Towards a noncommutative Picard-Vessiot theory.

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Abstract

A Chen generating series, along a path and with respect to m differential forms, is a noncommutative series on m letters and with coefficients which are holomorphic functions over a simply connected manifold in other words a series with variable (holomorphic) coefficients. Such a series satisfies a first order noncommutative differential equation which is considered, by some authors, as the universal differential equation, *i.e.*, in this case, universality can be seen by replacing each letter by constant matrices (resp. holomorphic vector fields) and then solving a system of linear (resp. nonlinear) differential equations.

Via rational series, on noncommutative indeterminates and with coefficients in rings, and their non-trivial combinatorial Hopf algebras, we give the first step of a noncommutative Picard-Vessiot theory and we illustrate it with the case of linear differential equations with singular regular singularities thanks to the universal equation previously mentioned.

Contents

1 Introduction

Combinatorial Picard-Vessiot (PV for short) theory of bilinear systems [1] was realized by Fliess and Reutenauer [29], as an application of differential algebra [42,47]. This theory allows to employ, with success, linear algebraic groups in control theory (*i.e.* as symmetry groups of linear differential equations), for which some questions were solved thanks to the theory of Hopf algebras [11] and some combinatorial and effective aspects were set in [46].

Let us, for instance, consider the following nonlinear dynamical system

$$\dot{q}(z) = A_0(q)u_0(z) + \ldots + A_m(q)u_m(z), q(z_0) = q_0, y(z) = f(q(z)), \tag{1}$$

where

(i) y is the output,

(ii) the vector state $q = (q_1, \ldots, q_n)$ belongs to a complex holomorphic manifold $\mathscr{M}$ of dimension n,

(iii) the observation f is defined within a fixed connected neighbourhood [2] U of of the initial state q_0.

(iv) the vector fields $(A_i)_{i=0,\ldots,m}$ are defined with respect to the coordinates as follows

[1] Namely - locally - linear of the states $q_1, \ldots, q_N$ and linear of the inputs $u_0, \ldots, u_m$ [29].

[2] In this introductive description the points are loosely identified with their coordinates through some chart $\varphi_U : U \to \mathbb{C}^n$ likewise, in [45], the space of holomorphic functions $\mathscr{H}(U)$ is described by $\mathbb{C}^{\mathrm{cv}}[[q_1, \ldots, q_n]]$.

$$A_i = \sum_{j=1}^{n} A_i^j(q)\frac{\partial}{\partial q_j}, \text{with} A_i^j(q) \in \mathscr{H}(U), \tag{2}$$

(v) the inputs $(u_i)_{i=0,\ldots,m}$, as well as their inverses $(u_i^{-1})_{i=0,\ldots,m}$, belong to a subring, $\mathscr{C}_0$, of the ring of holomorphic functions $\mathscr{H}(\Omega)$ with neutral element $1_{\mathscr{H}(\Omega)}$ over the simply connected manifold [3] Ω.

It is convenient (and possible) to separate the contribution of the vector fields $(A_i)_{i=0,\ldots,m}$ and that of the differential forms $(\omega_i)_{i=0,\ldots,m}$, defined by the inputs $\omega_i(z) = u_i(z)dz$, through the encoding alphabet $X = \{x_i\}_{i=0,\ldots,m}$ which generates the monoid X^* with neutral element 1_{X^*}. Indeed, the output y can be computed by

$$y(z_0,z) = \langle C_{z_0\rightsquigarrow z} \| \sigma f_{|_{q_0}} \rangle = \sum_{w\in X^*} \alpha_{z_0}^z(w)\mathscr{Y}(w)[f]_{|_{q_0}}, \tag{3}$$

as the pairing (under suitable convergence conditions [31,34,36,45]) between the Chen series [4] of $(\omega_i)_{i=0,\ldots,m}$ along the path $z_0 \rightsquigarrow z$ over Ω, $C_{z_0\rightsquigarrow z} \in \mathscr{H}(\Omega)\langle\langle X\rangle\rangle$ [10], and the generating series of (1), $\sigma f_{|_{q_0}} \in \mathscr{H}(U)\langle\langle X\rangle\rangle$ [31], defined as follows

$$C_{z_0\rightsquigarrow z} := \sum_{w\in X^*} \alpha_{z_0}^z(w)w \text{ and } \sigma f_{|_{q_0}} := \sum_{w\in X^*} \mathscr{Y}(w)[f]_{|_{q_0}} w, \tag{4}$$

where, in (3)–(4), the iterated integral $\alpha_{z_0}^z(w)$ and the differential operator $\mathscr{Y}(w)$, are decoded, from the word $w \in X^*$, recursively as follows

$$\begin{cases} \alpha_{z_0}^z(w) = 1_{\mathscr{H}(\Omega)} & \text{and } \mathscr{Y}(w) = \mathrm{Id}, & \text{for } w = 1_{X^*}, \\ \alpha_{z_0}^z(w) = \displaystyle\int_{z_0}^{z} \omega_i(s)\alpha_{z_0}^s(v) & \text{and } \mathscr{Y}(w) = A_i \circ \mathscr{Y}(v), & \begin{array}{l}\text{for } w = x_i v,\\ x_i \in X, v \in X^*.\end{array} \end{cases} \tag{5}$$

In this work, following this route, considering the differential ring $(\mathscr{H}(\Omega), \partial)$ and equipping $\mathscr{H}(\Omega)\langle\langle X\rangle\rangle$ with the derivation defined, for any $S \in \mathscr{H}(\Omega)\langle\langle X\rangle\rangle$, by

$$\mathbf{d}S = \sum_{w\in X^*} (\partial\langle S \mid w\rangle)w, \tag{6}$$

we can see that the Chen series satisfies the following noncommutative differential equation

$$\mathbf{d}S = MS \text{ with } M = u_0x_0 + \ldots + u_mx_m, \tag{7}$$

considered by many authors as the universal differential equation [10,14,17,18,39]. Universality can be seen by specialization, *i.e.* replacing the letters by constant matrices (resp. holomorphic vector fields) and therefore obtaining linear (resp. nonlinear) differential equations (see Remark 4.9 below) as well as their solutions.

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[3] This (usually one dimensional) manifold will be the support of the iterated integrals below.

[4] By a Ree's theorem [44], there is a primitive series $L_{z_0\rightsquigarrow z} = \sum_{n\geq 1} L_n \in \widehat{\mathscr{H}(\Omega)\langle X\rangle}$ s.t. $e^{L_{z_0\rightsquigarrow z}} = C_{z_0\rightsquigarrow z}$, meaning that $C_{z_0\rightsquigarrow z}$ is group-like and L_n is (homogenous of degree $n \geq 1$) primitive series.

From equation (7), it follows (see, for example, [9]) that a PV theory of nonlinear systems (1) should be intimately connected with (7) (the reader may remark that, due to the connectedness of Ω, the constants of $(\mathscr{H}(\Omega)\langle\langle X\rangle\rangle, \mathbf{d})$ are

$$\mathrm{Const}(\mathscr{H}(\Omega)\langle\langle X\rangle\rangle) = \ker \mathbf{d} = \mathbb{C}.1_{\mathscr{H}(\Omega)}\langle\langle X\rangle\rangle. \tag{8}$$

This culminates with the fact that the coefficients of any suitable [5] solution is group-like, *i.e.* satisfies [6], for any $u,v \in X^*$ and $x_i \in X$,

$$\partial\langle S \mid x_i u\rangle = u_i\langle S \mid v\rangle \text{ and } \langle S \mid u \sqcup\!\sqcup\, v\rangle = \langle S \mid u\rangle\langle S \mid v\rangle \ ; \ \langle S \mid 1_{X^*}\rangle = 1_{\mathscr{H}(\Omega)} \tag{9}$$

Due to the fact that Ω is simply connected, the coordinate values of this series only depend on the endpoints and not on paths drawn on Ω. Denoting the subalgebra of $(\mathscr{H}(\Omega), \partial)$ generated by the family $(f_i)_{i\in I}$ and derivatives by $\mathbb{C}\{\{(f_i)_{i\in I}\}\}$ [49] (*i.e.* the differential algebra generated by $(f_i)_{i\in I}$), it follows that [29]

$$\mathrm{span}_{\mathbb{C}}\{\langle \mathbf{d}^l S \mid w\rangle\}_{w\in X^*, l\geq 0} \subset \mathrm{span}_{\mathbb{C}\{\{(u_i)_{i=0,..,m}\}\}}\{\langle S \mid w\rangle\}_{w\in X^*} \tag{10}$$

$$\subset \mathrm{span}_{\mathbb{C}\{\{(u_i^{\pm 1})_{i=0,..,m}\}\}}\{\langle S \mid w\rangle\}_{w\in X^*} \tag{11}$$

and then, in Section 4, the isomorphism between $\mathrm{span}_{\mathbb{C}\{\{(u_i^{\pm 1})_{i=0,..,m}\}\}}\{\alpha_{z_0}^{z}(w)\}_{w\in X^*}$ and $\mathbb{C}\{\{(u_i^{\pm 1})_{i=0,..,m}\}\} \otimes_{\mathbb{C}} \mathrm{span}_{\mathbb{C}}\{\alpha_{z_0}^{z}(w)\}_{w\in X^*}$ will be examined (Theorem 4.4) via the PV-extension related to (7) and, on the other hand, the output of (1) will be computed (Theorem 4.8) by pairing the series given in (4). As example, this calculation will be achieved according to the algebraic combinatorics of rational series, established beforehand in Sections 2 (Theorems 2.2, 2.4) and 3 (Theorems 3.2, 3.7).

2 Combinatorial framework

In this section, coefficients are taken in a commutative ring [7] A and, unless explicitly stated, all tensor products will be considered over the ambient ring (or field).

2.1 Factorization in bialgebras

In section 1, the encoding alphabet X was already introduced. In particular, for $m = 1$ (*i.e.* $X = \{x_0, x_1\}$), let us note that there are one-to-one correspondences

$$(s_1,\ldots,s_r) \in \mathbb{N}_+^r \leftrightarrow x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1 \underset{\pi_X}{\overset{\pi_Y}{\rightleftharpoons}} y_{s_1}\ldots y_{s_r} \in Y^*, \tag{12}$$

where $Y := \{y_k\}_{k\geq 1}$ and π_X is the $\texttt{conc}$ morphism, from $A\langle Y\rangle$ to $A\langle X\rangle$, mapping y_k to $x_0^{k-1}x_1$. This morphism π_X admits an adjoint π_Y for the two standard scalar

[5] *i.e.* group-like at one - interior or frontier - point.

[6] In the first identity, also called Friedrichs criterion, is involved the shuffle product ($\sqcup\!\sqcup$) [10,30,46].

[7] although some of the properties already hold for a general commutative semiring [1].

products[8] which has a simple combinatorial description: the restriction of π_Y to the subalgebra $(A1_{X^*} \oplus A\langle Y\rangle x_1, \mathtt{conc})$, is an isomorphism given by $\pi_Y(x_0^{k-1}x_1) = y_k$ (and the kernel of the non-restricted π_Y is $A\langle X\rangle x_0$). For all matters concerning finite (X and similar) or infinite (Y and similar) alphabets, we will use a generic model noted $\mathscr{X}$ in order to state their common combinatorial features. Let us recall also that the coproduct $\Delta_{\mathtt{conc}}$ is defined, for any $w \in \mathscr{X}^*$, as follows

$$\Delta_{\mathtt{conc}} w = \sum_{u,v \in \mathscr{X}^*, uv=w} u \otimes v. \tag{13}$$

As an algebra the A-module $A\langle \mathscr{X}\rangle$ is equipped with the associative unital concatenation and the associative commutative and unital shuffle product. The latter being defined, for any $x, y \in \mathscr{X}$ and $u, v, w \in \mathscr{X}^*$, by the following recursion

$$w \sqcup\!\sqcup 1_{\mathscr{X}^*} = 1_{\mathscr{X}^*} \sqcup\!\sqcup w = w \text{ and } xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v) \tag{14}$$

or, equivalently, by its dual comultiplication (which is a morphism for concatenations[9]), defined, for each letter $x \in \mathscr{X}$, by

$$\Delta_{\sqcup\!\sqcup} x = 1_{\mathscr{X}^*} \otimes x + x \otimes 1_{\mathscr{X}^*}. \tag{15}$$

Once $\mathscr{X}$ has been totally ordered[10], the set of Lyndon words over $\mathscr{X}$ will be denoted by $\mathscr{L}yn\mathscr{X}$. A pair of Lyndon words (l_1, l_2) is called the standard factorization of a Lyndon l (and will be noted $(l_1, l_2) = st(l)$) if $l = l_1 l_2$ and l_2 is the longest nontrivial proper right factor of l or, equivalently, its smallest such (for the lexicographic ordering, see [43] for proofs and details). According to a theorem by Radford, the set of Lyndon words form a pure transcendence basis of the A-shuffle algebras $(A\langle \mathscr{X}\rangle, \sqcup\!\sqcup, 1_{\mathscr{X}^*})$.

