Linearly definable classes of Boolean functions

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Dedicated to Maurice Pouzet on the occasion of his 75th birthday

Abstract

In this paper we address the question "How many properties of Boolean functions can be defined by means of linear equations?" It follows from a result by Sparks that there are countably many such linearly definable classes of Boolean functions. In this paper, we refine this result by completely describing these classes. This work is tightly related with the theory of function minors and stable classes, a topic that has been widely investigated in recent years by several authors including Maurice Pouzet.

Keywords Functional equation · linear definability · clone · clonoid · Boolean function

1 Introduction and motivation

Functional equations are universally quantified first-order sentences in a certain algebraic syntax, with a single function symbol and no other predicate symbol than equality. More precisely, a *functional equation* for a function of several arguments from A to B is a formal expression

$$h_1(\mathbf{f}(g_1(\mathbf{v}_1, \dots, \mathbf{v}_p)), \dots, \mathbf{f}(g_m(\mathbf{v}_1, \dots, \mathbf{v}_p))) = h_2(\mathbf{f}(g'_1(\mathbf{v}_1, \dots, \mathbf{v}_p)), \dots, \mathbf{f}(g'_t(\mathbf{v}_1, \dots, \mathbf{v}_p))), \tag{1}$$

where $m, t, p \geq 1$, $h_1 \colon B^m \to C$, $h_2 \colon B^t \to C$, each g_i and g'_j is a map $A^p \to A$, the $\mathbf{v}_1, \dots, \mathbf{v}_p$ are p distinct symbols called *vector variables*, and $\mathbf{f}$ is a distinct symbol called *function symbol*. An n-ary function $f \colon A^n \to B$ is said to *satisfy* the equation (1) if, for all $\mathbf{a}_1, \dots, \mathbf{a}_p \in A^n$, we have

$$h_1(f(g_1(\mathbf{a}_1, \dots, \mathbf{a}_p)), \dots, f(g_m(\mathbf{a}_1, \dots, \mathbf{a}_p))) = h_2(f(g'_1(\mathbf{a}_1, \dots, \mathbf{a}_p)), \dots, f(g'_t(\mathbf{a}_1, \dots, \mathbf{a}_p))),$$

where the g_i and g'_i are applied componentwise. Well-known examples of functional properties definable by such functional equations include the linearity property of functions over fields, the monotonicity and convexity properties that are typically expressed by functional inequalities.

Such functional equations regained interest in 2000, due to the work of Ekin, Foldes, Hammer, and Hellerstein [8] who showed that the equational classes of Boolean functions are exactly those classes that are closed under introduction of fictitious variables, and identification and permutation of variables. These operations on functions give rise to a preorder on functions, the so-called *simple minor* relation, and equational classes are exactly the "initial segments" for this preorder [3, 7]. Alternatively, these classes appear naturally in a Galois theory proposed by Pippenger [18] that is based on the preservation relation between functions and relation pairs (also called "relational constraints"). Using this framework it was shown that, even in the case of Boolean functions, there are uncountably many classes of functions definable by functional equations. For instance, all Post's classes (clones of Boolean functions), traditionally characterized by relations, are definable by functional equations.

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This motivated several studies that considered syntactic restrictions on functional equations and relational constraints. Foldes and Pogosyan [10] considered a variant, the so-called *functional terms*, to define all Boolean clones and to give a criterion to determine whether a clone is finitely definable. In [4] the authors focused on linear equations and showed that the classes of Boolean functions definable by linear equations are exactly those that are stable under left and right compositions with the clone of constant-preserving linear functions or, equivalently, definable by affine constraints. This was later extended to arbitrary functions over fields [5], and to stability under compositions with arbitrary clones [6]: an equational class is definable by relation pairs in which the two relations are invariant for clones C_1 and C_2, respectively, if and only if the class is stable under left composition with C_1 and under right composition with C_2 (in short, (C_1, C_2)-stable). Instances of the idea of (C_1, C_2)-stability are present in various studies. The initial segments of so-called C-minor quasiorders, systematically studied in [12, 13, 14, 15, 16, 17], are exactly such (C_1, C_2)-stable classes where the first clone C_1 is the clone of projections. On the other hand, when C_2 is the clone of projections, we get clonoids, as studied by Aichinger, Mayr, and others [1, 2, 21]. The case when both C_1 and C_2 are clones of projections corresponds to minor-closed classes. As an example of recent work on (C_1, C_2)-stable classes that is closely related with the current paper, we would like to mention studies of function classes stable under left and right compositions with clones of linear functions by Fioravanti and Kreinecker [9, 11].

