DETERMINISTIC EQUIVALENCE FOR NOISY PERTURBATIONS

Martin Vogel, Ofer Zeitouni

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MARTIN VOGEL AND OFER ZEITOUNI

Abstract. We prove a quantitative deterministic equivalence theorem for the logarithmic potentials of deterministic complex $N \times N$ matrices subject to small random perturbations. We show that with probability close to 1 this log-potential is, up to a small error, determined by the singular values of the unperturbed matrix which are larger than some small N-dependent cut-off parameter.

1. Introduction and statement of results

In evaluating the limit of empirical measures of eigenvalues of (non-Hermitian) matrices, an important role is played by the evaluation of certain determinants. Specifically, for a sequence of matrices X_N of dimension $N \times N$ having eigenvalues $\lambda_i(X_N)$, let $L_N(X_N) = N^{-1} \sum_{i=1}^{N} \delta_{\lambda_i(X_N)}$ denote the empirical measure of eigenvalues of X_N and let $\mathcal{L}_{X_N}(z) = \int \log|z-x| L_N(X_N)(dx)$ denote its log-potential. Since a.e. convergence of log-potentials implies the weak convergence of the associated measures, the evaluation of limits of log-potentials has played an important role in the study of convergence of the spectrum of random matrices. We refer to [5, 3] for introductions to this vast topic.

Since

$$\mathcal{L}_{X_N}(z) = \frac{1}{2} \log \det(z - X_N)(z - X_N)^* = \log|\det(z - X_N)|,$$

evaluating logarithmic potentials amounts to computing determinants. In their study of the spectrum of small, noisy perturbations of non-normal matrices, the authors of [1] have identified a certain *deterministic equivalent* result, which we now present.

Theorem 1. [1, Theorem 2.1] *Let $A = A_N$ be a sequence of deterministic complex $N \times N$-matrix of uniformly bounded norm and singular values $s_1 \geq \dots s_N \geq 0$. Fix $\gamma > 1/2$ and $\eta > 0$. Set $\varepsilon_N = N^{-\eta}$, and set N^* to be the largest integer i so that*

$$s_{N-i+1} \leq \varepsilon_N^{-1} N^{-\gamma} (N - i + 1)^{1/2}. \tag{1.1}$$

If no such i exists then set $N^* = 1$. Let G_N be a matrix whose entries are i.i.d. standard complex Gaussian variables. Then, if $N^* \log N/N \to \alpha < \infty$,

$$\frac{1}{N} \log|\det(A_N + N^{-\gamma} G_N)| - \frac{1}{N} \sum_{i=1}^{N-N^*+1} \log s_i \to_{N\to\infty} 0, \tag{1.2}$$

in probability, as $N \to \infty$. If $\alpha = 0$, we may take $\varepsilon_N = N^{-\eta}$ for any $\eta > 0$.

The proof in [1] uses in an essential way the unitary invariance of G_N, and probabilistic arguments. However, it does not directly extend to other noise models, not even to the case where G_N is a matrix consisting of independent real standard Gaussian variables. The purpose of this note is to present a very general version of Theorem 1, based on the Grushin problem studied in [8]. It will be stated under the following assumption on the noise matrix. Here and throughout, for a matrix A, $s_1(A) \geq s_2(A) \geq \dots \geq s_N(A) \geq 0$ denote the singular values of A, and $\|A\|$ denotes the operator norm of G, i.e. $\|A\| = s_1(A)$,

Assumption 2. *$G = G_N$ is an $N \times N$ random matrix such that the following hold.*

(1) ***Norm bound*** *There exists a $\kappa_1 > 0$ such that*

$$\mathbb{E}[\|G\|] = \mathcal{O}(N^{\kappa_1}). \tag{1.3}$$

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(2) ***Anti-concentration bound*** *For each $\theta > 0$ there exists a $\beta > 0$ such that for any fixed deterministic complex $N \times N$ matrix D with $\|D\| = \mathcal{O}(N^{\kappa_2})$, $\kappa_2 \geq 0$, we have that*

$$\mathbf{P}(s_N(D+G) \leq N^{-\beta}) = \varepsilon_N(\theta) = o(1). \tag{1.4}$$

Theorem 3. *Let $A = A_N$ be a deterministic complex $N \times N$-matrix with $\|A\| = \mathcal{O}(N^{\kappa_2})$ for some fixed $\kappa_2 \geq 0$, and assume $G = G_N$ satisfies Assumption 2. Let $s_1 \geq \dots s_N \geq 0$ denote the singular values of A. Suppose that for some fixed $L > 0$ there exists*

$$CN^{-L} \leq \alpha \leq 1 \tag{1.5}$$

such that

$$\#\{j; s_j \in [0,\alpha]\} \leq \nu_N \frac{N}{\log N} =: M, \quad \nu_N = o(1). \tag{1.6}$$