It is well known that the enveloping algebra $\mathscr{U}(\mathscr{L}ie_A\langle \mathscr{X}\rangle)$ is isomorphic to the (connected, graded and co-commutative) bialgebra[11] $\mathscr{H}_{\sqcup\!\sqcup}(\mathscr{X}) = (A\langle \mathscr{X}\rangle, \mathtt{conc}, 1_{\mathscr{X}^*}, \Delta_{\sqcup\!\sqcup}, \mathtt{e})$ (the counit being here $\mathtt{e}(P) = \langle P \mid 1_{\mathscr{X}^*}\rangle$) and, via the pairing

$$A\langle\langle \mathscr{X}\rangle\rangle \otimes_A A\langle \mathscr{X}\rangle \longrightarrow A, \tag{16}$$

$$T \otimes P \longrightarrow \langle T \mid P\rangle := \sum_{w \in \mathscr{X}^*} \langle T \mid w\rangle \langle P \mid w\rangle, \tag{17}$$

we can, classically, endow $A\langle \mathscr{X}\rangle$ with the graded[12] linear basis $\{P_w\}_{w \in \mathscr{X}^*}$ (expanded after any homogeneous basis $\{P_l\}_{l \in \mathscr{L}yn\mathscr{X}}$ of $\mathscr{L}ie_A\langle \mathscr{X}\rangle$) and its graded

[8] That is to say $(\forall p \in A\langle X\rangle)\ (\forall q \in A\langle Y\rangle)\ (\langle \pi_Y p \mid q\rangle_Y = \langle p \mid \pi_X q\rangle_X)$.

[9] On $A\langle \mathscr{X}\rangle$ and $A\langle \mathscr{X}\rangle \otimes A\langle \mathscr{X}\rangle$, respectively.

[10] For technical reasons, the orders $x_0 < x_1$ (for X) and $y_1 > \ldots y_n > y_{n+1} > \ldots$ (for Y) are usual.

[11] In case A is a $\mathbb{Q}$-algebra, the isomorphism $\mathscr{U}(\mathscr{L}ie_A\langle \mathscr{X}\rangle) \simeq \mathscr{H}_{\sqcup\!\sqcup}(\mathscr{X})$ can also be seen as an easy application of the CQMM theorem.

[12] For $\mathscr{X} = X$ or $= Y$ the corresponding monoids are equipped with length functions, for X we consider the length of words and for Y the length is given by the weight $\ell(y_{i_1} \ldots y_{i_n}) = i_1 + \ldots + i_n$. This naturally induces a grading of $A\langle \mathscr{X}\rangle$ and $\mathscr{L}ie_A\langle \mathscr{X}\rangle$ in free modules of finite dimensions. For

dual basis $\{S_w\}_{w\in\mathscr{X}^*}$ (containing the pure transcendence basis $\{S_l\}_{l\in\mathscr{L}yn\mathscr{X}}$ of the A-shuffle algebra). In the case when A is a $\mathbb{Q}$-algebra, we also have the following factorization [13] of the diagonal series, *i.e.* [46] (here all tensor products are over A)

$$\mathscr{D}_{\mathscr{X}} := \sum_{w\in\mathscr{X}^*} w\otimes w = \sum_{w\in\mathscr{X}^*} S_w\otimes P_w = \prod_{l\in\mathscr{L}yn\mathscr{X}}^{\searrow} e^{S_l\otimes P_l} \tag{18}$$

and (still in case A is a $\mathbb{Q}$-algebra) dual bases of homogenous polynomials $\{P_w\}_{w\in\mathscr{X}^*}$ and $\{S_w\}_{w\in\mathscr{X}^*}$ can be constructed recursively as follows

$$\begin{cases} P_x = x, & S_x = x & \text{for } x\in\mathscr{X}, \\ P_l = [P_{l_1}, P_{l_2}], & S_l = yS_{l'}, & \text{for } l = yl' \in \mathscr{L}yn\mathscr{X} - \mathscr{X} \\ & & \quad st(l) = (l_1, l_2), \\ P_w = P_{l_1}^{i_1}\ldots P_{l_k}^{i_k}, & S_w = \dfrac{S_{l_1}^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup S_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1!\ldots i_k!}, & \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, \text{ with } l_1,\ldots, \\ & & \quad l_k\in\mathscr{L}yn\mathscr{X}, l_1 > \ldots > l_k. \end{cases} \tag{19}$$

The graded dual of $\mathscr{H}_{\sqcup\!\sqcup}(\mathscr{X})$ is $\mathscr{H}_{\sqcup\!\sqcup}^{\vee}(\mathscr{X}) = (A\langle\mathscr{X}\rangle, \sqcup\!\sqcup, 1_{\mathscr{X}^*}, \Delta_{\mathtt{conc}}, \varepsilon)$.

As an algebra, the module $A\langle Y\rangle$ is also equipped with the associative commutative and unital quasi-shuffle product defined, for $u,v,w\in Y^*$ and $y_i, y_j\in Y$, by

$$w \uplus 1_{Y^*} = 1_{Y^*} \uplus w = w, \tag{20}$$

$$y_i u \uplus y_j v = y_i(u\uplus y_j v) + y_j(y_i u\uplus v) + y_{i+j}(u\uplus v). \tag{21}$$

This product also can be dualized according to ($y_k\in Y$)

$$\Delta_{\uplus} y_k := y_k\otimes 1_{Y^*} + 1_{Y^*}\otimes y_k + \sum_{i+j=k} y_i\otimes y_j \tag{22}$$

which is also a conc-morphism (see [28]). We then get another (connected, graded and co-commutative) bialgebra which, in case A is a $\mathbb{Q}$-algebra, is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements,

$$\mathscr{H}_{\uplus}(Y) = (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus}, \mathtt{e}) \cong \mathscr{U}(\mathrm{Prim}(\mathscr{H}_{\uplus}(Y))), \tag{23}$$

where $\mathrm{Prim}(\mathscr{H}_{\uplus}(Y)) = \mathrm{Im}(\pi_1) = \mathrm{span}_A\{\pi_1(w)|w\in Y^*\}$ and π_1 is the eulerian projector defined, for any $w\in Y^*$, by [37,38]

$$\pi_1(w) = w + \sum_{k=2}^{(w)} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1\uplus\ldots\uplus u_k\rangle u_1\ldots u_k, \tag{24}$$

and, for any $w = y_{i_i}\ldots y_{i_k}\in Y^*$, (w) denotes the number $i_i+\ldots+i_k$.

Remark 2.1 By (13) and (15), any letter $x\in\mathscr{X}$ is primitive, for $\Delta_{\mathtt{conc}}$ and $\Delta_{\sqcup\!\sqcup}$. By (22), the polynomials $\{\pi_1(y_k)\}_{k\geq 2}$ and only the letter y_1 are primitive, for $\Delta_{\uplus}$.

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general $\mathscr{X}$, we consider the fine grading [46] *i.e.* the grading by all partial degrees which, as well, induces a grading of $A\langle\mathscr{X}\rangle$ and $\mathscr{L}ie_A\langle\mathscr{X}\rangle$ in free modules of finite dimensions.

[13] Also called MSR factorization after the names of Mélançon, Schützenberger and Reutenauer.

Now, let $\{\Pi_w\}_{w\in Y^*}$ be the linear basis, expanded by decreasing Poincaré-Birkhoff-Witt (PBW for short) after any basis $\{\Pi_l\}_{l\in\mathscr{L}ynY}$ of $\mathrm{Prim}(\mathscr{H}_{\stuffle}(Y))$ homogeneous in weight [14], and let $\{\Sigma_w\}_{w\in Y^*}$ be its dual basis which contains the pure transcendence basis $\{\Sigma_l\}_{l\in\mathscr{L}ynY}$ of the A-quasi-shuffle algebra. One also has the factorization of the diagonal series $\mathscr{D}_Y$, on $\mathscr{H}_{\stuffle}(Y)$, which reads [15] [37,38,39]

$$\mathscr{D}_Y := \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathscr{L}ynY}^{\searrow} e^{\Sigma_l\otimes\Pi_l}. \tag{25}$$

We are now in the position to state the following

Theorem 2.2 ([38,39]) *Let A be a $\mathbb{Q}$-algebra, then the endomorphism of algebras $\varphi_{\pi_1} : (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}) \longrightarrow (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*})$ mapping y_k to $\pi_1(y_k)$, is an automorphism of $A\langle Y\rangle$ realizing an isomorphism of bialgebras between $\mathscr{H}_{\shuffle}(Y)$ and*

$$\mathscr{H}_{\stuffle}(Y) \cong \mathscr{U}(\mathrm{Prim}(\mathscr{H}_{\stuffle}(Y))).$$

In particular, it can be easily checked that the following diagram commutes

$$\begin{tikzcd}
A\langle Y\rangle \arrow[r, hook, "\Delta_{\shuffle}"] \arrow[d, "\varphi_{\pi_1}"'] & A\langle Y\rangle\otimes A\langle Y\rangle \arrow[d, "\varphi_{\pi_1}\otimes\varphi_{\pi_1}"] \\
A\langle Y\rangle \arrow[r, hook, "\Delta_{\stuffle}"] & A\langle Y\rangle\otimes A\langle Y\rangle
\end{tikzcd}$$

Moreover, the bases $\{\Pi_w\}_{w\in Y^*}$ and $\{\Sigma_w\}_{w\in Y^*}$ of $\mathscr{U}(\mathrm{Prim}(\mathscr{H}_{\stuffle}(Y)))$ are images by φ_{π_1} and by the adjoint mapping of its inverse, $\check{\varphi}_{\pi_1}^{-1}$ of $\{P_w\}_{w\in Y^*}$ and $\{S_w\}_{w\in Y^*}$, respectively.

Algorithmically, by Remark 2.1, the dual bases of homogenous polynomials $\{\Pi_w\}_{w\in Y^*}$ and $\{\Sigma_w\}_{w\in Y^*}$ can be constructed directly and recursively as follows

$$\begin{cases}
\Pi_{y_s} = \pi_1(y_s), & \Sigma_{y_s} = y_s \qquad\qquad \text{for } y_s\in Y,\\
\Pi_l = [\Pi_{l_1},\Pi_{l_2}], & \Sigma_l = \displaystyle\sum_{(*)} \frac{y_{s_{k_1}+\ldots+s_{k_i}}}{i!}\Sigma_{l_1\ldots l_n}, \quad \begin{array}{l}\text{for } l\in\mathscr{L}ynY-Y\\ \quad st(l)=(l_1,l_2),\end{array}\\
\Pi_w = \Pi_{l_1}^{i_1}\ldots\Pi_{l_k}^{i_k}, & \Sigma_w = \dfrac{\Sigma_{l_1}^{\stuffle i_1}\stuffle\ldots\stuffle\Sigma_{l_k}^{\stuffle i_k}}{i_1!\ldots i_k!}, \quad \begin{array}{l}\text{for } w=l_1^{i_1}\ldots l_k^{i_k}, \text{ with } l_1,\ldots,\\ \quad l_k\in\mathscr{L}ynY, l_1>\ldots>l_k.\end{array}
\end{cases} \tag{26}$$

In $(*)$, the sum is taken over all $\{k_1,\ldots,k_i\}\subset\{1,\ldots,k\}$ and $l_1\geq\ldots\geq l_n$ such that $(y_{s_1},\ldots,y_{s_k}) \stackrel{*}{\Leftarrow} (y_{s_{k_1}},\ldots,y_{s_{k_i}},l_1,\ldots,l_n)$, where $\stackrel{*}{\Leftarrow}$ denotes the transitive closure of

[14] Factorization (25) will be true in particular for the basis (26) explicitly constructed there.