Getting back to linearly definable classes of Boolean functions, in [5] it was observed that, for each integer $k \geq 0$, the class of Boolean functions whose degree is upper bounded by k is definable by the following linear equation:

$$\sum_{I \subseteq \{1,\dots,k+1\}} \mathbf{f}(\sum_{i \in I} \mathbf{v}_i) = 0.$$

This shows that even in the case of Boolean functions, there are infinitely many linearly definable classes. Other examples were also provided, but it remained until recently an open problem to determine whether there are uncountably many linearly definable classes as is the case with classes definable by unrestricted functional equations. The answer follows from a result of Sparks [21, Theorem 1.3], namely, there are a countably infinite number of linearly definable classes.

In this paper we refine this result by explicitly describing the linearly definable classes of Boolean functions. After recalling some basic notions and results on function minors and stability under composition with clones in Section 2, we then completely describe the lattice of linearly definable classes (Section 3). Using this result and Post's classification of Boolean clones, we can easily determine the classes which are stable under right and left compositions with clones C_1 and C_2 containing the clone of constant-preserving linear functions (Section 4).

2 Basic notions and preliminary results

Throughout this paper, let $\mathbb{N}$ and $\mathbb{N}_+$ denote the set of all nonnegative integers and the set of all positive integers, respectively. For any $n \in \mathbb{N}$, the symbol $[n]$ denotes the set $\{\, i \in \mathbb{N} \mid 1 \leq i \leq n \,\}$.

Let A and B be sets. A mapping of the form $f\colon A^n \to B$ for some $n \in \mathbb{N}_+$ is called a *function of several arguments from A to B* (or simply a *function*). The number n is called the *arity* of f and denoted by $\operatorname{ar}(f)$. If $A = B$, then such a function is called an *operation on A*. We denote by $\mathcal{F}_{AB}$ and $\mathcal{O}_A$ the set of all functions of several arguments from A to B and the set of all operations on A, respectively. For any $n \in \mathbb{N}_+$, we denote by $\mathcal{F}_{AB}^{(n)}$ the set of all n-ary functions in $\mathcal{F}_{AB}$, and for any $C \subseteq \mathcal{F}_{AB}$, we let $C^{(n)} := C \cap \mathcal{F}_{AB}^{(n)}$ and call it the *n-ary part* of C.

Example 2.1. For $b \in B$ and $n \in \mathbb{N}$, the n-ary *constant function* $\mathrm{c}_b^{(n)}\colon A^n \to B$ is given by the rule $(a_1, \dots, a_n) \mapsto b$ for all $a_1, \dots, a_n \in A$.

Example 2.2. In the case when $A = B$, for $n \in \mathbb{N}$ and $i \in [n]$, the i-th n-ary *projection* $\mathrm{pr}_i^{(n)}\colon A^n \to A$ is given by the rule $(a_1, \dots, a_n) \mapsto a_i$ for all $a_1, \dots, a_n \in A$.

Let $f\colon A^n \to B$ and $i \in [n]$. The i-th argument is *essential* in f if there exist $a_1, \dots, a_n, a'_i \in A$ such that

$$f(a_1, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n).$$

An argument that is not essential is *fictitious.*

2.1 Minors and functional composition

Let $f\colon B^n \to C$ and $g_1, \dots, g_n\colon A^m \to B$. The *composition* of f with $g_1, \dots, g_n$ is the function $f(g_1, \dots, g_n)\colon A^m \to C$ given by the rule

$$f(g_1, \dots, g_n)(\mathbf{a}) := f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})), \quad \text{for all } \mathbf{a} \in A^m.$$

