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Filtering in Gaussian Linear Systems with Fuzzy Switches

Zied Bouyahia, Stéphane Derrode, and Wojciech Pieczynski

Abstract—This work extends recent results on Conditionally Gaussian Observed Markov Switching Models (CGOMSM) by incorporating fuzzy switches in the model, instead of hard ones. This kind of generalization is of interest for applications involving continuous switching regimes, such as tracking an object using cameras in intermittent sunlight and shadow conditions. The filter developed hereby is recursive, optimal and exact, as tracking an object using cameras in intermittent sunlight and shadow switches in the model, instead of hard ones. This kind of generalization is obtained by extending recent results on Conditionally Gaussian Observed Markov Switching Models (CGOMSM) by incorporating fuzzy measures.

Index Terms—Triplet Markov models, Fuzzy switching linear model, Fast filtering.

NOMENCLATURE

CMSHLM Conditionally Markov switching hidden linear model.

CGMSM Conditionally Gaussian Markov switching Model.

CGOMSM Conditionally Gaussian observed Markov switching model, a CGMSM where fuzzy sets are used, where

CGOFMSM Conditionally Gaussian observed fuzzy Markov switching model, a CGOMSM with fuzzy switches.

\( X_1^N \)

A stochastic process of size \( N \).

\( X_n, x_n \)

A random variable at time index \( n \), and a realization.

\( X^N, Y^N, R^N \)

State, observation and switches (also called jump) processes, respectively.

\( Z_n, T_n \)

Denotes \( (X_n, Y_n)^T \) and \( (X_n, R_n, Y_n)^T \), respectively.

\( \Omega = \{ 0, 1, \ldots, K – 1 \} \)

Conditionally Gaussian observed Markov Switching Models” (CGOMSM), considered as a natural way to extend Gaussian systems to Gaussian switching ones, do not allow for a filtering scheme that can be performed in a reasonable running time [1]–[5]. Classically, CGLSSMs rely on the following assumptions:

1) \( R^N \) is Markov;
2) \( X^N \) is Markov conditionally on \( R^N \);
3) \( (Y_n), 1 \leq n \leq N \), are Gaussian, independent conditionally on \( (R^N, X^N) \) and verify:

\[
p \left( y_n \mid r_n, x_n \right) = p \left( y_n \mid r_n, x_n \right).
\]

In CGLSSMs, \( R^N \) and \( (R^N, X^N) \) are both Markov, and \( p \left( y_n \mid r_n, x_n \right) \) is very simple. These assumptions do not allow for exact computation of recursive filters, since \( (R^N, Y^N) \) is not Markov and \( p \left( r_n \mid y_n \right) \) cannot be computed sequentially and exactly. This problem has been addressed in recent “conditionally Markov switching hidden linear models” (CMSHLMs [6]), in which both \( R^N \) and \( (R^N, Y^N) \) are Markov, and \( p \left( x_n \mid r_n, x_n \right) \) is pretty general. Here we consider particular Gaussian CMSHLMs called “Conditionally Gaussian Observed Markov Switching Models” (CGOMSMs [7]–[9]), which verify:

1) \( R^N, (R^N, Y^N) \) and \( (X^N, R^N, Y^N) \) are Markov;
2) \( (X^N, Y^N) \) is Gaussian conditionally on \( R^N \).

I. INTRODUCTION

Let us consider the problem of statistical optimal filtering in the presence of switches. Three stochastic sequences are involved: states \( X^N = (X_1, \ldots, X_N) \), switches \( R^N = (R_1, \ldots, R_N) \), and observations \( Y^N = (Y_1, \ldots, Y_N) \). For each \( n = 1, \ldots, N \), the random variables \( X_n \) and \( Y_n \) take their values in \( R^m \) and \( R^q \), respectively, while \( R_n \) takes its values in the finite discrete set \( \Omega = \{ 0, 1, \ldots, K – 1 \} \). For the sake of simplification, we will assume in the remainder of this paper that (i) \( m = q = 1 \), i.e. \( X^N \) and \( Y^N \) are scalar-valued processes, and (ii) that \( K = 2 \). We consider these hypotheses only to simplify the presentation of the filtering method; the algorithms proposed in the following can be extended to the cases of vectorial processes and for a number of switches greater than two.

The problem is to sequentially estimate each \( X_{n+1} \) from \( Y^N_{n+1} \). Fast recursive optimal filters compute the estimated \( \hat{x}_{n+1} = (\hat{x}^N_{n+1}) \) and verify:

\[
p \left( y_{n+1} \mid \hat{x}^N_{n+1} \right) = p \left( y_{n+1} \mid \hat{x}^N_{n+1} \right).
\]

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Let us set $Z_n = (X_n, Y_n)^T$, $T_n = (X_n, R_n, Y_n)^T$ and assume the following:

1) $T_n$ is Markov;
2) $p(r_{n+1} | t_n) = p(r_{n+1} | r_n)$, which implies the Markovianity of $R_n$;
3) $Z_1 = (Z_1, \ldots, Z_n)$ is Gaussian conditionally on $R_n$.

