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Determination of point-forces via extended boundary measurements using a game strategy approach

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ABSTRACT. In this work, we consider a game theory approach to deal with an inverse problem related to the Stokes system. The problem consists in detecting the unknown point-forces acting on the fluid from incomplete measurements on the boundary of a domain. The approach that we propose deals simultaneously with the reconstruction of the missing data and the determination of the unknown point-forces. The solution is interpreted in terms of Nash equilibrium between both problems. We develop a new point-force detection algorithm, and we present numerical results to illustrate the efficiency and robustness of the method.

KEYWORDS : Nash game, data completion, point-forces detection.

1. Introduction

Consider a bounded open domain $\Omega \subset \mathbf{R}^d$ (d=2, 3) occupied by an incompressible viscous fluid, such that its boundary is sufficiently smooth and composed of two parts Γ_c and Γ_i. We assume then, that this fluid flow is under the action of a finite number of particles, and we suppose that each particle is no larger than a single point. That point will exert a force on the fluid, which is mathematically expressed in terms of the Dirac delta distribution $\lambda_k \delta_{P_k}$, where $P_k \in \Omega$ is the position of the particle and $\lambda_k \in \mathbf{R}^d$ is a constant vector and refers to the magnitude and direction of the point-forces. Then, the source-term F is a linear combination of Dirac distributions representing the total collection of the point-forces :

$$F = \sum_{k=1}^{m} \lambda_k \delta_{P_k}, \tag{1}$$

where m is the number of point-forces, δ_{P_k} is the Dirac distribution. In this work, we assume that the vectors λ_k are nonzero and the positions P_k are well separated and satisfy :

$$\begin{array}{c} \lambda_k \neq 0 \; and \; P_k \neq P_{k'}, \; \forall k \neq k' \in \{1, ..., m\}, \\ dist(P_k, \partial\Omega) \geq d_0 > 0, \; \forall k \in \{1, ..., m\}. \end{array} \tag{2}$$

The source Cauchy-Stokes inverse problem here consists, from given velocity G and fluid stress forces Φ prescribed only on the accessible part Γ_c of the boundary, to identify the unknown source-term F^*, that is, to find the number, location and strength of these point-forces such that the fluid velocity u and the pressure p are solution of the following Stokes problem :

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\sigma(u,p)) & = F^* & \text{in} & \Omega, \\ \mathrm{div} u & = 0 & \text{in} & \Omega, \\ u & = G & \text{on} & \Gamma_c, \\ \sigma(u,p)n & = \Phi & \text{on} & \Gamma_c, \end{array} \right. \tag{3}$$

where n is the unit outward normal vector on the boundary, and $\sigma(u,p)$ the fluid stress tensor defined as follows :

$$\sigma(u,p) = -pI_d + 2\nu D(u)$$

with $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ being the linear strain tensor and I_d denotes the $d \times d$ identity matrix and $\nu > 0$ is the viscosity coefficient.

Additionally to the inverse problem of detecting the unknown point-force, one has to complete the boundary data, that is to recover the missing traces of the velocity u and of the normal stress $\sigma(u,p).n$ over Γ_i the inaccessible part of the boundary. This inverse problem is of Cauchy type, a family of problems known to be severely ill-posed in the sense of Hadamard [1], even without point-force identification, because the existence of solution is not guaranteed for arbitrary Cauchy data and depends on

the compatibility of the data, and even if exists, it is unstable with respect to small perturbation of the Cauchy data.

Our purpose is to extend the method introduced in [2], based on a game theory approach, to develop a new algorithm for simultaneous determination of the unknown point-forces and missing boundary data.