Let $\sigma\colon [n] \to [m]$. Define the function $f_\sigma\colon A^m \to B$ by the rule

$$f_\sigma(a_1, \dots, a_m) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)}),$$

for all $a_1, \dots, a_m \in A$. Such a function f_σ is called a *minor* of f. Intuitively, minors of f are all those functions that can be obtained from f by manipulation of its arguments: permutation of arguments, introduction of fictitious arguments, identification of arguments. It is clear from the definition that the minor f_σ can be obtained as a composition of f with m-ary projections on A:

$$f_\sigma = f(\mathrm{pr}^{(m)}_{\sigma(1)}, \dots, \mathrm{pr}^{(m)}_{\sigma(n)}).$$

We write $f \leq g$ if f is a minor of g. The minor relation $\leq$ is a quasiorder (a reflexive and transitive relation) on $\mathcal{F}_{AB}$, and it induces an equivalence relation $\equiv$ on $\mathcal{F}_{AB}$ and a partial order on the quotient $\mathcal{F}_{AB}/{\equiv}$ in the usual way: $f \equiv g$ if $f \leq g$ and $g \leq f$, and $f/{\equiv} \leq g/{\equiv}$ if $f \leq g$.

Functional composition can be extended to classes of functions. Let $C \subseteq \mathcal{F}_{BC}$ and $K \subseteq \mathcal{F}_{AB}$. The *composition* of C with K is defined as

$$CK := \{\, f(g_1, \dots, g_n) \mid f \in C^{(n)},\ g_1, \dots, g_n \in K^{(m)},\ n, m \in \mathbb{N}_+ \,\}.$$

It follows immediately from definition that function class composition is monotone, i.e., if $C, C' \subseteq \mathcal{F}_{BC}$ and $K, K' \subseteq \mathcal{F}_{AB}$ satisfy $C \subseteq C'$ and $K \subseteq K'$, then $CK \subseteq C'K'$.

2.2 Clones, minor closure and stability under compositions with clones

A class $C \subseteq \mathcal{O}_A$ is called a *clone* on A if $CC \subseteq C$ and C contains all projections. The set of all clones on A is a closure system in which the greatest and least elements are the clone $\mathcal{O}_A$ of all operations on A and the clone of all projections on A, respectively.

Definition 2.3. Let $K \subseteq \mathcal{F}_{AB}$, $C_1 \subseteq \mathcal{O}_B$, and $C_2 \subseteq \mathcal{O}_A$. We say that K is *stable under left composition* with C_1 if $C_1K \subseteq K$, and that K is *stable under right composition* with C_2 is $KC_2 \subseteq K$. If both $C_1K \subseteq K$ and $KC_2 \subseteq K$ hold, we say that K is *(C_1, C_2)-stable.* If $K, C \subseteq \mathcal{O}_A$ and K is (C, C)-stable, we say that K is *C-stable.* The set of all (C_1, C_2)-stable subsets of $\mathcal{F}_{AB}$ is a closure system.

Remark 2.4. A set $K \subseteq \mathcal{F}_{AB}$ is minor-closed if and only if it is stable under right composition with the set of all projections on A. Every clone is minor-closed. A clone C is (C, C)-stable.

Lemma 2.5. *Let C_1 and C'_1 be clones on B and C_2 and C'_2 clones on A such that $C_1 \subseteq C'_1$ and $C_2 \subseteq C'_2$. Then for every $K \subseteq \mathcal{F}_{AB}$, it holds that if K is (C'_1, C'_2)-stable then K is (C_1, C_2)-stable.*

Proof. Assume that K is (C'_1, C'_2)-stable. It follows from the monotonicity of function class composition that

$$C_1K \subseteq C'_1K \subseteq K \quad \text{and} \quad KC_2 \subseteq KC'_2 \subseteq K.$$

In other words, K is (C_1, C_2)-stable. □

3 The lattice of linearly definable classes of Boolean functions

Recall that operations on $\{0, 1\}$ are called *Boolean functions.* In this section we completely describe the lattice of linearly definable classes of Boolean functions. The starting point is the following characterization of these classes first obtained for Boolean functions in [4], and later extended to classes of functions defined on $\{0, 1\}$ and valued in rings [6].