[15] Again all tensor products will be taken over A. Note that this factorization holds for any enveloping algebra as announced in [46]. Of course, the diagonal series no longer exists and must be replaced by the identity $Id_{\mathscr{U}}$ (see [26], coda for details).

the relation on standard sequences, denoted by $\Leftarrow$ [7,46].

To end this section, let us extend $\mathtt{conc}$ and $\sqcup\!\sqcup$, for any series $S,R \in A\langle\langle \mathscr{X} \rangle\rangle$, by

$$SR = \sum_{w \in \mathscr{X}^*} \left(\sum_{u,v \in \mathscr{X}^*, uv=w} \langle S \mid u \rangle \langle R \mid v \rangle \right) w, \tag{27}$$

$$S \sqcup\!\sqcup R = \sum_{u,v \in \mathscr{X}^*} \langle S \mid u \rangle \langle R \mid v \rangle u \sqcup\!\sqcup v, \tag{28}$$

and $\uplus$, for any series $S,R \in A\langle\langle Y \rangle\rangle$, by

$$S \uplus R = \sum_{u,v \in Y^*} \langle S \mid u \rangle \langle R \mid v \rangle u \uplus v. \tag{29}$$

Let us also extend the coproduct $\Delta_{\uplus}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{\sqcup\!\sqcup}$) given in (22) (resp. (13) and (15)) over $A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathscr{X} \rangle\rangle$), for any series $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathscr{X} \rangle\rangle$), by linearity as follows

$$\Delta_{\uplus} S = \sum_{w \in Y^*} \langle S \mid w \rangle \Delta_{\uplus} w \in A\langle\langle Y^* \otimes Y^* \rangle\rangle, \tag{30}$$

$$\Delta_{\sqcup\!\sqcup} S = \sum_{w \in \mathscr{X}^*} \langle S \mid w \rangle \Delta_{\sqcup\!\sqcup} w \in A\langle\langle \mathscr{X}^* \otimes \mathscr{X}^* \rangle\rangle, \tag{31}$$

$$\Delta_{\mathtt{conc}} S = \sum_{w \in \mathscr{X}^*} \langle S \mid w \rangle \Delta_{\mathtt{conc}} w \in A\langle\langle \mathscr{X}^* \otimes \mathscr{X}^* \rangle\rangle. \tag{32}$$

The series S is said to be

(i) group like, for $\Delta_{\mathtt{conc}}$, if $\langle S \mid 1_{\mathscr{X}^*} \rangle = 1$ and $\Delta_{\mathtt{conc}} S = S \otimes S$,

(ii) primitive, for $\Delta_{\mathtt{conc}}$, if $\Delta_{\mathtt{conc}} S = S \otimes 1_{\mathscr{X}^*} + 1_{\mathscr{X}^*} \otimes S$.

Similarly for $\Delta_{\sqcup\!\sqcup}, \Delta_{\uplus}$ and then, letting $S \in A\langle\langle \mathscr{X} \rangle\rangle$ (resp. $A\langle\langle Y \rangle\rangle$), the Ree's theorem express that, for $\Delta_{\sqcup\!\sqcup}$ (resp. $\Delta_{\uplus}$), [44,46] (resp. [37,38])

$$S \text{ is primitive} \iff e^S \text{ is group like}, \tag{33}$$

$$\iff e^S \text{ satisfies the Friedrichs criterion}, \tag{34}$$

i.e it satisfies, for any $u,v \in \mathscr{X}^*$ (resp. Y^*), [44,46] (resp. [37,38])

$$\langle e^S \mid u \sqcup\!\sqcup v \rangle = \langle e^S \mid u \rangle \langle e^S \mid v \rangle \text{ (resp. } \langle e^S \mid u \uplus v \rangle = \langle e^S \mid u \rangle \langle e^S \mid v \rangle). \tag{35}$$

Or equivalently,

$$\langle \Delta_{\sqcup\!\sqcup} e^S \mid u \otimes v \rangle = \langle e^S \mid u \rangle \langle e^S \mid v \rangle \text{ (resp. } \langle \Delta_{\uplus} e^S \mid u \otimes v \rangle = \langle e^S \mid u \rangle \langle e^S \mid v \rangle). \tag{36}$$

We are going to see how all these combinatorics will operate over rational series and will be suitable, as illustration, to describe solutions of linear differential equations in Section 4 (see Theorems 4.4 and 4.8 bellow).

2.2 Representative series

Representative (or rational) series are the representative functions on the free monoid [16] [22] and their magic is that it rests on four (apparently distant) pillars:

- Separated coproduct (SC) [17] ,
- Finite orbit by shifts (FS),
- Result of a rational expression (RE),
- Linear representation (LR).

We first define what shifts, for (FS), and the Kleene star, for (RE) are, and then state the equivalence:

Definition 2.3 Let $S \in A\langle\langle \mathscr{X} \rangle\rangle$ (resp. $A\langle \mathscr{X} \rangle$) and $P \in A\langle \mathscr{X} \rangle$ (resp. $A\langle\langle \mathscr{X} \rangle\rangle$).

(i) The *left* (resp. *right*) *shift* [18] of S by P, is $P \triangleright S$ (resp. $S \triangleleft P$) defined by [19]

$$\forall w \in \mathscr{X}^*, \langle P \triangleright S \mid w \rangle = \langle S \mid wP \rangle \text{ (resp. } \langle S \triangleleft P \mid w \rangle = \langle S \mid Pw \rangle).$$

(ii) For any $S \in A\langle\langle \mathscr{X} \rangle\rangle$ such that $\langle S \mid 1_{\mathscr{X}^*} \rangle = 0$, the Kleene star of S is defined as [20] $S^* = (1-S)^{-1}$.

(iii) In case $A = K$ is a field, one can define also the Sweedler's dual $\mathscr{H}^{\circ}_{\sqcup\sqcup}(\mathscr{X})$ of $\mathscr{H}_{\sqcup\sqcup}(\mathscr{X})$ by $S \in \mathscr{H}^{\circ}_{\sqcup\sqcup}(\mathscr{X}) \iff \Delta_{\mathtt{conc}}(S) = \sum_{i \in I} G_i \otimes D_i$ [46], for some I finite, $\{G_i\}_{i \in I}; \{D_i\}_{i \in I}$ being series (as a matter of fact, it can be shown that they even can been choosen in $\mathscr{H}^{\circ}_{\sqcup\sqcup}(\mathscr{X})$, see [19,39])

Theorem 2.4 ([20,22,35,46]) *For $S \in A\langle\langle \mathscr{X} \rangle\rangle$, the following assertions are equivalent* [21]

(i) *The shifts $\{S \triangleleft w\}_{w \in \mathscr{X}^*}$ (resp. $\{w \triangleright S\}_{w \in \mathscr{X}^*}$) lie in a finitely generated shift-*

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[16] These functions were considered on groups in [11,12].

[17] Uniquely for fields.

[18] Some schools (as Jacob one, see [40,32]) used to call this a *residual*. These actions are none other than the shifts of functions of harmonic analysis.

[19] They are associative, commute with each other: $S \triangleleft (PR) = (S \triangleleft P) \triangleleft R, P \triangleright (R \triangleright S) = (P.R) \triangleright S$ and $(P \triangleleft S) \triangleright R = P \triangleleft (S \triangleright R)$ and, for $x, y \in \mathscr{X}, w \in \mathscr{X}^*$, $x \triangleright (wy) = (yw) \triangleleft x = \delta_x^y w$ (Kronecker delta).

[20] Using one of the topologies of section 4.2 (adapted with A replacing $\mathscr{H}(\Omega)$), we have $S^* = \sum_{n \geq 0} S^n$. We also get the fact that the space $\widehat{A.\mathscr{X}}$ (used below) of series of degree 1, *i.e.* the set $\{\sum_{x \in \mathscr{X}} \alpha(x) x\}_{\alpha \in A^X}$ is the closure of the A-module $A.\mathscr{X}$ generated by letters. In the case of a finite alphabet however (here $\mathscr{X} = X$) [22], $\widehat{A.\mathscr{X}} = A.\mathscr{X}$.

[21] When A is noetherian, first condition is equivalent to the fact that the module generated by $\{S \triangleleft w\}_{w \in \mathscr{X}^*}$ (resp. $\{w \triangleright S\}_{w \in \mathscr{X}^*}$) is finitely generated (and more precisely, in this case, by a finite number of those shifts). Unfortunately we are not in this case here, but our ring being without zero divisors (holomorphic functions), we can use the fraction field, here being realized by germs [15].

invariant A-module [22] .

(ii) *The series S belongs to the (algebraic) closure of $\widehat{A.\mathcal{X}}$ by the operations $\{\mathtt{conc},+,*\}$ (within $A\langle\langle\mathcal{X}\rangle\rangle$).*

(iii) *There is a linear representation (ν,μ,η), of rank n, for S with $\nu\in M_{1,n}(A)$, $\eta\in M_{n,1}(A)$ and a morphism of monoids $\mu:\mathcal{X}^*\to M_{n,n}(A)$ such that*
$$S=\sum_{w\in\mathcal{X}^*}(\nu\mu(w)\eta)w.$$

A series satisfying one of the conditions of Theorem 2.4 is called *rational*. The set of these series, a A-module [23] , is denoted by $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ and is closed by $\{\mathtt{conc},+,*\}$. We also have

Proposition 2.5 (see also [21,40]) *The module $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$) is closed by $\sqcup\!\sqcup$ (resp. $\uplus$). Moreover, for $i=1,2$, let $R_i\in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ and (ν_i,μ_i,η_i) be its representation of dimension n_i. Then the linear representation of* [24]

$$R_i^* \text{ is } \left(\begin{pmatrix}0 & 1\end{pmatrix},\left\{\begin{pmatrix}\mu_i(x)+\eta_i\nu_i\mu_i(x) & 0\\ \nu_i\eta_i & 0\end{pmatrix}\right\}_{x\in\mathcal{X}},\begin{pmatrix}\eta_i\\ 1\end{pmatrix}\right),$$

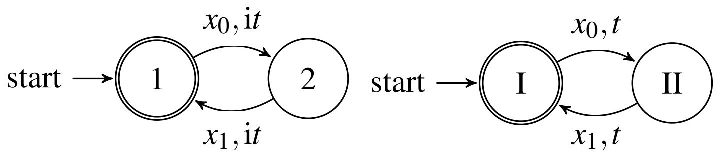
$$\text{that of } R_1+R_2 \text{ is } \left(\begin{pmatrix}\nu_1 & \nu_2\end{pmatrix},\left\{\begin{pmatrix}\mu_1(x) & \mathbf{0}\\ \mathbf{0} & \mu_2(x)\end{pmatrix}\right\}_{x\in\mathcal{X}},\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right),$$

$$\text{that of } R_1.R_2 \text{ is } \left(\begin{pmatrix}\nu_1 & 0\end{pmatrix},\left\{\begin{pmatrix}\mu_1(x) & \eta_1\nu_2\mu_2(x)\\ 0 & \mu_2(x)\end{pmatrix}\right\}_{x\in\mathcal{X}},\begin{pmatrix}\eta_1\mu_2\eta_2\\ \eta_2\end{pmatrix}\right),$$

$$\text{that of } R_1\sqcup\!\sqcup R_2 \text{ is } (\nu_1\otimes\nu_2,\{\mu_1(x)\otimes\mathrm{I}_{n_2}+\mathrm{I}_{n_1}\otimes\mu_2(x)\}_{x\in\mathcal{X}},\eta_1\otimes\eta_2),$$

$$\text{that of } R_1\uplus R_2 \text{ is } (\nu_1\otimes\nu_2,\{\mu_1(y_k)\otimes\mathrm{I}_{n_2}+\mathrm{I}_{n_1}\otimes\mu_2(y_k)$$
$$+\sum_{i+j=k}\mu_1(y_i)\otimes\mu_2(y_j)\}_{k\geq1},\eta_1\otimes\eta_2).$$