2. Nash game formulation of the coupled problem of point-forces detection and data completion

We assume that the Cauchy data G and Φ belongs to $(H_{00}^{\frac{1}{2}}(\Gamma_c)^d)' \times H^{\frac{1}{2}}(\Gamma_c)^d$. For given fluid velocity $\tau \in H^{\frac{1}{2}}(\Gamma_i)^d$, stress forces $\eta \in (H_{00}^{\frac{1}{2}}(\Gamma_i)^d)'$, and source-term $F \in H^s(\mathbf{R}^d)$ for $(s < 1)$, we define the states $(u_1, p_1) = (u_1(\eta, F), p_1(\eta, F)) \in L^2(\Omega)^d \times L^2(\Omega)$ and $(u_2, p_2) = (u_2(\tau, F), p_2(\tau, F)) \in L^2(\Omega)^d \times L^2(\Omega)$ as the unique solution of the following Stokes mixed boundary value problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$,

$$(\mathcal{P}_1)\begin{cases} -\mathrm{div}(\sigma(u_1,p_1)) = F & \text{in} \quad \Omega, \\ \mathrm{div} u_1 = 0 & \text{in} \quad \Omega, \\ u_1 = G & \text{on} \quad \Gamma_c, \\ \sigma(u_1,p_1)n = \eta & \text{on} \quad \Gamma_i, \end{cases} \qquad (\mathcal{P}_2)\begin{cases} -\mathrm{div}(\sigma(u_2,p_2)) = F & \text{in} \quad \Omega, \\ \mathrm{div} u_2 = 0 & \text{in} \quad \Omega, \\ u_2 = \tau & \text{on} \quad \Gamma_i, \\ \sigma(u_2,p_2)n = \Phi & \text{on} \quad \Gamma_c. \end{cases}$$

The regularity-lack of the source term prevent a variational formulation in $H^1(\Omega)^d \times L^2(\Omega)$. To work around the problem, we used a relaxation technique.

2.1. Relaxation step

We consider the classical approximation of a Dirac function at a point $P = \{P_1, ..., P_m\}$ by the characteristic function of a small ball centred at P divided by its volume. Thus, instead of the source term F given by (1), we consider the following :

$$F_\epsilon = \sum_{k=1}^{m} \frac{\lambda_k}{|\mathcal{S}_{k,\epsilon}|} \chi_{\mathcal{S}_{P_k,\epsilon}}$$

where $\chi_{\mathcal{S}_{P_k,\epsilon}}$ denotes the characteristic function of the ball $\mathcal{S}_{P_k,\epsilon} = P_k + \epsilon\omega_k$, with $\epsilon > 0$ is small enough and ω_k is bounded and smooth domain containing the origin.

Let $(u_{1,\epsilon}, p_{1,\epsilon}) \in H^1(\Omega)^d \times L^2(\Omega)$ and $(u_{2,\epsilon}, p_{2,\epsilon}) \in H^1(\Omega)^d \times L^2(\Omega)$ be the unique solutions of the respective relaxed boundary value problems $(\mathcal{P}_{1,\epsilon})$ and $(\mathcal{P}_{2,\epsilon})$, witch are the reformulation of the respective BVP $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$,

$$(\mathcal{P}_{1,\epsilon})\begin{cases} -\mathrm{div}(\sigma(u_{1,\epsilon},p_{1,\epsilon})) &= F_\epsilon \quad \text{in} \quad \Omega,\\ \mathrm{div}u_{1,\epsilon} &= 0 \quad \text{in} \quad \Omega,\\ u_{1,\epsilon} &= G \quad \text{on} \quad \Gamma_c,\\ \sigma(u_{1,\epsilon},p_{1,\epsilon})n &= \eta \quad \text{on} \quad \Gamma_i, \end{cases}$$

$$(\mathcal{P}_{2,\epsilon})\begin{cases} -\mathrm{div}(\sigma(u_{2,\epsilon},p_{2,\epsilon})) &= F_\epsilon \quad \text{in} \quad \Omega,\\ \mathrm{div}u_{2,\epsilon} &= 0 \quad \text{in} \quad \Omega,\\ \sigma(u_{2,\epsilon},p_{2,\epsilon})n &= \Phi \quad \text{on} \quad \Gamma_c,\\ u_{2,\epsilon} &= \tau \quad \text{on} \quad \Gamma_i. \end{cases}$$

