Robust control design of underactuated 2 x 2 PDE-ODE-PDE systems

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Jean Auriol[1], Ulf Jakob F. Aarsnes[2] , Florent Di Meglio[3] , Roman Shor[4]

***Abstract*—In this paper, we design a robust stabilizing controller for a system composed of two sets of linear heterodirectional hyperbolic PDEs, with actuation at one boundary of one of the PDEs, and couplings at the middle boundary with ODEs in a PDE-ODE-PDE configuration. The system is underactuated since only one of the PDE systems is actuated. The design approach employs a backstepping transformation to move the undesired system couplings to the proximal boundary (where the actuation is located). We can then express this target system as a time-delay neutral system for which we can design an appropriate control law to obtain an exponentially stable target system.**

***Index Terms*—Distributed parameter systems, Control of networks**

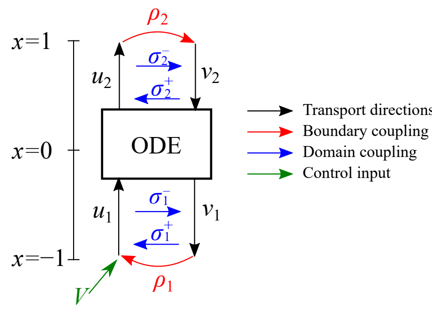

Fig. 1. Schematic representation of the considered system (4)-(12).

I. Introduction

We consider an interconnection of two sets of first-order linear hyperbolic Partial Differential Equations (PDE) coupled at their boundaries with Ordinary Differential Equations (ODE). For this problem, actuation acts only at one boundary of the first set of PDEs, which we will call the *proximal boundary* of the proximal PDEs. The proximal PDEs are coupled at their *distal boundary* with ODEs and a second set of PDEs, referred to as the *distal* PDEs.

The stabilization of interconnections of ODEs and PDEs has been the topic of numerous contributions since backstepping was first employed to re-interpret the delay-compensating Finite Spectrum Assignment controller in [16]. More involved interconnections have gradually been tackled through this method: non-linear ODEs with delays [8], ODEs coupled with a beam [22] or a heat equation [20]. Several contributions deal with the stabilization of coupled hyperbolic PDEs with ODEs in either a PDE-ODE structure [14], [5] or an ODE-PDE-ODE structure [21], [13]. The systems considered here arise when there is a lumped element in an otherwise distributed system such as heavy chain systems [19] or the Rijke tube [12]. Stabilizing controllers in particular instances of such systems have been designed in, e.g., [10], [12]. Most generally, these systems exist in situations where long continuous structures act as conduits for mass, energy, or signal transport constrained under some speed of propagation and acted on by a distributed source, or non-linear damping, term and a single point of actuation or control. In [12], a particular case of this problem is treated where the distal set of PDEs is disconnected through a backstepping transformation, and the proximal reflection is canceled. An extension is presented in [1], where the proximal PDEs feature coupling in the source terms. Using a backstepping transformation, the original system is mapped into a target delay system where stability is checked by Linear Matrix Inequality (LMI) conditions.

In this paper, we consider the more general case where we allow coupling in the source terms of both sets of PDEs and with less stringent conditions on the ODE. The main contribution is a robust stabilizing control law that relies on an easily computable stabilizability condition, analogous to that of a finite-dimensional linear system. The present result extends the results from [4], [6] and [1]. However, contrary to [4], only one PDE is actuated, which means that the system under consideration is underactuated. The presence of an ODE between the PDEs-subsystems is a major difference with [6]. The specific location of this ODE (sandwiched between two PDEs) and the in-domain couplings inside the PDEs can create unstable loops that do not appear in [1] and [4]. Then, although the proposed approach uses similar tools (backstepping transformations, neutral formulation of the target system), a deeper analysis must be used to deal with these new unstable loops.

Our approach is as follows. We perform a new backstepping transformation (different from the one used in [1]) that 1. moves certain coupling source terms from the two PDEs systems to the proximal boundary where they are canceled, and, 2. modifies the couplings between the ODE and the PDEs. Then, we show that the exponential stability of the resulting target system is equivalent to that of a delay equation of neutral type, which can be stabilized under a Kalman-like condition. The proposed control law is strictly proper, thus guaranteeing the existence of robustness margins. The paper is organized as follows. In Section II, we describe the control problem and introduce the notations for the rest of the paper. The backstepping transformation is derived in Section III. The system is then cast as a delay system in Section IV, and the stabilizing control law is given in Section V. Finally, some simulation results are given in Section VI.

II. Problem formulation

A. Definitions and notations

In this section, we define the various notations used in the rest of the paper. For any integer $p > 0$, $||\cdot||_{\mathbb{R}^p}$ is

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the classical euclidean norm on $\mathbb{R}^p$. We denote $L^2([0,1],\mathbb{R})$ the space of real-valued square-integrable functions defined on $[0,1]$ with the standard L^2 norm, *i.e.*, for any $f \in L^2([0,1],\mathbb{R})$, $||f||^2_{L^2} = \int_0^1 f^2(x)\mathrm{d}x$. We define $\chi = (L^2([-1,0],\mathbb{R}))^2 \times (L^2([0,1],\mathbb{R}))^2 \times \mathbb{R}^p$, with the associated norm $||(u_1,v_1,u_2,v_2,X)||_\chi = (||u_1||^2_{L^2}+||u_2||^2_{L^2}+||v_1||^2_{L^2}+||v_2||^2_{L^2}+||X||^2_{\mathbb{R}^p})^{\frac{1}{2}}$. The sets $\mathcal{T},\mathcal{T}_1$ $\mathcal{T}_2$ are defined as

$$\mathcal{T} = \{(x,\xi) \in [-1,0]\times[0,1]\}, \tag{1}$$
$$\mathcal{T}_1 = \{(x,\xi) \in [-1,0]^2 \text{ s.t. } \xi \geq x\}, \tag{2}$$
$$\mathcal{T}_2 = \{(x,\xi) \in [0,1]^2 \text{ s.t. } \xi \geq x\}. \tag{3}$$