For $\tau > 0$ and any fixed $\gamma \gg 1$ we let

$$N^{-\gamma} \leq \delta \ll N^{-\kappa_1} \alpha \tau^{-1}.$$

Then, we have that

$$\left| \frac{1}{N} \log|\det(A+\delta G)| - \frac{1}{N} \sum_{j:\ s_j > \alpha} \log s_j \right| = \mathcal{O}(1)\left(\nu_N + \alpha^{-1} N^{\kappa_1} \delta \tau\right)$$

with probability $\geq 1 - \varepsilon_N(\kappa_2 + \gamma) - \tau^{-1}$.

Remark 4. *Assumption 2 holds for a large class of noise matrices, including those with iid entries of zero mean and finite variance. We refer to* [2, Remark 1.3] *for details and references.*

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2. Grushin problem

We now present the proof of Theorem 3, based on [8, 4], see also [7, 6]. We begin by setting up a well-posed Grushin problem. Let $A = A_N$ be a deterministic complex $N \times N$-matrix and let

$$0 \leq t_1^2 \leq \dots \leq t_N^2 \tag{2.1}$$

denote the eigenvalues of A^*A with associated orthonormal basis of eigenvectors $e_1, \dots, e_N \in \mathbb{C}^N$. The spectra of A^*A and AA^* are equal and we can find an orthonormal basis $f_1, \dots, f_N \in \mathbb{C}^N$ of eigenvectors of AA^* associated with the eigenvalues (2.1) such that

$$A^* f_i = t_i e_i, \quad Ae_i = t_i f_i, \quad i = 1, \dots, N. \tag{2.2}$$

Recall α, M, see (1.5),(1.6), and let δ_i, $1 \leq i \leq M$, denote an orthonormal basis of $\mathbb{C}^M$. Put

$$R_+ = \sum_{i=1}^{M} \delta_i \circ e_i^*, \quad R_- = \sum_{i=1}^{M} f_i \circ \delta_i^*, \tag{2.3}$$

We claim that the Grushin problem

$$\mathcal{P} = \begin{pmatrix} A & R_- \\ R_+ & 0 \end{pmatrix} : \mathbb{C}^N \times \mathbb{C}^M \longrightarrow \mathbb{C}^N \times \mathbb{C}^M \tag{2.4}$$

is bijective. To see this we take $(v, v_+) \in \mathbb{C}^N \times \mathbb{C}^M$ and we want to solve

$$\mathcal{P}\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}. \tag{2.5}$$

We write $u = \sum_1^N u(j)e_j$ and $v = \sum_1^N v(j)f_j$. Similarly, we express u_-, v_+ in the basis $\delta_1, \dots, \delta_M$. The relation (2.2) then shows that (2.5) is equivalent to

$$\begin{cases} \sum_1^N t_i u_i f_i + \sum_1^M u_-(j) f_j = \sum_1^N v_j f_j \\ u_j = v_+(j), \quad j = 1, \dots, M, \end{cases}$$

which can be written as

$$\begin{cases} t_i u_i f_i = v_i f_i, & i = M+1, \dots, N, \\ \begin{pmatrix} t_i & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_i \\ u_-(i) \end{pmatrix} = \begin{pmatrix} v_i \\ v_+(i) \end{pmatrix}, & i = 1, \dots, M. \end{cases} \tag{2.6}$$

Since

$$\begin{pmatrix} t_i & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -t_i \end{pmatrix},$$

we see that

$$\mathcal{P}^{-1} = \mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \tag{2.7}$$

with

$$\begin{aligned} E &= \sum_{M+1}^N \frac{1}{t_i} e_i \circ f_i, \quad E_+ = \sum_1^M e_i \circ \delta_i^*, \\ E_- &= \sum_1^M \delta_i \circ f_i^*, \quad E_{-+} = -\sum_1^M t_j \delta_j \circ \delta_j^*, \end{aligned} \tag{2.8}$$

and the norm estimates

$$\|E(z)\| \le \frac{1}{\alpha}, \quad \|E_\pm\| = 1, \quad \|E_{-+}\| \le \alpha. \tag{2.9}$$