Let $\mathcal{I}_{\mathrm{ad}} = \{m \in \mathbf{N}^* \; and \; (\lambda, P) \in (\mathbf{R}^d \times \Omega)^m\}$ and let $\phi = (\lambda_k, P_k)_{1\leq k\leq m} \in \mathcal{I}_{\mathrm{ad}}$ be the elements defining the source F_ϵ. Let us present the following three cost functionals :

$$\mathcal{J}_1(\eta,\tau;\phi) = \frac{1}{2}||\sigma(u_{1,\epsilon},p_{1,\epsilon})n - \Phi||^2_{(H^{\frac{1}{2}}_{00}(\Gamma_c)^d)'} + \frac{1}{2}||u_{1,\epsilon} - u_{2,\epsilon}||^2_{H^{\frac{1}{2}}(\Gamma_i)^d}, \tag{4}$$

$$\mathcal{J}_2(\eta,\tau;\phi) = \frac{1}{2}||u_{2,\epsilon} - G||^2_{H^{\frac{1}{2}}(\Gamma_c)^d} + \frac{1}{2}||u_{1,\epsilon} - u_{2,\epsilon}||^2_{H^{\frac{1}{2}}(\Gamma_i)^d}, \tag{5}$$

$$\mathcal{J}_3(\eta,\tau;\phi) = ||\sigma(u_{1,\epsilon},p_{1,\epsilon}) - \sigma(u_{2,\epsilon},p_{2,\epsilon})||^2_{L^2(\Omega)^d}. \tag{6}$$

We are now in a position to formulate the three-player Nash game : Player (1) controls the strategy variable $\eta \in (H^{\frac{1}{2}}_{00}(\Gamma_i)^d)'$ and aims at minimizing the cost $\mathcal{J}_1$ and Player (2) controls the strategy variable $\tau \in H^{\frac{1}{2}}(\Gamma_i)^d$ and aims at minimizing the cost $\mathcal{J}_2$ and player (3) controls the strategy variable $\phi \in \mathcal{I}_{\mathrm{ad}}$ and aims at minimizing the Kohn-Vogelius type functional $\mathcal{J}_3$.

These players play a static Nash game with complete information. The most popular solution concept for such game is the one of a Nash equilibrium (NE) given by :

Definition 1 *A triplet $(\eta_N, \tau_N, \phi_N) \in (H^{\frac{1}{2}}_{00}(\Gamma_i)^d)' \times H^{\frac{1}{2}}(\Gamma_i)^d \times \mathcal{I}_{\mathrm{ad}}$ is a Nash equilibrium for the three players game if the following holds :*

$$\begin{cases} \mathcal{J}_1(\eta_N,\tau_N;\phi_N) \leq \mathcal{J}_1(\eta,\tau_N,\phi_N), & \forall \eta \in (H^{\frac{1}{2}}_{00}(\Gamma_i)^d)',\\ \mathcal{J}_2(\eta_N,\tau_N;\phi_N) \leq \mathcal{J}_2(\eta_N,\tau,\phi_N), & \forall \tau \in H^{\frac{1}{2}}(\Gamma_i)^d,\\ \mathcal{J}_3(\eta_N,\tau_N;\phi_N) \leq \mathcal{J}_3(\eta_N,\tau_N,\phi), & \forall \phi \in \mathcal{I}_{\mathrm{ad}}. \end{cases} \tag{7}$$

Next, the player (3) in charge of the inverse source-term problem will play in two steps in order to determine the triplet (m, λ_k, P_k) for $k = 1, ..., m$. Firstly, he enables the number and location of the point-forces support. Secondly, he uses the determined source position and computes the mean value of the source intensity $\Lambda = \{\lambda_1, ..., \lambda_m\}$.