We denote $L^\infty(\mathcal{T}_i)$ the space of real-valued L^∞ functions on $\mathcal{T}_i$. For any $(p,q) \in \mathbb{N}$, we denote $\mathcal{M}^{p\times q}(\mathbb{R})$ the set of real matrices with p rows and q columns. The symbol Id_p (or Id if no confusion arises) represents the $p \times p$ identity matrix. We denote $s \in \mathbb{C}$ the Laplace variable.

B. System under consideration

In this paper, we consider a PDE-ODE-PDE interconnection, which is schematically pictured in Figure 1. The different subsystems are only coupled between them through their boundaries. The global system is given by two sets of PDEs:

$$\partial_t u_1(t,x) + \lambda_1 \partial_x u_1(t,x) = \sigma_1^+(x) v_1(t,x), \tag{4}$$
$$\partial_t v_1(t,x) - \mu_1 \partial_x v_1(t,x) = \sigma_1^-(x) u_1(t,x), \tag{5}$$
$$\partial_t u_2(t,x) + \lambda_2 \partial_x u_2(t,x) = \sigma_2^+(x) v_2(t,x), \tag{6}$$
$$\partial_t v_2(t,x) - \mu_2 \partial_x v_2(t,x) = \sigma_2^-(x) u_2(t,x), \tag{7}$$

coupled at the boundary at $x = 0$

$$\dot{X}(t) = AX(t) + B_1 u_1(t,0) + B_2 v_2(t,0), \tag{8}$$
$$v_1(t,0) = C_1 X(t) + q_{11} u_1(t,0) + q_{12} v_2(t,0), \tag{9}$$
$$u_2(t,0) = C_2 X(t) + q_{21} u_1(t,0) + q_{22} v_2(t,0), \tag{10}$$

and finally we have the boundary conditions

$$u_1(t,-1) = \rho_1 v_1(t,-1) + V(t), \tag{11}$$
$$v_2(t,1) = \rho_2 u_2(t,1). \tag{12}$$

The scalar PDE states (u_1,v_1) are evolving in $\{(t,x) \text{ s.t. } \ t > 0, \ x \in [-1,0]\}$ while the states (u_2,v_2) are evolving in $\{(t,x) \text{ s.t. } \ t > 0, \ x \in [0,1]\}$. The ODE state X belongs to $\mathbb{R}^p$ (where $p \in \mathbb{N}\backslash\{0\}$). We assume that the transport velocities satisfy $-\mu_i < 0 < \lambda_i$ (where i is either 1 or 2). Note that the two subsystems can have identical velocities. The in-domain coupling terms σ_i^+ and σ_i^- are continuous functions. The different boundary couplings are constant. We have $A \in \mathcal{M}^{p\times p}(\mathbb{R})$, $B_1 \in \mathcal{M}^{p\times 1}(\mathbb{R})$, $B_2 \in \mathcal{M}^{p\times 1}(\mathbb{R})$, $C_1 \in \mathcal{M}^{1\times p}(\mathbb{R})$, $C_2 \in \mathcal{M}^{1\times p}(\mathbb{R})$. The function $V(t)$ is the control input. The subsystem (4)-(5) will be referred to as the *proximal PDEs*, while the subsystem (6)-(7) will be referred to as the *distal PDEs*. The initial condition $((u_i)_0,(v_i)_0,X_0)$ lies in χ and we consider weak solutions of (4)-(12). Such a system is well-posed [7]. This interconnected system features multiple couplings that can cause instabilities (in domain couplings, boundary couplings, unstable ODE). In what follows, for each set of PDEs, we denote $\tau_i = \frac{1}{\lambda_i} + \frac{1}{\mu_i}$, as the sum of the transport times in each direction.

Although the proximal subsystem is fully actuated at the boundary $x = -1$, the system (4)-(7) can be considered as an underactuated PDE-ODE system. The underactuation naturally arises when performing the change of variables $\bar{x} = -x$ on the distal subsystem. Then, system (4)-(10) rewrites as a PDE-ODE system where the PDE is composed of two leftward convecting equations and two rightward convecting ones. As only one of the rightward convecting equations is actuated (contrary to [4] where all the equations propagating in one direction are actuated), the system can be said underactuated. However, system (4)-(10) is a specific case of underactuated system as it presents a cascade structure between the different subsystems that simplifies its stabilization. See also the discussion in [1].

C. Stabilization problem and robustness aspects

The objective of this paper is to design a control law that guarantees the stabilization of the state (u_i,v_i,X) in the sense of the χ-norm. To design such a control law, we make the following assumption.

Assumption 1: The boundary couplings ρ_2 and q_{21} satisfy, $\rho_2 \neq 0$ and $q_{21} \neq 0$.

The first condition given in this Assumption is a consequence of the choice of the target system and backstepping transformation we make in this paper. However, we believe that the method described in this paper may be adjusted for the case $\rho_2 = 0$ following [11]. The condition $(q_{21} \neq 0)$ limits us to a specific class of systems, where $q_{21} = 0$ requires a different approach, such as [9] or a multi-step approach [13].