Furthermore, (2.6) shows that

$$|\det \mathcal{P}|^2 = \prod_{M+1}^N t_i^2. \tag{2.10}$$

2.1. **Grushin problem for the perturbed operator.** Now we turn to the perturbed operator

$$A^\delta = A + \delta G, \quad 0 \le \delta \ll 1. \tag{2.11}$$

where G is a complex $N \times N$-matrix. Let $R_\pm$ be as in (2.3), and put

$$\mathcal{P}^\delta = \begin{pmatrix} A^\delta & R_- \\ R_+ & 0 \end{pmatrix} : \mathbb{C}^N \times \mathbb{C}^M \longrightarrow \mathbb{C}^N \times \mathbb{C}^M \tag{2.12}$$

Then $\mathcal{P} = \mathcal{P}^0$. Applying $\mathcal{E}$, see (2.7), from the right to (2.12) yields

$$\mathcal{P}^\delta \mathcal{E} = I_{N+M} + \begin{pmatrix} \delta G E & \delta G E_+ \\ 0 & 0 \end{pmatrix} \tag{2.13}$$

Suppose that

$$\delta \|G\| \alpha^{-1} \le \frac{1}{2}. \tag{2.14}$$

Then, see (2.11), the matrix $\mathcal{P}^\delta \mathcal{E}$ is invertible by a Neumann series argument and we get that

$$\begin{aligned} \mathcal{E}^\delta = (\mathcal{P}^\delta)^{-1} &= \mathcal{E} + \sum_{n=1}^\infty (-\delta)^n \begin{pmatrix} E(GE)^n & (EG)^n E_+ \\ E_-(GE)^n & E_-(GE)^{n-1} G E_+ \end{pmatrix} \\ &\overset{\text{def}}{=} \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}, \end{aligned} \tag{2.15}$$

where by (2.14), (2.9),

$$\begin{aligned}
&\|E^\delta\| = \|E(1+\delta GE)^{-1}\| \leq 2\|E\| \leq 2\alpha^{-1},\\
&\|E_+^\delta\| = \|(1+\delta GE)^{-1}E_+\| \leq 2\|E_+\| \leq 2,\\
&\|E_-^\delta\| = \|E_-(1+\delta GE)^{-1}\| \leq 2\|E_-\| \leq 2,\\
&\|E_{-+}^\delta - E_{-+}\| = \|E_-(1+\delta Q_\omega E)^{-1}\delta GE_+\| \leq 2\|\delta G\| \leq \alpha.
\end{aligned} \tag{2.16}$$

The Schur complement formula applied to $\mathcal{P}^\delta$ and $\mathcal{E}^\delta$ shows that

$$\log|\det A^\delta| = \log|\det \mathcal{P}^\delta| + \log|\det E_{-+}^\delta|. \tag{2.17}$$

Notice that

$$\begin{aligned}
\Big|\log|\det \mathcal{P}^\delta| - \log|\det \mathcal{P}^0|\Big| &= \left|\Re\int_0^\delta \operatorname{tr}(\mathcal{E}^\tau \frac{d}{d\tau}\mathcal{P}^\tau)d\tau\right|\\
&= \left|\Re\int_0^\delta \operatorname{tr}\begin{pmatrix} E^\tau & E_+^\tau \\ E_-^\tau & E_{-+}^\tau\end{pmatrix}\cdot\begin{pmatrix} G & 0 \\ 0 & 0\end{pmatrix}d\tau\right|\\
&= \left|\Re\int_0^\delta \operatorname{tr}(E^\tau G)d\tau\right|\\
&\leq 2\alpha^{-1}\delta N\|G\|.
\end{aligned} \tag{2.18}$$

Here in the last line we used (2.16). Thus,

$$\left|\frac{1}{N}\log|\det \mathcal{P}^\delta| - \frac{1}{N}\log|\det \mathcal{P}|\right| \leq 2\alpha^{-1}\delta\|G\|. \tag{2.19}$$

Notice that by (2.16), (2.9), we have that $\|E_{-+}^\delta\| \leq 2\alpha$. Thus, by (2.17) and (2.19),

$$\log|\det A^\delta| \leq \log|\det \mathcal{P}| + M|\log 2\alpha| + 2\alpha^{-1}\delta N\|G\|. \tag{2.20}$$

2.2. **Random noise matrix.** We recall Assumption 2 on the noise matrix. By Markov's inequality,

$$\mathbf{P}(\|G\| > CN^{\kappa_1}\tau) \leq \tau^{-1}, \quad \tau > 0. \tag{2.21}$$

Since

$$0 \leq \delta \ll N^{-\kappa_1}\alpha\tau^{-1}, \tag{2.22}$$

we obtain that with probability $\geq 1-\tau^{-1}$, we have that (2.14) holds. Hence, the estimates (2.16) and (2.17) hold with the same probability. This together with (2.20), (1.6) and (1.5), implies that $\|G\| \leq CN^{\kappa_1}\tau$ and

$$\log|\det A^\delta| \leq \log|\det \mathcal{P}| + \mathcal{O}(1)\nu_N N + \mathcal{O}(1)\alpha^{-1}N^{1+\kappa_1}\delta\tau \tag{2.23}$$

with probability $\geq 1-\tau^{-1}$.