Example 2.6 [Identity $(-t^2x_0x_1)^*\sqcup\!\sqcup(t^2x_0x_1)^*=(-4t^4x_0^2x_1^2)^*$, [34,35]]



$$(-t^2x_0x_1)^*\leftrightarrow(\nu_2,\{\mu_2(x_0),\mu_2(x_1)\},\eta_2)\ \ (t^2x_0x_1)^*\leftrightarrow(\nu_1,\{\mu_1(x_0),\mu_1(x_1)\},\eta_1).$$

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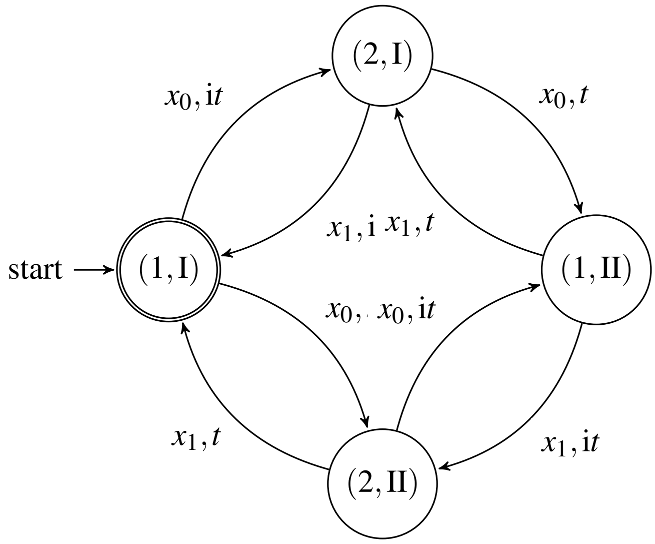
[22] see [41].

[23] In fact (we will see it) a unital A-algebra for $\mathtt{conc}$ and $\sqcup\!\sqcup$.

[24] The first constructions are already treated in [21,40], only the last one is new.

$$\nu_1 = \begin{pmatrix} 1 & 0 \end{pmatrix},\ \mu_1(x_0) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix},\ \mu_1(x_1) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix},\ \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\nu_2 = \begin{pmatrix} 1 & 0 \end{pmatrix},\ \mu_2(x_0) = \begin{pmatrix} 0 & \mathrm{i}t \\ 0 & 0 \end{pmatrix},\ \mu_2(x_1) = \begin{pmatrix} 0 & 0 \\ \mathrm{i}t & 0 \end{pmatrix},\ \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\begin{aligned}(-t^2x_0x_1)^* ⧢ (t^2x_0x_1)^* \leftrightarrow\ & (\nu, \{\mu(x_0), \mu(x_1)\}, \eta) \\ =\ & (\nu_1 \otimes \nu_2, \{\mu_1(x_0) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_0), \\ & \mu_1(x_1) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_1), \eta_1 \otimes \eta_2).\end{aligned}$$



$$\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},\quad \begin{aligned} \mu(x_0) &= \begin{pmatrix} 0&0&t&0 \\ 0&0&0&t \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&\mathrm{i}t&0&0 \\ 0&0&0&0 \\ 0&0&0&\mathrm{i}t \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&\mathrm{i}t&t&0 \\ 0&0&0&t \\ 0&0&0&\mathrm{i}t \\ 0&0&0&0 \end{pmatrix}, \\ \mu(x_1) &= \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ t&0&0&0 \\ 0&t&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0 \\ \mathrm{i}t&0&0&0 \\ 0&0&0&0 \\ 0&0&\mathrm{i}t&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ \mathrm{i}t&0&0&0 \\ t&0&0&0 \\ 0&t&\mathrm{i}t&0 \end{pmatrix}, \end{aligned} \quad \eta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

With the notations of Definition 2.3.(iii) and from Theorem 2.4, it follows that

Proposition 2.7 *Suppose A to be a field K. We have*

(a) *Assertions of Theorem 2.4 are equivalent to*

  (iv) *There exists a finite double family of series $(G_i, D_i)_{i \in F}$ such that*[25]

$$\Delta_{\mathtt{conc}} S = \sum_{i \in F} G_i \otimes D_i$$

(b) *For $S \in \mathscr{H}^{\circ}_{\shuffle}(\mathscr{X})$, since A is a field then the previous identity is equivalent to*

$$\forall P, Q \in \mathscr{H}_{\shuffle}(\mathscr{X}),\ \langle S \mid PQ \rangle = \sum_{i \in I} \langle G_i \mid P \rangle \langle D_i \mid Q \rangle.$$

Therefore, $(K^{\mathrm{rat}}\langle\langle \mathscr{X} \rangle\rangle, \shuffle, 1_{\mathscr{X}^*}, \Delta_{\mathtt{conc}}, \mathtt{e})$ (resp. $(K^{\mathrm{rat}}\langle\langle Y \rangle\rangle, \stuffle, 1_{\mathscr{X}^*}, \Delta_{\mathtt{conc}}, \mathtt{e})$) is the Sweedler's dual of $\mathscr{H}_{\shuffle}(\mathscr{X})$ (resp. $\mathscr{H}_{\stuffle}(Y)$).

Now, let us characterize characters of $(A\langle X \rangle, \mathtt{conc}, 1_{X^*})$.

Proposition 2.8 (Kleene stars of the plane) *Let $R \in A^{\mathrm{rat}}\langle\langle \mathscr{X} \rangle\rangle$, $\langle R \mid 1_{\mathscr{X}^*} \rangle = 1_A$. The following assertions are equivalent*

(i) *$\langle R \mid \bullet \rangle$ realizes a character*[26] *of $(A\langle X \rangle, \mathtt{conc}, 1_{X^*})$.*

(ii) *There is a family of coefficients $(c_x)_{x \in \mathscr{X}}$ such that $R = (\sum_{x \in \mathscr{X}} c_x x)^*$.*

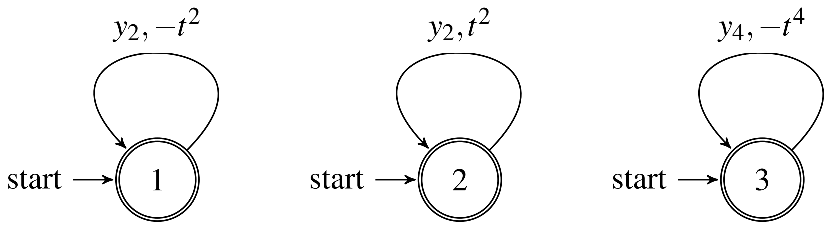
(iii) *The series R admits a linear representation of dimension one*[27].

Moreover, we have[28]

$$(\alpha_0 x_0 + \alpha_1 x_1)^* \shuffle (\beta_0 x_0 + \beta_1 x_1)^* = ((\alpha_0 + \beta_0) x_0 + (\alpha_1 + \beta_1) x_1)^*$$
$$\left(\sum_{s \geq 1} a_s y_s\right)^* \stuffle \left(\sum_{s \geq 1} b_s y_s\right)^* = \left(\sum_{s \geq 1} (a_s + b_s) y_s + \sum_{r,s \geq 1} a_s b_r y_{s+r}\right)^*,$$

where, for any $i = 0, 1$ and $s \geq 1$, $\alpha_i, \beta_i, a_s, b_s \in \mathbb{C}$.

Example 2.9 [Identity $(-t^2 y_2)^* \stuffle (t^2 y_2)^* = (-4t^4 y_4)^*$, [34,35]]



$$(-t^2 y_2)^* \leftrightarrow (\nu_2, \mu_2(y_2), \eta_2) = (1, -t^2, 1), \quad (t^2 y_2)^* \leftrightarrow (\nu_1, \mu_1(y_2), \eta_1) = (1, t^2, 1), \quad (-t^4 y_4)^* \leftrightarrow (\nu, \mu(y_4), \eta) = (1, -t^4, 1).$$

---

[25] See [39] for a way to obtain this finite double family of series $(G_i, D_i)_{i \in F}$.

[26] For $A = K$ being a field, this can be rephrased as "R is a group like element of $K^{\mathrm{rat}}\langle\langle \mathscr{X} \rangle\rangle$".

[27] The dimension is here (as in [1]) the size of the matrices.

[28] In particular, $(a_s y_s)^* \stuffle (a_r y_r)^* = (a_s y_s + a_r y_r + a_s a_r y_{s+r})^*$ and $(a_s y_s)^* \stuffle (-a_s y_s)^* = (-a_s^2 y_{2s})^*$.

3 Triangularity, solvability and rationality

3.1 Syntactically exchangeable rational series

Now, we have to study a special set of series in order to work with the rational series of this class: a series $S \in A\langle\langle \mathscr{X} \rangle\rangle$ is called *syntactically exchangeable* if and only if it is constant on multi-homogeneous classes, *i.e.*

$$(\forall u, v \in \mathscr{X}^*)([(\forall x \in \mathscr{X})(|u|_x = |v|_x)] \Rightarrow \langle S \mid u\rangle = \langle S \mid v\rangle). \tag{37}$$

A series $S \in A\langle\langle \mathscr{X} \rangle\rangle$ is syntactically exchangeable iff it is of the following form

$$S = \sum_{\alpha \in \mathbb{N}^{(\mathscr{X})}, \mathrm{supp}(\alpha) = \{x_1, \ldots, x_k\}} s_\alpha, x_1^{\alpha(x_1)} \sqcup\!\sqcup \ldots \sqcup\!\sqcup x_k^{\alpha(x_k)}. \tag{38}$$

The set of these series, a shuffle subalgebra of $A\langle\langle X \rangle\rangle$, will be denoted $A_{\mathrm{exc}}^{\mathrm{synt}}\langle\langle \mathscr{X} \rangle\rangle$.

When A is a field, the rational and exchangeable series are exactly those who admit a representation with commuting matrices (at least the minimal one is such, see Theorem 3.2 below). We will take this as a definition as, even for rings, this property implies syntactic exchangeability.

Definition 3.1 A series $S \in A^{\mathrm{rat}}\langle\langle \mathscr{X} \rangle\rangle$ will be called *rationally exchangeable* if it admits a representation (ν, μ, η) such that $\{\mu(x)\}_{x \in \mathscr{X}}$ is a set of commuting matrices, the set of these series, a shuffle subalgebra of $A\langle\langle X \rangle\rangle$, will be denoted $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle \mathscr{X} \rangle\rangle$.

Theorem 3.2 (See [24,39]) *Let $A_{\mathrm{exc}}^{\mathrm{synt}}\langle\langle \mathscr{X} \rangle\rangle$ denote the set of (syntactically) exchangeable series. Then*

(i) *In all cases, one has $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle \mathscr{X} \rangle\rangle \subset A^{\mathrm{rat}}\langle\langle \mathscr{X} \rangle\rangle \cap A_{\mathrm{exc}}^{\mathrm{synt}}\langle\langle \mathscr{X} \rangle\rangle$. The equality holds when A is a field and*

$$A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X \rangle\rangle = A^{\mathrm{rat}}\langle\langle x_0 \rangle\rangle \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle x_1 \rangle\rangle = \mathop{\sqcup\!\sqcup}_{x \in X} A^{\mathrm{rat}}\langle\langle x \rangle\rangle,$$

$$A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y \rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y \rangle\rangle = \bigcup_{k \geq 0} A^{\mathrm{rat}}\langle\langle y_1 \rangle\rangle \sqcup\!\sqcup \ldots \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle y_k \rangle\rangle \subsetneq A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y \rangle\rangle,$$

where $A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y \rangle\rangle = \cup_{F \subset_{finite} Y} A^{\mathrm{rat}}\langle\langle F \rangle\rangle$, the algebra of series over finite subalphabets[29].