Theorem 3.1. *A class of Boolean functions is linearly definable if and only if it is stable under left and right compositions with the clone of constant-preserving linear Boolean functions.*

Hence to completely describe the linearly definable classes it suffices to determine those that are stable under left and right compositions with the clone of constant-preserving linear Boolean functions. This will be presented in Subsection 3.2.

3.1 Some special classes of Boolean functions

The class of all Boolean functions is denoted by Ω. It is well known that every $f \in \Omega^{(n)}$ is represented by a unique multilinear polynomial over the two-element field, i.e., a polynomial with coefficients in $\{0, 1\}$ in which no variable appears with an exponent greater than 1. This polynomial is known as the *Zhegalkin polynomial* of f, and it can be written as

$$f = \sum_{S \in M_f} x_S,$$

where x_S is a shorthand for $\prod_{i \in S} x_i$ and where $M_f \subseteq \mathcal{P}([n])$ is the family of index sets corresponding to the monomials of f. Note that $x_\emptyset = 1$ and $\sum_{S \in \emptyset} x_S = 0$. The terms x_S with $S \neq \emptyset$ are called *monomials*. If $\emptyset \in M_f$, then we say that f has *constant term* 1; otherwise f has *constant term* 0. Without any risk of confusion, we will often denote functions by their Zhegalkin polynomials, and we refer to the set M_f as the *set of monomials* of f.

The *degree* of a Boolean function f, denoted $\deg(f)$, is the size of the largest monomial of f, i.e.,

$$\deg(f) := \max_{S \in M_f} |S|$$

for $f \neq 0$, and we agree that $\deg(0) := 0$. For $k \in \mathbb{N}$, we denote by D_k the class of all Boolean functions of degree at most k. Clearly $\mathsf{D}_k \subsetneq \mathsf{D}_{k+1}$ for all $k \in \mathbb{N}$. A Boolean function f is *linear* if $\deg(f) \leq 1$.[2] We denote by L the class of all linear functions. Thus $\mathsf{L} = \mathsf{D}_1$.

For $a \in \{0,1\}$, let $\mathsf{C}_a := \{\, f \in \Omega \mid f(0,\dots,0) = a \,\}$ and $\mathsf{E}_a := \{\, f \in \Omega \mid f(1,\dots,1) = a \,\}$. Clearly $\mathsf{C}_0 \cap \mathsf{C}_1 = \emptyset$ and $\mathsf{C}_0 \cup \mathsf{C}_1 = \Omega$; similarly, $\mathsf{E}_0 \cap \mathsf{E}_1 = \emptyset$ and $\mathsf{E}_0 \cup \mathsf{E}_1 = \Omega$. It is easy to see that C_a is the class of all Boolean functions with constant term a.

For $a \in \{0,1\}$, a Boolean function f is *a-preserving* if $f(a,\dots,a) = a$. A function is *constant-preserving* if it is both 0- and 1-preserving. We denote the classes of all 0-preserving, of all 1-preserving, and of all constant-preserving functions by T_0, T_1, and T_c, respectively. Note that $\mathsf{T}_\mathrm{c} = \mathsf{T}_0 \cap \mathsf{T}_1$. It follows from the definitions that $\mathsf{T}_0 = \mathsf{C}_0$, $\mathsf{T}_1 = \mathsf{E}_1$, and $\mathsf{T}_\mathrm{c} = \mathsf{C}_0 \cap \mathsf{E}_1$.

Remark 3.2. The reason why we have introduced multiple notation for the classes $\mathsf{T}_0 = \mathsf{C}_0$ and $\mathsf{T}_1 = \mathsf{E}_1$ is to facilitate writing certain statements in a parameterized form and to make reference, as the case may be, to either the classes C_a ($a \in \{0,1\}$), E_a ($a \in \{0,1\}$), or T_a ($a \in \{0,1\}$).