Besides, considering the robustness aspects, it has been shown in [18] that a necessary condition to guarantee the existence of robustness margins for the closed-loop system is that the open-loop transfer function must have a finite number of poles in the closed right half-plane. For the class of system we consider in this paper, using the condition given in [4], it means that the open-loop PDE system must be exponentially stable in the absence of the ODE and of in-domain coupling terms. This means that the ‘principal part’ of the system (4)-(12) must generate an exponentially stable semigroup. In the absence of the ODE and in-domain couplings (i.e. $\sigma_i^\pm \equiv 0, i \in \{1,2\}$), using the method of characteristics, we obtain

$$u_1(t,0) = \rho_1 q_{11} u_1(t-\tau_1,0) + \rho_1 q_{12} v_2(t-\tau_1,1), \tag{13}$$
$$v_2(t,1) = \rho_2 q_{21} u_1(t-\tau_2,0) + \rho_2 q_{22} v_2(t-\tau_2,1). \tag{14}$$

Using the transport structure of (4)-(12) (in the absence of in-domain coupling and of the ODE), it can easily be seen that the exponential stability of the states $u_1(t,0), v_2(t,1)$ is equivalent to the exponential stability of the whole system. Thus, in order to be able to design a *robust* feedback law for the system (4)-(12), we need (13)-(14) to be exponentially stable. More precisely, if this is not the case, the transfer function associated to (13)-(14) will have an infinite number of poles in the complex right half-plane. As the in-domain couplings and the ODE terms correspond to strictly proper terms that vanish at high frequencies, this will also be the case for the open-loop transfer function of (4)-(12) [3]. Consequently, as we want a robust control design, we are lead to the following *necessary* assumption.

Assumption 2: [4] The system (13)-(14) must be exponentially stable. In the case of rationally independent τ_1 and τ_2, this is equivalent to [15]

$$\sup_{\theta\in[0,2\pi]} \mathrm{Sp}\begin{pmatrix} \rho_1 q_{11} & \rho_1 q_{12} \\ \rho_2 q_{21}\mathrm{e}^{i\theta} & \rho_2 q_{22}\mathrm{e}^{i\theta} \end{pmatrix} < 1, \tag{15}$$

where Sp denotes the spectral radius.

III. Backstepping transformations: removal of in-domain couplings

In this section, we use a backstepping transformation to simplify the structure of the system (4)-(12) by removing most of the in-domain couplings of (4)-(7) and by changing the structure of the couplings between the ODE and the PDEs. This transformation is inspired by the ones given in [5], [6], but features several differences due to the specific structure of the considered problem.

A. Transformation and target system

Let us consider the following integral transformation

$$\alpha_1 = u_1 - \int_x^0 (K_1^{uu}(x,\xi)u_1(\xi) + K_1^{uv}(x,\xi)v_1(\xi))d\xi$$
$$- \int_0^1 (F^u(x,\xi)u_2(\xi) + F^v(x,\xi)v_2(\xi))d\xi + \gamma_u(x)X(t), \quad (16)$$

$$\beta_1 = v_1 - \int_x^0 (K_1^{vu}(x,\xi)u_1(\xi) + K_1^{vv}(x,\xi)v_1(\xi))d\xi, \quad (17)$$

$$\alpha_2 = u_2 - \int_x^1 (K_2^{uu}(x,\xi)u_2(\xi) + K_2^{uv}(x,\xi)v_2(\xi))d\xi, \quad (18)$$

$$\beta_2 = v_2 - \int_x^1 (K_2^{vu}(x,\xi)u_2(\xi) + K_2^{vv}(x,\xi)v_2(\xi))d\xi, \quad (19)$$

$$Y(t) = X(t) - \int_0^1 (M_u(\xi)u_2(\xi) + M_v(\xi)v_2(\xi))d\xi, \quad (20)$$

where the kernels $K_1^{\cdot\cdot}$ belong to $L^\infty(\mathcal{T}_1)$, the kernels $K_2^{\cdot\cdot}$ belong to $L^\infty(\mathcal{T}_2)$, the kernels F^u, F^v belong to $L^\infty(\mathcal{T})$. For all $x \in [-1,0]$ The function $\gamma_u(x)$ belongs to $\mathcal{M}^{1\times p}(\mathbb{R})$ while for all $x \in [0,1]$ the kernels $M_u(x)$ and $M_v(x)$ belong to $\mathcal{M}^{p\times 1}(\mathbb{R})$. All these kernels are defined below. Note that the arguments of the functions have been omitted when no confusion was possible. Although it appears to be of Fredholm type, the transformation (16)-(20) (provided its existence is guaranteed) is invertible. More precisely, since (18)-(19) is a Volterra transformation acting on the state (u_2, v_2), it is invertible [23]. This means that the states u_2, v_2 can be expressed as functions of α_2, β_2. Consequently, equation (20) is immediately invertible (as it only depends on (u_2, v_2)) and the state X can be expressed as a function of Y, α_2 and β_2. Finally, the transformation (16)-(17) is a Volterra transformation to which is added an affine part that only depends on X, u_2 and v_2. Using the aforementioned invertibility properties, this affine part of (16) can be expressed as a function of Y, α_2 and β_2. Thus, due to [23], the states u_1 and v_1 can be expressed as functions of α_1, β_1, Y, α_2 and β_2. Consequently, the whole transformation (16)-(20) is invertible, the inverse transformation presenting a structure similar to the one of (16)-(20). This inverse transformation is not given here due to space restrictions. On their domain of definition, the different kernels satisfy the following set of PDEs (with $i \in \{1,2\}$)