Such a model, introduced in [8], is called "Conditionally Gaussian Markov Switching Model" (CGMSM), and is defined by $p(t_1)$, transitions $p(r_{n+1} | r_n)$, and

\[ Z_{n+1} = A_{n+1}(r_{n+1}^1)Z_n + B_{n+1}(r_{n+1}^1)W_{n+1} + N_{n+1}(r_{n+1}^1), \]  

for $n = 1, \ldots, N - 1$, and where

- $W_n = (U_n, V_n)^T$ with $U_1, V_1, \ldots, U_N, V_N$ Gaussian zero-mean independent vectors with identity covariance matrices;
- Matrices $A_{n+1}(r_{n+1}^1)$ and $B_{n+1}(r_{n+1}^1)$:

\[ A_{n+1}(r_{n+1}^1) = \begin{bmatrix} a_{n+1}^1(r_{n+1}^1) & a_{n+1}^2(r_{n+1}^1) \\ a_{n+1}^3(r_{n+1}^1) \\ a_{n+1}^4(r_{n+1}^1) \end{bmatrix}, \]
\[ B_{n+1}(r_{n+1}^1) = \begin{bmatrix} b_{n+1}^1(r_{n+1}^1) \\ b_{n+1}^2(r_{n+1}^1) \\ b_{n+1}^3(r_{n+1}^1) \end{bmatrix}. \]

- Means $N_{n+1}(r_{n+1}^1) = (N_{n+1}^{X}(r_{n+1}^1), N_{n+1}^{Y}(r_{n+1}^1))$ are given by

\[ N_{n+1}(r_{n+1}^1) = M_{n+1}(r_{n+1}^1) - A_{n+1}(r_{n+1}^1)M_n(r_n), \]

with

\[ M_n(r_n) = \mathbb{E} \left( X_n | Y_n, r_n \right) = \begin{bmatrix} N_{n}^X(r_n) \\ N_{n}^Y(r_n) \end{bmatrix}. \]

Rewrite filtering is not workable in the general CGMSM. Besides, let us notice that the classic "Conditionally Gaussian linear state-space model" (CGLSSM) [2] is a particular CGMSM obtained by setting, for each $r_{n+1}^1 \in \Omega^2$

\[ a_{n+1}^2(r_{n+1}^1) = a_{n+1}^4(r_{n+1}^1) = b_{n+1}^4(r_{n+1}^1) = 0. \]

Another particular CGMSM, called "Conditionally Gaussian observed Markov switching model" (CGOMSM) obtained from CGMSM by setting, for each $r_{n+1}^1 \in \Omega^2$ [7-10]

\[ a_{n+1}^2(r_{n+1}^1) = 0, \]

allows for recursive optimal filtering even with switches [8]. Indeed, CGOMSM belongs to the category of "conditionally Markov switching hidden linear models" (CMSHLMs) in which recursive optimal filtering is workable [6].

The aim of this paper is to extend the CGOMSM defined by [1]-[4] and eq. [5] to a "fuzzy" CGOMSM (denoted by CGOFMSM) and to show how the related recursive optimal ‘fuzzy’ filter runs.

III. CONDITIONALLY GAUSSIAN OBSERVED Fuzzy Markov Switching Model

Let us begin by illustrating with three examples the interest of the new proposed model in real situations.

In the first example, let sequence $X_n^N$ model the positions at time index $1, \ldots, N$ of a flying object, and let sequence $Y_n^N$ model the measurements provided by some optical sensors situated on the ground. During the tracking process, the sunlight can be partially or totally hidden due to the presence of clouds, which gives two models for the distribution of $Y_n^N$. This can be modelled by a ‘hard’ model with $R_n$ such that each $R_n$ takes its value in $\Omega = \{0, 1\}$, with 0 corresponding to total sunlight exposure and 1 to shadow condition. In some situations, during cloudy weather conditions that hide the sun partially, the transition from sunlight to shadow is ‘continuous’, and the duration of ‘intermediary’ light can be of paramount importance to the tracking process. This motivates the introduction of ‘fuzzy’ model with each $R_n$ belonging to $\Omega = [0, 1]$ rather than to $\Omega = \{0, 1\}$.

However, the distribution of $R_n$ on $\Omega = [0, 1]$ has to verify some properties. Having to have non-null probability to have sunshine - and likewise for shadow - implies that there should be two Dirac masses on 0 and 1. Then one can complete the distribution of $R_n$.
on $\Omega = [0, 1]$ by setting continuous probability on $[0, 1]$. Finally, the distribution of $R_n$ is defined by its density $p : [0, 1] \rightarrow \mathbb{R}$ with respect to $\nu = \delta_0 + \delta_1 + \mu_{0,1}$, where $\delta_0, \delta_1$ are Dirac’s distributions on $0, 1$, and $\mu_{0,1}$ is the Lebesgue’s measure on $[0, 1]$.

Let us consider a second example dealing with pedestrian tracking for surveillance purposes which consists in tracking the movements of pedestrians using aggregated data acquired from deployed sensors in the monitored area [19]. Due to the dynamic aspect of pedestrian motion in the presence of several contextual information such as crowd, the use of a two-motion model (corresponding to crowded / uncrowded configurations) is necessary. However, the concept of ‘crowd’ can be seen as a fuzzy phenomenon. Hence, relying on an abrupt change of parameters within the two-jump scheme does not take into account the intermediate states of pedestrian motion and impacts the accuracy of tracking process. Another example of the same problem relates to car traffic speed and density in a road segment [20].