The *parity* of a Boolean function f, denoted $\operatorname{par}(f)$, is a number, either 0 or 1, which is given by

$$\operatorname{par}(f) := |M_f \setminus \{\emptyset\}| \bmod 2.$$

We call a function *even* or *odd* if its parity is 0 or 1, respectively. We denote by P_0 and P_1 the classes of all even and of all odd functions, respectively. Clearly $\mathsf{P}_0 \cap \mathsf{P}_1 = \emptyset$ and $\mathsf{P}_0 \cup \mathsf{P}_1 = \Omega$.

For $a \in \{0,1\}$, let $\overline{a}$ denote the *negation* of a, that is, $\overline{a} := 1 - a$. A function f is *self-dual* if

$$f(a_1,\dots,a_n) = \overline{f(\overline{a}_1,\dots,\overline{a}_n)}, \quad \text{for all } a_1,\dots,a_n \in \{0,1\}.$$

A function f is *reflexive* (or *self-anti-dual*) if $f(a_1,\dots,a_n) = f(\overline{a}_1,\dots,\overline{a}_n)$ for all $a_1,\dots,a_n \in \{0,1\}$. We denote by S the class of all self-dual functions. Let $\mathsf{S}_\mathrm{c} := \mathsf{S} \cap \mathsf{T}_\mathrm{c}$, the class of constant-preserving self-dual functions.

We also let $\mathsf{L}_0 := \mathsf{L} \cap \mathsf{T}_0$, $\mathsf{L}_1 := \mathsf{L} \cap \mathsf{T}_1$, $\mathsf{LS} := \mathsf{L} \cap \mathsf{S}$, and $\mathsf{L}_\mathrm{c} := \mathsf{L} \cap \mathsf{T}_\mathrm{c}$. It is easy to verify that $\mathsf{L}_0 = \mathsf{L} \cap \mathsf{C}_0$, $\mathsf{L}_1 = (\mathsf{L} \cap \mathsf{P}_0 \cap \mathsf{C}_1) \cup (\mathsf{L} \cap \mathsf{P}_1 \cap \mathsf{C}_0)$, $\mathsf{L}_\mathrm{c} = \mathsf{L} \cap \mathsf{P}_1 \cap \mathsf{C}_0$, and $\mathsf{LS} = \mathsf{L} \cap \mathsf{P}_1$.

It was shown by Post [19] that there are a countably infinite number of clones of Boolean functions. In this paper, we will only need a handful of them, namely the clones Ω, T_0, T_1, T_c, S, S_c, L, L_0, L_1, LS, and L_c that were defined above.

Let f be an n-ary Boolean function. The *characteristic* of a set $S \subseteq [n]$ in f is given by

$$\operatorname{ch}(S,f) := |\{\, A \in M_f \mid S \subsetneq A \,\}| \bmod 2.$$

The *characteristic rank* of f, denoted by $\chi(f)$, is the smallest integer m such that $\operatorname{ch}(S,f) = 0$ for all subsets $S \subseteq [n]$ with $|S| \geq m$. Clearly, $\chi(f) \leq n$ because $\operatorname{ch}([n],f) = 0$. For $k \in \mathbb{N}$, denote by X_k the class of all Boolean functions of characteristic rank at most k. For any $k \in \mathbb{N}$, we have $\mathsf{X}_k \subsetneq \mathsf{X}_{k+1}$. The inclusion is proper, as witnessed by the function $x_1 \dots x_{k+1} \in \mathsf{X}_{k+1} \setminus \mathsf{X}_k$. Moreover, for any $k \in \mathbb{N}$, we have $\mathsf{D}_k \subseteq \mathsf{X}_k$.

Reflexive and self-dual functions have a beautiful characterization in terms of the characteristic rank.

Lemma 3.3 (Selezneva, Bukhman [20, Lemmata 3.1, 3.5])**.**

1. *A Boolean function f is reflexive if and only if $\chi(f) = 0$.*

[2] Strictly speaking, functions of degree at most 1 are *affine* in the sense of linear algebra. We go along with the term *linear* that is common in the context of clone theory and especially in the theory of Boolean functions.