$$\lambda_i\partial_x K_i^{uu} + \lambda_i\partial_\xi K_i^{uu} = -\sigma_i^-(\xi)K_i^{uv}, \quad (21)$$
$$\lambda_i\partial_x K_i^{uv} - \mu_i\partial_\xi K_i^{uv} = -\sigma_i^+(\xi)K_i^{uu}, \quad (22)$$
$$\mu_i\partial_x K_i^{vu} - \lambda_i\partial_\xi K_i^{vu} = \sigma_i^-(\xi)K_i^{vv}, \quad (23)$$
$$\mu_i\partial_x K_i^{vv} + \mu_i\partial_\xi K_i^{vv} = \sigma_i^+(\xi)K_i^{vu}, \quad (24)$$
$$\lambda_1\partial_x F^u + \lambda_2\partial_\xi F^u = -\sigma_2^-(\xi)F^v, \quad (25)$$
$$\lambda_1\partial_x F^v - \mu_2\partial_\xi F^v = -\sigma_2^+(\xi)F^u, \quad (26)$$
$$\lambda_1\gamma_u'(x) = -\gamma_u(x)A + \mu_1 K_1^{uv}(x,0)C_1 + \lambda_2 F^u(x,0)C_2, \quad (27)$$
$$\lambda_2 M_u'(x) = -\sigma_2^- M_v(x) + \bar{A}M_u(x) + B_1F^u(0,x), \quad (28)$$
$$\mu_2 M_v'(x) = \sigma_2^+ M_u(x) - \bar{A}M_v(x) - B_1F^v(0,x), \quad (29)$$

with the boundary conditions

$$K_i^{uv}(x,x) = -\frac{\sigma_i^+}{\lambda_i + \mu_i}, \; K_i^{vu}(x,x) = \frac{\sigma_i^-}{\lambda_i+\mu_i}, \quad (30)$$

$$K_2^{uu}(x,1) = \frac{\mu_2}{\lambda_2}\rho_2 K_2^{uv}(x,1), \quad K_1^{vv}(x,0) = 0, \quad (31)$$

$$K_2^{vv}(x,1) = \frac{\lambda_2}{\rho_2\mu_2}K_2^{vu}(x,1), \quad (32)$$

$$K_1^{uu}(x,0) = \frac{1}{\lambda_1}(\lambda_2 q_{21}F^u(x,0) + \mu_1 q_{11}K_1^{uv}(x,0)$$
$$- \gamma_u(x)B_1), \quad (33)$$

$$F^u(0,\xi) = \frac{1}{q_{21}}(K_2^{uu}(0,\xi) - q_{22}K_2^{vu}(0,\xi)), \quad (34)$$

$$F^v(0,\xi) = \frac{1}{q_{21}}(K_2^{uv}(0,\xi) - q_{22}K_2^{vv}(0,\xi)), \quad (35)$$

$$F^u(x,1) = \frac{\mu_2}{\lambda_2}\rho_2 F^v(x,1), \quad (36)$$

$$F^v(x,0) = \frac{1}{\mu_2}(\mu_1 q_{12}K_1^{uv}(x,0) + \lambda_2 q_{22}F^u(x,0)$$
$$- \gamma_u(x)B_2), \quad (37)$$

$$\gamma_u(0) = \frac{1}{q_{21}}C_2, \quad M_u(0) = 0, \quad M_v(0) = \frac{-B_2}{\mu_2}, \quad (38)$$

where we have used the fact that ρ_2 and q_{21} are not equal to zero (Assumption 1) and where we set $\bar{A} = A - \frac{1}{q_{21}}B_1C_2$. Provided that the set of equations (21)-(38) admits a unique solution (which will be assessed in the next section), we can show that (16)-(20) maps the original system (4)-(12) to the following target system:

$$\partial_t\alpha_1(t,x) + \lambda_1\partial_x\alpha_1(t,x) = 0, \quad (39)$$
$$\partial_t\beta_1(t,x) - \mu_1\partial_x\beta_1(t,x) = \lambda_1 K_1^{vu}(x,0)u_1(t,0), \quad (40)$$
$$\partial_t\alpha_2(t,x) + \lambda_2\partial_x\alpha_2(t,x) = 0, \quad (41)$$
$$\partial_t\beta_2(t,x) - \mu_2\partial_x\beta_2(t,x) = 0, \quad (42)$$

with the boundary conditions

$$\alpha_1(t,-1) = \rho_1\beta_1(t,-1) + V(t)$$
$$+ I_1[u_1,v_1,u_2,v_2,X], \quad (43)$$
$$\dot{Y}(t) = \bar{A}Y(t) + B_1\alpha_1(t,0) + \bar{B}_2\alpha_2(t,1) \quad (44)$$
$$\beta_1(t,0) = q_{11}\alpha_1(t,0) + q_{12}\beta_2(t,0) \quad (45)$$
$$+ (C_1 - q_{11}\gamma_u(0))X(t) + I_2[u_2,v_2] \quad (46)$$
$$\alpha_2(t,0) = q_{21}\alpha_1(t,0) + q_{22}\beta_2(t,0), \quad (47)$$
$$\beta_2(t,1) = \rho_2\alpha_2(t,1), \quad (48)$$

where $\bar{B}_2 = \lambda_2 M_u(1) - \mu_2\rho_2 M_v(1)$ and

$$I_1[u_1,v_1,u_2,v_2,X] = \gamma_u(-1)X(t)$$
$$- \int_{-1}^0 (K_1^{uu}(-1,\xi) - \rho_1 K_1^{vu}(-1,\xi))u_1(t,\xi)d\xi$$
$$- \int_{-1}^0 (K_1^{uv}(-1,\xi) - \rho_1 K_1^{vv}(-1,\xi))v_1(t,\xi)d\xi$$
$$- \int_0^1 F^u(-1,\xi)u_2(t,\xi) + F^v(-1,\xi)v_2(t,\xi)d\xi. \quad (49)$$

$$I_2[u_2,v_2] = \int_0^1 (q_{11}F^u(0,\xi) + q_{12}K_2^{vu}(0,\xi))u_2(t,\xi)d\xi$$
$$+ \int_0^1 (q_{11}F^v(0,\xi) + q_{12}K_2^{vv}(0,\xi))v_2(t,\xi)d\xi. \quad (50)$$