A last example showing the potential interest of a fuzzy model appears when we want to study the phenomenon associated to outdoor air temperature. Typically, during one day, temperature reaches minimal values during the night and maximal during the afternoon. Between these two ranges, temperatures increase and decrease and can be represented by the fuzzy nature of the jumps considered in our model. An example of such a situation is detailed in Section VII.

From this perspective, the use of fuzzy transitions to model the transient change of parameters is more relevant than the salient switching model. The definition of the new “Conditionally Gaussian Observed Fuzzy Markov Switching Model” (CGOFMSM) we propose is similar to [1]–[3] and eq. [5], except that we limit our study to two hard classes and each $R_n$ takes its values in $\Omega = [0, 1]$.

**Definition 1.** Let $X^n_N, Y^n_N$ and $R^n_N$ be three stochastic sequences of random variables taking their values in $\mathbb{R}$, $\mathbb{R}$ and $[0, 1]$ respectively. The triplet $T^n_N = (T_1, \ldots, T_N)$, with $T_n = (X_n, R_n, Y_n)^\top$, will be said "Conditionally Gaussian observed fuzzy Markov switching model" (CGOFMSM) if:

1) $T^n_N$ verifies [1]–[3] and eq. [5];
2) The distribution of each $R_n$ is defined by a density (possibly depending on $n$) $p : [0, 1] \rightarrow \mathbb{R}$ with respect to $\nu = \delta_0 + \delta_1 + \mu_{0,1}$, where $\delta_0, \delta_1$ are Dirac’s distributions on $0, 1$, and $\mu_{0,1}$ is the Lebesgue’s measure on $[0, 1]$.

Let us recall some basic rules for integrating a function with respect to $\nu = \delta_0 + \delta_1 + \mu_{0,1}$. Such integration has two components: sum of its values on $0, 1$, and ‘classic’ integration over $[0, 1]$. More precisely, for any function $\phi : [0, 1] \rightarrow \mathbb{R}$, we have

\[
\int_0^1 \phi(r) dv(r) = \int_0^1 \phi(r)(\delta_0 + \delta_1 + \mu_{0,1}) = \phi(0) + \phi(1) + \int_0^1 \phi(r) dr.
\]  

(6)

In particular, the expectation of $\phi(R_n)$ is written

\[
E[\phi(r_n)] = \int_0^1 \phi(r)p(r)dv(r) = \phi(0)p(0) + \phi(1)p(1) + \int_0^1 \phi(r)p(r)dr.
\]

(7)

The distribution of a fuzzy Markov chain (FMC) $R^n_N$ is defined by the density $p(r_1)$ and the conditional densities $p(r_{n+1} | r_n)$. All of them are thus defined on $\Omega = [0, 1]$ and are densities w.r.t. $\nu$ as $\delta_0 + \delta_1 + \mu_{0,1}$. According to the general integration w.r.t. $\nu$ rule in eq. [6], we have

\[
\int_0^1 p(r_1) dv(r_1) = p(0) + p(1) + \int_0^1 p(r_1) dr_1 = 1,
\]  

(8)

and

\[
p(r_{n+1}) = \int_0^1 p(r_n) p(r_{n+1} | r_n) dv(r_n) = p(0)p(r_{n+1} | 0) + p(1)p(r_{n+1} | 1) + \int_0^1 p(r_n) p(r_{n+1} | r_n) dr_n,
\]

(9)

Finally, optimal filtering in ‘fuzzy’ CGOFMSM is not very different from that in ‘hard’ CGOMSM, the difference being that, in CGOMSM, integrating with respect to $r_n$ consists in summing, while in CGOFMSM it consists of integrating with respect to $\nu$.

**IV. OPTIMAL FUZZY SWITCHING RECURSIVE FILTER**

We wish to compute $p(r_{n+1} | y_{n+1}^n)$, $E[X_{n+1} | r_{n+1}, y_{n+1}^n]$, and $E[X_{n+2} | r_{n+1}, y_{n+1}^n]$ from $p(r_n | y_n^n)$, $E[X_n | r_n, y_n^n]$, $E[X_{n+1} | r_n, y_n^n]$, and $y_{n+1}$. According to eq. [2] and (5), $\{R^n_N, Y^n_N\}$ is a hidden Markov chain, which makes possible the computation of $p(r_{n+1} | y_{n+1}^n)$ as explained below.

First, let us note that the probabilities

\[
p(r_{n+1}^N, y_{n+1}^n | y_n^n) = p(n_{n+1} | r_{n+1}^N, y_n^n) p(r_n | y_n^n) p(r_{n+1} | r_n),
\]

(10)

can be calculated since

- $p(n_{n+1} | r_{n+1}^N)$ can be calculated from [13];
- $y_{n+1} | y_{n+1}^n$ are conditional densities of the multivariate Gaussians defined by [2]. Taking into account [5], by using [21] Section 8.13, page 40, we obtain means and variances of $a_{n+1}^N(r_{n+1}^N)(y_n - M_n^n(r_n)) + M_n^n(r_{n+1})$ and $(b_{n+1}^N(r_{n+1}^N))^2 + (b_n^N(r_{n+1}^N))^2$ respectively.