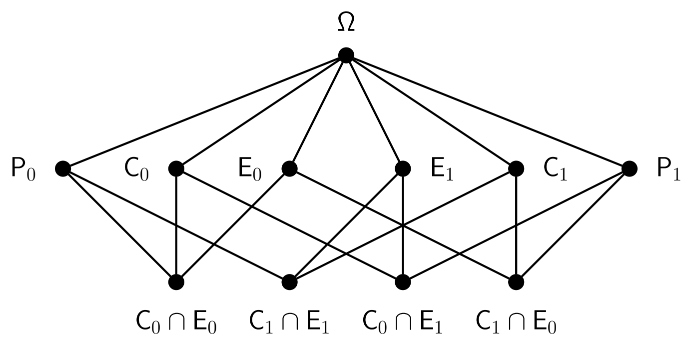

Figure 1: A block of eleven L_c-stable classes.

2. *A Boolean function f is self-dual if and only if $f + x_1$ is reflexive.*
3. *A Boolean function f is self-dual if and only if f is odd and $\chi(f) = 1$.*

In other words, $\mathsf{X}_0 = \mathsf{X}_1 \cap \mathsf{P}_0$ is the class of all reflexive functions, $\mathsf{X}_1 \cap \mathsf{P}_1$ is the class of all self-dual functions, and X_1 is the class of all self-dual or reflexive functions.

3.2 L_c-stable classes

We can now present the main result of the paper, namely, a complete description of the L_c-stable classes or, equivalently, of the linearly definable classes of Boolean functions. Of particular importance is the poset of the eleven classes Ω, P_0, P_1, C_0, C_1, E_0, E_1, $\mathsf{C}_0 \cap \mathsf{E}_0$, $\mathsf{C}_0 \cap \mathsf{E}_1$, $\mathsf{C}_1 \cap \mathsf{C}_0$, $\mathsf{C}_1 \cap \mathsf{E}_1$ that is shown in Figure 1. It is noteworthy that the four minimal classes of this poset are pairwise disjoint, and that the six lower covers of Ω are precisely the unions of the six different pairs of minimal classes.

Theorem 3.4. *The L_c-stable classes are*

Ω,	C_a,	E_a,	P_a,	$\mathsf{C}_a \cap \mathsf{E}_b$,
D_k,	$\mathsf{D}_k \cap \mathsf{C}_a$,	$\mathsf{D}_k \cap \mathsf{E}_a$,	$\mathsf{D}_k \cap \mathsf{P}_a$,	$\mathsf{D}_k \cap \mathsf{C}_a \cap \mathsf{E}_b$,
X_k,	$\mathsf{X}_k \cap \mathsf{C}_a$,	$\mathsf{X}_k \cap \mathsf{E}_a$,	$\mathsf{X}_k \cap \mathsf{P}_a$,	$\mathsf{X}_k \cap \mathsf{C}_a \cap \mathsf{E}_b$,
$\mathsf{D}_i \cap \mathsf{X}_j$,	$\mathsf{D}_i \cap \mathsf{X}_j \cap \mathsf{C}_a$,	$\mathsf{D}_i \cap \mathsf{X}_j \cap \mathsf{E}_a$,	$\mathsf{D}_i \cap \mathsf{X}_j \cap \mathsf{P}_a$,	$\mathsf{D}_i \cap \mathsf{X}_j \cap \mathsf{C}_a \cap \mathsf{E}_b$,
D_0,	$\mathsf{D}_0 \cap \mathsf{C}_a$,	$\emptyset$,		

for $a, b \in \{0, 1\}$ and $i, j, k \in \mathbb{N}_+$ with $i > j \geq 1$.

The lattice of L_c-stable classes is shown in Figure 2. In order to avoid clutter, we have used some shorthand notation. The diagram includes multiple copies of the 11-element poset of Figure 1 (the shaded blocks) connected by thick triple lines. Each thick triple line between a pair of such blocks represents eleven edges, each connecting a vertex of one poset to its corresponding vertex in the other poset. We have labeled in the diagram the meet-irreducible classes, as well as a few other classes of interest; the remaining classes are intersections of the meet-irreducible ones.