The target system (39)-(48) still contains some terms from the original system (the terms $u_1(t,0)$ in equations (40) and (46) and the integral terms I_1 and I_2 in the boundary conditions (43), (46)). This is not standard as target systems do not usually contain functions that depend on the original state. These terms could have been expressed in terms of the states α_i, β_i, Y by using the inverse of the transformation (16)-(20). As it will appear in the next sections, this is not necessary for the design of a stabilizing control law, as these terms will somehow be cancelled by the actuation. Finally, due to the boundedness and the invertibility of the transformation (16)-(20), the target system and the original system have equivalent stability properties for the norm χ. Note that this target system is different from the ones considered in [4] and [1]. In particular, the ODE (44) now depends on $\alpha_1(t,0)$ and $\alpha_2(t,1)$ (and not on $\beta_2(t,0)$). Moreover, unlike [6], only one transformation is required to obtain the target system. The advantage of the target system (39)-(48) is that we have a clear path between the actuation and the 'ODE-distal PDEs' subsystem.

B. Well-posedness of the system (30)-(38)

In this section, we prove the well-posedness of the system (30)-(38). We have the following lemma.

Lemma 1: Consider system (30)-(38). There exist a unique solution $K_1^{\cdot\cdot}$ in $L^\infty(\mathcal{T}_1)$, $K_2^{\cdot\cdot}$ in $L^\infty(\mathcal{T}_2)$, F^u, F^v in $L^\infty(\mathcal{T})$ and γ_u, M_u, M_v, differentiable vectors.

Proof: Due to space restrictions, we only give a sketch of the proof. The kernels $K_2^{\cdot\cdot}$ satisfy independent equations. Using [11], we immediately have the existence of these kernels on their corresponding domain of definition. Consider the kernels F^u and F^v. Note that the boundary conditions (34)-(35) are now perfectly defined. Let us consider the rectangular domain $\mathcal{R} = \{(x,\xi) \in [x_0, 0] \times [0,1]\}$ where $x_0 = \max(-1, -\frac{\lambda_1}{\mu_2})$. Let us define the two triangular domains $\mathcal{R}_u = \{(x,\xi) \in \mathcal{R},\ x \geq x_0\xi\}$ and $\mathcal{R}_l = \{(x,\xi) \in \mathcal{R},\ x \leq x_0\xi\}$. Applying [14, Theorem 3.2] we can prove that (25)-(26) admit a unique solution on $\mathcal{R}_u$. This allows us to compute the kernels F^u and F^v on the line $x = x_0\xi$ of $\mathcal{R}$. We now perform the change of variables $\bar{\xi} = \frac{1}{x_0}\xi$ in order to express the kernels $K_1^{\cdot\cdot}$ on the domain $\mathcal{R}_b$. Considering the extension of the ODE kernel γ_u on the triangular domain $\mathcal{R}_b$, we can apply [14, Theorem 3.2] to prove the existence of the kernels F^u, F^v on the domain $\mathcal{R}$ and the kernels $\gamma_u, K_1^{\cdot\cdot}$ on the domain $\{(x,\xi) \in [x_0, 0]^2 \mid x \leq \xi\}$. Iterating such an approach (on the intervals $[(k+1)x_0, x_0]$) we have the existence of the kernels $\gamma_u, K_1^{\cdot\cdot}, F^u, F^v$ on their corresponding domains of definition. Finally, Cauchy-Lipschitz's (aka Picard–Lindelöf's) theorem applied on (28)-(29) concludes the proof as it allows us to assess the existence of the kernels M_u and M_v. ∎

IV. Delay form of the target system

Although the structure of the target system (39)-(48) is simpler compared to the one of the original system (4)-(12), there are still multiple couplings between the different subsystems that make the analysis difficult and different from the one done in [1] and [6]. In this section, we first rewrite the target system (39)-(48) as a time-delay system. We then show that, with a specific choice of structure for V, the state $Y(t)$ actually satisfies an autonomous delay-equation of neutral type which is directly actuated. This simpler system will be of specific interest for the design of a stabilizing control.