Secondly, the following probabilities

\[
p(y_{n+1} | y_n^n) = \int_0^1 p(r_{n+1}^N, y_{n+1} | y_n^n) (dv(r_n) \otimes dv(r_{n+1})),
\]

(11)

can also be computed accordingly:

\[
p(r_{n+1}^N | y_n^n) = \frac{p(r_{n+1}^N, y_{n+1} | y_n^n)}{p(y_{n+1} | y_n^n)}.
\]

(12)

Finally, using (10), eq. (12) gives the so-called forward probabilities

\[
p(r_{n+1} | y_{n+1}^n) = \int_0^1 p(r_{n+1}^N | y_{n+1}^n) dv(r_n) = \int_0^1 \frac{p(r_{n+1}^N, y_{n+1} | y_n^n) dv(r_n)}{p(y_{n+1} | y_n^n)}.
\]

(13)

Also, for the later use, note that

\[
p(r_n | r_{n+1}, y_{n+1}^n) = \frac{p(r_{n+1}^N | y_{n+1}^n)}{p(r_{n+1}^N | y_{n+1}^n)}.
\]

(14)

The “optimal fuzzy switching recursive filter” (OFSRF) we propose consists of five steps outlined as follows. To start the iterations, we first use the distribution of $T_1$. It is then possible to run the OFSRLF iterations, assuming that all quantities have been computed for sample $n$:

1) Compute $p(r_{n+1} | y_{n+1}^n)$ with [12]–[13];
2) Compute $E[Z_{n+1} | r_{n+1}^N, y_{n+1}^n]$ and $\text{Var}[Z_{n+1} | r_{n+1}^N, y_{n+1}^n]$.

From [3], we have

\[
E[Z_{n+1} | r_{n+1}^N, y_{n+1}^n] = A_{n+1}(r_{n+1}^N)E[Z_n | r_{n+1}^N, y_{n+1}^n] + N_{n+1}(r_{n+1}^N),
\]

(15)
Recalling that $R_{n+1}$ and $Z_n$ are independent conditionally on $R_n$ (Condition 2 in the definition of CGFMSM), we have
\[
E \left[ Z_n | r_{n+1}, y^n_1 \right] = E \left[ X_n | r_n, y_1 \right].
\]
Also, using (3) and from classical calculations detailed in (3) we have
\[
\begin{align*}
\text{Var} \left[ Z_{n+1} | r_{n+1}, y^n_1 \right] &= B_{n+1} \left( r_{n+1} \right) B_{n+1}^\top \left( r_{n+1} \right) \\
&= A_{n+1} \left( r_{n+1} \right) \text{Var} \left[ Z_n | r_{n+1}, y^n_1 \right] A_{n+1}^\top \left( r_{n+1} \right) \\
&= A_{n+1} \left( r_{n+1} \right) B_{n+1}^N \left( r_{n+1} \right) \\
&= A_{n+1} \left( r_{n+1} \right) \text{Var} \left[ Z_n, y_1^n \right] A_{n+1}^\top \left( r_{n+1} \right). 
\end{align*}
\] (16)

For the later convenience, let us note:
\[
\text{Var} \left[ Z_{n+1} | r_{n+1}, y^n_1 \right] = \left[ \alpha_{n+1} \left( r_{n+1} \right) - \delta_{n+1} \left( r_{n+1} \right) \right] \cdot \left( \frac{\beta_{n+1} \left( r_{n+1} \right)}{\delta_{n+1} \left( r_{n+1} \right)} \right),
\]
with $E \left[ X_{n+1} | r_{n+1}, y^n_1 \right]$ and $E \left[ Y_{n+1} | r_{n+1}, y^n_1 \right]$ given by eq. (15) and (17).

4) Compute $E \left[ X_{n+1} | r_{n+1}, y^n_1 \right]$ and $E \left[ Y_{n+1} | r_{n+1}, y^n_1 \right]$ using (15) with
\[
\begin{align*}
E \left[ X_{n+1} | r_{n+1}, y^n_1 \right] &= \int_0^1 E \left[ X_{n+1} | r_{n+1}, y^n_1 \right] p \left( r_n | r_{n+1}, y^n_1 \right) \, dr (r_n), \\
E \left[ Y_{n+1} | r_{n+1}, y^n_1 \right] &= \int_0^1 E \left[ Y_{n+1} | r_{n+1}, y^n_1 \right] p \left( r_n | r_{n+1}, y^n_1 \right) \, dr (r_n). 
\end{align*}
\] (20)