Thanks to the relaxation step, the minimization problem of $\mathcal{J}_3$ w.r.t. P can be formulated as a topological optimization one [4]. Then, the unknown point-forces support $\mathcal{S} = \cup_{k=1}^{m} \mathcal{S}_{P_k,\epsilon}$, can be characterized as the solution to the following topological problem, for fixed $(\eta, \tau, \Lambda) \in (H_{00}^{\frac{1}{2}}(\Gamma_i)^d)' \times H^{\frac{1}{2}}(\Gamma_i)^d \times \mathbf{R}^{dm}$,

$$\left\{ \begin{array}{l} Find\ \mathcal{S}^* = \cup_{k=1}^{m} \mathcal{S}_{P_k^*,\epsilon} \subset \Omega,\ such\ that \\ \mathcal{S}^* = \mathrm{argmin}_{\mathcal{S}\subset\Omega} \left\{ \mathcal{J}(\mathcal{S}) := \displaystyle\int_\Omega |\sigma(u_{1,\epsilon}, p_{1,\epsilon}) - \sigma(u_{2,\epsilon}, p_{2,\epsilon})|^2 dx \right\}, \end{array} \right.$$

where $(u_{1,\epsilon}, p_{1,\epsilon})$ and $(u_{2,\epsilon}, p_{2,\epsilon})$ are the solutions to respectively $(\mathcal{P}_{1,\epsilon})$ and $(\mathcal{P}_{2,\epsilon})$. In order to solve this problem, we shall use a topological sensitivity analysis method. The latter consists in studying the variation of the cost function with respect to the small modification of the topology of the domain.

2.2. Numerical procedure

We describe in Algorithm 1 below the steps of the method, with a version where the Cauchy data of the Dirichlet type G are possibly perturbed by a noise with some magnitude σ, yielding for the Cauchy problem a noisy Dirichlet data G^σ :

Algorithm 1: Computation of the Nash equilibrium

Set $k = 0$ and choose an initial guess $(\eta^{(0)}, \tau^{(0)})$ and $F_\epsilon = 0$:
Compute $\overline{\eta}^{(0)}$ solution of $\min_\eta \mathcal{J}_1(\eta, \tau^{(0)}; F_\epsilon)$.
Compute $\overline{\tau}^{(0)}$ solution of $\min_\tau \mathcal{J}_2(\eta^{(0)}, \tau; F_\epsilon)$.
Evaluate $(\eta^{(1)}, \tau^{(1)}) = \alpha(\eta^{(0)}, \tau^{(0)}) + (1-\alpha)(\overline{\eta}^{(0)}, \overline{\tau}^{(0)})$.
• Step I. Determine of point-forces location :
Set $F_\epsilon = 0$, find

$$\mathcal{S}_{x_0,\epsilon}{}^{(k)} = \underset{\mathcal{S}_{x,\epsilon}\in\Omega}{\mathrm{argmin}}\, \mathcal{J}_3(\eta^{(k+1)}, \tau^{(k+1)}; (\lambda^{(k)}, x)),$$

by using the topological sensitivity.
• Step II. Solve the Nash game between η, τ and λ : Set $p = 1$,

– Step 1 : Evaluate $F_\epsilon^{(p)} = \frac{1}{|\mathcal{S}_{x_0,\epsilon}|} \lambda^{(p-1)} \chi_{\mathcal{S}_{x_0,\epsilon}}$

– Step 2 : Compute $\overline{\eta}^{(p)}$ solution of $\min_\eta \mathcal{J}_1(\eta, \tau^{(p-1)}; (\lambda^{(p-1)}, x_0))$

– Step 3 : Compute $\overline{\tau}^{(p)}$ solution of $\min_\tau \mathcal{J}_2(\eta^{(p-1)}, \tau; (\lambda^{(p-1)}, x_0))$

– Step 4 : Compute $\overline{\lambda}^{(p)}$ solution of $\min_\lambda \mathcal{J}_3(\eta^{(p-1)}, \tau^{(p-1)}; (\lambda, x_0))$

– Step 5 : Set $S^{(p)} = (\eta^{(p)}, \tau^{(p)}, \lambda^{(p)}) = \alpha(\eta^{(p-1)}, \tau^{(p-1)}, \lambda^{(p-1)}) + (1-\alpha)(\overline{\eta}^{(p)}, \overline{\tau}^{(p)}, \overline{\lambda}^{(p)})$, with $0 \leq \alpha < 1$.