A. A neutral system

Since (41)-(42) are transport equations, using the method of characteristics, we immediately have for any $t \geq \tau_2$:

$$\alpha_2(t,1) = \rho_{22}q_{22}\alpha_2(t-\tau_2, 0) + q_{21}\alpha_1(t - \frac{1}{\lambda_2}, 0). \quad (51)$$

Similarly, we obtain for $t \geq \frac{1}{\mu_1}$:

$$\beta_1(t,-1) = q_{11}\alpha_1(t - \frac{1}{\mu_1}, 0) + q_{12}\beta_2(t - \frac{1}{\mu_1}, 0) + I_3[u_1, v_1, u_2, v_2, X], \quad (52)$$

where

$$I_3[u_1, v_1, u_2, v_2, X] = (C_1 - q_{11}\gamma_u(0))X(t - \frac{1}{\mu_1}) + I_2(u_2(t - \frac{1}{\mu_1}, \cdot), v_2(t - \frac{1}{\mu_1}, \cdot))d\xi + \int_0^{\frac{1}{\mu_1}} \lambda_1 K_1^{vu}(\nu\mu_1 - 1, 0)u_1(t-\nu, 0)d\nu. \quad (53)$$

Inserting (52) into (43) we get $\alpha_1(t,-1) = \rho_1\left(q_{11}\alpha_1(t - \frac{1}{\mu_1}, 0) + q_{12}\beta_2(t - \frac{1}{\mu_1}, 0)\right) + V(t) + I_1[u_1, v_1, u_2, v_2, X] + \rho_1 I_3[u_1, v_1, u_2, v_2, X]$. Using the actuation to cancel the strictly proper terms

$$V(t) = \tilde{V}(t) - I_1[u_1, v_1, u_2, v_2, X] - \rho_1 I_3[u_1, v_1, u_2, v_2, X], \quad (54)$$

and using the transport structure of the system, we get the following (neutral) time-delay system

$$\alpha_1(t,0) = \tilde{V}(t - \frac{1}{\lambda_1}) + \rho_1\rho_2 q_{12}\alpha_2(t - \tau_1 - \frac{1}{\mu_2}, 1) + \rho_1 q_{11}\alpha_1(t - \tau_1, 0), \quad (55)$$

$$\alpha_2(t,1) = \rho_2 q_{22}\alpha_2(t - \tau_2, 1) + q_{21}\alpha_1(t - \frac{1}{\lambda_2}, 0), \quad (56)$$

$$\dot{Y}(t) = \bar{A}Y(t) + B_1\alpha_1(t,0) + \bar{B}_2\alpha_2(t,1), \quad (57)$$

where the last equation is just a copy of (44). The proposed actuation design does not cancel the pointwise delay terms in (55) (or equivalently $\rho_1\beta_1(t,-1)$). Although this would simplify equation (55), such a cancellation can be the origin of potential robustness issues (see [3] for details).

B. A time-delay equation satisfied by Y

The state $Y(t)$ can be expressed as the solution of a time-delay equation of neutral type that does not depend on $\alpha_1(t,0)$ nor $\alpha_2(t,1)$. To do so, we will use (55)-(56) and Assumption 2. Taking the Laplace transform of (55)-(56), we obtain

$$(Id - F(s))(\alpha_1(s,0), \alpha_2(s,1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\frac{1}{\lambda_1}s}\tilde{V}(s), \quad (58)$$

where $F(s)$ is the holomorphic function defined by

$$F(s) = \begin{pmatrix} \rho_1 q_{11}\mathrm{e}^{-\tau_1 s} & \rho_1\rho_2 q_{12}\mathrm{e}^{-(\tau_1 + \frac{1}{\mu_2})s} \\ q_{21}\mathrm{e}^{-\frac{1}{\lambda_2}s} & \rho_2 q_{22}\mathrm{e}^{-\tau_2 s} \end{pmatrix}. \quad (59)$$

Let us define $F_1(s) = \begin{pmatrix} \rho_1 q_{11}\mathrm{e}^{-\tau_1 s} & \rho_1 q_{12}\mathrm{e}^{-\tau_1 s} \\ \rho_2 q_{21}\mathrm{e}^{-\tau_2 s} & \rho_2 q_{22}\mathrm{e}^{-\tau_2 s} \end{pmatrix}$. Due to Assumption 2, the system (13)-(14) is exponentially stable. Thus, the determinant of $Id - F_1(s)$ cannot vanish on the complex closed right half-plane [15]. Since the matrices $Id - F(s)$ and $Id - F_1(s)$ have the same determinants, we obtain that $Id - F(s)$ is invertible on the right-half plane. We can consequently invert equation (58) to obtain

$$D(s)\alpha_1(s,0) = (1 - \rho_2 q_{22}\mathrm{e}^{-\tau_2 s})\mathrm{e}^{-\frac{1}{\lambda_1}s}\tilde{V}(s), \quad (60)$$

$$D(s)\alpha_2(s,1) = q_{21}\mathrm{e}^{-(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})s}\tilde{V}(s), \tag{61}$$

where $D(s) = (1 - \rho_2 q_{22}\mathrm{e}^{-\tau_2 s})(1 - \rho_1 q_{11}\mathrm{e}^{-\tau_1 s}) - \rho_1\rho_2 q_{12}q_{21}\mathrm{e}^{-(\tau_1+\tau_2)s} \neq 0$. Thus, taking the Laplace transform of (57) and multiplying it by $D(s)$, we obtain

$$\begin{aligned} D(s)sY(s) = \bar{A}D(s)Y(s) + B_1(1-\rho_2 q_{22}\mathrm{e}^{-\tau_2 s})\mathrm{e}^{-\frac{1}{\lambda_1}s}\tilde{V}(s) \\ + \bar{B}_2 q_{21}\mathrm{e}^{-(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})s}\tilde{V}(s). \end{aligned} \tag{62}$$

Note that we have omitted the effect of the initial condition of Y when taking the Laplace transform of (57) since it does not modify the stability analysis (it only impacts the transient) [15]. Due to the fact that the operator $D(s)$ does not vanish on the right-half plane, we can define $Z(s) = D(s)Y(s)$. We then have the detectability of Y from the new variable Z (i.e. if Z goes to zero, so does Y), [15]. This yields

$$\begin{aligned} \dot{Z}(t) = \bar{A}Z(t) + B_1\tilde{V}(t-\frac{1}{\lambda_1}) + \bar{B}_2 q_{21}\tilde{V}(t-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}) \\ - \rho_2 q_{22}B_1\tilde{V}(t-\frac{1}{\lambda_1}-\tau_2). \end{aligned} \tag{63}$$

We can now design a feedback law $\tilde{V}$ to stabilize (63).