5) Finally, compute the filtering equations
\[
\begin{align*}
E \left[ X_{n+1} | y^n_1 \right] &= \int_0^1 E \left[ X_{n+1} | r_{n+1}, y^n_1 \right] p \left( r_{n+1} | y^n_1 \right) \, dr (r_{n+1}), \\
E \left[ Y_{n+1} | y^n_1 \right] &= \int_0^1 E \left[ Y_{n+1} | r_{n+1}, y^n_1 \right] p \left( r_{n+1} | y^n_1 \right) \, dr (r_{n+1}). 
\end{align*}
\] (21)

\[ \text{Remark 1. Integration with respect to } \nu \text{ above cannot be written in a closed-form formula. It is, then, approximated by numerical integration. Let } F \text{ denote the number of discrete steps used to compute integrals on } [0, 1]. \text{ The impact of } F \text{ on the restoration results will be discussed in the experimental section.} \]

V. Model Parametrization

In order to assess the interest of the filtering algorithm on CGOFMSM simulated data, we will consider stationary CGOFMSM models, with distribution defined by $p \left( x_1^n, r_1, y^n_1 \right) = p \left( r_1 \right) p \left( x_1^n \right) p \left( y^n_1 | r_1 \right).$ Thus, we have to define $p \left( r_1 \right)$ (Section V-A) and $p \left( x_1^n, y^n_1 | r_1 \right)$ (Section V-B). Thanks to the particular structure of $\nu$, this can be done in such a way that when the fuzziness disappears, it is to say when $p \left( r_n \right) = 0$ on $[0, 1]$ for $n = 1, \ldots, N$, a CGOFMSM becomes a classical CGFMSM.

A. Distribution of $(R_1, R_2)$

Let us notice that in the ‘hard’ case with two possible switches, the distribution $P_{R_1, R_2}$ is simply a proportion over $\{0, 1\}$. In the fuzzy case we deal with, it is a distribution on $[0, 1]^2$, which provides a wide range of possibilities for choosing its shape. We next describe two possible shapes of interest for $P_{R_1, R_2}$ (called FMC1 and FMC2 models, where FMC stands for ‘Fuzzy Markov Chain’, that will be experimented in next Section.

1) First case (FMC1 model): The density $p \left( r_1 \right)$ of $P_{R_1}$ w.r.t. $\nu \otimes \nu$ - where $\nu = \delta_0 + \delta_1 + \mu_{0,1} - \mu_{1,0}$ - is of the form:
\[
p(0, 0) = p(1, 1) = \alpha, \quad p(1, 0) = p(0, 1) = \beta, \quad p(r_1, r_2) = \eta + (\delta - \eta) |r_1 - r_2|,
\]
with $r_1, r_2 \in [0, 1]^2$ \ $\{0, 1\}^2$, \ \ $\int_0^1 p(r_1, r_2) \, dr (r_1) \, dr (r_2) = 1$. A possible shape for this density is illustrated in Fig 3.

2) Second case (FMC2 model): The density $p \left( r_1 \right)$ of $P_{R_1}$ is computed as follows:
\[
p(r_1) = \begin{cases}
\alpha + \beta + \delta + \eta & \text{if } r_1 = 0, \\
\alpha + \beta + \delta - \eta & \text{if } r_1 = 1,
\end{cases}
\]
\[
p(r_1, r_2) = \eta + (\delta - \eta) (r_1^2 - r_1), \quad \text{if } r_1 \in [0, 1].
\]

Knowing that $\int_0^1 p(r_1) \, dr (r_1) = 1$, we get
\[
\beta = 1 - \frac{1}{2} (\delta + \eta) + \frac{1}{2} (\delta - \eta) - \alpha.
\]

Hence, this model is only parametrized by $\{\alpha, \delta, \eta\}$ (the calculations are detailed in Appendix A).

The limit proportion of hard data $(p_H)$ with respect to fuzzy ones $(p_F)$ in a sampled sequence is
\[
p_H = p(0) + p(1) = 2 (\alpha + \beta) + (\delta + \eta), \quad p_F = 1 - p_H = \frac{3}{2} (\delta + \eta) - \frac{1}{2} (\delta - \eta). \quad (25)
\]

The density $p(r_2 | r_1)$ of distribution $P_{R_2 | R_1}$ w.r.t. $\nu$ is the ratio between the joint density and the marginal density. We have to distinguish between different cases, according to the value of $r_1$:
\[
p(r_2 | r_1 = 0) = \begin{cases}
\frac{1}{r_1} & \text{if } r_2 = 0, \\
\frac{1}{r_1 + \delta - \eta - r_2} & \text{if } r_2 \in [0, 1],
\end{cases}
\]
\[
p(r_2 | r_1 = 1) = \begin{cases}
\frac{1}{r_1} & \text{if } r_2 = 0, \\
\frac{1}{r_1 + \delta - \eta - r_2} & \text{if } r_2 \in [0, 1],
\end{cases}
\]
\[
p(r_2 | r_1 \in [0, 1]) = \frac{\delta + \eta - r_1}{r_1 + \delta - \eta - r_2}, \quad \text{if } r_2 = 1. 
\] (28)
with $D_1 = \alpha + \beta + \frac{\delta + \eta}{2}$ and $D_2 = \frac{3}{2}(\delta + \eta) + (\delta - \eta)(r_2^2 - r_1)$.