– While $\|S^{(p)} - S^{(p-1)}\| > \varepsilon_S$ and $p < N_{max}$, set $p = p + 1$, return back to step 1.
• Step III. Compute $r_k = ||u_{2,\epsilon} - G^\sigma||_{0,\Gamma_c}$, where $(u_{2,\epsilon}, p_{2,\epsilon})$ is the solution of $(\mathcal{P}_{2,\epsilon})$.
While $r_k > \rho(\sigma)\epsilon$ and $k < K_{max}$, set $k = k + 1$, go back to step I.

The topological gradient and the gradient with a fixed step methods are used to solve step I and II respectively. In step II, in order to solve the problems of partial optimization of $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$, we need to calculate the gradient of these costs with respect to their strategies η, τ and λ. The fast computation of the latter is classical, and led by means of an adjoint state method [3].

3. Numerical results

The computations of the topological gradient and the minimization algorithm as well, were implemented using the finite elements method under FreeFem++ Software environment. To evaluate the effectiveness of the proposed approach, we tested it for exact and noisy data. We consider the centred square $]-\frac{1}{2},\frac{1}{2}[\times]-\frac{1}{2},\frac{1}{2}[$, where the overspecified Cauchy data are prescribed on the boundaries $y \in \{-\frac{1}{2},\frac{1}{2}\}$ and $x = -1/2$ of the square and the underspecified boundary data $u_{|\Gamma_i} = u(1/2, y)$ and stress force $\sigma(u,p)_{|\Gamma_i}$ are sought. We take the case of a single point-forces located at $P_{ex} = (-0.3, -0.25)$ with intensity $\lambda_{ex} = (0.25, 0.2)$.
To show the performance of the Nash Equilibrium Algorithm described in the previous section, we present in figure 1 the relative errors of missing data and the relative errors on the identified source position and its intensity. The results in Figure 2 and Table 1 show the efficiency and stability of our method. The numerical Dirichlet solution is a good approximation for the exact solution, see (a) and (b) in Figure 2. Then, concerning the numerical Neumann solution, see (c) and (d) in Figure 2, it can be seen that the estimates deviate from the exact one, especially near the endpoints of the boundary which is the region of singularities, in the corners.

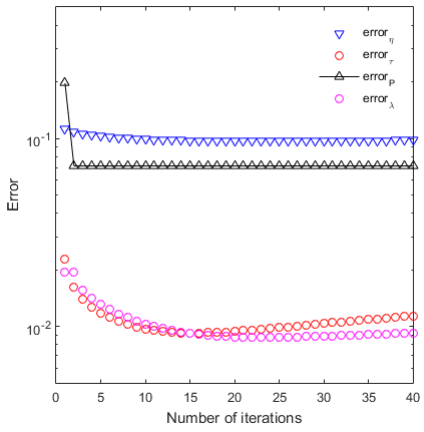


Figure 1. *L^2 relative errors on missing data on Γ_i (on Dirichlet-red balls- and Neumann-blue triangles-data), and the relative errors on the identified source position-black triangles- and its intensity-pink balls- as function of Nash iterations for unnoisy data.*

4. Conclusion

In this paper, we have presented a new approach to solve the coupled problem of the data completion and detection of the unknown point-forces acting on the fluid using a Nash game strategy. Using the relaxation formulation, this problem was formulated as a three-players Nash game. The two first targeting the data completion while the third one targets the source-term identification. Then, we introduced three objective functions, where each player tries to minimize his own cost by seeking to converge towards a Nash equilibrium, which is expected to approximate the solution of the original coupled problem. Thanks to the relaxation step, we have used the topological sensitivity analysis method in order to determine the point-force support. Finally, we obtained numerical results that corroborate the efficiency of our algorithm.

5. References

[1] J. Hadamard, The Cauchy Problem and the Linear Hyperbolic Partial Differential Equations, *Dover, New York*, (1953).