V. Stabilization

A. Stabilization of the state $Z(t)$

Equation (63) is a linear system with delayed controls. It has been broadly studied in the literature [2], [24]. Let us consider the following change of variables:

$$\begin{aligned} \bar{Z} = Z + \int_{t-\frac{1}{\lambda_1}}^{t} \mathrm{e}^{(t-\nu-\frac{1}{\lambda_1})\bar{A}}B_1\tilde{V}(\nu)d\nu \\ + q_{21}\int_{t-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}}^{t} \mathrm{e}^{(t-\nu-\frac{1}{\lambda_1}-\frac{1}{\lambda_2})\bar{A}}\bar{B}_2\tilde{V}(\nu)d\nu, \\ - \rho_2 q_{22}\int_{t-\frac{1}{\lambda_1}-\tau_2}^{t} \mathrm{e}^{(t-\nu-\frac{1}{\lambda_1}-\tau_2)\bar{A}}B_1\tilde{V}(\nu)d\nu, \end{aligned} \tag{64}$$

with which we immediately obtain

$$\begin{aligned} \dot{\bar{Z}}(t) = \bar{A}\bar{Z}(t) + (\mathrm{e}^{-\frac{1}{\lambda_1}\bar{A}}B_1 + q_{21}\mathrm{e}^{-(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})\bar{A}}\bar{B}_2 \\ - \rho_2 q_{22}\mathrm{e}^{-(\tau_2+\frac{1}{\lambda_1})\bar{A}}B_1)\tilde{V}(t). \end{aligned} \tag{65}$$

A necessary and sufficient [24] condition for the stabilization of equation (65) is given by the following assumption

Assumption 3: Let us define

$$\bar{B} = \mathrm{e}^{-\frac{1}{\lambda_1}\bar{A}}B_1 - \rho_2 q_{22}\mathrm{e}^{-(\tau_2+\frac{1}{\lambda_1})\bar{A}}B_1 + q_{21}\mathrm{e}^{-(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})\bar{A}}\bar{B}_2,$$

where we recall that $\bar{A} = A - \frac{1}{q_{21}}B_1C_2$ and $\bar{B}_2 = \lambda_2 M_u(1) - \mu_2\rho_2 M_v(1)$ (M_u and M_v being defined by (28)-(29)). We assume that the pair $(\bar{A},\bar{B})$ is stabilizable, i.e. there exists K such that $\bar{A}+\bar{B}K$ is Hurwitz.

Then, choosing

$$\begin{aligned} \tilde{V}(t) = K\bar{Z}(t) = KZ(t) + K\int_{t-\frac{1}{\lambda_1}}^{t} \mathrm{e}^{(t-\nu-\frac{1}{\lambda_1})\bar{A}}B_1\tilde{V}(\nu)d\nu \\ + q_{21}K\int_{t-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}}^{t} \mathrm{e}^{(t-\nu-\frac{1}{\lambda_1}-\frac{1}{\lambda_2})\bar{A}}\bar{B}_2\tilde{V}(\nu)d\nu. \\ - \rho_2 q_{22}K\int_{t-\frac{1}{\lambda_1}-\tau_2}^{t} \mathrm{e}^{(t-\nu-\frac{1}{\lambda_1}-\tau_2)\bar{A}}B_1\tilde{V}(\nu)d\nu, \end{aligned} \tag{66}$$

where K is defined in Assumption 3, exponentially stabilizes $Z(t)$ in (65).

B. Stabilization of (4)-(12)

This analysis enables us to state the main result of the paper.

Theorem 1: Consider system (4)-(12) and let Assumptions 1, 2 and 3 be satisfied. Let us choose the control law $V(t)$ as,

$$\begin{aligned} V(t) = \tilde{V}(t) - I_1[u_1,v_1,u_2,v_2,X] \\ - \rho_1 I_3[u_1,v_1,u_2,v_2,X], \end{aligned} \tag{67}$$

where I_1, I_3 and $\tilde{V}$ are defined by (49), (53) and (66), respectively. Then, the zero equilibrium of the system is exponentially stable in the sense of the $||\cdot||_\chi$ norm.

Proof: We have already shown that the state Z exponentially converges to zero. Using the detectability of Y from Z, so does the state Y (and $\tilde{V}$). Thus, using the variations of constants formula on (55)-(56) (see [15, Section 9.5]), the state $(\alpha_1(t,0),\alpha_2(t,1))$ exponentially converges to zero. The rest of the proof is a consequence of the transport structure of (39)-(42) and of the invertibility of the transformation (16)-(20). We first show the convergence of (α_2,β_2) which implies the stabilization of (u_2,v_2,X) and $u_1(t,0)$. Then, we consider the remaining states (u_1,v_1). ■

Remark that the control law $V(t)$ defined by (67) has been chosen as strictly proper (we do not cancel the PDEs reflection) which means that it is robust to small delays in the input and uncertainties on the parameters. The proof follows the same ideas as that in [3].