(ii) *(Kronecker's theorem [1,51]) One has $A^{\mathrm{rat}}\langle\langle x \rangle\rangle = \{P(1 - xQ)^{-1}\}_{P,Q \in A[x]}$ (for $x \in \mathscr{X}$) and if $A = K$ is an algebraically closed field of characteristic zero one*

[29] The last inclusion is strict as shows the example of the following identity [6]

$$(ty_1 + t^2 y_2 + \ldots)^* = \lim_{k \to +\infty} (ty_1 + \ldots + t^k y_k)^* = \lim_{k \to +\infty} (ty_1)^* \sqcup\!\sqcup \ldots \sqcup\!\sqcup (t^k y_k)^* = \mathop{\sqcup\!\sqcup}_{k \geq 1} (t^k y_k)^*$$

which lives in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y \rangle\rangle$ but not in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y \rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y \rangle\rangle$.

also has $K^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_K\{(ax)^* \sqcup\!\sqcup K\langle x\rangle | a \in K\}$.

(iii) *The rational series* $(\sum_{x\in\mathscr{X}} \alpha_x x)^*$ *are* conc*-characters and any* conc*-character is of this form.*

(iv) *Let us suppose that* A *is without zero divisors and let* $(\varphi_i)_{i\in I}$ *be a family within* $\widehat{A\mathscr{X}}$ *which is* $\mathbb{Z}$*-linearly independent then, the family* $\mathscr{L}yn(\mathscr{X}) \uplus \{\varphi_i^*\}_{i\in I}$ *is algebraically free over* A *within* $(A^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathscr{X}^*})$.

(v) *In particular, if* A *is a ring without zero divisors* $\{x^*\}_{x\in\mathscr{X}}$ *(resp.* $\{y^*\}_{y\in Y}$*) are algebraically independent over* $(A\langle\mathscr{X}\rangle, \sqcup\!\sqcup, 1_{\mathscr{X}^*})$ *(resp.* $(A\langle Y\rangle, \uplus, 1_{Y^*})$*) within* $(A^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathscr{X}^*})$ *(resp.* $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{Y^*})$*).*

Proof.

(i) The inclusion is obvious in view of (38). For the equality, it suffices to prove that, when A is a field, every rational and exchangeable series admits a representation with commuting matrices. This is true of any minimal representation as shows the computation of shifts (see [20,24,39]).

Now, if $\mathscr{X}$ is finite, as all matrices commute, we have

$$\sum_{w\in\mathscr{X}^*} \mu(w)w = \Big(\sum_{x\in\mathscr{X}} \mu(x)x\Big)^* = \underset{x\in\mathscr{X}}{\sqcup\!\sqcup} (\mu(x)x)^*$$

and the result comes from the fact that R is a linear combination of matrix elements. As regards the second equality, inclusion $\supset$ is straightforward. We remark that the union $\bigcup_{k\geq 1} A^{\mathrm{rat}}\langle\langle y_1\rangle\rangle \sqcup\!\sqcup \ldots \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle y_k\rangle\rangle$ is directed as these algebras are nested in one another. With this in view, the reverse inclusion comes from the fact that every $S \in A^{\mathrm{rat}}_{\mathrm{fin}}\langle\langle Y\rangle\rangle$ is a series over a finite alphabet and the result follows from the first equality.

(ii) Let $\mathscr{A} = \{P(1-xQ)^{-1}\}_{P,Q\in A[x]}$. Since $P(1-xQ)^{-1} = P(xQ)^*$ then it is obvious that $\mathscr{A} \subset A^{\mathrm{rat}}\langle\langle x\rangle\rangle$. Next, it is easy to check that $\mathscr{A}$ contains $A\langle x\rangle (= A[x])$ and it is closed by $+$, conc as, for instance,

$$(1-xQ_1)(1-xQ_2) = (1-x(Q_1+Q_2-xQ_1Q_2)).$$

We also have to prove that $\mathscr{A}$ is closed for $*$. For this to be applied to $P(1-xQ)^{-1}$, we must suppose that $P(0) = 0$ (as, indeed, $\langle P(1-xQ)^{-1} \mid 1_{x^*}\rangle = P(0)$) and, in this case, $P = xP_1$. Now

$$\Big(\frac{P}{1-xQ}\Big)^* = \Big(1-\frac{P}{1-xQ}\Big)^{-1} = \frac{1-xQ}{1-x(Q+P_1)} \in \mathscr{A}.$$

(iii) Let $S = (\sum_{x\in\mathscr{X}} \alpha_x x)^*$ and note that $S = 1 + (\sum_{x\in\mathscr{X}} \alpha_x, x)S$. Then $\langle S \mid 1_{\mathscr{X}^*}\rangle = 1_A$ and, if $w = xu$, we have $\langle S \mid xu\rangle = \alpha_x\langle S \mid u\rangle$, then by recurrence on the length, $\langle S \mid x_1 \ldots x_k\rangle = \prod_{i=1}^k \alpha_{x_i}$ which shows that S is a conc-character. For the converse, we have Schützenberger's reconstruction lemma which says that, for every series S

$$S = \langle S \mid 1_{\mathscr{X}^*}\rangle.1_A + \sum_{x\in\mathscr{X}} x.x^{-1}S$$

but, if S is a conc-character, $\langle S \mid 1_{\mathscr{X}^*}\rangle = 1$ and $x^{-1}S = \langle S \mid x\rangle S$, then the previous expression reads

$$S = 1_A + \left(\sum_{x\in\mathscr{X}} \langle S \mid x\rangle x\right) S$$

this last equality being equivalent to $S = (\sum_{x\in\mathscr{X}} \langle S \mid x\rangle.x)^*$ proving the claim.

(iv) As $(A\langle\mathscr{X}\rangle, ⧢, 1_{\mathscr{X}^*})$ and $(A\langle Y\rangle, \uplus, 1_{Y^*})$ are enveloping algebras, this property is an application of the fact that, on an enveloping $\mathscr{U}$, the characters are linearly independent w.r.t. to the convolution algebra $\mathscr{U}_\infty^*$ (see the general construction and proof in [25] or [27]). Here, this convolution algebra $(\mathscr{U}_\infty^*)$ contains the polynomials (is equal in case of finite $\mathscr{X}$). Now, consider a monomial

$$(\varphi_{i_1}^*)^{⧢\,\alpha_1} \dots (\varphi_{i_n}^*)^{⧢\,\alpha_n} = \left(\sum_{k=1}^{n} \alpha_{i_k}\varphi_{i_k}\right)^*$$

The $\mathbb{Z}$-linear independence of the monomials in $(\varphi_i)_{i\in I}$ implies that all these monomials are linearly independent over $A\langle\mathscr{X}\rangle$ which proves algebraic independence of the family $(\varphi_i)_{i\in I}$.

To end with, the fact that $\mathscr{L}yn(\mathscr{X}) \uplus \{\varphi_i^*\}_{i\in I}$ is algebraically free comes from Radford theorem $(A\langle\mathscr{X}\rangle, ⧢, 1_{\mathscr{X}^*}) \simeq A[\mathscr{L}yn(\mathscr{X})]$ and the transitivity of polynomial algebras (see [3] ch III.2 Proposition 8).

(v) Comes directly as an application of the preceding point.

□

Remark 3.3 (Point (ii) of Theorem 3.2 above) Kronecker's theorem which can be rephrased in terms of stars as $A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(xQ)^*\}_{P,Q\in A[x]}$ holds for every ring and is therefore characteristic free, unlike the shuffle version requiring algebraic closure and denominators.

3.2 Exchangeable rational series and their linear representations

As examples, one can consider the following forms $(F_0), (F_1)$ and (F_2) of rational series in $A^{\mathrm{rat}}\langle\langle X\rangle\rangle$ [33,39]:

(F_0) $E_1x_{i_1}\dots E_jx_{i_j}E_{j+1}$, where $x_{i_1},\dots,x_{i_j} \in X, E_1,\dots,E_j \in A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle$,

(F_1) $E_1x_{i_1}\dots E_jx_{i_j}E_{j+1}$, where $x_{i_1},\dots,x_{i_j} \in X, E_1,\dots,E_j \in A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$,

(F_2) $E_1x_{i_1}\dots E_jx_{i_j}E_{j+1}$, where $x_{i_1},\dots,x_{i_j} \in X, E_1,\dots,E_j \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$.

Using linear representations, we also have

Theorem 3.4 (Triangular sub bialgebras of $(A^{\rm rat}\langle\langle\mathcal{X}\rangle\rangle, ш, 1_{X^*}, \Delta_{\tt conc}, {\tt e})$, [39]) *Let $\rho = (\nu, \mu, \eta)$ a representation of $R \in A^{\rm rat}\langle\langle\mathcal{X}\rangle\rangle$. Then*

(i) *If the matrices $\{\mu(x)\}_{x\in\mathcal{X}}$ commute between themselves and if the alphabet is finite, every rational exchangeable series decomposes as*

$$R = \sum_{i=1}^{n} \mathop{ш}_{x\in\mathcal{X}} R_x^{(i)} \text{ with } R_x^{(i)} \in A^{\rm rat}\langle\langle x\rangle\rangle.$$

(ii) *If $\mathcal{L}$ consists of upper-triangular matrices then $R \in A^{\rm rat}_{\rm exc}\langle\langle\mathcal{X}\rangle\rangle ш A\langle\mathcal{X}\rangle$.*

(iii) *For any $x \in \mathcal{X}$, letting $M(x) := \mu(x)x$ and then extending, in the obvious way, this representation to $A\langle\langle\mathcal{X}\rangle\rangle$ by $M(S) = \sum_{w\in\mathcal{X}^*}\langle S \mid w\rangle\mu(w)w$, we have*

$$R = \nu M(\mathcal{X}^*)\eta.$$

Moreover, we have

(a) *If $\{\mu(x)\}_{x\in\mathcal{X}}$ are upper-triangular then $M(\mathcal{X}) = D(\mathcal{X}) + N(\mathcal{X})$, where $D(\mathcal{X})$ and $N(\mathcal{X})$ are diagonal and strictly upper-triangular letter matrices, respectively, such that* [30]

$$M(\mathcal{X}^*) = ((D(\mathcal{X}^*)N(\mathcal{X}))^* D(\mathcal{X}^*)).$$

(b) *We get* [31] *(for $\mathcal{X} = X$)*

$$M((x_0+x_1)^*) = (M(x_1^*)M(x_0))^* M(x_1^*) = (M(x_0^*)M(x_1))^* M(x_0^*)$$

and the modules generated by the families (F_0), (F_1) and (F_2) are closed by conc, $ш$ *(and coproducts if $A = K$ is a field). From this, it follows that R is a linear combination of expressions in the form (F_0) (resp. (F_1)) if $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is strictly upper-triangular.*

(c) *If A is a $\mathbb{Q}$-algebra then*

$$M(\mathcal{X}^*) = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l \mu(P_l)}.$$

Remark 3.5 (i) The point (i) of Theorem 3.4 is no longer true for an infinite alphabet as shows the example of the series $S = \sum_{k\geq 1} y_k$ in $A^{\rm rat}\langle\langle Y\rangle\rangle$.

(ii) On a general ring it can happen that R is exchangeable, ρ minimal and nevertheless $\mathcal{L}$ is noncommutative, as shows the case of $A = \mathbb{Q}[x,t]/t^3\mathbb{Q}[x,t]$ and

$$X = \{a, b\},\ \mu(a) = t\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},\ \mu(b) = t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ \nu = \begin{pmatrix} 1 & 1 \end{pmatrix},\ \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

[30] by Lazard factorization [43,50].

[31] *idem.*

With these data, $R=2+(xt+2t)(a+b)+(x^2t^2+2xt^2+2t^2)(ab+ba)$ which is an exchangeable polynomial but

$$\mu(a)\mu(b)=\begin{pmatrix} t^2 & xt^2 \\ xt^2 & x^2t^2+t^2 \end{pmatrix}, \mu(b)\mu(a)=\begin{pmatrix} x^2t^2+t^2 & xt^2 \\ xt^2 & t^2 \end{pmatrix}$$

Now the representation is minimal because if it were of dimension 1, $\frac{1}{2}R$ would be a conc-character, which is not the case. Otherwise, if it were of dimension 0, R would be zero.