[2] A. Habbal and M. Kallel, Neumann-Dirichlet Nash strategies for the solution of elliptic cauchy problems, *SIAM Journal on Control and Optimization*, **51** (2013), 4066–4083.

[3] A. Habbal, M. Kallel and M. Ouni, Nash strategies for the inverse inclusion Cauchy-Stokes problem, *Inverse problems and imaging*, **13** (2019), 827–862.

[4] J. Ferchichi, M. Hassine and H. Khenous, Detection of point-forces location using topological algorithm in Stokes flows, *Applied Mathematics and Computation,* (2013), 7056–7074.

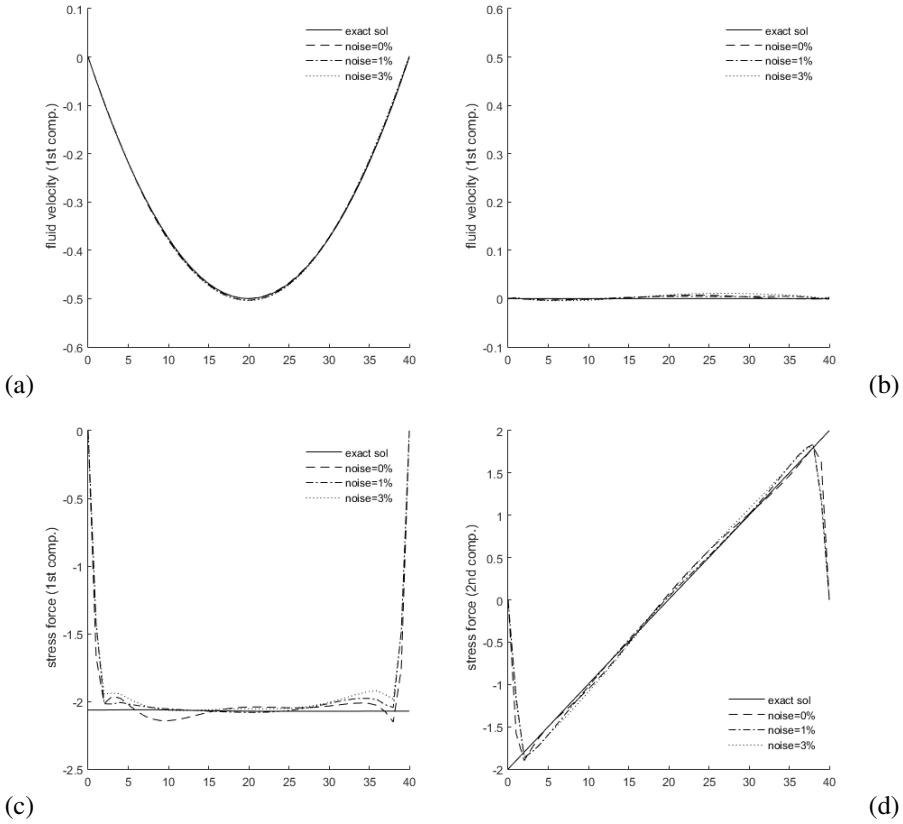


Figure 2. *Reconstruction of the missing boundary data with noisy Dirichlet data over Γ_c with noise levels $\sigma = \{0\%, 1\%, 3\%\}$. **(a)** exact and computed first components of the velocity over Γ_i **(b)** exact and computed second components of the velocity over Γ_i **(c)** exact and computed first components of the normal stress over Γ_i **(d)** exact and computed second components of the normal stress over Γ_i.*

Noise level	$\sigma = 0\%$	$\sigma = 1\%$	$\sigma = 3\%$
P_{op}	(-0.326, -0.259)	(-0.339, -0.285)	(-0.36, -0.29)
λ_{op}	(0.248, 0.197)	(0.24, 0.199)	(0.231, 0.196)
err_P	0.071	0.136	0.184
err_λ	0.009	0.029	0.058

Tableau 1. *Identified source position P_{op} and their intensity λ_{op} for various noise levels, respectively to their relative errors err_P and err_λ for various noise levels.*