Remark 1: The reader should be aware that due to the proposed design, the convergence rate of the closed-loop system will be bounded by the natural dissipative rate of (13)-(14) (since the system (4)-(12) is equivalent to (55)-(57)). Thus, even by choosing a large convergence rate for $\bar{Z}$, the proposed design does not allow an arbitrarily large convergence rate. It is worth mentioning that such a convergence rate can be increased by canceling a part of the reflection terms in (43). However, as mentioned in [3], this may raise robustness issues.

Remark 2: In the absence of the ODE (i.e. $p = 0$), the proposed control law is identical to the one developed in [6].

Remark 3: Under Assumptions 1 and 2, Assumption 3 is a *necessary* condition for the existence of a feedback law stabilizing system (4)–(12). We only give here a sketch of the proof because of space constraints. System (4)–(12) and (55)–(57) have equivalent stabilizability properties since they are related by an invertible change of state and control coordinates. The latter rewrites

$$\mathcal{A}(s)\begin{pmatrix}\alpha_1(s,0) & \alpha_2(s,1) & Y(s)\end{pmatrix}^\top = \mathcal{B}(s)\tilde{V}(s) \tag{68}$$

in the Laplace domain, for some $\mathcal{A}$, $\mathcal{B}$. Straightforward computation shows that if Assumption 3 is not satisfied, then $\mathcal{M}(s) = \begin{pmatrix}\mathcal{A}(s) & \mathcal{B}(s)\end{pmatrix}$ is rank deficient in the Right-Half Plane, i.e. $\exists s^* \in \mathbb{C}^-, v^*, \in \mathbb{R}^{p+2}$ s.t. $v^{*\top}\mathcal{M}(s^*) = 0$ This, in turn[1], implies that there is no controller stabilizing (55)–(57). This means that under Assumptions 1 and 2, Assumption 3 is necessary and sufficient for the stabilization of (4)–(12) with non-zero delay robustness.

VI. Simulation results

The proposed control law has been tested in simulations using Matlab. The PDE system is simulated using a classical finite volume method based on a Godunov scheme [17]. We

[1] one can, e.g., construct an unstable output unaffected by the control input using v^* and s^*

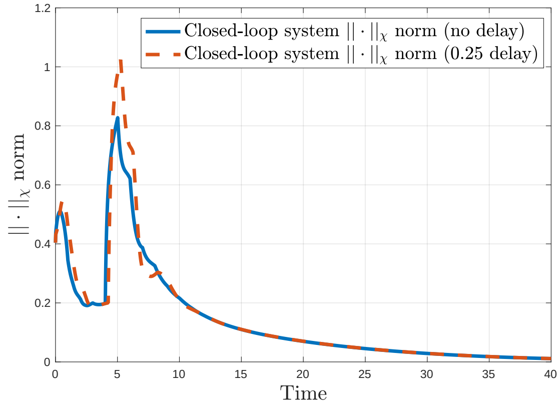

Fig. 2. Evolution of the χ-norm of the closed-loop system without delay and with an input delay of 0.25.

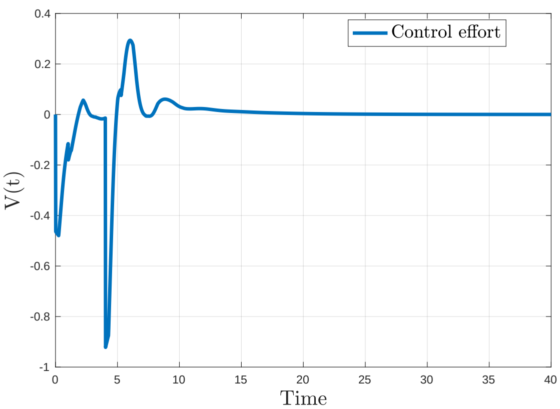

Fig. 3. Evolution of the control input V in closed-loop.

used 101 spatial discretization points (and a CFL number of 1). The algorithm we use to compute the different kernels is adapted from the one proposed in [5]. Using the method of characteristics, we write the integral equations associated to the kernel PDE-systems. These integral equations are solved using a fixed-point algorithm. The numerical values used are: $\lambda_1 = \mu_1 = 1$, $\lambda_2 = 2$, $\mu_2 = 0.7$, $\sigma_1^+ = 1, \sigma_1^- = 0.8, \sigma_2^+ = 0.8, \sigma_2^- = 0.6, q_{11} = 0.2, q_{12} = 0.7, q_{21} = 0.4, q_{22} = 0.6, \rho_1 = 0.5, \rho_2 = 0.3$, $C_1 = (1 \quad 0)$, $C_2 = (0.5 \quad 1)$

$$A = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 \\ 0.3 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}.$$

These coefficients are chosen such that the ODE system and the PDEs system are independently unstable in open-loop (and remain so when interconnected). However, it can be shown after some numerical computations that Assumptions 1, 2 and 3 are satisfied. Also, an input delay of 0.25 was introduced in the control action to show the robustness of the design to small delays in the loop. We have pictured in Figure 2 the evolution of the χ-norm of the closed-loop system without any delay and with an input delay of 0.25. As expected, it exponentially converges to zero and is robust to delays. We have not plotted the open-loop behavior since it was diverging extremely fast. Note that there is a transient phase which is due to the control design we have chosen, that require to have old values of the ODE state available to obtain the control law. The control effort has been plotted in Figure 3.

VII. Concluding remarks

In this paper we have designed a full-state feedback control law for the stabilization of an interconnected hyperbolic PDE-ODE-PDE system. The proposed approach is based on an original backstepping transformation that allows us to reformulate the stabilization problem in terms of a time-delay system of neutral type. We have given a necessary and sufficient robust stabilizability condition for the considered interconnection. Future works will attempt to generalize the proposed results to arbitrary cascade networks.

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