In order to establish Theorem 3.7 below, we will use the following

Lemma 3.6 *Let (ν,τ,η) a representation of S of dimension r such that, for all $x\in\mathscr{X}$, $(\tau(x)-c(x)I_r)$ is strictly upper triangular, then $S\in K_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle \sqcup\!\sqcup K\langle\mathscr{X}\rangle$.*

Proof. Let $(e_i)_{1\le i\le r}$ be the canonical basis of $K^{1\times r}$. We construct the representations $\rho_1=(\nu,(x\mapsto\tau(x)-c(x)I_r),\eta)$, $\rho_2=(e_1,(x\mapsto c(x)I_r),e_1^*)$ of S_1 and S_2 and remark that $S_1\sqcup\!\sqcup S_2$ admits the representation

$$\rho_3=(\nu\otimes e_1,((\tau(x)-c(x)I_r)\otimes I_r+I_r\otimes c(x)I_r)_{x\in\mathscr{X}},\eta\otimes e_1^*)$$

as $I_r\otimes c(x)I_r=c(x)I_r\otimes I_r$, ρ_3 is, in fact, $(\nu\otimes e_1,(\tau(x)\otimes I_r)_{x\in\mathscr{X}},\eta\otimes e_1^*)$ which represents S, the result now comes from the fact that $S_1\in K\langle\mathscr{X}\rangle$ and $S_2=(\sum_{x\in\mathscr{X}}c(x)x)^*\in K_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle$. □

We first begin by properties essentially true over algebraically closed fields.

Theorem 3.7 (Triangular sub bialgebras of $(K^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle,\sqcup\!\sqcup,1_{X^*},\Delta_{\mathtt{conc}},\mathtt{e})$, [39]) *We suppose that K is an algebraically closed field and that $\rho=(\nu,\mu,\eta)$ is a linear representation of $R\in K^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle$ of minimal dimension n, we note $\mathscr{L}=\mathscr{L}(\mu)\subset K^{n\times n}$ the Lie algebra generated by the matrices $(\mu(x))_{x\in\mathscr{X}}$. Then*

(i) *$\mathscr{L}$ is commutative iff $R\in K_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle$,*
(ii) *$\mathscr{L}$ is nilpotent iff $R\in K_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle\sqcup\!\sqcup K\langle\mathscr{X}\rangle$,*
(iii) *$\mathscr{L}$ is solvable iff R is a linear combination of expressions in the form (F_2).*

Moreover, denoting $K_{\mathrm{nil}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle$ (resp. $K_{\mathrm{sol}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle$), the set of rational series such that $\mathscr{L}(\mu)$ is nilpotent (resp. solvable), we get a tower of sub Hopf algebras of the Sweedler's dual, $K_{\mathrm{nil}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle\subset K_{\mathrm{sol}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle\subset\mathscr{H}_{\sqcup\!\sqcup}^{\circ}(\mathscr{X})$.

Proof.

(i) Let us remark that, for $x,y\in\mathscr{X},p,s\in\mathscr{X}^*$, we have $\langle R\mid pxys\rangle=\langle R\mid pyxs\rangle$ which is due to the commutation of matrices. Conversely, since ρ is minimal then there is $P_i,Q_i\in K\langle\mathscr{X}\rangle,i=1\ldots n$ such that (see [1,20,48])

$$\forall u \in \mathscr{X}^*,\ \mu(u) = (\langle P_i \triangleright R \triangleleft Q_i \mid u\rangle)_{1\le i,j\le n} = (\langle R \mid Q_i u P_i\rangle)_{1\le i,j\le n}.$$

Now, for $x, y \in \mathscr{X}$, we have

$$\mu(xy) = (\langle R \mid Q_i xy P_i\rangle)_{1\le i,j\le n} \stackrel{*}{=} (\langle R \mid Q_i yx P_i\rangle)_{1\le i,j\le n} = \mu(yx)$$

equality $\stackrel{*}{=}$ being due to exchangeability.

(ii) Let us consider K^n as the space of the representation of $\mathscr{L}$ given by μ. Let $K^n = \bigoplus_{j=1}^m V_j$ be a decomposition of K^n into indecomposable $\mathscr{L}$-modules (see [16], Theorem 1.3.19 where it is done for $ch(K) = 0$, or [5] Chapter VII §1 Propopsition 9 for arbitrary characteristic), we know that each V_j is a $\mathscr{L}$-module and that the action of $\mathscr{L}$ is triangularizable with constant diagonals inside each sector V_j. Thus, it is an invertible matrix $P \in \mathrm{GL}(n, K)$ such that

$$\forall x \in \mathscr{X},\ P\mu(x)P^{-1} = \mathrm{blockdiag}(T_1, T_2 \dots, T_k) = \begin{pmatrix} T_1 & 0 & 0 & \dots & 0 \\ 0 & T_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & T_k \end{pmatrix}$$

where the T_j are upper triangular matrices with scalar diagonal *i.e.* is of the form $T_j(x) = \lambda(x)I + N(x)$ where $N(x)$ is strictly upper-triangular [32]. Set d_j to be the dimension of T_j (so that $n = \sum_{j=1}^m d_j$), partitioning $\nu P^{-1} = \nu'$ (resp. $P\eta = \eta'$) with these dimensions we get blocks so that each (ν'_j, T_j, η'_j) is the representation of a series R_j and $R = \sum_{j=1}^m R_j$. It suffices then to prove that, for all j, $R_j \in K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle \mathscr{X} \rangle\rangle \sqcup\!\sqcup K\langle \mathscr{X} \rangle$. This is a consequence of Lemma 3.6.

Conversely, if $\rho_i = (\nu_i, \tau_i, \eta_i), i = 1, 2$, are two representations then $[\tau_1(x) \otimes I_r + I_r \otimes \tau_2(x), \tau_1(y) \otimes I_r + I_r \otimes \tau_2(y)] = [\tau_1(x), \tau_1(y)] \otimes I_r + I_r \otimes [\tau_2(x), \tau_2(y)]$ and a similar formula holds for m-fold brackets (Dynkin combs), so that if $\mathscr{L}(\tau_i)$'s are nilpotent, the Lie algebra $\mathscr{L}(\tau_1 \otimes I_r + I_r \otimes \tau_2)$ is also nilpotent. The point here comes from the fact that series in $K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle \mathscr{X} \rangle\rangle$ as well as in $K\langle \mathscr{X} \rangle$ admit nilpotent representations, so, let (α, τ, β) such a representation and (α', τ', β') its minimal quotient (obtained by minimization, see [1]), then $\mathscr{L}(\tau')$ is nilpotent as a quotient of $\mathscr{L}(\tau)$. Now two minimal representations being isomorphic, $\mathscr{L}(\mu)$ is isomorphic to $\mathscr{L}(\tau)$ and then it is nilpotent.

(iii) As $\mathscr{L}$ is solvable and K algebraically closed, using Lie's theorem, we can find a conjugate form of $\rho = (\nu, \mu, \eta)$ such that the matrices $\mu(x)$ are upper-triangular. Since this form also represents R, letting $D(\mathscr{X})$ (resp. $N(\mathscr{X})$) be

[32] Even, as K is infinite, there is a global linear form on $\mathscr{L}$, λ_{lin} such that, for all $g \in \mathscr{L}$, $PgP^{-1} - \lambda_{lin}(g)I$ is strictly upper-triangular.

the diagonal (rep. strictly upper-triangular) letter matrice such that $M(\mathscr{X}) = D(\mathscr{X}) + N(\mathscr{X})$ then

$$R = \nu M(\mathscr{X}^*)\eta = \nu(D(\mathscr{X}^*)N(\mathscr{X}))^* D(\mathscr{X}^*)\eta.$$

Since $D(\mathscr{X}^*)N(\mathscr{X})$ being nilpotent of order n then $(D(\mathscr{X}^*)N(\mathscr{X}))^* = \sum_{j=0}^{n}(D(\mathscr{X}^*)N(\mathscr{X}))^j$. Hence, letting $\mathscr{S}$ be the vector space generated by forms of type (F_2) which is closed by concatenation, we have $D(\mathscr{X}^*)N(\mathscr{X}) \in \mathscr{S}^{n\times n}$ and then $(D(\mathscr{X}^*)N(\mathscr{X}))^* \in \mathscr{S}^{n\times n}$. Finally, $R = \nu M(\mathscr{X}^*)\eta \in \mathscr{S}$ which is the claim.

Conversely, as sums and quotients of solvable representations are solvable is suffices to show that a single form of type F_2 admits a solvable representation and end by quotient and isomorphism as in (ii). From Proposition (2.5), we get the fact that, if R_i admit solvable representations so does R_1R_2, then the claim follows from the fact that, firstly, single letters admit solvable (even nilpotent) representations and secondly series of $\sqcup\!\sqcup\{K^{\mathrm{rat}}\langle\langle x\rangle\rangle\}_{x\in\mathscr{X}}$ admit solvable representations. Finally, we choose (or construct) a solvable representation of R, call it (α,τ,β) and (α',τ',β') its minimal quotient, then $\mathscr{L}(\tau')$ is solvable as a quotient of $\mathscr{L}(\tau)$. Now two minimal representations being isomorphic, $\mathscr{L}(\mu)$ is isomorphic to $\mathscr{L}(\tau)$, hence solvable.

Moreover and ff.] Comes from the computation of the coproduct by insertion of identity $\sum_{i=1}^{n} e_i^* e_i$.

$\square$

Remark 3.8 For an example of series S with solvable representation but such that $S \notin K_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle \sqcup\!\sqcup K\langle\mathscr{X}\rangle$. One can take $\mathscr{X} = \{a,b\}$ and $S = a^*b(-a)^*$.

To end this section (of combinatorial framework), for a need of the proof of Theorem 4.8 below, let us extend the pairing (16) as a partially defined map

$$\mathrm{Dom}(\langle ?\|?\rangle) \longrightarrow A, \tag{39}$$

$$T\otimes S \longrightarrow \langle T\|S\rangle := \sum_{w\in\mathscr{X}^*}\langle T \mid w\rangle\langle S \mid w\rangle. \tag{40}$$

where $\mathrm{Dom}(\langle ?\|?\rangle) \subset A\langle\langle\mathscr{X}\rangle\rangle \otimes A\langle\langle\mathscr{X}\rangle\rangle$.

Here, the family $\sum_{w\in\mathscr{X}^*}\langle T \mid w\rangle\langle S \mid w\rangle$ is summable, for some topology on A. Its sum is denoted by $\langle T\|S\rangle$ and the set of these series S is denoted by $\mathrm{Dom}_{word}(T)$.

This proof will also use the following lemma as a consequence of Theorem 2.4

Lemma 3.9 *For any ring A without zero divisors, let $R \in A^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle$ of linear representation (ν,μ,η) of dimension n. Then any family $\{R\triangleleft P_i | P_i \in A\langle\mathscr{X}\rangle\}_{i=1\ldots m>n}$ is linearly dependent, i.e. there are $\{\alpha_i\}_{i=1\ldots m}$ in A, not all zero, such that $\sum_{I=1}^{m}\alpha_i(R\triangleleft P_i) = 0$.*

4 Towards a noncommutative Picard-Vessiot theory

Let $(\mathscr{A},d)$ be a commutative associative differential ring ($\ker(d)=k$ being a field), $\mathscr{C}_0$ be a differential subring of $\mathscr{A}$ ($d(\mathscr{C}_0)\subset\mathscr{C}_0$) which is an integral domain containing the field of constants and $\mathbb{C}\{\{(g_i)_{i\in I}\}\}$ be the differential subalgebra of $\mathscr{A}$ generated by $(g_i)_{i\in I}$, *i.e.* the k-algebra generated by g_i's and their derivatives [49].