The proof of Theorem 3.4 is omitted for space constraints. The proof has two parts. First we need to verify that the classes listed in Theorem 3.4 are L_c-stable. Since intersections of L_c-stable classes are L_c-stable, it suffices to show this for the meet-irreducible classes; this is rather straightforward. Secondly, we need to verify that there are no other L_c-stable classes. This is a more difficult task and can be accomplished by proving that each class K is generated by any subset of K that contains for each proper subclass C of K an element in $K \setminus C$.

4 Stability under clones containing L_c

Using Theorem 3.4 together with Lemma 2.5 it is straightforward to determine the (C_1, C_2)-stable classes for any clones C_1 and C_2 containing L_c. Such classes must occur among the L_c-stable classes by Lemma 2.5, so it is just a matter of deciding which ones are (C_1, C_2)-stable. In particular, we obtain the C-stable classes for every clone C containing L_c, i.e., the clones Ω, T_0, T_1, T_c, S, S_c, L, L_0, L_1, LS and L_c.

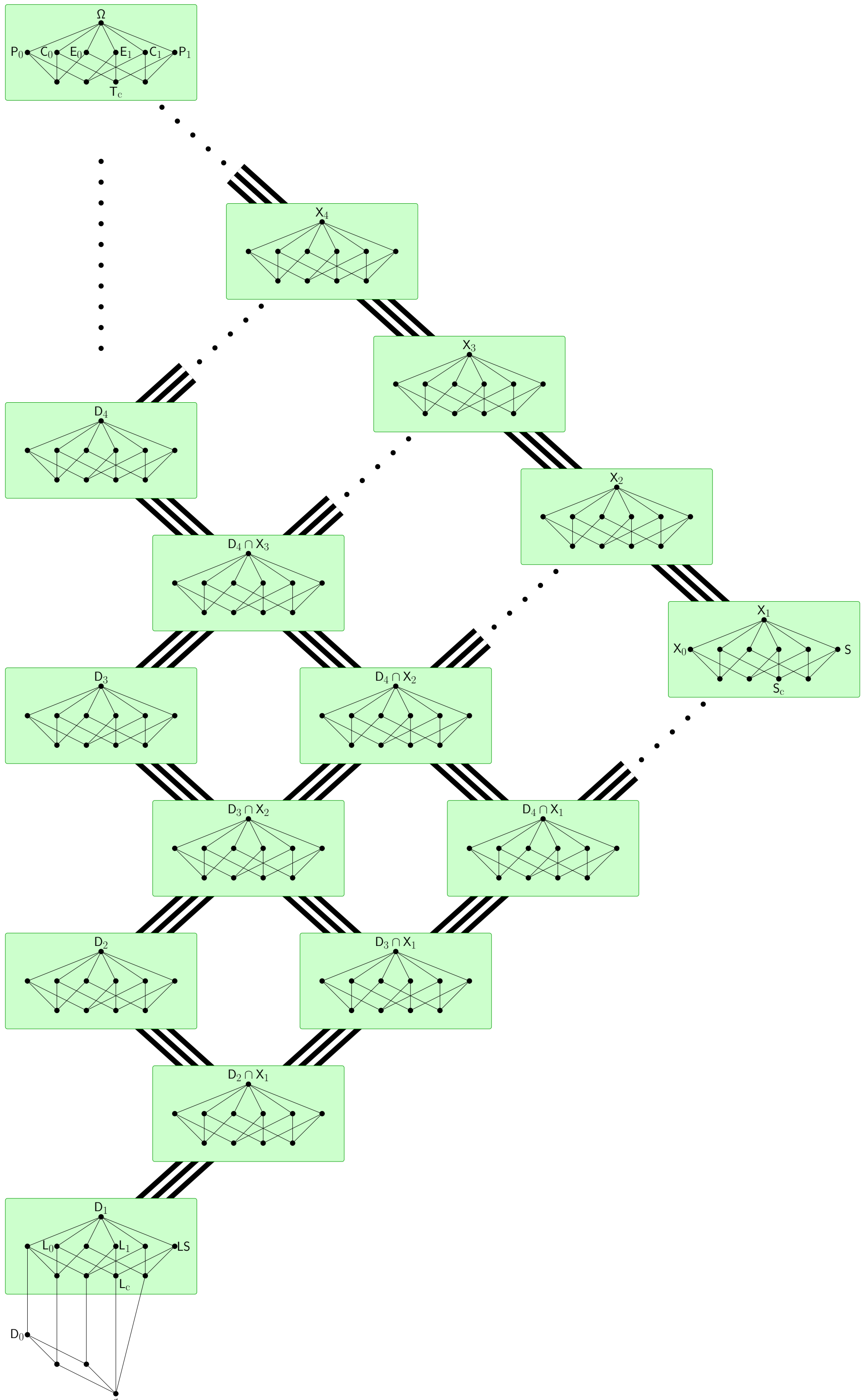

Figure 2: L_c-stable classes.

Theorem 4.1.

(i) *The* L_{c}*-stable classes are* Ω, C_a, E_a, P_a, $\mathsf{C}_a \cap \mathsf{E}_b$, D_k, $\mathsf{D}_k \cap \mathsf{C}_a$, $\mathsf{D}_k \cap \mathsf{E}_a$, $\mathsf{D}_k \cap \mathsf{P}_a$, $\mathsf{D}_k \cap \mathsf{C}_a \cap \mathsf{E}_b$, X_k, $\mathsf{X}_k \cap \mathsf{C}_a$, $\mathsf{X}_k \cap \mathsf{E}_a$, $\mathsf{X}_k \cap \mathsf{P}_a$, $\mathsf{X}_k \cap \mathsf{C}_a \cap \mathsf{E}_b$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \mathsf{C}_a$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \mathsf{E}_a$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \mathsf{P}_a$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \mathsf{C}_a \cap \mathsf{E}_b$, D_0, $\mathsf{D}_0 \cap \mathsf{C}_a$, $\emptyset$, *for* $a, b \in \{0,1\}$ *and* $i, j, k \in \mathbb{N}_+$ *with* $i > j \geq 1$.