It remains to find a lower bound on $\log|\det E_{-+}^\delta|$. We begin by recalling a classical result on Grushin problems, see for instance [8, Lemma 18].

Lemma 5. *Let $\mathcal{H}$ be an N-dimensional complex Hilbert spaces, and let $N \geq M > 0$. Suppose that*

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0\end{pmatrix} : \mathcal{H}\times\mathbb{C}^M \longrightarrow \mathcal{H}\times\mathbb{C}^M$$

is a bijective matrix of linear operators, with inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+}\end{pmatrix}.$$

Let $0 \leq t_1(P) \leq \cdots \leq t_N(P)$ denote the eigenvalues of $(P^*P)^{1/2}$, and let $0 \leq t_1(E_{-+}) \leq \cdots \leq t_M(E_{-+})$ denote the eigenvalues of $(E_{-+}^*E_{-+})^{1/2}$. Then,

$$\frac{t_n(E_{-+})}{\|E\|t_n(E_{-+}) + \|E_-\|\|E_+\|} \leq t_n(P) \leq \|R_+\|\|R_-\|t_n(E_{-+}), \quad 1 \leq n \leq M.$$

By (2.3) we know that $\|R_\pm\| = 1$, and by (2.16) we then get

$$t_n(A^\delta) \leq t_n(E^\delta_{-+}), \quad 1 \leq n \leq M. \tag{2.24}$$

Next note that, for any $\delta \geq N^{-\gamma}$ and $\beta > 0$ and any deterministic matrix A,

$$\begin{aligned} \mathbf{P}\left(s_N(A + \delta G) \leq N^{-\gamma-\beta}\right) &= \mathbf{P}\left(s_N(A/\delta + G) \leq N^{-\gamma-\beta}/\delta\right) \\ &\leq \mathbf{P}\left(s_N(A/\delta + G) \leq N^{-\beta}\right). \end{aligned}$$

Thus, from (1.4), there exists a $\beta > 0$ such for any fixed deterministic matrix A with $\|A\| = \mathcal{O}(N^{\kappa_2})$ and any $\delta \geq N^{-\gamma}$, we have that

$$\mathbf{P}\left(s_N(A + \delta G) \leq N^{-\gamma-\beta}\right) \leq \varepsilon_N(\kappa_2 + \gamma). \tag{2.25}$$

We recall that

$$N^{-\gamma} \leq \delta \ll N^{-\kappa_1}\alpha\tau^{-1}. \tag{2.26}$$

By combining (2.24), (2.25) and (2.21), we obtain that

$$\mathbf{P}\left(s_M(E^\delta_{-+}) > N^{-\gamma-\beta} \text{ and } \|G\| \leq CN^{\kappa_1}\tau\right) \geq 1 - \varepsilon_N(\kappa_2 + \gamma) - \tau^{-1}. \tag{2.27}$$

Provided that this event holds and using (1.6), we get hat

$$\begin{aligned} \log|\det E^\delta_{-+}| &= \sum_1^M \log s_j(E^\delta_{-+}) \\ &\geq M \log s_M(E^\delta_{-+}) \\ &\geq -(\gamma + \beta)M \log N \\ &\geq -(\gamma + \beta)\nu_N N, \end{aligned} \tag{2.28}$$

which in combination with (2.17), (2.19) yields that

$$\log|\det A^\delta| \geq \log|\det \mathcal{P}| - \mathcal{O}(1)\nu_N N - \mathcal{O}(1)\alpha^{-1}N^{1+\kappa_1}\delta\tau \tag{2.29}$$

with probability $\geq 1 - \varepsilon_N(\kappa_2 + \gamma) - \tau^{-1}$. This, in view of (2.23), (2.22) and (2.10) concludes the proof of the theorem.

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(Martin Vogel) Institut de Recherche Mathématique Avancée - UMR 7501, Université de Strasbourg et CNRS, 7 rue René-Descartes, 67084 Strasbourg Cedex, France.

(Ofer Zeitouni) Department of Mathematics, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel and Courant Institute, New York University, 251 Mercer St, New York, NY 10012, USA.