hyperedges containing y. Then*

$$q(\mathscr{H}) \leq \Delta([\mathscr{H}]_2) + \sum_{2 \leq i \leq k} (|e_1| - |e_i| + 1) - (|e_1| - 2) + \left(d_{[\mathscr{H}]_2}(y) - \Delta([\mathscr{H}]_2)\right).$$

Proof. By Lemma 4.1 we get

$$q(\mathscr{H}) - 1 \leq d(e_1) = \sum_{x \in e_1} (d(x) - 1) \leq |e_1|(d(y) - 1) = |e_1|(k-1).$$

If follows that

$$\begin{aligned} q(\mathscr{H}) - 1 &\leq \sum_{i=2}^{k} (|e_i| - 1) + \sum_{i=2}^{k} (|e_1| - |e_i| + 1) \\ &= d_{[\mathscr{H}]_2}(y) - (|e_1| - 1) + \sum_{i=2}^{k} (|e_1| - |e_i| + 1). \end{aligned}$$

This yields

$$q(\mathscr{H}) \leq d_{[\mathscr{H}]_2}(y) + \sum_{2\leq i\leq k} (|e_1| - |e_i| + 1) - (|e_1| - 2)$$

and the results follows. ☐

Theorem 7.5. *Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a well ordered linear loopless hypergraph. Then $q(\mathscr{H}) \leq \Delta([\mathscr{H}]_2)$.*

Proof. By removing -if necessary- hyperedges of $\mathscr{H}$, we get a partial well ordered linear critical hypergraph $\mathscr{H}' = (\mathscr{V}';\mathscr{E}')$ with $\mathrm{ar}(\mathscr{H}') > 1$ such that $q(\mathscr{H}') = q(\mathscr{H})$. Let $e_1 \in \mathscr{E}'$ be a hyperedge with $|e_1| = \mathrm{ar}(\mathscr{H}')$ and $y \in e_1$ be a vertex such that $d(y) = \max\{d(x) : x \in e_1\}$.
We order all the hyperedges containing y according to their cardinalities: $1 < |e_1| < |e_2| < |e_3| < \cdots < |e_k|$. Then by Lemma 7.4

$$q(\mathscr{H}) = q(\mathscr{H}') \leq d_{[\mathscr{H}']_2}(y) + \sum_{2\leq i\leq k} (|e_1| - |e_i| + 1) - (|e_1| - 2).$$

Since $|e_i| \geq |e_1| + i - 1$, $i = 2,\ldots,k$, we get

$$q(\mathscr{H}) \leq d_{[\mathscr{H}']_2}(y) - \sum_{i=2}^{k} (i-2) - (|e_1| - 2).$$

Since $|e_1| = \mathrm{ar}(\mathscr{H}') \geq 2$ and $d_{[\mathscr{H}']_2}(y) \leq \Delta([\mathscr{H}']_2)$ we finally obtain

$$q(\mathscr{H}) \leq \Delta([\mathscr{H}']_2) \leq \Delta([\mathscr{H}]_2).$$ ☐

Corollary 7.6. *Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a well ordered linear hypergraph with $\mathrm{ar}(\mathscr{H}) > 1$ and $|\mathscr{V}| = n$. Then $q(\mathscr{H}) \leq n-1$.*

8. Hyperedge coloring divergence

Let $\mathscr{H} = (\mathscr{V};\mathscr{E})$ be a linear hypergraph. For any $x \in \mathscr{V}$ let

$$\Theta(x) = \sum_{2\leq i\leq k} (|e_1| - |e_i| + 1) - (|e_1| - 2) + \left(d_{[\mathscr{H}]_2}(x) - \Delta([\mathscr{H}]_2\right)$$

where the e_i's denote all the hyperedges containing x and $|e_1| \leq \cdots \leq |e_k|$. Let

$$\Theta(\mathscr{H}) = \min_{\substack{e\in\mathscr{E}\\ |e|=\mathrm{ar}(\mathscr{H})}} \max_{x\in e} \Theta(x)$$

be the so-called *hyperedge coloring divergence of $\mathscr{H}$*.

By Lemma 7.4 we have:

Proposition 8.1. *Let $\mathscr{H}$ be a critical linear loopless hypergraph. Then*

$$q(\mathscr{H}) \leq \Delta([\mathscr{H}]_2) + \Theta(\mathscr{H}).$$

Corollary 8.2. *Let $\mathscr{H}$ be a critical linear loopless hypergraph. If $\Theta(\mathscr{H}) \leq 1$, then $\mathscr{H}$ satisfies Berge-Füredi Conjecture 7.2 hence also Erdős-Faber-Lovász Conjecture 3.1.*

Corollary 8.3 (Berge [3])**.** *Let $r \geq 2$ and $\mathscr{H}$ be a linear r-uniform hypergraph such that $\Delta(\mathscr{H}) \leq r$. Then $\mathscr{H}$ satisfies Conjecture 7.2 .*

Proof. If $\mathscr{H}$ is critical then

$$\Theta(\mathscr{H}) = \sum_{2 \leq i \leq \Delta(\mathscr{H})} (r - r + 1) - (r - 2) + \Delta([\mathscr{H}]_2) - \Delta([\mathscr{H}]_2) = \Delta(\mathscr{H}) - r + 1.$$

Hence the have the result by Corollary 8.2.
Otherwise, we let $\mathscr{H}'$ be an arbitrary partial critical hypergraph of $\mathscr{H}$. Obviously $\mathscr{H}'$ is r-uniform and $\Delta(\mathscr{H}') \leq r$. We deduce from the previous case

$$q(\mathscr{H}) = q(\mathscr{H}') \leq \Delta([\mathscr{H}']_2) + 1 \leq \Delta([\mathscr{H}]_2) + 1$$

yielding the result. □

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