(ii) *The* LS*-stable classes are* Ω, X_k, $\mathsf{X}_1 \cap \mathsf{P}_a$, D_k, $\mathsf{D}_1 \cap \mathsf{P}_a$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_1 \cap \mathsf{P}_a$, D_0, $\emptyset$, *for* $a \in \{0,1\}$ *and* $i, j, k \in \mathbb{N}_+$ *with* $i > j \geq 1$.

(iii) *The* L_0*-stable classes are* Ω, C_0, D_k, $\mathsf{D}_k \cap \mathsf{C}_0$, D_0, $\mathsf{D}_0 \cap \mathsf{C}_0$, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(iv) *The* L_1*-stable classes are* Ω, E_1, D_k, $\mathsf{D}_k \cap \mathsf{E}_1$, D_0, $\mathsf{D}_0 \cap \mathsf{C}_1$, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(v) *The* L*-stable classes are* Ω, D_k, D_0, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(vi) *The* S_{c}*-stable classes are* Ω, C_a, E_a, P_a, $\mathsf{C}_a \cap \mathsf{E}_b$, $\mathsf{X}_1 \cap \mathsf{P}_a$, $\mathsf{X}_1 \cap \mathsf{C}_a \cap \mathsf{E}_b$, D_0, $\mathsf{D}_0 \cap \mathsf{C}_a$, $\emptyset$, *for* $a, b \in \{0,1\}$.

(vii) *The* S*-stable classes are* Ω, $\mathsf{X}_1 \cap \mathsf{P}_a$, D_0, $\emptyset$, *for* $a \in \{0,1\}$.

(viii) *The* T_{c}*-stable classes are* Ω, C_a, E_a, P_0, $\mathsf{C}_a \cap \mathsf{E}_b$, D_0, $\mathsf{D}_0 \cap \mathsf{C}_a$, $\emptyset$, *for* $a, b \in \{0,1\}$

(ix) *The* T_0*-stable classes are* Ω, C_0, D_0, $\mathsf{D}_0 \cap \mathsf{C}_0$, $\emptyset$.

(x) *The* T_1*-stable classes are* Ω, E_1, D_0, $\mathsf{D}_0 \cap \mathsf{C}_1$, $\emptyset$.

(xi) *The* Ω*-stable classes are* Ω, D_0, $\emptyset$.

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