For the particular case where $\delta = \eta = 0$, we have a classical Markov chain with 2 states, and $p(r_2|r_1)$ writes

$$p(r_2|r_1 = 0) = \begin{cases} \frac{\alpha}{\alpha + \beta} & \text{if } r_2 = 0, \\ \frac{\beta}{\alpha + \beta} & \text{if } r_2 = 1. \end{cases}$$

$$p(r_2|r_1 = 1) = \begin{cases} \frac{\alpha}{\alpha + \beta} & \text{if } r_2 = 0, \\ \frac{\beta}{\alpha + \beta} & \text{if } r_2 = 1. \end{cases}$$  

2) Second case (FMC2 model): The density $p(r_1^2)$ of $P_{R|T}$ w.r.t. $\nu \otimes \nu$ is of the form:

$$p(0,0) = p(1,1) = \alpha,$$

$$p(1,0) = p(0,1) = \beta,$$

$\gamma$ for $r_1, r_2 \in [0,1]^2 \setminus \{0,1\}^2$ and $-\delta \leq r_2 - r_1 \leq \delta$, $0$ elsewhere.

with $\alpha, \beta \geq 0$, $0 \leq \delta < \frac{1}{2}$, and under the constraint that $\int_{0}^{1} p(r_1, r_2) \, dv(r_1) \, dv(r_2) = 1$. A possible shape for this density is illustrated in Fig. 2. By varying $\delta$, this model allows expressing transient fuzzy changes.

**Remark 3.** If $\alpha + \beta = \frac{1}{2}$, then $\gamma = 0$, and the joint law is only made of the four Dirac’s distributions at the four corners, which gives a CGOMSM.

The density $p(r_1)$ of $P_{R|T}$ is computed as follows:

$$p(r_1) = \begin{cases} \alpha + \beta + \gamma \delta & \text{if } r_1 = 0, \\ \gamma(\delta + r_1 + 1) & \text{if } r_1 \in [0,\delta], \\ 2\gamma \delta & \text{if } r_1 \in [\delta, 1 - \delta], \\ \gamma(2 + \delta - r_1) & \text{if } r_1 \in [1 - \delta, 1], \\ \alpha + \beta + \gamma \delta & \text{if } r_1 = 1. \end{cases}$$  

Since $\int_{0}^{1} p(r_1) \, dv(r_1) = 1$, we get

$$\beta = \frac{1 - \gamma M}{2} - \alpha,$$

with $M = \delta(6 - \delta)$, and under the constraint that $\gamma \leq \frac{1 - 2\alpha}{2\delta}$. Hence this model is only parametrized by $(\alpha, \gamma, \delta)$.

The limit proportion of hard data ($p_H$) to fuzzy ones ($p_F$) in a sampled sequence is

$$p_H = p(0) + p(1) = 2(\alpha + \beta + \gamma \delta),$$

$$p_F = 1 - p_H = \gamma \delta(4 - \delta).$$

The density $p(r_2|r_1)$ of distribution $P_{R_2|R_1}$, w.r.t. $\nu$, is the ratio between the joint density and the marginal density. Similarly to the first case, we have to distinguish between different configurations, according to the value of $r_1$:

$$p(r_2|r_1 = 0) = \begin{cases} \frac{\alpha}{\alpha + \beta + \gamma \delta} & \text{if } r_2 = 0, \\ \frac{\alpha + \beta + \gamma \delta}{\alpha + \beta} & \text{if } r_2 = 1, \\ 0 & \text{elsewhere}. \end{cases}$$

$$p(r_2|r_1 \in [0,\delta]) = \begin{cases} \frac{1}{\alpha + \beta + \gamma \delta} & \text{if } r_2 \in [0, r_1 + \delta], \\ 0 & \text{elsewhere}. \end{cases}$$

$$p(r_2|r_1 \in [1 - \delta, 1]) = \begin{cases} \frac{\beta}{\alpha + \beta + \gamma \delta} & \text{if } r_2 \in [r_1 - \delta, 1], \\ \frac{\alpha}{\alpha + \beta + \gamma \delta} & \text{if } r_2 = 1, \\ 0 & \text{elsewhere}. \end{cases}$$

**Remark 4.** In the two examples of fuzzy Markov models, we assume that the distribution $P_{R_1,R_2}$ is defined by four masses on the corners, i.e. at locations $\{0,1\} \times \{0,1\}$, and a density on the remaining $[0,1]^2 \setminus \{0,1\}^2$. It is to say that we assume the distributions on the sides of the square $[0, 1] \times [0, 1]$, i.e. the distributions $P_{\{0,1\}, R}$, $P_{\{1, R\}, 0}$ and $P_{R, \{0,1\}}$ for $R$ in $[0, 1]$, to be identical to the inner density of the square. This is a particular case, used here to simplify the parametrization, since we can set them independently.
null
B. Restoration results

The restoration of simulated data is performed according to the OFSRF algorithm detailed in Section IV, from simulated observations only. Fig. 4 shows an example of the restoration of \( r_{1}^N \) and \( x_{1}^N \) for the FMC2 law and when the number of discrete fuzzy jumps are set to \( F = 5 \). The restoration of \( r_{1}^N \) was obtained by applying Maximum Posteriori Mode (MPM) principle:

- If \( p(r_n = 0|y_{n}^N) + p(r_n = 1|y_{n}^N) > 0.5 \) then the restoration will be \"hard\", else it will be \"fuzzy\".
- If the restoration is \"hard\", set \( \hat{r}_n \) to be \"0\" if \( p(r_n = 0|y_{n}^N) > p(r_n = 1|y_{n}^N) \), else set \( \hat{r}_n \) to be \"1\";
- If the restoration is \"fuzzy\", set \( \hat{r}_n \) to the discrete fuzzy jump which maximizes \( p(r_n|y_{n}^N) \).