4.1 Noncommutative differential equations

Let us consider the following differential equation, with homogeneous series of degree 1 as multiplier (a polynomial in the case of finite alphabet).

$$\mathbf{d}S=MS;\langle S\mid 1\rangle=1,\text{ where }M=\sum_{x\in\mathscr{X}}u_x x\in\mathscr{C}_0\langle\langle\mathscr{X}\rangle\rangle \tag{41}$$

Example 4.1 [Drinfel'd equation] $X=\{x_0,x_1\}$ and $\Omega=\mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$.

$$(KZ_3)\quad \mathbf{d}S=(x_0u_{x_0}+x_1u_{x_1})S\text{ with }u_{x_0}(z)=z^{-1},u_{x_1}(z)=(1-z)^{-1}.$$

This equation was introduced in [17,18] and a complete study was presented in [39] (solutions via polylogarithms and their special values, polyzetas).

Example 4.2 $Y=\{y_i\}_{i\geq1}$ and $\Omega=\{z\in\mathbb{C}\mid\,|z|<1\}$.

$$\mathbf{d}S=\left(\sum_{i\geq1}y_iu_{y_i}\right)S\text{ with }u_{y_i}(z)=\partial\ell_i(z).$$

where, denoting γ the Euler's constant and ζ the Riemann zeta function,

$$\ell_1(z):=\gamma z-\sum_{k\geq2}\zeta(k)\frac{(-z)^k}{k}\text{ and for }r\geq2,\ell_r(z):=-\sum_{k\geq1}\zeta(kr)\frac{(-z^r)^k}{k}.$$

This equation was introduced in [9] to study the independence of a family of eulerian functions.

Let us also recall the following useful result for proving Theorem 4.8 bellow.

Proposition 4.3 ([34,35,37]) *Let $S\in\mathscr{A}\langle\langle\mathscr{X}\rangle\rangle$ be solution of* (41). *Then S satisfies the differential equations $\mathbf{d}^lS=Q_lS$, for $l\geq0$, where $\mathbb{Q}_l\in\mathbb{C}\{\{(u_i)_{i\geq0}\}\}\langle\mathscr{X}\rangle$ satisfying the recursion $Q_0=1$ and $Q_l=Q_{l-1}M+\mathbf{d}Q_{l-1}$.*

More explicitly, Q_l can be computed as follows (suming over words $w=x_{i_1}\ldots x_{i_l}$ and derivation multi-indices $\mathbf{r}=(r_1,\ldots,r_l)$ of degree $\deg\mathbf{r}=|w|=l$ and of weight $\mathtt{wgt}\;\mathbf{r}=l+r_1+\ldots+r_l$)

$$Q_l=\sum_{\substack{\mathtt{wgt}\;\mathbf{r}=l\\ w\in\mathscr{X}^{\deg\mathbf{r}}}}\prod_{l=1}^{\deg\mathbf{r}}\binom{\sum_{j=1}^{l}r_j+j-1}{r_l}\tau_{\mathbf{r}}(w)\;and\;\begin{cases}\tau_{\mathbf{r}}(w)=\tau_{r_1}(x_{i_1})\ldots\tau_{r_l}(x_{i_l})=\\ (\partial_z^{r_1}u_{x_{i_1}})x_{i_1}\ldots(\partial_z^{r_l}u_{x_{i_l}})x_{i_l}.\end{cases}$$

Theorem 4.4 *Suppose that the $\mathbb{C}$-commutative ring $\mathscr{A}$ is without zero divisors and equipped with a differential operator ∂ such that $\mathbb{C} = \ker \partial$.*

Let $S \in \mathscr{A}\langle\langle\mathscr{X}\rangle\rangle$ be a group-like solution of (41), in the following form

$$S = 1_{\mathscr{X}^*} + \sum_{w \in \mathscr{X}^*\mathscr{X}} \langle S \mid w \rangle w = 1_{\mathscr{X}^*} + \sum_{w \in \mathscr{X}^*\mathscr{X}} \langle S \mid S_w \rangle P_w = \prod_{l \in \mathscr{L}yn\mathscr{X}}^{\searrow} e^{\langle S|S_l \rangle P_l}.$$

Then

(i) *If $H \in \mathscr{A}\langle\langle\mathscr{X}\rangle\rangle$ is another group-like solution of (41) then there exists $C \in \mathscr{L}ie_{\mathscr{A}}\langle\langle\mathscr{X}\rangle\rangle$ such that $S = He^C$ (and conversely).*

(ii) *The following assertions are equivalent*

(a) $\{\langle S \mid w \rangle\}_{w \in \mathscr{X}^*}$ *is $\mathscr{C}_0$-linearly independent,*

(b) $\{\langle S \mid l \rangle\}_{l \in \mathscr{L}yn\mathscr{X}}$ *is $\mathscr{C}_0$-algebraically independent,*

(c) $\{\langle S \mid x \rangle\}_{x \in \mathscr{X}}$ *is $\mathscr{C}_0$-algebraically independent,*

(d) $\{\langle S \mid x \rangle\}_{x \in \mathscr{X} \cup \{1_{\mathscr{X}^*}\}}$ *is $\mathscr{C}_0$-linearly independent,*

(e) *The family $\{u_x\}_{x \in \mathscr{X}}$ is such that, for $f \in \mathrm{Frac}(\mathscr{C}_0)$ and $(c_x)_{x \in \mathscr{X}} \in \mathbb{C}^{(\mathscr{X})}$,*

$$\sum_{x \in \mathscr{X}} c_x u_x = \partial f \Longrightarrow (\forall x \in \mathscr{X})(c_x = 0).$$

(f) *The family $(u_x)_{x \in \mathscr{X}}$ is free over $\mathbb{C}$ and $\partial \mathrm{Frac}(\mathscr{C}_0) \cap \mathrm{span}_{\mathbb{C}}\{u_x\}_{x \in \mathscr{X}} = \{0\}$.*

Proof. [Sketch] The first item has been treated in [35]. The second is a group-like version of the abstract form of Theorem 1 of [15]. It goes as follows

- due to the fact that $\mathscr{A}$ is without zero divisors, we have the following embeddings $\mathscr{C}_0 \subset \mathrm{Frac}(\mathscr{C}_0) \subset \mathrm{Frac}(\mathrm{A})$, $\mathrm{Frac}(\mathrm{A})$ is a differential field, and its derivation can still be denoted by ∂ as it induces the previous one on $\mathscr{A}$,
- the same holds for $\mathscr{A}\langle\langle\mathscr{X}\rangle\rangle \subset \mathrm{Frac}(\mathrm{A})\langle\langle\mathscr{X}\rangle\rangle$ and $\mathbf{d}$
- therefore, equation (41) can be transported in $\mathrm{Frac}(\mathrm{A})\langle\langle\mathscr{X}\rangle\rangle$ and M satisfies the same condition as previously.
- Equivalence between a-d comes from the fact that $\mathscr{C}_0$ is without zero divisors and then, by denominator chasing, linear independances w.r.t $\mathscr{C}_0$ and $\mathrm{Frac}(\mathscr{C}_0)$ are equivalent. In particular, supposing condition d, the family $\{\langle S \mid x \rangle\}_{x \in \mathscr{X} \cup \{1_{\mathscr{X}^*}\}}$ (basic triangle) is $\mathrm{Frac}(\mathscr{C}_0)$-linearly independent which imply, by the Theorem 1 of [15], condition e,
- still by Theorem 1 of [15], e is equivalent to f, implying that $\{\langle S \mid w \rangle\}_{w \in \mathscr{X}^*}$ is $\mathrm{Frac}(\mathscr{C}_0)$-linearly independent which induces $\mathscr{C}_0$-linear independence (*i.e.* a).

□

Now, let us go back to notations of Section 1 and equip the differential rings of

(i) holomorphic functions over a simply connected domain Ω, $(\mathcal{H}(\Omega),\partial)$, with the topology of compact convergence (CC),

(ii) formal series over $\mathcal{X}$ and with coefficients in $\mathcal{H}(\Omega)$, $(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle,\mathbf{d})$, with the ultrametric distance defined by [33] $\delta(S,T)=2^{-\varpi(S-T)}$.

Let us also consider again the Chen series of the differential forms $(\omega_i)_{i\geq 1}$ defined by the inputs $\omega_i=u_{x_i}dz$ along a path $z_0\rightsquigarrow z$ on Ω. By (18), it follows that

$$C_{z_0\rightsquigarrow z}=\sum_{w\in\mathcal{X}^*}\alpha_{z_0}^z(w)w=(\alpha_{z_0}^z\otimes\mathrm{Id})\mathcal{D}_{\mathcal{X}}=\prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow}e^{\alpha_{z_0}^z(S_l)P_l}. \tag{42}$$

This series satisfies (41) and is obtained as the limit, for the topology of (discrete) pointwise convergence over the words, of Picard iteration process initialized at $\langle C_{z_0\rightsquigarrow z}\mid 1_{\mathcal{X}^*}\rangle=1_{\mathcal{H}(\Omega)}$.

Let us illustrate Theorem 4.4, with simple examples, for which $\mathcal{C}_0$ contains $\mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}=\mathbb{C}[u_x^{\pm 1},\partial^i u_x]_{i\geq 1,x\in\mathcal{X}}\subset\mathcal{A}=(\mathcal{H}(\Omega),\partial)$. In these examples, we use

Proposition 4.5 ([33]) *For $\mathcal{X}=\{x\}$, since $x^n=x^{\sqcup\!\sqcup\, n}/n!$ then*

$$\alpha_{z_0}^z(x^n)=\frac{(\alpha_{z_0}^z(x))^n}{n!},C_{0\rightsquigarrow z}=\sum_{n\geq 0}\alpha_{z_0}^z(x^n)x^n=e^{\alpha_{z_0}^z(x)x},\alpha_0^z(x^*)=e^{\alpha_{z_0}^z(x)}.$$

Example 4.6 Let us consider two positive cases over $\mathcal{X}=\{x\}$.

(i) $\Omega=\mathbb{C},u_x(z)=1_\Omega,\mathcal{C}_0=\mathbb{C}$. Since $\alpha_0^z(x^n)=z^n/n!$ then, by Proposition 4.5,

$$C_{0\rightsquigarrow z}=e^{zx}\text{ and }\mathbf{d}C_{0\rightsquigarrow z}=xC_{0\rightsquigarrow z}.$$

Moreover, $\alpha_0^z(x)=z$ which is transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is $\mathcal{C}_0$-free. Now, let $f\in\mathcal{C}_0$ then $\partial=0$. Hence, if $\partial f=cu_x$ then $c=0$.

(ii) $\Omega=\mathbb{C}\backslash]-\infty,0],u_x(z)=z^{-1},\mathcal{C}_0=\mathbb{C}[z^{\pm 1}]\subset\mathbb{C}(z)$. Since $\alpha_1^z(x^n)=\log^n(z)/n!$ then, by Proposition 4.5,

$$C_{1\rightsquigarrow z}=z^x\text{ and }\mathbf{d}C_{1\rightsquigarrow z}=z^{-1}xC_{1\rightsquigarrow z}.$$

Moreover, $\alpha_1^z(x)=\log(z)$ which is transcendent over $\mathbb{C}(z)$ then over $\mathcal{C}_0$ and $\{\alpha_1^z(x^n)\}_{n\geq 0}$ is $\mathcal{C}_0$-free. Now, let $f\in\mathcal{C}_0$ then $\partial f\in\mathrm{span}_{\mathbb{C}}\{z^{-n}\}_{n\in\mathbb{Z},n\neq 1}$. Hence, if $\partial f=cu_x$ then $c=0$.

Example 4.7 Let us consider two negative cases over $\mathcal{X}=\{x\}$.

(i) $\Omega=\mathbb{C},u_x(z)=e^z,\mathcal{C}_0=\mathbb{C}[e^{\pm z}]$. Since $\alpha_0^z(x^n)=(e^z-1)^n/n!$ then, by Proposition 4.5,

[33] $\forall S\in\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, if $S=0$ then $\varpi(S)=-\infty$ else $\min_{w\in\mathrm{supp}(S)}\{|w|\text{ or }(w)\}$ [1].

$$C_{0\rightsquigarrow z}=e^{(e^z-1)x} \text{ and } \mathbf{d}C_{0\rightsquigarrow z}=e^z x C_{0\rightsquigarrow z}.$$

Moreover, $\alpha_0^z(x)=e^z-1$ which is not transcendent over $\mathscr{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathscr{C}_0$-free. If $f(z)=ce^z\in\mathscr{C}_0$ $(c\neq 0)$ then $\partial f(z)=ce^z=cu_x(z)$.