Note however that the restoration of \( r_{1}^N \) is only performed for illustration purpose and is not required for the restoration of \( x_{1}^N \).

We can observe the numerical effect of \( F \) with the presence of stair-steps in the restoration of jumps in (b). Fig. (a) shows the restoration of \( x_{1}^N \) assuming that the jumps are known (i.e. using the simulated fuzzy Markov chain for \( r_{1}^N \)). Fig. (c) assumes that the jumps are unknown (OFSRF algorithm). Also, fig. (c) allows to observe the restoration difference between the \‘classic\’ CGOFMSM and the \‘fuzzy\’ CGOFMSM, the latter follows the simulated states better than the first one for \( n \leq 30 \). This behaviour must be compared with that of the fuzzy jumps in Fig. (b) which shows a large difference in the restoration of jumps when \( n \leq 30 \).

To measure the quality of restorations with respect to \( F \), we compute the mean MSE of 50 independent experiments of \( N = 300 \) samples, for the FMC1 Markov law. Fig. 5 shows the MSE evolution for increasing values of \( F \) for both \( x_{1}^N \) and \( r_{1}^N \). The result of applying the CGOFMSM filter on the data is reported in the same graph (horizontal black dotted line, denoted as ‘Hard filter - UJ’).

The parameters used for the hard filters are the same than the ones used for the fuzzy filter, except that we ‘harden’ the Markov laws by integrating the fuzzy laws on all 4 quadrants of \([0,1]\) to get...
The last experiment, whose results are reported in Table II, for the FMC1 and FMC2 laws, shows the restoration MSE of the CGOMSM and the CGOFMSM filters when the number of fuzzy samples in the simulated data is decreasing (by adjusting parameter’ values). We can observe that, for both fuzzy Markov models, the hard filter reaches the performance of the fuzzy filter only when the percent of hard jumps is near 100%. Elsewhere the fuzzy filter provides lower MSE, and the difference can be very large for hard sample rates lower than 50%. This result confirms the interest of the fuzzy filter in comparison with the hard one in the presence of transient changes in observation data.

VII. ILLUSTRATION ON REAL DATA

This section intends to illustrate the behaviour of the proposed algorithm when confronted to real data. The experimental data are time series representing the energy power (in kilo-Watt) consumed by some building along with the outdoor temperature (in Fahrenheit)\(^1\).

We try to understand to what extent the fuzzy model is able to infer the outdoor temperature from the consumed energy for some building during the first week of June 2010, cf. Fig. 6.

Thus, energy consumption is considered as the observation \((Y^N_1)\), and the outside air temperature as the state \((X^N_1)\), to be estimated, with \(N = 672\). The fuzzy filter requires to know the jumps \(R^N_1\) to estimate the best suited parameters for data. Regardless the data, we have fixed that the lowest temperature appears between 1am and 5am (corresponding to hard jump ‘0’), and that the highest appear between 1pm and 5pm (corresponding to hard jump ‘1’). Between these ranges, temperatures increase and decrease linearly and are represented by the fuzzy nature of the jumps considered in this model. The shape of the observations suggests that a fuzzy model is better suited than a hard model (we choose to use the FMC1 model).

From this pseudo ground-truth for jumps, using classical empirical estimators, it is possible to estimate all the required parameters of the model: on the one hand, mean values and covariance matrices of \((X^N_1, Y^N_1)\) conditionally on jumps for the two hard jumps, and, on the other hand, the \(\alpha, \beta, \eta, \delta\) parameters required to define the law of \(R^N_1\).

The results of fuzzy and hard filtering are reported in cyan in Fig. 7 and 8 respectively, with the measured air temperature given in red. For the fuzzy filtering, we considered \(F = 5\) because it gives good results while maintaining low computation times. The MSE of estimated states with respect to the true outdoor air temperature is 8701 for the hard model, and 6759 for the fuzzy model. Regarding the jumps, the MSE is 0.13 for the hard model, and 0.07 for the fuzzy one. The better results obtained with the fuzzy model w.r.t. the hard one are illustrated by both the estimated jumps and the estimated outdoor air temperature.

VIII. CONCLUSION

We proposed a new jump Markov model made of a triplet random process (observations, hidden states, hidden fuzzy switches), and designed the related optimal recursive fuzzy filter which is able to restore switches and states from observations. We called this model “Conditionally Gaussian Observed Fuzzy Markov switching Model” (CGOFMSM). The work is based on two key ideas:

- A recursive and exact filter to deal with hard jumps, called CGOMSM, is available, see [7]–[10]. This filter only assumes the presence of a zero-term \((5)\) in the transition matrix of the very general Conditionally Gaussian Markov Switching model defined by \(F\).

\(^{1}\)This time series, collected by the American Department of Energy, is open-data and can be downloaded for free from https://openepi.org/datasets/dataset/consumption-outdoor-air-temperature-11-commercial-buildings. The experiments focus on building #5.
The MSE for jumps is

\[
\text{MSE for jumps} = \int_{0}^{1} \eta + (\delta - \eta) |r_1 - r_2| \, dr_2, 
\]

with

\[
A(r_1) = \eta + (\delta - \eta) \int_{0}^{1} |r_1 - r_2| \, dr_2, 
\]

and

\[
B_1(r_1) = \int_{0}^{1} (r_1 - r_2) \, dr_2 = r_1^2 - \frac{1}{2}r_2^2 = \frac{1}{2}r_1^2, 
\]

\[
B_2(r_1) = \int_{0}^{1} (2r_2 - r_1) \, dr_2 
= 2(1-r_1^2) - r_1(1-r_1) = \frac{1}{2}((1+r_1^2) - r_1). 
\]

Thus we have

\[
B(r_1) = \frac{1}{2} + r_1^2 - r_1, \quad \text{and} \quad A(r_1) = \eta + (\delta - \eta)(\frac{1}{2} + r_1^2 - r_1). 
\]

So, for \( r_1 = 0 \), \( p(r_1) = \alpha + \beta + A(0) = \alpha + \beta + \frac{3+\eta}{2} \), for \( r_1 = 1 \), \( p(r_1) = \alpha + \beta + A(1) = \alpha + \beta + \frac{1+\eta}{2} \), and for \( r_1 \in [0, 1] \),

\[
p(r_1) = \eta + (\delta - \eta) r_1 + \eta + (\delta - \eta)(1-r_1) + \eta + (\delta - \eta) \left( \frac{1}{2} + r_1^2 - r_1 \right) 
= 3\eta + (\delta - \eta) \frac{3}{2} + (\delta - \eta)(r_1^2 - r_1) 
= \frac{3}{2}(\delta + \eta) + (\delta - \eta)(r_1^2 - r_1). 
\]

Hence, we get eq. [23].

### B. Calculation of \( \beta \)

We have

\[
\int_{0}^{1} p(r_1) \, dr_1 = p(0) + p(1) + \int_{0}^{1} p(r_1) \, dr_1 
= 2(\alpha + \beta) + (\delta + \eta) + \int_{0}^{1} \frac{1}{2}(\delta + \eta) + (\delta - \eta)(r_1^2 - r_1) \, dr_1, 
\]

\[
c = c_1 + c_2 
\]
with
\[ C_1 = \int_0^t \frac{3}{2} (\delta + \eta) \, dr_1 = \frac{3}{2} (\delta + \eta), \]
\[ C_2 = (\delta - \eta) \int_0^t (r_1^2 - r_1) \, dr_1 = -\frac{1}{6} (\delta - \eta). \]
Knowing that \( \int_0^t p(r_1) \, dr_1 = 1 \), we get \( 2(a + \beta) + (\delta + \eta) + \frac{3}{2} (\delta + \eta) - \frac{1}{6} (\delta - \eta) = 1 \), and find result in eq. (24).

C. Calculation to get eq. (16)

From
\[ \text{Var} \left[ Z_{n+1} | r_n^{n+1}, y_n^r \right] = \mathbb{E} \left[ Z_{n+1} Z_{n+1}^T | r_n^{n+1}, y_n^r \right] - \mathbb{E} \left[ Z_{n+1} | r_n^{n+1}, y_n^r \right] \mathbb{E} \left[ Z_{n+1}^T | r_n^{n+1}, y_n^r \right], \]
we have
\[ \mathbb{E} \left[ Z_{n+1} Z_{n+1}^T | r_n^{n+1}, y_n^r \right] = A_{n+1} (r_n^{n+1}) \mathbb{E} \left[ Z_n Z_n^T | r_n^{n+1}, y_n^r \right] A_{n+1}^T (r_n^{n+1}) + N_{n+1} (r_n^{n+1}) \mathbb{E} \left[ Z_n r_n^{n+1} y_n^r \right] A_{n+1}^T (r_n^{n+1}) + B_{n+1} (r_n^{n+1}) B_n^T (r_n^{n+1}) + N_{n+1} (r_n^{n+1}) N_{n+1}^T (r_n^{n+1}). \]
and, using (15),
\[ \mathbb{E} \left[ Z_{n+1} | r_n^{n+1}, y_n^r \right] = A_{n+1} (r_n^{n+1}) \mathbb{E} \left[ Z_n | r_n^{n+1}, y_n^r \right] A_{n+1}^T (r_n^{n+1}) + N_{n+1} (r_n^{n+1}) \mathbb{E} \left[ Z_n r_n^{n+1} y_n^r \right] A_{n+1}^T (r_n^{n+1}) + N_{n+1} (r_n^{n+1}) N_{n+1}^T (r_n^{n+1}). \]
Subtracting (47) and (48) gives (16).

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