(ii) $\Omega=\mathbb{C}\setminus]-\infty,0], u_x(z)=z^a, a\in\mathbb{C}\setminus\mathbb{Q}, \mathscr{C}_0=\mathbb{C}\{\{z,z^{\pm a}\}\}=\mathrm{span}_{\mathbb{C}}\{z^{ka+l}\}_{k,l\in\mathbb{Z}}$. Since $\alpha_0^z(x^n)=(a+1)^{-n}z^{n(a+1)}/n!$ then, by Proposition 4.5,

$$C_{0\rightsquigarrow z}=e^{(a+1)^{-1}z^{(a+1)}x} \text{ and } \mathbf{d}C_{0\rightsquigarrow z}=z^a x C_{0\rightsquigarrow z}.$$

Moreover, $\alpha_0^z(x)=z^{a+1}/(a+1)$ which is not transcendent over $\mathscr{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathscr{C}_0$-free. If $f(z)=cz^{a+1}/(a+1)\in\mathscr{C}_0$ $(c\neq 0)$ then $\partial f(z)=cz^a=cu_x(z)$.

4.2 First step of a noncommutative Picard-Vessiot theory

Let us recall that the vector space of solutions of (41) is a free ($\mathbb{C}\langle\langle\mathscr{X}\rangle\rangle$-right) module of dimension one [34] generated by $C_{z_0\rightsquigarrow z}$ [35]. Hence, by Theorem 4.4, we have common traits with the ordinary case of first order differential equations,

(i) the differential Galois group of (41) + group-like is the Hausdorff group $\{e^C\}_{C\in\mathscr{L}ie_{\mathbb{C}.1_{\mathscr{H}(\Omega)}}\langle\langle\mathscr{X}\rangle\rangle}$ (group of characters of $\mathscr{H}_{\sqcup\!\sqcup}(\mathscr{X})$).

(ii) the PV extension related to (41) is $\mathscr{C}\langle\langle\mathscr{X}\rangle\rangle(C_{z_0\rightsquigarrow z})$, where $\mathscr{C}\subset\mathscr{A}=(\mathscr{H}(\Omega),\partial)$ such that $\mathrm{Const}(\mathscr{C}\langle\langle\mathscr{X}\rangle\rangle)=\ker\mathbf{d}=\mathbb{C}.1_{\mathscr{H}(\Omega)}\langle\langle\mathscr{X}\rangle\rangle$.

Theorem 4.8 ([34,35,37]) *Let $R\in\mathbb{C}.1_{\mathscr{H}(\Omega)}^{\mathrm{rat}}\langle\langle\mathscr{X}\rangle\rangle$. Then, for any path $z_0\rightsquigarrow z$ over Ω, we have* [35] *$R\in\mathrm{Dom}_{word}(C_{z_0\rightsquigarrow z})$ and the output of (1) can be computed by*

$$y(z_0,z)=\alpha_{z_0}^z(R)=\sum_{w\in\mathscr{X}^*}(\nu\mu(w)\eta)\alpha_{z_0}^z(w)=\langle C_{z_0\rightsquigarrow z}\|R\rangle.$$

Now, let N be the least integer n such that y satisfies a (non-trivial) differential equation of order N (with coefficents in $\mathscr{C}$), the family $\{\partial y\}_{0\leq k\leq N-1}$ is $\mathscr{C}$-linearly independent, i.e.

$$(a_n\partial^N+\ldots+a_1\partial+a_0)y=0, \text{with } a_N,\ldots,a_0\in\mathscr{C}.$$

and, from what precedes, we have $N\leq n=\mathrm{rk}(R)$.

Proof. Due to this strong convergence condition, we have

(i) for any $T\in\mathscr{H}(\Omega)\langle\langle\mathscr{X}\rangle\rangle$ and $P\in\mathscr{H}(\Omega)\langle\mathscr{X}\rangle, S\in\mathrm{Dom}_{word}(T)$, we have $S\in\mathrm{Dom}_{word}(PT), S\triangleleft P\in\mathrm{Dom}_{word}(T)$ and $\langle PT\|S\rangle=\langle T\|S\triangleleft P\rangle$,

[34] In fact, we will see that it is the $\mathbb{C}\langle\langle\mathscr{X}\rangle\rangle$-right module $C_{z_0\rightsquigarrow z}.\mathbb{C}.1_{\mathscr{H}(\Omega)}\langle\langle\mathscr{X}\rangle\rangle$.

[35] Once (z,z_0) is fixed on Ω, $\mathrm{Dom}_{word}(C_{z_0\rightsquigarrow z})$ is the subset of $A\langle\langle\mathscr{X}\rangle\rangle$ of series R such that $\sum_{n\geq 0}\alpha_{z_0}^z(R_n)$ is convergent for the standard topology, where $R_n=\sum_{|w=n|}\langle R\mid w\rangle w$ is a homogeneous component (we need to check that this series is convergent via *majoration morphisms* [34,35,37]).

(ii) from the continuity of ∂, for any $T \in \mathscr{H}(\Omega)\langle\langle\mathscr{X}\rangle\rangle$ and $S \in \mathrm{Dom}_{word}(T)$, we have $\partial(\langle T \| S\rangle) = \langle \mathbf{d}T \| S\rangle + \langle T \| \mathbf{d}S\rangle$.

Now, let (ν, μ, η) be a representation of $R \in \mathbb{C}.1^{\mathrm{rat}}_{\mathscr{H}(\Omega)}\langle\langle\mathscr{X}\rangle\rangle$ of rank n. Let us see that the family $(\langle C_{z_0 \rightsquigarrow z} \mid w\rangle\langle R \mid w\rangle)_{w\in\mathscr{X}^*}$ is summable in $\mathscr{H}(\Omega)$. Indeed, since the matrix norm is multiplicative then, for any $w \in \mathscr{X}^*$ and $B_1 > 0$, we have [36]

$$\|\mu(w)\| \le B_1^{|w|} \text{ and } |\, \nu\mu(w)\eta \,| \le k_1 \|\nu\|_r \|\mu(w)\| \|\eta\|_c.$$

The Chen series $C_{z_0 \rightsquigarrow z}$ is exponentially bounded from above [37] , *i.e.* for all compact $\kappa \subset \Omega$, there is $k_2, B_2 > 0$ such that [38] [34,35,37]

$$\forall w \in \mathscr{X}^*,\ \|\langle C_{z_0 \rightsquigarrow z} \mid w\rangle\|_\kappa \le k_2 B_2^{|w|}/|\,w\,|!.$$

Hence, choosing a compact $\kappa \subset \Omega$, we obtain

$$\begin{aligned}\sum_{w\in\mathscr{X}^*} \|\langle C_{z_0 \rightsquigarrow z} \mid w\rangle\langle R \mid w\rangle\|_\kappa &\le \sum_{w\in\mathscr{X}^*} \|\langle C_{z_0 \rightsquigarrow z} \mid w\rangle\|_\kappa \,|\,\langle R \| w\rangle\,| \\ &\le \sum_{w\in\mathscr{X}^*} k_2 \frac{B_2^{|w|}}{|\,w\,|!}(k_1 \|\nu\|_r B_1^{|w|} \|\eta\|_c) < +\infty.\end{aligned}$$

Since $y = y(z_0, z) = \alpha_{z_0}^z(R)$ and ∂ is continuous for (CC) then, by Proposition 4.3,

$$\partial^l y(z_0, z) = \langle \mathbf{d}^l C_{z_0 \rightsquigarrow z} \| R\rangle \text{ and for } l \le n, \mathbf{d}^l C_{z_0 \rightsquigarrow z} = Q_l(z) C_{z_0 \rightsquigarrow z}$$

and then [34,35,37]

$$\partial^l y(z_0, z) = \langle Q_l(z) C_{z_0 \rightsquigarrow z} \| R\rangle = \langle C_{z_0 \rightsquigarrow z} \| R \triangleleft Q_l(z)\rangle.$$

By Lemma 3.9, there is $\{a_k\}_{k=0,..,n}$ in $\mathscr{C}$, not all zero, such that $\sum_{k=0}^n a_k (R \triangleleft Q_k) = 0$ yielding the expected result. This linear independence holds in any module whatever the ring. □

Remark 4.9 (i) The rational series in Theorem 4.8 is the generating series of the first order linear differential system, $\partial q = (u_0\mu(x_0) + \ldots + u_m\mu(x_m))q, y = \nu q$, initialized at $y(z_0) = \eta$. From [30], $y(z) = \alpha_{z_0}^z(R)$. The Nth-order differential equation in Theorem 4.8 is then the result, obtained by eliminating the states $\{q_i\}_{i=0,..,m}$ in this system.

(ii) The converse process is also possible thanks to the *compagnion form*.

(iii) Analogue results for nonlinear equations can be found in [34,35,37].

[36] We choose a matrix norm (*i.e.* multiplicative) on $\mathbb{C}^{n\times n}$, denoted $\|M\|$, and two norms $\|\nu\|_r, \|\eta\|_c$ on $\mathbb{C}^{1\times n}, \mathbb{C}^{n\times 1}$, respectively, and there is classically $k_1 > 0$ such that $|\,\nu.M.\eta\,| \le k_1 \|\nu\|_r \|M\| \|\eta\|_c$.

[37] In the references the bounding is finer and adapted as well to infinite alphabet.

[38] For any $f \in \mathscr{H}(\Omega)$, we denote $\|f\|_\kappa := \sup_{s\in\kappa} |\,f(s)\,|$.

5 Conclusion

In this work, we gave a first step to construct a Picard-Vessiot theory for a class of noncommutative differential equations satisfied by the Chen series $C_{z_0 \rightsquigarrow z}$ over the alphabet $\mathscr{X} = \{x_i\}_{i \geq 0}$ (along paths $z_0 \rightsquigarrow z$ belonging to a simply connected manifold Ω and with respect to the differential forms $(u_i dz)_{i \geq 0}$):

(i) The coefficients of these noncommutative generating series belong to the differential ring $\mathbb{C}\{\{(u_i)_{i\geq 0}\}\}\{\langle C_{z_0 \rightsquigarrow z} \mid w\rangle\}_{w \in \mathscr{X}^*}$ which is closed by integration with respect to $(u_i dz)_{i\geq 0}$.

(ii) The Picard-Vessiot extension of these noncommutative differential equations is defined as the module $C_{z_0 \rightsquigarrow z}\mathbb{C}1_{\mathscr{H}(\Omega)}$ and the Haussdorf group $\{e^C\}_{C \in \mathscr{L}ie_{\mathbb{C}}\langle\langle \mathscr{X} \rangle\rangle}$ plays the rôle of differential Galois group associated with this extension.

(iii) These differential equations were considered as universal differential equations [10,14,17,18,39] by many authors. Universality can be seen by replacing each letter by constant matrices (resp. holomorphic vector field, given in (2)) and then solving a system of linear (resp. nonlinear) differential equations, given in (1).

(iv) These solutions are obtained as a pairing between the series $C_{z_0 \rightsquigarrow z}$ and the generating series of finite Hankel (resp. Lie-Hankel) rank [31,29,30,45], for linear (resp. nonlinear) differential equations explaned by Remark 4.9.

(v) Via rational series (on noncommutative indeterminates and with coefficients in rings) [1,46] and their non-trivial combinatorial Hopf algebras (Theorems 2.2, 2.4, 3.2, 3.4 and 3.7), we illustrated this theory, still under construction, with the case of linear differential equations with singular regular singularities (Theorem 4.8) thanks to an equation satisfied by the Chen generating series.

This practical study allowed also to treat the noncommutative generating series of multiindexed polylogarithms and harmonic sums and as well as those of their special values (polyzetas). In particular, we proved the existence of well defined infinite sums of these polylogarithms and harmonic sums [9] in order to describe solutions of differential equations (Theorem 4.8).

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